Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:4 (2025) 2143–2177
DOI: 10.2140/agt.2025.25.2143
Published: 11 August 2025

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The volume conjecture relates the quantum invariant and the hyperbolic geometry. Bonahon, Wong and Yang introduced a new version of the volume conjecture by using the intertwiners between two isomorphic irreducible representations of the skein algebra. The intertwiners are built from surface diffeomorphisms; they formulated the volume conjecture when diffeomorphisms are pseudo-Anosov. We explicitly calculate all the intertwiners for the closed torus using an algebraic embedding from the skein algebra of the closed torus to a quantum torus, and show the limit superior related to the trace of these intertwiners is zero. Moreover, we consider the periodic diffeomorphisms for surfaces with negative Euler characteristic, and conjecture the corresponding limit is zero because the simplicial volume of the mapping tori for periodic diffeomorphisms is zero. For the once punctured torus, we make precise calculations for intertwiners and prove our conjecture.

14H10, 14H30, 14H45, 14H50, 14L10

1 Introduction

We first discuss irreducible representations for skein algebras of the closed torus and the once punctured torus, which is related to Bonahon and Wong's work [3; 4; 5; 6]. They explored the connection between irreducible representations of skein algebras and the character variety related to the fundamental group of a surface. In Section 3, we give more detailed discussions about this connection for the closed torus and the once punctured torus.

A profound result of the skein algebra is the unicity theorem, which was conjectured by Bonahon and Wong and was proved by Frohman, Kania-Bartoszynska and Lê [15]. Based on this result there is an increased focus on the Azumaya locus. Ganev, Jordan and Safronov proved that the smooth part of the character variety lives in the Azumaya locus when the surface is closed [16]. In Section 3, we give an explicit description for the Azumaya locus for the skein algebra of the closed torus.

Let S be an oriented surface, let φ be a diffeomorphism for S, and let $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. Using these data, Bonahon, Wong and Yang built a sequence of intertwiners between irreducible representations of the skein algebra of S [7; 8]. When S has negative Euler characteristic and φ is pseudo-Anosov, they formulated the volume conjecture using these intertwiners:

$$\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\mathrm{Trace}\, \Lambda^{q_n}_{\varphi,\gamma}| = \frac{1}{4\pi} \mathrm{vol}_{\mathrm{hyp}}(M_\varphi),$$

where $\mathrm{vol}_{\mathrm{hyp}}(M_\varphi)$ is the volume of the complete hyperbolic metric of the mapping torus M_φ.

We explicitly compute the intertwiners corresponding to all diffeomorphisms of the closed torus using an algebraic embedding from the skein algebra of the closed torus to a quantum torus [14]; see Section 3 for more details. The representation theory for this quantum torus is well studied. We prove almost all the irreducible representations of this quantum torus can be restricted to irreducible representations of the skein algebra of the closed torus. So intertwiners between two isomorphic irreducible representations of this quantum torus are also the intertwiners between irreducible representations for the skein algebra of the closed torus. These intertwiners are built when the quantum parameter q for the skein algebra is a primitive root of unity of odd order. We use Λ_n to denote the intertwiner obtained as above when the quantum parameter is $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. We also normalize Λ_n such that $|\det(\Lambda_n)| = 1$. Then we prove the following theorem; please refer to Theorem 4.15 for a more detailed version.

Theorem 1.1 *Let Λ_n be defined as above; then*

$$\limsup_{\text{odd } n\to\infty} \frac{\log(|\operatorname{Trace}\Lambda_n|)}{n} = 0.$$

The volume conjecture was first introduced by Kashaev [19], and was rewritten and generalized to the nonhyperbolic case by Hitoshi Murakami and Jun Murakami [22] using the simplicial volume.

Bonahon, Wong and Yang only formulated the conjecture when the diffeomorphisms are pseudo-Anosov for surfaces with negative Euler characteristic. In this paper, we broaden the scope of the conjecture to include periodic diffeomorphisms. When φ is a periodic diffeomorphism for the surface S, the corresponding mapping torus M_φ is a Seifert manifold whose simplicial volume is zero. So we conjecture the limits are zero for periodic diffeomorphisms. We prove our conjecture for the once punctured torus, which serves as an example to confirm the limit is the simplicial volume of the corresponding mapping torus.

Let S be an oriented surface with negative Euler characteristic, and let φ be a periodic diffeomorphism for S. According to [13, page 371], φ fixes a point in the Teichmüller space of S. This fixed point in the Teichmüller space offers a smooth φ-invariant character γ (that is γ is a group homomorphism from $\pi_1(S)$ to $\mathrm{SL}(2,\mathbb{C})$ such that $\gamma\varphi_*$ and γ have the same character, where φ_* is the isomorphism from $\pi_1(S)$ to $\pi_1(S)$ induced by φ). Suppose the quantum parameter for the skein algebra is $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$ and n odd. For each puncture v of S, we choose a complex number p_v such that $p_v = p_{\varphi(v)}$ and $T_n(p_v) = -\operatorname{Trace}(\gamma(\alpha_v))$, where T_n is the n^{th} Chebyshev polynomial of the first type and α_v is the element in $\pi_1(S)$ going around puncture v. According to Theorem 2.1, we know γ and p_v uniquely determine an irreducible representation ρ of the skein algebra. Let $\varphi_\sharp$ be the isomorphism from the skein algebra of S to itself induced by φ. Since both γ and p_v are φ-invariant, we have ρ and $\rho\varphi_\sharp$ are isomorphic according to Theorem 2.2. Thus there exists the intertwiner $\Lambda^{q_n}_{\varphi,\gamma}$ between these two isomorphic irreducible representations. We normalize it such that $|\det(\Lambda^{q_n}_{\varphi,\gamma})| = 1$. Then we formulate the following conjecture, please refer to Conjecture 5.4 for a more detailed version.

Conjecture 1.2 *Let S be a surface with negative Euler characteristic, let φ be a periodic diffeomorphism for S, and let $\Lambda^{q_n}_{\varphi,\gamma}$ be defined as above; then*

$$\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\text{Trace}\, \Lambda^{q_n}_{\varphi,\gamma}| = 0.$$

In Theorem 5.5, we prove the limit in Conjecture 1.2 is less than or equal to zero if it exists by using the periodic property. It seems like we are halfway there to prove our conjecture. But proving that the limit is greater than or equal to zero is harder, which is actually related to an interesting question raised by Gerald Myerson [24] and Terry Tao [27]. By direct calculations and using some conclusions in [20; 24], we prove the above conjecture for some special cases:

Theorem 1.3 *For any surface with negative Euler characteristic, if φ is of order 2^m where m is any positive integer, then*

$$\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\text{Trace}\, \Lambda^{q_n}_{\varphi,\gamma_\varphi}| = 0.$$

Theorem 1.4 *Conjecture 1.2 holds if S is the once punctured torus.*

Plan of the paper: In Section 2, we introduce the Kauffman bracket skein algebra, the classical shadow, the volume conjecture, and the Chekhov–Fork algebra. Section 3 is about the discussion on the irreducible representations of skein algebras of the closed torus and the once punctured torus. In Section 4, we calculate the intertwiners for the closed torus and prove Theorem 1.1. In Section 5, we formulate our conjecture for periodic diffeomorphisms and prove Theorems 1.3 and 1.4.

Acknowledgements The idea of considering the periodic case was suggested by my supervisor Andrew James Kricker. We would like to thank Andrew James Kricker, Jeffrey Weenink, Roland van der Veen and Xiaoming Yu for constructive discussion and help. We wish to thank the referee most warmly for numerous suggestions that have improved the exposition of this paper. The research is supported by the NTU research scholarship.

2 Preliminaries

2.1 The $\mathrm{SL}(2,\mathbb{C})$ character variety and the Kauffman bracket skein algebra

Let S be an oriented surface of finite type. The corresponding character variety

$$\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{SL}(2,\mathbb{C}))/\!/\mathrm{SL}(2,\mathbb{C})$$

is the set of the group homomorphisms from the fundamental group of S to $\mathrm{SL}(2,\mathbb{C})$ with the equivalence relation that two homomorphisms are equivalent if and only if they have the same character [9; 12; 25].

The Kauffman bracket skein algebra $\mathrm{SK}_{q^{1/2}}(S)$ of a surface S, as a vector space over the complex field $\mathbb{C}$, is generated by all isotopic framed links in $S\times[0,1]$, subject to the skein relation

$$K_1 = q^{-1/2} K_\infty + q^{1/2} K_0,$$

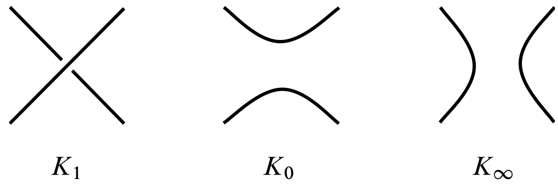

Figure 1: The Kauffman bracket skein relation.

where K_1, K_∞ and K_0 are three links that differ in a small neighborhood as shown in Figure 1, and the trivial knot relation $K \coprod \bigcirc = -(q+q^{-1})K$, where $\bigcirc$ is a simple knot bounding a disk that has no intersection with K. For any two links $[L_1]$ and $[L_2]$, the multiplication $[L_1][L_2]$ is defined by stacking L_2 above L_1. Here $q^{1/2}$ is a nonzero complex number. The skein algebra $\mathrm{SK}_{q^{1/2}}(S)$ is a quantization for the regular ring of the character variety $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ [9].

2.2 Classical shadow and unicity theorem

We recall some notation and constructions for the classical shadow [3]. When q is a primitive n^{th} root of unity with n odd and $(q^{1/2})^n = -1$, Bonahon and Wong found a fascinating algebra homomorphism $T^{q^{1/2}}$ from $\mathrm{SK}_{-1}(S)$ to $\mathrm{SK}_{q^{1/2}}(S)$, called the Chebyshev homomorphism. Bonahon and Wong proved that $\mathrm{Im}(T^{q^{1/2}})$ is contained in the center of $\mathrm{SK}_{q^{1/2}}(S)$. If K is a simple knot with vertical framing, then $T^{q^{1/2}}([K]) = T_n([K])$ where T_n is the n^{th} Chebyshev polynomial of the first type.

Let $\rho\colon \mathrm{SK}_{q^{1/2}}(S) \to \mathrm{End}(V)$ be an irreducible representation of $\mathrm{SK}_{q^{1/2}}(S)$. Then there exists an algebra homomorphism κ_ρ from $\mathrm{SK}_{-1}(S)$ to $\mathbb{C}$ such that $\rho \circ T^{q^{1/2}}(X) = \kappa_\rho(X)\mathrm{Id}_V$ for any X in $\mathrm{SK}_{-1}(S)$. According to [9], there exists a unique character $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ such that $\mathrm{Tr}^\gamma = \kappa_\rho$. Recall that Tr^γ is an algebra homomorphism from $\mathrm{SK}_{-1}(S)$ to $\mathbb{C}$ defined by $\mathrm{Tr}^\gamma([K]) = -\operatorname{Trace}\gamma(K)$ where $[K]$ is a simple knot. For every puncture v, we use P_v to denote the loop going around this puncture. There is a complex number p_v such that $\rho([P_v]) = p_v \mathrm{Id}_V$. Then an irreducible representation of $\mathrm{SK}_{q^{1/2}}(S)$ gives a character $[\gamma]$, called the classical shadow of this irreducible representation, and puncture weights $\{p_v\}_v$, with the relation that $-\operatorname{Trace}\gamma(\alpha_v) = T_n(p_v)$ where α_v denotes the element in the fundamental group of S going around the puncture v.

Theorem 2.1 [3; 5; 6; 7] *Let q be a primitive n^{th} root of unity with n odd and $(q^{1/2})^n = -1$. Then an irreducible representation $\rho\colon \mathrm{SK}_{q^{1/2}} \to \mathrm{End}(V)$ uniquely determines*

(1) *a character $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$, represented by a group homomorphism $\gamma : \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$;*

(2) *a weight p_v associated to each puncture v of S such that $T_n(p_v) = -\operatorname{Trace}\gamma(\alpha_v)$.*

Conversely, every data of a character $\gamma \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ and of puncture weights $p_v \in \mathbb{C}$ satisfying the above condition is realized by an irreducible representation $\rho\colon \mathrm{SK}_{q^{1/2}}(S) \to \mathrm{End}(V)$.

It turns out that every character in an open dense subset of $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ corresponds to a unique irreducible representation of the skein algebra.

Theorem 2.2 [7; 15; 16] *Suppose that* $[\gamma]$ *is in the smooth part of* $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ *or, equivalently, that it is realized by an irreducible homomorphism* $\gamma\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$. *Then the irreducible representation* $\rho\colon \mathrm{SK}_{q^{1/2}}(S) \to \mathrm{End}(V)$ *in Theorem 2.1 is unique up to isomorphism of representations. This representation has dimension* $\dim V = n^{3g+p-3}$ *if* S *has genus* g *and* p *punctures.*

2.3 Volume conjecture for surface diffeomorphisms

Bonahon, Wong and Yang constructed the so called Kauffman bracket intertwiners [7; 8]. They used these intertwiners to formulate the volume conjecture for surface diffeomorphisms. Here we recall their construction for Kauffman bracket intertwiners.

For a surface S, let φ be a diffeomorphism of S. Obviously φ induces an isomorphism φ_* from $\pi_1(S)$ to $\pi_1(S)$. Then φ_* induces an action on $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ defined by $\varphi^*([\gamma]) = [\gamma\varphi_*]$ where γ is a representative for $[\gamma]$. Although φ_* is only defined up to conjugation, φ^* is well defined. Actually the mapping class group $\mathrm{Mod}(S)$ acts on $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$. We say an element $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ is invariant under a diffeomorphism φ, or the element it represents in $\mathrm{Mod}(S)$, if $\varphi^*([\gamma]) = [\gamma]$.

The algebra isomorphism induced by φ from $\mathrm{SK}_{q^{1/2}}(S)$ to itself is defined by $\varphi_\sharp([K]) = [\varphi \times \mathrm{Id}_{[0,1]}(K)]$ where K is a framed link in $S \times [0,1]$. Actually the mapping class group $\mathrm{Mod}(S)$ acts on $\mathrm{SK}_{q^{1/2}}(S)$.

Let φ be any diffeomorphism for surface S, and let $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ be a φ-invariant smooth character. For each puncture v, select a complex number θ_v such that $\mathrm{Trace}\,\gamma(\alpha_v) = -e^{\theta_v} - e^{-\theta_v}$. Since $[\gamma]$ is φ-invariant, we can choose θ_v to be φ-invariant, that is, $\theta_v = \theta_{\varphi(v)}$. Then set $p_v = e^{\theta_v/n} + e^{-\theta_v/n}$; we have that $T_n(p_v) = -\mathrm{Trace}(\alpha_v)$ and $\{p_v\}_v$ are invariant under the action of φ. Suppose ρ is an irreducible representation associated to $[\gamma]$ and puncture weights p_v. Then $\rho \circ \varphi_\sharp$ is also an irreducible representation associated to $[\gamma]$ and puncture weights p_v. By the unicity theorem, we know there exists an intertwiner $\Lambda^q_{\varphi,\gamma}$ such that

$$\rho \circ \varphi_\sharp(X) = \Lambda^q_{\varphi,\gamma} \circ \rho(X) \circ (\Lambda^q_{\varphi,\gamma})^{-1}$$

for every $X \in \mathrm{SK}_{q^{1/2}}(S)$. We normalize the intertwiner such that $|\det(\Lambda^q_{\varphi,\gamma})| = 1$.

Conjecture 2.3 [7; 8] *Let the pseudo-Anosov surface diffeomorphism* $\varphi\colon S \to S$, *the* φ*-invariant smooth character* $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$ *and the* φ*-invariant puncture weights* p_v *as above be given. For every odd* n, *consider the primitive* n^{th} *root of unity* $q_n = e^{2\pi i/n}$ *and choose* $(q_n)^{1/2} = e^{\pi i/n}$. *Then*

$$\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,\gamma}| = \frac{1}{4\pi}\,\mathrm{vol}_{\mathrm{hyp}}(M_\varphi),$$

where $\mathrm{vol}_{\mathrm{hyp}}(M_\varphi)$ *is the volume of the complete hyperbolic metric of the mapping torus* M_φ.

