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We look at isometric actions on arbitrary hyperbolic spaces of generalised Baumslag–Solitar groups of any rank (the rank of the free abelian vertex and edge subgroups). It is known that being a hierarchically hyperbolic group is not a quasiisometric invariant. We show that virtually being a hierarchically hyperbolic group is not invariant under quasiisometry either, and nor is property (QT).

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1 Introduction

Gromov's notion of a word hyperbolic group encapsulates what it means for a finitely presented group to be negatively curved. In particular this property is not just a commensurability invariant (that is, if H is a finite index subgroup of G, for which we write $H \leq_f G$, then H has the property if and only if G has the property) but it is also invariant under quasiisometry.

However trying to come up with an equivalent notion of nonpositive curvature for finitely presented groups which is also invariant under quasiisometry seems less clear. One definition that is invariant is that of having (at most) quadratic Dehn function but this contains groups that would not naturally be thought of as being nonpositively curved. To give but one example, de Cornulier and Tessera [13] showed that for each $n \geq 2$ there exists a metabelian group with quadratic Dehn function but which contains the Baumslag–Solitar group $\mathrm{BS}(1,n)$, whereas one might hope that any finitely generated soluble subgroup of a nonpositively curved group is virtually abelian.

Even if invariance under quasiisometry fails, one could at least hope for a definition of nonpositive curvature which is invariant under commensurability. However for the property of being CAT(0) (a group which acts geometrically on a CAT(0) space), which is known not to be invariant under quasiisometry, it is currently open whether H being a CAT(0) group and $H \leq_f G$ implies that G is a CAT(0) group. It is also still open whether all hyperbolic groups are CAT(0).

A more recent property that also aims to encapsulate nonpositive curvature is that of being a hierarchically hyperbolic group (a HHG for short). This does include all hyperbolic groups but it was recently shown by Petyt and Spriano [34] that this is not a commensurability invariant as there are groups which are not HHGs but which have a finite index subgroup equal to $\mathbb{Z}^2$, which is. Note that neither CAT(0) groups or HHGs contain each other.

However there is an easy way to turn a group theoretic property $\mathcal{P}$ into a commensurability invariant provided it is preserved by finite index subgroups (which many properties are, including being CAT(0) and an HHG). We merely alter it to being virtually $\mathcal{P}$, that is it has some finite index subgroup with $\mathcal{P}$. We can thus ask instead: for finitely generated groups, is being virtually $\mathcal{P}$ a quasiisometric invariant? For instance this question has been considered for the properties of being virtually free, virtually cyclic, virtually abelian, virtually nilpotent, virtually polycyclic and virtually soluble. None of these would be commensurability invariants if virtually were removed, but the first four are known to be quasiisometry invariants by various deep results. The fifth is unknown and the sixth is false for finitely generated groups but unknown for finitely presented groups (see de la Harpe [23, IV.B.50]).

Therefore it is natural to ask in this context whether virtually being an HHG is a quasiisometry invariant. Note that virtually being a CAT(0) group is not. This is demonstrated by the well known example of a group G which is a central extension of $\mathbb{Z}$ by a closed surface group S_g for $g \geq 2$ which does not virtually split and so is not a CAT(0) group (see Bridson and Haefliger [9, II.7.26]) but which is quasiisometric to $\mathbb{Z} \times S_g$. This group G has all of its finite index subgroups of the same form, so it is not virtually a CAT(0) group either. We also note here that $\mathbb{Z} \times S_g$ acts geometrically not just on a CAT(0) space but on a CAT(0) cube complex too, thus the property $\mathcal{P}$ of acting geometrically on a CAT(0) cube complex is not a quasiisometric invariant and nor is being virtually $\mathcal{P}$ (note that here virtually $\mathcal{P}$ is not equal to $\mathcal{P}$ as shown by Hagen [21]).

However the group G, and more generally central extensions of $\mathbb{Z}$ by any nonelementary hyperbolic group, was recently shown to be a HHG by Hagen, Russell, Sisto and Spriano [22, Corollary 4.3], so will not provide a counterexample to virtually HHGs being invariant under quasiisometry. Nor will the examples which are not HHGs in [34, Theorem 4.4] work because these are all virtually $\mathbb{Z}^k$ for some k.

In this paper we show in Corollary 5.4 that there exist finitely presented groups with no finite index subgroup being a HHG but which are quasiisometric to an HHG. Thus even adding virtually to the property of being an HHG will not make it a quasiisometric invariant. The particular group used in Corollary 5.4 is the Leary–Minasyan group first appearing in [28]. A Leary–Minasyan group denotes any group formed by starting with $A = \mathbb{Z}^n$ for $n \geq 1$ which is regarded as sitting naturally in $\mathbb{Q}^n$ and then taking a matrix $M : \mathbb{Q}^n \to \mathbb{Q}^n$ which is in $\mathrm{GL}(n, \mathbb{Q})$ and a finite index subgroup B of $A \cap M^{-1}(A)$. The group is then defined to be the HNN extension with base A and associated subgroups B and $M(B)$ using the matrix M. We refer to THE Leary–Minasyan group L as the case where $n = 2$,

$$M = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

and $B = A \cap M^{-1}(A)$ which is a CAT(0) group. It was shown in [28, Theorem 1.1] that a Leary–Minasyan group is biautomatic if and only if it is virtually biautomatic which occurs if and only if the matrix M has finite order. (It is unknown whether there exist virtually biautomatic groups which are not biautomatic.) In particular the group L is CAT(0) but not (virtually) biautomatic: this was the first known example. As for whether every HHG is biautomatic, this was answered negatively by Hughes and Valiunas [25]. In their introduction, it is asked whether any nonbiautomatic Leary–Minasyan group is an HHG, with the answer expected to be no. In this paper we confirm this in Corollary 5.2 by showing that a Leary–Minasyan group is virtually an HHG if and only if the matrix M has finite order.

It turns out that our arguments work in a somewhat wider class of groups than Leary–Minasyan groups, that of generalised Baumslag–Solitar groups of arbitrary rank. A generalised Baumslag–Solitar group of rank n, or GBS_n group for short, is a finite graph of groups where all vertex and edge groups are isomorphic to $\mathbb{Z}^n$ (sometimes the term generalised Baumslag–Solitar group refers only to the case $n = 1$ but we will use the term in its wider context). A GBS_n group G gives rise to a homomorphism from G to $\mathrm{GL}(n, \mathbb{Q})$, defined up to conjugacy, which we call the modular homomorphism of G. (Strictly speaking this should be the modular homomorphism of the decomposition of G as a GBS_n group but in all but some basic cases this homomorphism is unique, even if the decomposition is not.) In the case of a Leary–Minasyan group the graph of groups has one vertex and one edge with the modular homomorphism sending the stable letter to the matrix M and all of the vertex subgroup $\mathbb{Z}^n$ to the identity.

We begin in Section 2 with some introductory material on how groups act on hyperbolic spaces. In Section 3 we introduce generalised Baumslag–Solitar groups G of arbitrary rank n and define the aforementioned modular homomorphism. We use this to examine the free abelianisation of G, that is the usual abelianisation G/G' but with the torsion of G/G' quotiented out. This gives us a dichotomy between groups with finite and infinite monodromy, where the monodromy is the image of G in $\mathrm{GL}(n, \mathbb{Q})$ under the modular homomorphism. Indeed it is shown in Theorem 3.4 that a GBS_n group with finite monodromy is virtually equal to $\mathbb{Z}^n \times F_r$ for some finite rank free group F_r (and hence in this case G is virtually a HHG), whereas a GBS_n group with infinite monodromy cannot be of this form. (As a consequence, whether a GBS_n group has finite or infinite monodromy depends in all cases only on the group and not the decomposition.)

In Section 3.3 we use this information to examine the possible actions by isometries of a GBS_n group G on an arbitrary hyperbolic space. Whilst a group of the form $\mathbb{Z}^n \times F_r$ will have many different actions on hyperbolic spaces, we show in Theorem 3.5 that if G has infinite monodromy then there is a nontrivial element of the base $\mathbb{Z}^n$-subgroup which never acts loxodromically in any action of G on a hyperbolic space.

As for particular actions on hyperbolic spaces, the concept of an acylindrical action of a group G on an arbitrary hyperbolic space X was introduced by Bowditch [8]. However if X is bounded then any action by isometries is acylindrical, so we require acylindrical actions of G with unbounded orbits in order to obtain anything useful. This was developed by Osin [33] to obtain the much studied concept of an acylindrically hyperbolic group G, in which G has an acylindrical action on some hyperbolic space that is nonelementary, which is equivalent to saying that the action has unbounded orbits and G is not virtually cyclic. In the theory of HHGs, such a group comes equipped with various hyperbolic spaces, including one on which G acts by isometries called the maximal domain S. It was shown by Behrstock, Hagen and

Sisto [4, Theorem 14.3] that if G is an HHG then the resulting action of G on S is acylindrical. However it is perfectly possible that S is a bounded metric space even if G is an infinite HHG (for instance for groups acting geometrically on the product of two trees).

Now it can be seen using standard results that GBS_n groups are never acylindrically hyperbolic but this does not allow us to conclude that they are not HHGs as well, because it could be that in some potential HHG structure on such a group, the maximal domain S is bounded. However this can be avoided by appealing to recent results by Petyt and Spriano [34] which show that if G is any HHG then there is a finite collection of unbounded domains, invariant under the action of G and which are pairwise orthogonal, such that any unbounded domain in the HHG structure is nested in one of these. If G is an infinite group then this collection will be nonempty. Moreover the action of G on each of these domains can be combined to give an action of G on their product and this action is also acylindrical (as well as containing a loxodromic element). Of course a product of at least two unbounded hyperbolic spaces will not itself be hyperbolic but the definition of an acylindrical action makes sense for an arbitrary metric space. Whilst we do not know of any group theoretic consequences if there exists an acylindrical action with a loxodromic element on a product P of hyperbolic spaces, in Section 4 we look at such actions which also preserve the factors of P, which we call a product acylindrical action. We show in Theorem 4.3 that every element in such an action must either be elliptic or loxodromic, as was shown by Bowditch [8, Lemma 2.2] to be the case for a single hyperbolic space.

