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Line bundle twists for unitary bordism are ghosts

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We prove that the canonical twist $\zeta\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(M\mathrm{Spin}^c)$ does not extend to a twist for unitary bordism by showing that every continuous map $f\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(MU)$ loops to a nullhomotopic map.

55N22, 55P42, 55R37; 14L05, 19L50

1 Introduction

Twisted (co)homologies capture geometric information and invariants that are only available for untwisted (co)homologies under additional orientation assumptions. Prominent examples of this phenomenon are Thom classes. A vector bundle $E\to X$ of rank d has a Thom class $\mathfrak{t}_E\in H^d(E,E\setminus 0;\mathbb{Z})$ if and only if E is orientable. However, it always has a Thom class in $H^d(E,E\setminus 0;\mathbb{Z}^{w_1(E)})$ if the coefficients are twisted by the first Stiefel–Whitney class $w_1(E)\in H^1(X;\mathbb{Z}_2)=\mathrm{Hom}(\pi_1(X),\mathrm{Aut}(\mathbb{Z}))$.

Orientability of E, however, does not suffice to guarantee a Thom class in K-theory. Atiyah, Bott and Shapiro showed in [4] that an oriented vector bundle E has a Thom class in K-theory if the structure group reduces to $\mathrm{Spin}^c(d)$. They also constructed a Thom class in this case, which is referred to as *Atiyah–Bott–Shapiro* orientation. To produce K-theoretic Thom classes for all oriented vector bundles, Donovan and Karoubi [10], later generalised by Atiyah and Segal [5], constructed twisted K-theory groups using operator-theoretic methods. Since $K(\mathbb{Z},3)=B\mathrm{PU}(H)$ is the classifying space of the projective unitary group of a separable Hilbert space H, each $\alpha\in H^3(X,\mathbb{Z})$ gives rise to a bundle of Fredholm operators over X. The group $K_\alpha(X)$ is the set of homotopy classes of sections of this bundle. We have now, for each oriented bundle $E\to X$, a Thom isomorphism

$$K^*(X)\cong K^{*+\mathrm{rk}(E)}_{W_3(E)}(E,E\setminus 0),$$

where $W_3(E)\in H^3(X;\mathbb{Z})$ is the third integral Stiefel–Whitney class; see for example Carey and Wang [8] for details.

A homotopy-theoretic approach to twisted (co)homology in terms of twisted spectra is provided by the work of Ando, Blumberg and Gepner [1] and Ando, Blumberg, Gepner, Hopkins and Rezk [2; 3]. Their construction goes roughly as follows: An A_∞-ring spectrum R has a space of units $\mathrm{GL}_1(R)$, which deloops to a classifying space $B\mathrm{GL}_1(R)$. A *twist* is a map $\xi\colon X\to B\mathrm{GL}_1(R)$ from which we can construct a Thom spectrum $R_{X,\xi}$, which is an R-module spectrum. The associated twisted (co)homology groups are then given by

$$R_k(X,\xi) := \pi_0 R\text{-mod}(\Sigma^k R, R_{X,\xi}) \quad \text{and} \quad R^k(X,\xi) := \pi_0 R\text{-mod}(R_{X,\xi}, \Sigma^k R).$$

If the twist is nullhomotopic, then this construction gives the usual (untwisted) (co)homology theory associated to R. In practice, twists of interest factor through specific reference spaces; in the case of K-theory, this reference space is $K(\mathbb{Z},3)$.

Our motivation for the work presented here arose from the results of Hebestreit and Joachim [11]; see also Hebestreit and Savage [12]. The authors construct point–set models for twisted Spin^c bordism and twisted K-theory over $K(\mathbb{Z},3)$ as well as a model for a twisted Atiyah–Bott–Shapiro orientation. It is shown in [11, Appendix C] that the point–set model also arises from the twist $\zeta\colon K(\mathbb{Z},3) \to B\mathrm{GL}_1(M\mathrm{Spin}^c)$ that comes from the fibre inclusion $U(1) \to \mathrm{Spin}^c$; see Section 4 for a detailed description of this twist.

One application of these models concerns the bordism-determines-homology question. The classical results of Conner and Floyd [9] and Hopkins and Hovey [13] state that K-theory is completely described by the Todd orientation $\mathrm{Td}\colon MU \to K$ and by the Atiyah–Bott–Shapiro orientation $\alpha_{\mathrm{ABS}}\colon M\mathrm{Spin}^c \to K$ in the sense that

$$MU_*(X) \otimes_{MU_*(\mathrm{pt})} K_*(\mathrm{pt}) \cong K_*(X) \cong M\mathrm{Spin}^c_*(X) \otimes_{M\mathrm{Spin}^c_*(\mathrm{pt})} K_*(\mathrm{pt})$$

for all spectra X. In the forthcoming preprint of Baum, Joachim, Khorami and Schick [7] the latter isomorphism is proved by geometric means in the twisted setup using the $K(\mathbb{Z},3)$-twisted ABS orientation.

Since the Todd orientation factors through the ABS orientation under the canonical map $MU \to M\mathrm{Spin}^c$, it is natural to ask whether one can also extend the first isomorphism to the $K(\mathbb{Z},3)$-twisted setup so that it reduces to the isomorphism of [7]. The first step would be to construct a twisted unitary bordism spectrum $MU_{K(\mathbb{Z},3),?}$ over $K(\mathbb{Z},3)$ and a comparison map $MU_{K(\mathbb{Z},3),?} \to M\mathrm{Spin}^c_{K(\mathbb{Z},3),\zeta}$ that extends the twisted ABS orientation "to the left". Our first result shows that this is impossible:

Theorem A *The twist $\zeta\colon K(\mathbb{Z},3) \to B\mathrm{GL}_1(M\mathrm{Spin}^c)$ does not factor through unitary bordism. More precisely, there are no continuous maps S and T that make the following diagram homotopy commutative:*

$$\begin{array}{ccc} K(\mathbb{Z},3) & \xrightarrow{\ \zeta\ } & B\mathrm{GL}_1(M\mathrm{Spin}^c) \\ & {\scriptstyle T}\searrow & \uparrow{\scriptstyle S} \\ & & B\mathrm{GL}_1(MU) \end{array}$$

Theorem A is a consequence of the following purely homotopy-theoretical result:

Theorem B *Every continuous map $f\colon K(\mathbb{Z},3) \to B\mathrm{GL}_1(MU)$ loops to a nullhomotopic map*

$$\Omega f\colon K(\mathbb{Z},2) \to \mathrm{GL}_1(MU).$$

Its (slightly more general) stable counterpart reads as follows:

Lemma C *Each map of spectra* $\Sigma^\infty_+ K(\mathbb{Z},2) \to MU$ *that is a ring map up to homotopy induces a homomorphism between the Pontrjagin rings* $MU_*(K(\mathbb{Z},2))$ *and* $MU_*(MU)$ *that vanishes on* $\widetilde{MU}_*(K(\mathbb{Z},2))$.

