Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:4 (2025) 1917–1944
DOI: 10.2140/agt.2025.25.1917
Published: 11 August 2025

The zero stability for the one-row colored $\mathfrak{sl}_3$-Jones polynomial

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The stability of coefficients of colored ($\mathfrak{sl}_2$-)Jones polynomials $\{J^{\mathfrak{sl}_2}_{K,n}(q)\}_n$ was discovered by Dasbach and Lin. This stability is now called the zero stability of $J^{\mathfrak{sl}_2}_{K,n}(q)$. Armond showed zero stability for a B-adequate link by using the linear skein theory based on the Kauffman bracket. We prove the zero stability of one-row colored $\mathfrak{sl}_3$-Jones polynomials $\{J^{\mathfrak{sl}_3}_{K,n}(q)\}_n$ for B-adequate links L with antiparallel twist regions by using the linear skein theory based on Kuperberg's $\mathfrak{sl}_3$-webs. This implies the existence of many q-series obtained from a quantum invariant associated with $\mathfrak{sl}_3$.

57K10, 57K14, 57K16

1 Introduction

The colored $\mathfrak{g}$-Jones polynomial of a knot K is a quantum invariant obtained from an irreducible representation of a simple Lie algebra $\mathfrak{g}$. We will discuss some stability of coefficients of the one-row colored $\mathfrak{sl}_3$-Jones polynomials $\{J^{\mathfrak{sl}_3}_{K,n}(q) \in \mathbb{Z}[q^{\pm 1/2}] \mid n \in \mathbb{N}\}$, which is a quantum invariant of K associated with irreducible representations of $\mathfrak{sl}_3$ corresponding to the one-row Yang diagram (n). This kind of stability for the colored ($\mathfrak{sl}_2$-)Jones polynomials was discovered by Dasbach and Lin [5; 6]. They showed that some leading coefficients, concerning the degree of q, of $\{J^{\mathfrak{sl}_2}_{K,n}(q)\}_n$ are independent of the colorings n (where $n+1$ is the dimension of an irreducible representation) if a knot K is alternating. They also conjectured that the first n coefficients of $J^{\mathfrak{sl}_2}_{K,N}(q)$ are constant for all N greater than n if K is alternating. Armond [1] proved this conjecture for a larger class of links, called adequate links, which contain alternating links. Independently, Garoufalidis and Lê [9] proved more general stability, called *k-stability*, for alternating links, where k is a nonnegative integer. In the sense of Garoufalidis and Lê, the stability proved in [1] corresponds to the zero stability. The k-stability also ensures the existence of a q-series called the k-limit, which is closely related to quantum modular forms. The 0-limit is also known as the *tail* of K.

Definition 1.1 Let $\hat{J}^{\mathfrak{sl}_2}_{K,n}(q) := \pm q^d J^{\mathfrak{sl}_2}_{K,n}(q) = a_0 + \sum_{i=1}^{\infty} a_i q^i$ be a normalization of the colored Jones polynomial $J^{\mathfrak{sl}_2}_{K,n}(q)$ of a knot K, where the sign is chosen so that a_0 is positive. The *tail* of K is a q-series $\Phi_K(q) \in \mathbb{Z}[\![q]\!]$ satisfying

$$\Phi_K(q) - \hat{J}^{\mathfrak{sl}_2}_{K,n}(q) \in q^{n+1}\mathbb{Z}[\![q]\!]$$

for any positive integer n.

Note that the integrality theorem for the colored Jones polynomial proved by Lê [20] claims that the coefficients of $\widehat{J}_{K,n}(q)$ become integral, and therefore its tail $\Phi_K(q)$ belongs to $\mathbb{Z}[\![q]\!]$. Armond and Dasbach [3] showed that the tail of an adequate knot is determined by its reduced B-graph.[1] A similar result was obtained by Garoufalidis, Norin and Vuong [11]. They stated that the first three coefficients of $\Phi_K(q)$ of an alternating link K are described in terms of its reduced Tait graph. From these results, we can see that the tail is not useful in distinguishing links. However, tails of knots and links give us interesting q-series related to quantum modular forms. For example, Garoufalidis and Lê [9] showed that tails of alternating links are described as a generalization of Nahm sums. In particular, the tail of a $(2,m)$-torus link is the (false) theta series. In work of Armond and Dasbach [2], Hajij [13] and Yuasa [25], an Andrews–Gordon-type identity for the (false) theta series was derived from two explicit formulas for the tail of a $(2,m)$-torus link. Explicit formulas for tails of other knots and links have been studied by Garoufalidis and Lê [9], Elhamdadi and Hajij [7], Keilthy and Osburn [15] and Beirne and Osburn [4], and for quantum spin networks by Hajij [13].

Our goal is to develop a study of the stability and tails for $J^{\mathfrak{sl}_2}_{K,n}(q)$ to quantum invariants $J^{\mathfrak{g}}_{K,\lambda}(q)$ associated with a higher-rank simple Lie algebra $\mathfrak{g}$. Many problems arise when we consider higher-rank cases. For example, we have to choose a sequence of irreducible representations to consider the stability because the colored $\mathfrak{g}$-Jones polynomial of a knot is parametrized by dominant weights. Moreover, the explicit computation of the colored $\mathfrak{g}$-Jones polynomials of a given knot is much more difficult than in the $\mathfrak{sl}_2$ case.

The aim of this paper is to show zero stability of the one-row colored $\mathfrak{sl}_3$-Jones polynomial $\{J^{\mathfrak{sl}_3}_{K,n}(q)\}_n$ of a B-adequate link K with antiparallel twist regions. The one-row coloring n for K means that all components of K are colored by the irreducible representation of the highest weight $n\varpi_1$ (or we write it as $(n,0)$), where $\{\varpi_i\}_{i=1,2}$ correspond to the fundamental weights of $\mathfrak{sl}_3$. There are some studies on the explicit computation of the colored $\mathfrak{sl}_3$ Jones polynomial: for the trefoil knot by Lawrence [19], for the $(2,2m{+}1)$- and $(4,5)$-torus knots with general coloring by Garoufalidis, Morton and Vuong [10; 12], for 2-bridge links with one-row coloring by Yuasa [24], and for pretzel links with one-row coloring by Kawasoe [14]. These explicit formulas give tails of the colored $\mathfrak{sl}_3$-Jones polynomial of some links; see Garoufalidis and Vuong [12] and Yuasa [25; 26]. For the λ-colored $\mathfrak{g}$-Jones polynomial of the (a,b)-torus knot when $\mathfrak{g}$ is a simple Lie algebra of rank 2, Garoufalidis and Vuong [12] proved the k-stability for any k. They used the formula of Rosso and Jones in [22] to prove it. Here we prove the zero stability of the one-row colored $\mathfrak{sl}_3$-Jones polynomial for any antiparallel B-adequate link using the linear skein theory for $\mathfrak{sl}_3$ developed by Kuperberg [18]. Our proof is inspired by work of Armond [1] using the Kauffman bracket.

Theorem 1 (zero stability for the one-row colored $\mathfrak{sl}_3$-Jones polynomial; see Theorem 4.16) *For any antiparallel B-adequate link L, there exists $\Phi^{\mathfrak{sl}_3}_L(q)$ in $\mathbb{Z}[\![q]\!]$ such that*

$$\Phi^{\mathfrak{sl}_3}_L(q) - \widehat{J}^{\mathfrak{sl}_3}_{L,n}(q) \in q^{n+1}\mathbb{Z}[\![q]\!].$$

[1] They consider A-graphs. However, this corresponds to B-graphs in our convention. That is, our q is q^{-1} in [3].

An antiparallel B-adequate link is an oriented link whose representative is a B-adequate link diagram with only antiparallel twist regions; see Definition 4.1.

This result and its proof are an extension of the work on the zero stability of colored $\mathfrak{sl}_2$-Jones polynomials in [1] to $\mathfrak{sl}_3$. We will discuss the zero stability for general B-adequate links in a forthcoming paper.

This paper is organized as follows. In Section 2, we introduce the $\mathfrak{sl}_3$ version of the linear skein theory and review properties of $\mathfrak{sl}_3$-webs and $\mathfrak{sl}_3$-clasps. In Section 3, we discuss a lower bound on the minimum degree of a clasped $\mathfrak{sl}_3$-web. In Section 4, we prove the zero stability of the one-row colored $\mathfrak{sl}_3$-Jones polynomials by calculating clasped $\mathfrak{sl}_3$-webs. In the appendix, we prove some new formulas for the clasped $\mathfrak{sl}_3$-webs used in this paper.

Acknowledgments

The author gratefully thanks the referees for their careful reading, helpful comments and great efforts to improve this paper. This work was supported by Grant-in-Aid for Early-Career Scientists grants JP19K14528, JP23K12972.

2 $\mathfrak{sl}_3$-webs and $\mathfrak{sl}_3$-clasps

We mainly work with a space of $\mathfrak{sl}_3$-webs, which is a linear combination of oriented planar trivalent graphs with coefficients in $\mathcal{R} = \mathbb{Z}((q^{1/6}))$. Let us introduce some useful symbols for elements in $\mathcal{R}$. We define

- a *quantum integer* by $[n] := (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$ for any nonnegative integer $n \in \mathbb{Z}_{\geq 0}$, and
- a *quantum binomial coefficient* by $\begin{bmatrix} n \\ k \end{bmatrix} := [n]!/([k]![n-k]!)$ for $0 \leq k \leq n$, and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $k > n$, where $[n]! := [n][n-1]\cdots[1]$.

Let us define $\mathfrak{sl}_3$-web spaces based on [18]. We consider a surface Σ equipped with signed marked points (P, s), where $P \subset \partial\Sigma$ is a finite set and $s\colon P \to \{+,-\}$ a map.

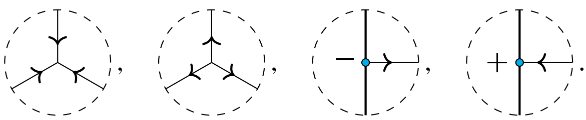
A *tangled trivalent graph* on Σ is an immersion of a directed graph G into Σ satisfying:

(1) The valency of a vertex of G is 1 or 3.

(2) All crossing points are transversal double points of two edges with under/over-passing information.

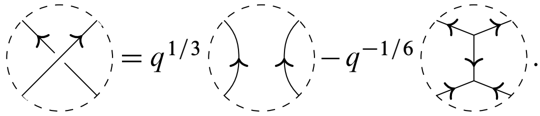
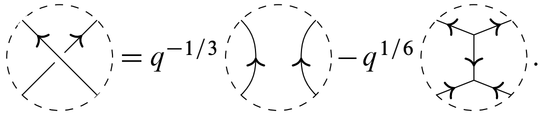
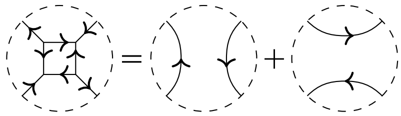
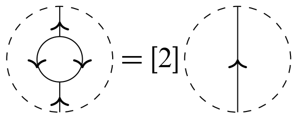
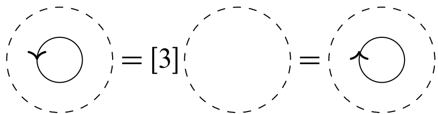
(3) The set of univalent vertices of G coincides with P.

(4) A neighborhood of a vertex in Σ is one of



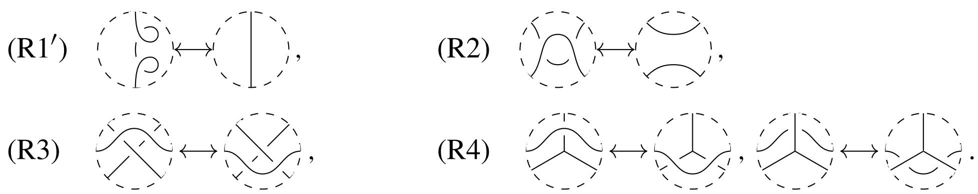
A tangled trivalent graph is *flat* if it has no crossings. An *elliptic face* of a flat trivalent graph G is a 0-gon (ie a disk), 2-gon or 4-gon in the set of connected components of $\Sigma \setminus G$ which does not touch the boundary of Σ.

Definition 2.1 ($\mathfrak{sl}_3$-web spaces [18]) Let $\mathcal{G}(\boldsymbol{s};\Sigma)$ be the set of the boundary-fixing isotopy classes of tangled trivalent graphs on Σ. The $\mathfrak{sl}_3$*-web space* $\mathcal{W}(\boldsymbol{s};\Sigma)$ is the quotient of the $\mathcal{R}$-module spanned by $\mathcal{G}(\boldsymbol{s};\Sigma)$ modulo the following $\mathfrak{sl}_3$*-skein relations*:

- 
- 
-  (the 4-gon relation).
-  (the 2-gon relation).
-  (the trivial loop relation).