This is then used to obtain Theorem 4.4 which states that a GBS_n group with infinite monodromy has no product acylindrical action. In Section 5 we give some background on the properties of HHGs that we will need, as opposed to giving the full definition. We then conclude in Corollary 5.2 that a GBS_n group is virtually an HHG if and only if it has finite monodromy, by virtue of the (non)existence of some product acylindrical action. However having finite monodromy is not a quasiisometric invariant, as the Leary–Minasyan group shows.

Bestvina, Bromberg and Fujiwara [6] introduced the property (QT), which applies to finitely generated groups. Such a group has (QT) if it acts isometrically on a finite product of quasitrees such that the orbit map is a quasiisometric embedding. They show there that many groups do have (QT). We show in Corollary 6.4, once again using the Leary–Minasyan group L, that (QT) is not preserved under quasiisometry. For this we do not need knowledge of possible product acylindrical actions of generalised Baumslag–Solitar groups G. We only need to know which elements will be elliptic in all actions of G on a hyperbolic space.

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2 Actions on hyperbolic spaces

In this paper we consider groups G acting by isometries on a hyperbolic space (or later a finite product of hyperbolic spaces). Here a hyperbolic space X will always mean that X is a geodesic metric space satisfying any of the equivalent definitions of δ-hyperbolicity. Note that no further conditions such as properness of the space will be assumed. As X is hyperbolic we can look at the action of G by homeomorphisms on the (Gromov) boundary ∂X of X to obtain the limit set $\partial_G X$ which is a subset of ∂X. This subset is G-invariant and we have $\partial_H X \subseteq \partial_G X$ if H is a subgroup of G.

We have the standard classification of the individual elements $g \in G$ as follows:

Definition 2.1 (i) The element g is *elliptic* under the given action if the subgroup $\langle g\rangle$ has bounded orbits. This happens if and only if $\partial_{\langle g\rangle} X = \varnothing$.

(ii) The element g is *loxodromic* under the given action if the subgroup $\langle g\rangle$ embeds quasiisometrically in X under the (or an) orbit map, namely there is $c > 0$ (which exists independently of $x \in X$) such that for all $n \in \mathbb{Z}$ we have $d(g^n(x), x) \geq |n|c$. This occurs if and only if $\partial_{\langle g\rangle} X$ consists of exactly 2 points $\{g^{\pm}\}$ for any $x \in X$ and this is the fixed point set of g on ∂X.

(iii) The element g is *parabolic* exactly when it is not elliptic or loxodromic, which occurs if and only if $\partial_{\langle g\rangle} X$ consists of exactly 1 point and again this is the fixed point set of g on ∂X.

Note that an element $g \in G$ is elliptic/loxodromic/parabolic if and only if g^n is (for some/all $n \in \mathbb{Z} \setminus \{0\}$) and if and only if some conjugate of g is.

Moving back now to arbitrary groups, if G acts by isometries on an arbitrary hyperbolic space X then we have the Gromov classification dividing possible actions into five very different classes (for these facts and related references, see [1; 15]):

(1) The action has *bounded orbits*.

(2) The action is *parabolic* (or horocyclic), meaning that $\partial_G X$ has exactly one point p. In this case the action can never be cobounded.

(3) The action is *lineal*, meaning that $\partial_G X = \{p, q\}$ has exactly 2 points (which in general can be swapped or fixed pointwise by the action of G). In this case there will exist some loxodromic element in G with limit set $\{p, q\}$.

(4) The action is *quasiparabolic* (or focal). This says that the limit set has at least 3 points, so is infinite, but there is some point $p \in \partial_G X$ which is globally fixed by G. This implies that G contains a pair of loxodromic elements with limit sets $\{p, q\}$ and $\{p, r\}$ for p, q, r distinct points.

(5) The action is *general*: the limit set is infinite and we have two loxodromic elements with disjoint limit sets.

We will be interested in the question: given a specific group G and a particular element $g \in G$, can we find an action of G on some hyperbolic space X where g acts loxodromically? If we first consider obstructions, the most obvious is if g has finite order or if G is finitely generated and $g \in G$ is distorted in G. As for sufficient conditions, if we have a homomorphism $\theta\colon G \to \mathbb{R}$ with $\theta(g) \neq 0$ then we can realise θ as an action of G on $\mathbb{R}$ by translations in which g acts loxodromically. Indeed this also works for a homomorphism $\theta\colon G \to \mathrm{Isom}(\mathbb{R}) \cong \mathbb{R} \rtimes C_2$ if $\theta(g)$ has infinite order, as here the elements swapping the ends of $\mathbb{R}$ will have order 2. However, much more generally, this also works by [1, Proposition 4.9] if we have a homogeneous quasimorphism $q\colon G \to \mathbb{R}$ with $q(g) \neq 0$, whereupon X is a quasiline.

We also have a type of converse which we will use later: given an action of a locally compact group G on a hyperbolic space X where there is a point $p \in \partial X$ on the boundary fixed by all of G, there exist *Busemann functions* on G. Here all groups with such an action that we consider are countable and discrete, hence locally compact. In particular, see [10] where it is shown that there exists a homogeneous quasimorphism $q\colon G \to \mathbb{R}$, the Busemann quasicharacter, where the quasikernel $\{g \in G \mid q(g) = 0\}$ consists exactly of those elements of G which are not acting loxodromically on X. In particular, although this results in q being trivial in actions of the first or second type, in the third (assuming the two limit points are fixed pointwise) or the fourth types of actions we do obtain a nontrivial homogeneous quasimorphism on G. Moreover [10] also shows that this Busemann quasicharacter is a genuine homomorphism if either G is amenable or X is proper. Indeed the first point follows from the well known fact that the only homogeneous quasimorphisms on an amenable group are the homomorphisms.

3 Generalised rank n Baumslag–Solitar groups

Let G be the fundamental group of a finite graph of groups where all edge and vertex groups are isomorphic to $\mathbb{Z}$. The classical Baumslag–Solitar groups are obtained when the underlying graph has a single vertex and one loop. For an arbitrary finite graph, these groups go back at least to [27] and at some later point they became known as generalised Baumslag–Solitar groups. However one can replace $\mathbb{Z}$ with $\mathbb{Z}^n$ for a fixed integer $n \geq 1$, which are also sometimes known as generalised Baumslag–Solitar groups. Therefore for clarity we introduce the following definition.

Definition 3.1 A *generalised rank n Baumslag–Solitar group* (GBS_n group) G is the fundamental group of a finite graph Γ of groups where all vertex and edge groups are isomorphic to $\mathbb{Z}^n$.

This notation occurs in [3], where also the term vGBS group is used (with v standing for variable) for when the rank of the various vertex and edge groups $\mathbb{Z}^k$ need not be constant. However vGBS groups need not enjoy the finite valence property mentioned below and so we will not consider those groups further.

3.1 The modular homomorphism

We can obtain a finite presentation for a GBS_n group G by taking each vertex $v_i \in \Gamma$ and corresponding vertex group V_i and fixing a free basis $a_{i,1}, \dots, a_{i,n}$ for V_i. We take a maximal tree T_0 in the finite graph

Γ defining G and form the amalgamation of the V_i over the finite index edge subgroups. We then add a stable letter t_j to the generators for each of the r (say) edges e_j of $\Gamma \setminus \{T_0\}$ and the corresponding relations where t_j conjugates the inclusion of the edge group E_j at one end of e_j to the inclusion of E_j at the other end. Note this means that $G = N \rtimes F_r$, where N is given by the normal closure of the vertex groups V_i in G and $F_r = \langle t_1, \dots, t_r \rangle$.

For $n = 1$ a modular homomorphism from G to $\mathbb{Q}^\times$ was considered in [29]. This was generalised to arbitrary n in [14], where the homomorphism is now from G to $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$. We present our own more basic version here for arbitrary n which has the advantage that it works for the finite index subgroups of G as well. It is equivalent to the modular representation in [12, Section 3] but is a more concrete version, making it easier to calculate in specific examples.

As a GBS_n group is formed from a graph of groups, it acts (by automorphisms without inverting edges) coboundedly on a simplicial tree T. The crucial property here is that this tree has finite valence because a subgroup of $\mathbb{Z}^n$ which is itself isomorphic to $\mathbb{Z}^n$ must be a finite index subgroup. Now given any GBS_n group G, we first make arbitrary choices of a base vertex v_0 in the tree T on which G acts, a finite index subgroup $A \cong \mathbb{Z}^n$ of the stabiliser $\mathrm{Stab}_G(v_0) \cong \mathbb{Z}^n$ and an ordered basis $a_1, \dots, a_n$ of A. We refer to A as our *base* $\mathbb{Z}^n$*-subgroup* for G. But given any element $g \in G$, we have that A and $g^{-1}Ag$ are commensurable subgroups of G, that is their intersection has finite index in both. This is because $g^{-1}Ag \leq_f \mathrm{Stab}_G(g^{-1}(v_0)) = g^{-1}\,\mathrm{Stab}(v_0)g$ and the finite valence of the tree T implies that any two vertex stabilisers are commensurable so this holds for $\mathrm{Stab}_G(v_0)$ and $g^{-1}\,\mathrm{Stab}(v_0)g$, hence also for A and $g^{-1}Ag$.

Our modular homomorphism $\mathcal{M}$ which we now define will be a homomorphism from G to $\mathrm{GL}(n, \mathbb{Q})$ and it is easily checked that changing any of these choices results in a homomorphism that is conjugate in $\mathrm{GL}(n, \mathbb{Q})$ to $\mathcal{M}$. We first note that for any finite index subgroup H of $\mathbb{Z}^n$, there is $m \in \mathbb{N}$ (depending on H) such that $m\mathbb{Z}^n \leq_f H \leq_f \mathbb{Z}^n$ (here using additive notation).

