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Let Y be a closed, orientable 3-manifold with Heegaard genus 2. We prove that if $H_1(Y;\mathbb{Z})$ has order 1, 3, or 5, then there is a representation $\pi_1(Y) \to \mathrm{SU}(2)$ with nonabelian image. Similarly, if $H_1(Y;\mathbb{Z})$ has order 2 then we find a nonabelian representation $\pi_1(Y) \to \mathrm{SO}(3)$. We also prove that a knot K in S^3 is a trefoil if and only if there is a unique conjugacy class of irreducible representations $\pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ sending a fixed meridian to $\mathrm{diag}(i, -i)$.

57R58; 57K31, 57M12

1 Introduction

The instanton Floer homology of a homology 3-sphere Y is generated as a chain complex by conjugacy classes of irreducible representations $\pi_1(Y) \to \mathrm{SU}(2)$, so it is natural to ask whether these exist. This question has been studied from many different perspectives in recent years, including surgeries on knots [25], branched double covers [46], Stein fillings [3], and incompressible tori [4; 32]; in every case so far the answer has been yes, unless $Y \cong S^3$.

In this article, we add small Heegaard genus to the list. If Y has a Heegaard splitting of genus at most 1 then $\pi_1(Y)$ is abelian, so the first interesting case is Heegaard genus 2. In this case we can apply work of Birman and Hilden [7] to identify such manifolds as branched covers of links, and thus reduce the problem to one about representations of link groups. Our main result is the following.

Theorem 1.1 *Let Y be a rational homology 3-sphere with Heegaard genus 2, and with first homology $H_1(Y;\mathbb{Z})$ of order 1, 3, or 5. Then there is a representation $\pi_1(Y) \to \mathrm{SU}(2)$ with nonabelian image.*

In general, we will say that Y is SU(2)-*abelian* if every homomorphism $\pi_1(Y) \to \mathrm{SU}(2)$ has abelian image; then manifolds satisfying the hypotheses of Theorem 1.1 are *not* SU(2)-abelian.

One could ask the same question about rational homology 3-spheres with other homology groups: for fixed $n \geq 1$, if Y has Heegaard genus 2 and $|H_1(Y)| = n$, must there be a nonabelian representation $\pi_1(Y) \to \mathrm{SU}(2)$? The answer is negative for many other values of n. For example:

- If $n \geq 4$ is even, then the connected sum $\mathbb{RP}^3 \# L(n/2, 1)$ is SU(2)-abelian.

- For $n = |24g + 11|$, where $g \neq 0, -1$, the toroidal manifolds $Y = Y(T_{2,3}, T_{2,2g+1})$ studied by Zentner [46] (and originally constructed by Motegi [35, Section 3]) are SU(2)-abelian. These have Heegaard genus 2 because they are branched double covers of 3-bridge knots $L(T_{2,3}, T_{2,2g+1})$ [46, Theorem 4.14]; see [42, Figure 1]. For example, when $n = 35$ the branch locus is identified in [46, Section 5] as the knot 8_{16}.
- The smallest known hyperbolic example, pointed out to us by Nathan Dunfield, is Vol3 = m007(3, 1), with $n = 18$. It has Heegaard genus 2 because it is the branched double cover of the 3-bridge link $L10n46$.
- The smallest known hyperbolic example with n odd is m036(−3, 2), with $n = 45$; it is the branched double cover of the 3-bridge knot 8_{18}, or equivalently the 4-fold branched cyclic cover of the figure eight knot. (See [11, Section 6], or [42, Example 5.6] for more discussion.) It is also the unique double cover of Vol3.

Unfortunately, we do not expect that the strategy we use to prove Theorem 1.1 will work for larger odd values of $|H_1(Y)|$, because the order-5 case already requires several special results that do not apply to higher orders. One is a theorem of Li and Liang [31, Theorem 1.4] asserting that if the instanton knot homology $KHI(K)$ of Kronheimer and Mrowka [28] has rank 1 in Alexander gradings $\pm g$, $\pm(g-1)$, and 0, and it vanishes everywhere else, then either K or its mirror is an *instanton L-space knot* of genus g. The analogous claim for $\widehat{HFK}(K)$ is certainly true but does not seem to exist in the Heegaard Floer literature. The second theorem, by Farber, Reinoso, and Wang [16], says that the $(2,5)$ torus knot is the only instanton L-space knot of genus 2. We would like to generalize the first to larger values of $\dim KHI(K)$, and the second to genera $g \geq 3$, but both of these seem entirely out of reach.

Theorem 1.1 makes use of some facts about SU(2)-simple knots, whose definition we recall now. Here and in the sequel we view SU(2) as the unit quaternions; this identifies the trace of a 2×2 matrix with twice the real part of the corresponding quaternion.

Definition 1.2 [46] Given a link $L \subset S^3$, a representation

$$\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2)$$

is *meridian-traceless* if $\mathrm{Re}(\rho(\mu)) = 0$ for every meridian μ of L. We then say that L is SU(2)-*simple* if every meridian-traceless representation is conjugate to one with image in the binary dihedral group $\{e^{i\theta}\} \cup \{e^{i\theta} j\}$.

What we really show en route to proving Theorem 1.1 is that the only SU(2)-simple knots of determinant 1 or 3 are the unknot and the trefoils $T_{\pm 2,3}$, and that the only SU(2)-simple knots of determinant 5 and bridge index at most 3 are the figure eight and the cinquefoils $T_{\pm 2,5}$.

Building on this, we can study the representation variety

$$\mathcal{R}(K,i) = \{\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2) \mid \rho(\mu) = i\},$$

which consists of representations that send a fixed meridian μ of K to the quaternion i. Kronheimer and Mrowka [28, Theorem 7.17] proved that $\mathscr{R}(K,i) \cong \{*\}$ if and only if K is the unknot. For the trefoils,

$$\mathscr{R}(T_{\pm 2,3}, i) \cong \{*\} \sqcup S^1,$$

and [5, Conjecture 1.9] says that this should uniquely characterize the trefoils. The classification of SU(2)-simple knots of determinant 3 allows us to prove that this is indeed the case.

Theorem 1.3 *Suppose for a knot $K \subset S^3$ that $\mathscr{R}(K,i) \cong \{*\} \sqcup S^1$. Then K is a trefoil.*

Partial results toward Theorem 1.3 were known under the assumption that the S^1 component consists of *nondegenerate* representations: Kronheimer and Mrowka [28, Corollary 7.20] first proved that K must be fibered, and then in [5, Theorem 1.10] we proved that K must be a trefoil. Theorem 1.3 removes the nondegeneracy assumption, establishing the conjecture in full generality.