2.4 Ideal triangulation and intertwiners obtained from Chekhov–Fock algebras

Let S be an oriented surface with punctures, and let $\tau = \{e_1,\ldots,e_m\}$ be an ideal triangulation for S, where $e_1,\ldots,e_m$ are nonisotopic disjoint embedded arcs in S connecting punctures such that all these arcs cut S into triangles. We call $e_1,\ldots,e_m$ the edges of τ. An edge weight system for τ is an m-tuple,

$a = (a_1, \dots, a_m)$, where a_i is a nonzero complex number for each $1 \le i \le m$. The pair (τ, a) determines a character $[\bar{\gamma}]$ in $\mathscr{X}_{\mathrm{PSL}(2,\mathbb{C})}(S)$; please refer to [1, Section 8] or [7, Section 3] for more details.

For each ideal triangulation τ, there is a Chekhov–Fock algebra $\mathcal{T}_\tau^q$ corresponding to τ, where q is a nonzero complex number. As an algebra over $\mathbb{C}$, the Chekhov–Fock algebra $\mathcal{T}_\tau^q$ is generated by $X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_m^{\pm 1}$ subject to the relations

$$X_i X_i^{-1} = X_i^{-1} X_i = 1, \quad X_i X_j = q^{2\sigma_{ij}} X_j X_i.$$

Each X_i is associated to the i^{th} edge in the ideal triangulation τ, and σ_{ij} is an integer determined by τ; see [1; 2; 21] for more details. If we replace q with $q^{1/4}$, we get the so called Chekhov–Fock square root algebra $\mathcal{T}_\tau^{q^{1/4}}$. It is well known that $\mathcal{T}_\tau^q$ is an Ore domain. We will use $\widehat{\mathcal{T}}_\tau^q$ to denote the ring of fractions of $\mathcal{T}_\tau^q$ (that is the localization over all nonzero elements).

Let τ and τ' be any two ideal triangulations for S. Then there is an algebra isomorphism $\Phi_{\tau\tau'}^q : \widehat{\mathcal{T}}_{\tau'}^q \to \widehat{\mathcal{T}}_\tau^q$, called be the Chekhov–Fock coordinate change isomorphism [21].

For an ideal triangulation τ, there are two operations.

(1) **Reindexing** Obtain a new ideal triangulation τ' by reindexing all the edges in τ.

(2) **Diagonal exchange** For any $1 \le i \le m$, define a new ideal triangulation $\tau' = \{e_1', \dots, e_m'\}$, where $e_j' = e_j$ for every $j \ne i$ and e_i' is the other diagonal of the square formed by the two faces of τ that are adjacent to e_i.

Let τ be an ideal triangulation, let $a = (a_1, \dots, a_m)$ be an edge weight system for τ. Suppose $\tau' = \{e_1', \dots, e_m'\}$ is obtained from τ by reindexing such that $e_i' = e_{\sigma(i)}$ for $1 \le i \le m$, where σ is a permutation for $\{1, \dots, m\}$. Then we define an edge weight system $a' = (a_1', \dots, a_m')$ for τ' by setting $a_i' = a_{\sigma(i)}$ for $1 \le i \le m$. If τ' is obtained from τ by the diagonal exchange, we define an edge weight system a' for τ' using formulas in [21, Proposition 3]. We will say a' is an edge weight system for τ' derived from the pair (τ, a). Then (τ', a') determines the same character in $\mathscr{X}_{\mathrm{PSL}(2,\mathbb{C})}(S)$ as (τ, a) [1; 7].

A sequence of ideal triangulations $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$ is called an ideal triangulation sweep if, for each $1 \le i \le k-1$, we have that $\tau^{(i+1)}$ is obtained from $\tau^{(i)}$ by reindexing or the diagonal exchange. A sequence of edge weight systems $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ is called an edge weight system sweep for the ideal triangulation sweep $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$, if the edge weight system $a^{(i+1)}$ for $\tau^{(i+1)}$ is derived from $(\tau^{(i)}, a^{(i)})$ for each $0 \le i \le k-1$. Note that the sequence $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ is completely determined by $a^{(0)}$. If in addition $a^{(0)} = a^{(k)}$, we call the sequence $a^{(0)}, a^{(1)}, \dots, a^{(k)}$ a periodic edge weight system for the ideal triangulation sweep $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$.

Suppose q is a primitive n^{th} root of unity with n odd. Let φ be an orientation preserving diffeomorphism for surface S, and let $\tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau)$ be an ideal triangulation sweep. Suppose $a = a^{(0)}, a^{(1)}, \dots, a^{(k)} = a$ is a periodic edge weight system for $\tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)}$ (the existence of the periodic edge weight system is guaranteed by [7, Lemma 11]), which defines a φ-invariant character

$[\bar{\gamma}] \in \mathscr{X}_{\mathrm{PSL}(2,\mathbb{C})}(S)$. Then, for each puncture v, we can choose a nonzero complex number h_v such that $h_v = h_{\varphi(v)}$ for every puncture v and $(h_v)^n = a_{i_1} a_{i_2} \cdots a_{i_j}$ for every puncture v adjacent to the edges $e_{i_1}, e_{i_2}, \dots, e_{i_j}$. From [7, Proposition 13], we know a and puncture weights h_v uniquely determine an irreducible representation $\bar{\rho}\colon \mathcal{T}^q_\tau \to \mathrm{End}(V)$ for the Chekhov–Fock algebra $\mathcal{T}^q_\tau$ such that $\bar{\rho}(X_i^n) = a_i$ for $1 \le i \le m$ and $\bar{\rho}(H_v) = h_v$ for each puncture v, where H_v is a central element in $\mathcal{T}^q_\tau$ associated to each puncture v. Let $\Phi^q_{\tau\varphi(\tau)}\colon \widehat{\mathcal{T}}^q_{\varphi(\tau)} \to \widehat{\mathcal{T}}^q_\tau$ be the Chekhov–Fock coordinate change isomorphism, and let $\Psi^q_{\varphi(\tau)\tau}\colon \widehat{\mathcal{T}}^q_\tau \to \widehat{\mathcal{T}}^q_{\varphi(\tau)}$ be the algebra isomorphism induced by φ. Then $\bar{\rho} \simeq \bar{\rho} \circ \Phi^q_{\tau\varphi(\tau)} \circ \Psi^q_{\varphi(\tau)\tau}$, so there exists an intertwiner $\overline{\Lambda}^q_{\varphi,\bar{\gamma}}$ such that

$$\bar{\rho} \circ \Phi^q_{\tau\varphi(\tau)} \circ \Psi^q_{\varphi(\tau)\tau}(X) = \overline{\Lambda}^q_{\varphi,\bar{\gamma}} \circ \bar{\rho}(X) \circ (\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^{-1}$$

for every $X \in \mathcal{T}^q_\tau$.

Under certain conditions, the trace of intertwiners obtained from Chekhov–Fock algebras equals the trace of intertwiners obtained from skein algebras; see [7, Theorem 16]. We will use this equality to calculate the trace of intertwiners obtained from skein algebras for the once punctured torus in Section 5.

From now on, we will always assume $q^{1/2}$ is a primitive n^{th} root of -1 with n odd.

3 Irreducible representation construction for $\mathrm{SK}_{q^{1/2}}(T^2)$ and $\mathrm{SK}_{q^{1/2}}(S_{1,1})$

In order to get Kauffman bracket intertwiners, we want to find the explicit irreducible representations associated to given characters and puncture weights. Here we construct irreducible representations for skein algebras of the closed torus T^2 and the once punctured torus $S_{1,1}$. In Section 4, we will use these irreducible representations to calculate intertwiners for the closed torus.

3.1 An algebraic embedding for $\mathrm{SK}_{q^{1/2}}(T^2)$

Let $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ be the algebra generated by X, X^{-1}, Y and Y^{-1}, subject to the relations $XY = qYX$, $XX^{-1} = X^{-1}X = 1$ and $YY^{-1} = Y^{-1}Y = 1$. Frohman and Gelca [14] built an algebraic embedding

$$G_{q^{1/2}}\colon \mathrm{SK}_{q^{1/2}}(T^2) \to \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}, \quad (a,b)_T \mapsto \theta_{(a,b)} + \theta_{(-a,-b)},$$

where $(a,b)_T$ is the simple link associated to two integers a and b, and $\theta_{(a,b)} = q^{-ab/2} X^a Y^b$. If $\gcd(a,b) = 1$ (with the convention that $\gcd(\pm 1, 0) = \gcd(0, \pm 1) = 1$), then $(a,b)_T$ is represented by the simple knot (a,b) in $\mathbb{R}^2/\mathbb{Z}^2$ with vertical framing. If $\gcd(a,b) = k$, $a = a'k$ and $b = b'k$, then $(a,b)_T = T_k((a',b'))$ where T_k is the k^{th} Chebyshev polynomial of the first type. We have

$$\theta_{(a,b)}\theta_{(c,d)} = q^{1/2\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}\theta_{(a+c,b+d)} \quad \text{and} \quad (\theta_{(a,b)})^{-1} = \theta_{(-a,-b)}.$$

Since $\theta_{(a,b)} + \theta_{(-a,-b)} = q^{-ab/2}(X^a Y^b + X^{-a} Y^{-b})$,

$$\operatorname{Im} G_{q^{1/2}} = \operatorname{span}\langle X^a Y^b + X^{-a} Y^{-b} \mid (a,b) \in \mathbb{Z} \times \mathbb{Z} \rangle.$$

Let $T_{q^{1/2}}$ be the Chebyshev homomorphism from the skein algebra $\mathrm{SK}_{-1}(T^2)$ to $\mathrm{SK}_{q^{1/2}}(T^2)$ defined in [3], and let $F_{q^{1/2}}\colon \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{-1} \to \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ be defined by $X \mapsto X^n$ and $Y \mapsto Y^n$. It is easy to check that we have $F_{q^{1/2}} G_{-1} = G_{q^{1/2}} T_{q^{1/2}}$.

3.2 Irreducible representations for $\mathrm{SK}_{q^{1/2}}(T^2)$

Bonahon and Liu described the irreducible representations of $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$ [1]. Let V denote the n-dimensional vector space over the complex field with basis $e_0, e_1, \dots, e_{n-1}$, and let u and v be any two nonzero complex numbers. Set $\rho_{u,v}(X)e_i = uq^i e_i$ and $\rho_{u,v}(Y)e_i = ve_{i+1}$, where the indices are considered modulo n; then $\rho_{u,v}$ is an irreducible representation. Any irreducible representation of $\mathbb{C}[X^{\pm}, Y^{\pm}]_{q^{1/2}}$ is isomorphic to a representation $\rho_{u,v}$, and $\rho_{u,v} \simeq \rho_{u',v'}$ if and only if $u^n = (u')^n$ and $v^n = (v')^n$.

It is well known that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$ where $\alpha = (1,0)$ and $\beta = (0,1)$. For any $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$, $[\gamma]$ has a representative γ such that

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

because $\pi_1(T^2)$ is commutative.

For any given character $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$, the following theorem offers a representation of $\mathrm{SK}_{q^{1/2}}(T^2)$ whose classical shadow is $[\gamma]$. For this theorem, we use the fact that $ab + a + b \equiv \gcd(a,b) \pmod 2$ for any two integers a and b (recall that $\gcd(\pm 1, 0) = \gcd(0, \pm 1) = 1$).

Theorem 3.1 *Choose $u, v \in \mathbb{C}$ such that $u^n = -\lambda_1$, $v^n = -\lambda_2$ or $u^n = -\lambda_1^{-1}$, $v^n = -\lambda_2^{-1}$; then the classical shadow of $\rho_{u,v} G_{q^{1/2}}$ is $[\gamma]$.*

Proof To show the classical shadow of $\rho_{u,v} G_{q^{1/2}}$ is $[\gamma]$, it suffices to show that

$$\rho_{u,v} G_{q^{1/2}}(T_{q^{1/2}}((a,b)_T)) = \mathrm{Tr}^{\gamma}((a,b)_T)\mathrm{Id}_V$$

for all $(a,b)_T \in \mathrm{SK}_{-1}(T^2)$. First we have

$$\begin{aligned}
\rho_{u,v} G_{q^{1/2}}(T_{q^{1/2}}((a,b)_T)) &= \rho_{u,v}(F_{q^{1/2}} G_{-1}((a,b)_T)) \\
&= \rho_{u,v}(F_{q^{1/2}}(\theta_{(a,b)} + \theta_{(-a,-b)})) \\
&= \rho_{u,v}(\theta_{(na,nb)} + \theta_{(-na,-nb)}) \\
&= \rho_{u,v}((-1)^{ab} X^{na} Y^{nb} + (-1)^{ab} X^{-na} Y^{-nb}) \\
&= (-1)^{ab}[(\rho_{u,v}(X))^{na}(\rho_{u,v}(Y))^{nb} + (\rho_{u,v}(X))^{-na}(\rho_{u,v}(Y))^{-nb}] \\
&= (-1)^{ab}[(u^n)^a (v^n)^b + (u^n)^{-a}(v^n)^{-b}]\mathrm{Id}_V \\
&= (-1)^{ab+a+b}[\lambda_1^a \lambda_2^b + \lambda_1^{-a}\lambda_2^{-b}]\mathrm{Id}_V.
\end{aligned}$$

Suppose $\gcd(a,b)=d$ and $a=a'd, b=b'd$; then we have

$$\begin{aligned}
\mathrm{Tr}^\gamma((a,b)_T)=\mathrm{Tr}^\gamma(T_d((a',b')))&=T_d(\mathrm{Tr}^\gamma((a',b')))\\
&=T_d(-\mathrm{Trace}(\gamma((a',b'))))\\
&=T_d(-\mathrm{Trace}(\gamma(a'\alpha+b'\beta)))\\
&=T_d(-\mathrm{Trace}((\gamma(\alpha))^{a'}(\gamma(\beta))^{b'}))\\
&=T_d((-\lambda_1^{a'}\lambda_2^{b'})+(-\lambda_1^{-a'}\lambda_2^{-b'}))\\
&=(-\lambda_1^{a'}\lambda_2^{b'})^d+(-\lambda_1^{-a'}\lambda_2^{-b'})^d\\
&=(-1)^d[\lambda_1^{da'}\lambda_2^{db'}+\lambda_1^{-da'}\lambda_2^{-db'}]\\
&=(-1)^{ab+a+b}[\lambda_1^a\lambda_2^b+\lambda_1^{-a}\lambda_2^{-b}]. \qquad \square
\end{aligned}$$

We can easily get the following theorem by using the representation theory.

Theorem 3.2 *Under the same assumption as in Theorem 3.1, we have the following conclusions*:

(a) *If $\lambda_1\neq\pm1$ or $\lambda_2\neq\pm1$, the representation $\rho_{u,v}G_{q^{1/2}}$ is irreducible.*

(b) *If $\lambda_1=\pm1$ and $\lambda_2=\pm1$, then V has only two irreducible subrepresentations, V_1 and V_2, $V=V_1\oplus V_2$, $\dim(V_1)=(n+1)/2$, and $\dim(V_2)=(n-1)/2$; especially*

$$V_1=\mathrm{span}\langle e_0, e_1+e_{n-1}, e_2+e_{n-2},\dots,e_{(n-1)/2}+e_{(n+1)/2}\rangle,$$
$$V_2=\mathrm{span}\langle e_1-e_{n-1}, e_2-e_{n-2},\dots,e_{(n-1)/2}-e_{(n+1)/2}\rangle$$

if $u=\pm1$ and $v=\pm1$.

Remark 3.3 The Azumaya locus of $\mathrm{SK}_{q^{1/2}}(T^2)$ is a subset of $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$; an element in $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$ lives in the Azumaya locus if it corresponds to a unique irreducible representation of $\mathrm{SK}_{q^{1/2}}(T^2)$ (the correspondence is the one in Theorem 2.1). We know the PI-dimension of $\mathrm{SK}_{q^{1/2}}(T^2)$ is n. Then a character $[\gamma]\in\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$ lives in the Azumaya locus if and only if there exists an irreducible representation of $\mathrm{SK}_{q^{1/2}}(T^2)$ of dimension n whose classical shadow is $[\gamma]$. So by Theorem 3.2, $[\gamma]$ lives in the Azumaya locus if and only if $\lambda_1\neq\pm1$ or $\lambda_2\neq\pm1$, where

$$\gamma(\alpha)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}\quad\text{and}\quad\gamma(\beta)=\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_2^{-1}\end{pmatrix}.$$

A contemporaneous paper by Karuo and Korinman [18] considered instead the case when $q^{1/2}$ is an odd order root of unity; both cases were studied through similar methods. They proved the character lives in the Azumaya locus of the skein algebra of a closed surface if and only if the character is noncentral.