Definition 3.2 Let G be a GBS_n group for $n \geq 1$ with some base subgroup $A \cong \mathbb{Z}^n$. We define the *modular homomorphism* $\mathcal{M}\colon G \to \mathrm{GL}(n, \mathbb{Q})$ in the following way. For any $g \in G$, we have just remarked that (taking $\mathbb{Z}^n$ to be our base subgroup A and H to be $A \cap g^{-1}Ag$) there is $m > 0$ such that

$$mA = \langle a_1^m, \dots, a_n^m \rangle \leq_f A \cap g^{-1}Ag \leq_f A.$$

In particular for any $a = a_1^{l_1} \cdots a_n^{l_n} \in A$ with $l_1, \dots, l_n \in \mathbb{Z}$, we have that $a^m \in g^{-1}Ag$ and so the element $ga^m g^{-1}$ of G is actually in A. This means that on taking a to be each of our basis elements $a_1, \dots, a_n$ in turn, we have uniquely defined integer coefficients g_{ij} with

$$ga_j^m g^{-1} = a_1^{g_{1j}} \cdots a_n^{g_{nj}}$$

and our definition of the modular map $\mathcal{M}$ is that it sends g to the matrix whose ij^{th} entry is $g_{ij}/m \in \mathbb{Q}$.

The *monodromy* of a GBS_n group G is the image $\mathcal{M}(G)$.

Note that this definition of $\mathcal{M}(g)$ is independent of the value of m taken as if we replace m with m' for any appropriate $m' > 0$ then $(ga^m g^{-1})^{m'}$ is equal to $(ga^{m'} g^{-1})^m$. Moreover for any $g, h \in G$ we have $\mathcal{M}(g)\mathcal{M}(h) = \mathcal{M}(gh)$ so that $\mathcal{M}$ maps to $\mathrm{GL}(n, \mathbb{Q})$ and is a homomorphism. Clearly the subgroup A is in the kernel of $\mathcal{M}$. We also note here that any element g of G acting elliptically on the tree T is sent to the identity by $\mathcal{M}$ because g lies in some vertex stabiliser $\mathrm{Stab}(v) \cong \mathbb{Z}^n$ and by finite valence of T there will be a finite index subgroup $B \leq_f A$ with $B \leq \mathrm{Stab}(v)$, whereupon $gbg^{-1} = b$ for all $b \in B$. Thus $\mathcal{M}$ factors through the decomposition of G into $N \rtimes F_r$ above and so can also been thought of as a homomorphism from F_r to $\mathrm{GL}(n, \mathbb{Q})$. In particular if $r = 0$ so that the underlying graph Γ is actually a finite tree then $\mathcal{M}$ is the trivial homomorphism. This can also happen for $r > 0$, for instance the HNN extension $\langle a, b, t \mid [a, b],\ tat^{-1} = a,\ tbt^{-1} = b \rangle$ when $n = 2$.

Given our modular homomorphism $\mathcal{M}\colon G \to \mathrm{GL}(n, \mathbb{Q})$, we can restrict $\mathcal{M}$ to a subgroup H of G. If $H \leq_f G$ then H is also a GBS_n group because the restriction of the action of G on the tree T to H is also cobounded, with edge and vertex subgroups which are finite index subgroups of $\mathbb{Z}^n$, hence are all isomorphic to $\mathbb{Z}^n$ too. This description of H gives rise to its own modular homomorphism $\mathcal{M}_H$ but we can regard this, up to conjugacy, as the restriction of the modular homomorphism $\mathcal{M}$ for G. This is because as G and H are acting on the same tree T, we can first take the same base vertex v_0 in T. Then since $H \leq_f G$, we have that $H \cap \mathrm{Stab}_G(v_0) \leq_f \mathrm{Stab}_G(v_0)$. Thus we can choose our subgroup A to be $H \cap \mathrm{Stab}_G(v_0)$ both when defining $\mathcal{M}$ and $\mathcal{M}_H$ and we also choose the same ordered basis for A in both cases. We are now in the position that the definition of $\mathcal{M}_H$ is exactly the definition of $\mathcal{M}$ but just for elements $h \in H$.

3.2 The free abelianisation

If G is a finitely generated group and G' is its commutator subgroup then G/G' is the abelianisation of G. It is a finitely generated abelian group and so is of the form $\mathbb{Z}^k \oplus T$, where the torsion subgroup T is finite. Moreover every abelian quotient of G factors through G/G'.

Here we will consider the *free abelianisation* $\overline{G}$ where we further quotient out by the torsion in the abelianisation to obtain $\overline{G} = \mathbb{Z}^k$ for some k. This has the corresponding universal property that any homomorphism from G to a torsion free abelian group factors through $\overline{G}$.

If we are given a finite presentation for G with m generators then it is easy to calculate G/G' and $\overline{G}$ by abelianising these relations and considering them as defining a subgroup S of $\mathbb{Z}^m$ so that G/G' is the quotient abelian group $\mathbb{Z}^m/S$. In fact the process is even easier for $\overline{G}$ because of the lack of torsion: we can work over $\mathbb{Q}$ to get that the rank k of $\overline{G}$ is the dimension of the quotient space $\mathbb{Q}^m/R$, where $\mathbb{Q}^m$ is the vector space spanned by the given generators for G and R is the subspace spanned by the relators for G, once these relators have been abelianised and regarded as elements of $\mathbb{Q}^m$. This also says that an element $g \in G$ has infinite order (equivalently is nontrivial) in $\overline{G}$ if on expressing g as a word in the generators and abelianising this word, the corresponding $\mathbb{Q}^m$-vector is not in the subspace R.

This process works out especially well for a GBS_n group G. As before, we take our base vertex v_0, finite index subgroup A of $\mathrm{Stab}_G(v_0)$ and basis $a_1,\dots,a_n$ for A. On considering the $\mathbb{Q}$-vector space W of dimension n spanned by this basis, let us consider how this relates to forming the group G as the fundamental group of a finite graph of $\mathbb{Z}^n$ groups and how it also relates to $\overline{G}$. Each time we introduce a new vertex group V_i and form its amalgamation with the previous vertex groups over the appropriate edge group, we are giving an identification of the $\mathbb{Q}$-vector space spanned by V_i with our original vector space W. Thus if there are no stable letters then $\overline{G}=\mathbb{Z}^n$. However on taking one of the stable letters t_j with its edge running from the vertex v_j to the vertex $v_{j'}$, this introduces n new relations in G of the form

$$t_j x_1^{l_1}\cdots x_n^{l_n} t_j^{-1} = y_1^{m_1}\cdots y_n^{m_n},$$

where $x_1,\dots,x_n$ is a basis for the vertex group V_j and $y_1,\dots,y_n$ a basis for $V_{j'}$. Thus in $\overline{G}$ we obtain the abelianised relation $l_1x_1+\cdots+l_nx_n=m_1y_1+\cdots+m_ny_n$, so that the corresponding relator can be expressed using our identifications above as an element of W which is trivial in $\overline{G}$. Thus the span of these rn relators forms a subspace R of $W\cong\mathbb{Q}^n$ and our free abelianisation $\overline{G}$ can be described over $\mathbb{Q}$ as the quotient vector space $\mathbb{Q}^r\oplus(W/R)$, where the first summand comes from the stable letters. Note that we can again work out easily whether an element g of G is nontrivial in $\overline{G}$ by writing g in terms of the generators obtained from the graph of groups decomposition and abelianising. Indeed g will be trivial in $\overline{G}$ if and only if each stable letter appears in g with exponent sum 0 and such that the resulting abelianisation of the word representing g, which will now lie in W, also lies in R. Thus in particular we have from this discussion:

Theorem 3.3 *Suppose that G is a GBS_n group with base $\mathbb{Z}^n$-subgroup A. Then A embeds in the free abelianisation $\overline{G}$ if and only if $R=\{0\}$. Moreover this happens if and only if the monodromy $\mathcal{M}(G)$ is trivial because the modular homomorphism is defined by what it does on the stable letters.*

Note: from this, we see that the set of elements in A which are trivial (equivalently have finite order) in the free abelianisation of G form a subgroup of A. This is because these are the elements of A which, when considered as elements of $\mathbb{Q}^n$, lie in the subspace R of W.

We can now present a dichotomy in the behaviour of GBS_n groups (which for $n=1$ is shown in [29, Proposition 2.6] and for general n was stated in [12, Section 3]).

Theorem 3.4 *If G is any GBS_n group with finite monodromy then G is virtually $\mathbb{Z}^n\times F_r$ for some $r\geq 0$.*

Proof First drop down to a finite index subgroup H of G where H has trivial monodromy, for instance we could take H to be the kernel of the modular homomorphism $\mathcal{M}$. As H is the fundamental group of a finite graph of groups where all of the finitely many edge and vertex groups are commensurable, we can intersect them to get a subgroup B of H which can be used as a base $\mathbb{Z}^n$-subgroup H. Now let us consider the presentation we obtain for H from this graph of groups decomposition. Our finite generating set consists of generators g_i of the vertex groups along with the stable letters t_j. Now any element $b \in B$ will also lie in any vertex group and so will commute with every element g_i. On taking a stable letter t which is obtained from the edge joining the vertices v_1 and v_2 (possibly the same vertex) with vertex groups V_1, V_2 say and respective edge inclusions $E_1 \leq V_1$ and $E_2 \leq V_2$, we have that $tE_1t^{-1} = E_2$.

But as the monodromy is trivial, there is $M > 0$ (depending on $b \in B$) such that $b^M \in B \cap t^{-1}Bt$ and $tb^Mt^{-1} = b^M$. However B lies in every edge and vertex group, so that $b \in E_1$ and hence $tbt^{-1} \in E_2 \cong \mathbb{Z}^n$. Thus tbt^{-1} must be an element in E_2 such that $(tbt^{-1})^M = b^M \in E_2$. Clearly the element b has this property as $B \leq E_2$ as well. Moreover M^{th} roots are unique in $\mathbb{Z}^n$, thus $tbt^{-1} = b$.

Hence we conclude that B is normal and indeed central in H. As it lies in (and is normal in) every vertex and edge group, we can consider H/B. This group itself admits a graph of groups decomposition with the same underlying finite graph, but with vertex groups V_i/B and edge groups E_j/B. These are all finite groups so H/B is virtually free. Hence we can pull back a finite index free subgroup of H/B to obtain a subgroup $L \leq_f H$ with $L/B \cong F_r$. But as this quotient is free the extension splits, so there is a copy of F_r in L with $L = B \rtimes F_r$. As B is central this is simply a direct product, so that G has the finite index subgroup $L \cong \mathbb{Z}^n \times F_r$. □

3.3 Actions of GBS$_n$ groups on arbitrary hyperbolic spaces

We can now use the above to see how a given GBS$_n$ group G and its finite index subgroups H act on hyperbolic spaces. Certainly we have the action of G on its Bass–Serre tree where all elements of the base $\mathbb{Z}^n$-subgroup A act elliptically. However the point is that if some element a of A is loxodromic when G acts on a hyperbolic space, the fact that the base subgroup A is commensurated in all of G means that the action must be very restricted.

