AGT

Algebraic & Geometric Topology

Volume 25 (2025)

The generalized Kauffman–Harary conjecture is true

Rhea Palak Bakshi
Huizheng Guo
Gabriel Montoya-Vega
Sujoy Mukherjee
Józef H Przytycki

msp

Algebraic & Geometric Topology **25**:4 (2025) 2067–2081
DOI: 10.2140/agt.2025.25.2067
Published: 11 August 2025

The generalized Kauffman–Harary conjecture is true

RHEA PALAK BAKSHI
HUIZHENG GUO
GABRIEL MONTOYA-VEGA
SUJOY MUKHERJEE
JÓZEF H PRZYTYCKI

For a reduced alternating diagram of a knot with a prime determinant p, the Kauffman–Harary conjecture states that every nontrivial Fox p-coloring of the knot assigns different colors to its arcs. We prove a generalization of the conjecture, stated nineteen years ago by Asaeda, Przytycki and Sikora: for every pair of distinct arcs in the reduced alternating diagram of a prime link with determinant δ, there exists a Fox δ-coloring that distinguishes them.

57K10; 57M12

1 History of the alternation conjecture

In 1998, Louis H Kauffman and Frank Harary formulated the following conjecture [7]:

Alternation conjecture *Let D be a reduced, alternating diagram of a knot K having determinant p, where p is prime. Then every nontrivial p-coloring of D assigns different colors to different arcs.*

This conjecture is now known as the Kauffman–Harary conjecture. It was proved for rational knots [9; 13], Montesinos knots [1], some Turk's head knots [6], and algebraic knots [5]. In 2009, Thomas W Mattman and Pablo Solis proved this conjecture using the notion of pseudocolorings. A generalization of this conjecture, known as the generalized Kauffman–Harary (GKH) conjecture, was formulated by Marta M Asaeda, Adam S Sikora and the fifth author in 2004 [1]. They proved this conjecture for Montesinos links in the same paper. In this paper, we prove it in full generality.

The paper is structured as follows. In the next section we introduce the GKH conjecture and we prove it in Section 3. In Section 4, we reformulate and prove the conjecture for nonprime alternating links. We illustrate the results with some examples in Section 5. In the last section, we discuss pseudocolorings, followed by some open questions.

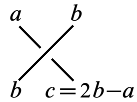
2 Preliminaries

In this section, we state the original GKH conjecture and then some alternative versions. The difference between the original and generalized Kauffman–Harary conjecture is that the former is restricted to links with prime determinant. Note that the only link whose determinant is prime is the Hopf link.

Generalized Kauffman–Harary conjecture *If D is a reduced alternating diagram of a prime link L, then different arcs of D represent different elements of $H_1(M_L^{(2)}, \mathbb{Z})$, where $M_L^{(2)}$ denotes the double branched cover of S^3 branched along L.*

The GKH conjecture was formulated in [1] using the homology of the double branched cover of S^3 branched along L. We use a diagrammatic version of this conjecture by using the universal[1] group of Fox colorings $\mathrm{Col}(D)$ for a prime link L with diagram D.

Definition 2.1 The group $\mathrm{Col}(D)$ is the abelian group whose generators are indexed by the arcs of D, denoted by $\mathrm{arcs}(D)$, and whose relations are $2b-a-c=0$, given by the crossings of D. More precisely,

$$\mathrm{Col}(D) = \left\{ \mathrm{arcs}(D) \;\middle|\; \right.$$


$$\left. \right\}.$$

It is known that $\mathrm{Col}(D) = \mathbb{Z} \oplus H_1(M_L^{(2)}, \mathbb{Z})$ (see for example [14]).

Definition 2.2 Let $\mathrm{Col}^{\mathrm{trivial}}(D) \cong \mathbb{Z}$ be the group of trivial colorings of D. This group is embedded in $\mathrm{Col}(D)$ and the quotient group $\mathrm{Col}(D)/\mathrm{Col}^{\mathrm{trivial}}(D)$ is called the *reduced group of Fox colorings*. We denote it by $\mathrm{Col}^{\mathrm{red}}(D)$.

Notice that, for a diagram D of a link L, $\mathrm{Col}^{\mathrm{red}}(D) = H_1(M_L^{(2)}, \mathbb{Z})$, and, for nonsplit alternating links, this group is finite with nonzero determinant.

The first two statements of the following conjecture are equivalent to the original GKH conjecture, while part (c) offers an extension.

Conjecture 2.3 (alternative forms of the generalized Kauffman–Harary conjecture) *Let D be a reduced alternating diagram of an alternating prime link and let $\delta(D)$ denote the absolute value of its determinant.*

(a) *Let $\mathbb{Z}^{|\mathrm{arcs}|}$ denote the free abelian group with basis set $\mathrm{arcs}(D)$. The map $\mathbb{Z}^{|\mathrm{arcs}|} \xrightarrow{\beta} \mathrm{Col}(D)$ is injective on the arcs of D, that is, $\beta(a_i) \neq \beta(a_j)$ for $i \neq j$.*

(b) *The diagram D has t Fox $\delta(D)$-colorings $y_1, y_2, \dots, y_t$, such that, for every pair of distinct arcs a_i, a_j, there exists y_k such that $y_k(a_i) \neq y_k(a_j)$.*

(c) *If $\mathrm{Col}^{\mathrm{red}}(D) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ with $n_{i+1} | n_i$, then there are s Fox n_1-colorings such that, for each pair of arcs, one of these colorings distinguishes them.*

Remark 2.4 Parts (a) and (b) of Conjecture 2.3 are equivalent to each other, since, for a finite group G, we have $G \cong \mathrm{Hom}(G, \mathbb{Z}_{n_1}) \cong \mathrm{Hom}(G, \mathbb{Z}_{\delta(D)})$, where $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ with $n_{i+1} | n_i$ and $\delta(D) = n_1 n_2 \cdots n_s$. In particular, $\mathrm{Hom}(\mathrm{Col}^{\mathrm{red}}(D), \mathbb{Z}_{\delta(D)}) \cong \mathrm{Hom}(\mathrm{Col}^{\mathrm{red}}(D), \mathbb{Z}_{n_1}) \cong \mathrm{Col}^{\mathrm{red}}(D)$. Thus, we can work with a group or its dual. To distinguish elements in the group, we often analyze its homomorphisms (dual elements) into the given ring. See [10], for example.