In [4], Bonahon and Wong proved the Witten–Reshetikhin–Turaev representation of the Kauffman bracket skein algebra is irreducible and whose classical shadow is the trivial character. For the closed torus T^2, we use V_{T^2} to denote the Witten–Reshetikhin–Turaev representation of $\mathrm{SK}_{q^{1/2}}(T^2)$. We know $\dim V_{T^2}=\frac{1}{2}(n-1)$ with basis $v_1,v_2,\dots,v_{(n-1)/2}$ where v_k is the skein in the solid torus represented by

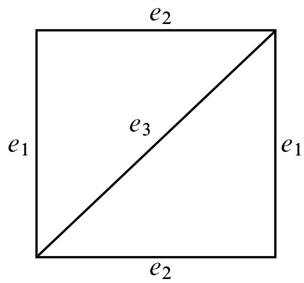

Figure 2

$2k-2$ nontrivial parallel closed curves, which are parallel to the core of the solid torus, with the $(2k-2)^{\text{nd}}$ Jones–Wenzel idempotent inserted. In Theorem 3.2 with $\lambda_1 = \lambda_2 = 1$, V_2 and V_{T^2} are isomorphic as representations for $\mathrm{SK}_{q^{1/2}}(T^2)$, and the isomorphism is given by

$$e_{2k-1} - e_{n-2k+1} \mapsto v_k \quad \text{for all } 1 \le k \le \tfrac{1}{2}(n-1).$$

3.3 Irreducible representations for $\mathrm{SK}_{q^{1/2}}(S_{1,1})$

We want to find the explicit irreducible representations of $\mathrm{SK}_{q^{1/2}}(S_{1,1})$ corresponding to given characters and puncture weights. Let $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ be the algebra generated by X_1, X_2 and X_3 subject to the relations

$$X_1X_2 = qX_2X_1, \quad X_2X_3 = qX_3X_2, \quad X_3X_1 = qX_1X_2, \quad X_iX_i^{-1} = X_i^{-1}X_i = 1.$$

We have $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}] = \mathcal{T}_\tau^{q^{1/4}}(S_{1,1})$ where τ is the ideal triangulation in Figure 2.

We define the skeins K_1, K_2 and K_3 in the skein algebra $\mathrm{SK}_{q^{1/2}}(S_{1,1})$ using Figure 3.

According to [10], the algebra $\mathrm{SK}_{q^{1/2}}(S_{1,1})$ is generated by K_1, K_2 and K_3 subject to the relations

$$\begin{aligned} q^{-1/2}K_1K_2 - q^{1/2}K_2K_1 &= (q^{-1}-q)K_3, \\ q^{-1/2}K_2K_3 - q^{1/2}K_3K_2 &= (q^{-1}-q)K_1, \\ q^{-1/2}K_3K_1 - q^{1/2}K_1K_3 &= (q^{-1}-q)K_2. \end{aligned}$$

Let P be the loop around the puncture in $S_{1,1}$. Then

$$P = q^{-1/2}K_1K_2K_3 - q^{-1}K_1^2 - qK_2^2 - q^{-1}K_3^2 + q + q^{-1}.$$

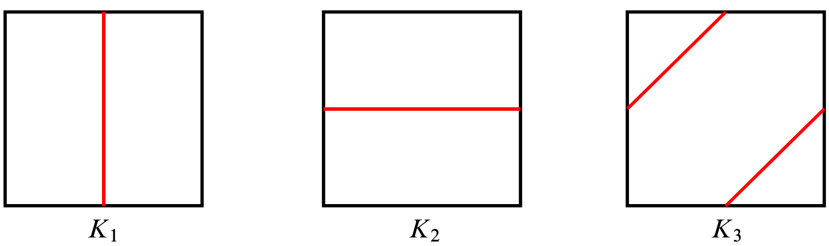

Figure 3

Lemma 3.4 *There is an algebraic embedding* $F\colon S_{q^{1/2}}(S_{1,1}) \to \mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ *such that*

$$\begin{aligned} F(K_1) &= [X_2 X_3] + [X_2^{-1} X_3^{-1}] + [X_2 X_3^{-1}], \\ F(K_2) &= [X_3 X_1] + [X_3^{-1} X_1^{-1}] + [X_3 X_1^{-1}], \\ F(K_3) &= [X_1 X_2] + [X_1^{-1} X_2^{-1}] + [X_1 X_2^{-1}], \\ F(P) &= [X_1^2 X_2^2 X_3^2] + [X_1^{-2} X_2^{-2} X_3^{-2}]. \end{aligned}$$

Proof Actually, F is just the quantum trace map constructed in [2, Theorem 11] if we regard

$$\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$$

as the Chekhov–Fock square root algebra associated to the ideal triangulation in Figure 2 where X_i corresponds to e_i for $i = 1, 2, 3$. □

Let V be an n dimensional vector space over the complex field with basis $w_0, w_1, \dots, w_{n-1}$. We can define a representation $\rho_{r_1,r_2,r_3}\colon \mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}] \to \mathrm{End}(V)$ such that

$$\begin{aligned} \rho_{r_1,r_2,r_3}(X_1) w_i &= r_1 q^i w_i, \\ \rho_{r_1,r_2,r_3}(X_2) w_i &= r_2 q^{-i} w_{i+1}, \\ \rho_{r_1,r_2,r_3}(X_3) w_i &= r_3 w_{i-1}, \end{aligned}$$

where r_1, r_2 and r_3 are nonzero complex numbers. We can get $\rho_{r_1,r_2,r_3}([X_1 X_2 X_3]) = r_1 r_2 r_3 q^{1/2} \mathrm{Id}_V$.

Lemma 3.5 *For any three nonzero complex numbers* r_1, r_2 *and* r_3, ρ_{r_1,r_2,r_3} *is an irreducible representation of* $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$. *Furthermore, every irreducible representation of* $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ *is isomorphic to a representation* ρ_{r_1,r_2,r_3}, *and* $\rho_{r_1,r_2,r_3} \simeq \rho_{s_1,s_2,s_3}$ *if and only if*

$$r_1^n = s_1^n, \quad r_2^n = s_2^n, \quad r_3^n = s_3^n, \quad r_1 r_2 r_3 = s_1 s_2 s_3.$$

For any $\gamma \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S_{1,1})$ and a nonzero complex number p such that $T_n(p) = -\operatorname{Trace}\gamma(P)$ where P is the loop going around the only puncture in $S_{1,1}$, let $t_i = -\operatorname{Trace}\gamma(K_i)$ for $i = 1, 2, 3$. According to [26], we have

$$T_n(p) = -t_1 t_2 t_3 - t_1^2 - t_2^2 - t_3^2 + 2.$$

Lemma 3.6 *Let* x *and* y *be two indeterminates such that* $xy = q^{-2} yx$. *Then*

$$T_n(x + x^{-1} + y) = x^{-n} + x^n + y^n$$

for $n \geq 1$.

For any given character $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S_{1,1})$, the following theorem offers a representation of $\mathrm{SK}_{q^{1/2}}(S_{1,1})$ whose classical shadow is $[\gamma]$.

Theorem 3.7 *With the above notation, $\rho_{r_1,r_2,r_3}F$ is a representation of $\mathrm{SK}_{q^{1/2}}(S_{1,1})$. The classical shadow of $\rho_{r_1,r_2,r_3}F$ is γ and $\rho_{r_1,r_2,r_3}F(P) = p\mathrm{Id}_V$ if and only if we have the following equations*:

$$\begin{aligned} r_2^n r_3^n + r_2^{-n} r_3^{-n} + r_2^n r_3^{-n} &= -t_1, \\ r_3^n r_1^n + r_3^{-n} r_1^{-n} + r_3^n r_1^{-n} &= -t_2, \\ r_1^n r_2^n + r_1^{-n} r_2^{-n} + r_1^n r_2^{-n} &= -t_3, \\ r_1^2 r_2^2 r_3^2 q + r_1^{-2} r_2^{-2} r_3^{-2} q^{-1} &= p. \end{aligned} \tag{1}$$

Proof It is easy to see

$$\begin{aligned} [X_2X_3][X_2X_3^{-1}] &= q^{-2}[X_2X_3^{-1}][X_2X_3], \\ [X_3X_1][X_3X_1^{-1}] &= q^{-2}[X_3X_1^{-1}][X_3X_1], \\ [X_1X_2][X_1X_2^{-1}] &= q^{-2}[X_1X_2^{-1}][X_1X_2]. \end{aligned}$$

From Lemma 3.6, we get

$$\begin{aligned} T_n(\rho_{r_1,r_2,r_3}F(K_1)) &= T_n(\rho_{r_1,r_2,r_3}([X_2X_3] + [X_2^{-1}X_3^{-1}] + [X_2X_3^{-1}])) \\ &= \rho_{r_1,r_2,r_3}(T_n([X_2X_3] + [X_2^{-1}X_3^{-1}] + [X_2X_3^{-1}])) \\ &= \rho_{r_1,r_2,r_3}([X_2^nX_3^n] + [X_2^{-n}X_3^{-n}] + [X_2^nX_3^{-n}]) \\ &= \rho_{r_1,r_2,r_3}(-(X_2^nX_3^n + X_2^{-n}X_3^{-n} + X_2^nX_3^{-n})) \\ &= -(r_2^nr_3^n + r_2^{-n}r_3^{-n} + r_2^nr_3^{-n})\mathrm{Id}_V. \end{aligned} \tag{2}$$

Similarly we can get

$$\begin{aligned} T_n(\rho_{r_1,r_2,r_3}F(K_2)) &= -(r_3^nr_1^n + r_3^{-n}r_1^{-n} + r_3^nr_1^{-n})\mathrm{Id}_V, \\ T_n(\rho_{r_1,r_2,r_3}F(K_3)) &= -(r_1^nr_2^n + r_1^{-n}r_2^{-n} + r_1^nr_2^{-n})\mathrm{Id}_V. \end{aligned} \tag{3}$$

And

$$\begin{aligned} \rho_{r_1,r_2,r_3}F(P) &= \rho_{r_1,r_2,r_3}([X_1^2X_2^2X_1^2] + [X_1^{-2}X_2^{-2}X_1^{-2}]) \\ &= \rho_{r_1,r_2,r_3}([X_1^2X_2^2X_1^2]) + \rho_{r_1,r_2,r_3}([X_1^{-2}X_2^{-2}X_1^{-2}]) \\ &= (r_1^2r_2^2r_3^2q + r_1^{-2}r_2^{-2}r_3^{-2}q^{-1})\mathrm{Id}_V. \end{aligned} \tag{4}$$

From (2), (3), (4) and the fact that K_1, K_2 and K_3 generate the algebra $\mathrm{SK}_{-1}(S_{1,1})$, we can get the conclusions in Theorem 3.7. □

Remark 3.8 At first glance, it seems like, in (1), we may not be able to get solutions, but actually the forth one is a consequence of first three equations because we have the relation $T_n(p) = -t_1t_2t_3 - t_1^2 - t_2^2 - t_3^2 + 2$. In fact, to get solutions, we only need to solve the equations

$$\begin{aligned} yz + y^{-1}z^{-1} + yz^{-1} &= -t_1, \\ zx + z^{-1}x^{-1} + zx^{-1} &= -t_2, \\ xy + x^{-1}y^{-1} + xy^{-1} &= -t_3. \end{aligned}$$

Let $Y_i = X_i^2$ for $i = 1, 2, 3$; then

$$Y_1 Y_2 = q^4 Y_2 Y_1, Y_2 Y_3 = q^4 Y_3 Y_2, Y_3 Y_1 = q^4 Y_1 Y_3, Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1.$$

The subalgebra of $\mathbb{C}_q[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}]$ generated by $Y_1^{\pm 1}$, $Y_2^{\pm 1}$ and $Y_3^{\pm 1}$ is $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$.

Here we recall a lemma from [7] for irreducible representations for $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$. Let V be an n dimensional vector space over $\mathbb{C}$ with basis $w_0, w_1, \dots, w_{n-1}$. For any three nonzero complex numbers y_1, y_2 and y_3, define $\rho_{y_1,y_2,y_3} \colon \mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}] \to \mathrm{End}(V)$ such that

$$\begin{aligned} \rho_{y_1,y_2,y_3}(Y_1)(w_i) &= y_1 q^{4i} w_i, \\ \rho_{y_1,y_2,y_3}(Y_2)(w_i) &= y_2 q^{-2i} w_{i+1}, \\ \rho_{y_1,y_2,y_3}(Y_3)(w_i) &= y_3 q^{-2i} w_{i-1}. \end{aligned} \tag{5}$$

Lemma 3.9 [7] (1) *For any three nonzero complex numbers y_1, y_2 and y_3, the representation ρ_{y_1,y_2,y_3} is irreducible.*

(2) *Every irreducible representation of $\mathbb{C}_{q^4}[Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}]$ is isomorphic to a representation ρ_{y_1,y_2,y_3}.*

(3) *The representations ρ_{y_1,y_2,y_3} and $\rho_{y_1',y_2',y_3'}$ are isomorphic if and only if $y_1^n = (y')^n$, $y_2^n = (y_2')^n$, $y_3^n = (y_3')^n$ and $y_1 y_2 y_3 = y_1' y_2' y_3'$.*

4 Calculation of intertwiners for the closed torus

4.1 Construction of intertwiners for the closed torus

The mapping class group of the closed torus is $\mathrm{SL}(2, \mathbb{Z})$ [13]. For any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, we hope to find invariant characters under A. For a $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$, we choose a representative with

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

where α and β denote loops $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2/\mathbb{Z}^2$ respectively. We have $[\gamma]$ is invariant under A if and only if $\mathrm{Trace}(\gamma(A_*(z))) = \mathrm{Trace}(\gamma(z))$ for all $z \in \pi_1(T^2)$. For any $(k_1, k_2) \in \mathbb{Z} \oplus \mathbb{Z}$, we have

$$A_*(k_1\alpha + k_2\beta) = (k_1, k_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (k_1 a + k_2 c, k_1 b + k_2 d) = (k_1 a + k_2 c)\alpha + (k_1 b + k_2 d)\beta,$$

$$\begin{aligned} \gamma(A_*(k_1\alpha + k_2\beta)) &= \gamma[(k_1 a + k_2 c)\alpha + (k_1 b + k_2 d)\beta] \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}^{k_1 a + k_2 c} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}^{k_1 b + k_2 d} \\ &= \begin{pmatrix} \lambda_1^{k_1 a + k_2 c} \lambda_2^{k_1 b + k_2 d} & 0 \\ 0 & \lambda_1^{-(k_1 a + k_2 c)} \lambda_2^{-(k_1 b + k_2 d)} \end{pmatrix}, \end{aligned}$$

$$\gamma(k_1\alpha + k_2\beta) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}^{k_1} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}^{k_2} = \begin{pmatrix} \lambda_1^{k_1} \lambda_2^{k_2} & 0 \\ 0 & \lambda_1^{-k_1} \lambda_2^{-k_2} \end{pmatrix}.$$

Then it is easy to show that $[\gamma]$ is A-invariant if and only if $\lambda_1=\lambda_1^a\lambda_2^b$, $\lambda_2=\lambda_1^c\lambda_2^d$ or $\lambda_1=\lambda_1^{-a}\lambda_2^{-b}$, $\lambda_2=\lambda_1^{-c}\lambda_2^{-d}$.