Along the way to classifying SU(2)-simple knots of small determinant, we deduce the following result which may be independently useful, as part of Theorem 3.3.

Theorem 1.4 *Let $K \subset S^3$ be an* SU(2)*-simple knot whose determinant is prime, and write its Alexander polynomial as*

$$\Delta_K(t) = \sum_{i=-g}^{g} a_i t^i$$

where g is the Seifert genus of K. Then $(-1)^{i+\sigma(K)/2} a_i \geq 0$ for all i, where $\sigma(K)$ is the signature of K, and $|a_g| \geq 1$ with equality if and only if K is fibered.

In fact, in Theorem 3.3 we show that $(-1)^{i+\sigma(K)/2} a_i$ is nonnegative because it is equal to the dimension of the summand $\mathit{KHI}(K,i)$ of the instanton knot homology of K.

The reason we insist that $\det(K)$ be prime in Theorem 1.4 is that it allows us to verify a nondegeneracy condition for binary dihedral representations of $\pi_1(S^3 \setminus K)$, following work of Heusener and Klassen [19]; we do not know whether the same conclusions should hold when K is SU(2)-simple but $\det(K)$ is composite. These nondegeneracy conditions allow us to show that $\dim \mathit{KHI}(K) = \det(K)$, and then in the determinant-3 case, we know from [5] that the only such knots are the trefoils. When $\det(K) = 5$ this is not quite enough, and we also need to understand Stein fillings of SU(2)-abelian manifolds; see Proposition 4.3 for details.

Finally, the reader might wonder what happens when $|H_1(Y)|$ is even. In this case we have to consider SU(2)-simplicity for links rather than knots, and less is known in this setting, but recent work of Xie and Zhang [45] provides a starting point. Even so, it is hard to guarantee the existence of SU(2) representations, because our methods might lead to SO(3) representations that do not lift to SU(2). (When $|H_1(Y)|$ is odd, there is no obstruction to lifting.) But we can often at least find nonabelian SO(3) representations, and in particular we prove the following.

Theorem 1.5 *If Y has Heegaard genus 2 and satisfies $|H_1(Y;\mathbb{Z})| = 2$, then there is a representation*

$$\pi_1(Y) \to \mathrm{SO}(3)$$

with nonabelian image.

As in the case of determinants 1 and 3, what we really prove is Proposition 7.5, asserting that the Hopf link is the only SU(2)-simple link of determinant 2, and then Theorem 1.5 follows from the special case of 3-bridge links.

Organization

In Section 2 we prove Theorem 1.1 for integral homology spheres. In Section 3 we study the instanton knot homology of SU(2)-simple knots, and in Section 4 we prove Proposition 4.3, asserting that simply connected Stein fillings of many SU(2)-abelian manifolds must have negative definite intersection form. In Section 5 we apply these to prove Theorem 1.1 in the cases where $H_1(Y)$ has order 3 or 5. Section 6 uses this work to prove Theorem 1.3, that the variety $\mathcal{R}(K,i)$ detects the trefoils. Finally, Section 7 is devoted to the proof of Theorem 1.5.

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2 From SU(2)-simple knots to SU(2)-abelian 3-manifolds

Given a knot $K \subset S^3$, we will let $\Sigma_2(K)$ denote the branched double cover of K. The proof of Theorem 1.1 then relies on the following proposition.

Proposition 2.1 *For each odd integer $d \geq 1$, the map $K \mapsto \Sigma_2(K)$ gives a well-defined surjection*

$$\left\{ \begin{matrix} \mathrm{SU}(2)\text{-simple 3-bridge knots} \\ K \subset S^3 \text{ of determinant } d \end{matrix} \right\} \to \left\{ \begin{matrix} \mathrm{SU}(2)\text{-abelian 3-manifolds } Y \text{ with} \\ \text{Heegaard genus 2 and } |H_1(Y)| = d \end{matrix} \right\}. \tag{2-1}$$

Proof We first check that this map is well defined. If K is an SU(2)-simple 3-bridge knot, then its branched double cover is SU(2)-abelian by [42, Theorem 1.6]. Moreover, $\Sigma_2(K)$ certainly has Heegaard genus at most 2, since we can split K in half using a bridge sphere, and the branched double cover of the 3-ball over either half will be a genus-2 handlebody. In fact, the Heegaard genus of $\Sigma_2(K)$ must be exactly 2, since otherwise $\Sigma_2(K)$ would be S^3 or a lens space, and then Hodgson and Rubinstein [20] proved that K would have bridge index at most 2. So this proves that $\Sigma_2(K)$ belongs to the codomain, as claimed.

We now wish to show that the map is surjective, so let Y be an element of the codomain. Since Y has Heegaard genus 2, Birman and Hilden [7, Theorem 1] proved that it is the branched double cover of a 3-bridge link $L \subset S^3$. Writing $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, we have a short exact sequence of chain complexes

$$0 \to C_*(S^3, L; \mathbb{F}) \to C_*(\Sigma_2(L); \mathbb{F}) \to C_*(S^3; \mathbb{F}) \to 0,$$

where the first map is a transfer map and the second is induced by projection, and this leads to an isomorphism

$$H_1(\Sigma_2(L); \mathbb{F}) \cong H_1(S^3, L; \mathbb{F}) \cong \widetilde{H}_0(L; \mathbb{F}) \tag{2-2}$$

using the long exact sequence of the pair (S^3, L). Thus if L has ℓ components then

$$0 = \dim_{\mathbb{F}} H_1(Y; \mathbb{F}) = \dim_{\mathbb{F}} H_1(\Sigma_2(L); \mathbb{F}) = \ell - 1,$$

so $\ell = 1$ and we can write $Y = \Sigma_2(K)$ where $K = L$ is a 3-bridge knot.

We know from [46, Lemma 3.2] and the fact that $H_1(\Sigma_2(K); \mathbb{F}) = 0$ that $\Sigma_2(K)$ is SU(2)-abelian if and only if it is SO(3)-abelian, and the latter implies that K is SU(2)-simple by [46, Proposition 3.1], so if Y is SU(2)-abelian then K must be SU(2)-simple. Thus K belongs to the preimage of Y, proving the desired surjectivity. □

Remark 2.2 We note from the proof of Proposition 2.1 (and in particular from [46, Section 3]) the fact that if a knot $K \subset S^3$ has an SU(2)-abelian branched double cover, then K must be SU(2)-simple. This holds regardless of the bridge index of K or the Heegaard genus of $\Sigma_2(K)$. By contrast, an SU(2)-simple knot of bridge index greater than 3 need not have an SU(2)-abelian branched double cover.

The following fact constrains the knots considered in Proposition 2.1, though we will not use it in this paper.