[1] Analogous to the fundamental group and the fundamental quandle, this group is often called the fundamental group of Fox colorings.

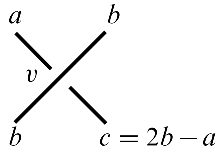


Figure 1: Fox coloring relation at crossing v.

3 Proof of the generalized Kauffman–Harary conjecture

The proof of the GKH conjecture is organized as follows. First we define the crossing matrix $C'(D)$ and coloring matrix $L(D)$ of a link diagram D. Following [11], we prove that every column of the coloring matrix represents a nontrivial Fox $\delta(D)$-coloring. Then, using the fact that the coloring matrix of the mirror image of D is the transpose of L, we prove part (b), and equivalently, part (a) of Conjecture 2.3. Additionally, we show that the columns of the coloring matrix generate the group $\mathrm{Col}^{\mathrm{red}}(D)$ and use this fact to prove part (c) of Conjecture 2.3.

Definition 3.1 A *Fox k-coloring* of a diagram D is a function $f\colon \mathrm{arcs}(D) \to \mathbb{Z}_k$ satisfying the property that every arc is colored by an element of $\mathbb{Z}_k = \{0, 1, 2, 3, \dots, k-1\}$ in such a way that, at each crossing, the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo k. That is, if at a crossing v the overcrossing is colored by b and the undercrossings are colored by a and c, then $2b - a - c \equiv 0$ modulo k. See Figure 1 for an illustration. The group of Fox k-colorings of a diagram D is denoted by $\mathrm{Col}_k(D)$ and the number of Fox k-colorings is denoted by $\mathrm{col}_k(D)$. Analogously to Definition 2.2, we quotient $\mathrm{Col}_k(D)$ by the group of trivial colorings and denote this by $\mathrm{Col}_k^{\mathrm{red}}(D)$.

The matrix describing the space of colorings $\mathrm{Col}(D)$ is referred to by Mattman and Solis [11] as the crossing matrix for a fixed arbitrary ordering of crossings and arcs. If the diagram D of n crossings has no trivial components, then this is an $n \times n$ matrix. For an alternating link diagram, we can order arcs in agreement with the ordering of crossings. Thus we have:

Definition 3.2 Fix an ordering of the crossings of a reduced, connected, alternating link diagram D of a nontrivial link. Then the set of arcs inherits the order of the set of crossings. In this way, the over-arc has the same index as the crossing. The *crossing matrix*[2] of D, denoted by $C'(D)$, is an $n \times n$ matrix such that each row corresponds to a crossing that gives the relation $2b - a - c = 0$ (see Figure 1). The entries of the matrix are defined as[3]

$$C'_{ij} = \begin{cases} 2 & \text{if } a_i \text{ is the over-arc at } c_i, \\ -1 & \text{if } a_j \text{ is an under-arc at } c_i \ (i \neq j), \\ 0 & \text{otherwise.} \end{cases}$$

[2]The alternative, more descriptive name could be *unreduced fundamental Fox colorings matrix.*

[3]It is possible that two under-arcs at a crossing are not distinct. Then the relation $2b - a - c = 0$ becomes $2b - 2a = 0$. For instance, this may occur for the Hopf link.

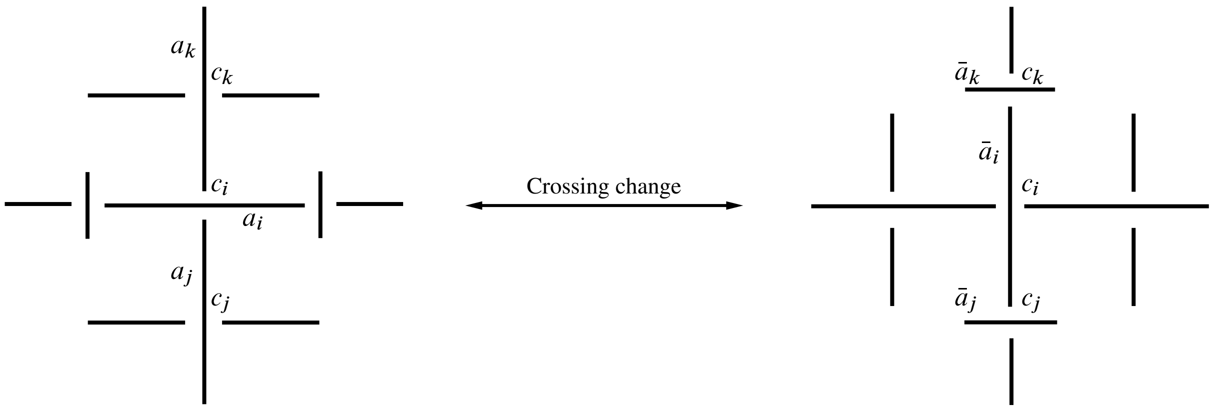


Figure 2: Neighborhood of the crossing c_i in D (on the left) and $\overline{D}$.

The following lemma holds only for alternating links and plays an important role in the proof of the GKH conjecture:

Lemma 3.3 *Let D be a reduced alternating link diagram with crossing matrix $C'(D)$ and let $\overline{D}$ be its mirror image. Then the matrix C'^T is a crossing matrix for $\overline{D}$.*

Proof Denote the crossings of the diagram D by $c_1, \dots, c_n$ and let the over-arc at the crossing c_i be denoted by a_i. Notice that, in the matrix $C'(D)$, all entries on the diagonal are 2. We obtain $\overline{D}$ by crossing-change operations and we keep the ordering and names of the crossings. Now let $\bar{a}_i$ denote the over-arc at the crossing c_i in the diagram $\overline{D}$. In the row corresponding to the crossing c_i, suppose the columns corresponding to the arcs a_j and a_k have -1 as entries. Then, in the matrix $C'(\overline{D})$, the column corresponding to $\bar{a}_i$ must have entries -1 in the rows corresponding to the crossings c_j and c_k; see Figure 2. □

Recall that, if $\delta(D) \neq 0$, then $\mathrm{Col}^{\mathrm{red}}(D)$ is a finite group whose invariant factor decomposition is $\mathrm{Col}^{\mathrm{red}}(D) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, with $n_{i+1} | n_i$ for all i. Notice that s is the minimum number of generators of this group and n_1 is the annihilator of the group. Let $C(D)$ denote the reduced crossing matrix of D, which is the matrix obtained from $C'(D)$ by removing its last row and last column. We call the arc corresponding to the last column of $C'(D)$ the *base arc*. This matrix describes the group $\mathrm{Col}^{\mathrm{red}}(D)$. The matrix $C^{-1}(D)$ is a matrix with rational entries. However, $n_1 C^{-1}(D)$ is an integral matrix, which we denote by $L_{n_1}(D)$. Observe that the columns of $L_{n_1}(D)$ modulo n_1 represent Fox n_1-colorings of the diagram D after coloring the base arc by color 0.