Theorem 3.5 *Suppose the monodromy of a* GBS$_n$ *group G with given base $\mathbb{Z}^n$-subgroup A has infinite order. Then there is a nontrivial element $z \in A$ such that for any isometric action of G on a hyperbolic space, the element z does not act loxodromically.*

Proof Consider any isometric action of G on some hyperbolic space X and suppose that there is some element $a \in A$ which is acting loxodromically. We have the two limit points $p^\pm \in \partial X$ for the action of $\langle a \rangle$ and these are the only points in ∂X fixed by any nontrivial power of a. Now for any $g \in G$, we have that A and $g^{-1}Ag$ are commensurable subgroups. As $a \in A$, we can find $j > 0$ such that a^j is in both of these subgroups. In particular $ga^jg^{-1} \in A$ and it acts loxodromically on X because it is a conjugate in G of the element a^j. But as A is abelian, this means ga^jg^{-1} sends the fixed point set of a^j to itself and so swaps or fixes the two points p^+ and p^-. However the case where p^+ and p^- are swapped cannot occur: if so then the loxodromic element ga^jg^{-1} would have two other fixed points q^+, q^- say on ∂X. But then the square of ga^jg^{-1}, which is also loxodromic, would fix at least four distinct points $p^\pm, q^\pm$ on ∂X, which loxodromic elements cannot do. Thus for every $g \in G$ we have that a^j and ga^jg^{-1} both have the two fixed points p^+ and p^-. But the latter element actually has fixed points $g(p^+)$ and $g(p^-)$ in the action of G on ∂X, so $\{p^+, p^-\}$ is preserved by every element of G and hence the action must be lineal of type (3). Here we can have elements of G which swap the two points, but if so then we can avoid this by dropping down to a subgroup of index 2.

Therefore let G_0 be the index-1 or index-2 subgroup of G where the action preserves p^+ and p^- pointwise. We can now take the Busemann quasicharacter $q\colon G_0 \to \mathbb{R}$ (at p^+ say) which is a homogeneous quasimorphism on G_0 that restricts to a genuine homomorphism $\theta = q|_{G_0 \cap A}$ on $G_0 \cap A$ (which is amenable). Note that θ is nontrivial because $a \in G_0 \cap A$ is acting loxodromically.

We will now show that we can "lift" θ to G_0, in that there is some homomorphism $\Theta\colon G_0 \to \mathbb{R}$ which restricts to θ on $G_0 \cap A$. As G is a GBS_n group with a graph of groups decomposition giving us a presentation for G, we can do the same for G_0 by restricting the action of G on the Bass–Serre tree T to G_0 and then taking a quotient. We already have Θ defined on $G_0 \cap A$, which is a finite index subgroup of the vertex group at the base vertex in the finite graph $\Gamma = G_0 \backslash T$ and we can extend Θ to the group generated by all vertex groups but without the relations from the stable letters. This is because all other defining relations are each given by an isomorphism between finite index subgroups of vertex groups, so we proceed inductively by extending Θ to the image of the edge group in the new vertex group, then we can extend over this vertex group.

However we must also consider the defining relations for G_0 coming from the stable letters. These are all of the form

$$t x_1^{l_1} \cdots x_n^{l_n} t^{-1} = y_1^{m_1} \cdots y_n^{m_n},$$

where t is one of these stable letters. We allow Θ to send the stable letters anywhere but $h_1 := x_1^{l_1} \cdots x_n^{l_n}$ is an element in the vertex group at one end of the edge defining t and $h_2 := y_1^{m_1} \cdots y_n^{m_n}$ is in the vertex group at the other end. We must now show that Θ is still well defined when these relations of the form $t h_1 t^{-1} = h_2$ are added. Now h_1 and h_2 both lie in vertex stabilisers so by commensurability there will be $M > 0$ such that h_1^M and h_2^M are both in our base $\mathbb{Z}^n$-subgroup $G_0 \cap A$. But t, h_1, h_2 all lie in G_0 which is the domain of the homogeneous quasimorphism q, thus we have $q(t h_1^M t^{-1}) = q(h_2^M)$. Now homogeneous quasimorphisms are invariant under conjugation (because $|q(yxy^{-1}) - q(x)|$ is bounded independently of x, so we can replace x with x^m and let m tend to infinity), thus this becomes $q(h_1^M) = q(h_2^M)$. But $q(h_i^M) = \theta(h_i^M)$ for $i = 1, 2$ as h_1^M, h_2^M are in the domain of θ and we have $\Theta(h_i) = (1/M)\Theta(h_i^M)$ by definition of Θ above. Also $\Theta(t h_1 t^{-1}) = \Theta(h_1)$ as we are mapping to $\mathbb{R}$, so this is also equal to

$$(1/M)\Theta(h_1^M) = (1/M)\theta(h_1^M) = (1/M)\theta(h_2^M) = (1/M)\Theta(h_2^M) = \Theta(h_2)$$

so Θ is indeed well defined.

We can now finish the proof. The subgroup G_0 obtained above has index 1 or 2 in G and there are only finitely many index-2 subgroups in G as it is finitely generated. Let G_2 be the intersection of all of these index-2 subgroups, which will have finite index in G. As G has infinite monodromy, the same is true for G_2 where we can take our base $\mathbb{Z}^n$-subgroup to be $G_2 \cap A$. Thus by Theorem 3.3 applied to G_2 with base $\mathbb{Z}^n$-subgroup $G_2 \cap A$, there is some nonidentity $z \in G_2 \cap A$ which is trivial in the free abelianisation of G_2. Let us now suppose that G does have an action on some hyperbolic space X in which z is loxodromic, so we can run through the above argument with z equal to a. But then we will obtain a homomorphism Θ from some index-2 subgroup of G to $\mathbb{R}$ with $\Theta(z) \neq 0$ and this subgroup will contain G_2. Thus we can restrict Θ to G_2 which is a contradiction because z is trivial in the free abelianisation of G_2. □

Note that this theorem can be used to complement Theorem 3.4 in that if a GBS_n group G has infinite monodromy then it can have no finite index subgroup H which is isomorphic to a direct product of a free group and copies of $\mathbb{Z}$. This is because H would have infinite monodromy as well, thus by Theorem 3.5 there would be a nontrivial element of H which cannot be loxodromic in any action of H on a hyperbolic space. But if H has the above form then we can use actions on trees to make any given element loxodromic.

This at least deals with a potential ambiguity which we have previously glossed over: that G could have different decompositions as a generalised Baumslag–Solitar group. For instance $\mathbb{Z}^{n+1}$ is both a GBS_{n+1} group in a trivial way where the underlying graph is a single vertex and also a GBS_n group $\mathbb{Z}^n \times \mathbb{Z}$ with graph a vertex and an edge, where the stable letter acts trivially by conjugation and where we have the freedom to take any primitive element as the stable letter. Here the monodromy is trivial in all cases, but if we take the fundamental group of the Nil torus bundle

$$\langle t, a, b \mid [a,b] = e,\ tat^{-1} = ba,\ tbt^{-1} = b \rangle,$$

which is of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$, we see that we have a decomposition with base group $\langle a, b\rangle$ and stable letter t, or base group $\langle t, b\rangle$ and stable letter a. The respective modular homomorphisms of these two decompositions are different but both have infinite monodromy.

The above tells us that whether the monodromy is finite or infinite (which is our main concern in this paper) is an invariant only of the group G, not of how it decomposes as a generalised Baumslag–Solitar group. In fact in nearly all cases the modular homomorphism itself and thus the monodromy (up to conjugation in $\mathrm{GL}(n, \mathbb{Q})$) is well defined. We state this as:

Proposition 3.6 *Take any two decompositions of the same group G as a generalised Baumslag–Solitar group. If neither of the two actions on the corresponding Bass–Serre trees have an invariant line or point, the modular homomorphisms corresponding to the two decompositions are the same.*

Proof Here we adapt the result in [20, Corollary 6.10] which achieves this for $n = 1$. Suppose that a group G has two decompositions as a generalised Baumslag–Solitar group with the first of rank n, say. By considering cohomological dimension, the second is also of rank n unless one decomposition is trivial (in which case G is abelian and so the modular homomorphism obtained from any possible decomposition will be trivial too). Hence we have two actions of G on trees T_1 and T_2, say, and hence two partitions $\{E_1, H_1\}$ and $\{E_2, H_2\}$ of G into elliptic and hyperbolic elements according to these actions. But if these partitions are the same then the resulting modular homomorphisms are the same. This is because elliptic elements are sent to the identity and the image of a hyperbolic element g is determined by how it conjugates elements in any base $\mathbb{Z}^n$-subgroup. But under any action of G on a tree (by automorphisms without inverting edges), a finitely generated purely elliptic subgroup will fix some vertex, so we can use the same base $\mathbb{Z}^n$-subgroup in either decomposition.

Thus we are done if we have a characterisation of the hyperbolic and elliptic elements of G which does not depend on the particular action. In [20] this is achieved (in all but these exceptional cases) for $n = 1$ by showing that the elliptics are the elements of G which are commensurable with all their conjugates. Here we can instead argue: suppose that $g \in G$ lies in $E_2 \setminus E_1$ so that it fixes a vertex v_2 when G acts on T_2 but which is hyperbolic when G acts on T_1. The action of G on T_2 gives rise to a generalised Baumslag–Solitar decomposition of G where we can take the base group to be $\mathrm{Stab}_G(v_2)$. Thus on taking g equal to a in the proof of Theorem 3.5 where we use the action of G on T_1 with g loxodromic, we conclude by the same argument that G fixes setwise the axis of g. This means that this GBS_n decomposition of G gives rise to an invariant line when G acts on the Bass–Serre tree which is ruled out by hypothesis. (In fact a brief check reveals that in the exceptional case, G can only be equal to $\mathbb{Z}^n$, $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ for α some automorphism of $\mathbb{Z}^n$, or $\mathbb{Z}^n *_C \mathbb{Z}^n$, where C has index 2 in both copies of $\mathbb{Z}^n$.) Thus we see that g was not loxodromic and so $E_2 \subseteq E_1$. We can now swap the actions and argue again to conclude that $E_1 = E_2$ and $H_1 = H_2$. □

We finish this section by noting though that even though the modular homomorphism is well defined (away from these exceptional cases), a group might have many decompositions as a generalised Baumslag–Solitar group which are not obviously related. For instance the isomorphism problem is open just amongst GBS_1 groups (see [11] for some recent progress on this question). Moreover [7] exhibits GBS_4 groups with unsolvable conjugacy problem.