An $\mathfrak{sl}_3$*-web* is an element in $\mathcal{W}(\boldsymbol{s};\Sigma)$ and a *basis web* is an $\mathfrak{sl}_3$-web represented by a graph in $\mathcal{G}(\boldsymbol{s};\Sigma)$ with no elliptic faces.

The $\mathfrak{sl}_3$-skein relations realize the *Reidemeister moves* (R1′) and (R2)–(R4):



The above means that $\mathfrak{sl}_3$-webs on the left and right sides represent the same element in an $\mathfrak{sl}_3$-web space for any choice of orientation of the edges.

It is easy to see that any tangled trivalent graph decomposes into a sum of basis webs by using the $\mathfrak{sl}_3$-skein relations. In fact, the set of basis webs consists of a basis of the $\mathfrak{sl}_3$-web space:

Theorem 2.2 [18; 23] *The set of basis webs on a surface Σ with signed marked points $\boldsymbol{s}\colon P \to \{+,-\}$ is a basis of $\mathcal{W}(\boldsymbol{s};\Sigma)$ as a $\mathbb{Z}[q^{\pm 1/6}]$-module.*

In some cases, one can give the set of basis webs via an argument concerning the Euler characteristic.

Example 2.3 Let D be a disk with a basepoint $* \in \partial D$. We identify signed marked points on $\partial D \setminus \{*\}$ with a sequence of signs. Then the following isomorphisms hold for any $\epsilon \in \{+, -\}$:

(1) $\mathcal{W}(\varnothing; D)$ of a disk D with no marked points is isomorphic to a free $\mathcal{R}$-module spanned by the empty diagram $\varnothing$.

(2) $\mathcal{W}(\epsilon; D) = 0$.

(3) $\mathcal{W}(\epsilon\epsilon; D) = 0$.

(4) $\mathcal{W}(\epsilon\bar{\epsilon}; D)$ is a free $\mathcal{R}$-module spanned by an oriented simple arc.

(5) $\mathcal{W}(\epsilon\epsilon\bar{\epsilon}; D) = 0$.

(6) $\mathcal{W}(\epsilon\epsilon\epsilon; D)$ is a free $\mathcal{R}$-module spanned by a tripod with a sink or source vertex.

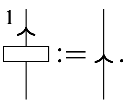
In the above, $\bar{\epsilon}$ means the opposite sign of ϵ.

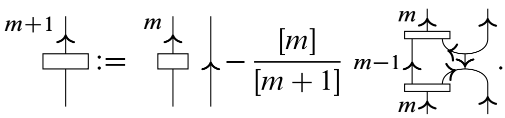
We review a diagrammatic definition of an $\mathfrak{sl}_3$-clasp introduced in [18; 21; 17; 26] and note some useful properties. The $\mathfrak{sl}_3$-clasp plays a similar role to the Jones–Wenzl projector in the Kauffman bracket skein theory.

In what follows, we will mainly consider tangled trivalent graphs or $\mathfrak{sl}_3$-webs in a rectangle $D = [0,1] \times [0,1]$. We assume that the set of marked points lies in the top edge $I_1 = [0,1] \times \{1\}$ and the bottom edge $I_0 = [0,1] \times \{0\}$, and a basepoint $*$ at $(0,0)$. In this situation, the set of marked points is divided into $P^{(0)}$ and $P^{(1)}$, where $P^{(j)} := P \cap I_j$, and we denote the assignment of signs by $\boldsymbol{s}^{(j)} \colon P^{(j)} \to \{+,-\}$ for $j = 0, 1$. One can identify $\boldsymbol{s}^{(j)}$ with a sequence of signs on $[0,1] \times \{j\}$ arranged from 0 to 1. We abbreviate $\mathcal{G}(\boldsymbol{s}^{(0)} \sqcup \boldsymbol{s}^{(1)}; D)$ and $\mathcal{W}(\boldsymbol{s}^{(0)} \sqcup \boldsymbol{s}^{(1)}; D)$ by $\mathcal{G}(\boldsymbol{s}^{(0)}, \bar{\boldsymbol{s}}^{(1)})$ and $\mathcal{TL}(\boldsymbol{s}^{(0)}, \bar{\boldsymbol{s}}^{(1)})$, respectively, where $\bar{\boldsymbol{s}}^{(1)}$ is a sequence consisting of the opposite signs of $\boldsymbol{s}^{(1)}$.[2] When we describe diagrams representing $\mathfrak{sl}_3$-webs in D, we omit from the notation the signs, the basepoint, and the boundary of D. The composition $\mathcal{TL}(\boldsymbol{s}_1, \boldsymbol{s}_2) \otimes \mathcal{TL}(\boldsymbol{s}_0, \boldsymbol{s}_1) \to \mathcal{TL}(\boldsymbol{s}_0, \boldsymbol{s}_2)$ is defined by gluing the top side of an $\mathfrak{sl}_3$-web in $\mathcal{TL}(\boldsymbol{s}_0, \boldsymbol{s}_1)$ and the bottom side of an $\mathfrak{sl}_3$-web in $\mathcal{TL}(\boldsymbol{s}_1, \boldsymbol{s}_2)$.

We first define the $\mathfrak{sl}_3$-clasp in $\mathcal{TL}(-^m, -^m)$.

Definition 2.4 (one-row colored $\mathfrak{sl}_3$-clasps) The $\mathfrak{sl}_3$-clasp JW_{-^m} described by a white box with $m \in \mathbb{Z}_{>0}$ is defined as follows:

(1) $\mathrm{JW}_{-} =$ 

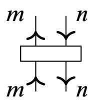
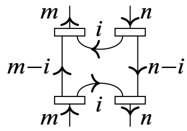
(2) $\mathrm{JW}_{-^{m+1}} =$ 

In the above, an edge labeled by m represents the m-parallelization of the edge. JW_{+^m} is defined by the same diagram with oppositely directed edges.

[2] We take the opposite sign $\bar{\boldsymbol{s}}^{(1)}$ to be consistent with the composition.

We next introduce the $\mathfrak{sl}_3$-clasp in $\mathcal{TL}(-^m+^n, -^m+^n)$.

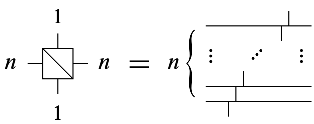
Definition 2.5 (two-row colored $\mathfrak{sl}_3$-clasps) We define

$$\mathrm{JW}_{-^m+^n} =$$  $$= \sum_{i=0}^{\min\{m,n\}} (-1)^i \frac{\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} n \\ i \end{bmatrix}}{\begin{bmatrix} m+n+1 \\ i \end{bmatrix}}$$  .

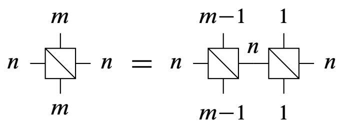
One can define $\mathrm{JW}_{+^m-^n}$ in the same way.

For convenience in the computation of $\mathfrak{sl}_3$-webs, we will introduce "stair-step" and "triangle" webs in Definitions 2.6 and 2.7, which were also used in [16; 17; 24; 8]. In these definitions, the orientation of edges of $\mathfrak{sl}_3$-webs is not explicitly given because it is uniquely determined according to your choice of the orientation of an edge around the box.

Definition 2.6 For positive integers n and m, a *stair-step web* is defined recursively by

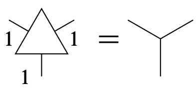


and

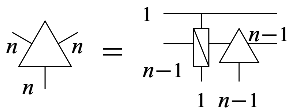


for $m > 1$.

Definition 2.7 For positive integers n, define the *triangle webs* by

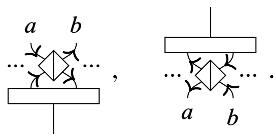


and

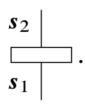


for $n > 1$.

Definition 2.8 (general type of the $\mathfrak{sl}_3$-clasps) Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be two sequences of signs which consist of m pluses and n minuses. We define an $\mathfrak{sl}_3$-clasp in $\mathcal{TL}(\boldsymbol{s}_1, \bar{\boldsymbol{s}}_2)$ by gluing stair-step webs to the top and the bottom of $\mathrm{JW}_{-^m+^n}$ as follows:



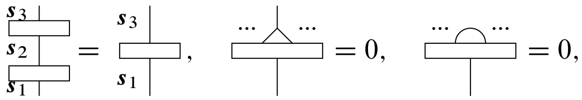
Repeating these operations, we can arbitrarily exchange signs in sequences on the top and bottom of the disk, respectively, and we obtain an $\mathfrak{sl}_3$-web in $\mathcal{TL}(\boldsymbol{s}_1, \bar{\boldsymbol{s}}_2)$. We denote it by $\mathrm{JW}_{\boldsymbol{s}_1}^{\bar{\boldsymbol{s}}_2}$ and depict it as

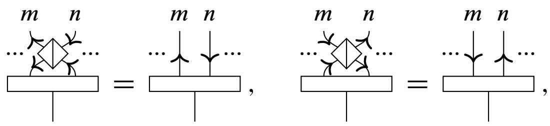


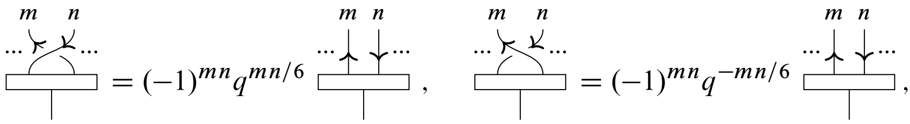
The resulting $\mathfrak{sl}_3$-clasp is independent of choice of sequences of stair-step webs; see [26]. Thus, $\mathrm{JW}_{\boldsymbol{s}_1}^{\bar{\boldsymbol{s}}_2}$ is uniquely determined by $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

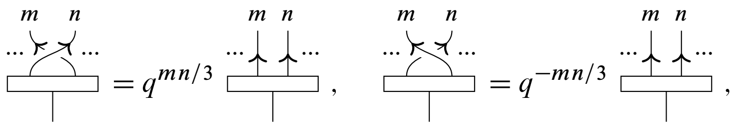
One can prove the following useful formulas for $\mathfrak{sl}_3$-clasps by straightforward computation:

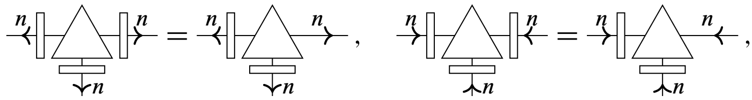
Lemma 2.9 *Let s_1, s_2 and s_3 be sequences of signs. An arc labeled by a positive integer m (resp. n) denotes m-parallelization (resp. n-parallelization) of the arc.*

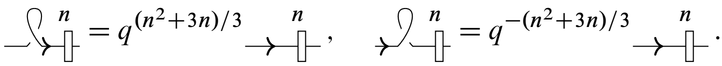
(1) 

(2) 

(3) 

(4) 

(5) 

(6) 

Proof One can prove (1)–(6) by induction on labels and the skein relations. See [24; 26], for example. □

We give a definition of the one-row colored $\mathfrak{sl}_3$-Jones polynomial of oriented framed links via an $\mathfrak{sl}_3$-web. First we introduce a normalization of a Laurent series by shifting the q-degree and changing the sign.

Definition 2.10 (minimum degree) We define the *minimum degree* $\mathrm{d}_*\colon \mathcal{R} = \mathbb{Z}((q^{1/6})) \to \frac{1}{6}\mathbb{Z} \cup \infty$ by $\mathrm{d}_*(f(q)) := \frac{1}{6}d$ for a nonzero series $f(q) = \sum_{i=d}^{\infty} a_i q^{i/6}$ in $\mathbb{Z}((q^{1/6}))$ such that $a_d \neq 0$. For the zero polynomial, we define its minimum degree as ∞. We also define a normalization $\hat{f}(q)$ of a nonzero Laurent series $f(q)$ as

$$\hat{f}(q) := \pm q^{-\mathrm{d}_*(f(q))} f(q) = \pm \sum_{i=0}^{\infty} a_{i+d} q^{i/6} \in \mathbb{Z}[\![q^{1/6}]\!].$$

In the above, we choose the sign so that the constant term $\pm a_d$ is positive.