The following result also holds for reduced nonalternating diagrams:

Theorem 3.4 *Let D be a reduced diagram of a link with nonzero determinant. Then the columns of $L_{n_1}(D)$ modulo n_1 generate the space of Fox n_1-colorings of D.*

Proof Let $C(D)$ be the reduced crossing matrix of D and let $\mathrm{Col}^{\mathrm{red}}(D) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$ with $n_{i+1} | n_i$ for all i. After row and column operations, $C(D)$ can be reduced to its Smith normal form

$$C_{\mathrm{SNF}}(D) = \begin{pmatrix} n_1 & & & & & & \\ & n_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & n_s & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Its inverse matrix, $C^{-1}_{\mathrm{SNF}}(D)$, with entries in $\mathbb{Q}$ has the form

$$C^{-1}_{\mathrm{SNF}}(D) = \begin{pmatrix} 1/n_1 & & & & & & \\ & 1/n_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1/n_s & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Thus, we obtain the integral matrix

$$L^{\mathrm{SNF}}_{n_1}(D) = n_1 C^{-1}_{\mathrm{SNF}}(D) = \begin{pmatrix} n_1/n_1 & & & & & & \\ & n_1/n_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & n_1/n_s & & & \\ & & & & n_1 & & \\ & 0 & & & & \ddots & \\ & & & & & & n_1 \end{pmatrix}$$

Now the i^{th} column $(0, 0, \ldots, n_1/n_i, \ldots, 0)^T$ of $L^{\mathrm{SNF}}_{n_1}(D)$ modulo n_1 with $i \le s$ generates the subgroup $\mathbb{Z}_{n_i}$ of $\mathbb{Z}_{n_1}$. Since $\mathrm{Col}^{\mathrm{red}}_{n_1}(D) = \mathrm{Hom}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}, \mathbb{Z}_{n_1})$, the columns of $L^{\mathrm{SNF}}_{n_1}$ generate $\mathrm{Col}^{\mathrm{red}}_{n_1}(D)$, as desired. □

For alternating diagrams we can prove the following stronger result, which proves part (b) and, equivalently, part (a) of Conjecture 2.3:

Theorem 3.5 *Let D be a reduced alternating diagram of a prime link. For any two arcs a_i and a_j, there exists a column of $L_{n_1}(D)$ which distinguishes them.*

Proof Suppose the arcs (indexing rows) of the coloring matrix $n_1 C^{-1}(D) = L_{n_1}(D)$ are given by $a_1, a_2, \ldots, a_{n-1}$ as shown below:

$$L_{n_1}(D) = n_1 C^{-1}(D) = \begin{array}{r} a_1: \\ a_2: \\ \vdots \\ a_{n-1}: \\ a_n: \end{array} \begin{array}{c} \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-1} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} \end{pmatrix}. \\ \begin{matrix} 0 & \quad 0 & \quad \cdots & \quad 0 \end{matrix} \end{array}$$

Recall that the reduced crossing matrix $C(D)$ is obtained from $C'(D)$ by removing its last row and last column. Now each column of $L_{n_1}(D)$ colors the remaining first $n-1$ arcs of the diagram. For a complete Fox n_1-coloring of D, we color the last (base) arc a_n by color 0. If any $c_{i,1} = 0 \bmod n_1$ for $i < n$, then column C_1 modulo n_1 cannot distinguish between the arcs a_n and a_i. If all the entries of the rows corresponding to a_i and a_j are not identical modulo n_1, then they can be automatically distinguished by the column in which they are different.

Step 1 If there is no column C_j of $L_{n_1}(D)$ such that $c_{i,j} \neq 0 \bmod n_1$, then every entry in the i^{th} row is $0 \bmod n_1$. It follows that, in the transpose matrix $L^T_{n_1}(D)$, the column C^T_i is the zero column modulo n_1. This would result in the existence of a pseudocoloring of $\overline{D}$ (see Definition 6.1 below [11]), which is a contradiction.[4] Thus, the base arc a_n can be distinguished from any other arc by some column in $L_{n_1}(D)$ modulo n_1.

Step 2 Furthermore, if there are two arcs a_i and a_j with the same color in every column of $L_{n_1}(D)$, then we choose arc a_j as the base arc, which implies that the colors of a_i are equal to zero. So we are back to Step 1. □

The next theorem proves part (c) of Conjecture 2.3, which is a more general version of Theorem 3.5.

Theorem 3.6 *If* $\mathrm{Col}^{\mathrm{red}}(D) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_s}$, *with* $n_{i+1} | n_i$, *then there are* s *Fox* n_1*-colorings* (*not necessarily corresponding to the columns of the coloring matrix*) *which distinguish all arcs. That is*, *for each pair of arcs*, *one of these colorings distinguishes them.*

Proof Denote the generators of the group $\mathrm{Col}^{\mathrm{red}}(D)$ by $a_1, a_2, \dots, a_s$. Every generator a_i is a linear combination of some columns of the coloring matrix $L_{n_1}(D)$ modulo n_1 (see Theorem 3.4). Therefore, they correspond to some coloring of the diagram D. Hence, for every pair of arcs there is a column of $L_{n_1}(D)$ modulo n_1 that distinguishes them. □

Remark 3.7 Let $\mathrm{Col}^{\mathrm{red}}(D)$ be the cyclic group $\mathbb{Z}_{n_1}$.

(a) There exists a nontrivial Fox n_1-coloring that distinguishes all arcs. This follows directly from Theorem 3.6 for $s = 1$.