A also induces two algebra isomorphisms, $F_{A,+}$ and $F_{A,-}$, from $\mathbb{C}[X^{\pm 1},Y^{\pm 1}]_{q^{1/2}}$ to itself defined by

$$F_{A,+}(\theta_{(i,j)})=\theta_{(i,j)A},\quad F_{A,-}(\theta_{(i,j)})=\theta_{(-i,-j)A}.$$

$F_{A,+}$ and $F_{A,-}$ are well defined because

$$\theta_{(i,j)A}\theta_{(k,l)A}=q^{1/2\left[\begin{smallmatrix}(i,j)A\\(k,l)A\end{smallmatrix}\right]}\theta_{(i+k,j+l)A}=q^{1/2\left[\begin{smallmatrix}i&j\\k&l\end{smallmatrix}\right][A]}\theta_{(i+k,j+l)A}=q^{1/2\left[\begin{smallmatrix}i&j\\k&l\end{smallmatrix}\right]}\theta_{(i+k,j+l)A},$$

and similarly

$$\theta_{(-i,-j)A}\theta_{(-k,-l)A}=q^{1/2\left[\begin{smallmatrix}i&j\\k&l\end{smallmatrix}\right]}\theta_{(-i-k,-j-l)A}.$$

From Section 3, we know there is an embedding

$$G_{q^{1/2}}\colon \mathrm{SK}_{q^{1/2}}(T^2)\to\mathbb{C}[X^{\pm 1},Y^{\pm 1}]_{q^{1/2}}.$$

For the following discussion we will omit the subscript for $G_{q^{1/2}}$ when there is no confusion.

Lemma 4.1 *The following diagram is commutative*:

$$\begin{tikzcd}
\mathrm{SK}_{q^{1/2}}(T^2) \arrow[r, "G"] \arrow[d, "A_\sharp"] & \mathbb{C}[X^{\pm 1},Y^{\pm 1}]_{q^{1/2}} \arrow[d, "F_A"] \\
\mathrm{SK}_{q^{1/2}}(T^2) \arrow[r, "G"] & \mathbb{C}[X^{\pm 1},Y^{\pm 1}]_{q^{1/2}}
\end{tikzcd}$$

where F_A is $F_{A,+}$ or $F_{A,-}$.

Proof We only prove the case when $F_A=F_{A,+}$.

First we show $A_\sharp((k,l)_T)=((k,l)A)_T$. Assume $\gcd(k,l)=j$, $k=k'j$ and $l=l'j$; then

$$A_\sharp((k,l)_T)=A_\sharp(T_j((k',l')))=T_j(A_\sharp((k',l')))=T_j((k',l')A)=T_j(ak'+cl',bk'+dl').$$

There exist integers u and v such that $uk+vl=j$, so $\left[\begin{smallmatrix}k&l\\-v&u\end{smallmatrix}\right]=j$ and $\det\left(\left(\begin{smallmatrix}k&l\\-v&u\end{smallmatrix}\right)A\right)=j$. Then

$$\det\left(\begin{pmatrix}k&l\\-v&u\end{pmatrix}A\right)=\begin{bmatrix}ak+cl&bk+dl\\-v'&u'\end{bmatrix}=(ak+cl)u'+(bk+dl)v'=j.$$

We also have $j\mid(ak+cl)$ and $j\mid(bk+dl)$. Thus $\gcd(ak+cl,bk+dl)=j$, $ak+cl=j(ak'+cl')$ and $bk+dl=j(bk'+dl')$. Then

$$A_\sharp((k,l)_T)=T_j(ak'+cl',bk'+dl')=(ak+cl,bk+dl)_T=((k,l)A)_T.$$

So

$$GA_\sharp((k,l)_T)=G(((k,l)A)_T)=\theta_{(k,l)A}+\theta^{-1}_{(k,l)A},$$
$$F_AG((k,l)_T)=F_A(\theta_{(k,l)}+\theta^{-1}_{(k,l)})=\theta_{(k,l)A}+\theta^{-1}_{(k,l)A}.$$

We have $GA_\sharp=F_AG$ because all $(k,l)_T$ span the skein algebra. □

The following two theorems give the intertwiners for the closed surface for all the diffeomorphisms. We will give explicit formulas for these intertwiners and their trace in the following subsections.

Theorem 4.2 *In the diagram*

$$\begin{tikzcd} \mathrm{SK}_{q^{1/2}}(T^2) \arrow[r, "G"] \arrow[d, "A_\sharp"] & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \arrow[r, "\rho_{u,v}"] \arrow[d, "F_A"] & \mathrm{End}(V) \arrow[d, dotted, "G_\Lambda"] \\ \mathrm{SK}_{q^{1/2}}(T^2) \arrow[r, "G"] & \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}} \arrow[r, "\rho_{u,v}"] & \mathrm{End}(V) \end{tikzcd} \tag{6}$$

suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ *and* $F_A = F_{A,+}$. *Let* $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$ *with*

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

where $\lambda_1 = \lambda_1^a \lambda_2^b$ *and* $\lambda_2 = \lambda_1^c \lambda_2^d$, *and let* u *and* v *be two complex numbers such that* $u^n = -\lambda_1$ *and* $v^n = -\lambda_2$. *We have the following conclusions*:

(a) $[\gamma]$ *is invariant under* A.

(b) *The classical shadow of* $\rho_{u,v} G$ *is* $[\gamma]$.

(c) $\rho_{u,v} F_A \simeq \rho_{u,v}$.

(d) *From* (c), *we know there exists an intertwiner* $\Lambda_{n,+}$ *such that* $\rho_{u,v} F_A(Z) = \Lambda_{n,+} \rho_{u,v}(Z) \Lambda_{n,+}^{-1}$ *for all* $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$. *Then this intertwiner induces an intertwiner between two irreducible representations of the skein algebra.*

Proof Items (a) and (b) are already shown in the previous discussion.

To prove (c), we have

$$\begin{aligned} \rho_{u,v} F_A(X^n) = \rho_{u,v} F_A(\theta_{(n,0)}) &= \rho_{u,v}(\theta_{(n,0)A}) \\ &= \rho_{u,v}(\theta_{(na,nb)}) \\ &= \rho_{u,v}((-1)^{ab} X^{na} Y^{nb}) \\ &= (-1)^{ab} u^{na} v^{nb} \mathrm{Id}_V \\ &= (-1)^{ab} (-\lambda_1)^a (-\lambda_2)^b \mathrm{Id}_V \\ &= (-1)^{ab+a+b} \lambda_1^a \lambda_2^b \mathrm{Id}_V = -\lambda_1 \mathrm{Id}_V = u^n \mathrm{Id}_V. \end{aligned}$$

Similarly we can show $\rho_{u,v} F_A(Y^n) = v^n \mathrm{Id}_V$, thus $\rho_{u,v} F_A \simeq \rho_{u,v}$.

For (d), if $\lambda_1 \neq \pm 1$ or $\lambda_1 \neq \pm 1$, Theorem 3.2 implies that $\Lambda_{n,+}$ itself is the intertwiner between two irreducible representations of the skein algebra. If $\lambda_1 = \pm 1$ and $\lambda_1 = \pm 1$, Theorem 3.2 implies that V has only two irreducible subrepresentations, V_1 and V_2, with $\dim(V_1) = (n+1)/2$ and $\dim(V_2) = (n-1)/2$. We have that $\Lambda_{n,+}(V_1)$ is an irreducible subrepresentation of V and $\dim(\Lambda_{n,+}(V_1)) = (n+1)/2$. Then $\Lambda_{n,+}(V_1) = V_1$. Thus $\Lambda_{n,+}|_{V_1}$ is an intertwiner for V_1. Similarly $\Lambda_{n,+}|_{V_2}$ is an intertwiner for V_2. □

Theorem 4.3 *In the diagram* (6), *suppose* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ *and* $F_A = F_{A,-}$. *Let* $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$ *with*

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$$

where $\lambda_1 = \lambda_1^{-a}\lambda_2^{-b}$ *and* $\lambda_2 = \lambda_1^{-c}\lambda_2^{-d}$, *and let* u *and* v *be two complex numbers such that* $u^n = -\lambda_1^{-1}$ *and* $v^n = -\lambda_2^{-1}$. *We have the following conclusions*:

(a) $[\gamma]$ *is invariant under* A.

(b) *The classical shadow of* $\rho_{u,v}G$ *is* $[\gamma]$.

(c) $\rho_{u,v}F_A \simeq \rho_{u,v}$.

(d) *From* (c), *we know there exists an intertwiner* $\Lambda_{n,-}$ *such that* $\rho_{u,v}F_A(Z) = \Lambda_{n,-}\rho_{u,v}(Z)\Lambda_{n,-}^{-1}$ *for all* $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$. *Then this intertwiner induces an intertwiner between two irreducible representations of the skein algebra.*

Proof The proof is the same as in Theorem 4.2. □

Note that a rescaling of the intertwiner $\Lambda_{n,+}$ in Theorem 4.2 such that $|\det(\Lambda_{n,+})| = 1$ makes $|\mathrm{Trace}\,\Lambda_{n,+}|$ independent of the choice of u and v. The same thing holds for the intertwiner in Theorem 4.3.

For the following discussion, we always require F_A to be $F_{A,+}$ unless specified otherwise (parallel results hold for $F_{A,-}$). From the above discussion, we know there exists an intertwiner $\Lambda_n \in \mathrm{End}(V)$ such that the diagram (6) commutes, where $G_\Lambda(B) = \Lambda_n B \Lambda_n^{-1}$ for all $B \in \mathrm{End}(V)$. Next we are going to find an intertwiner $\overline{\Lambda}_n$ under the assumption in Theorem 4.2.

4.2 Calculation for intertwiners

Under the assumption of Theorem 4.2, we have $\rho_{u,v}F_A \simeq \rho_{u,v}$. For any $a \in V$ and $Z \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]_{q^{1/2}}$, we use $Z \cdot a$ and $Z \star a$ to denote $\rho_{u,v}(Z)(a)$ and $\rho_{u,v}F_A(Z)(a)$ respectively. Then we are trying to find $\overline{\Lambda}_n \in \mathrm{End}(V)$ such that $\overline{\Lambda}_n(X \cdot a) = X \star (\overline{\Lambda}_n(a))$ and $\overline{\Lambda}_n(Y \cdot a) = Y \star (\overline{\Lambda}_n(a))$ for all $a \in V$.

Remark 4.4 Assume $\gcd(b,n) = m$ and $n = n'm$. There exist two integers r and s such that $br + sn = m$. Then we have

$$\begin{aligned} (7)\qquad (v^{n'b}u^{n'(a-1)}q^{ab(n')^2/2})^m &= v^{mn'b}u^{mn'(a-1)}q^{abm(n')^2/2} \\ &= v^{nb}u^{n(a-1)}q^{abnn'/2} \\ &= (-\lambda_2)^b(-\lambda_1)^{a-1}(-1)^{abn'} = (-1)^{ab+b+a-1}\lambda_2^b\lambda_1^{a-1} = 1, \end{aligned}$$

and $q^{an'}$ is a primitive m^{th} root of unity. Then there exists a unique integer $0 \le k_0 \le m-1$ such that $(v^{n'b}u^{n'(a-1)}q^{ab(n')^2/2})q^{an'k_0} = 1$ and $(v^{n'b}u^{n'(a-1)}q^{ab(n')^2/2})q^{an'k} \neq 1$ for $k \neq k_0, 0 \le k \le m-1$. We set $r_{k_0} = 1$ and $r_k = 0$ for $k \neq k_0, 0 \le k \le m-1$, and define $r_{k+tb} = r_k v^{tb} u^{t(a-1)} q^{a(tk+bt^2/2)}$ for all $0 \le k \le m-1$ and $t \in \mathbb{Z}$, where we consider all indices modulo n. Since $\gcd(b,n) = m$ and $n = mn'$, we can reach all the indices. It is an easy check that $r_{k_1+t_1 b} = r_{k_2+t_2 b}$ if $k_1 + t_1 b \equiv k_2 + t_2 b \pmod{n}$. Then r_k is well defined for each $0 \le k \le n-1$.

It is easy to check that we have $r_{k+tb} = r_k v^{tb} u^{t(a-1)} q^{atk+abt^2/2}$ for all $k, t \in \mathbb{Z}$. Actually we have

$$r_{k_0+tb} = v^{tb} u^{t(a-1)} q^{a(tk_0+bt^2/2)} \quad \text{for all } 0 \le t \le n'-1,$$

and all other r_k are 0. We have

$$(v^b u^{a-1})^n = (v^n)^b (u^n)^{a-1} = (-\lambda_2)^b (-\lambda_1)^{a-1} = (-1)^{a+b-1} {\lambda_1}^{a-1} {\lambda_2}^b = (-1)^{a+b-1}.$$

Then we get $|r_k| = 0$ or 1 for all $k \in \mathbb{Z}$. From $br + sn = m$, we get $tm = tbr + tsn$ for all $t \in \mathbb{Z}$. Then we have

$$r_{k_0+tm} = r_{k_0+trb} = v^{trb} u^{tr(a-1)} q^{a(trk_0+bt^2r^2/2)} \quad \text{for all } 0 \le t \le n'-1,$$

and all other r_k are 0.

The following lemma offers an explicit formula for the intertwiner constructed in Theorem 4.2(d).

Lemma 4.5 *Under the assumption of Theorem 4.2, suppose $\overline{\Lambda}_n \in \mathrm{End}(V)$ and*

$$\overline{\Lambda}_n(e_t) = \sum_{0 \le k \le n-1} (\overline{\Lambda}_n)_{k,t} e_k$$

for all $0 \le t \le n-1$, where

$$(\overline{\Lambda}_n)_{k,t} = r_{k-td} (v^{(d-1)} u^c)^t q^{c(tk-dt^2/2)}.$$

Then $\overline{\Lambda}_n$ satisfies the conditions in Theorem 4.2(d).

Proof From direct calculations, we can get $\overline{\Lambda}_n(X \cdot e_t) = X \star (\overline{\Lambda}_n(e_t))$ and $\overline{\Lambda}_n(Y \cdot e_t) = Y \star (\overline{\Lambda}_n(e_t))$ for all $0 \le t \le n-1$. □

We have $(v^{d-1}u^c)^n = (v^n)^{d-1}(u^n)^c = (-\lambda_2)^{d-1}(-\lambda_1)^c = (-1)^{c+d-1}{\lambda_1}^c{\lambda_2}^{d-1} = (-1)^{c+d-1}$. Then we can get $|(\overline{\Lambda}_n)_{k,t}| = 0$ or 1. We have $(\overline{\Lambda}_n)_{k,t} = 0$ if and only if $r_{k-td} = 0$. Then it is easy to show that $(\overline{\Lambda}_n)_{ld+km+k_0,l+tm}$, for $0 \le l \le m-1$ and $0 \le k,t \le n'-1$, are the only nonzero entries.

For each $0 \le l \le m-1$, we define an $n' \times n'$ matrix B^l such that $(B^l)_{k,t} = (\overline{\Lambda}_n)_{ld+km+k_0,l+tm}$ for all $0 \le k,t \le n'-1$. Then by Laplace expansion, we know $|\det(\overline{\Lambda}_n)| = \prod_{0 \le l \le m} |\det(B^l)|$.

From pure calculations, we can get $|\det(B^l)| = (n')^{n'/2}$. Then we have

$$|\det(\overline{\Lambda}_n)| = \prod_{0 \le l \le m} |\det(B^l)| = ((n')^{n'/2})^m = (n')^{mn'/2} = (n')^{n/2}.$$

Furthermore, $|\det((n')^{-1/2}\overline{\Lambda}_n)| = 1$.

Remark 4.6 If $\widetilde{\Lambda}_n$ is the intertwiner in Theorem 4.3, and we still suppose $\gcd(b,n)=m$, $br+sn=m$ and $n=n'm$, then

$$r_{k_0-tm}=(v^{-b}u^{-a-1})^{tr}q^{-trak_0+abt^2r^2/2}\quad \text{for all } t\in\mathbb{Z}$$

with all other $r_k=0$, and

$$(\widetilde{\Lambda}_n)_{k,t}=r_{k+td}(v^{-d-1}u^{-c})^t q^{-tck-cdt^2/2}\quad \text{for all } 0\le k,t\le n-1,$$

where $0\le k_0\le m-1$ such that

$$(v^{-b}u^{-a-1})^{n'}q^{ab(n')^2/2}q^{-n'ak_0}=1.$$

Also we can get $|\det(\widetilde{\Lambda}_n)|=(n')^{n/2}$.