Partial results strongly indicating Theorem A were obtained by Joel Meier in his unpublished Master's thesis [16]. He showed that there is no H-map $K(\mathbb{Z},2) \to \mathrm{SO}/U$ whose composition with the fibre comparison map $\mathrm{SO}/U \to K(\mathbb{Z},2) = \mathrm{hofib}(B\mathrm{Spin}^c \to B\mathrm{SO})$ is homotopic to the identity. Such a map, when it deloops, would produce a nontrivial map T in Theorem A. In this case, S would arise from a map of spectra, a condition we do not assume.

Lemma C has a purely algebraic application. A *formal group law* over a ring R is a formal power series $F \in R[\![x,y]\!]$ that satisfies the same conditions as the Taylor series expansion of the multiplication of a one-dimensional Lie group. Let L be the Lazard ring, the ring over which the universal formal group law F_{univ} is defined. Lemma C and Quillen's result [17] that L is isomorphic to $MU_*(\mathrm{pt})$ imply the next result:

Theorem D *There is no nontrivial homomorphism between the universal formal group law* F_{univ} *and the multiplicative formal group* $F_{\mathrm{mult}}(x,y) := x + y + xy$ *on* L. *More precisely, if* $g \in L[\![x]\!]$ *is a formal power series with* $g(0) = 0$ *and*

$$g(F_{\mathrm{univ}}(x,y)) = F_{\mathrm{mult}}(g(x), g(y)),$$

then $g = 0$.

Outline of the proof Theorem A follows from Theorem B once we know that ζ is sufficiently nontrivial. We will translate the statement of Theorem B into a stable homotopy-theoretical one and use the Adams–Novikov spectral sequence to deduce it from Lemma C. The main task is therefore the proof of Lemma C, which is purely algebraic. We will show that $MU_*(\mathrm{pt})$-algebra homomorphisms between the Pontrjagin rings $MU_*(K(\mathbb{Z},2))$ and $MU_*(MU)$ are uniquely determined by the image of a single element in low degrees. Algebraic arguments then imply that any algebra homomorphism $MU_*(\mathbb{C}P^\infty) \to MU_*(MU)$ factors through $MU_*(\mathrm{pt})$. Using the natural homology co-operations, we derive that no such algebra homomorphism can come from a map of spectra.

Organisation We first give a short summary of the theory of orthogonal spectra and set up conventions in Section 2. Section 3 is devoted to the proof of Theorem B and Lemma C. There, we also outline how the questions we address are related to the question of whether the K-theory spectrum is a retract as a ring spectrum of the periodic complex bordism spectrum MUP. The proof of Theorem A will be carried out in Section 4. Section 5, in which we prove Theorem D, is logically independent of Section 4.

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2 Foundations and conventions

Throughout the entire article we work in the category of compactly generated Hausdorff spaces. We assume that the reader is familiar with the notion of orthogonal spectra at the level of [6]. The fact important to us is that orthogonal ring spectra form a symmetric monoidal model category [6, Theorem 6.4.8] with the sphere spectrum $\mathbb{S}$ as unit. The monoid objects are precisely (unital) ring spectra. The category of ring spectra also carries a model structure such that forgetting the ring structure is a Quillen right adjoint [6, Theorem 6.6.12]. Suspension and evaluation at the initial space form a strong symmetric monoidal adjunction

(1) $$\Sigma^\infty_+ \colon \mathbf{Top} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathbf{Sp}^{\mathrm{orth}} : \Omega^\infty.$$

See [6, Lemmas 6.3.21 and 6.3.18]. We refer to the natural bijection between $[X, \Omega^\infty Y]$ and $[\Sigma^\infty_+ X, Y]$ with Ad. Consequently, the suspension spectrum of a topological monoid is a ring spectrum and the subspace $R_0^\times$ of all elements of the initial space R_0 of a ring spectrum R that induce invertible elements in $\pi_0(R)$ is a topological monoid. The suspension of a monoid homomorphism is a ring map and, conversely, each ring map $f\colon R \to S$ restricts to a monoid homomorphism $f_0\colon R_0^\times \to S_0^\times$.

The left adjoint Σ^∞_+ maps the canonical cylinder object $(\text{-}) \times [0,1]$ to $(\text{-}) \wedge [0,1]_+$ so that H-spaces give rise to homotopy ring spectra and H-maps give rise to homotopy ring maps.

If R is a cofibrant and fibrant object in the model category of orthogonal ring spectra, then its *space of units* $\mathrm{GL}_1(R)$ is the set of all path components of $R_0 = \Omega^\infty R$ that represent invertible elements in $\pi_0(R) = \pi_0(R_0)$. It is a group-like topological monoid.

For a topological monoid G, we denote by $|G_\bullet|$ the geometric realisation of the singular simplicial set $S_\bullet(G)$. It is again a topological monoid and the counit $\varepsilon\colon |G_\bullet| \to G$ is a continuous monoid-homomorphism and a weak homotopy equivalence.

For a group-like topological monoid, we define its classifying space $BG := |B_\bullet(e, |G_\bullet|, e)|$ to be the (thin) geometric realisation of the bar construction of $|G_\bullet|$ in sense of [15, Section 11]. It comes with a quasifibration $EG \to BG$, where EG is a pointed contractible free $|G_\bullet|$-right space. In this setup, Sugawara [20] constructed a *group completion map* $\iota\colon G \to \Omega BG$, which is an H-map [20, Lemma 11] and hence a weak homotopy equivalence (because G is group-like).