(b) Additionally, if n_1 is a prime number, then the original Kauffman–Harary conjecture holds. That is, every nontrivial Fox n_1-coloring distinguishes all arcs. This follows because every nonzero element of $\mathbb{Z}_{n_1}$ is its generator.

4 Nonprime alternating links

Theorems 3.5 and 3.6 do not hold as stated for the connected sum of alternating links[5] (see part (a) of Lemma 4.1). In Theorem 4.2, we present a version of the GKH conjecture which holds for nonprime alternating links.

[4] We work with links but the method of [11] can be extended to this case as well.

[5] The connected sum of alternating links is an alternating link; see for example [16].

$$C(D) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

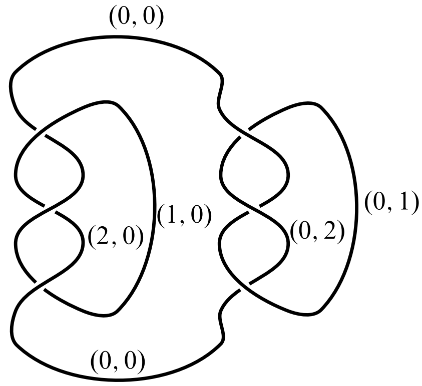


Figure 3: Left: the reduced crossing matrix for the square knot. Right: the square knot $\overline{3}_1 \# 3_1$ with two Fox 3-colorings distinguishing every pair of arcs.

Lemma 4.1 [14] *Let $D = D_1 \# D_2$ be the connected sum of two link diagrams. Then*:

(a) *The arcs connecting the two components represent the same element in* $\mathrm{Col}(D)$.

(b) $\mathrm{Col}^{\mathrm{red}}(D_1 \# D_2) \cong \mathrm{Col}^{\mathrm{red}}(D_1) \oplus \mathrm{Col}^{\mathrm{red}}(D_2)$.

Theorem 4.2 *Let $D = D_1 \# D_2 \# \cdots \# D_n$, where D_i is a reduced alternating diagram of a prime link L_i for $i = 1, 2, \dots, n$. Then*:

(a) *For any pair of arcs different from arcs joining D_i with D_{i+1}, there exists a Fox n_1-coloring which distinguishes them.*

(b) *There are t ($t \le s$) Fox n_1-colorings such that any pair of arcs different from the ones joining D_i with D_{i+1} is distinguished by one of them.*

Proof This result follows from Theorems 3.4 and 3.5 and Lemma 4.1. □

Remark 4.3 Theorem 4.2 was formulated for connected sums of diagrams. However, from William W Menasco's result (see [12; 8]), it follows that, if an alternating diagram represents the connected sum of alternating links, then it is already a connected sum of diagrams.

Example 4.4 Let D be an alternating diagram of the square knot; that is, $D = \overline{3}_1 \# 3_1$, with reduced crossing matrix $C(D)$ (see Figure 3). Then $\mathrm{Col}^{\mathrm{red}}(\overline{3}_1 \# 3_1) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Observe that columns 3 and 5 of $L_3(D)$ modulo 3 (Figure 4) distinguish all pairs of arcs except the ones connecting $\overline{3}_1$ with 3_1. Also, the

$$L_3(D) = 3C^{-1}(D) = \begin{pmatrix} 3 & 2 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}, \quad L_3(D) \bmod 3 = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Figure 4: Matrices $L_3(D)$ and $L_3(D)$ modulo 3 for the square knot.

$$L(7_7)=\begin{pmatrix}24&20&12&10&16&11\\12&24&6&12&15&9\\15&16&18&8&17&13\\6&5&3&13&4&8\\18&22&9&11&26&10\\12&10&6&5&8&16\end{pmatrix},\quad L(7_7)\bmod 21=\begin{pmatrix}3&20&12&10&16&11\\12&3&6&12&15&9\\15&16&18&8&17&13\\6&5&3&13&4&8\\18&1&9&11&5&10\\12&10&6&5&8&16\end{pmatrix}$$

Figure 5: Matrices $L(7_7)$ and $L(7_7)$ modulo 21. Some nontrivial Fox 21-colorings of 7_7 do not distinguish all arcs, for example columns 1 or 3 of L modulo 21. However, columns 2, 4, 5 and 6 distinguish all arcs.

third row (corresponding to the third crossing in the chosen ordering and, therefore, to the third arc) has all zero entries. That is, the third arc cannot be distinguished from the base arc.

5 Examples of Fox colorings

In this section we study examples of alternating link diagrams and their Fox colorings. For the structure of the group $\mathrm{Col}^{\mathrm{red}}(D)=H_1(M_D^{(2)},\mathbb{Z})$ for knots up to 10 crossings, see [2, Appendix C].

Example 5.1 Kauffman and Harary showed that the knot 7_7 is a counterexample to their conjecture for a knot with nonprime determinant [7]. We have $\det(7_7)=21$ and $\mathrm{Col}^{\mathrm{red}}(7_7)=\mathbb{Z}_{21}$.[6] See Figure 6 for a Fox 21-coloring distinguishing all arcs.

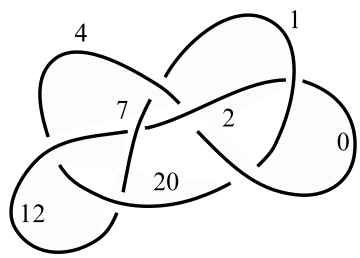


Figure 6: The knot 7_7 with a Fox 21-coloring which distinguishes all arcs.

Example 5.2 Consider the family of links obtained by closing the braids $(\sigma_1\sigma_2^{-1})^n$. These links are sometimes called Turk's head links and can also be obtained by drawing the Tait diagrams of the wheel graphs W_n. The closed formula for the determinant of the $D(W_n)$ is given in [15]. Examples for $n=5$ and $n=6$ are drawn in Figure 7 and their reduced groups of Fox colorings are as follows:

(a) For $n=5$, $D(W_5)$ is 10_{123} in Rolfsen's table [17]. $\mathrm{Col}^{\mathrm{red}}(D(W_5))=\mathbb{Z}_{11}\oplus\mathbb{Z}_{11}$.

(b) For $n=6$, $D(W_6)$ is the link 12^3_{474} in Thistlethwaite's tables [15; 18]. $\mathrm{Col}^{\mathrm{red}}(D(W_6))=\mathbb{Z}_{40}\oplus\mathbb{Z}_8$.