We note some properties of the minimum degree and useful examples:

Lemma 2.11 *For any $f(q), g(q) \in \mathcal{R}$:*

(1) $\mathrm{d}_*(f(q) + g(q)) \geq \min\{\mathrm{d}_*(f(q)), \mathrm{d}_*(g(q))\}$.

(2) $\mathrm{d}_*(f(q)g(q)) = \mathrm{d}_*(f(q)) + \mathrm{d}_*(g(q))$.

The equality in (1) *holds if and only if $\mathrm{d}_*(f(q)) \neq \mathrm{d}_*(g(q))$ or $\mathrm{d}_*(f(q)) = \mathrm{d}_*(g(q)) =: d$ with $a_d + b_d \neq 0$, where $f(q) = \sum_{i=d}^{\infty} a_i q^{i/6}$ and $g(q) = \sum_{i=d}^{\infty} b_i q^{i/6}$.*

Example 2.12 For any positive integer n and $1 \le k \le n$,

$$\mathrm{d}_*([n]) = -\tfrac{1}{2}(n-1), \quad \mathrm{d}_*([n]^{-1}) = \tfrac{1}{2}(n-1), \quad \mathrm{d}_*\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = -\tfrac{1}{2}k(n-k), \quad \mathrm{d}_*\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1}\right) = \tfrac{1}{2}k(n-k).$$

These follow from the identity $(1-q^m)^{-1} = 1 + q^m + q^{2m} + \cdots \in \mathcal{R} = \mathbb{Z}((q^{1/6}))$ and Lemma 2.11.

Definition 2.13 Let L be a link diagram of a framed link whose framing is given by the blackboard framing. One can replace arcs of the link diagram with n parallelized arcs and put white boxes on the n parallelized arcs. The resulting diagram, denoted by $L^{(n)}$, represents an $\mathfrak{sl}_3$-web in a disk D with no marked points.[3] The *one-row colored* $\mathfrak{sl}_3$*-Jones polynomial* $J^{\mathfrak{sl}_3}_{L,n}(q)$ with $(n,0)$-coloring (or n boxes) is defined by $L^{(n)} = J^{\mathfrak{sl}_3}_{L,n}(q)\varnothing$. We also define a variation of the one-row colored $\mathfrak{sl}_3$-Jones polynomial as $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$, according to Definition 2.10.

Remark 2.14

- Skein relations realize the Reidemeister moves (R1′)–(R4) for arcs with one-row colored clasps because clasped arcs are expressed as linear combinations of $\mathfrak{sl}_3$-webs. Hence, $J^{\mathfrak{sl}_3}_{L,n}(q)$ is an invariant of framed links.
- The choice of framing of L appears as multiplication by $\pm q^{\bullet}$; see Lemma 2.9(6). This difference is ignored in the normalization $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$; thus, it is an invariant of links.
- Lê [20] showed the integrality theorem for a quantum $\mathfrak{g}$ invariant of links. It says that $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ belongs to $\mathbb{Z}[q]$.

We will discuss *zero stability* of $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ for a certain class of links in the following sections. Let us recall the definition of zero stability and tails of the one-row colored $\mathfrak{sl}_3$-Jones polynomials.

Definition 2.15 (one-row colored $\mathfrak{sl}_3$-tail) The one-row colored $\mathfrak{sl}_3$-Jones polynomial $\{\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)\}_n$ of a link L is *zero stable* if there exists a formal power series $\Phi^{\mathfrak{sl}_3}_L(q)$ in $\mathbb{Z}[\![q]\!]$ such that

$$\Phi^{\mathfrak{sl}_3}_L(q) - \hat{J}^{\mathfrak{sl}_3}_{L,n}(q) \in q^{n+1}\mathbb{Z}[\![q]\!]$$

for all $n \ge 1$. We call $\Phi^{\mathfrak{sl}_3}_L(q)$ the *one-row colored* $\mathfrak{sl}_3$*-tail* of L, or simply the $\mathfrak{sl}_3$*-tail* of L when $\{\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)\}_n$ is zero stable.

3 The minimum degree of clasped $\mathfrak{sl}_3$-webs

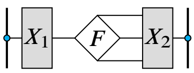
We will prove the existence of the $\mathfrak{sl}_3$-tail of the one-row colored $\mathfrak{sl}_3$-Jones polynomial by developing an $\mathfrak{sl}_3$ analog of Armond's argument [1] using the Kauffman bracket. In this section, we will discuss a lower bound on the minimum degree of a clasped $\mathfrak{sl}_3$-web with no crossings in a disk.

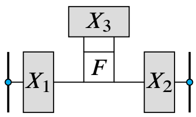
First we prepare a lemma that studies the isomorphisms in Example 2.3 in detail:

[3] Such an $\mathfrak{sl}_3$-web space $\mathcal{W}(\varnothing; D)$ is spanned by the empty diagram $\varnothing$; see Example 2.3.

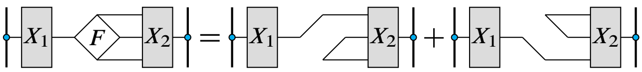
Lemma 3.1 *Let $G \in \mathcal{G}(-+; D)$ be a connected flat trivalent graph in a disk D with two marked points such that $G \neq 0$. There exists a sequence, composed only of 2- and 4-gon relations, that reduces G to an oriented simple arc γ without increasing the number of connected components of intermediate graphs.*

Proof Let us assume that G has at least one elliptic face and $G \neq 0$. We only have to attend to the 4-gon relation because the 2-gon relation does not change the number of connected components of a graph. Let us prove the claim by induction on the number $v(G)$ of vertices. A connected flat trivalent graph G with $v(G) = 2$ has to be the diagram in the left-hand side of the 2-gon relation in Definition 2.1, and G with $v(G) = 3$ does not exist because a trivalent vertex is a sink or source. Let G be a connected flat trivalent graph with $v(G) \geq 4$ and assume that it has at least one internal 4-gon F. Example 2.3(2) claims that we cannot describe a circle in D which intersects G at a single edge. Example 2.3(5) claims that no circle intersects two incoming (resp. outgoing) edges and one outgoing (resp. incoming) edge of G. This fact requires that four edges incident to corners of F connect to other parts of G like either

(i) 

or (ii) 

where the subgraphs X_1, X_2, and X_3 of G are connected. We apply the 4-gon relation at F in (i):



Graphs after applying the 4-gon relation are divided into the left and right parts containing X_1 and X_2, respectively, by cutting along a vertical line in D. Right and left subgraphs are considered $\mathfrak{sl}_3$-web in a disk with two marked points whose number of vertices is smaller than $v(G)$. The right subgraph containing X_2 is connected due to Example 2.3(2). Hence these subgraphs satisfy the induction hypothesis. For case (ii), a sequence of the 2- and 4-gon relations changes X_3 into an arc without increasing the number of components because $v(X_3) < v(G)$. Then one can obtain a graph consisting of X_1 and X_2 by applying the 2-gon relation twice. The resulting graph also satisfies the induction hypothesis. □

Proposition 3.2 *Let G be a flat trivalent graph in a disk with no marked points, and we identify G with its value in $\mathcal{R}$ (see Example 2.3). Then*

$$\mathrm{d}_*(G) \geq -\tfrac{1}{4}v(G) - c(G),$$

where $v(G)$ is the number of trivalent vertices of G and $c(G)$ is the number of connected components of G. Moreover, $\mathrm{d}_*(G) = -c(G)$ holds when $v(G) = 0$.

Proof We first prove $\mathrm{d}_*(G) = -c(G)$ when $v(G) = 0$. If G has no trivalent vertices, then it consists only of loop components. By using an innermost argument and the trivial loop relation in Definition 2.1, it is easy to see that $G = [3]^{c(G)}$. We obtain $\mathrm{d}_*(G) = c(G)\mathrm{d}_*([3]) = -c(G)$ because $[3] = q + 1 + q^{-1}$. Let us consider when $G = \bigsqcup_{i=1}^{c(G)} G_i$ is a nontrivial flat trivalent graph with $v(G) > 0$, where the G_i are connected components of G. Choose a point p_i on the outermost edge of G_i and a small interval I_{p_i} for each $i = 1, 2, \ldots, c(G)$. Then one can take disks $\{D_i\}_i^{c(G)}$ with two marked points all $i = 1, 2, \ldots, c(G)$

such that $G_i \setminus \mathrm{int}(I_{p_i}) \subset D_i$, ∂I_{p_i} is precisely the marked points of D_i, and $D_i \cap D_j = \varnothing$ for $i \neq j$. Apply Lemma 3.1 to $G_i \cap D_i$ for all $i \in \{1, 2, \dots, c(G)\}$ to obtain a disjoint union $\Gamma := \bigsqcup_{i=1}^{c(G)} \gamma_i$ of simple loops from G. Each component γ_i is obtained from G_i by a sequence of 2- and 4-gon relations preserving the number of connected components. Let $G = G' + G''$ be a 4-gon relation appearing in the above sequence; we can assume that $G' \neq 0$ and $\mathrm{d}_*(G') \leq \mathrm{d}_*(G'')$ without loss of generality. Then

- $\mathrm{d}_*(G) \geq \min\{\mathrm{d}_*(G'), \mathrm{d}_*(G'')\} = \mathrm{d}_*(G')$,
- $v(G) = v(G') + 4$, and
- $c(G) = c(G')$.

Thus,

$$\mathrm{d}_*(G) + \tfrac{1}{4}v(G) + c(G) \geq \mathrm{d}_*(G') + \tfrac{1}{4}v(G') + c(G') + 1.$$

Suppose instead that G' is obtained by a 2-gon relation, that is, $G = [2]G'$. Then

- $\mathrm{d}_*(G) = \mathrm{d}_*(G') - \frac{1}{2}$,
- $v(G) = v(G') + 2$, and
- $c(G) = c(G')$.

Thus,

$$\mathrm{d}_*(G) + \tfrac{1}{4}v(G) + c(G) = \mathrm{d}_*(G') + \tfrac{1}{4}v(G') + c(G').$$

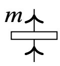
As mentioned above, we can choose a reduction sequence from G to Γ so that flat trivalent graphs in this sequence satisfy the above inequality for the minimum degree. Note that $\mathrm{d}_*(\Gamma) = -c(\Gamma)$ because $v(\Gamma) = 0$. Hence, G and Γ should satisfy

$$\mathrm{d}_*(G) + \tfrac{1}{4}v(G) + c(G) \geq \mathrm{d}_*(\Gamma) + \tfrac{1}{4}v(\Gamma) + c(\Gamma) = 0. \qquad \square$$

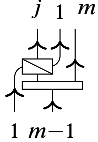
Next we give a lower bound on the minimum degree of a flat trivalent graph with $\mathfrak{sl}_3$-clasps. Let us consider the minimum degree of coefficients appearing in expansion formulas of $\mathfrak{sl}_3$-clasps.

Lemma 3.3 (the single clasp expansion formula [17, Proposition 3.1]) *For any positive integer m,*

$$\mathrm{JW}_{-m} =$$



$$= \sum_{j=0}^{m-1} f_j^{(m)}(q)$$



,

where $f_j^{(m)}(q) := (-1)^j [m-j]/[m]$.

One can obtain the following lemma from the single clasp expansion formula and induction on m:

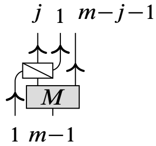
Lemma 3.4 *The one-row colored $\mathfrak{sl}_3$-clasp has an expansion*

$$\mathrm{JW}_{-m} = \sum_M f_M(q) M$$

with $\mathrm{d}_*(f_M(q)) = \frac{1}{4}v(M)$, where the sum runs over finitely many flat trivalent graphs M, and $v(M)$ is the number of trivalent vertices in M. We remark that M may contain 4-gons or 2-gons.

Proof It is obvious that the claim is true for $m=1,2$. We prove it by induction on m. A flat trivalent graph in the right-hand side of the single clasp expansion of JW_{-^m} has a stair-step web with $2j$ vertices and $\mathrm{JW}_{-^{m-1}}$. We know that $\mathrm{d}_*(f_j^{(m)}(q))=\frac{1}{2}j$ by Example 2.12. Lemma 3.3 and the induction hypothesis derive an expansion

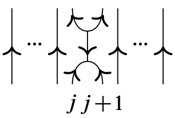
$$\mathrm{JW}_{-^m}=\sum_{j=0}^{m-1}\sum_{M} f_j^{(m)}(q)\, f_M(q)$$



where $\mathrm{d}_*(f_M(q))=\frac{1}{4}v(M)$. The flat trivalent graph in the right-hand side has $2j+v(M)$ vertices, and $\mathrm{d}_*(f_j^{(m)}(q)\,f_M(q))=\mathrm{d}_*(f_j^{(m)}(q))+\mathrm{d}_*(f_M(q))=\frac{1}{4}(2j+v(M))$ by Lemma 2.11. This expansion satisfies the condition of our claim. □

Remark 3.5 The proof of Lemma 3.4 can be used to show that M constructed by composing webs

$$I_j :=$$



$$\in \mathcal{TL}(-^m,-^m)$$

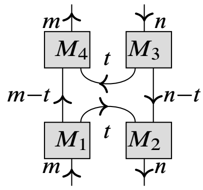
for $j=1,2,\dots,m-1$. Labels j and $j+1$ in the bottom mean the j^{th} and $(j+1)^{\text{st}}$ marked points, respectively.