3 Proof of Theorem B

We first prove Theorem B under the assumption of Lemma C. To this end, we need good models for the involved players so that the spaces of units are strict monoids.

A model for $K(\mathbb{Z}, 2) \simeq BU(1)$ is given by $\mathbb{C}P^\infty$. The H-structure induced by the group structures is modelled by polynomial multiplication

$$\mu\colon ([\zeta_0 : \zeta_1 : \cdots], [\omega_0 : \omega_1 : \cdots]) \mapsto ([\zeta_0\omega_0 : \zeta_1\omega_0 + \zeta_0\omega_1 : \zeta_2\omega_0 + \zeta_1\omega_1 + \zeta_0\omega_2 : \cdots]).$$

It turns $\mathbb{C}P^\infty$ into a group-like monoid, whose homotopy inversion is modelled by complex conjugation.

Pick an orthogonal ring spectrum that represents unitary bordism and denote its cofibrant-fibrant replacement in the model category of ring spectra by MU.

Lemma 3.1 *Adjunction (1) induces an injection*

$$[\mathbb{C}P^\infty, \mathrm{GL}_1\, MU] \hookrightarrow [\Sigma^\infty_+ \mathbb{C}P^\infty; MU]$$

and the image of the constant maps are precisely those homotopy classes that induce the zero map on $\widetilde{MU}_*(\mathbb{C}P^\infty)$.

Proof Since $\mathrm{GL}_1(MU) \subseteq MU_0 = \Omega^\infty MU$ is a collection of path components, this inclusion induces an injective map on homotopy classes. As adjunction (1) is a Quillen adjunction, we get an injective map

$$[\mathbb{C}P^\infty, \mathrm{GL}_1(MU)] \hookrightarrow [\mathbb{C}P^\infty, MU_0] \xrightarrow[\cong]{\mathrm{Ad}} [\Sigma^\infty_+ \mathbb{C}P^\infty; MU].$$

From the Adams–Novikov spectral sequence it follows that the Hurewicz homomorphism

$$[\Sigma^\infty_+ \mathbb{C}P^n; MU] \to \mathrm{Hom}_{MU_* MU}(MU_* \mathbb{C}P^n; MU_* MU),$$

whose target is the set of all MU_*MU-comodule maps, is bijective. Indeed, as described in Switzer's book [21, Chapter 19], the edge homomorphism of the Adams–Novikov spectral sequence agrees with the Hurewicz homomorphism. Since MU_*MU is a free MU_*MU-comodule and $MU_*(\mathbb{C}P^n)$ is a free $MU_*(\mathrm{pt})$-module [21, Proposition 16.30], the Adams–Novikov spectral sequence is concentrated in the column

$$E^2_{0,t} = \mathrm{Hom}^t_{MU_*MU}(MU_* \mathbb{C}P^n; MU_* MU),$$

by [21, Proposition 19.7], which forces the spectral sequence to collapse for algebraic reasons. As the spectral sequence converges to $[\Sigma^\infty_+ \mathbb{C}P^n; MU]_*$, the edge homomorphism must be an isomorphism.

Since $[\Sigma^\infty_+ \mathbb{C}P^\infty; MU] = MU^0(\mathbb{C}P^\infty) = \lim_{n\to\infty} MU^0(\mathbb{C}P^n)$, see [21, Proof of Proposition 16.29], we deduce that the Hurewicz-homomorphism

$$[\Sigma^\infty_+ \mathbb{C}P^\infty; MU] \to \mathrm{Hom}_{MU_* MU}(MU_* \mathbb{C}P^\infty; MU_* MU),$$

is bijective, too.

The constant maps

$$\mathbb{C}P^\infty \xrightarrow{\text{const}} \mathrm{pt} \xrightarrow{c} MU_0 = \Omega^\infty MU$$

correspond under the adjunction to the maps of spectra

$$\Sigma^\infty_+ \mathbb{C}P^\infty \xrightarrow{\Sigma^\infty_+ \text{const}} \Sigma^\infty_+ \mathrm{pt} = \mathbb{S} \xrightarrow{\mathrm{Ad}(c)} MU,$$

so they induce the zero map on $\widetilde{MU}_*(\mathbb{C}P^\infty) = MU_*(\mathbb{C}P^\infty, \mathrm{pt}) = \ker \mathrm{const}_*$. The bijection of the Hurewicz map conversely shows that every $MU_*(MU)$ comodule map that vanishes on $\widetilde{MU}_*(\mathbb{C}P^\infty) \subseteq MU_*(\mathbb{C}P^\infty)$ must come from a map of spectra that is homotopic to the adjoint of a constant map. □

Proof of Theorem B Let $f \colon B\mathbb{C}P^\infty \to B\mathrm{GL}_1(MU)$ be a twist and let Ωf be its induced map on loop spaces. The monoids $\mathbb{C}P^\infty$ and $\mathrm{GL}_1(MU)$ are group-like, so the group completion maps are weak homotopy equivalences. Since postcomposition with weak equivalences induces bijections between homotopy classes (see [21, Theorem 6.31]) there are unique homotopy classes such that each representative makes the following diagram homotopy commutative:

$$\begin{array}{ccc} \mathbb{C}P^\infty & \dashrightarrow & \mathrm{GL}_1(MU) \\ \varepsilon \uparrow \simeq & & \simeq \uparrow \varepsilon \\ |\mathbb{C}P^\infty_\bullet| & \dashrightarrow & |\mathrm{GL}_1(MU)_\bullet| \\ \iota \downarrow \simeq & & \simeq \downarrow \iota \\ \Omega B\mathbb{C}P^\infty & \xrightarrow{\Omega f} & \Omega B\mathrm{GL}_1(MU) \end{array}$$

The maps of the group-competition zigzags and Ωf are H-maps, so the homotopy class of the lift $\mathbb{C}P^\infty \dashrightarrow \mathrm{GL}_1(MU)$ must consist of H-maps. We denote any representative of the lifted homotopy class again by Ωf.