A closed formula for $\mathrm{Col}^{\mathrm{red}}(\sigma_1\sigma_2^{-1})^n$ is given in [4].

[6] It was noticed in [9] that the Kauffman–Harary conjecture holds for any rational (2-bridge) knot without restrictions on the determinant of the knot. However, as they note, the formulation of the conjecture needs to be changed from "every nontrivial Fox D-coloring" to "there exists a Fox D-coloring". See Remark 3.7.

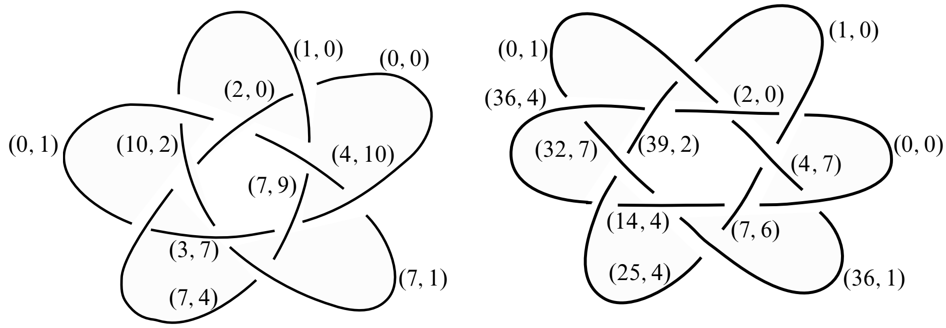


Figure 7: The knot $D(W_5)$ with two Fox colorings distinguishing all arcs (on the left), and the link $D(W_6)$ with two Fox colorings distinguishing all arcs (on the right).

$$\begin{pmatrix}
5 & 1 & 1 & 1 & 12 & 12 & 7 & 8 & 3 & 3 & 14 & 14 & 14 & 10 \\
10 & 5 & 10 & 10 & 10 & 10 & 5 & 10 & 0 & 10 & 10 & 10 & 10 & 10 \\
10 & 1 & 11 & 1 & 12 & 12 & 7 & 8 & 13 & 3 & 4 & 14 & 14 & 10 \\
10 & 12 & 12 & 7 & 14 & 14 & 9 & 6 & 11 & 11 & 13 & 3 & 3 & 10 \\
10 & 1 & 1 & 1 & 7 & 12 & 7 & 8 & 13 & 13 & 4 & 4 & 14 & 10 \\
10 & 12 & 12 & 12 & 14 & 9 & 9 & 6 & 11 & 11 & 13 & 13 & 3 & 0 \\
10 & 8 & 8 & 8 & 6 & 6 & 11 & 4 & 9 & 9 & 7 & 7 & 7 & 5 \\
5 & 7 & 7 & 7 & 9 & 9 & 4 & 11 & 6 & 6 & 8 & 8 & 8 & 10 \\
0 & 3 & 13 & 13 & 11 & 11 & 6 & 9 & 9 & 14 & 12 & 12 & 12 & 10 \\
10 & 14 & 4 & 4 & 13 & 13 & 8 & 7 & 12 & 7 & 1 & 1 & 1 & 10 \\
10 & 3 & 3 & 13 & 11 & 11 & 6 & 9 & 14 & 14 & 7 & 12 & 12 & 10 \\
10 & 14 & 14 & 4 & 3 & 13 & 8 & 7 & 12 & 12 & 1 & 11 & 1 & 10 \\
10 & 10 & 10 & 10 & 10 & 0 & 10 & 5 & 10 & 10 & 10 & 10 & 5 & 10 \\
10 & 14 & 14 & 14 & 3 & 3 & 8 & 7 & 12 & 12 & 1 & 1 & 1 & 5
\end{pmatrix}
\qquad
\begin{pmatrix}
9 & 5 & 1 & 10 & 11 & 18 & 18 & 5 & 7 & 3 & 20 & 1 & 1 & 14 \\
1 & 6 & 11 & 12 & 9 & 16 & 16 & 20 & 14 & 19 & 3 & 18 & 18 & 14 \\
14 & 7 & 0 & 14 & 7 & 14 & 14 & 14 & 0 & 14 & 7 & 14 & 14 & 14 \\
6 & 8 & 10 & 16 & 5 & 12 & 12 & 8 & 7 & 9 & 11 & 10 & 10 & 14 \\
15 & 13 & 11 & 5 & 16 & 9 & 9 & 13 & 14 & 12 & 10 & 11 & 11 & 7 \\
13 & 15 & 17 & 9 & 12 & 12 & 19 & 15 & 14 & 16 & 18 & 17 & 3 & 0 \\
11 & 17 & 2 & 13 & 8 & 15 & 8 & 17 & 14 & 20 & 5 & 2 & 16 & 14 \\
13 & 15 & 17 & 9 & 12 & 19 & 19 & 8 & 14 & 16 & 18 & 3 & 3 & 14 \\
5 & 16 & 6 & 11 & 10 & 17 & 17 & 2 & 0 & 11 & 1 & 20 & 20 & 14 \\
18 & 17 & 16 & 13 & 8 & 15 & 15 & 17 & 7 & 6 & 5 & 16 & 16 & 14 \\
10 & 18 & 5 & 15 & 6 & 13 & 13 & 11 & 14 & 1 & 9 & 12 & 12 & 14 \\
12 & 16 & 20 & 11 & 10 & 17 & 3 & 2 & 14 & 18 & 1 & 13 & 20 & 14 \\
14 & 14 & 14 & 7 & 14 & 0 & 14 & 14 & 14 & 14 & 14 & 14 & 7 & 14 \\
12 & 16 & 20 & 11 & 10 & 3 & 3 & 16 & 14 & 18 & 1 & 20 & 20 & 7
\end{pmatrix}$$

Figure 8: Left: $L(P(3,3,3,3,3))$ modulo 15. The colorings given by columns 3, 4 and 10 distinguish all arcs. Right: $L(P(3,3,3,6))$ modulo 21. The colorings given by columns 1and 6 distinguish all arcs.