Proposition 2.3 *All* SU(2)*-simple* 3*-bridge knots are hyperbolic.*

Proof Suppose that K is an SU(2)-simple 3-bridge knot, but that it is not hyperbolic. Then K is either a torus knot or a satellite [43, Corollary 2.5], so we address these cases separately below.

Suppose first that K is a torus knot. Schubert [40, Satz 10] showed that the only 3-bridge torus knots are $T(3, q)$ where $|q| > 3$, so $K = T(3, q)$ for some such q. Then $\det(K)$ is either 1 or 3 depending on whether q is odd or even, so $\Sigma_2(K)$ is an SU(2)-abelian Seifert fibered space, with first homology of order 1 or 3. The only such manifolds are S^3 and lens spaces [42, Theorem 1.2], contradicting the fact that $\Sigma_2(K)$ has Heegaard genus 2.

Now suppose that K is a satellite knot, say with pattern $P \subset S^1 \times D^2$ and companion C. Let $\alpha > 0$ be the wrapping number of P, meaning the minimal number of times that some meridional disk of $S^1 \times D^2$ intersects P. Then Schubert [40, Satz 3] proved that the bridge index satisfies

$$b(K) \geq \alpha \cdot b(C).$$

Since $b(K) = 3$ and the knottedness of C implies that $b(C) \geq 2$, we must have $\alpha = 1$; hence K is a connected sum $C \# C'$. But then $b(C \# C') = b(C) + b(C') - 1$ by [40, Satz 7], so C and C' are both 2-bridge knots, and thus the SU(2)-abelian manifold

$$\Sigma_2(K) \cong \Sigma_2(C) \# \Sigma_2(C')$$

is a connected sum of two lens spaces of odd order. Now $\pi_1(\Sigma_2(K)) \cong (\mathbb{Z}/p\mathbb{Z}) * (\mathbb{Z}/q\mathbb{Z})$ for some odd integers $p, q \geq 3$, and there is an irreducible representation $\pi_1(\Sigma_2(K)) \to \mathrm{SU}(2)$ defined by sending the generators of the $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ factors to $e^{2\pi i/p}$ and $e^{2\pi j/q}$ respectively, so $\Sigma_2(K)$ is not SU(2)-abelian after all and we have a contradiction. □

Proposition 2.1, together with some gauge-theoretic input from Kronheimer and Mrowka [28], suffices to establish the homology sphere case of Theorem 1.1.

Proof of Theorem 1.1 in the case $H_1(Y;\mathbb{Z}) = 0$ We wish to show that there are no SU(2)-abelian integer homology 3-spheres of Heegaard genus 2. By Proposition 2.1, it suffices to show that there are no nontrivial SU(2)-simple knots of determinant 1, so let us suppose that such a knot K exists. Then Klassen [24, Theorem 10] proved that there are exactly $\frac{1}{2}(\det(K) - 1) = 0$ conjugacy classes of nonabelian representations $\pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ with binary dihedral image. By assumption, every representation

$$\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$$

such that $\operatorname{Re} \rho(\mu) = 0$ is binary dihedral, so now every such ρ has abelian image. But Kronheimer and Mrowka [28, Corollary 7.17] proved that since K is a nontrivial knot, there is at least one nonabelian ρ with $\operatorname{Re} \rho(\mu) = 0$, a contradiction. □

Remark 2.4 Zentner [46, Proposition 9.1] gave a slightly different proof of the fact that a homology sphere which is the branched double cover of a knot cannot be SU(2)-abelian. We argue differently here because the idea of counting binary dihedral representations will turn out to be useful for manifolds with nontrivial first homology.

3 Instanton knot homology and SU(2)-simple knots

When Y is not an integer homology sphere, we can still occasionally say something about representations $\pi_1(Y) \to \mathrm{SU}(2)$, though it will require substantially more machinery. In preparation, we study the instanton knot homology [28] of SU(2)-simple knots with prime determinant.

Proposition 3.1 *Let $K \subset S^3$ be an* SU(2)*-simple knot, with determinant an odd prime p. Then its signature satisfies $|\sigma(K)| \leq p - 1$ and $\sigma(K) \equiv p - 1 \pmod 4$, and its instanton knot homology satisfies $\dim_{\mathbb{Q}} KHI(K) = p$.*

Proof Since K is SU(2)-simple, all of the irreducible SU(2) representations $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ with $\mathrm{Re}(\rho(\mu)) = 0$ are conjugate to binary dihedral ones, and again by [24, Theorem 10] there are $n = \frac{1}{2}(\det(K) - 1)$ of these up to conjugacy. Thus by [46, Proposition 7.3], we will have

$$\dim_{\mathbb{Q}} I^{\natural}(K) = 2n + 1 = \det(K),$$

where $I^{\natural}(K)$ is the reduced singular instanton knot homology of [29], if each such representation ρ satisfies the following two conditions:

(a) $\dim_{\mathbb{R}} H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) = 1$, and

(b) the restriction map $H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) \to H^1(\mu; \mathfrak{su}(2)_\rho)$ is onto.

If ρ is one of these irreducible, binary dihedral representations, with image in $\{e^{i\theta}\} \cup \{e^{i\theta} j\}$, then one can extract from the proof of [24, Theorem 10] that it has finite image, of order dividing $4\det(K) = 4p$. (For completeness, we will work out the details explicitly in Lemma 7.3 and Remark 7.4.) Since $\rho(\mu)$ is traceless it has order 4, so then the order of $\mathrm{Im}(\rho)$ is also a multiple of 4, and it is not equal to 4 or else $\mathrm{Im}(\rho)$ would be abelian. Then $\mathrm{Im}(\rho)$ has order exactly $4p$ since p is prime. Now Proposition 3.2 below asserts that ρ satisfies both of the desired conditions.

We deduce from this that $\dim_{\mathbb{Q}} I^{\natural}(K) = \det(K) = p$ after all, and then the isomorphism

$$I^{\natural}(K; \mathbb{Q}) \cong \mathit{KHI}(K; \mathbb{Q})$$

of [29, Proposition 1.4] tells us that $\dim_{\mathbb{Q}} \mathit{KHI}(K) = p$ as claimed. These computations of the twisted cohomology of $S^3 \setminus K$ also suffice to ensure that the Casson–Lin invariant $h(K)$, defined in [34] and reinterpreted using gauge theory by Herald [18], is in fact a signed count of the conjugacy classes of irreducible representations with $\mathrm{Re}(\rho(\mu)) = 0$. It follows that

$$|h(K)| \le \tfrac{1}{2}(\det(K) - 1),$$

but Lin [34, Corollary 2.10] proved that $h(K) = \frac{1}{2}\sigma(K)$, so this establishes the inequality

$$|\sigma(K)| \le \det(K) - 1.$$

Finally, the congruence $\sigma(K) \equiv \det(K) - 1 \pmod 4$ actually holds for arbitrary knots $K \subset S^3$, as an immediate consequence of the evenness of $\sigma(K)$ and the relation

$$\det(K) \equiv (-1)^{\sigma(K)/2} \pmod 4 \tag{3-1}$$

of [36, Theorem 5.6], so the proof is complete. □

The following proposition completes the missing part of Proposition 3.1. The hard technical details of its proof are essentially contained in [19], though we have to do some work to extract them.