Under adjunction (1) the H-map Ωf corresponds to a homotopy ring map

$$\mathrm{Ad}(\Omega f)\colon \Sigma^\infty_+ \mathbb{C}P^\infty \to MU.$$

Lemma 3.1 implies that Ωf is nullhomotopic if $\mathrm{Ad}(\Omega f)$ is. But $\mathrm{Ad}(\Omega f)$ is nullhomotopic by the second half of Lemma 3.1 and Lemma C. □

It remains to prove Lemma C, which is reformulated in Lemma 3.4 below. Before we carry out the proof, we recall and develop the required structural results.

We recall from Switzer [21, Proposition 16.29] that $MU^*(\mathbb{C}P^\infty) \cong MU^*[\![c]\!]$, where $c \in \widetilde{MU}^2(\mathbb{C}P^\infty)$ is the universal first Chern class. As a $MU_*(\mathrm{pt})$-left module, $MU_*(\mathbb{C}P^\infty)$ is isomorphic to the free module $MU_*(\mathrm{pt})\{1, \beta_1, \beta_2, \dots\}$, where β_j is dual to c^j; see [21, Proposition 16.30]. Since $MU_*(\mathrm{pt})$ is a polynomial ring over $\mathbb{Z}$ (see [21, Theorem 20.25]) it is torsion free.

We recall further that $(MU_*MU, MU_*(\mathrm{pt}))$ is a Hopf-algebroid. This means, in particular, that the multiplication of MU turns MU_*MU into a commutative ring, that the diagonal map $\Delta\colon MU \to MU \wedge MU$ induces a coaction on MU_*MU, and that these two structures are compatible. However, $MU_*(\mathrm{pt})$ acts canonically from the left and from the right on $MU_*(MU)$ and these actions are not the same. There are therefore a left unit η_L and a right unit η_R on MU_*MU. The left unit sends $\lambda \in MU_*(\mathrm{pt})$ to $\lambda \cdot 1$, where 1 is the unit of the Pontrjagin ring. For more details, see [21, page 414 ff] or [14, Proposition 4.5.3]. We will consider MU_*MU as a left-module over $MU_*(\mathrm{pt})$. As an $MU_*(\mathrm{pt})$-algebra, on which the ring $MU_*(\mathrm{pt})$ acts from the left, $MU_*(MU)$ is a polynomial algebra

$$MU_*(MU) \cong MU_*(\mathrm{pt})[b_1, b_2, \dots],$$

where $b_j \in MU_{2j}(MU)$; see [21, Theorem 17.16] or [14, Proposition 4.4.4]. The next lemma on the Pontrjagin ring structure of $MU_*(\mathbb{C}P^\infty)$ is a direct consequence of [18, Theorem 3.4].

Lemma 3.2 *The additive generators $\{1, \beta_1, \beta_2, \dots\}$ of the Pontrjagin ring $MU_*(\mathbb{C}P^\infty)$ satisfy the multiplicative relations*

$$\beta_i \bullet \beta_j := MU_*(\mu)(\beta_i \otimes \beta_j) = \sum_{k \geq 0} \Bigg(\sum_{\substack{a_1+\dots+a_k=i \\ b_1+\dots+b_k=j}} \prod_{r=1}^{k} \alpha_{a_r b_r} \Bigg) \beta_k,$$

where the $\alpha_{ij} \in MU_{2(i+j-1)}(\mathrm{pt})$ are the coefficients in the unitary bordism formal group law.

To make calculations manageable, we eventually will carry them out in complex K-theory and integral homology. Let us first recall the connection between unitary bordism and complex K-theory following [21, page 423 ff and page 433 ff]. If $t \in K_2(\mathrm{pt}) = K^{-2}(\mathrm{pt})$ denotes the Bott-element, then Bott periodicity yields $K_*(\mathrm{pt}) \cong \mathbb{Z}[t, t^{-1}]$. The Pontrjagin ring $K_*(K)$ embeds into $K_*(K) \otimes \mathbb{Q} \cong \mathbb{Q}[u, v, u^{-1}, v^{-1}]$. It is a Hopf-algebroid with left unit $\eta_L(t) = u$ and right unit $\eta_R(t) = v$. We do not need its coaction here.

As in the case of unitary bordism, $K_*(\mathbb{C}P^\infty)$ is a free $K_*(\mathrm{pt})$-left module and $K_*(MU)$ is isomorphic to a polynomial ring over $K_*(\mathrm{pt})$. More precisely, we have an isomorphism of $K_*(\mathrm{pt})$-left modules

$$K_*(\mathbb{C}P^\infty) \cong K_*(\mathrm{pt})\{1, t^1 Y_1, t^2 Y_2, \dots\},$$

with $Y_j \in K_0(\mathbb{C}P^\infty)$ and an isomorphism of $K_*(\mathrm{pt})$-algebras

$$K_*(MU) \cong K_*(\mathrm{pt})[b_1^K, b_2^K, \dots],$$

where the underlying ring $K_*(\mathrm{pt})$ acts from the left. In [21] the elements b_j^K are denoted by Y_j'.

The Todd orientation is a map of spectra $\mathrm{Td}\colon MU \to K$ that classifies the Thom class of the universal complex vector bundle; see [21, page 434] or [9, page 29]. It induces a natural transformation of homology theories $MU_* \to K_*$ that maps β_j to $t^j Y_j$ and b_j to b_j^K, respectively [21, page 434]. It follows from the geometric description that Td maps the generator $[\mathbb{C}P^1] \in \Omega_2^U(\mathrm{pt})$ to the Bott element $t \in K_2(\mathrm{pt})$. Alternatively, it follows from [9, Corollary 6.5] and the fact that the Todd-genus of $\mathbb{C}P^1$ is 1.

Like Lemma 3.2, the next lemma is also a consequence of [18, Theorem 3.4], but the given formula is now explicit.