Example 5.3 The group $\mathrm{Col}^{\mathrm{red}}$ for pretzel links is given in [1, Proposition 7] and its generalization to Montesinos links in [1, Proposition 8]. Here are two examples with their coloring matrices modulo n_1:

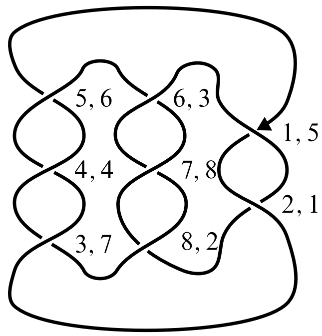
(a) Let $P(3,3,3,3,3)$ be a pretzel knot with 15 crossings. $\mathrm{Col}^{\mathrm{red}} = \mathbb{Z}_{15} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. See its coloring matrix $L(P(3,3,3,3,3))$ modulo 15 in Figure 8, left.

(b) Let $P(3,3,3,6)$ be a pretzel knot with 15 crossings. $\mathrm{Col}^{\mathrm{red}} = \mathbb{Z}_{21} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. See its coloring matrix $L(P(3,3,3,6))$ modulo 21 in Figure 8, right.

6 Odds and ends

6.1 Pseudocolorings

An important tool in our proof of Theorem 3.5 is the idea of pseudocolorings. In [11] and in this paper, it is shown that no pseudocolorings exist for reduced, prime, alternating link diagrams. However, the



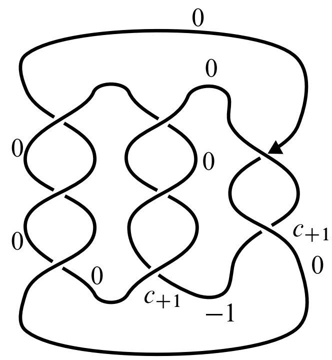


Figure 9: Left: the torus knot $T(3,4)$ (8_{19} in Rolfsen's table [17]) depicted as the pretzel knot $P(3,3,-2)$ showing the ordering of crossings and arcs. Right: the pseudocoloring given by the second column; compare Remark 6.5.

existence of pseudocolorings can be used to see how far a diagram is from being an alternating link diagram. In this section, we briefly explore this concept. Proposition 3.2 in [11] depends on the fact that, for reduced alternating diagrams, the rows of the crossing matrix add to zero. This does not hold for nonalternating diagrams, as we illustrate in the following examples.

Definition 6.1 Let D be a link diagram and $\epsilon \in \{-1,+1\}$. Following Mattman and Solis [11], we define an *ϵ-pseudocoloring* of D as a coloring of the arcs of D such that, at all but two crossings, the Fox coloring convention $2b-a-c=0$ is satisfied. We denote the other two crossings by c_{+1} and c_ϵ, where the coloring conventions are $2b-a-c=+1$ and $2b-a-c=\epsilon$, respectively. To obtain the pseudocolorings as defined in [11], put $\epsilon=-1$.

For an alternating link diagram D, our convention was to order crossings first, and then the set of arcs inherits the order of the set of crossings. Compare Figure 1. The reason for such a choice is that $C'(\overline{D})$ is the same as $C'(D)^T$. This does not work for nonalternating link diagrams.

In general, we can arbitrarily order crossings and arcs. In Figure 9 we give an example of ordering crossings and arcs for the knot 8_{19}. We first choose a basepoint and an orientation (shown by an arrow in Figure 9, left). Starting at this basepoint, we move along the knot and order crossings. Next, arcs can be ordered arbitrarily with the base arc always being the last one. In Figure 9, the first coordinate gives the number of the crossing and the second one gives the number of the arc.

In the following example, we analyze nonsplit, nonprime alternating diagrams:

Example 6.2 Let $D=D_1 \# D_2$ be a nonsplit, nonprime alternating link diagram. D always has a -1-pseudocoloring using color 1 on D_1 and color 0 on D_2. We illustrate this idea for the square knot $\overline{3}_1 \# 3_1$ in Figure 10.

On the other hand, nonalternating link diagrams often have -1-pseudocolorings and $+1$-pseudocolorings. See Examples 6.3 and 6.4. If the determinant of a knot with diagram D is equal to 1, we have

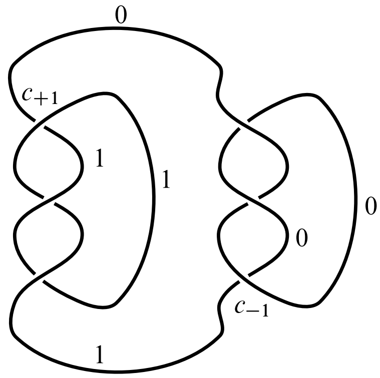


Figure 10: -1-pseudocoloring of the square knot with the $+1$-crossing denoted by c_{+1} and the -1-crossing denoted by c_{-1}.

$$C'(D) = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Figure 11: The crossing matrix of the Conway knot. Notice that the rows of the crossing matrix satisfy the linear equation $R_1 - R_2 - R_3 - R_4 - R_5 - R_6 - R_7 + R_8 + R_9 + R_{10} + R_{11} = 0$.

$L(D) = C^{-1}(D)$ and every column of $C^{-1}(D)$ colors the first $n-1$ arcs of the diagram. Then, for a complete ϵ-pseudocoloring of D, we color the last (base) arc a_n by color 0.

Example 6.3 Consider the braid word $\sigma_2^3\sigma_1\sigma_3^{-1}\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1\sigma_3^{-1}$ whose closure is the Conway knot. The determinant of this knot is 1 and its crossing matrix $C'(D)$ is given in Figure 11. The $+1$-pseudocoloring given by column 4 and the -1-pseudocoloring given by column 1 in the matrix shown in Figure 12 are illustrated in Figure 13, left and right, respectively.

Example 6.4 Consider the torus knot $T(3,4)$ with diagram D and crossings and arcs ordered as illustrated in Figure 9. Its crossing matrix is shown in Figure 15. Three columns of $C^{-1}(D)$ (Figure 16) are integral and they yield ϵ-pseudocolorings. Column 5 gives a -1-pseudocoloring (Figure 14, right) and columns 1 and 2 give $+1$-pseudocolorings. The $+1$-pseudocoloring corresponding to column 1 is shown in Figure 14, left.