Lemma 3.6 *The two-row colored $\mathfrak{sl}_3$-clasp has the expansion*

$$\mathrm{JW}_{-^m+^n}=\sum_{t=0}^{\min\{m,n\}}\sum_{M_1,M_2,M_3,M_4} f_{(m,n;t)}(M_1,M_2,M_3,M_4;q)\begin{vmatrix} M_4 & M_3\\ M_1 & M_2\end{vmatrix}_{(m,n;t)}$$

with $\mathrm{d}_*(f_{(m,n;t)}(M_1,M_2,M_3,M_4;q))=\frac{1}{2}t(t+1)+\sum_{i=1}^{4}\frac{1}{4}v(M_i)$, *where*

$$\begin{vmatrix} M_4 & M_3\\ M_1 & M_2\end{vmatrix}_{(m,n;t)} :=$$



Proof Apply Lemma 3.4 to each one-row colored $\mathfrak{sl}_3$-clasp in the right-hand side of Definition 2.5. Then we obtain an expansion as in the statement such that

$$f_{(m,n;t)}(M_1,M_2,M_3,M_4;q):=(-1)^t\frac{\begin{bmatrix} m\\ t\end{bmatrix}\begin{bmatrix} n\\ t\end{bmatrix}}{\begin{bmatrix} m+n+1\\ t\end{bmatrix}} f_{M_1}(q)\,f_{M_2}(q)\,f_{M_3}(q)\,f_{M_4}(q).$$

One can calculate the minimum degree as

$$\begin{aligned}\mathrm{d}_*(f_{(m,n;t)}(M_1,M_2,M_3,M_4;q))&=\mathrm{d}_*\left((-1)^t\frac{\begin{bmatrix} m\\ t\end{bmatrix}\begin{bmatrix} n\\ t\end{bmatrix}}{\begin{bmatrix} m+n+1\\ t\end{bmatrix}}\right)+\sum_{i=1}^{4}\mathrm{d}_*(f_{M_i}(q))\\ &=\tfrac{1}{2}t(t+1)+\sum_{i=1}^{4}\tfrac{1}{4}v(M_i).\end{aligned}$$ □

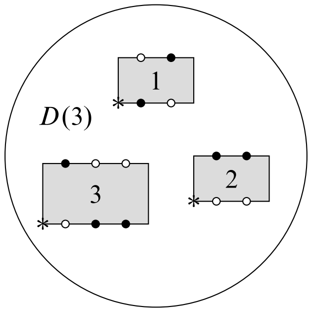


Figure 1: It is a k-holed disk $D(k)$ with $k=3$. A shaded rectangle labeled by i is D_i. Marked points with sign $+$ (resp. $-$) are described as black (resp. white) dots. In this case, $\boldsymbol{s}_1^{(0)} = \bar{\boldsymbol{s}}_1^{(1)} = +-$, $\boldsymbol{s}_2^{(0)} = \bar{\boldsymbol{s}}_2^{(1)} = --$, and $\boldsymbol{s}_3^{(0)} = \bar{\boldsymbol{s}}_3^{(1)} = -++$. A tangled trivalent graph G defines $\mathcal{TL}(-+,-+)\otimes\mathcal{TL}(++,++)\otimes\mathcal{TL}(+--,+--)\to\mathcal{R}$.

We introduce notation for a planar algebra specialized to our situation because it is useful for writing $\mathfrak{sl}_3$-webs in the form of an equation. Let D be a disk, and D_i for $i=1,2,\dots,k$ disjoint rectangles in $D\setminus\partial D$. Each D_i is homeomorphic to $[0,1]\times[0,1]$ and it has a basepoint at $(0,0)$ and marked points $P_i = P_i^{(0)} \sqcup P_i^{(1)}$. The set $P_i^{(j)}$ of marked points lies in the edge of D_i corresponding to $[0,1]\times\{j\}$. We have a k-holed disk $D(k) := D\setminus\bigcup_{i=1}^k \operatorname{int}(D_i)$ with marked points $\bigcup_{i=1}^k P_i$. A small disk D_i share P_i with $D(k)$ for $i=1,2,\dots,k$, see Figure 1. Let a sequence of signs $\boldsymbol{s}_i^{(0)}$ (resp. $\boldsymbol{s}_i^{(1)}$) be an assignment of signs to a set of marked points $P_i^{(0)}$ (resp. $P_i^{(1)}$) of $D(k)$ for each $i=1,2,\dots,k$. Then we consider the $\mathfrak{sl}_3$-web spaces

- $\mathcal{W}\big(\bigcup_{i=1}^k \boldsymbol{s}_i; D(k)\big)$, where $\boldsymbol{s}_i = \boldsymbol{s}_i^{(0)}\cup\boldsymbol{s}_i^{(1)}$;
- $\mathcal{TL}(\bar{\boldsymbol{s}}_i^{(0)}, \boldsymbol{s}_i^{(1)}) = \mathcal{W}(\bar{\boldsymbol{s}}_i^{(0)}\cup\bar{\boldsymbol{s}}_i^{(1)}; D_i)$ for $i=1,2,\dots,k$.

As I mentioned above, $D(k)$ and D_i share the set of marked points $P_i = P_i^{(0)}\sqcup P_i^{(1)}$. Then the sequence of signs $\boldsymbol{s}_i^{(j)}$ of $P_i^{(j)}$ in $D(k)$ is consistent with $\bar{\boldsymbol{s}}_i^{(j)}$ of $P_i^{(j)}$ in D_i for $j=1,2$. For example, an edge terminating at $p\in P_i$ in $D(k)$ can be composed with an edge starting from p in D_i. Thus, the sign of p in $D(k)$ and D_i are different. For a tangled trivalent graph $\mathsf{G}\in\mathcal{G}\big(\bigcup_{i=1}^k \boldsymbol{s}_i; D(k)\big)$, a linear map

$$\mathsf{G}\colon \bigotimes_{i=1}^{k}\mathcal{TL}(\bar{\boldsymbol{s}}_i^{(0)}, \boldsymbol{s}_i^{(1)}) \to \mathcal{W}(\varnothing, D)\cong\mathbb{Z}((q^{1/6}))$$

is induced by a map $D(k)\sqcup(D_1\sqcup D_2\sqcup\cdots\sqcup D_k)\to D$. This map composes $\mathfrak{sl}_3$-webs in D_i for $i=1,2,\dots,k$ with G in $D(k)$.

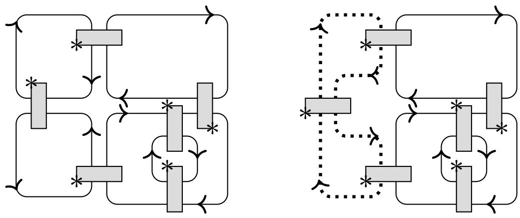
In this paper, we only consider segregated sign sequences $\boldsymbol{s}_i^{(0)} = \epsilon^{m_i}\bar{\epsilon}^{n_i}$ and $\boldsymbol{s}_i^{(1)} = \bar{\boldsymbol{s}}_i^{(0)} = \bar{\epsilon}^{m_i}\epsilon^{n_i}$, where ϵ is $+$ or $-$ and $m_i, n_i\in\mathbb{Z}_{\geq 0}$ satisfy $m_i+n_i = \#P_i^{(0)} = \#P_i^{(1)}$. The *identity web* $\mathbb{1}_{\boldsymbol{s}_i^{(1)}}$ in $\mathcal{TL}(\bar{\boldsymbol{s}}_i^{(0)}, \boldsymbol{s}_i^{(1)})$ is m_i+n_i parallel strands in D_i. The identity web $\mathbb{1}_{\boldsymbol{s}_i^{(1)}}$ and the $\mathfrak{sl}_3$-clasp $\mathrm{JW}_{\boldsymbol{s}_i^{(1)}}$ in D_i are simply denoted by $\mathbb{1}_{D_i}$ and JW_{D_i}, respectively.

Definition 3.7 Set $s_i^{(0)} = \bar{s}_i^{(1)} = \epsilon^{m_i}\bar{\epsilon}^{n_i}$ for all $i = 1, 2, \dots, k$. $\mathsf{G} \in \mathcal{G}(\bigcup_{i=1}^k s_i; D(k))$ is *adequate* if

- G is a disjoint union of oriented simple arcs, and
- For every $j = 1, 2, \dots, k$, any pair of strands in $\mathbb{1}_{D_j}$ belong to different connected components of the graph $\mathsf{G}(\bigotimes_{i=1}^k \mathbb{1}_{D_i})$ which is composed of oriented simple loops.

See Example 3.8. We also call the clasped $\mathfrak{sl}_3$-web $\mathsf{G}(\bigotimes_{i=1}^k \mathrm{JW}_{D_i})$ *adequate* when G is adequate.

Example 3.8 The left $\mathfrak{sl}_3$-web below is adequate, but the right is not adequate because of the dotted arc:



Proposition 3.9 *Let $D(k) = D \setminus \bigsqcup_{i=1}^k \mathrm{int}(D_i)$ be a k-holed disk with signed marked point $s_i^{(0)} = \bar{s}_i^{(1)} = \epsilon^{m_i}\bar{\epsilon}^{n_i}$ on the i^{th} boundary component for $i = 1, 2, \dots, k$. For any flat trivalent graph G in $\mathcal{G}(\bigcup_{i=1}^k s_i; D(k))$,*

$$\mathrm{d}_*\Bigl(\mathsf{G}\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr)\Bigr) \geq -\tfrac{1}{4}v(\mathsf{G}) - c\Bigl(\mathsf{G}\Bigl(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\Bigr)\Bigr).$$

In particular, $\mathrm{d}_*(\mathsf{G}(\bigotimes_{i=1}^k \mathrm{JW}_{D_i})) \geq \mathrm{d}_*(\mathsf{G}(\bigotimes_{i=1}^k \mathbb{1}_{D_i}))$ holds when G has no trivalent vertices due to Proposition 3.2. Moreover, $\mathrm{d}_*(\mathsf{G}(\bigotimes_{i=1}^k \mathrm{JW}_{D_i})) = \mathrm{d}_*(\mathsf{G}(\bigotimes_{i=1}^k \mathbb{1}_{D_i}))$ holds when G is adequate.

Proof Lemma 3.6 expands all $\mathfrak{sl}_3$-clasps in D_i for $i = 1, 2, \dots, k$ as

$$\mathsf{G}\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr) = \sum_{t_1=0}^{\min\{m_1,n_1\}} \cdots \sum_{t_k=0}^{\min\{m_k,n_k\}} \sum_{M^{(1)}} \cdots \sum_{M^{(k)}} \Bigl(\prod_{i=1}^k f_{t_i}(M^{(i)}; q)\Bigr)\mathsf{G}\Bigl(\bigotimes_{i=1}^k \uparrow M^{(i)} \downarrow_{t_i}\Bigr),$$

where

- $\sum_{M^{(i)}}$ means a summation over $M_1^{(i)}$, $M_2^{(i)}$, $M_3^{(i)}$ and $M_4^{(i)}$;
- $\uparrow M^{(i)} \downarrow_{t_i}$ is the $\mathfrak{sl}_3$-web defined in Lemma 3.6; and
- $f_{t_i}(M^{(i)}; q) := f_{(m_i,n_i;t_i)}(M_1^{(i)}, M_2^{(i)}, M_3^{(i)}, M_4^{(i)}; q)$ with $\mathrm{d}_*(f_{t_i}(M^{(i)}; q)) = \frac{1}{2}t_i(t_i+1) + \sum_{j=1}^4 \frac{1}{4}v(M_j^{(i)})$.