Lemma 3.3 *The additive generators $\{1, tY_1, t^2 Y_2, \dots\}$ of the Pontrjagin ring $K_*(\mathbb{C}P^\infty)$ satisfy the multiplicative relations*

$$t^i Y_i \bullet t^j Y_j = t^{i+j} \cdot \sum_{k=\max\{i,j\}}^{i+j} \binom{k}{2k-(i+j)} \binom{2k-(i+j)}{k-j} Y_k.$$

Proof The (standard) complex orientation of complex K-theory is given by

$$y^K := t^{-1} \cdot (\tau - 1) \in \widetilde{K}^2(\mathbb{C}P^\infty),$$

where $1 = \varepsilon^1$ denotes the trivial complex line bundle. Since the monoid map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifies the tensor product of vector bundles, we have under the isomorphism $K^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong K^*(\mathrm{pt})[\![x, y]\!]$ the equality

$$K^*(\mu)(y^K) = x + y + t \cdot xy.$$

See [14, page 148]. Using [18, Theorem 3.4], we deduce the following identity of formal power series:

$$\sum_{i,j\geq 0}(t^iY_i\bullet t^jY_j)x^iy^j=\sum_{k\geq 0}t^kY_k(x+y+txy)^k=\sum_{k\geq 0}\sum_{l=0}^{k}\sum_{m=0}^{l}\binom{k}{l}\binom{l}{m}t^{2k-l}Y_k\cdot x^{k-l+m}y^{k-m}.$$

The system of equations $k-l+m=i$ and $k-m=j$ is solvable under the constrains $0\leq m\leq l\leq k$ if and only if $0\leq 2k-(i+j)\leq k$, or equivalently if $\lceil\frac{1}{2}(i+j)\rceil\leq k\leq i+j$. In this case the solutions are $m=k-j$ and $l=2k-(i+j)$. Comparing coefficients now yields

$$t^iY_i\bullet t^jY_j=t^{i+j}\cdot\sum_{k=\lceil(i+j)/2\rceil}^{i+j}\binom{k}{2k-(i+j)}\binom{2k-(i+j)}{k-j}Y_k.$$

The latter binomial coefficient is zero if $k-j>2k-(i+j)$ or $k-i>2k-(i+j)$. Thus the formula can be shortened to the claimed formula. □

Lastly, the integral homology groups of $\mathbb{C}P^\infty$ and MU are given by $H_*(\mathbb{C}P^\infty;\mathbb{Z})\cong\mathbb{Z}\{1,\beta_1,\dots\}$ and $H_*(MU;\mathbb{Z})\cong\mathbb{Z}[b_1,b_2,\dots]$, where the latter is an isomorphism of rings. The multiplicative structure on the Pontrjagin ring $H_*(\mathbb{C}P^\infty;\mathbb{Z})$ is given by

$$\beta_i\bullet\beta_j=\binom{i+j}{i}\beta_{i+j}.$$

The (generalised) Hurewicz homomorphism that sends a singular stable complex manifold $[f\colon M^n\to X]$ to the image of the fundamental class $f_*[M]\in H_n(X,\mathbb{Z})$ is a natural transformation of homology theories $\mathrm{Hur}\colon MU_*(\text{-})\to H_*(\text{-};\mathbb{Z})$. It has the property that it sends β_i^{MU} to β_i^H and b_i^{MU} to b_i^H.

Lemma 3.4 *Every map of homotopy ring spectra $\varphi\colon\Sigma^\infty_+\mathbb{C}P^\infty\to MU$ induces a homomorphism that vanishes on $\widetilde{MU}_*(\mathbb{C}P^\infty)$.*

Proof Since $\widetilde{MU}_*(\mathbb{C}P^\infty)=MU_*(\mathrm{pt})\{\beta_1,\beta_2,\dots\}\subseteq MU_*(\mathbb{C}P^\infty)$ [21, Proposition 16.30] it suffices to show that $MU_*(\varphi)$ sends all β_n to zero. Since $MU_*(\varphi)$ is degree preserving, there are elements $\lambda_j\in MU_j(\mathrm{pt})$ such that

$$MU_*(\varphi)(\beta_1)=\lambda_2 1+\lambda_0 b_1=:Q\in MU_2(MU).$$

From Lemma 3.2 we derive $\beta_1\bullet\beta_1=2\beta_2+\alpha_{11}\beta_1$ and inductively

$$\beta_1^{\bullet n}=n!\beta_n+{}_n\zeta_{n-1}\beta_{n-1}+\dots+{}_n\zeta_1\beta_1$$

for some ${}_n\zeta_j\in MU_{2(n-j)}(\mathrm{pt})$. Since $MU_*(MU)$ is torsion-free it suffices to show $Q:=MU_*(\varphi)(\beta_1)=0$.

To show that $\lambda_0=0$, we consider the image of Q in integral homology. The following diagram commutes:

$$\begin{array}{ccc} MU_*(\mathbb{C}P^\infty) & \xrightarrow{MU_*(\varphi)} & MU_*MU \\ \downarrow{\scriptstyle\mathrm{Hur}} & & \downarrow{\scriptstyle\mathrm{Hur}} \\ H_*(\mathbb{C}P^\infty;\mathbb{Z}) & \xrightarrow{H_*(\varphi)} & H_*(MU;\mathbb{Z}) \end{array}$$

Since Hur is a natural transformation, we deduce

$$n!\cdot H_{2n}(\varphi)(\beta_n) = \mathrm{Hur}(f(\beta^{\bullet n})) = \mathrm{Hur}(Q^n) = \mathrm{Hur}(Q)^n = \lambda_0^n,$$

which implies that $n \mid \lambda_0$ for all $n \in \mathbb{N}$, and hence $\lambda_0 = 0$. Thus $Q = \lambda_2 \cdot 1 \in MU_2(\mathrm{pt})$.