4.3 On the trace of intertwiners

Bonahon, Wong and Yang only formulated the conjecture when the mapping tori are hyperbolic. So they considered surfaces with negative Euler characteristic because the mapping tori for the closed torus can never be hyperbolic. Since the simplicial volume of mapping tori for the closed torus is zero — see [13, page 380] — we expect the corresponding limit to be zero. In Theorem 4.15, we can show the limit superior is zero for any diffeomorphism. But the limits are not zero for some cases; see Example 4.12. Some diffeomorphisms even do not have invariant characters that live in the Azumaya locus, but the intertwiners in Theorems 4.2 and 4.3 are very close to intertwiners constructed in [7].

When we consider the intertwiners in Theorems 4.2 and 4.3, we fix the mapping class A and the A-invariant character $[\gamma]$. In this subsection we will use (l,s) to denote $\gcd(l,s)$ for any two integers l and s.

Theorem 4.7 *If we require* $|\det(\Lambda_n)|=1$ *for the intertwiner in Theorem 4.2, then* $|\operatorname{Trace}\Lambda_n|\le n^{3/2}$.

Proof Since any two intertwiners in Theorem 4.2 are different by a scalar multiplication or by conjugation and we require $|\det(\Lambda_n)|=1$, the absolute value $|\operatorname{Trace}\Lambda_n|$ is independent of the choice of intertwiners. Let $\Lambda_n=(n')^{-1/2}\overline{\Lambda}_n$; then $|\det(\Lambda_n)|=1$. Since $|(\overline{\Lambda}_n)_{k,t}|=0$ or 1 for all $0\le k,t\le n-1$ and each row has exactly n' nonzero entries, we have the absolute value of every eigenvalue of $\overline{\Lambda}_n$ is not more than n'. Then

$$|\operatorname{Trace}(\Lambda_n)|=|\operatorname{Trace}((n')^{-1/2}\overline{\Lambda}_n)|\le (n')^{-1/2}(nn')=(n')^{1/2}n\le n^{3/2}. \qquad\square$$

Theorem 4.8 *If we require* $|\det(\Lambda_n)|=1$ *for the intertwiner in Theorem 4.3, then* $|\operatorname{Trace}\Lambda_n|\le n^{3/2}$.

Proof It is similar to the proof for Theorem 4.7. $\square$

Lemma 4.9 *Let* k *be any integer; then we have*

$$\left|\sum_{0\le t\le n-1}(-q^{1/2})^{kt^2}\right|=\sqrt{(k,n)n}.$$

Recall that $q^{1/2}$ *is a primitive* n^{th} *root of* -1.

Proof In [23] this result is proved for $k = 2$. Using the same trick, we can prove this generalized lemma. $\square$

In the following of this section, we always assume $q^{1/2} = e^{\pi i/n}$ unless especially specified. Next we are going to calculate $\operatorname{Trace}\Lambda_n$, where Λ_n is the intertwiner in Theorem 4.2 or 4.3 with $|\det\Lambda_n| = 1$. First we give detailed discussion on the invariant character. Recall that for any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and a character $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$ with

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix},$$

we have that $[\gamma]$ is A-invariant if and only if $1 = \lambda_1^{a-1}\lambda_2^{b}$, $1 = \lambda_1^{c}\lambda_2^{d-1}$ or $1 = \lambda_1^{a+1}\lambda_2^{b}$, $1 = \lambda_1^{c}\lambda_2^{d+1}$.

Remark 4.10 We will provide a detailed discussion only for the case when $1 = \lambda_1^{a-1}\lambda_2^{b}$ and $1 = \lambda_1^{c}\lambda_2^{d-1}$. Suppose $\lambda_1 = \alpha_1 e^{i\theta_1}$ and $\lambda_2 = \alpha_2 e^{i\theta_2}$; then we get

$$\begin{gathered} 1 = \alpha_1^{a-1}\alpha_2^{b}, \quad 1 = \alpha_1^{c}\alpha_2^{d-1}, \\ (a-1)\theta_1 + b\theta_2 = 2k_1\pi, \quad c\theta_1 + (d-1)\theta_2 = 2k_2\pi. \end{gathered} \tag{8}$$

Since $u^n = -\lambda_1$ and $v^n = -\lambda_2$, we can suppose $u = -\alpha_1^{1/n} e^{i\theta_1/n} q^{r_1}$ and $v = -\alpha_2^{1/n} e^{i\theta_2/n} q^{r_2}$ where both r_1 and r_2 are integers. Then we have

$$\begin{aligned} u^{a-1}v^{b} &= (-1)^{a+b-1} q^{r_1(a-1)+r_2 b} e^{(i/n)((a-1)\theta_1+b\theta_2)} = (-1)^{a+b-1} q^{r_1(a-1)+r_2 b+k_1}, \\ u^{c}v^{d-1} &= (-1)^{c+d-1} q^{r_1 c+r_2(d-1)} e^{(i/n)(c\theta_1+(d-1)\theta_2)} = (-1)^{c+d-1} q^{r_1 c+r_2(d-1)+k_2}. \end{aligned} \tag{9}$$

Define $s_1 = r_1(a-1) + r_2 b + k_1$ and $s_2 = r_1 c + r_2(d-1) + k_2$. Then $u^{a-1}v^{b} = (-1)^{a+b-1}q^{s_1}$ and $u^{c}v^{d-1} = (-1)^{c+d-1}q^{s_2}$.

From $1 = \lambda_1^{a-1}\lambda_2^{b}$ and $1 = \lambda_1^{c}\lambda_2^{d-1}$, we can get $1 = \lambda_1^{(a-1)c}\lambda_2^{bc}$ and $1 = \lambda_1^{(a-1)c}\lambda_2^{(a-1)(d-1)}$. Thus we have

$$\lambda_2^{bc} = \lambda_2^{(a-1)(d-1)} = \lambda_2^{ad-(a+d)+1} \implies 1 = \lambda_2^{ad-bc-(a+d)+1} = \lambda_2^{2-(a+d)}.$$

If $a + d \neq 2$, then λ_2 is a root of unity with $\lambda_2^{2-(a+d)} = 1$. Similarly we can show λ_1 is also a root of unity, with $\lambda_1^{2-(a+d)} = 1$, under the assumption $a + d \neq 2$.

We look at the case when $(b, n) = 1$, and suppose $br + sn = 1$. Then we have

$$q^{br} = q, \quad q^{br/2} = (-1)^{s} q^{1/2}, \quad (-1)^{s} = (-1)^{br+1}.$$

When $(b, n) = 1$, we can choose $k_0 = 1$ and set $r_{k_0} = r_0 = 1$. Then we have

$$r_{tb} = v^{tb} u^{t(a-1)} q^{abt^2/2} \quad \text{for all } t \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$, we have $k = krb + ksn$. Then

$$r_k = r_{krb} = v^{krb} u^{kr(a-1)} q^{abk^2r^2/2}.$$

From the above discussion, we know we can choose Λ_n to be $n^{-1/2}\overline{\Lambda}_n$. We have

$$
\begin{aligned}
(\overline{\Lambda}_n)_{t,t} &= r_{t-td}(v^{d-1}u^c)^t q^{ct^2-cdt^2/2} \\
&= (-1)^{cdt} r_{t-td} q^{s_2 t} q^{ct^2-cdt^2/2} \\
&= (-1)^{cdt} (v^b u^{a-1})^{r(t-td)} q^{abr^2(t-td)^2/2} q^{s_2 t} q^{ct^2-cdt^2/2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} q^{s_1 r(t-td)} q^{abr^2(t-td)^2/2} q^{s_2 t} q^{ct^2-cdt^2/2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (q^{s_1 r(1-d)} q^{s_2})^t ((q^{br/2})^{ar(1-d)^2} q^{c-cd/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (q^{s_1 r(1-d)} q^{s_2})^t (((-1)^s q^{1/2})^{ar(1-d)^2} q^{c-cd/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{ar/2} q^{-ard} q^{ard^2/2} q^{c-cd/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{ar/2} q^{-r} q^{-rbc} q^{rd/2} q^{rbcd/2} q^{c-cd/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (q^{s_1 r(1-d)} q^{s_2})^t (q^{ar/2} q^{-r} q^{-c} q^{rd/2} ((-1)^s q^{1/2})^{cd} q^{c-cd/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{tsar(1-d)^2} (-1)^{tscd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{(a+d-2)r/2})^{t^2} \\
&= (-1)^{cdt} (-1)^{abr(t-td)} (-1)^{t(br+1)ar(1-d)} (-1)^{t(br+1)cd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{(a+d-2)r/2})^{t^2} \\
&= (-1)^{tar} (-1)^{tard} (-1)^{trd(ad+1)} (q^{s_1 r(1-d)} q^{s_2})^t (q^{(a+d-2)r/2})^{t^2} \\
&= (-1)^{tar} (-1)^{trd} (q^{s_1 r(1-d)} q^{s_2})^t (q^{(a+d-2)r/2})^{t^2} \\
&= ((-1)^{(a+d-2)r})^t (q^{s_1 r(1-d)} q^{s_2})^t (q^{(a+d-2)r/2})^{t^2}.
\end{aligned}
$$

Since $q^{s_2} = q^{rs_2 b}$, we have

$$(\overline{\Lambda}_n)_{t,t} = ((-1)^{(a+d-2)r})^t q^{(r/2)((a+d-2)t^2+2(s_1(1+d)-s_2 b)t)}, \tag{10}$$

and

$$\operatorname{Trace} \Lambda_n = n^{-1/2} \sum_{0\le t\le n-1} ((-1)^{(a+d-2)r})^t q^{(r/2)((a+d-2)t^2+2(s_1(1+d)-s_2 b)t)}. \tag{11}$$

Remark 4.11 Here we state the parallel results for $1 = \lambda_1^{a+1}\lambda_2^b$, $1 = \lambda_1^c\lambda_2^{d+1}$, $u^n = -\lambda_1^{-1}$, and $v^n = -\lambda_2^{-1}$.

Suppose $\lambda_1 = \alpha_1 e^{i\theta_1}$ and $\lambda_2 = \alpha_2 e^{i\theta_2}$; then we get

$$
\begin{gathered}
1 = \alpha_1^{a+1}\alpha_2^b, \quad 1 = \alpha_1^c \alpha_2^{d+1}, \\
(a+1)\theta_1 + b\theta_2 = 2k_1\pi, \quad c\theta_1 + (d+1)\theta_2 = 2k_2\pi.
\end{gathered}
$$

Since $u^n = -\lambda_1^{-1}$ and $v^n = -\lambda_2^{-1}$, we can suppose $u = -\alpha_1^{-1/n} e^{-i\theta_1/n} q^{r_1}$ and $v = -\alpha_2^{-1/n} e^{-i\theta_2/n} q^{r_2}$ where both r_1 and r_2 are integers. Then we have

$$
\begin{aligned}
u^{a+1}v^b &= (-1)^{a+b+1} q^{r_1(a+1)+r_2 b-k_1}, \\
u^c v^{d+1} &= (-1)^{c+d+1} q^{r_1 c+r_2(d+1)-k_2}.
\end{aligned}
$$

Similarly we set $s_1 = r_1(a+1)+r_2b-k_1$ and $s_2 = r_1c+r_2(d+1)-k_2$. Then $u^{a+1}v^b = (-1)^{a+b+1}q^{s_1}$ and $u^c v^{d+1} = (-1)^{c+d+1}q^{s_2}$. If $a+d \neq -2$, then $\alpha_1 = \alpha_2 = 1$.

For the case when $(b,n)=1$ and $br+sn=1$, we have

$$(\widetilde{\Lambda}_n)_{t,t} = ((-1)^{(a+d+2)r})^t q^{(r/2)((a+d+2)t^2+2(s_1(1+d)-s_2b)t)}$$

and

$$\text{Trace}\,\Lambda_n = n^{-1/2} \sum_{0\le t\le n-1} ((-1)^{(a+d+2)r})^t q^{(r/2)((a+d+2)t^2+2(s_1(1+d)-s_2b)t)}. \tag{12}$$

Example 4.12 Let $A = \left(\begin{smallmatrix} 2 & 1 \\ -7 & -3 \end{smallmatrix}\right)$. If we try to solve $1 = \lambda_1^{a-1}\lambda_2^b$ and $1 = \lambda_1^c\lambda_2^{d-1}$, we get

$$\theta_1+\theta_2 = 2\pi, \quad -7\theta_1 - 4\theta_2 = 2\pi. \tag{13}$$

We have $\theta_1 = -\frac{10\pi}{3}$ and $\theta_2 = \frac{16\pi}{3}$; thus $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{4\pi i/3}$. So if we set $u = -e^{-10\pi i/3n}$ and $v = -e^{16\pi i/3n}$, then $u^{a-1}v^b = q$ and $u^c v^{d-1} = -q$. We have $s_1 = s_2 = 1$. Since $b=1$, we get $r=1$. Then from (11), we have

$$\text{Trace}\,\Lambda_n = n^{-1/2} \sum_{0\le t\le n-1} (-1)^t q^{(-3t^2+10t)/2}.$$

Note that when n is a multiple of 3, we have $\text{Trace}\,\Lambda_n = 0$.

Example 4.13 Let A be the same matrix as above. But this time we try to solve $1 = \lambda_1^{a+1}\lambda_2^b$ and $1 = \lambda_1^c\lambda_2^{d+1}$; then we get $\lambda_1^{2+a+d} = \lambda_2^{2+a+d} = 1$. Since $2+a+d = 1$, we have $\lambda_1 = \lambda_2 = 1$. If we set $u = v = -1$, then $s_1 = s_2 = 0$. From (11), we have

$$|\text{Trace}\,\Lambda_n| = n^{-1/2}\left|\sum_{0\le t\le n-1} (-1)^t q^{t^2/2}\right| = 1.$$

Lemma 4.14 *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $(b,n)=1$ and $|a+d|=2$. Then we have the following statements*:

(1) *If $a+d=2$ and Λ_n is the intertwiner obtained in Theorem 4.2 such that $|\det(\Lambda_n)|=1$, then $|\text{Trace}\,\Lambda_n| = \sqrt{n}$.*

(2) *If $a+d=-2$ and Λ_n is the intertwiner obtained in Theorem 4.3 such that $|\det(\Lambda_n)|=1$, then $|\text{Trace}\,\Lambda_n| = \sqrt{n}$.*

Proof We only prove the statement (1) (the proof for statement (2) is similar). Let $[\gamma] \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(T^2)$, with

$$\gamma(\alpha) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \quad \text{and} \quad \gamma(\beta) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix},$$

be any A-invariant character.

We use the same notation as in Remark 4.10. Then we have

$$\begin{aligned} s_1(1-d)+s_2b &= r_1(a-1)(1-d)+r_2b(1-d)+k_1(1-d)+r_1cb+r_2(d-1)b+k_2b \\ &= k_1(1-d)+k_2b. \end{aligned}$$

From (8), we can get

$$\begin{aligned}2\pi((1-d)k_1+k_2b)&=(1-d)2\pi k_1+b2\pi k_2\\&=(1-d)(a-1)\theta_1+b(1-d)\theta_2+bc\theta_1+b(d-1)\theta_2=0.\end{aligned}$$

Thus we have

$$(1-d)k_1+k_2b=s_1(1-d)+s_2b=0.$$

From (11), we know

$$\operatorname{Trace}\Lambda_n=n^{-1/2}\sum_{0\le t\le n-1}((-1)^{(a+d-2)r})^t q^{(r/2)((a+d-2)t^2+2(s_1(1-d)+s_2b)t)}=n^{-1/2}n=\sqrt{n}.\quad\square$$

The following theorem shows the limit superior related to the trace of intertwiners for any diffeomorphism of the closed torus is zero, which equals the simplicial volume of the corresponding mapping torus.