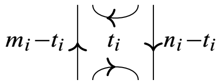
We remark that $v(\mathsf{G}(\bigotimes_{i=1}^k \uparrow M^{(i)} \downarrow_{t_i})) = \sum_{i=1}^k \sum_{j=1}^4 v(M_j^{(i)}) + v(\mathsf{G})$ by definition. Now

$$\mathrm{d}_*\Bigl(\Bigl(\prod_{i=1}^k f_{t_i}(M^{(i)}; q)\Bigr)\mathsf{G}\Bigl(\bigotimes_{i=1}^k \uparrow M^{(i)} \downarrow_{t_i}\Bigr)\Bigr) = \sum_{i=1}^k \tfrac{1}{2}t_i(t_i+1) + \sum_{i=1}^k \sum_{j=1}^4 \tfrac{1}{4}v(M_j^{(i)}) + \mathrm{d}_*\Bigl(\mathsf{G}\Bigl(\bigotimes_{i=1}^k \uparrow M^{(i)} \downarrow_{t_i}\Bigr)\Bigr). \tag{3-1}$$

Proposition 3.2 gives a lower bound on $\mathrm{d}_*\big(\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\big)\big)$ using the number of vertices and connected components:

$$\mathrm{d}_*\left(\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\right)\right) \geq -\frac{1}{4}\left(\sum_{i=1}^k\sum_{j=1}^4 v(M_j^{(i)}) + v(\mathsf{G})\right) - c\left(\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\right)\right).$$

Let $\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)$ be an $\mathfrak{sl}_3$-web whose intersection with $D(k)$ is G and with each D_i is $\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}$, where

$\uparrow\mathbb{1}^{(i)}\downarrow_{t_i} :=$ 

.

In other words, $\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)$ is obtained by replacing all $M_j^{(i)}$ in $\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\big)$ with identity webs.

Remark 3.5 says that $\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\big)$ is obtained from $\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)$ by a sequence of *zip cobordisms* which replace parallel strands with the I_j. Then $c\big(\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\big)\big) \leq c\big(\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)$ holds because a zip cobordism reduces the number of connected components. Combining these inequalities with (3-1), we obtain

$$\mathrm{d}_*\left(\left(\prod_{i=1}^k f_{t_i}(M^{(i)};q)\right)\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\right)\right) \geq \sum_{i=1}^k \tfrac{1}{2}t_i(t_i+1) - \tfrac{1}{4}v(\mathsf{G}) - c\left(\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right). \tag{3-2}$$

One can make a similar argument when the JW_{D_i} have one-row colored $\mathfrak{sl}_3$-clasps by using Lemma 3.4. If the i^{th} disk D_i has a one-row colored $\mathfrak{sl}_3$-clasp, then we read $\uparrow M^{(i)}\downarrow_{t_i}$ as $M^{(i)}$ and $f_{t_i}(M^{(i)};q)$ as $f_{M^{(i)}}(q)$, and replace $\sum_{j=1}^4 \frac{1}{4}v(M_j^{(i)})$ with $\frac{1}{4}v(M^{(i)})$.

Finally, we observe how the right-hand side of (3-2) changes by a single *orientable saddle cobordism* which transforms t_i to t_i+1. One can see that the single orientable saddle cobordism changes $c\big(\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)$ by ± 1 by considering the orientation of strands in $\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}$. Hence, the right-hand side of (3-2) is a monotonically increasing function on $0 \leq t_i \leq \min\{m_i, n_i\}$. Consequently,

$$\begin{aligned}\mathrm{d}_*\left(\left(\prod_{i=1}^k f_{t_i}(M^{(i)};q)\right)\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow M^{(i)}\downarrow_{t_i}\right)\right) &\geq -\tfrac{1}{4}v(\mathsf{G}) - c\left(\mathsf{G}\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_0\right)\right)\\ &= -\tfrac{1}{4}v(\mathsf{G}) - c\left(\mathsf{G}\left(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\right)\right),\end{aligned}$$

because $\uparrow\mathbb{1}^{(i)}\downarrow_0 = \mathbb{1}_{D_i}$ by definition.

When $v(\mathsf{G}) = 0$, the right-hand side becomes $\mathrm{d}_*\big(\mathsf{G}\big(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\big)\big)$ by Proposition 3.2. Moreover, if G is adequate, $c\big(\mathsf{G}\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)$ strictly decreases by the orientable saddle cobordism changing t_i from 0 to 1 for some i. Hence, we obtain $\mathrm{d}_*\big(\mathsf{G}\big(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\big)\big) = \mathrm{d}_*\big(\mathsf{G}\big(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\big)\big)$. One can make a similar argument when all the JW_{D_i} are one-row colored $\mathfrak{sl}_3$-clasps and G is adequate. □

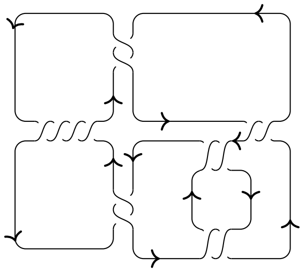
4 Zero stability of the one-row colored $\mathfrak{sl}_3$-Jones polynomial

We show the zero stability of the one-row colored $\mathfrak{sl}_3$-tail for a certain class of B-adequate oriented links.

Definition 4.1 Let D be a disk equipped with pairwise disjoint twist regions D_i for $i = 1, \dots, k$ in $\mathrm{int}(D)$ such that each D_i is isomorphic to $[0,1] \times [0,1]$ with a basepoint at $(0,0)$ and an assignment $\boldsymbol{l} \colon \{D_i\}_{i=1}^k \to 2\mathbb{Z}_{>0}$. An *antiparallel B-adequate link* is an oriented link represented by an oriented link diagram L in D satisfying the following conditions:

- $L \cap D(k)$ is an adequate graph G, where $D(k) := D \setminus \bigsqcup_{i=1}^k \mathrm{int}(D_i)$.
- $L \cap D_i$ is a twist region R_{l_i} with negative $l_i := \boldsymbol{l}(D_i)$ half twists of antiparallel strands for each i; see Figure 2.

Example 4.2 The diagram below represents an antiparallel B-adequate link:



Example 4.3 (plumbed-like links) Let X be a planar embedded graph equipped with a weight $\boldsymbol{l} \colon E(X) \to 2\mathbb{Z}_{\geq 0}$ for the edge set $E(X)$. Then we obtain an antiparallel B-adequate link diagram from X by replacing all vertices with positively oriented circles and then adding a twist $R_{\boldsymbol{l}(e)}$ between two circles connected by an edge $e \in E(X)$.

Before we prove the zero stability for antiparallel B-adequate links, let us introduce some symbols for values of special $\mathfrak{sl}_3$-webs and coefficients. We can describe the one-row colored $\mathfrak{sl}_3$-Jones polynomial by using these values.

Lemma 4.4 *We have*

$$\Delta^{(n)}(j) = \frac{[n-j+1]^2[2n-2j+2]}{[2]}, \quad \Theta^{(n)}(j) = \frac{\begin{bmatrix} 2n-j+2 \\ 2n-2j+2 \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}^2} \Delta^{(n)}(j),$$

$$\gamma^{(n)}(j) = (-1)^{n-j} q^{-n^2/6} q^{-j^2/2+(n+1)j},$$

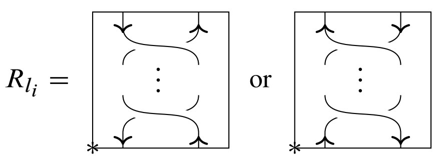
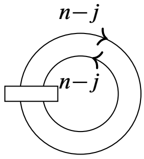
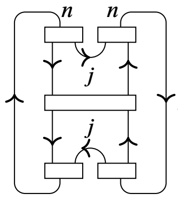
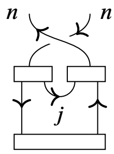


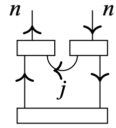
Figure 2: The diagram R_{l_i} has l_i crossings, where $l_i \in 2\mathbb{Z}_{>0}$.

where

$\Delta^{(n)}(j) =$ 

, $\Theta^{(n)}(j) =$ 

, 

$= \gamma^{(n)}(j)$ 

.

Proof It is well known that the value of a closure of $\mathrm{JW}_{-^m+^n}$ is obtained by quantum dimension $[m+1][n+1][m+n+2]/[2]$, and it is just $\Delta_j^{(n)}$. One can compute the value γ by using Lemma 2.9(3), (4) and (6). $\Theta^{(n)}(j)$ was computed in [25]. □

Lemma 4.5 *We have*

$$\mathrm{d}_*(\Delta^{(n)}(j)) = \mathrm{d}_*(\Delta^{(n)}(j-1)) + 2, \quad \mathrm{d}_*(\Theta^{(n)}(j)) = \mathrm{d}_*(\Theta^{(n)}(j-1)) + 1,$$
$$\mathrm{d}_*(\gamma^{(n)}(j)) = \mathrm{d}_*(\gamma^{(n)}(j-1)) + \left(n - j + \tfrac{3}{2}\right).$$

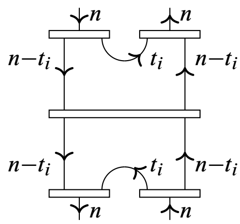
Proof From Lemma 2.11 and Example 2.12, one can compute the minimum degrees of $\Delta^{(n)}(j)$ and $\Theta^{(n)}(j)$ as $\mathrm{d}_*(\Delta^{(n)}(j)) = -2n + 2j$ and $\mathrm{d}_*(\Theta^{(n)}(j)) = -2n + j$. □

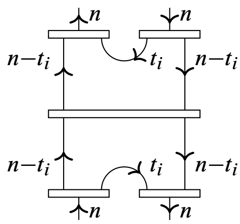
An n-parallelization L_n of the antiparallel B-adequate link diagram L defines an adequate graph $\mathsf{G}_n := L_n \cap D(k) \in \left(\bigcup_{i=1}^k s_i; D(k)\right)$, where $s_i^{(0)} = \bar{s}_i^{(1)} = \epsilon_i^n \bar{\epsilon}_i^n$ for some $\epsilon_i \in \{\pm\}$, and an n-parallelization $(R_{l_i})_n := L_n \cap D_i$ of l_i half twists for any i. Then we define a clasped $\mathfrak{sl}_3$-web $R_{l_i}^{(n)}$ in D_i by inserting one-row colored $\mathfrak{sl}_3$-clasps for each n parallelized strands of $(R_{l_i})_n$. The one-row colored $\mathfrak{sl}_3$-Jones polynomial $J_{L,n}^{\mathfrak{sl}_3}(q)$ of the antiparallel B-adequate link L is obtained by replacing $(R_{l_i})_n$ of L_n with $R_{l_i}^{(n)}$. Using a linear map defined by G_n, this replacement is described as $\mathsf{G}_n\left(\bigotimes_{i=1}^k ({R_{l_i}}^{(n)})\right)$.

Lemma 4.6 *For an antiparallel B-adequate link diagram L with twist regions D_i,*

$$L^{(n)} = \mathsf{G}_n\left(\bigotimes_{i=1}^{k} ({R_{l_i}}^{(n)})\right) = \sum_{t_1,t_2\dots,t_k=0}^{n} \prod_{i=1}^{k} \Gamma^{(n)}(t_i; l_i)\mathsf{G}_n\left(\bigotimes_{i=1}^{k} M(t_i;n)\right),$$

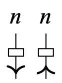
where

$$\Gamma^{(n)}(t_i; l_i) = \gamma^{(n)}(t_i)^{l_i} \frac{\Delta^{(n)}(t_i)}{\Theta^{(n)}(t_i)} \quad \text{and} \quad M(t_i;n) =$$ 

or 

.