Nonalternating link diagrams always have ϵ-pseudocolorings, as we describe in the following remark:

$$L(D)=\begin{pmatrix} \mathbf{6} & -6 & -2 & \mathbf{2} & -4 & -10 & -3 & 4 & 8 & 12 \\ \mathbf{-33} & 32 & 12 & \mathbf{-8} & 22 & 52 & 17 & -22 & -44 & -66 \\ \mathbf{-9} & 9 & 4 & \mathbf{-2} & 6 & 14 & 5 & -6 & -12 & -18 \\ \mathbf{21} & -21 & -8 & \mathbf{5} & -14 & -34 & -11 & 14 & 28 & 42 \\ \mathbf{-21} & 21 & 8 & \mathbf{-5} & 14 & 33 & 11 & -14 & -28 & -42 \\ \mathbf{3} & -3 & -1 & \mathbf{1} & -2 & -5 & -1 & 2 & 4 & 6 \\ \mathbf{39} & -39 & -15 & \mathbf{9} & -27 & -63 & -21 & 26 & 52 & 78 \\ \mathbf{13} & -13 & -5 & \mathbf{3} & -9 & -21 & -7 & 9 & 18 & 26 \\ \mathbf{-13} & 13 & 5 & \mathbf{-3} & 9 & 21 & 7 & -9 & -18 & -27 \\ \mathbf{26} & -26 & -10 & \mathbf{6} & -18 & -42 & -14 & 18 & 35 & 52 \end{pmatrix}$$
$$\begin{matrix} \mathbf{0} & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Figure 12: Coloring matrix for the Conway knot. The last row of zeroes correspond to the coloring of the base arc.

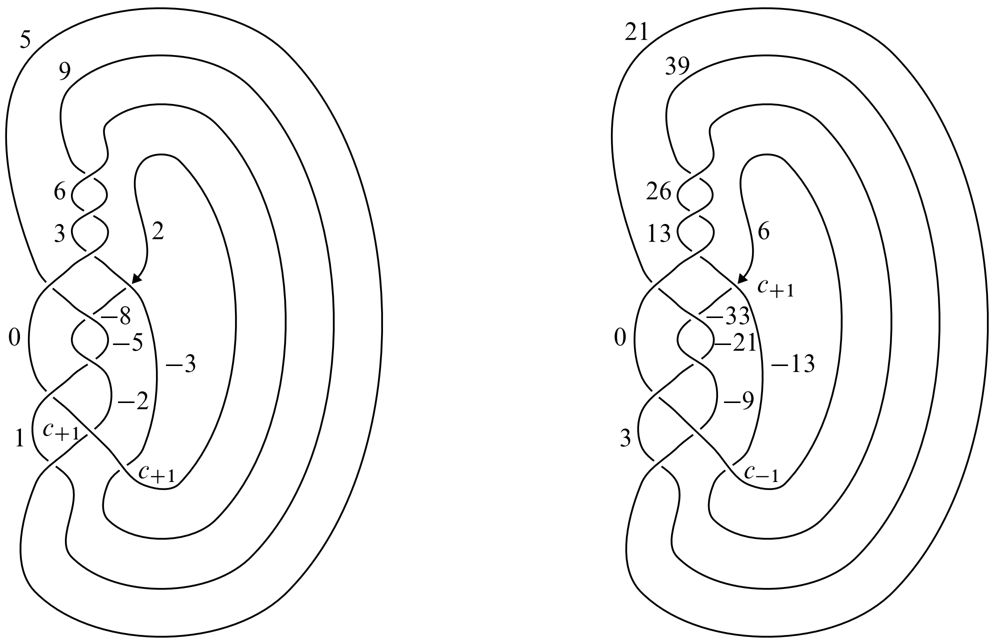


Figure 13: The Conway knot with $+1$-pseudocoloring (left) and -1-pseudocoloring (right). The last crossing c_{+1} on the left changes to c_{-1} on the right.

Remark 6.5 Let D be a nonalternating link diagram.

(1) Every integral column of $C^{-1}(D)$ leads to some ϵ-pseudocoloring.

(2) D has an ϵ-pseudocoloring. This follows from the fact that every nonalternating diagram has a tunnel of length at least two. Now we can color D by coloring one of the arcs of the tunnel by color -1 and all other arcs by color 0 to get the $+1$-pseudocoloring. An example of such a coloring is shown in Figure 9, right.

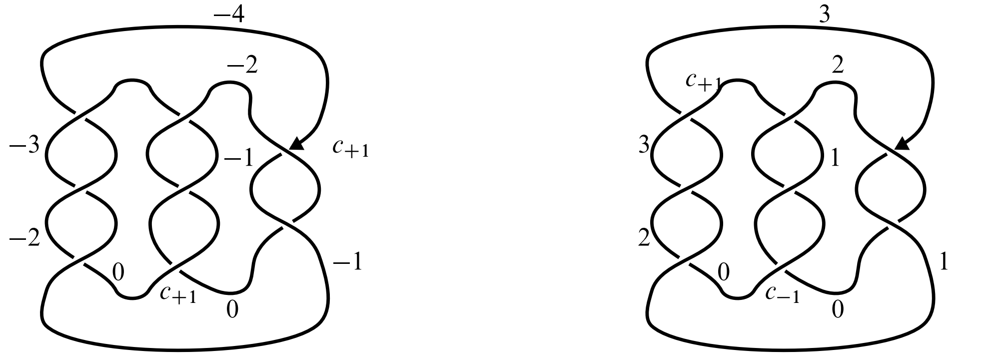


Figure 14: The torus knot $T(3,4)$ (8_{19} in Rolfsen's table [17]) depicted as the pretzel knot $P(3,3,-2)$.

$$C'(D)=\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & -1\\ 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0\\ 0 & 0 & 0 & 2 & 0 & -1 & -1 & 0\\ 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 2 & 0 & 0 & 0 & 0 & -1\\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Figure 15: $C'(P(3,3,-2))$. The rows satisfy $R_1+R_2-R_3-R_4-R_5-R_6-R_7-R_8=0$.

$$C^{-1}(D)=\begin{pmatrix} -2 & 0 & -\frac{2}{3} & \frac{2}{3} & 2 & \frac{10}{3} & \frac{5}{3}\\ 0 & -1 & \frac{4}{3} & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{3}\\ -1 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 & \frac{5}{3} & \frac{4}{3}\\ -3 & 0 & -1 & 1 & 3 & 4 & 2\\ -1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{4}{3} & \frac{2}{3}\\ -4 & 0 & -\frac{5}{3} & \frac{2}{3} & 3 & \frac{16}{3} & \frac{8}{3}\\ -2 & 0 & -\frac{1}{3} & \frac{4}{3} & 2 & \frac{8}{3} & \frac{4}{3} \end{pmatrix}$$

Figure 16: $C^{-1}(D)$ corresponding to $T(3,4)$ with three integral columns.