Proposition 3.2 *Let $K \subset S^3$ be a knot, and let $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ be a representation whose image is binary dihedral of order $4p$ for some odd prime p. Then*

(a) $\dim_{\mathbb{R}} H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) = 1$, *and*

(b) *the restriction map* $H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) \to H^1(\mu; \mathfrak{su}(2)_\rho)$ *is onto.*

Proof Since $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ has binary dihedral image of order 4 times an odd prime, Heusener and Klassen [19, Section 3] proved that $H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ is 1-dimensional, verifying (a). In fact, their proof also implies that the restriction map

$$H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) \to H^1(\mu; \mathfrak{su}(2)_\rho) \cong \mathbb{R} \tag{3-2}$$

is onto, as we will now explain. To set the stage, we assume without loss of generality that $\rho(\mu) = j$. Then $H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ is the group $Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ of cocycles, which consists of functions

$$\xi\colon \pi_1(S^3 \setminus K) \to \mathfrak{su}(2)$$

such that

$$\xi(gh) = \xi(g) + \mathrm{Ad}_{\rho(g)}\,\xi(h) \quad \text{for all } g, h,$$

modulo the group $B^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ of coboundaries, which have the form

$$\zeta^{\#}\colon \pi_1(S^3 \setminus K) \to \mathfrak{su}(2), \quad g \mapsto \zeta - \mathrm{Ad}_{\rho(g)}\,\zeta$$

for some $\zeta \in \mathfrak{su}(2)$. (Here we view $\mathfrak{su}(2)$ as the vector space of purely imaginary quaternions.) If $\zeta = ai + bj + ck$, then since $\rho(\mu) = j$ we have

$$\zeta^{\#}(\mu) = (ai + bj + ck) - \mathrm{Ad}_j(ai + bj + ck) = (ai + bj + ck) - (-ai + bj - ck) = 2(ai + ck),$$

so the j-component of $\zeta^{\#}(\mu)$ is zero. It follows that if $\xi \in Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ is a 1-cocycle for which $\xi(\mu)$ has nonzero j-component, then its restriction to an element of $Z^1(\mu; \mathfrak{su}(2)_\rho)$ is not a coboundary, so this is how we will prove the surjectivity of (3-2).

In order to determine $Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$, Heusener and Klassen take a Wirtinger presentation [19, Equation (5)] of the form

$$\pi_1(S^3 \setminus K) = \langle S_1, \dots, S_n \mid S_{j_\ell}^{\epsilon_\ell} S_\ell S_{j_\ell}^{-\epsilon_\ell} = S_{\ell+1},\ \ell = 1, \dots, n-1 \rangle$$

with each $\epsilon_\ell \in \{\pm 1\}$, and turn this into a system of $n-1$ linear equations in $n+1$ real variables $s_1, \dots, s_n, t$, given in [19, Equation (14)] as

$$\alpha(j_\ell, \ell) s_{j_\ell} - s_\ell - s_{\ell+1} - \epsilon_{j_\ell} \beta(j_\ell, \ell) t = 0, \quad 1 \le \ell \le n-1. \tag{3-3}$$

Writing $\rho(S_\ell) = \zeta_\ell j$ for a complex unit ζ_ℓ as in [19, Equation (6)], they show that a collection of values $\xi(S_\ell) = x_\ell i + y_\ell j + z_\ell k$ determines a cocycle if and only if

- we have $x_1 = x_2 = \dots = x_n$ (coming from [19, Equation (9)]), and
- if we write $y_\ell + z_\ell i = \zeta_\ell(t_\ell + s_\ell i)$, then $t_1 = t_2 = \dots = t_\ell$ have some common value t, and $(s_1, \dots, s_n, t)$ is a solution to (3-3).

Now $\dim H^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) = 1$ implies that $\dim Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho) = 4$, as explained after [19, Remark 5], so the space of solutions to (3-3) is 3-dimensional.

Suppose we fix $\mu = S_1$ and $\rho(\mu) = j$ as above, and let $\xi \in Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ be a cocycle corresponding to a solution $(s_1, \dots, s_n, t)$ of (3-3). Following the above notation, $\zeta_1 = 1$ and so $y_1 + z_1 i = t_1 + s_1 i$; this means that the j-coordinate y_1 of $\xi(\mu)$ is nonzero if and only if $t_1 = t$ is. Thus it will suffice to find a solution of (3-3) with $t \neq 0$, since this will give a cocycle $\xi \in Z^1(S^3 \setminus K; \mathfrak{su}(2)_\rho)$ whose restriction to $Z^1(\mu; \mathfrak{su}(2)_\rho)$ is not a coboundary.

The proof of [19, Theorem 1] concludes by determining that the simpler system

$$\alpha(j_\ell, \ell) s_{j_\ell} - s_\ell - s_{\ell+1} = 0, \quad 1 \leq \ell \leq n-1 \tag{3-4}$$

of $n-1$ equations in n variables has at most a 2-dimensional solution space (its rank is at least $n-2$). If every solution to (3-3) has $t = 0$, then it follows that the space of solutions to (3-3) is also at most 2-dimensional, since (by forgetting the t coordinate) it injects into the space of solutions to (3-4). But we know that (3-3) actually has a 3-dimensional solution space, so there must be a solution with $t \neq 0$. Thus the map (3-2) must be onto after all, verifying (b). □

We can use Proposition 3.1 to say a bit more about the instanton knot homology of knots which satisfy its hypotheses. We first recall that if K has genus g, then its instanton knot homology comes with a symmetric Alexander decomposition

$$KHI(K) = \bigoplus_{i=-g}^{g} KHI(K, i),$$

each of whose summands has a natural $\mathbb{Z}/2\mathbb{Z}$-grading. Kronheimer and Mrowka [27] and Lim [33] proved that this recovers the Alexander polynomial $\Delta_K(t)$, by the formula

$$\pm\Delta_K(t) = \sum_{i=-g}^{g} \chi(KHI(K, i)) \cdot t^i, \tag{3-5}$$

where the sign ambiguity comes from using different conventions for the grading.