Proof Apply the formula



$$= \sum_{t=0}^{n} \frac{\Delta^{(n)}(t)}{\Theta^{(n)}(t)} M(t;n)$$

shown in [25] to all twist regions, and resolve twists by definition of $\gamma^{(n)}(j)$ in Lemma 4.4. We obtain the desired formula. □

Lemma 4.7 $\quad \mathrm{d}_*(\Gamma^{(n)}(t_i;l_i)) = \mathrm{d}_*(\Gamma^{(n)}(t_i-1;l_i)) + l_i\big(n-t_i+\frac{3}{2}\big)+1.$

Proof By Lemmas 2.11 and 4.5,

$$\begin{aligned}\mathrm{d}_*(\Gamma^{(n)}(t_i;l_i)) &= l_i\mathrm{d}_*(\gamma^{(n)}(t_i)) + \mathrm{d}_*(\Delta^{(n)}(t_i)) - \mathrm{d}_*(\Theta^{(n)}(t_i))\\ &= l_i\mathrm{d}_*(\gamma^{(n)}(t_i-1)) + \mathrm{d}_*(\Delta^{(n)}(t_i-1)) - \mathrm{d}_*(\Theta^{(n)}(t_i-1)) + l_i\big(n-t_i+\tfrac{3}{2}\big)+1\\ &= \mathrm{d}_*(\Gamma^{(n)}(t_i-1;l_i)) + l_i\big(n-t_i+\tfrac{3}{2}\big)+1.\end{aligned}$$

□

Proposition 4.8 *Let L be an antiparallel B-adequate link diagram with disjoint twist regions D_i, $\boldsymbol{l}\colon\{D_i\}_{i=1}^k \to 2\mathbb{Z}_{>0}$ and $\mathsf{G} = L\cap D(k)$. Then*

$$\mathsf{G}_n\Big(\bigotimes_{i=1}^k (R_{l_i}{}^{(n)})\Big) - \prod_{i=1}^k \gamma^{(n)}(0)^{l_i}\,\mathsf{G}_n\Big(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Big) \in q^{2(n+2)+\mathrm{d}_*(L^{(n)})}\mathbb{Z}[q].$$

To prove Proposition 4.8, we prepare several lemmas. First of all, let us introduce an operation $\boldsymbol{S}_j$ corresponding to the single orientable saddle cobordism at D_j for $i=1,2,\dots,k$. More precisely, $\boldsymbol{S}_j$ acts on the set $\big\{\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i} \mid 0\geq t_i\geq n\big\}$ of an $\mathfrak{sl}_3$-web in $\bigsqcup_{i=1}^k D_i$ as follows: $\boldsymbol{S}_j$ replaces $\uparrow\mathbb{1}^{(j)}\downarrow_{t_j}$ with $\uparrow\mathbb{1}^{(j)}\downarrow_{t_j-1}$, and acts as the identity on $\uparrow\mathbb{1}^{(j)}\downarrow_0$ in D_j or elements in D_i with $i\neq j$. Note that $\bigotimes_{i=1}^k\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}$ changes to $\bigotimes_{i=1}^k \mathbb{1}_{D_i}$ by a composition of orientable saddle cobordisms. For instance, $\boldsymbol{S}_k^{t_k}\cdots\boldsymbol{S}_2^{t_2}\boldsymbol{S}_1^{t_1}$ realizes this deformation.

Lemma 4.9 *For an adequate graph G of the antiparallel B-adequate link diagram L in Proposition 4.8 and any fixed tuple $(t_1,\dots,t_k)\in\{0,\dots,n\}^k$,*

$$\mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^k M(t_i;n)\Big)\Big) \geq \mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^k\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big).$$

Proof A clasped $\mathfrak{sl}_3$-web $M(t_i;n)$ has five $\mathfrak{sl}_3$-clasps in D_i. For all $i=1,2,\dots,k$, one can choose five small disks $D_{(i,1)},\dots,D_{(i,5)}$ in D_i so that each small disk surrounds a single $\mathfrak{sl}_3$-clasp. The intersection of L and $D\setminus\bigsqcup\{D_{(i,j)} \mid 1\leq i\leq k,\ 1\leq j\leq 5\}$ is a graph G' with no trivalent vertices. Then $\mathsf{G}_n\big(\bigotimes_{i=1}^k M(t_i;n)\big) = \mathsf{G}'_n\big(\bigotimes_{i=1}^k\bigotimes_{j=1}^5 \mathrm{JW}_{D_{(i,j)}}\big)$ by the construction of G', where $\mathrm{JW}_{D_{(i,j)}}$ is a one- or two-row colored clasp. Apply Proposition 3.9 to $\mathsf{G}'_n\big(\bigotimes_{i=1}^k\bigotimes_{j=1}^5 \mathrm{JW}_{D_{(i,j)}}\big)$ to obtain

$$\begin{aligned}\mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^k M(t_i;n)\Big)\Big) &= \mathrm{d}_*\Big(\mathsf{G}'_n\Big(\bigotimes_{i=1}^k\bigotimes_{j=1}^5 \mathrm{JW}_{D_{(i,j)}}\Big)\Big)\\ &\geq \mathrm{d}_*\Big(\mathsf{G}'_n\Big(\bigotimes_{i=1}^k\bigotimes_{j=1}^5 \mathbb{1}_{D_{(i,j)}}\Big)\Big) = \mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^k\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big).\end{aligned}$$

□

Lemma 4.10 *For an adequate graph G of the antiparallel B-adequate link diagram L in Proposition 4.8 and any fixed tuple $(t_1,\dots,t_k)\in\{0,\dots,n\}^k$ with $0<t_j\leq n$,*

$$\mathrm{d}_*\Big(\Gamma^{(n)}(t_j;l_j)\mathsf{G}_n\Big(\bigotimes_{i=1}^k\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big) > \mathrm{d}_*\Big(\Gamma^{(n)}(t_j-1;l_j)\mathsf{G}_n\Big(\boldsymbol{S}_j\Big(\bigotimes_{i=1}^k\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big)\Big).$$

Proof Note that $\mathrm{d}_*\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big) = -c\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)$ and $\mathrm{d}_*\big(\mathsf{G}_n\big(\boldsymbol{S}_j\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)\big) = -c\big(\mathsf{G}_n\big(\boldsymbol{S}_j\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)\big)$ by Proposition 3.2. The orientable saddle cobordism $\boldsymbol{S}_j$ changes the number of connected components by ± 1 or 0. Hence $\mathrm{d}_*\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big) \geq \mathrm{d}_*\big(\mathsf{G}_n\big(\boldsymbol{S}_j\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)\big) - 1$. This inequality and Lemma 4.7 imply

$$\begin{aligned}
\mathrm{d}_*\left(\Gamma^{(n)}(t_j;l_j)\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right) &= \mathrm{d}_*(\Gamma^{(n)}(t_j;l_j)) + \mathrm{d}_*\left(\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right) \\
&\geq \mathrm{d}_*(\Gamma^{(n)}(t_j;l_j)) + \mathrm{d}_*\left(\mathsf{G}_n\left(\boldsymbol{S}_j\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right)\right) - 1 \\
&= \mathrm{d}_*(\Gamma^{(n)}(t_j-1;l_j)) + \mathrm{d}_*\left(\mathsf{G}_n\left(\boldsymbol{S}_j\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right)\right) + l_j\big(n-t_j+\tfrac{3}{2}\big) \\
&= \mathrm{d}_*\left(\Gamma^{(n)}(t_j-1;l_j)\mathsf{G}_n\left(\boldsymbol{S}_j\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\right)\right)\right) + l_j\big(n-t_j+\tfrac{3}{2}\big).
\end{aligned}$$

One can easily see that $l_j\big(n - t_j + \frac{3}{2}\big) \geq 3$ because $l_j \in 2\mathbb{Z}_{>0}$ and $0 < t_j \leq n$. □

Lemma 4.11 *For an adequate graph* G *of the antiparallel* B*-adequate link diagram* L *in Proposition 4.8 and any* $0 < j \leq k$,

$$\mathrm{d}_*\left(\left(\prod_{i=1}^k \Gamma^{(n)}(\delta_{ij};l_i)\right)\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\right)\right) - \mathrm{d}_*\left(\left(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\right)\mathsf{G}_n\left(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\right)\right) \geq 2n+3,$$

where δ_{ij} *is the Kronecker delta function.*

Proof The adequacy of G_n says that $c\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\big)\big) - c\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\big)\big) = -1$. We know $\mathrm{d}_*(\Gamma^{(n)}(1;l_j)) - \mathrm{d}_*(\Gamma^{(n)}(0;l_j)) = l_j\big(n+\frac{1}{2}\big) + 1$ by Lemma 4.7, while $\mathrm{d}_*\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\big)\big) = -c\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\big)\big)$ and $\mathrm{d}_*\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\big)\big) = -c\big(\mathsf{G}_n\big(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\big)\big)$ by Proposition 3.2. Hence,

$$\begin{aligned}
&\mathrm{d}_*\left(\left(\prod_{i=1}^k \Gamma^{(n)}(\delta_{ij};l_i)\right)\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\right)\right) - \mathrm{d}_*\left(\left(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\right)\mathsf{G}_n\left(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\right)\right) \\
&\quad = \mathrm{d}_*(\Gamma^{(n)}(1;l_j)) - \mathrm{d}_*(\Gamma^{(n)}(0;l_j)) + \mathrm{d}_*\left(\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\right)\right) - \mathrm{d}_*\left(\mathsf{G}_n\left(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\right)\right) \\
&\quad = l_j\big(n+\tfrac{1}{2}\big) + 1 - c\left(\mathsf{G}_n\left(\bigotimes_{i=1}^k \uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\right)\right) + c\left(\mathsf{G}_n\left(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\right)\right) \\
&\quad = l_j\big(n+\tfrac{1}{2}\big) + 2 \geq 2n+3.
\end{aligned}$$

The last inequality holds because l_j is a positive even integer. □

Lemma 4.12 *For an adequate graph* G *of the antiparallel* B*-adequate link diagram* L *in Proposition 4.8,*

$$\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(0;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathbb{1}_{D_i}\Big)\Big)=\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(0;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathrm{JW}_{D_i}\Big)\Big).$$

Proof This assertion comes from the adequacy of G and Proposition 3.9. □

Proof of Proposition 4.8 By Lemmas 2.11 and 4.9, we obtain

$$\begin{aligned}\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(t_i;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}M(t_i;n)\Big)\Big)&=\Big(\sum_{i=1}^{k}\mathrm{d}_*(\Gamma^{(n)}(t_i;l_i))\Big)+\mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}M(t_i;n)\Big)\Big)\\&\geq\Big(\sum_{i=1}^{k}\mathrm{d}_*(\Gamma^{(n)}(t_i;l_i))\Big)+\mathrm{d}_*\Big(\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big)\\&=\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(t_i;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big).\end{aligned}$$

Choose a sequence of orientable saddle cobordisms that changes $\mathsf{G}_n\big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)$ to

$$\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{0}\Big)=\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathbb{1}_{D_i}\Big),$$

and apply Lemma 4.10 to $\mathrm{d}_*\big(\big(\prod_{i=1}^{k}\Gamma^{(n)}(t_i;l_i)\big)\mathsf{G}_n\big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\big)\big)$ along the sequence until just before the last step. We can apply Lemma 4.11 to the last orientable saddle cobordism given by $\boldsymbol{S}_j$. This operation gives

$$\begin{aligned}\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(t_i;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{t_i}\Big)\Big)&>\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(\delta_{ij};l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\uparrow\mathbb{1}^{(i)}\downarrow_{\delta_{ij}}\Big)\Big)\\&\geq\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(0;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathbb{1}_{D_i}\Big)\Big)+2n+3.\end{aligned}$$

The above two inequalities and Lemma 4.12 imply

$$\begin{aligned}\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(t_i;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}M(t_i;n)\Big)\Big)&>\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(0;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathbb{1}_{D_i}\Big)\Big)+2n+3\\&=\mathrm{d}_*\Big(\Big(\prod_{i=1}^{k}\Gamma^{(n)}(0;l_i)\Big)\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}\mathrm{JW}_{D_i}\Big)\Big)+2n+3\end{aligned}\tag{4-1}$$

for any $(t_1,t_2,\dots,t_k)\neq(0,0,\dots,0)$.