4 Acylindrical actions on products of hyperbolic spaces

4.1 Product acylindrical actions

Given a group G acting by isometries on a metric space X, a well known definition is that of G acting *acylindrically*: that is, given any $\epsilon \geq 0$ we have N, R such that if $x, y \in X$ are two points which are at least distance R apart then the set of group elements moving both x and y by at most ϵ has cardinality at most N. This definition is generally used when X is a hyperbolic metric space whereupon it gives rise to the concept of a group being *acylindrically hyperbolic*. This is where there exists an acylindrical action which is of type (5) in the list on page 2257 (in fact actions of types (2) and (4) can never be acylindrical on hyperbolic spaces) and which implies a number of consequences, for instance such a group must be SQ-universal.

Observe that the definition of an acylindrical action makes sense for any group action by isometries on any metric space. However such a concept is not useful in this generality. First of all if X is bounded then any action is trivially acylindrical, so any suitable notion needs to avoid this case. (Indeed even if X is unbounded then an acylindrical action can still have bounded orbits, but not all actions with bounded orbits need be acylindrical.) But even this is problematic because any geometric action on any metric space is uniformly metrically proper, which in turn implies that the action is acylindrical. Therefore any finitely generated group acts acylindrically on its own Cayley graph and consequently there is no chance that an acylindrical action automatically implies any group theoretic consequences, even if we could agree on what a suitable acylindrical action meant in this context.

But one option is to restrict the metric space X to being hyperbolic-like or to have nonpositive curvature in some sense, whilst making it more general than just a hyperbolic space. In this section we look at what happens when we allow X to be a finite product of hyperbolic spaces rather than just one. This will allow us to create obstructions to GBS_n groups having such an action, which we will then compare to the class of hierarchically hyperbolic groups in the next section. Suppose we have a product of r metric spaces $P = X_1 \times \cdots \times X_r$ (where P is equipped with the ℓ_1 product metric) and an isometric action of a group G on P. Note that $\mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_r)$ is naturally a subgroup of $\mathrm{Isom}(X)$ using the diagonal action. We say that G acts on P *preserving factors* if the image of this action lies inside $\mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_r)$, whereupon we can think of any element $g \in G$ as having an expression $(g_1, \dots, g_r)$ with g_i an isometry of $\mathrm{Isom}(X_i)$. We now introduce our main definition of this section.

Definition 4.1 If a group G acts isometrically on $P = X_1 \times \cdots \times X_r$, where each X_i is a hyperbolic space, then we say that the action is *product acylindrical* if the action on P is acylindrical, preserves the factors of P and such that there is an element $g = (g_1, \dots, g_r)$ in G where some g_i acts as a loxodromic element on the space X_i.

Note that if $r = 1$ then, as opposed to the standard definition of an acylindrically hyperbolic group, we allow actions of type (3) (whereupon our group will be virtually cyclic) with an infinite order element acting loxodromically as we will not need to treat this as a special case. However in common with the standard definition, we rule out any action with bounded orbits.

We make some points about the above definition. First it need not be the case that if G acts acylindrically on a space X and arbitrarily on another space Y then the product action on $X \times Y$ is acylindrical. (One could take any acylindrical action of some group G on X where a point has an infinite stabiliser and then set $Y = \mathbb{R}$ with G acting as the identity on Y.) However if G acts uniformly metrically properly on X and arbitrarily on Y then the product action on $X \times Y$ will also be uniformly metrically proper and hence will be an acylindrical action. Thus if G does have a product acylindrical action then we can say nothing in general about the action on any individual factor. Moreover we do obtain groups which have a product acylindrical action but which are not themselves acylindrically hyperbolic, for instance $F_2 \times F_2$

or Burger–Mozes–Wise groups acting geometrically on a product of two trees. As these last groups can be virtually simple, we also note that we do not have any result for product acylindrical actions of the form: a group with such an action is SQ-universal or virtually cyclic which we do have for unbounded acylindrical actions on a single hyperbolic space.

In Definition 2.1 we saw a division of isometries into a trichotomy of elliptic/loxodromic/parabolic elements when the space is hyperbolic. Note that we still have this trichotomy for a group acting on an arbitrary metric space X: elliptic elements have bounded orbits, loxodromic elements have orbits which quasiisometrically embed in X and everything else is a parabolic element. In general we might not have a nice description of how these isometries act on the boundary of X (indeed there might not even be a suitable boundary of X). However we do have a simple way of classifying an element of a group acting on a finite product of metric spaces preserving factors if we know how this element is behaving on each of the factors.

Lemma 4.2 *Let G act on the product of metric spaces $P = X_1 \times \cdots \times X_r$ by isometries preserving factors and take $g = (g_1, \ldots, g_r) \in G$, where g_i acts isometrically on X_i. Then:*

(i) *g acts elliptically on P if and only if g_i acts elliptically on X_i for each $1 \le i \le r$.*

(ii) *g acts loxodromically on P if and only if there is some i where g_i acts loxodromically on X_i.*

(iii) *g acts parabolically on P if and only if no g_i acts loxodromically on X_i but some g_i acts parabolically on X_i.*

Proof The first case is straightforward to establish in both directions, just by using the fact that if in the product space $P = X_1 \times \cdots \times X_r$ we have points $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$ then $d_{X_i}(x_i, y_i) \le d_P(x, y)$. The same is true for the reverse implication of the second case.

Thus let us now suppose without loss of generality that g_1 acts parabolically on X_1 and g_i does not act loxodromically on X_i for any $1 \le i \le r$. Then the orbit of any point $x_1 \in X_1$ under $\langle g_1 \rangle$ will be unbounded so the orbit of any $x \in P$ under $\langle g \rangle$ will be too. Thus in order to establish the forward direction of (ii) and the reverse direction of (iii), we just need to show that g is not loxodromic, for which we can use stable translation length. In particular for any point $x = (x_1, \ldots, x_r) \in P$, we have $d_i(g_i^m(x_i), x_i)/m$ tending to zero as m tends to infinity because no g_i acts loxodromically. But then by adding and using inequalities we have that $d_P(g^m(x), x)/m$ tends to zero too, thus we cannot have $c > 0$ and $\epsilon \ge 0$ with $d_P(g^m(x), x) \ge cm - \epsilon$ for all $m \in \mathbb{N}$ and so g does not act loxodromically on P. Finally if g acts parabolically on P then we cannot be in cases (i) or (ii) by what we have already shown. □

It was shown in [8, Lemma 2.2] that if G acts acylindrically on a hyperbolic space X then no element can act parabolically. For a general metric space X, we can certainly have acylindrical actions with elements acting parabolically. For instance take any finitely generated group G with a distorted infinite cyclic subgroup $\langle g \rangle$. Then G acts geometrically and hence acylindrically on its own Cayley graph, but the action of g here will be parabolic.

We now show however that we cannot have parabolic elements in a product acylindrical action.

Theorem 4.3 *Suppose that a group G acts acylindrically and preserving factors on the product P of hyperbolic spaces $X_1 \times \cdots \times X_r$. Then no element of G acts as a parabolic element on P.*

Proof Suppose otherwise, so that by Lemma 4.2 we have $g = (g_1, \dots, g_r)$ acting on P where without loss of generality the action of g_1 on X_1 is parabolic and none of the actions of g_i on X_i are loxodromic for $1 \le i \le r$. We now invoke [26, Proposition 3.2]. This states that if X is a geodesic δ-hyperbolic space and S is any finite set of isometries of X such that neither S nor S^2 contains a loxodromic element, then the joint minimum displacement $L(S)$ is at most 100δ. Here $L(S)$ is defined to be the infimum over all points $x \in X$ of the joint displacement $\max_{s \in S} d(x, s(x))$.

For each i we apply this result to the δ_i-hyperbolic space X_i. We have no loxodromic elements in $\langle g_i \rangle$, thus for any finite subset S of $\langle g_i \rangle$ there is some point $x_i \in X_i$ which is moved by at most distance $101\delta_i$ by any $s \in S$. In particular, for any $N > 0$ this applies to the set $S_N = \{g_i, \dots, g_i^N\}$. We will label the point obtained above when this result is applied with $S = S_N$ as $x_i^{(N)} \in X_i$.

We now show that this action of $\langle g \rangle$ on P is not acylindrical and hence nor is the action of G on P. Set ϵ to be $101\delta r$, where $\delta = \max(\delta_1, \dots, \delta_r)$ and suppose we are given R and N. Now as g_1 does act parabolically on X_1, the orbit of $x_1^{(N)}$ under $\langle g_1 \rangle$ is not bounded and so we can find g_1^K where, on setting $y_1^{(N)} := g_1^K(x_1^{(N)})$, we have $d_1(x_1^{(N)}, y_1^{(N)}) \ge R$, where d_i is the distance in X_i. Let $y_i^{(N)}$ be $g_i^K(x_i^{(N)})$. Now for all $1 \le i \le r$ and $1 \le j \le N$ we have that $d_i(g_i^j(x_i^{(N)}), x_i^{(N)}) \le 101\delta$ and also

$$d_i(g_i^j(y_i^{(N)}), y_i^{(N)}) = d_i(g_i^{K+j}(x_i^{(N)}), g_i^K(x_i^{(N)})) = d_i(g_i^j(x_i^{(N)}), x_i^{(N)})$$

which therefore is at most 101δ as well. Thus if we take the two points $x = (x_1^{(N)}, \dots, x_r^{(N)})$ and $y = (y_1^{(N)}, \dots, y_r^{(N)})$ of P then we have $d(x, y) \ge d_1(x_1^{(N)}, y_1^{(N)}) \ge R$ but the N distinct elements g^j for $1 \le j \le N$ satisfy

$$d(g^j(x), x) = d_1(g_1^j(x_1^{(N)}), x_1^{(N)}) + \cdots + d_r(g_r^j(x_r^{(N)}), x_r^{(N)}) = d(g^j(y), y) \le 101\delta r$$

so that each of these elements moves both x and y by at most ϵ. □

4.2 Acylindrical actions of generalised Baumslag–Solitar groups

A generalised Baumslag–Solitar group can never be acylindrically hyperbolic. This can be seen by using [32, Theorem 3.7], which states that if G is acylindrically hyperbolic and has a subgroup H where $H \cap gHg^{-1}$ is infinite for all $g \in G$ then H must itself be acylindrically hyperbolic. For a GBS_n group G we can of course take H to be a base $\mathbb{Z}^n$-subgroup for a contradiction.