Theorem 3.3 *Let $K \subset S^3$ be an* SU(2)*-simple knot for which* $\det(K)$ *is prime. Then each Alexander summand $KHI(K, i)$ is supported in a single $\mathbb{Z}/2\mathbb{Z}$ grading. Moreover, if K has genus g and Alexander polynomial*

$$\Delta_K(t) = \sum_{i=-g}^{g} a_i t^i,$$

then the following must be true:

- $a_i = (-1)^{i+\sigma(K)/2} \dim_{\mathbb{Q}} KHI(K, i)$ *for all i, and*
- $|a_g| \geq 1$, *with equality if and only if K is fibered.*

Proof Proposition 3.1 says that $\dim_{\mathbb{Q}} KHI(K) = \det(K)$. Now substituting $t = -1$ into (3-5), taking absolute values, and applying the triangle inequality gives

$$\det(K) = \left| \sum_{i=-g}^{g} \chi(KHI(K,i)) \cdot (-1)^i \right| \le \sum_{i=-g}^{g} \left|(-1)^i \chi(KHI(K,i))\right| \le \sum_{i=-g}^{g} \dim_{\mathbb{Q}} KHI(K,i) = \det(K).$$

In particular, equality must hold in both of the inequalities above. We deduce from the second one that $|\chi(KHI(K,i))| = \dim_{\mathbb{Q}} KHI(K,i)$ for each i, so each $KHI(K,i)$ is supported in a single mod 2 grading. The first one says that all of the nonzero terms $(-1)^i \chi(KHI(K,i))$ must have the same sign, say $(-1)^{\delta_0}$, and then the mod 2 grading of each $KHI(K,i)$ is $i + \delta_0$. Then

$$\chi(KHI(K,i)) = (-1)^{i+\delta_0} \dim_{\mathbb{Q}} KHI(K,i),$$

so we substitute this information back into (3-5) to get

$$\pm\Delta_K(t) = \sum_{i=-g}^{g} \bigl((-1)^{i+\delta_0} \dim_{\mathbb{Q}} KHI(K,i)\bigr) t^i.$$

Let δ be either δ_0 or $\delta_0 + 1$, depending on whether the left side is $+\Delta_K(t)$ or $-\Delta_K(t)$. Then

$$a_i = (-1)^{i+\delta} \dim_{\mathbb{Q}} KHI(K,i)$$

for all i, and so we have

$$\Delta_K(-1) = \sum_{i=-g}^{g} (-1)^i a_i = (-1)^{\delta} \sum_{i=-g}^{g} \dim_{\mathbb{Q}} KHI(K,i) = (-1)^{\delta} \cdot \det(K).$$

But we also know in general that $\Delta_K(-1) = (-1)^{\sigma(K)/2} \det(K)$: this follows from the congruence (3-1) and the fact that

$$\Delta_K(-1) = a_0 + \sum_{i=1}^{g} ((-1)^i + (-1)^{-i}) a_i = a_0 + 2\sum_{i=1}^{g} (-1)^i a_i$$

is congruent modulo 4 to $a_0 + 2\sum_{i=1}^{g} a_i = \Delta_K(1) = 1$. Thus we have $(-1)^{\delta} = (-1)^{\sigma(K)/2}$, completing the determination of a_i in terms of $KHI(K,i)$.

The last detail is the claim about $|a_g|$, which is equal to $\dim_{\mathbb{Q}} KHI(K,g)$. We know that this dimension is strictly positive [28, Proposition 7.6], and that equality holds if and only if K is fibered [27, Proposition 4.1], so the claim follows immediately. □

4 Symplectic fillings of SU(2)-abelian manifolds

An *instanton L-space* is a rational homology 3-sphere Y whose framed instanton homology has the smallest possible rank, namely $\dim_{\mathbb{Q}} I^{\#}(Y) = |H_1(Y)|$. By analogy with Heegaard Floer homology [38, Theorem 1.4], we expect that any symplectic filling of an instanton L-space should have negative definite intersection form. We do not prove this here, but in this section we will prove this claim for a restricted class of instanton L-spaces which will suffice for our purposes. We begin with the following definition.

Definition 4.1 [10] An SU(2)-abelian rational homology 3-sphere Y has *cyclically finite* fundamental group if for every representation

$$\rho\colon \pi_1(Y) \to \mathrm{SU}(2),$$

the finite cover $\widetilde{Y}$ whose fundamental group $\pi_1(\widetilde{Y})$ is the kernel of

$$\operatorname{ad}\rho\colon \pi_1(Y) \to \mathrm{SO}(3)$$

is a rational homology sphere.

For example, in Definition 4.1 it suffices to know that the universal abelian cover of Y is a rational homology sphere. The following are Proposition 4.9 and Corollary 4.10 of [3].

Lemma 4.2 [3] *Suppose that Y is an* SU(2)*-abelian rational homology sphere, and that $|H_1(Y)| \le 5$ or that $H_1(Y)$ is cyclic of order a prime power. Then $\pi_1(Y)$ is cyclically finite.*

In particular, [3, Theorem 4.6] says that if Y is an SU(2)-abelian rational homology sphere and $\pi_1(Y)$ is cyclically finite, then Y is an instanton L-space. We now prove an analogue of [38, Theorem 1.4] for such manifolds.

Proposition 4.3 *Let Y be an* SU(2)*-abelian rational homology sphere, and suppose that $\pi_1(Y)$ is cyclically finite. Then any simply connected, weak symplectic filling (X,ω) of a contact structure on Y must satisfy $b_2^+(X) = 0$.*

Proof Let (X,ω) be a weak symplectic filling of (Y,ξ), with $\pi_1(X) = 1$, and suppose that $b_2^+(X) \ge 1$. We attach a Weinstein 2-handle H along a Legendrian torus knot $T_{2,5}$ with Thurston–Bennequin invariant 3, inside a Darboux ball $B^3 \subset Y$, to get a symplectic cobordism to a new (Y',ξ'). Then following Eliashberg [14], we can construct a concave symplectic filling (Z_0,ω_0) of (Y',ξ') from an open book decomposition, by first performing 0-surgery on the binding B and then extending the symplectic form across a Lefschetz fibration with concave boundary $Y'_0(B)$. Kronheimer and Mrowka [26, Lemma 11] observed that one can take $H_1(Z_0) = 0$ by choosing a collection of vanishing cycles which generate the homology of the base, but the same argument with π_1 instead of H_1 allows us to take Z_0 to be simply connected.