Finally, we will compare $\mathrm{d}_*\bigl(\mathsf{G}_n\bigl(\bigotimes_{i=1}^k (R_{l_i}{}^{(n)})\bigr)\bigr)$ to $\mathrm{d}_*\bigl(\prod_{i=1}^k \gamma^{(n)}(0)^{l_i}\,\mathsf{G}_n\bigl(\bigotimes_{i=1}^k M(0;n)\bigr)\bigr)$ by using the expansion in Lemma 4.6 and (4-1). By Lemma 4.6,

$$\begin{aligned}\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k (R_{l_i}{}^{(n)})\Bigr)\\ &= \Bigl(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k M(0;n)\Bigr) + \sum_{(t_1,t_2\dots,t_k)\neq(0,0,\dots,0)} \Bigl(\prod_{i=1}^k \Gamma^{(n)}(t_i;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k M(t_i;n)\Bigr)\\ &= \Bigl(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr) + \sum_{(t_1,t_2\dots,t_k)\neq(0,0,\dots,0)} \Bigl(\prod_{i=1}^k \Gamma^{(n)}(t_i;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k M(t_i;n)\Bigr).\end{aligned}$$

By Lemma 2.11 and (4-1), one can obtain

$$\begin{aligned}\mathrm{d}_*\Bigl(\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k (R_{l_i}{}^{(n)})\Bigr) - \Bigl(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr)\Bigr)\\ &= \min_{(t_1,t_2\dots,t_k)\neq(0,0,\dots,0)} \Bigl\{\mathrm{d}_*\Bigl(\Bigl(\prod_{i=1}^k \Gamma^{(n)}(t_i;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k M(t_i;n)\Bigr)\Bigr)\Bigr\}\\ &> \mathrm{d}_*\Bigl(\Bigl(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr)\Bigr) + 2n+3.\end{aligned}$$

Note that

$$\mathrm{d}_*\Bigl(\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k (R_{l_i}{}^{(n)})\Bigr)\Bigr) = \mathrm{d}_*\Bigl(\Bigl(\prod_{i=1}^k \Gamma^{(n)}(0;l_i)\Bigr)\mathsf{G}_n\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr)\Bigr)$$

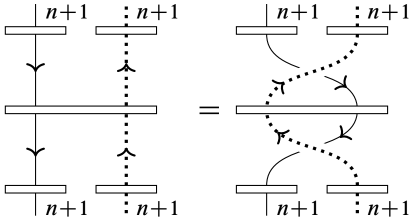
by (4-1) and $\Gamma^{(n)}(0;l_i) = \prod_{i=1}^k \gamma^{(n)}(0)^{l_i}$ by definition. □

Definition 4.13 For $f(q)$ and $g(q)$ in $\mathbb{Z}((q))$, we define $f(q) \equiv_n g(q)$ if $\mathrm{d}_*(\hat{f}(q) - \hat{g}(q)) \geq n+1$.

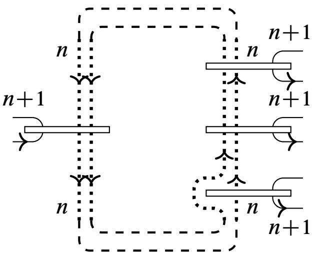
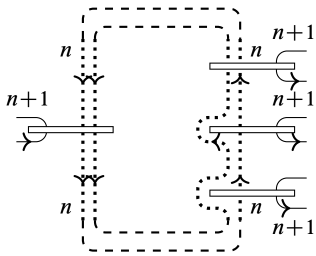
Proposition 4.14 *Let* JW_{D_i} *represent* $M(0;n)$ *in* D_i, *as defined in Lemma 4.6. If* G *is adequate, then*

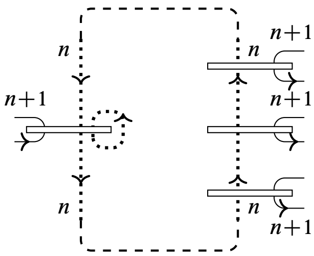
$$\mathsf{G}_{n+1}\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr) \equiv_{n+1} \mathsf{G}_n\Bigl(\bigotimes_{i=1}^k \mathrm{JW}_{D_i}\Bigr).$$

We will prove this proposition using a similar strategy to [1]. Let us explain it in our situation. Choose a one-row colored $\mathfrak{sl}_3$-clasp with $n+1$ strands in $\mathsf{G}_{n+1}\bigl(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\bigr)$. It corresponds to $n+1$ parallel circles in $\mathsf{G}\bigl(\bigotimes_{i=1}^k \mathbb{1}_{D_i}\bigr)$. First we move the $n+1$ parallel strands to the left side of the two-row colored $\mathfrak{sl}_3$-clasps at the center of D_i. The "left side" is determined by the orientation of $n+1$ parallel strands at each $\mathfrak{sl}_3$-clasp:

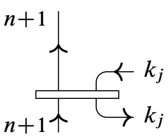
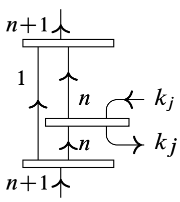
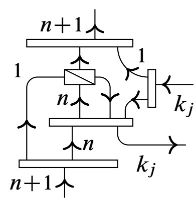


In the above, the chosen $n+1$ parallel strands are expressed as a dotted arc labeled by $n+1$. We remark that this deformation of $\mathfrak{sl}_3$-webs does not change the coefficients. We assume that the chosen $n+1$ parallel strands pass through m two-row colored $\mathfrak{sl}_3$-clasps $\mathrm{JW}_{D_1}, \mathrm{JW}_{D_2}, \ldots, \mathrm{JW}_{D_m}$ in this order by replacing labels of twist regions if necessary. We denote the initial $\mathfrak{sl}_3$-web by $\mathsf{G}^{(n+1)}(0)$, and a clasped $\mathfrak{sl}_3$-web obtained by unclasping the leftmost strand of the $n+1$ strands from $\mathrm{JW}_{D_1}, \ldots, \mathrm{JW}_{D_{j-1}}$, and JW_{D_j} in $\mathsf{G}^{(n+1)}(0)$ by $\mathsf{G}^{(n+1)}(j)$ for $j = 1, 2, \ldots, m-1$. If one could unclasp the leftmost strand from $\mathrm{JW}_{D_1}, \ldots, \mathrm{JW}_{D_{m-1}}$, then the $\mathfrak{sl}_3$-web would become $\mathsf{G}^{(n+1)}(m-1)$. One can shrink the unclasped strand in $\mathsf{G}^{(n+1)}(m-1)$ to JW_{D_m} as follows:

$\mathsf{G}^{(n+1)}(0) \to \mathsf{G}^{(n+1)}(1) =$  $\to$ 

$\to \cdots \to$  $= \mathsf{G}^{(n+1)}(m-1).$

We will see that the above sequence of $\mathfrak{sl}_3$-webs can be realized by computing $\mathsf{G}^{(n+1)}(j)$ modulo $q^{n+1}\mathbb{Z}[\![q]\!]$.

Lemma 4.15  $=$  $+ (-1)^{n+1}\dfrac{[k_j]}{[n+k_j+2]}$  .

We defer the proof of this lemma to the appendix; see Proposition A.6.

Proof of Proposition 4.14 Let us do the unclasping operation that we explained. Choose $n+1$ parallel circles passing the left side of $\mathfrak{sl}_3$-clasps $\mathrm{JW}_{D_1}, \ldots, \mathrm{JW}_{D_m}$. We apply Lemma 4.15 to the j^{th} $\mathfrak{sl}_3$-clasp

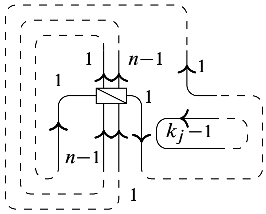


Figure 3: A flat $\mathfrak{sl}_3$-web $\widetilde{\mathsf{H}}^{(n+1)}(j)$ obtained from $\mathsf{H}^{(n+1)}(j)$.

JW_{D_j} in $\mathsf{G}^{(n+1)}(j-1)$ for $j=1,2,\dots,m-2$. Then we obtain

$$\mathsf{G}^{(n+1)}(j-1)=\mathsf{G}^{(n+1)}(j)+(-1)^{n+1}\frac{[k_j]}{[n+k_j+2]}\mathsf{H}^{(n+1)}(j),$$

where $\mathsf{H}^{(n+1)}(j)$ is a clasped $\mathfrak{sl}_3$-web corresponding to the second term in Lemma 4.15. We use k_j above although $k_j=n+1$ in this situation because it will be useful for later discussion. We compare $\mathrm{d}_*(\mathsf{G}^{(n+1)}(j))$ to $\mathrm{d}_*(\mathsf{H}^{(n+1)}(j))$. Let $\widetilde{\mathsf{G}}^{(n+1)}(j)$ and $\widetilde{\mathsf{H}}^{(n+1)}(j)$ denote flat trivalent graphs obtained by replacing all $\mathfrak{sl}_3$-clasps in $\mathsf{G}^{(n+1)}(j)$ and $\mathsf{H}^{(n+1)}(j)$, respectively, with identity webs. Proposition 3.9 says that the lower bound on the minimum degree $\mathrm{d}_*(\mathsf{H}^{(n+1)}(j))$ is given by the number of vertices and connected components of $\widetilde{\mathsf{H}}$. By tracing strands of $\widetilde{\mathsf{H}}^{(n+1)}(j)$ as in Figure 3, one can see that $c(\widetilde{\mathsf{G}}^{(n+1)}(j-1))-c(\widetilde{\mathsf{H}}^{(n+1)}(j))=n+1$ and $v(\widetilde{\mathsf{H}}^{(n+1)}(j))=2n$. By Proposition 3.9,

$$\begin{aligned}\mathrm{d}_*(\mathsf{H}^{(n+1)}(j))&\geq -\tfrac{1}{4}v(\widetilde{\mathsf{H}}^{(n+1)}(j))-c(\widetilde{\mathsf{H}}^{(n+1)}(j))=-\tfrac{1}{4}(2n)-\big(c(\widetilde{\mathsf{G}}^{(n+1)}(j))-(n+1)\big)\\&=\tfrac{1}{2}(n+2)-c(\widetilde{\mathsf{G}}^{(n+1)}(j)).\end{aligned}$$

Proposition 3.2 and $v(\mathsf{G}^{(n+1)}(j))=0$ give $\mathrm{d}_*(\widetilde{\mathsf{G}}^{(n+1)}(j))=-c(\widetilde{\mathsf{G}}^{(n+1)}(j))$. Moreover, $\mathrm{d}_*(\mathsf{G}^{(n+1)}(j))=\mathrm{d}_*(\widetilde{\mathsf{G}}^{(n+1)}(j))$ holds by adequacy of $\mathsf{G}^{n+1}(j)$ and Proposition 3.9. Using Lemma 2.11 and Example 2.12, the above facts lead to the inequality

$$\mathrm{d}_*\left((-1)^{n+1}\frac{[k_j]}{[n+k_j+2]}\widetilde{\mathsf{H}}^{(n+1)}(j)\right)\geq\tfrac{1}{2}(n+2)+\big(\tfrac{1}{2}(n+2)+\mathrm{d}_*(\mathsf{G}^{(n+1)}(j))\big)=(n+2)+\mathrm{d}_*(\mathsf{G}^{(n+1)}(j)).$$

It also holds that $\mathrm{d}_*(\mathsf{G}^{(n+1)}(j-1))=\mathrm{d}_*(\mathsf{G}^{(n+1)}(j))$ due to Propositions 3.2 and 3.9. In fact, the adequacy of these clasped $\mathfrak{sl}_3$-webs and $\widetilde{\mathsf{G}}^{(n+1)}(j-1)=\widetilde{\mathsf{G}}^{(n+1)}(j)$ imply

$$\begin{aligned}\mathrm{d}_*(\mathsf{G}^{(n+1)}(j-1))&=\mathrm{d}_*(\widetilde{\mathsf{G}}^{(n+1)}(j-1))=-c(\widetilde{\mathsf{G}}^{(n+1)}(j-1))\\&=-c(\widetilde{\mathsf{G}}^{(n+1)}(j))=\mathrm{d}_*(\widetilde{\mathsf{G}}^{(n+1)}(j))=\mathrm{d}_*(\mathsf{G}^{(n+1)}(j)).\end{aligned}$$

Thus we obtain

$$\mathsf{G}^{(n+1)}(j)-\mathsf{G}^{(n+1)}(j-1)=(-1)^{n+1}\frac{[k_j]}{[n+k_j+2]}\widetilde{\mathsf{H}}^{(n+1)}(j)\in q^{(n+2)+\mathrm{d}_*(\mathsf{G}^{(n+1)})}\mathbb{Z}[\![q^{1/6}]\!],$$

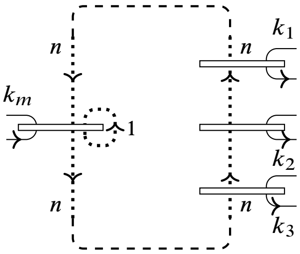
where $\mathrm{d}_*(\mathsf{G}^{(n+1)}):=\mathrm{d}_*(\mathsf{G}^{(n+1)}(j-1))=\mathrm{d}_*(\mathsf{G}^{(n+1)}(j))$.

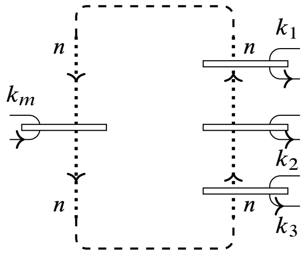
This result holds independently of the number $m-j$ of $\mathfrak{sl}_3$-clasps that the $n+1$ parallel strands pass through, and also the number k_j of the oppositely oriented strands adjacent to the $n+1$ parallel strands.