6.2 Future directions

The Kauffman–Harary conjecture was extended to the case of virtual knots by Mathew Williamson [19] and proved by Zhiyun Cheng [3]. A natural question is to ask whether the conjecture in [1] holds for virtual links whose determinants are not prime. Another path of further research is to look for a natural generalization to nonalternating diagrams using a set-theoretic Yang–Baxter operator or a general Yang–Baxter operator.

An interesting prospect is to approach the generalized Kauffman–Harary conjecture from the perspective of incompressible surfaces in the double branched cover $M_L^{(2)}$ of S^3 branched along L. This was outlined in [1] with the hope of proving the GKH conjecture. Now that the GKH conjecture is proved, we can proceed in the opposite direction and analyze incompressible surfaces in $M_L^{(2)}$.

Acknowledgements

Bakshi acknowledges the support of Dr Max Rössler, the Walter Haefner Foundation and the ETH Zürich Foundation. Montoya-Vega acknowledges the support of the National Science Foundation through grant DMS-2212736. Mukherjee was supported by the American Mathematical Society and the Simons Foundation through the AMS–Simons Travel Grant. Przytycki was partially supported by Simons Collaboration Grant 637794.

References

[1] **M M Asaeda**, **J H Przytycki**, **A S Sikora**, *Kauffman–Harary conjecture holds for Montesinos knots*, J. Knot Theory Ramifications 13 (2004) 467–477 MR Zbl

[2] **G Burde**, **H Zieschang**, *Knots*, 2nd edition, De Gruyter Studies in Mathematics 5, de Gruyter, Berlin (2003) MR Zbl

[3] **Z Cheng**, *Kauffman–Harary conjecture for alternating virtual knots*, J. Knot Theory Ramifications 24 (2015) art. id. 1550046 MR Zbl

[4] **A Christiana**, **H Guo**, **J H Przytycki**, *Using Fibonacci numbers and Chebyshev polynomials to express Fox coloring groups and Alexander–Burau–Fox modules of diagrams of wheel graphs*, preprint (2023) arXiv 2310.19979 To appear in Contemp. Math.

[5] **D B Damiano**, **E M Sennott**, *Coloring algebraic knots and links*, J. Knot Theory Ramifications 17 (2008) 553–578 MR Zbl

[6] **N E Dowdall**, **T W Mattman**, **K Meek**, **P R Solis**, *On the Harary–Kauffman conjecture and Turk's head knots*, Kobe J. Math. 27 (2010) 1–20 MR Zbl

[7] **F Harary**, **L H Kauffman**, *Knots and graphs, I: Arc graphs and colorings*, Adv. in Appl. Math. 22 (1999) 312–337 MR Zbl

[8] **J Hoste**, *The enumeration and classification of knots and links*, from "Handbook of knot theory" (W Menasco, M Thistlethwaite, editors), Elsevier, Amsterdam (2005) 209–232 MR Zbl

[9] **L H Kauffman**, **S Lambropoulou**, *On the classification of rational tangles*, Adv. in Appl. Math. 33 (2004) 199–237 MR Zbl

[10] **S Lang**, *Algebra*, 3rd edition, Graduate Texts in Math. 211, Springer (2002) MR Zbl

[11] **T W Mattman**, **P Solis**, *A proof of the Kauffman–Harary conjecture*, Algebr. Geom. Topol. 9 (2009) 2027–2039 MR Zbl

[12] **W Menasco**, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984) 37–44 MR Zbl

[13] **L Person**, **M Dunne**, **J DeNinno**, **B Guntel**, **L Smith**, *Colorings of rational*, *alternating knots and links*, preprint (2002)

[14] **J H Przytycki**, *3-coloring and other elementary invariants of knots*, from "Knot theory" (V F R Jones, J Kania-Bartoszyńska, J H Przytycki, P Traczyk, V G Turaev, editors), Banach Center Publ. 42, Polish Acad. Sci. Inst. Math., Warsaw (1998) 275–295 MR Zbl

[15] **J H Przytycki**, *From Goeritz matrices to quasi-alternating links*, from "The mathematics of knots" (M Banagl, D Vogel, editors), Contrib. Math. Comput. Sci. 1, Springer (2011) 257–316 MR Zbl

[16] **J H Przytycki**, **R P Bakshi**, **D Ibarra**, **G Montoya-Vega**, **D Weeks**, *Lectures in knot theory: an exploration of contemporary topics*, Springer (2024) MR Zbl

[17] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR Zbl

[18] **M B Thistlethwaite**, *Knot tabulations and related topics*, from "Aspects of topology" (I M James, E H Kronheimer, editors), London Math. Soc. Lecture Note Ser. 93, Cambridge Univ. Press (1985) 1–76 MR Zbl

[19] **M Williamson**, *Kauffman–Harary conjecture for virtual knots*, master's thesis, University of South Florida (2007) Available at `https://digitalcommons.usf.edu/etd/3916`

RPB: *Institute for Theoretical Studies, ETH Zurich*
Zurich, Switzerland

Current address: *Department of Mathematics, University of California, Santa Barbara*
Santa Barbara, CA, United States

GM-V: *Department of Mathematics, The Graduate Center CUNY*
New York, NY, United States

Department of Mathematics, University of Puerto Rico at Río Piedras
San Juan, PR, United States

SM: *Department of Mathematics, University of Denver*
Denver, CO, United States

HG, JHP: *Department of Mathematics, The George Washington University*
Washington, DC, United States

JHP: *Department of Mathematics, University of Gdańsk*
Gdańsk, Poland

`rheapalak@math.ucsb.edu, hguo30@gwu.edu, gabriel.montoyavega55@gc.cuny.edu, sujoymukherjee.math@gmail.com, przytyck@gwu.edu`

Received: 13 January 2023 Revised: 24 November 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 4 (pages 1917–2526) 2025