To show that $\lambda_2 = 0$, we consider the image of Q in K-theory. The following diagram commutes:

$$\begin{array}{ccc} MU_*(\mathbb{C}P^\infty) & \xrightarrow{MU_*(\varphi)} & MU_*MU \\ \downarrow{\scriptstyle \mathrm{Td}} & & \downarrow{\scriptstyle \mathrm{Td}} \\ K_*(\mathbb{C}P^\infty) & \xrightarrow{K_*(\varphi)} & K_*(MU) \end{array}$$

Set $g := K_*(\varphi)$. Since Td is an isomorphism in degree two, the previous calculation implies

$$g(tY_1) = g(\mathrm{Td}_2(\beta_1)) = \mathrm{Td}_2(f(\beta_1)) = \mathrm{Td}_2(\lambda_2 1_{MU_*MU}) = \mathrm{Td}_2(\lambda_2)1_{K_*(MU)} = \lambda_2 t \cdot 1_{K_*MU},$$

where we abuse notation and denote by λ_2 an element in $MU_2(\mathrm{pt})$ and the multiple $\lambda_2 \in \mathbb{Z}$ of the generator $[\mathbb{C}P^1] \in \Omega_2^U(\mathrm{pt})$ such that $\lambda_2 \cdot [\mathbb{C}P^1] = \lambda_2$. The Pontrjagin ring structure of $K_*(\mathbb{C}P^\infty)$ gives

$$2g(t^2Y_2) = g(tY_1 \bullet tY_1) - g(t^2Y_1) = g(tY_1)^2 - tg(tY_1) = t^2(\lambda_2^2 - \lambda_2).$$

The formulas for the coactions

$$\psi_{\mathbb{C}P^\infty} \colon K_*(\mathbb{C}P^\infty) \to K_*(K) \otimes_{K_*(\mathrm{pt})} K_*(\mathbb{C}P^\infty) \quad \text{and} \quad \psi_{MU} \colon K_*(MU) \to K_*(K) \otimes_{K_*(\mathrm{pt})} K_*(MU)$$

can be found in [21, Proposition 17.38]. Since the coactions are natural with respect to morphisms induced by maps of spectra, we must have

$$(\mathrm{id} \otimes g)(\psi_{\mathbb{C}P^\infty}(t^2Y_2)) = \psi_{MU}(g(t^2Y_2)).$$

The following calculations,[1] in which $P = 1 + p_1 + p_2 + \cdots$ denotes the formal series with

$$p_n = \frac{(v-u)(v-2u)\cdots(v-nu)}{(n+1)!} \in K_{2n}(K) \subseteq \mathbb{Q}[u, v, u^{-1}, v^{-1}]_{(2n)}$$

and $\cdot_r$ denotes the right action,

$$\begin{aligned} 2(\mathrm{id} \otimes g)(\psi_{\mathbb{C}P^\infty}(t^2Y_2)) &= \sum_{i+j=2} (P^j)_{2i} \otimes 2 \cdot g(t^jY_j) \\ &= 1 \otimes t^2(\lambda_2^2 - \lambda_2) \cdot 1_{K_*MU} + p_1 \otimes 2\lambda_2 t \cdot 1_{K_*MU} + 0 \otimes 2 \\ &= 1_{K_*K} \cdot_r t^2(\lambda_2^2 - \lambda_2) \otimes 1_{K_*MU} + \tfrac{1}{2}(v-u) \cdot_r 2\lambda_2 t \otimes 1_{K_*MU} \\ &= (\lambda_2^2 - \lambda_2)v^2 \otimes 1_{K_*MU} + \lambda_2(v-u)v \otimes 1_{K_*MU} = (\lambda_2^2 v^2 - \lambda_2 uv) \otimes 1_{K_*MU} \end{aligned}$$

and

$$\begin{aligned} 2\psi_{MU}(g(t^2Y_2)) &= \psi_{MU}(t^2(\lambda_2^2 - \lambda_2) \cdot 1_{K_*MU}) = t^2(\lambda_2^2 - \lambda_2) \cdot \psi_{MU}(1) \\ &= t^2(\lambda_2^2 - \lambda_2) \cdot 1_{K_*K} \otimes 1_{K_*MU} = (\lambda_2^2 - \lambda_2)u^2 \otimes 1_{K_*MU} \end{aligned}$$

show that these two elements are equal only if $\lambda_2 = 0$. Thus $Q = 0$, and so $MU_*(\varphi)$ vanishes on $\widetilde{MU}_*(\mathbb{C}P^\infty)$ because $MU_*(MU)$ has no torsion. □

[1] Here $\lambda_2 \in \mathbb{Z}$ denotes only the multiple of a generator.

Remark 3.5 One might be tempted to prove the previous lemma using the $\ker\varepsilon$-adic filtration of the augmentation map $\varepsilon\colon MU_*(\mathrm{pt}) \to \mathbb{Z} = H_*(\mathrm{pt};\mathbb{Z})$ (which is nothing but the Hurewicz homomorphism) because the associated graded rings $\mathrm{gr}^\bullet MU_*(\mathbb{C}P^\infty)$ and $\mathrm{gr}^\bullet MU_*MU$ are isomorphic (as rings) to $\mathrm{gr}^\bullet MU_*(\mathrm{pt}) \otimes H_*(\mathbb{C}P^\infty;\mathbb{Z})$ and $\mathrm{gr}^\bullet MU_*(\mathrm{pt}) \otimes H_*(MU)$, respectively, and the Pontrjagin ring structure induced by the formal group law is the $\mathrm{gr}^\bullet MU_*(\mathrm{pt})$-linear extension of the Pontrjagin ring structure of $H_*(\mathbb{C}P^\infty;\mathbb{Z})$. However, under these identification, the associated map $\mathrm{gr}^\bullet(MU_*(\varphi))$ corresponds to $\mathrm{id}_{MU_*(\mathrm{pt})} \otimes H_*(\varphi)$, so it cannot rule out the possibility that $MU_*(\varphi)(\beta_1) = \lambda_2$.

Remark 3.6 The weaker statement that every homotopy ring map $\varphi\colon \Sigma^\infty_+\mathbb{C}P^\infty \to MU$ induces the zero map on all homotopy groups of positive degree has close connection to the question of whether K is a retract (as a ring spectrum) of the periodic complex bordism spectrum MUP.

Both statements can be proved using the fact that $H_*(K;\mathbb{Z}) \cong \mathbb{Q}[t,t^{-1}]$, where $t \in H_2(K;\mathbb{Z})$ is the image of the Bott element $\beta \in \pi_2(K)$ under the Hurewicz homomorphism; see [21, Theorem 16.25]. Since $H_*(MUP;\mathbb{Z}) \cong \bigoplus_{i\in\mathbb{Z}} H_*(MU;\mathbb{Z})$, it is clear that there are no unital ring maps $f\colon K \to MUP$ and $g\colon MUP \to K$ such that $gf \simeq \mathrm{id}_K$.