However certainly there are GBS_n groups which possess a product acylindrical action, for instance $\mathbb{Z}^n \times F_r$ has an obvious geometric (and hence acylindrical) action on $\mathbb{R} \times \cdots \times \mathbb{R} \times T_{2r}$, where T_d is the regular tree of degree d. We are interested in when a GBS_n group G has a finite index subgroup H possessing a product acylindrical action. (Note that even for acylindrical hyperbolicity, it is not known whether H having this property and $H \leq_f G$ implies that G has this property too. See the correction to [32] which applies to Lemma 3.9.) However if we consider the property of virtually having a proper acylindrical action then we can now give a complete answer in the case of GBS_n groups.

Theorem 4.4 *For any $n \geq 1$, a GBS_n group G has a finite index subgroup H possessing a product acylindrical action if and only if the monodromy of G is finite.*

Proof First if the monodromy of G is finite then by Theorem 3.4 we have $H \leq_f G$ with H of the form $\mathbb{Z}^n \times F_r$ and this has a geometric, hence acylindrical, action on the product of $n+1$ hyperbolic spaces which preserves factors.

Now say that the monodromy of G is infinite. Then so will the monodromy of H for any finite index subgroup H, hence we just need to rule out that G has a product acylindrical action.

We thus suppose that G acts on the product P of hyperbolic spaces $X_1 \times \cdots \times X_r$ by isometries, and that this action preserves factors and is acylindrical. By Theorem 3.5 we have an infinite order element z of G which lies in our base $\mathbb{Z}^n$-subgroup A and which cannot be loxodromic in any action of G on a hyperbolic space. Hence on splitting z into its component parts $(z_1, \dots, z_r)$ with each z_i acting as an isometry of the hyperbolic space X_i, we have that z_i must act parabolically or elliptically on X_i. By Lemma 4.2, z must be a parabolic or elliptic element in the action of G on P. But by Theorem 4.3 there are no parabolic elements if we have a product acylindrical action of G.

Thus z must be an elliptic element. Whilst we can certainly have acylindrical actions with elliptic infinite order elements, or indeed with every element acting elliptically, we are not in the latter case here because in the definition of a product acylindrical action we must have some other element of G which is acting loxodromically.

Hence let $g \in G$ be this element, with $g = (g_1, \dots, g_r)$ and without loss of generality g_1 acts loxodromically on X_1. Let us set G_2 to be the intersection of the index-2 subgroups of G as before and note that by the proof of Theorem 3.5, we can take for z any nonidentity element of $A \cap G_2$ which is trivial in the free abelianisation of G_2. Consider the subset Z of $A \cap G_2$ consisting of elements with this property along with the identity and recall that this forms a subgroup of $A \cap G_2$ and so it is a finitely generated free abelian group. It is also infinite (for instance it contains all powers of a given $z \in Z$). Now for the element g as above, we can replace it by a power (which we will continue to call g) which lies in G_2.

We will show that this action of G on P cannot in fact be acylindrical. Pick any point $x_0 \in P$ and let D be an upper bound for the set $\{d_P(z(x_0), x_0) \mid z \in Z\}$. This exists because every element z is acting elliptically and so the orbit of x under Z is bounded, because Z is finitely generated and abelian.

Set $\epsilon = D$ and suppose we are given any $R > 0$. As g acts loxodromically, there will exist $K > 0$ such that $d_P(g^K(x_0), x_0) \geq R$ and set $y_R := g^K(x_0)$. Now by definition of the modular homomorphism, there will be an integer $m > 0$ (depending on R) such that $g^K z^m g^{-K}$ is in our base subgroup A and hence in $G_2 \cap A$ as $g^K \in G_2$. But note that if z^m is trivial in the free abelianisation of G_2 then so is $g^K z^m g^{-K}$ as $g^K \in G_2$, so z^m and $g^K z^m g^{-K}$ will have the same image in the free abelianisation of G_2. Notice this also works for powers z^{im} and $g^K z^{im} g^{-K}$ for any $i > 0$.

Thus we have infinitely many elements $\{g^K z^{im} g^{-K} \mid i \in \mathbb{N}\}$ (which are distinct because z^{im} are) lying in Z and so they each move x_0 by at most a distance D in P. But clearly we also have

$$d_P(g^K z^{im} g^{-K}(y_R), y_R) = d_P(g^K z^{im}(x_0), g^K(x_0)) = d_P(z^{im}(x_0), x_0),$$

which is at most D, and so this infinite set of elements moves both x_0 and y_R a distance at most ϵ, but $d_P(x_0, Y_R) \geq R$. Thus G is not acting acylindrically on P. □

5 Hierarchically hyperbolic groups and generalised Baumslag–Solitar groups

5.1 Hierarchically hyperbolic groups

The notion of a hierarchically hyperbolic space (HHS) was introduced in [4] as a way of generalising hyperbolic spaces to include mapping class groups and many CAT(0) cube complexes. We do not give a definition here but roughly speaking an HHS is a quasigeodesic metric space X together with a structure given in terms of projections $\pi_i \colon X \to U_i$ to a family (infinite in general) of hyperbolic spaces $\{U_i \mid i \in I\}$ called the domains (these spaces need not be proper in general and they can also be bounded). A hierarchically hyperbolic group (HHG) G is not merely a finitely generated group quasiisometric to a HHS but one where there is a HHS X where the group G acts (or rather quasiacts) geometrically on X and permutes the family of hyperbolic spaces by isometries. This family also has a nesting and an orthogonality relation. If G is a HHG then we can take X to be the Cayley graph of G with respect to a finite generating set along with the usual action of G on itself by left multiplication (thus without loss of generality we do have a genuine action of G on X rather than a quasiaction). We can also assume without loss of generality that the image $\pi_i(G)$ is coarsely dense in U_i and that this is uniform over $i \in I$.

At this point we might wonder what properties are possessed by HHGs. We have:

(i) Hyperbolic groups are HHGs (this is seen by taking the family of hyperbolic spaces to be a single hyperbolic space).

(ii) If G is an HHG and $H \leq_f G$ then H is an HHG [2, Lemma 2.25].

(iii) If G_1 and G_2 are HHGs then so is $G_1 \times G_2$ [5, Corollary 8.28].

We might also wonder about obstructions for a given group G to be an HHG. Here we have:

(1) Any HHG G is finitely presented and has quadratic isoperimetric inequality [5, Corollary 7.5].

(2) Given any finitely generated subgroup H of an HHG G, either H is virtually abelian or $F_2 \le H$ ([17, Theorem 9.15] where the condition of being finitely generated is not used, but see also the correction in [18]).

(3) Every infinite order element $g \in G$ is undistorted in G with respect to word length of a finite generating set (see [17, Theorem 7.1] but again see also the correction in [18]).

Note that for each ordered pair of these three statements, there exists a group satisfying the first but not the second.

More recently another obstruction was found in [34]. We take the following result from Remark 4.9 of that paper.

Theorem 5.1 *If G is an infinite HHG then there exist finitely many unbounded hyperbolic spaces $X_1, \dots, X_r$ for $r \ge 1$ and an isometric action of some finite index subgroup H of G on the product space $P = X_1 \times \cdots \times X_r$ (with the l_1 metric) such that H preserves each factor X_i and H acts acylindrically on the product P. Moreover there exists a loxodromic element in the action of H on X_i, thus this action of H is product acylindrical.*

Proof Theorem 3.2 of [34] shows that for any HHG G there is a finite G-invariant set $\mathcal{W} = \{W_1, \dots, W_r\}$ of unbounded domains which are pairwise orthogonal and such that any unbounded domain U_i is nested in one of these W_j. In the case when G is finite $\mathcal{W}$ is necessarily empty as all domains will be bounded. Conversely an HHS with all domains bounded is itself bounded, so whenever G is infinite there will be some unbounded domain and therefore $\mathcal{W}$ is nonempty.

Consequently there is a finite index subgroup H of G with

$$H(W_j) = W_j \quad \text{for all } 1 \le j \le r$$

and H acts by isometries on each domain W_j. Moreover the action of H on each W_j is cobounded. This is because we can assume for any domain U_i and any $g \in G,\ x \in X(= G)$ that $g(\pi_i(x)) = \pi_i(gx)$ by Remark 2.1 of [18]. But we mentioned above that $\pi_i(G) = G(\pi_i(\mathrm{id}))$ is coarsely dense and therefore so is $\pi_i(H) = H(\pi_i(\mathrm{id}))$ as H has finite index in G. However any cobounded action of a group on an unbounded hyperbolic space must contain a loxodromic element, as mentioned above when we listed the five types of action.

We can now get H to act on the product $W_1 \times \cdots \times W_r$ with the ℓ_1 metric of these unbounded domains using the diagonal action. This action clearly preserves factors and we have said that it will contain a loxodromic element. But as pointed out in [34, Remark 4.9], the proof in [4, Theorem 14.3] that G acts acylindrically on S in the case where $\mathcal{W}$ consists of the single unbounded domain S applies equally to the above action of H on $W_1 \times \cdots \times W_r$ which preserves factors and contains a loxodromic element in the action, thus this action of H is product acylindrical. □

5.2 Generalised Baumslag–Solitar groups and HHGs

We can now use the results above for our main application. Our initial question might be which generalised Baumslag–Solitar groups are HHGs. However it is possible for a group G to have a finite index subgroup which is an HHG but for G not to be. This was shown in [34, Corollary 4.5] by using Theorem 3.2 in that paper which we have already quoted above. Specifically they show that the (orientation preserving) $(3,3,3)$ triangle group is not an HHG but of course it has the finite index subgroup $\mathbb{Z}^2$ which is. A group G which is virtually an HHG will still have good group theoretic and geometric properties as these will be inherited from the finite index subgroup, so we now give a complete answer to which generalised Baumslag–Solitar groups are virtually HHGs.