Theorem 4.15 *Let $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ be any fixed element in the mapping class group for the closed torus, and let $[\gamma]$ be any fixed A-invariant character with*

$$\gamma(\alpha)=\begin{pmatrix}\lambda_1&0\\0&\lambda_1^{-1}\end{pmatrix}\quad\text{and}\quad\gamma(\beta)=\begin{pmatrix}\lambda_2&0\\0&\lambda_2^{-1}\end{pmatrix}.$$

If $1=\lambda_1^{a-1}\lambda_2^{b}$ and $1=\lambda_1^{c}\lambda_2^{d-1}$, let $\{\Lambda_n\}_{n\in 2\mathbb{Z}_{\ge 0}+1}$ be intertwiners obtained in Theorem 4.2 such that $|\det(\Lambda_n)|=1$ for all $n\in 2\mathbb{Z}_{\ge 0}+1$. If $1=\lambda_1^{a+1}\lambda_2^{b}$ and $1=\lambda_1^{c}\lambda_2^{d+1}$, let $\{\Lambda_n\}_{n\in 2\mathbb{Z}_{\ge 0}+1}$ be intertwiners obtained in Theorem 4.3 such that $|\det(\Lambda_n)|=1$ for all $n\in 2\mathbb{Z}_{\ge 0}+1$. Then we have

$$\limsup_{\text{odd } n\to\infty}\frac{\log(|\operatorname{Trace}\Lambda_n|)}{n}=0.$$

Proof Since $[\gamma]$ is A-invariant, we have $1=\lambda_1^{a-1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d-1}$ or $1=\lambda_1^{a+1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d+1}$. We look at the case when $1=\lambda_1^{a-1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d-1}$. Then we can set Λ_n to be $a_n\overline{\Lambda}_n$, where $a_n=|\det(\overline{\Lambda}_n)|^{-1/n}$.

Case I Suppose $b=0$. In this case we know $|\det(\overline{\Lambda}_n)|=1$ since $n'=1$.

We have $A=\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}-1&0\\c&-1\end{smallmatrix}\right)$. We first consider the case when $A=\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$. If $c=0$, it is trivial. So suppose $c\ne 0$. Since we have $\lambda_1^{a-1}\lambda_2^{b}=1$, $\lambda_1^{c}\lambda_2^{d-1}=1$, we get $\lambda_1^{c}=1$. Suppose $\lambda_1=e^{i\theta}$; then we get $\theta c=2k\pi$ where k is an integer. Since $u^n=-\lambda_1$, we can choose $u=-e^{i\theta/n}q^r$ where r is an integer. Then we have

$$u^c=(-1)^c e^{i\theta c/n}q^{cr}=(-1)^c e^{2k\pi i/n}q^{cr}=(-1)^c q^{k+cr}.$$

Note that $|\operatorname{Trace}\Lambda_n|$ is independent of the choice of r.

From Remark 4.4 and Lemma 4.5, Λ_n is a diagonal matrix, and

$$(\overline{\Lambda}_n)_{t,t}=(v^{(d-1)}u^c)^t q^{c(t^2-dt^2/2)}.$$

Then we have

$$\operatorname{Trace}\Lambda_n=\sum_{0\le t\le n-1}v^{t(d-1)}u^{tc}q^{cdt^2/2}=\sum_{0\le t\le n-1}(-1)^{ct}q^{ct^2/2}q^{(k+cr)t}.$$

Let $\{n_i\}_{i\in\mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\geq 0}+1$ such that $(n_i,c)=1$ for all i. Then for every i there exists r such that $k+cr\equiv 0 \pmod{n_i}$; thus

$$|\mathrm{Trace}\,\Lambda_{n_i}| = \left|\sum_{0\leq t\leq n_i-1}(-1)^{ct}q^{ct^2/2}\right| = \sqrt{(n_i,c)n_i} = \sqrt{n_i}\geq 1.$$

Thus we have

$$0\leq \limsup_{\text{odd } n\to\infty}\frac{\log(|\mathrm{Trace}\,\Lambda_n|)}{n}.$$

According to Theorem 4.7, we also have

$$\limsup_{\text{odd } n\to\infty}\frac{\log(|\mathrm{Trace}\,\Lambda_n|)}{n}\leq \limsup_{\text{odd } n\to\infty}\frac{\log(n^{3/2})}{n}=0.$$

We look at the case when $A=\left(\begin{smallmatrix}-1 & 0\\ c & -1\end{smallmatrix}\right)$. Then we get $(\lambda_1)^2=1$ and $\lambda_1=\pm 1$. We can choose $u=\pm 1$. From Remark 4.4 and Lemma 4.5 we get

$$(\overline{\Lambda}_n)_{t,k} = r_{k-td}(v^{(d-1)}u^c)^t q^{c(tk-dt^2/2)},$$

where $r_k=1$ if k is a multiple of n and it is zero otherwise. Then $(\overline{\Lambda}_n)_{t,t}\neq 0$ if and only if $r_{2t}\neq 0$ if and only if $n\mid (2t)$, which means there is only one nonzero diagonal element. Then we get $|\mathrm{Trace}\,\Lambda_n|=1$ for any n, which proves this special case.

Case II Suppose $b\neq 0$.

We first consider the subcase when $a+d\neq 2$. From the above discussion we know λ_1 and λ_2 are both roots of unity; thus we can suppose $\lambda_1=e^{i\theta_1}$, $\lambda_2=e^{i\theta_2}$ and we can get (8) where θ_1, θ_2, k_1 and k_2 are determined by γ. Since $u^n=-\lambda_1$ and $v^n=-\lambda_2$, we can write $u=-e^{i\theta_1/n}q^{r_1}$ and $v=-e^{i\theta_2/n}q^{r_1}$. Then we have (9). Note that $|\mathrm{Trace}\,\Lambda_n|$ is independent of the choice of r_1 and r_2.

Since $b\neq 0$ and $2-(a+d)\neq 0$, let $\{n_j\}_{j\in\mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\geq 0}+1$ such that

$$(n_j,b)=(n_j,2-(a+d))=1.$$

Since $\left[\begin{smallmatrix}a-1 & b\\ c & d-1\end{smallmatrix}\right]=2-(a+d)$ and $(n_j,2-(a+d))=1$, the following equations always have solutions in $\mathbb{Z}_{n_j}$:

$$r_1(a-1)+r_2b+k_1=0,\quad r_1c+r_2(d-1)+k_2=0.$$

Thus for every j, there always exist integers r_1 and r_2 such that $s_1=s_2=0$ in $\mathbb{Z}_{n_j}$. Then, from (11),

$$\begin{aligned}|\mathrm{Trace}\,\Lambda_{n_j}| &= n_j^{-1/2}\left|\sum_{0\leq t\leq n_j-1}((-1)^{(a+d-2)r})^t q^{(r/2)((a+d-2)t^2+2(s_1(1-d)+s_2b)t)}\right|\\ &= n_j^{-1/2}\left|\sum_{0\leq t\leq n_j-1}((-1)^{(a+d-2)r})^t q^{(r/2)((a+d-2)t^2)}\right|\\ &= n_j^{-1/2}\left|\sum_{0\leq t\leq n_j-1}(-q^{1/2})^{r(a+d-2)t^2}\right| = n_j^{-1/2}\sqrt{(r(a+d-2),n_j)n_j}=1.\end{aligned}$$

Then we have

$$0 \le \limsup_{\text{odd } n\to\infty} \frac{\log(|\mathrm{Trace}\,\Lambda_n|)}{n}.$$

By Theorem 4.7,

$$\limsup_{\text{odd } n\to\infty} \frac{\log(|\mathrm{Trace}\,\Lambda_n|)}{n} = 0.$$

The other subcase is when $a+d=2$. Since $b\neq 0$, let $\{n_k\}_{k\in\mathbb{N}}$ be a subsequence of $2\mathbb{Z}_{\ge 0}+1$ such that $(n_k,b)=1$ for all k. From Lemma 4.14, we have $|\mathrm{Trace}\,\Lambda_{n_k}|=\sqrt{n_k}\ge 1$. Similarly, we get

$$\limsup_{\text{odd } n\to\infty} \frac{\log(|\mathrm{Trace}\,\Lambda_n|)}{n} = 0. \qquad \square$$

From now on we discuss the periodic mapping class. Recall that $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is periodic if and only if $|a+d|\in\{0,1\}$. Suppose $[\gamma]$ is an A-invariant character with

$$\gamma(\alpha)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix} \quad\text{and}\quad \gamma(\beta)=\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_2^{-1}\end{pmatrix}.$$

Then we have $1=\lambda_1^{a-1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d-1}$ or $1=\lambda_1^{a+1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d+1}$. For the case when $1=\lambda_1^{a-1}\lambda_2^{b}$, $1=\lambda_1^{c}\lambda_2^{d-1}$, the above discussion implies $1=\lambda_1^{2-(a+d)}=\lambda_2^{2-(a+d)}$. So if $a+d=0$, we have $\lambda_1^2=\lambda_2^2=1$; thus $\lambda_1=\pm1$ and $\lambda_2=\pm1$. Hence there is no A-invariant character living in the Azumaya locus if $\lambda_1=\pm1$ and $\lambda_2=\pm1$. But we can still get intertwiners in Theorems 4.2 and 4.3, although $\lambda_1=\pm1$ and $\lambda_2=\pm1$. Now we consider intertwiners if we choose $\lambda_1=\lambda_2=1$.

Theorem 4.16 *Let A be a periodic mapping class, and let Λ_n be the intertwiner obtained in Theorem 4.2 or 4.3 by using the trivial A-invariant character, that is $\lambda_1=\lambda_2=1$, and we require $|\det(\Lambda_n)|=1$. We have the following conclusions*:

(1) *If $a+d=1$ and Λ_n is obtained in Theorem 4.2, then $|\mathrm{Trace}(\Lambda_n)|=1$ for any odd n.*

(2) *If $a+d=-1$ and Λ_n is obtained in Theorem 4.3, then $|\mathrm{Trace}(\Lambda_n)|=1$ for any odd n.*

(3) *If $a+d=0$ and Λ_n is obtained in Theorem 4.2 or 4.3, then $|\mathrm{Trace}(\Lambda_n)|=1$ for any odd n.*

Proof Suppose $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, $(b,n)=m$, $br+sn=m$ and $n=mn'$. Since $\lambda_1=\lambda_2=1$, we can set $u=v=-1$. From the previous discussion we know

$$(\overline{\Lambda}_n)_{k,t}=(-1)^{cdt}r_{k-td}q^{c(tk-dt^2/2)}$$

for all $0\le k,t\le n-1$, where

$$r_{tm}=(-1)^{abrt}q^{abt^2r^2/2} \quad \text{for all } 0\le t\le n'-1,$$

and all other r_k are 0. For the l^{th} column, we have $\{(\overline{\Lambda}_n)_{ld+km,l}\}_{0\le k\le n'-1}$ are the only nonzero entries. Then the l^{th} column contains a nonzero diagonal entry if and only if $ld+km\equiv l \pmod n$ for some $0\le k\le n'-1$. It is easy to show $ld+km\equiv l \pmod n$ for some $0\le k\le n'-1$ if and only if $m\mid(ld-l)$.

Now we suppose $(m, d-1) = 1$. Then the l^{th} column of $\overline{\Lambda}_n$ contains a nonzero diagonal entry if and only if $m \mid l$. Thus $(\overline{\Lambda}_n)_{tm,tm}$, $0 \le t \le n'-1$, are the only nonzero diagonal entries, and

$$\begin{aligned}(\overline{\Lambda}_n)_{tm,tm} &= (-1)^{cdtm} r_{tm-tmd} q^{c(t^2m^2-dt^2m^2/2)} \\ &= (-1)^{cdtm}(-1)^{abr(t-td)} q^{abr^2(1-d)^2t^2/2} q^{c(t^2m^2-dt^2m^2/2)}.\end{aligned}$$

After a similar calculation as in Remark 4.10, we can get

$$(\overline{\Lambda}_n)_{tm,tm} = (-1)^{(ar+dr-2r)t}(q^{m/2})^{(ar+dr-2r)t^2}.$$

Then we have

$$\text{Trace}\,\overline{\Lambda}_n = \sum_{0\le t\le n'-1} (-1)^{(ar+dr-2r)t}(q^{m/2})^{(ar+dr-2r)t^2}.$$

Similarly if $(m, d+1) = 1$, then

$$\text{Trace}\,\widetilde{\Lambda}_n = \sum_{0\le t\le n'-1} (-1)^{(ar+dr+2r)t}(q^{m/2})^{(ar+dr+2r)t^2}.$$

(1) Since the intertwiner is obtained in Theorem 4.2, we can set $\Lambda_n = (n')^{-1/2}\overline{\Lambda}_n$. We have $d-1 = -a$ because $a+d = 1$. Then we get $(d-1, m) = 1$ because $(a, b) = 1$ and m is a divisor of b. Then from the above discussion, we get

$$\begin{aligned}|\text{Trace}\,\Lambda_n| &= (n')^{-1/2}\left|\sum_{0\le t\le n'-1} (-1)^{(ar+dr-2r)t}(q^{m/2})^{(ar+dr-2r)t^2}\right| \\ &= (n')^{-1/2}\left|\sum_{0\le t\le n'-1} (-1)^{-rt}(q^{m/2})^{-rt^2}\right| \\ &= (n')^{-1/2}\sqrt{(-r, n')n'} = 1.\end{aligned}$$

(2) The proof is similar to that of (1).

(3) First we show $(m, d-1) = (m, d+1) = 1$ if $a+d = 0$. From $ad-bc = 1$, we get $-bc = d^2+1$. Suppose $p \mid m$ and $p \mid d-1$. Then $p \mid (d^2+1)$ and $p \mid (d^2-1)$. Thus we get $p \mid 2$, which means $p = 1$ because $(m, 2) = 1$. Similarly we can show $(m, d+1) = 1$. If $\Lambda_n = (n')^{-1/2}\overline{\Lambda}_n$, then

$$\begin{aligned}|\text{Trace}\,\Lambda_n| &= (n')^{-1/2}\left|\sum_{0\le t\le n'-1} (-1)^{(ar+dr-2r)t}(q^{m/2})^{(ar+dr-2r)t^2}\right| \\ &= (n')^{-1/2}\left|\sum_{0\le t\le n'-1} (q^{m})^{-rt^2}\right| \\ &= (n')^{-1/2}\sqrt{(-r, n')n'} = 1.\end{aligned}$$

If $\Lambda_n = (n')^{-1/2}\widetilde{\Lambda}_n$, we can similarly show that $|\text{Trace}\,\Lambda_n| = 1$. □

5 The volume conjecture for surface diffeomorphisms: periodic case

5.1 Preliminaries for the volume conjecture for periodic surface diffeomorphisms

If we want to formulate the parallel conjecture for periodic diffeomorphisms as in [7; 8], we have to find a good invariant character that lives in the smooth part of $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$.

Lemma 5.1 [11] *Let $A, B \in \mathrm{SL}(2,\mathbb{C})$. If $\mathrm{Trace}([A,B]) = 2$ where $[A,B] = ABA^{-1}B^{-1}$, then $G = \langle A, B\rangle \le \mathrm{SL}(2,\mathbb{C})$ is not free of rank two where $\langle A, B\rangle$ is the group generated by A and B.*

Lemma 5.2 *Let G be a subgroup of $\mathrm{SL}(2,\mathbb{C})$ freely generated by two elements, and let R be the subalgebra of $\mathrm{Mat}(2,\mathbb{C})$ generated by G, where $\mathrm{Mat}(2,\mathbb{C})$ is the algebra of all 2 by 2 complex matrices. Then $R = \mathrm{Mat}(2,\mathbb{C})$.*

Proof Suppose G is freely generated by A and B. We know there exists $X \in \mathrm{GL}(2,\mathbb{C})$ such that $XAX^{-1} = \left(\begin{smallmatrix} u & v \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $XBX^{-1} = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Then XGX^{-1} is a free subgroup generated by XAX^{-1} and XBX^{-1}, and XRX^{-1} is the subalgebra generated by XGX^{-1}. Since $XRX^{-1} = \mathrm{Mat}(2,\mathbb{C})$ if and only if $R = \mathrm{Mat}(2,\mathbb{C})$, we can assume $A = \left(\begin{smallmatrix} u & v \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$.

(I) Suppose $v = 0$. Then $A = \left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}\end{smallmatrix}\right)$ and $u^2 \neq 1$, otherwise $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ or $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, which contradicts the fact that G is freely generated by A and B. We also get $b \neq 0$ and $c \neq 0$, since otherwise $\mathrm{Trace}([A,B]) = 2$, which contradicts the fact that A and B freely generate G by Lemma 5.1. Since $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right) \in R$ and $u \neq \pm 1$, we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right) \in R$. Then $\left(\begin{smallmatrix} 0 & b \\ c & 0\end{smallmatrix}\right) \in R$. By multiplication, $\left(\begin{smallmatrix} 0 & b \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ c & 0\end{smallmatrix}\right) \in R$, which implies $R = \mathrm{Mat}(2,\mathbb{C})$ since $b \neq 0$ and $c \neq 0$.