The proof that there are no ring maps that induce nontrivial homomorphisms on all homotopy groups is a bit more elaborate: Since $\pi_*(MU)$ is torsion-free, $\pi_*(\varphi)$ factors through $\pi_*(\Sigma^\infty_+\mathbb{C}P^\infty)/\mathrm{torsion}$. The Hurewicz homomorphisms

$$\pi_*(\Sigma^\infty_+\mathbb{C}P^\infty)/\mathrm{torsion} \to H_*(\mathbb{C}P^\infty;\mathbb{Z}) \quad \text{and} \quad \pi_*(MU) \to H_*(MU;\mathbb{Z})$$

are injective because the rational Hurewicz homomorphism is a natural equivalence between the homology theories $\pi_*(\text{-})\otimes\mathbb{Q}$ and $H_*(\text{-};\mathbb{Q})$. Naturality implies that $\pi_*(\varphi)$ is uniquely determined by $H_*(\varphi)$.

By the Pontrjagin ring structure of $H_*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}\{1,\beta_1,\dots\}$ the homomorphism $H_*(\varphi)$ is uniquely determined by the image of β_1. Thus there is a $\lambda \in \mathbb{Z}$ such that $H_*(\varphi)(\beta_1) = \lambda b_1$ and if $\pi_*(\varphi) \neq 0$, then $\lambda \neq 0$, too.

By Snaith's theorem [19], $MUP \simeq \Sigma^\infty_+ BU[b_1^{-1}]$, so we have the following factorisation:

$$\begin{array}{ccccccc}
\Sigma^\infty_+\mathbb{C}P^\infty & \xrightarrow{\varphi} & MU & \hookrightarrow & MUP & \longrightarrow & MUP[\lambda^{-1}] \\
\downarrow & & & & & & \| \\
\Sigma^\infty_+\mathbb{C}P^\infty[\beta_1^{-1}] & & & \overset{\bar\varphi}{\dashrightarrow} & & & \Sigma^\infty_+ BU[b_1^{-1},\lambda^{-1}]
\end{array}$$

As the second and third upper horizontal maps induce injective maps on integral homology, the ring homomorphism $H_*(\bar\varphi)$ is not trivial. But this is a contradiction, because, by Snaith's theorem [19], $\Sigma^\infty_+\mathbb{C}P^\infty[\beta_1^{-1}] \simeq K$ and every ring homomorphism

$$\mathbb{Q}[t,t^{-1}] \cong H_*(K,\mathbb{Z}) \to H_*(MUP[\lambda^{-1}];\mathbb{Z}) \cong H_*(MUP;\mathbb{Z}) \otimes \mathbb{Z}[\lambda^{-1}]$$

is necessarily the zero map.

4 Proof of Theorem A

Theorem B immediately implies Theorem A once we show that the twist $\zeta\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(M\mathrm{Spin}^c)$ in Theorem A induces nontrivial morphisms on homology groups. According to [11, page 52 ff] the twist $\zeta\colon K(\mathbb{Z},3)\to B\mathrm{GL}_1(\mathrm{Spin}^c)$ is the delooping of the map $BU(1)=K(\mathbb{Z},2)\to \mathrm{GL}_1(M\mathrm{Spin}^c)$ that is the adjoint of

$$M\iota\colon \Sigma^\infty_+ BU(1)\to M\mathrm{Spin}^c,$$

the map between Thom spectra induced by the fibre inclusion $\iota\colon U(1)\to \mathrm{Spin}^c$.

Recall the canonical homomorphism $\xi\colon \mathrm{Spin}^c \xrightarrow{2:1} \mathrm{SO}\times U(1)$. The representation that is used in the construction of the universal $\mathrm{Spin}^c(n)$ vector bundle factors through $\mathrm{SO}(n)$. That means, by definition,

$$M\mathrm{Spin}^c_n=\mathrm{Th}(E\mathrm{Spin}^c(n)\times_{\mathrm{pr}_1\circ\xi}\mathbb{R}^n).$$

Thus the homomorphism $\xi\colon \mathrm{Spin}^c(n)\to \mathrm{SO}(n)\times U(1)$ induces maps

$$\begin{array}{ccc}
\mathrm{Th}(E\mathrm{Spin}^c(n)\times_{\mathrm{pr}_1\circ\xi}\mathbb{R}^n) & \longrightarrow & \mathrm{Th}\big(E(\mathrm{SO}(n)\times U(1))\times_{\mathrm{pr}_1}\mathbb{R}^n\big)\\
\| & & \|\\
 & & \mathrm{Th}\big((E\mathrm{SO}(n)\times_{\text{torsion}}\mathbb{R}^n)\times BU(1)\big)\\
\| & & \|\\
M\mathrm{Spin}^c_n & \xrightarrow{\quad M_n\xi\quad} & M\mathrm{SO}_n\wedge BU(1)_+
\end{array}$$

These maps are compatible with the inclusion and therefore give rise to a map of spectra

$$M\mathrm{Spin}^c \xrightarrow{M\xi} M\mathrm{SO}\wedge BU(1)_+.$$

We will show that the composition

$$\Sigma^\infty_+ BU(1)\xrightarrow{M\iota} M\mathrm{Spin}^c\xrightarrow{M\xi} M\mathrm{SO}\wedge BU(1)_+$$

induces nontrivial morphisms on rational homology groups. By the Thom isomorphism theorem, we obtain the commutative diagram

$$\begin{array}{ccc}
H_*(\Sigma^\infty_+ BU(1);\mathbb{Q}) & \xrightarrow{H_*(M\xi\circ M\iota)} & H_*(M\mathrm{SO}\wedge BU(1)_+;\mathbb{Q})\\
 & & \cong\big\downarrow\Phi\\
\Phi\big\downarrow\cong & & H_*(B\mathrm{SO}\times BU(1);\mathbb{Q})\\
 & & \big\downarrow H_*(\mathrm{pr}_2)\\
H_*(BU(1);\mathbb{Q}) & \xrightarrow{H_*(\mathrm{pr}_2\circ B\xi\circ B\iota)} & H_*(BU(1);\mathbb{Q})
\end{array}$$

where Φ denotes the Thom isomorphism maps. Note that $\mathrm{pr}_2\circ B\xi\circ B\iota$ is homotopic to $B\varphi$, where φ denotes the composition

$$U(1)\xrightarrow{\iota}\mathrm{Spin}^c\xrightarrow{\xi}\mathrm{SO}\times U(1)\xrightarrow{\mathrm{pr}_2} U(1).$$

The homomorphism φ is given by $\varphi(z)=z^2$.