Corollary 5.2 *If G is a GBS_n group then G is virtually a HHG if and only if G has finite monodromy.*

Proof If G has finite monodromy then G is virtually $\mathbb{Z}^n \times F_r$ by Theorem 3.4 and the latter group is an HHG as it is a direct product of HHGs.

Now suppose the GBS_n group G has a finite index subgroup L which is a HHG. By Theorem 5.1 there is some finite index subgroup H of L which has a product acylindrical action (unless L is finite, in which case G was not a generalised Baumslag–Solitar group). But H is then a finite index subgroup of G too and so by Theorem 4.4 we have that the monodromy of G must be finite. □

The result above in [34] that the $(3,3,3)$ triangle group is not a HHG even though it is virtually $\mathbb{Z}^2$ established that being a HHG is not a quasiisometry invariant and indeed not even a commensurability invariant. However, whenever we have a property $\mathcal{P}$ of abstract groups which is invariant under taking finite index subgroups, as is the case for being a HHG, we can recover a commensurability invariant by using the property of being virtually $\mathcal{P}$. Thus although [34] tells us that being a HHG is not a commensurability invariant and thus not a quasiisometric invariant, being virtually a HHG is certainly a commensurability invariant so the obvious question now is whether it is preserved under quasiisometry. By our results here along with famous recent examples of Leary and Minasyan, we can show the answer is no. The next theorem is an adaptation of [28, Theorem 7.2] to GBS_n groups.

Theorem 5.3 *Suppose that G is a GBS_n group with monodromy $\mathcal{M} \le \mathrm{GL}(n,\mathbb{R})$ which is conjugate in $\mathrm{GL}(n,\mathbb{R})$ to a subgroup of the orthogonal group $O(n)$. Then G is a CAT(0) group acting properly, freely and cocompactly on $T \times \mathbb{R}^n$, where T is a locally finite tree. Conversely if G is virtually a CAT(0) group then its monodromy can be conjugated into $O(n)$.*

Proof We take for T the Bass–Serre tree of the decomposition of G as a GBS_n group, so that G acts on T by tree automorphisms. We also pick some base vertex v_0 of T and we set our base $\mathbb{Z}^n$-subgroup A to be the intersection of all vertex and edge subgroups obtained from this decomposition, which will have finite index in each of these subgroups, along with a free generating set $\langle a_1, \dots, a_n \rangle$ for A. We will now define a homomorphism θ from G to $\mathrm{Isom}(\mathbb{R}^n)$ and then let G act on $T \times \mathbb{R}^n$ via the diagonal action, which will be also be by isometries. (This homomorphism is similar to the version of the modular homomorphism used in [14], which maps to $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ but which does not specifically consider orthogonal matrices.)

Suppose that the matrix $X \in \mathrm{GL}(n, \mathbb{R})$ is such that $X\mathcal{M}(G)X^{-1}$ lies in $O(n)$. If $e_1, \dots, e_n$ is the standard orthogonal basis for $\mathbb{R}^n$ then set α_i to be the vector $X(e_i)$ in $\mathbb{R}^n$. We now let the base subgroup A act via θ on $\mathbb{R}^n$ as a group of translations, where we define $\theta(a_i)$ to be the map $x \mapsto x + \alpha_i$. Recalling how we obtain a finite presentation for G, we next extend θ to all vertex subgroups by using the appropriate translations, where we map consistently across edge subgroups for the edges in some maximal tree, by working out from v_0. This works because any edge subgroup E is a copy of $\mathbb{Z}^n$ with finite index in its neighbouring vertex subgroup V, so a homomorphism from E to $\mathbb{R}^n$ has a unique extension from V to $\mathbb{R}^n$.

We now need to set $\theta(t)$ for all stable letters t. Once we have chosen $\theta(t)$ to be some isometry of $\mathbb{R}^n$, θ will be well defined if t acts by conjugation on A in the correct way because A has finite index in any vertex group.

Therefore we send t to the orthogonal matrix $X\mathcal{M}(t)X^{-1}$. Let us write the matrix $\mathcal{M}(t)$ as $\mathcal{T}$ with ij^{th} entry equal to τ_{ij}. We then have that $\theta(t)\theta(a_j)\theta(t^{-1})$ is the translation by

$$X\mathcal{T}X^{-1}(\alpha_j) = X\mathcal{T}(e_j) = X(\tau_{1j}e_1 + \cdots + \tau_{nj}e_n) = \tau_{1j}\alpha_1 + \cdots + \tau_{nj}\alpha_n.$$

As for the translation given by $\theta(ta_jt^{-1})$, we can assume ta_jt^{-1} is in A by replacing a_j with a power if necessary, whereupon we have

$$ta_jt^{-1} = a_1^{\tau_{1j}} \cdots a_n^{\tau_{nj}}$$

by definition of the modular homomorphism and so $\theta(ta_jt^{-1})$ will be the translation of $\mathbb{R}^n$ by

$$\tau_{1j}\alpha_1 + \cdots + \tau_{nj}\alpha_n$$

too. Thus θ is well defined and so G acts on $T \times \mathbb{R}^n$ by isometries.

This action of G on $T \times \mathbb{R}^n$ is (metrically) proper: for instance see [14, Lemma 4.5]. This implies that the action is free (and in particular faithful) because G is torsion free. It is also cocompact, as we can take the compact set $C \times D \subseteq T \times \mathbb{R}^n$, where C is a closed ball centred at v_0 and containing a fundamental domain for the action of G on T, whereas D is a closed parallelopiped containing a fundamental domain for the action of A on $\mathbb{R}^n$. Given any point $(t, x) \in T \times \mathbb{R}^n$, we can find $g \in G$ which moves t into C, and then take some element $a \in A$ which moves $g(x)$ into D. As $a \in \mathrm{Stab}(v_0)$ we have $a(C) = C$ so that $ag(t, x) \in C \times D$.

For the converse, first suppose that G is a CAT(0) group, acting geometrically (or even properly and semisimply) on the CAT(0) space X. Then by [28, Theorem 6.4(4')] (itself based on the flat torus theorem of [9, Theorem II.7.1], but replacing elements that normalise a copy of $\mathbb{Z}^n$ with elements that commensurate it), there is an inner product on $A \otimes \mathbb{R} \cong \mathbb{R}^n$ such that the monodromy of any isometry of X that commensurates A preserves this inner product, so that we can conjugate the monodromy of G to lie in $O(n)$.

Finally if G has a finite index subgroup H that is a CAT(0) group then by the above we can conjugate the monodromy $\mathcal{M}(G)$ so that the finite index subgroup $\mathcal{M}(H)$ lies in $O(n)$. We can now further conjugate so that all of $\mathcal{M}(G)$ is in $O(n)$ too (either by mimicking the proof when $\mathcal{M}(H)$ is trivial, or by $O(n)$ being a maximal compact subgroup of $\mathrm{GL}(n,\mathbb{R})$). □

We can now bring in examples from [28].

Corollary 5.4 *The Leary–Minasyan group given by the finite presentation*

$$L = \langle t, a, b \mid [a,b], ta^2b^{-1}t^{-1} = a^2b, tab^2t^{-1} = a^{-1}b^2 \rangle$$

is a GBS_2 *group which is not virtually a HHG but which is quasiisometric to a HHG.*

Proof We see that L is a GBS_2 group with base $\mathbb{Z}^2$-subgroup $\langle a, b\rangle$ by using the graph of groups decomposition of one vertex and one edge associated to the HNN extension, with the edge subgroup having index 5 in the vertex group at both ends. Thus the Bass–Serre tree is the regular tree T_{10}. The monodromy is determined by the conjugation action of the single stable letter and we can work this out explicitly by noting that the above relations imply that

$$ta^5t^{-1} = a^3b^4 \quad \text{and} \quad tb^5t^{-1} = a^{-4}b^3$$

both hold in G. Thus the monodromy of L is generated by the matrix

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

which has infinite order. Hence L has infinite monodromy and is not virtually a HHG by Corollary 5.2.

However this matrix is orthogonal so Theorem 5.3 says that L acts properly and cocompactly by isometries on $T_{10}\times\mathbb{R}^2$, as does $M = F_5\times\mathbb{Z}^2$. Thus, by the Švarc–Milnor lemma, L and M are both quasiisometric to this space and hence to each other, but M is certainly a HHG. □

6 Property (QT) is not invariant under quasiisometry

Another property of groups which, like HHGs, considers how a group can act on different hyperbolic spaces, is Bestvina, Bromberg and Fujiwara's property (QT) from [6]. Here a quasitree will always be a graph equipped with its path metric which is quasiisometric to a simplicial tree but which need not be locally finite. A finitely generated group G (equipped with the word metric with respect to some finite generating set) is said to have (QT) if it acts by isometries on a finite product P of quasitrees equipped with the ℓ_1 product metric such that the orbit map (using an arbitrary basepoint of P) is a quasiisometric embedding from G to P. This is a strong definition: for instance it implies that this action is metrically proper. Nevertheless it is shown in [6] that mapping class groups and all residually finite hyperbolic groups have (QT). It is also a consequence of [16] that every Coxeter group has (QT).

Moreover property (QT) has good closure properties, in fact these are better than for HHGs. It is certainly the case that if G has (QT) and H has finite index in G then H also has (QT) but the definition ensures that it also holds more generally when H is an undistorted finitely generated subgroup of G. Moreover if G_1 has (QT) via an action on the space P_1 and G_2 on the space P_2 then it can be checked directly that $G_1 \times G_2$ has (QT) by letting it act on the direct product $P_1 \times P_2$ (with the ℓ_1 product metric) using the action on each factor and summing the word metrics on G_1 and G_2, which is the word metric on $G_1 \times G_2$ with the obvious generating set.