With this in mind, we write $Z = H \cup_{Y'} Z_0$, and note that Z contains a surface of positive self-intersection: this is the core of the handle H, glued to a Seifert surface for the attaching curve $T_{2,5}$ in Y. Thus we have a closed symplectic manifold $W = X \cup_Y Z$ such that

- both X and Z are simply connected, and
- both $b_2^+(X)$ and $b_2^+(Z)$ are strictly positive.

We choose a class $w \in H^2(W;\mathbb{Z})$ and a symplectic form on W with integral cohomology class, and if the latter class has Poincaré dual $h \in H_2(W;\mathbb{Z})$ then we know from [28; 41] that the Donaldson invariants

$$D^w_W(h^k)$$

are nonzero for all large enough k in some residue class mod 4.

On the other hand, the assumption that Y is SU(2)-abelian means that all critical points of the Chern–Simons functional on Y are reducible, and in fact the Chern–Simons functional on Y is Morse–Bott since $\pi_1(Y)$ is cyclically finite [3, Corollary 4.5]. Under these circumstances, Austin and Braam [1, Proposition 6.3] proved that any Donaldson invariant of the form $D^w_W(x^k)$ must vanish, where $x \in H_2(W;\mathbb{Z})$ is a class whose restriction to $H_1(Y)$ is zero. Letting $n = |H_1(Y)|$, the restriction of $x = nh$ to $H_1(Y)$ is certainly zero, so the Donaldson invariant

$$D^w_W((nh)^k) = n^k \cdot D^w_W(h^k)$$

must be zero as well, but this contradicts the nonvanishing result mentioned above. It follows that we must have had $b_2^+(X) = 0$ after all. □

Remark 4.4 In the proof of Proposition 4.3, we can choose the class $w \in H^2(W;\mathbb{Z})$ to be nonzero mod 2 on each of X and Z, and then the desired vanishing for the invariants $D^w_W(h^k)$ follows equally well from the "dimension counting argument" outlined by Donaldson following the statement of [12, Theorem 4.9]. The key facts we need to carry out this argument are that every flat SU(2) connection A on Y is reducible and nondegenerate (meaning that $H^1_A(Y) = 0$), or in other words that Y is SU(2)-abelian and $\pi_1(Y)$ is cyclically finite. This allows us to glue the relevant moduli spaces of instantons over X and Z, each with the same asymptotic limit at Y, as described for example in [13, Section 4.4.1].

We remark that Proposition 4.3 should apply equally well to fillings that are not simply connected, but we do not need this more general statement here. We also observe the following corollary; cf [30, Theorem 2.1] or [38, Theorem 1.4].

Corollary 4.5 *Let Y be an* SU(2)*-abelian rational homology sphere. If $\pi_1(Y)$ is cyclically finite, then Y does not admit a coorientable taut foliation.*

Proof If Y has a coorientable taut foliation then we can perturb it to a weakly fillable contact structure [9; 15; 23], and use [14] to construct a simply connected weak symplectic filling of (Y,ξ) with $b^+ > 0$, exactly as in the proof of [26, Proposition 15]. This contradicts Proposition 4.3. □

5 Homology of orders 3 and 5

We can now prove the second case of Theorem 1.1, which asserts that if Y has Heegaard genus 2 and first homology of order 3, then it is not SU(2)-abelian. Here we will need the fact, proved in [5], that the dimension of *KHI* is enough to detect the trefoils.

Lemma 5.1 *If $K \subset S^3$ is an* SU(2)*-simple knot of determinant* 3*, then K is a trefoil.*

Proof Proposition 3.1 says that $\dim_{\mathbb{Q}} \mathit{KHI}(K) = 3$, and by [5, Theorem 1.6] the only knots with three-dimensional instanton knot homology are the trefoils. □

Proof of Theorem 1.1 in the case $|H_1(Y)| = 3$ Suppose that Y has Heegaard genus 2 and is SU(2)-abelian, and that $H_1(Y)$ has order 3. By Proposition 2.1, we can write $Y \cong \Sigma_2(K)$ for an SU(2)-simple 3-bridge knot K of determinant 3. But Lemma 5.1 says that there are no such knots, since the trefoils have bridge index 2, so such Y cannot exist. □

We cannot quite prove the analogue of Lemma 5.1 for knots of determinant 5, but we can achieve a partial characterization. We recall a bit of terminology first: an *instanton L-space knot* is a knot in S^3 on which some positive Dehn surgery produces an instanton L-space. Such knots are known to be fibered and strongly quasipositive [6, Theorem 1.15].

Lemma 5.2 *Let K be an* SU(2)*-simple knot of determinant* 5 *other than the figure eight or $T_{\pm 2,5}$. Then, after possibly replacing it with its mirror, K is an instanton L-space knot, hence fibered and strongly quasipositive, with Seifert genus $g = g(K)$ at least* 3 *and signature $\sigma(K) \in \{-4, 0, 4\}$. Moreover, its branched double cover $Y = \Sigma_2(K)$ bounds a simply connected Stein domain X with*

$$b_2^+(X) = g + \tfrac{1}{2}\sigma(K) \geq 1.$$

Proof Proposition 3.1 says that $|\sigma(K)| \leq 4$ and $\sigma(K) \equiv \det(K) - 1 \equiv 0 \pmod 4$, so $\sigma(K)$ must be either 0 or ± 4; and that $\dim_{\mathbb{Q}} \mathit{KHI}(K) = 5$. We will use the Alexander decomposition of $\mathit{KHI}(K)$ to understand more about K.

First, we suppose that $\dim_{\mathbb{Q}} \mathit{KHI}(K, g) > 1$. Then since $\dim_{\mathbb{Q}} \mathit{KHI}(K) = 5$ we must have

$$\mathit{KHI}(K; \mathbb{Q}) \cong \mathbb{Q}_g^2 \oplus \mathbb{Q}_0 \oplus \mathbb{Q}_{-g}^2,$$

where the subscripts denote the Alexander grading. But then Theorem 3.3 and the fact that $\frac{1}{2}\sigma(K)$ is even would tell us that

$$\Delta_K(t) = 2(-t)^g + 1 + 2(-t)^{-g},$$

which is impossible since $\Delta_K(1) = 1$. It follows that $\dim_{\mathbb{Q}} \mathit{KHI}(K, g) = 1$ and that K is fibered. Moreover, if K were fibered of genus 1 then it would be the figure eight (the trefoils having determinant 3), but we have excluded this possibility. Therefore K must have genus $g \geq 2$.