(II) Suppose $v \neq 0$. In this case we should have $c \neq 0$, otherwise $\mathrm{Trace}([A,B]) = 2$, which is a contradiction.

If $u = \pm 1$, then $A = \left(\begin{smallmatrix} u & v \\ 0 & u\end{smallmatrix}\right) \in R$. Remember we also have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right) \in R$, which implies $\left(\begin{smallmatrix} 0 & v \\ 0 & 0\end{smallmatrix}\right) \in R$. Furthermore, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right) \in R$ because $v \neq 0$. Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in R$, we have $\left(\begin{smallmatrix} a & 0 \\ c & d\end{smallmatrix}\right) \in R$, and also $\left(\begin{smallmatrix} a-d & 0 \\ c & 0\end{smallmatrix}\right) \in R$. By multiplication, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} a-d & 0 \\ c & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} c & 0 \\ 0 & 0\end{smallmatrix}\right) \in R$, which implies $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right) \in R$ because $c \neq 0$. We have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right) \in R$, so $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right) \in R$. Remember we also have $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in R$ and $c \neq 0$, which implies $R = \mathrm{Mat}(2,\mathbb{C})$.

If $u \neq \pm 1$, then $\left(\begin{smallmatrix} u-u^{-1} & v \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & v \\ 0 & u^{-1}-u\end{smallmatrix}\right) \in R$, so $\left(\begin{smallmatrix} 1 & k \\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -k \\ 0 & 1\end{smallmatrix}\right) \in R$ where $k = v/(u-u^{-1})$. Then from multiplication, we get

$$\begin{pmatrix} 1 & k \\ 0 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a+kc & b+kd \\ 0 & 0\end{pmatrix} \in R.$$

Next we want to show $b + kd \neq k(a+kc)$. Suppose the contrary. Then $b + kd = k(a+kc) = ka + k^2c$. With $k = v/(u-u^{-1})$, we can get

$$b + \frac{dv}{u-u^{-1}} = \frac{av}{u-u^{-1}} + \frac{cv^2}{(u-u^{-1})^2} \implies 2b = bu^2 + bu^{-2} + dvu - dvu^{-1} - avu + avu^{-1} - cv^2.$$

Then we get

$$\begin{aligned}\operatorname{Trace}([A,B]) &= ad + acuv + cdvu^{-1} + c^2v^2 - cbu^2 - cduv - cbu^{-2} - cau^{-1}v + ad \\ &= 2ad - c(-auv - dvu^{-1} - cv^2 + bu^2 + duv + bu^{-2} + au^{-1}v) = 2ad - 2cb = 2.\end{aligned}$$

Since $\operatorname{Trace}([A,B]) = 2$ is a contradiction, we have $b + kd \neq k(a + kc)$. We then get $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \in R$ because $\left(\begin{smallmatrix}a+kc&b+kd\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&k\\0&0\end{smallmatrix}\right) \in R$. We also have $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in R$, so $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in R$. Thus, $R = \operatorname{Mat}(2,\mathbb{C})$ because $c \neq 0$. □

Proposition 5.3 *Let $\gamma\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$ be a representative of an element in the character variety $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$. Then γ is irreducible if $\operatorname{Im}\gamma$ contains a subgroup of $\mathrm{SL}(2,\mathbb{C})$ freely generated by two elements. In particular, γ is irreducible if S has negative Euler characteristic and γ is injective.*

Proof This is a direct consequence of Lemma 5.2. □

5.2 Statement of the conjecture

To get the intertwiner, we first have to get a φ-invariant smooth character $\gamma \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$. In [13, page 371] it is proved that every periodic diffeomorphism fixes a point in the Teichmüller space. This means there is a discrete and faithful group homomorphism $\bar\gamma\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ such that $\bar\gamma\varphi_*$ is conjugate to $\bar\gamma$ by an element in $\mathrm{PSL}(2,\mathbb{R})$, where φ_* is the isomorphism from $\pi_1(S)$ to $\pi_1(S)$ induced by φ.

Since $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$, we can regard $\bar\gamma\varphi_*$ and $\bar\gamma$ as two elements in $\mathscr{X}_{\mathrm{PSL}(2,\mathbb{C})}(S)$. Then $\bar\gamma\varphi_*$ is conjugate to $\bar\gamma$ by an element in $\mathrm{PSL}(2,\mathbb{C})$. Thus $\bar\gamma$ can be extended to a group homomorphism from $\pi_1(M_\varphi)$ to $\mathrm{PSL}(2,\mathbb{C})$; we use $\hat\gamma$ to denote this homomorphism. Then we can lift $\hat\gamma$ to a group homomorphism $\tilde\gamma$ from $\pi_1(M_\varphi)$ to $\mathrm{SL}(2,\mathbb{C})$. The restriction of $\tilde\gamma$ to $\pi_1(S)$ is φ-invariant, and we use γ to denote this group homomorphism. Note that γ is a group homomorphism from $\pi_1(S)$ to $\mathrm{SL}(2,\mathbb{C})$. Let ε be the projection from $\mathrm{SL}(2,\mathbb{C})$ be $\mathrm{PSL}(2,\mathbb{C})$; then we have $\varepsilon\tilde\gamma = \hat\gamma$. Furthermore,

$$\varepsilon\gamma = \varepsilon\tilde\gamma|_{\pi_1(S)} = \hat\gamma|_{\pi_1(S)} = \bar\gamma.$$

Since $\bar\gamma$ is injective, γ is injective. From Proposition 5.3, we know γ is irreducible. Thus we get a φ-invariant smooth character $\gamma \in \mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$. From now on, we use γ_φ to denote γ and $\bar\gamma_\varphi$ to denote $\bar\gamma$.

For every puncture v in S, we know $\operatorname{Trace}\gamma_\varphi(\alpha_v) = \pm 2$ where α_v is the loop going around puncture v. If $\operatorname{Trace}\gamma_\varphi(\alpha_v) = 2$, we choose $p_v = -(q + q^{-1})$. Then

$$T_n(p_v) = (-q)^n + (-q^{-1})^n = -1 - 1 = -\operatorname{Trace}\gamma_\varphi(\alpha_v).$$

If $\operatorname{Trace}\gamma_\varphi(\alpha_v) = -2$, we choose $p_v = 1 + 1$. Then

$$T_n(p_v) = 1^n + 1^n = 1 + 1 = -\operatorname{Trace}\gamma_\varphi(\alpha_v).$$

Since $\operatorname{Trace}\gamma_\varphi(\alpha_v) = \operatorname{Trace}\gamma_\varphi(\varphi(\alpha_v)) = \operatorname{Trace}\gamma_\varphi(\alpha_{\varphi(v)})$, we have $p_v = p_{\varphi(v)}$. So now we have everything we want. Then we obtain the Kauffman bracket intertwiner $\Lambda^q_{\varphi,\gamma_\varphi}$ associated to these data. We require $|\det(\Lambda^q_{\varphi,\gamma_\varphi})| = 1$. With the fixed S, φ, γ_φ and $\{p_v\}_v$, we have $|\operatorname{Trace}\Lambda^q_{\varphi,r}|$ is only related to q.

Conjecture 5.4 *Suppose S is an oriented surface with negative Euler characteristic, and φ is a periodic diffeomorphism for S. Let γ_φ be the φ-invariant smooth character defined as in the second paragraph of this subsection. For each puncture v, let p_v be the complex number defined as in the third paragraph of this subsection. Let $q_n = e^{2\pi i/n}$ with $(q_n)^{1/2} = e^{\pi i/n}$. Then*

$$\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\operatorname{Trace} \Lambda^{q_n}_{\varphi,r}| = 0.$$

5.3 Proofs for the conjecture for some special cases

In the remaining part of this paper, we will present some results related to our conjecture. Especially, we will give a proof for our conjecture when the surface S is the once punctured torus.

In the following theorem, we use the periodic property of the diffeomorphisms to prove that the limit in Conjecture 5.4 is less than or equal to zero if it exists.

Theorem 5.5 *If $\lim_{\text{odd } n\to\infty} \frac{1}{n} \log|\operatorname{Trace} \Lambda^{q_n}_{\varphi,r_\varphi}|$ exists, the limit is less than or equal to zero.*

Proof Let $\rho\colon \mathrm{SK}_{q_n^{1/2}}(S) \to \mathrm{End}(V)$ be an irreducible representation of the skein algebra associated to γ_φ and weight system $\{p_v\}_v$. From the definition of intertwiners $\Lambda^{q_n}_{\varphi,r_\varphi}$, we know

$$\rho\varphi_\sharp(X) = \Lambda^{q_n}_{\varphi,r_\varphi} \circ \rho(X) \circ (\Lambda^{q_n}_{\varphi,r_\varphi})^{-1}$$

for all $X \in \mathrm{SK}_{q_n^{1/2}}(S)$. We have

$$\rho(\varphi^2)_\sharp(X) = \rho\varphi_\sharp(\varphi_\sharp(X)) = \Lambda^{q_n}_{\varphi,r_\varphi} \circ \rho\varphi_\sharp(X) \circ (\Lambda^{q_n}_{\varphi,r_\varphi})^{-1} = (\Lambda^{q_n}_{\varphi,r_\varphi})^2 \circ \rho\varphi_\sharp(X) \circ (\Lambda^{q_n}_{\varphi,r_\varphi})^{-2}.$$

Then it is easy to show that, with any integer j, we have

$$\rho(\varphi^j)_\sharp(X) = (\Lambda^{q_n}_{\varphi,r_\varphi})^j \circ \rho\varphi_\sharp(X) \circ (\Lambda^{q_n}_{\varphi,r_\varphi})^{-j}.$$

Since φ is periodic, there exists a positive integer k such that $\varphi^k = \mathrm{Id}_S$. Then we have

$$\rho(X) = \rho(\varphi^k)_\sharp(X) = (\Lambda^{q_n}_{\varphi,r_\varphi})^k \circ \rho(X) \circ (\Lambda^{q_n}_{\varphi,r_\varphi})^{-k}$$

for all $X \in \mathrm{SK}_{q^{1/2}}(S)$. We must have $(\Lambda^{q_n}_{\varphi,r_\varphi})^k = \lambda I$ because ρ is irreducible, where I is the identity matrix and λ is a nonzero complex number. But we require $|\det(\Lambda^{q_n}_{\varphi,r_\varphi})| = 1$; thus $|\lambda| = 1$. Actually we can always choose a good $\Lambda^{q_n}_{\varphi,r_\varphi}$ such that $(\Lambda^{q_n}_{\varphi,r_\varphi})^k = I$. Since $x^k - 1$ has no multiple roots, $\Lambda^{q_n}_{\varphi,r_\varphi}$ is always diagonalizable. All its eigenvalues are k^{th} roots of unity. Then

$$\operatorname{Trace} \Lambda^{q_n}_{\varphi,r_\varphi} = \sum_{0\le i\le n-1} \lambda_i,$$

where $\lambda_i^k = 1$ for all $0 \le i \le n-1$.

We have $|\operatorname{Trace} \Lambda^{q_n}_{\varphi,r_\varphi}| \le n$. So if the limit exists, the limit is less than or equal to zero. $\square$

From the proof of Theorem 5.5, we know $|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}|$ is simply the absolute value of the sum of roots of unity. We are only concerned with how small $|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}|$ can be because of Theorem 5.5. Actually this problem was already asked by Myerson [24] and Tao [27]. For any two positive integers k and n, let $f(n,k)$ be the least absolute value of a nonzero sum of n (not necessarily distinct) k^{th} roots of unity. Myerson gave the lower bound for all positive integers k and n:

$$f(n,k) \geq n^{-k}. \tag{14}$$

According to [20], we know $\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r} \neq 0$ if the order of φ is 2^m for some positive integer m.

Theorem 5.6 *If φ is of order 2^m where m is any positive integer, then for any surface with negative Euler characteristic,*

$$\lim_{\text{odd } n\to\infty} \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| = 0.$$

Proof Since for any odd n, we have $\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r} \neq 0$. Then

$$n^{-k} \leq f(n,k) \leq |\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r}|,$$

where $k = 2^m$. So we get

$$\frac{1}{n}\log n^{-k} \leq \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| \leq \frac{1}{n}\log n.$$

Then $\lim_{\text{odd } n\to\infty} \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| = 0$. □

Proposition 5.7 *If φ is of order p^m where p is any positive prime number and m is any positive integer, then for any surface with negative Euler characteristic,*

$$\limsup_{\text{odd } n\to\infty} \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| = 0.$$

Proof The proof is similar to that of Theorem 5.6. □

For the following discussion, we will use some notation and terminology in [7]. Suppose the surface S has at least one puncture, that is, it has ideal triangulations. Let τ be an ideal triangulation of S, and let φ be any periodic map of S. Suppose $\tau = \tau^{(0)}, \tau^{(1)}, \dots, \tau^{(k)} = \varphi(\tau)$ is an ideal triangulation sweep. Since φ fixes a point in the Teichmüller space, there exists a periodic edge weight system $a = a^{(0)}, a^{(1)}, \dots, a^{(k)} = a \in (\mathbb{R}_{>0})^e$ where a is the shear parameter corresponding to this fixed point in the Teichmüller space. Then $[\bar{\gamma}_\varphi] \in \mathscr{X}_{\mathrm{PSL}(2,\mathbb{C})}(S)$ is the character associated to the weight system a. From the above discussion, we know $[\bar{\gamma}_\varphi]$ can be lift to a smooth φ-invariant character $[\gamma_\varphi]$ in $\mathscr{X}_{\mathrm{SL}(2,\mathbb{C})}(S)$.

We also have $a_{i_1}a_{i_2}\cdots a_{i_j} = 1$, where $e_{i_1}, e_{i_2}, \dots, e_{i_j}$ are all the edges connecting to a common vertex, because a corresponds to a complete hyperbolic structure. If $\mathrm{Trace}\,\gamma_\varphi(\alpha_v) = 2$, set $h_v = q^2$. Then $h_v^n = 1$ and $p_v^2 = h_v + h_v^{-1} + 2$. If $\mathrm{Trace}\,\gamma_\varphi(\alpha_v) = -2$, set $h_v = 1$. Then $h_v^n = 1$ and $p_v^2 = h_v + h_v^{-1} + 2$. Obviously $h_v = h_{\varphi(v)}$ for any puncture v. Proposition 15 of [7] implies that we can obtain an intertwiner $\overline{\Lambda}^q_{\varphi,\bar{r}_\varphi}$ with $|\det(\overline{\Lambda}^q_{\varphi,\bar{r}_\varphi})| = 1$. According to [7, Theorem 16], we have $|\mathrm{Trace}\,\overline{\Lambda}^q_{\varphi,\bar{r}_\varphi}| = |\mathrm{Trace}\,\Lambda^q_{\varphi,r_\varphi}|$.

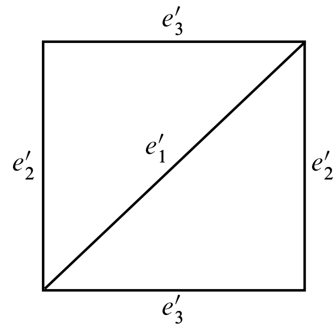

Figure 4

For the once punctured torus $S_{1,1}$, we only have one puncture v. Let $\alpha = K_2$ and $\beta = K_1$ denote two elements in $\pi_1(S_{1,1})$; see Figure 3. It is well known that α and β freely generate $\pi_1(S_{1,1})$. Let α_v be the loop around v. Then $\alpha_v = \beta\alpha\beta^{-1}\alpha^{-1}$. From Lemma 5.1, we have

$$\operatorname{Trace}\gamma_\varphi(\alpha_v) = \operatorname{Trace}\gamma_\varphi(\beta\alpha\beta^{-1}\alpha^{-1}) = \operatorname{Trace}\gamma_\varphi(\beta)\gamma_\varphi(\alpha)\gamma_\varphi(\beta)^{-1}\gamma_\varphi(\alpha)^{-1} \neq 2$$

because γ_φ is injective. Thus we must have $\operatorname{Trace}\gamma_\varphi(\alpha_v) = -2$, which means $h_v = 1$.