However property (QT) is also a commensurability invariant because if H has index i in G and H acts isometrically on the product P of quasitrees then we can induce an isometric action of G on the product P^i of copies of P. This will also turn the orbit map under G into a quasiisometric embedding. In particular a group which virtually has (QT) does itself have (QT). Therefore the questions of whether possessing (QT) and virtually possessing (QT) are quasiisometry invariants are in fact the same question. Here we will answer this by first considering how an isometry on a product of graphs $\Gamma_1 \times \cdots \times \Gamma_m$ (each equipped with the path metric and then using the ℓ_1 product metric) breaks up, or at least virtually breaks up, into individual isometries on each Γ_i. That this can be done using the ℓ_∞ metric is a result of W Malone in [30] and we will now mimic his proof for the (easier) ℓ_1 case.

Theorem 6.1 *Suppose that $X = \Gamma_1 \times \cdots \times \Gamma_m$ is a finite product of connected graphs, where each Γ_i has the induced path metric and X has the l_1 or the l_∞ product metric. Suppose that G is any group acting by isometries on X. Then G has a finite index subgroup H which preserves factors and acts as an isometry on each factor.*

Proof On giving a graph the induced path metric, it becomes not just a metric space which is geodesic but a geodesic metric space which is locally uniquely geodesic. Thus if X is given the l_∞ metric then this result is a direct consequence of [30], which is established by using the fact that for certain directions in the product space, geodesics are unique if they are unique in the factors. This is certainly true in an l_1 product space if we travel "horizontally/vertically", so in this case we proceed as follows:

Lemma 6.2 *Let $X = X_1 \times \cdots \times X_m$ have the l_1 product metric d_X for geodesic metric spaces $(X_1, d_1), \ldots, (X_m, d_m)$. Suppose we have two points $\boldsymbol{x}, \boldsymbol{y} \in X$ of the form*

$$\boldsymbol{x} = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) \quad \text{and} \quad \boldsymbol{y} = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m)$$

for $1 \le i \le m$. Then any geodesic between $\boldsymbol{x}$ and $\boldsymbol{y}$ only varies in the i^{th} coordinate. Conversely if we have two points $\boldsymbol{x}, \boldsymbol{y} \in X$ which differ in at least two coordinates then there is more than one geodesic from $\boldsymbol{x}$ to $\boldsymbol{y}$ in X.

Proof Suppose we have a (unit speed) geodesic $\boldsymbol{\gamma}\colon [0, d] \to X$ from $\boldsymbol{x}$ to $\boldsymbol{y}$, where $d = d_X(\boldsymbol{x}, \boldsymbol{y})$. If there is $k \neq i$ and $t \in (0, d)$ such that the k^{th} coordinate u_k of $\boldsymbol{\gamma}(t) = (u_1, \ldots, u_m)$ is not equal to x_k

then

$$\begin{aligned} d_i(x_i, y_i) = d_X(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{x}, \boldsymbol{\gamma}(t)) + d(\boldsymbol{\gamma}(t), \boldsymbol{y}) \\ &= \Sigma_{j=1, j\neq i}^{m} (d_j(x_j, u_j) + d_j(u_j, x_j)) + d_i(x_i, u_i) + d_i(u_i, y_i) \\ &\geq d_k(x_k, u_k) + d_k(u_k, x_k) + d_i(x_i, u_i) + d_i(u_i, y_i) \\ &> d_i(x_i, y_i) \quad \text{since } x_k \neq u_k, \end{aligned}$$

which is a contradiction.

Now suppose we have two points $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_m)$ in X which differ in (without loss of generality) at least the first two coordinates. Take constant speed geodesics γ_i in each X_i running from x_i to y_i and set

$$\boldsymbol{\delta}_1(t) = (x_1, \gamma_2(t), \dots, \gamma_m(t)), \quad \boldsymbol{\delta}_2(t) = (\gamma_1(t), y_2, \dots, y_m))$$

and

$$\boldsymbol{\delta}_3(t) = (\gamma_1(t), x_2, \gamma_3(t), \dots, \gamma_m(t)), \quad \boldsymbol{\delta}_4(t) = (y_1, \gamma_2(t), y_3, \dots, y_m)),$$

where in $\boldsymbol{\delta}_1$ the geodesics $\gamma_2, \dots, \gamma_m$ are reparametrised to have domain $[0, d_X(\boldsymbol{x}, \boldsymbol{y}) - d_1(x_1, y_1)]$ but γ_1 and hence $\boldsymbol{\delta}_2$ remain unit speed. We also do the same for $\boldsymbol{\delta}_3$ and $\boldsymbol{\delta}_4$. Then it is easily checked that following $\boldsymbol{\delta}_1$ then $\boldsymbol{\delta}_2$ and also following $\boldsymbol{\delta}_3$ then $\boldsymbol{\delta}_4$ are both unit speed geodesics from $\boldsymbol{x}$ to $\boldsymbol{y}$. Moreover they are distinct as the first geodesic passes through a point that projects to (x_1, y_2) in the first two coordinates, whereas the second geodesic never does since $x_1 \neq y_1$ and $x_2 \neq y_2$. □

Now we return to the setting in Theorem 6.1. Each Γ_i is a geodesic metric space (which we assume without loss of generality is not a single point). Thus given any isometry g of X, take any point $\boldsymbol{x} = (x_1, \dots, x_i, \dots, x_m)$ in X and for a given $1 \leq i \leq m$, let z_i be another point in Γ_i near x_i such that there is only one geodesic γ_i in Γ_i from x_i to z_i. Then by Lemma 6.2 we have that $\boldsymbol{\gamma}(t)$, which is equal to $(x_1, \dots, \gamma_i(t), x_{i+1}, \dots, x_m)$, is the unique geodesic in X between $\boldsymbol{x}$ and $\boldsymbol{z} = (x_1, \dots, x_{i-1}, z_i, x_{i+1}, \dots, x_m)$ and so under g it must map to a unique geodesic between $g(\boldsymbol{x})$ and $g(\boldsymbol{z})$, by considering g^{-1}. Thus by Lemma 6.2 again these two points (which are not the same because g is a bijection) differ in only one coordinate, say the j^{th}.

Now take y_i to be an arbitrary point in Γ_i. Given any geodesic in Γ_i from x_i to y_i, we can split it into a finite number of subgeodesics, with some overlap that is more than a point, where each subgeodesic is the unique geodesic between its own endpoints. Thus the image of each of these subgeodesics under g is a subset of X where only one coordinate varies, and as these subgeodesics overlap this will always be the j^{th} coordinate. In other words given $1 \leq i \leq m$ we have $1 \leq j \leq m$ such that for fixed

$$x_1 \in \Gamma_1, \quad \dots, \quad x_{i-1} \in \Gamma_{i-1}, \quad x_{i+1} \in \Gamma_{i+1}, \quad \dots, \quad x_m \in \Gamma_m$$

there is

$$y_1 \in \Gamma_1, \quad \dots, \quad y_{j-1} \in \Gamma_{j-1}, \quad y_{j+1} \in \Gamma_{j+1}, \quad \dots, \quad y_m \in \Gamma_m$$

and a function $f:\Gamma_i\to\Gamma_j$ with

$$g(x_1,\dots,x_{i-1},x,x_{i+1},\dots,x_m)=(y_1,\dots,y_{j-1},f(x),y_{j+1},\dots,y_m)$$

for all $x\in\Gamma_i$. Moreover f is a bijection and an isometry because g is. By varying over all m coordinates we see that g permutes $\Gamma_1,\dots,\Gamma_m$ and for each $1\le k\le m$ it acts as an isometry from Γ_k to $\Gamma_{g(k)}$ (but clearly only isometric factors can be permuted). We can therefore set H to be the kernel of this finite permutation action, so that H has finite index in G, preserves factors and acts as an isometry on each factor. □

We can now use the same example as for HHGs to show that property (QT) is not a quasiisometric invariant.

Theorem 6.3 *A GBS_n G group with infinite monodromy does not possess any metrically proper action by isometries on any finite product of graphs (using the path metric) which are quasitrees with the l_1 (or l_∞) product metric. In particular G does not have property (QT).*

Proof If G did have such an action then by Theorem 6.1 G would have a finite index subgroup H (also acting metrically properly) which preserves factors and acts by isometries on each factor. But H is also a GBS_n group with infinite monodromy. Thus by Theorem 3.5 applied to H, there is a nontrivial element z of H which does not act loxodromically in any action of H on a hyperbolic space. But as isometries of quasitrees can only be loxodromic or elliptic by [31] the action of the infinite order element z is elliptic on each quasitree. Hence the action of $\langle z\rangle$ on X is also bounded, so we have an infinite subgroup of G acting on X with bounded orbits, meaning that the action of G is not metrically proper. Therefore the orbit map from G to X cannot be a quasiisometric embedding because there are only finitely many elements in G with word length at most a given value. □

Corollary 6.4 *The property of having (QT) is not preserved under quasiisometries.*

Proof We take the Leary–Minasyan group L as given in Corollary 5.4 and again note that it has infinite monodromy, so it does not have (QT) by Theorem 6.3. However it is quasiisometric to $M=\mathbb{Z}^2\times F_5$ which acts geometrically on the product $P=\mathbb{R}\times\mathbb{R}\times T_{10}$ of simplicial trees and hence the orbit map from M to P is a quasiisometry by Švarc–Milnor. □

Note that as P is a product not just of arbitrary quasitrees but of bounded valence simplicial trees, changing the property (QT) by replacing quasitrees with trees or inserting the bounded valence condition, in any combination, will still not result in a quasiisometry invariant.

Note also that if we equip our product spaces with the ℓ_2 product metric then the Leary–Minasyan group does indeed act geometrically on a product of quasitrees with the orbit map a quasiisometric embedding. Here the equivalent of Theorem 6.1 does not hold as we can have a de Rham factor (see [19] for this case). However the definition of (QT) uses the ℓ_1 product metric.

Finally we also note that the Leary–Minasyan group acts geometrically on the CAT(0) cube complex P by *isometries* using the CAT(0) metric but it has no geometric action on any CAT(0) cube complex by *cubical automorphisms* and nor does any finite index subgroup. This is noted in [28] as a consequence of [24].

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