We repeatedly apply Lemma 4.15 to $\mathrm{JW}_{D_1}, \dots, \mathrm{JW}_{D_{m-1}}$ and obtain

$$\mathsf{G}^{(n+1)}(0) \equiv_{n+1} \mathsf{G}^{(n+1)}(1) \equiv_{n+1} \cdots \equiv_{n+1} \mathsf{G}^{(n+1)}(m-1).$$

Let $\mathsf{G}^{(n+1)}(m)$ be an $\mathfrak{sl}_3$-web removing the small circle from $\mathsf{G}^{(n+1)}(m-1)$:

$\mathsf{G}^{(n+1)}(m-1) =$ 

, $\mathsf{G}^{(n+1)}(m) =$ 

.

Then we obtain

$$\mathsf{G}^{(n+1)}(m-1) = \frac{[n+2][n+k_m+3]}{[n+1][n+k_m+2]}\mathsf{G}^{(n+1)}(m)$$

by Proposition A.1. From the above equality, it is easily seen that

$$\mathsf{G}^{(n+1)}(1) \equiv_{n+1} \mathsf{G}^{(n+1)}(0)$$

holds for any k_m. Next, we consider the leftmost strand of the other $n+1$ parallel circles. One can unclasp the leftmost strand from $\mathfrak{sl}_3$-clasps exactly in the same way. The label k_j in this argument might be n. However, it works independently of k_j as I mentioned above. We repeatedly apply this argument until all $n+1$ parallel circles passing through $\mathfrak{sl}_3$-clasps become n parallel strands. Consequently, we obtain $\mathsf{G}_{n+1}\big(\bigotimes_{i=1}^{k} \mathbb{1}_{D_i}\big) \equiv_{n+1} \mathsf{G}_n\big(\bigotimes_{i=1}^{k} \mathbb{1}_{D_i}\big)$. □

Theorem 4.16 *Let L be an antiparallel B-adequate link. Then*

$$\hat{J}^{\mathfrak{sl}_3}_{L,n+1}(q) - \hat{J}^{\mathfrak{sl}_3}_{L,n}(q) \in q^{n+1}\mathbb{Z}[\![q]\!].$$

In other words, the one-row colored $\mathfrak{sl}_3$-Jones polynomial $\{\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)\}_n$ of L is zero stable.

Proof Let us take a link diagram $\mathsf{G}\big(\bigotimes_{i=1}^{k} R_{l_i}\big)$ of L with an adequate graph G and twist regions $\boldsymbol{l}\colon \{D_i\}_{i=1}^{k} \to 2\mathbb{Z}_{>0}$. The one-row colored $\mathfrak{sl}_3$-Jones polynomial $\hat{J}^{\mathfrak{sl}_3}_{L,n}(q)$ is given by the normalization in Definition 2.10 of the clasped $\mathfrak{sl}_3$-web $\mathsf{G}_n\big(\bigotimes_{i=1}^{k} (R_{l_i}{}^{(n)})\big)$. Lemma 4.6 and Proposition 4.8 claim that

$$\mathsf{G}_n\Big(\bigotimes_{i=1}^{k} (R^{(n)}_{l_i})\Big) \equiv_{2n+1} \prod_{i=1}^{k} \gamma^{(n)}(0)^{l_i}\, \mathsf{G}_n\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big),$$

$$\mathsf{G}_{n+1}\Big(\bigotimes_{i=1}^{k} (R^{(n+1)}_{l_i})\Big) \equiv_{2n+3} \prod_{i=1}^{k} \gamma^{(n+1)}(0)^{l_i}\, \mathsf{G}_{n+1}\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big),$$

and Proposition 4.14 claims

$$\mathsf{G}_n\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big) \equiv_{n+1} \mathsf{G}_{n+1}\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big).$$

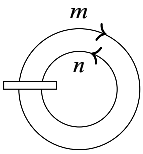
It is easy to see that $f(q) \equiv_n g(q)$ if $f(q) \equiv_N g(q)$ for some $N \geq n$, and $(-1)^{k_1} q^{k_2} f(q) \equiv_n (-1)^{l_1} q^{l_2} g(q)$ if $f(q) \equiv_n g(q)$ for any k_1, k_2, l_1 and l_2. Hence, the above equivalence relations derive

$$\mathsf{G}_n\Big(\bigotimes_{i=1}^{k}(R_{l_i}^{(n)})\Big) \equiv_{n+1} \prod_{i=1}^{k} \gamma^{(n)}(0)^{l_i}\, \mathsf{G}_n\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big) \equiv_{n+1} \prod_{i=1}^{k} \gamma^{(n+1)}(0)^{l_i}\, \mathsf{G}_{n+1}\Big(\bigotimes_{i=1}^{k} \mathrm{JW}_{D_i}\Big)$$
$$\equiv_{n+1} \mathsf{G}_{n+1}\Big(\bigotimes_{i=1}^{k}(R_{l_i}^{(n+1)})\Big).$$

This means $\hat{J}^{\mathfrak{sl}_3}_{L,n+1}(q) - \hat{J}^{\mathfrak{sl}_3}_{L,n}(q) \in q^{n+1}\mathbb{Z}[\![q^{1/6}]\!]$. □

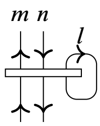
Appendix Formulas for clasped $\mathfrak{sl}_3$-webs

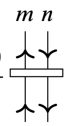
It is well known that the closure of $\mathrm{JW}_{-^m+^n}$ is given by

$\Delta(m,n) =$ 

$= \dfrac{[m+1][n+1][m+n+2]}{[2]}.$

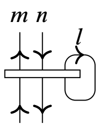
Proposition A.1

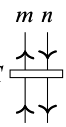


$= \dfrac{\Delta(m+l,n)}{\Delta(m,n)}$ 

.

Proof It is known that this clasped $\mathfrak{sl}_3$-web space is one-dimensional and it is spanned by $\mathrm{JW}_{-^m+^n}$. Thus, we set



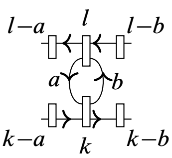
$= C$ 

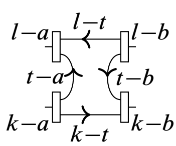
.

The closures of the diagrams in the left- and right-hand sides are given by $\Delta(m+l,n)$ and $\Delta(m,n)$, respectively. Hence, $C = \Delta(m+l,n)/\Delta(m,n)$. □

In order to prove Lemma 4.15, we prepare some lemmas.

Lemma A.2 (the bubble skein expansion formula [24])



$= \displaystyle\sum_{t=\max\{a,b\}}^{\min\{a+b,k,l\}} \frac{\begin{bmatrix} k \\ t \end{bmatrix}\begin{bmatrix} l \\ t \end{bmatrix}\begin{bmatrix} t \\ a \end{bmatrix}\begin{bmatrix} t \\ b \end{bmatrix}\begin{bmatrix} k+l-t+2 \\ k+l-a-b+2 \end{bmatrix}}{\begin{bmatrix} k \\ a \end{bmatrix}\begin{bmatrix} l \\ a \end{bmatrix}\begin{bmatrix} k \\ b \end{bmatrix}\begin{bmatrix} l \\ b \end{bmatrix}}$ 

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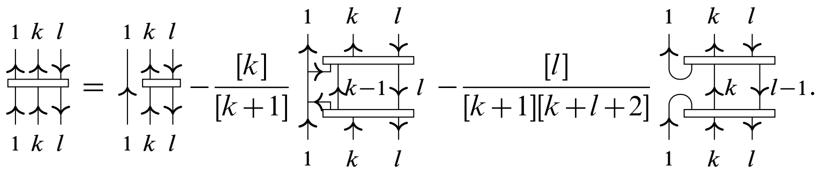
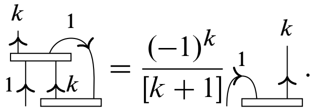
Lemma A.3 [17, Theorem 3.3]

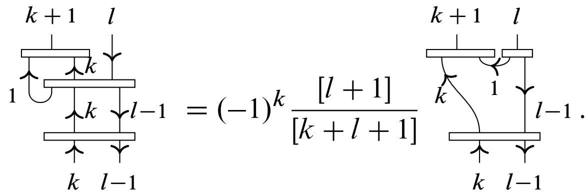


Lemma A.4

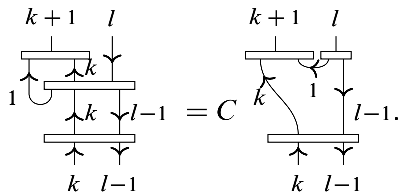


Proof Apply Lemma 3.3 to an $\mathfrak{sl}_3$-clasp in the left side above. Then one can see that the diagrams in the expansion vanish except for the last term, due to the bottom $\mathfrak{sl}_3$-clasp. This becomes the right-hand side by Lemma 2.9(2). □

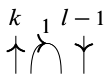
Lemma A.5



Proof It is known that the $\mathfrak{sl}_3$-web space on a disk with clasped endpoints JW_{-^l}, $\mathrm{JW}_{+^{k+1}}$ and $\mathrm{JW}_{-^k+^{l-1}}$ is spanned by one clasped $\mathfrak{sl}_3$-web in the right-hand side. See for example [18; 17] for details. Hence we only have to determine the coefficient C such that

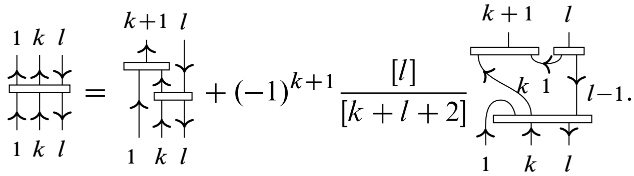


Attach an $\mathfrak{sl}_3$-web

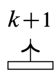


to the top of the $\mathfrak{sl}_3$-webs in both sides. The left-hand side becomes $(-1)^k\Delta(k,l)/([k+1]\Delta(k,l-1))\cdot \mathrm{JW}_{-^k+^{l+1}}$ by Lemma A.4 and Proposition A.1. The right-hand side becomes $C[k+l+2]/([k+1][l])\cdot \mathrm{JW}_{-^k+^{l+1}}$ by Lemma A.2. We obtain the value C in the assertion by solving this equation. □

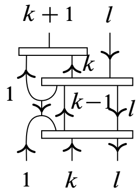
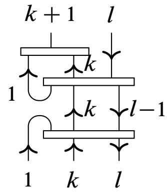
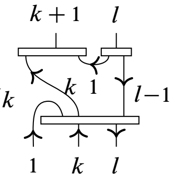
Proposition A.6



Proof Let us denote the second coefficient in the assertion by $a_k := (-1)^{k+1}[l]/[k+l+2]$. We first attach an $\mathfrak{sl}_3$-clasp



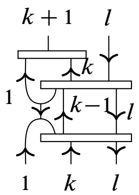
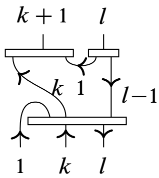
to the top left of the diagrams in Lemma A.3. Next we calculate the second and the third terms in the right-hand side of the resulting equation. More precisely, we will show

(A-1) $-\frac{[k]}{[k+1]}$  $-\frac{[l]}{[k+1][k+l+2]}$  $= a_k$  .

The second $\mathfrak{sl}_3$-web on the left-hand side was already done in Lemma A.5, and it provides a coefficient $(-1)^k[l+1]/[k+l+1]$. Equation (A-1) holds if the first $\mathfrak{sl}_3$-web provides a coefficient a_{k-1} because the summation of these coefficients is calculated as

$$-\frac{[k]}{[k+1]}a_{k-1} + \left(-\frac{[l]}{[k+1][k+l+2]}\right)\left((-1)^k\frac{[l+1]}{[k+l+1]}\right)$$
$$= (-1)^{k+1}\frac{[l]}{[k+1][k+l+1][k+l+2]}([k][k+l+2]+[l+1])$$
$$= (-1)^{k+1}\frac{[l]}{[k+l+2]} = a_k.$$

We used $[k+1][k+l+1]-[k][k+l+2]=[l+1]$ in the equation above. Hence, let us prove

(A-2)  $= a_{k-1}$ 

by induction on k. One can easily prove (3-2) at $k=1$ by expanding the middle $\mathfrak{sl}_3$-clasp. We assume that (A-2) holds when $k=n$. This means that Proposition A.6 also holds when $k=n$ by the above argument. Thus, we apply Proposition A.6 to the middle $\mathfrak{sl}_3$-clasp on the left-hand side of (A-2) at $k=n+1$, and one can confirm that this gives the right-hand side of (A-2). □

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Received: 27 August 2020 Revised: 7 October 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY
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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 4 (pages 1917–2526) 2025