Lemma 5.8 *Let the surface be $S_{1,1}$. Then Conjecture 5.4 holds if φ is $\left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1 & 1\\ -1 & 0\end{smallmatrix}\right)$.*

Proof We only prove the case when $\varphi = \left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right)$ (the proof for the other one is similar). Let τ be the ideal triangulation in Figure 2. Then $\varphi(\tau)$ is the ideal triangulation in Figure 4.

Thus from τ to $\varphi(\tau)$ is relabeling. Suppose the shear parameter for τ is $a^\tau = (a_1, a_2, a_3)$; then the shear parameter for $\varphi(\tau)$ is $a^{\varphi(\tau)} = (a_3, a_1, a_2)$. From $a^\tau = a^{\varphi(\tau)}$, we get $a_1 = a_2 = a_3$. Since we also have $a_1^2a_2^2a_3^2 = 1$ and $a_i \in \mathbb{R}_{>0}$, we have $a_1 = a_2 = a_3 = 1$.

Recall that the Chekhov–Fock algebra associated to the ideal triangulation τ is $\mathbb{C}_{q^4}[Y_1^{\pm1}, Y_2^{\pm1}, Y_3^{\pm1}]$, where Y_i corresponds to edge e_i for $i = 1, 2, 3$. The algebra $\mathbb{C}_{q^4}[Y_1^{\pm1}, Y_2^{\pm1}, Y_3^{\pm1}]$ is generated by Y_1, Y_2 and Y_3, and subject to the relations

$$Y_1Y_2 = q^4Y_2Y_1, \quad Y_2Y_3 = q^4Y_3Y_2, \quad Y_3Y_1 = q^4Y_1Y_3, \quad Y_iY_i^{-1} = Y_i^{-1}Y_i = 1.$$

Define the irreducible representation ρ of $\mathbb{C}_{q^4}[Y_1^{\pm1}, Y_2^{\pm1}, Y_3^{\pm1}]$ as $\rho_{1,1,1}$ in Lemma 3.9, that is, set $y_1 = y_2 = y_3 = 1$ in (5). Then

$$\rho(Y_1^n) = \mathrm{Id}_V = a_1\mathrm{Id}_V, \quad \rho(Y_2^n) = \mathrm{Id}_V = a_2\mathrm{Id}_V, \quad \rho(Y_3^n) = \mathrm{Id}_V = a_3\mathrm{Id}_V$$

and

$$\rho(H_v) = \rho([Y_1^2Y_2^2Y_3^2]) = \mathrm{Id}_V = h_v\mathrm{Id}_V.$$

It is easy to calculate that $\Phi^{q_n}_{\tau\varphi(\tau)}\Psi^{q_n}_{\varphi(\tau)\tau}$ is actually an isomorphism from $\mathbb{C}_{q^4}[Y_1^{\pm1}, Y_2^{\pm1}, Y_3^{\pm1}]$ to itself and

$$\Phi^{q_n}_{\tau\varphi(\tau)}\Psi^{q_n}_{\varphi(\tau)\tau}(Y_1) = Y_3, \quad \Phi^{q_n}_{\tau\varphi(\tau)}\Psi^{q_n}_{\varphi(\tau)\tau}(Y_2) = Y_1, \quad \Phi^{q_n}_{\tau\varphi(\tau)}\Psi^{q_n}_{\varphi(\tau)\tau}(Y_3) = Y_2.$$

We use ρ' to denote the irreducible representation $\rho\Phi^{q_n}_{\tau\varphi(\tau)}\Psi_{\varphi(\tau)\tau}$. Then ρ is isomorphic to ρ'.

For each $0 \le k \le n-1$, set

$$v_k = \sum_{0 \le i \le n-1} q_n^{k^2+i^2+4ik+i-k} w_i.$$

Then we have

$$\rho'(Y_1)(v_k) = q_n^{4k} v_k, \quad \rho'(Y_2)(v_k) = q_n^{-2k} v_{k+1}, \quad \rho'(Y_3)(v_k) = q_n^{-2k} v_{k-1}.$$

Define invertible operator Λ for V such that $\Lambda(w_k) = v_k$ for all $0 \le k \le n-1$. Then, for all $0 \le k \le n-1$, we have

$$\rho'(Y_1)(\Lambda(w_k)) = \rho'(Y_1)(v_k) = q_n^{4k} v_k = \Lambda(q_n^{4k} w_k) = \Lambda(\rho(Y_1) w_k).$$

Thus we get $\rho'(Y_1) = \Lambda \circ \rho(Y_1) \circ \Lambda^{-1}$. Similarly, $\rho'(Y_2) = \Lambda \circ \rho(Y_2) \circ \Lambda^{-1}$ and $\rho'(Y_3) = \Lambda \circ \rho(Y_3) \circ \Lambda^{-1}$. Thus Λ is the intertwiner. As a matrix, we have $\Lambda_{i,k} = q_n^{k^2+i^2+4ik+i-k}$.

From direct calculation, we get $|\det(\Lambda)| = n^{n/2}$. Thus we can set $\overline{\Lambda}^{q_n}_{\varphi,\bar{r}_\varphi} = n^{-1/2}\Lambda$. Then

$$|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| = |\mathrm{Trace}\,\overline{\Lambda}^{q_n}_{\varphi,\bar{r}_\varphi}| = n^{-1/2} \sum_{0 \le i \le n-1} q_n^{6i^2} = n^{-1/2}\sqrt{(6,n)n} = \sqrt{(6,n)}.$$

Obviously we get

$$\lim_{\text{odd } n \to \infty} \frac{1}{n} \log|\mathrm{Trace}\,\Lambda^{q_n}_{\varphi,r_\varphi}| = 0. \qquad \square$$

Remark 5.9 In the proof of Lemma 5.8, when we try to find the periodic edge weight system for the triangulation sweep $\tau, \varphi(\tau)$, we require $a_i \in \mathbb{R}_{>0}$ because we want to get the fixed character corresponding to a point in the Teichmüller space. Actually we still get the same intertwiner Λ as in Lemma 5.8 without requiring $a_i \in \mathbb{R}_{>0}$, that is, for any periodic edge weight system, the intertwiner we get is Λ. This means Lemma 5.8 still holds when we choose any other φ-invariant smooth character (without the restriction for only choosing the one corresponding to a fixed point in the Teichmüller space). Readers can check the same arguments hold for Theorems 5.6 and 5.13.

Let ϕ be a pseudo-Anosov map for S, and let f be any diffeomorphism for surface S. Then $f\phi f^{-1}$ is also a pseudo-Anosov map. Then we have the following conclusion:

Lemma 5.10 *Let ϕ be any pseudo-Anosov map for S, and let f be any diffeomorphism for S. If Conjecture 2.3 holds for ϕ, then it also holds for $f\phi f^{-1}$.*

Proof We will use the same notation as in Conjecture 2.3. Let f_*^{-1} be the isomorphism from $\pi_1(S)$ to itself induced by f^{-1}. Then $[\gamma f_*^{-1}]$ is a smooth $f\phi f^{-1}$-invariant character. Set $\theta'_v = \theta_{f^{-1}(v)}$; then θ'_v are invariant under the action of $f\phi f^{-1}$ and

$$\mathrm{Trace}\,\gamma f_*^{-1}(\alpha_v) = \mathrm{Trace}\,\gamma(\alpha_{f^{-1}(v)}) = -e^{\theta_{f^{-1}(v)}} - e^{-\theta_{f^{-1}(v)}} = -e^{\theta'_v} - e^{-\theta'_v}.$$

Set $p'_v = e^{\theta'_v/n} + e^{-\theta'_v/n} = e^{\theta_{f^{-1}(v)}/n} + e^{-\theta_{f^{-1}(v)}/n} = p_{f^{-1}(v)}$; then

$$T_n(p'_v) = -\,\mathrm{Trace}\,\gamma f_*^{-1}(\alpha_v).$$

Recall that we use $f_\sharp^{-1}$ to denote the isomorphism from $\mathrm{SK}_{q^{1/2}}(S)$ to itself induced by f^{-1}. Let ρ be the irreducible representation associated to $[\gamma]$ and puncture weights p_v. Then $\rho f_\sharp^{-1}$ is an irreducible representation associated to the character $[\gamma f_*^{-1}]$ and puncture weights p'_v.

With the assumption for Conjecture 2.3, we have

$$\rho\phi_\sharp(X) = \Lambda^{q_n}_{\phi,r} \circ \rho(X) \circ (\Lambda^{q_n}_{\phi,r})^{-1}$$

for any element $X \in \mathrm{SK}_{q^{1/2}}(S)$ and $|\det(\Lambda^{q_n}_{\phi,r})| = 1$. Then we get

$$\rho f_\sharp^{-1}(f\phi f^{-1})_\sharp(X) = \rho f_\sharp^{-1} f_\sharp \phi_\sharp f_\sharp^{-1}(X) = \rho\phi_\sharp f_\sharp^{-1}(X) = \Lambda^{q_n}_{\phi,r} \circ \rho(f_\sharp^{-1}(X)) \circ (\Lambda^{q_n}_{\phi,r})^{-1}.$$

Thus we get $\Lambda^{q_n}_{f\phi f^{-1}, r f_*^{-1}} = \Lambda^{q_n}_{\phi,r}$, and

$$\begin{aligned}\lim_{\text{odd } n\to\infty} \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{f\phi f^{-1}, r f_*^{-1}}| &= \lim_{\text{odd } n\to\infty} \frac{1}{n}\log|\mathrm{Trace}\,\Lambda^{q_n}_{\phi,r}| \\ &= \frac{1}{4\pi}\mathrm{vol}_{\mathrm{hyp}}(M_\phi) = \frac{1}{4\pi}\mathrm{vol}_{\mathrm{hyp}}(M_{f\phi f^{-1}}). \qquad \square\end{aligned}$$

From [8], we know Conjecture 2.3 holds for $\phi = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1\end{smallmatrix}\right)$.

Corollary 5.11 *Conjecture 2.3 holds for all $f\phi f^{-1}$ where f is any element in* $\mathrm{GL}(2,\mathbb{Z})$.

Let φ be a periodic map for S, and let g be any diffeomorphism for S. Then $g\varphi g^{-1}$ is also a periodic map. The same discussion as in Lemma 5.10 implies the following conclusion.

Lemma 5.12 *Let φ be any periodic map for S, and let g be any diffeomorphism for S. If Conjecture 5.4 holds for φ, then it also holds for $g\varphi g^{-1}$.*

The following theorem shows Conjecture 5.4 holds for the once punctured torus. This confirms the relation between the intertwiner and the simplicial volume of the corresponding mapping torus.

Theorem 5.13 *Conjecture 5.4 holds for the once punctured torus.*

Proof Let φ be any periodic map for $S_{1,1}$. Then the order of φ could be 1, 2, 3, 4 or 6. According to Theorem 5.6, Conjecture 5.4 holds if the order of φ is 2 or 4.

If the order of φ is 1, then φ is just the identity map. In this case, we can just choose the intertwiner to be the identity operator. Then Conjecture 5.4 holds trivially.

We look at the case when the order of φ is 3 or 6. For these two cases, we have $|\mathrm{Trace}\,\varphi| = 1$. According to [17], we know there exists an element $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $\varphi = g\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1\end{smallmatrix}\right)g^{-1}$ or $\varphi = g\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0\end{smallmatrix}\right)g^{-1}$. By Lemmas 5.8 and 5.12, Conjecture 5.4 holds for these two cases. $\square$

Remark 5.14 From the proof of Theorem 5.6, we know if we can show $\operatorname{Trace}\Lambda^{q_n}_{\varphi,r}\neq 0$ after n is big enough, then we can prove

$$\lim_{\text{odd } n\to\infty}\frac{1}{n}\log|\operatorname{Trace}\Lambda^{q_n}_{\varphi,r_\varphi}|=0.$$

Remark 5.15 From Section 2.4, we know the periodic edge weight system $a=a^{(0)},a^{(1)},\dots,a^{(k)}=a$ for the ideal triangulation sweep $\tau=\tau^{(0)},\tau^{(1)},\dots,\tau^{(k)}=\varphi(\tau)$ and φ-invariant puncture weights h_v can give us the intertwiner $\overline{\Lambda}^q_{\varphi,\bar{\gamma}}$ such that

$$\bar{\rho}\circ\Phi^q_{\tau\varphi(\tau)}\circ\Psi^q_{\varphi(\tau)\tau}(X)=\overline{\Lambda}^q_{\varphi,\bar{\gamma}}\circ\bar{\rho}(X)\circ(\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^{-1}$$

for every $X\in\mathcal{T}^q_\tau$.

It is easy to verify that $(\Phi^q_{\tau\varphi(\tau)}\circ\Psi^q_{\varphi(\tau)\tau})^m=(\Phi^q_{\tau\varphi^m(\tau)}\circ\Psi^q_{\varphi^m(\tau)\tau})$, and

$$a=a^{(0)},a^{(1)},\dots,a^{(k)},\dots,a^{(0)},a^{(1)},\dots,a^{(k)}=a$$

is the periodic edge weight system for the ideal triangulation sweep

$$\tau=\tau^{(0)},\tau^{(1)},\dots,\tau^{(k)}=\varphi(\tau),\varphi(\tau^{(1)}),\dots,\varphi(\tau^{(k)})=\varphi^2(\tau),\dots$$
$$\dots,\varphi^{m-1}(\tau)=\varphi^{m-1}(\tau^{(0)}),\varphi^{m-1}(\tau^{(1)}),\dots,\varphi^{m-1}(\tau^{(k)})=\varphi^m(\tau)$$

and $h_{\varphi^m(v)}=h_v$.

Suppose φ is periodic with order m, then

$$(\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^m\circ\bar{\rho}(X)\circ(\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^{-m}=\bar{\rho}\circ\Phi^q_{\tau\varphi^m(\tau)}\circ\Psi^q_{\varphi^m(\tau)\tau}(X)=\bar{\rho}(X)$$

for every $X\in\mathcal{T}^q_\tau$. Then $(\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^m$ is a scalar matrix since $\bar{\rho}$ is irreducible. Actually we can choose good $\overline{\Lambda}^q_{\varphi,\bar{\gamma}}$ such that $(\overline{\Lambda}^q_{\varphi,\bar{\gamma}})^m$ is the identity matrix. We have all the eigenvalues of $\overline{\Lambda}^q_{\varphi,\bar{\gamma}}$ are m^{th} roots of unity, and $|\operatorname{Trace}\overline{\Lambda}^q_{\varphi,\bar{\gamma}}|=0$ or $|\operatorname{Trace}\overline{\Lambda}^q_{\varphi,\bar{\gamma}}|\geq n^{-m}$.

From [7, Lemma 11], we know the complex dimension of the space of all periodic edge weight systems for the fixed ideal triangulation is more than or equal to 1. Thus this space is connected. In a local open subset of this space, we can choose φ-invariant puncture weights such that these puncture weights smoothly vary according to periodic edge weight systems. Then $|\operatorname{Trace}\overline{\Lambda}^q_{\varphi,\bar{\gamma}}|$ smoothly varies according to periodic edge weight systems in a local open subset by using the similar argument in [6, Complement 10]. Since this space is connected and 0 is an isolated point in the image, we have $|\operatorname{Trace}\overline{\Lambda}^q_{\varphi,\bar{\gamma}}|=0$ for all periodic edge weight systems with the chosen puncture weights, or $|\operatorname{Trace}\overline{\Lambda}^q_{\varphi,\bar{\gamma}}|\geq n^{-m}$ for all periodic edge weight systems with the chosen puncture weights.

If we can find one periodic edge weight system with the chosen puncture weights such that

$$\lim_{\text{odd } n\to\infty}\frac{1}{n}\log|\operatorname{Trace}\overline{\Lambda}^{q_n}_{\varphi,\bar{\gamma}}|=0,$$

we can conclude that the above equation is true for every periodic edge weight system with the chosen puncture weights.

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Received: 9 February 2023 Revised: 15 February 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 4 (pages 1917–2526) 2025