Now since K is fibered we can apply [5, Theorem 1.7] to show that $\mathit{KHI}(K, g-1) \neq 0$, and since $\mathit{KHI}(K)$ has total dimension 5, we have

$$\mathit{KHI}(K; \mathbb{Q}) \cong \mathbb{Q}_g \oplus \mathbb{Q}_{g-1} \oplus \mathbb{Q}_0 \oplus \mathbb{Q}_{1-g} \oplus \mathbb{Q}_{-g}.$$

According to work of Li and Liang [31, Theorem 1.4], this tells us that K is an instanton L-space knot, up to mirroring; we replace K with its mirror as needed to guarantee that it has a positive instanton L-space surgery. Then K is also strongly quasipositive [6, Theorem 1.15], as claimed. Moreover, Farber, Reinoso, and Wang [16, Corollary 1.4] proved (building on a partial characterization in [2, Section 2]) that $T_{2,5}$ is the only genus-2 instanton L-space knot, and we have assumed that K is not $T_{2,5}$, so in fact $g \geq 3$.

Boileau, Boyer, and Gordon [8, Proposition 6.1] proved that since K is strongly quasipositive, if its branched double cover is a Heegaard Floer L-space, then $|\sigma(K)| = 2g(K)$. We adapt part of their argument to the instanton setting: they push a genus-g Seifert surface F for $K \subset S^3$ into the 4-ball so that F is the intersection of a smooth complex curve with that ball, and then observe that $X = \Sigma_2(F)$ is a Stein manifold [17; 39] with boundary $\Sigma_2(K)$. Then X is simply connected [8, Section 3.1.1], and has $b_3(X) = 0$ because it is Stein, and this implies that $b_2(X) = 2g$ by [8, Remark 3.7]. Its signature $\sigma(X)$ is equal to $\sigma(K)$ [22, Lemma 1.1], so we determine that X gives a Stein filling of $\Sigma_2(K)$ satisfying

$$b_2^+(X) = g + \tfrac{1}{2}\sigma(K), \quad b_2^-(X) = g - \tfrac{1}{2}\sigma(K),$$

as claimed. Moreover, since $g \geq 3$ and $\sigma(K) \geq -4$ we conclude that $b_2^+(X)$ is positive. □

We can now apply Proposition 2.1 to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 We have already proved the cases where $H_1(Y;\mathbb{Z})$ has order 1 (in Section 2) or 3, so now we suppose that Y has Heegaard genus 2, that $|H_1(Y;\mathbb{Z})| = 5$, and that Y is SU(2)-abelian. Then Proposition 2.1 says that Y is the branched double cover of an SU(2)-simple 3-bridge knot K, with $\det(K) = 5$. Since K is not a 2-bridge knot, it is neither the figure eight nor the torus knot $T_{\pm 2,5}$.

We now apply Lemma 5.2, replacing K with its mirror (and hence Y with $-Y$) if necessary to ensure that K is an instanton L-space knot. The conclusion of this lemma says that Y has a simply connected Stein filling (X, J) satisfying $b_2^+(X) \geq 1$. But Lemma 4.2 guarantees that $\pi_1(Y)$ is cyclically finite, so this contradicts Proposition 4.3, and thus such Y cannot exist after all. □

Remark 5.3 Unlike in the cases $\det(K) = 1$ and $\det(K) = 3$, we have not shown that there are no SU(2)-simple knots of determinant 5 and bridge index greater than 2. What we have shown is that if such a knot K exists, then its branched double cover $Y = \Sigma_2(K)$ cannot be SU(2)-abelian. We saw in the proof of Proposition 2.1 that this cannot happen if K is a 3-bridge knot, but K could still have bridge index 4 or greater, as per Remark 2.2.

6 Meridian-traceless representation varieties and the trefoils

Given a knot $K \subset S^3$, we define the representation variety

$$\mathcal{R}(K, i) = \{\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2) \mid \rho(\mu) = i\},$$

where μ is a fixed meridian. This carries an action of

$$U(1)/(\mathbb{Z}/2\mathbb{Z}) = \{e^{i\theta}\}/\{\pm 1\}$$

by conjugation, which fixes the unique reducible representation in $\mathscr{R}(K,i)$ and partitions the irreducible representations into S^1 orbits. In this section, we prove Theorem 1.3, which asserts that there is a single S^1 orbit of irreducible representations if and only if K is a trefoil.

Theorem 6.1 *If $\mathscr{R}(K,i) \cong \{*\} \sqcup S^1$, then K is a trefoil.*

Proof Suppose for now that K is SU(2)-simple, so that every irreducible, meridian-traceless representation $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2)$ has binary dihedral image. We recall that according to Klassen [24, Theorem 10], there are exactly $\frac{1}{2}(\det(K)-1)$ such ρ up to conjugacy. But by hypothesis there is a unique conjugacy class of such ρ, so now K is an SU(2)-simple knot with $\det(K)=3$, and Lemma 5.1 says that K must therefore be a trefoil.

We now show that K is indeed SU(2)-simple. Fixing the nontrivial central character

$$\chi\colon \pi_1(S^3 \setminus K) \twoheadrightarrow H_1(S^3 \setminus K) \cong \mathbb{Z} \to \{\pm 1\},$$

we can take any representation in $\mathscr{R}(K,i)$, say

$$\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{SU}(2),$$

and consider the representation

$$\rho'(\gamma) = j \cdot \chi(\gamma)\rho(\gamma) \cdot j^{-1}.$$

We note that $\rho' \in \mathscr{R}(K,i)$, since $\chi(\mu) = -1$ and $\rho(\mu) = i$ imply that

$$\rho'(\mu) = j \cdot (-i) \cdot j^{-1} = i.$$

The operation $\rho \mapsto \rho'$ defines an involution not just on $\mathscr{R}(K,i)$ but on the entire meridian-traceless $\mathrm{SL}_2(\mathbb{C})$ character variety of K, and Nagasato and Yamaguchi [37, Proposition 3] proved that the irreducible characters fixed by this involution are precisely the metabelian ones, which are the same as the binary dihedral characters [37, Proposition 2].

If ρ is conjugate to ρ' then they have the same characters, so the above says that $\operatorname{tr}\rho$ is equal to the character of an irreducible binary dihedral representation ρ_0, which we can take to have image in SU(2). Facts (1) and (2) in the proof of [24, Proposition 15] say that ρ and ρ_0 are conjugate in $\mathrm{SL}_2(\mathbb{C})$, and then that they are conjugate by an element of SU(2). Thus ρ can only be conjugate to ρ' if ρ is itself a binary dihedral representation. But if K is not SU(2)-simple then we can choose some irreducible $\rho \in \mathscr{R}(K,i)$ which is not binary dihedral, and the corresponding $\rho' \in \mathscr{R}(K,i)$ will not be conjugate to ρ. In this case $\mathscr{R}(K,i)$ has at least two distinct S^1 orbits, which contradicts the hypothesis that $\mathscr{R}(K,i) \cong \{*\} \sqcup S^1$. We conclude that K must be SU(2)-simple after all, and hence K is a trefoil as argued above. □

7 Homology of order 2

In this section we will prove Proposition 7.5, which classifies SU(2)-simple links of determinant 2. We will then use this to prove Theorem 1.5, constructing nonabelian SO(3)-representations of arbitrary 3-manifolds Y with Heegaard genus 2 and first homology of order 2.