Lemma 4.1 *The map $B\varphi$ induces nonvanishing homomorphisms on all rational homology groups in even degree.*

Proof Note that $B\varphi$ is an H-map. It induces the multiplication with two on $H_2(BU(1);\mathbb{Q})$ by the Hurewicz theorem. Recall that the Pontrjagin ring structure of $H_*(BU(1);\mathbb{Q}) \cong \mathbb{Q}\{1,\beta_1,\beta_2,\dots\}$ is given by

$$\beta_i \bullet \beta_j = \binom{i+j}{i}\beta_{i+j}.$$

Using this formula, we derive inductively

$$H_{2n}(B\varphi;\mathbb{Q})(\beta_n) = 2^n\beta_n,$$

which is obviously nonzero. □

Corollary 4.2 *The map $M\iota\colon \Sigma^\infty_+ BU(1) \to M\mathrm{Spin}^c$ induces nontrivial homomorphisms on homology groups in even degree.*

This corollary together with Theorem B immediately implies the main theorem.

Proof of Theorem A Assume that we would have a factorisation up to homotopy

$$\begin{array}{ccc} K(\mathbb{Z},3) & \xrightarrow{\zeta} & B\mathrm{GL}_1(M\mathrm{Spin}^c) \\ & \searrow^{T} & \uparrow S \\ & & B\mathrm{GL}_1(MU) \end{array}$$

Then their loopings were homotopic, too. The adjoint maps then would satisfy

$$M\iota = \mathrm{Ad}(\Omega\zeta) \simeq \mathrm{Ad}(\Omega S \circ \Omega T).$$

But we know from Theorem B that ΩT, and hence $M\iota = \mathrm{Ad}(\Omega S \circ \Omega T)$ too, is nullhomotopic. Thus, $M\iota$ would be trivial on all homology groups, which is a contradiction. □

5 Proof of Theorem D

Let R be a commutative ring with unit. Recall that a (one-dimensional commutative) *formal group law* over R is a formal power series $F \in R[\![x,y]\!]$ that satisfies

$$F(x,0) = F(0,x) = x, \quad F(x,y) = F(y,x) \quad \text{and} \quad F(x,F(y,z)) = F(F(x,y),z).$$

A *homomorphism* between formal group laws F and G is a formal power series $g \in R[\![x]\!]$ with $g(0) = 0$ and $g(F(x,y)) = G(g(x),g(y))$.

By the work of Lazard, there is a ring L that carries a universal group law $F_{\mathrm{univ}} \in L[\![x,y]\!]$ on it in the sense that, for any other formal group law F over a ring R, there is a unique ring homomorphism $\phi\colon L \to R$ such that $\phi(F_{\mathrm{univ}}(x,y)) = F(x,y)$. Quillen [17] identified L as the unitary cobordism ring

$MU^*(\mathrm{pt})$. Under this identification, the universal group law agrees with the unitary bordism formal group law, meaning it is given by the formal power series

$$MU^*(\mu)(c) = \sum_{i,j\geq 0} \alpha_{ij} c^i d^j \in MU^*(\mathrm{pt})[\![c,d]\!],$$

where we use the same notation as in Lemma 3.2.

Quillen's result allows us to deduce Theorem D from Lemma C by showing that these two statements are, in fact, equivalent.

Proof of Theorem D As $MU^*(\mathrm{pt})[\![c]\!]_0 = MU^0(\mathbb{C}P^\infty) = [\Sigma^\infty_+\mathbb{C}P^\infty; MU]$, we can represent every formal power series of pure degree zero

$$a = a(c) = \sum_{k\geq 0} a_k c^k \quad \text{with } a_k \in MU^{-2k}(\mathrm{pt}),$$

by a map of spectra $f_a\colon \Sigma^\infty_+\mathbb{C}P^\infty \to MU$ that is unique up to homotopy. The map f_a is, by definition, a homotopy ring map if and only if the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \Sigma^\infty_+\mathbb{C}P^\infty \wedge \Sigma^\infty_+\mathbb{C}P^\infty & \xrightarrow{f_a\wedge f_a} & MU\wedge MU \\ \downarrow{\scriptstyle \Sigma^\infty_+\mu} & & \downarrow{\scriptstyle m} \\ \Sigma^\infty_+\mathbb{C}P^\infty & \xrightarrow{f_a} & MU \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{S} & & \\ \downarrow{\scriptstyle \eta_{\Sigma^\infty_+\mathbb{C}P^\infty}} & \searrow{\scriptstyle \eta_{MU}} & \\ \Sigma^\infty_+\mathbb{C}P^\infty & \xrightarrow{f_a} & MU \end{array}$$

Here m denotes the ring structure of MU and $\eta_{(-)}$ denotes the corresponding unit. Translating these diagrams into algebra gives

$$\mu^*(a) = a(c)\cdot a(d) \quad \text{and} \quad a(0) = 1,$$

because m represents the cohomology cross product; see [21, page 270]. Quillen's theorem [17] yields

$$\mu^*(a) = \sum_{k\geq 0} a_k \mu^*(c)^k = \sum_{k\geq 0} a_k (F_{\mathrm{univ}}(c,d))^k = a(F_{\mathrm{univ}}(c,d)),$$

so that $a(F_{\mathrm{univ}}(c,d)) = a(c)\cdot a(d)$. If we set $g = a-1$, then we get

$$g(F_{\mathrm{univ}}(c,d)) = g(c) + g(d) + g(c)g(d) = F_{\mathrm{mult}}(g(c), g(d)) \quad \text{and} \quad g(0) = 0.$$

In conclusion, an element $g \in MU^*(\mathrm{pt})[\![c]\!]$ is a homomorphism between the universal formal group law and the multiplicative formal group law if and only if $g+1$ is represented by a homotopy ring map. But Lemma C applied to the representing homotopy ring map of g implies that $g+1 = 1$, so that $g = 0$. □

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