Lemma 7.1 *If $L \subset S^3$ is a link of $\ell \geq 1$ components, then $\det(L)$ is a multiple of $2^{\ell-1}$. If $\ell = 1$ then $\det(L)$ is odd, while if $\ell = 2$ and we write $L = L_1 \cup L_2$, then*

$$\det(L) \equiv 2\mathrm{lk}(L_1, L_2) \pmod 4.$$

Proof Let $Y = \Sigma_2(L)$. We will assume $\det(L) \neq 0$ for now, since otherwise $2^{\ell-1}$ divides it anyway. Then $H_1(Y;\mathbb{Z})$ is finite of order $\det(L)$, and according to the universal coefficient theorem and (2-2),

$$H_1(Y;\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z}) \cong H_1(Y;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\ell-1}.$$

Writing $H_1(Y;\mathbb{Z}) \cong \bigoplus_i \mathbb{Z}/d_i\mathbb{Z}$ as a sum of cyclic groups, so that $\det(L) = \prod_i d_i$, each $\mathbb{Z}/d_i\mathbb{Z}$ summand contributes a $\mathbb{Z}/2\mathbb{Z}$ summand to the tensor product on the left if d_i is even, and the trivial group otherwise. Thus there must be exactly $\ell - 1$ even values of d_i, and it follows that $2^{\ell-1}$ divides $\prod_i d_i = \det(L)$. If $\ell = 1$ then all of the d_i are odd, so their product $\det(L)$ is odd as well.

Now suppose that $\ell = 2$ and that $\det(L)$ is arbitrary. In this case it equals

$$|H_1(Y;\mathbb{Z})| = 2\Delta_L(-1,-1),$$

where Δ_L is the multivariable Alexander polynomial of $L = L_1 \cup L_2$, as shown by Hosokawa and Kinoshita [21]. On the other hand, Torres [44] proved that the linking number $\mathrm{lk}(L_1, L_2)$ satisfies

$$\Delta_L(1,1) = \pm\mathrm{lk}(L_1, L_2),$$

so in particular

$$\Delta_L(-1,-1) \equiv \Delta_L(1,1) \equiv \mathrm{lk}(L_1, L_2) \pmod 2.$$

Thus $\det(L) \equiv 2\mathrm{lk}(L_1, L_2) \pmod 4$ as claimed. □

In what follows we will let

$$D = \{e^{i\theta}\} \cup \{e^{i\theta} j\} \subset \mathrm{SU}(2)$$

denote the binary dihedral subgroup of SU(2).

Lemma 7.2 *Let $L \subset S^3$ be a link with $\det(L) \equiv 2 \pmod 4$, and fix a meridian-traceless representation*

$$\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2)$$

with nonabelian image in the binary dihedral group D. Then ρ sends every meridian of L to an element of the form $e^{i\theta} j$.

Proof Lemma 7.1 tells us that L must have exactly two components, since $\det(L)$ is neither odd nor a multiple of 4, and that if we write $L = L_1 \cup L_2$ then $\mathrm{lk}(L_1, L_2)$ is odd. We will consider the Wirtinger presentation associated to some diagram of L, which is generated by the meridians around each strand. From this presentation we know that all of the meridians of L_1 are conjugate to each other, and likewise the meridians of L_2 are all mutually conjugate.

Both the normal subgroup $N = \{e^{i\theta}\}$ of D and its coset $Nj = \{e^{i\theta} j\}$ are fixed by conjugation in D, so either ρ sends all the meridians of L_1 to elements of N, or it sends them all to elements of Nj. The same is true for L_2. If ρ sends both sets of meridians to Nj then we are done; if instead both sets are sent to N then all of $\mathrm{Im}(\rho)$ lies in N, contradicting our assumption that $\mathrm{Im}(\rho)$ is nonabelian. Thus we will assume without loss of generality that the meridians of L_1 are sent to $N = \{e^{i\theta}\}$, while the meridians of L_2 are sent to $Nj = \{e^{i\theta} j\}$.

We now walk along L_1 in a circle and observe the sequence of values of ρ at each strand. Each of these values is either i or $-i$, since these are the only purely imaginary elements of N. If two such strands μ and μ' of L_1 meet at a crossing c, where the overcrossing strand has meridian μ_c, then the Wirtinger presentation says that

$$\rho(\mu') = \rho(\mu_c^{\pm 1}) \cdot \rho(\mu) \cdot \rho(\mu_c^{\mp 1}).$$

If μ_c belongs to L_1 then its image lies in N, hence commutes with $\rho(\mu) = \pm i$, and so $\rho(\mu') = \rho(\mu)$. On the other hand, if μ_c belongs to L_2 then we can write $\rho(\mu_c^{\pm 1}) = e^{i\theta} j$ for some θ, and we have

$$e^{i\theta} j \cdot (\pm i) \cdot (e^{i\theta} j)^{-1} = e^{i\theta} j \cdot (\pm i) \cdot (-j e^{-i\theta}) = \mp i,$$

so then $\rho(\mu') = -\rho(\mu)$. In particular, as we travel along L_1, the value of ρ at each meridian changes sign every time we cross under L_2, and it stays the same otherwise.

In order to assign $\pm i$ to each strand of L_1 in a consistent way, it follows that L_1 must cross under L_2 an even number of times. Changing one of these undercrossings to an overcrossing changes $\mathrm{lk}(L_1, L_2)$ by ± 1; if instead we change all of them, then the total change must be even. But doing this causes L_1 and L_2 to be unlinked from each other, since now L_1 always crosses over L_2. Therefore $\mathrm{lk}(L_1, L_2)$ differs from 0 by an even number. In other words, the linking number $\mathrm{lk}(L_1, L_2)$ is even, but we saw at the beginning of the proof that it must also be odd and therefore we have a contradiction. □

Lemma 7.3 *Let $L \subset S^3$ be a link of at least two components, with $\det(L) \neq 0$, and let*

$$\rho\colon \pi_1(S^3 \setminus L) \to D \subset \mathrm{SU}(2)$$

be a meridian-traceless representation with nonabelian image. Suppose that ρ sends every meridian of L to an element of the form $e^{i\theta} j$. Then $\mathrm{Im}(\rho)$ is a finite group of order dividing $2\det(L)$.

Proof We argue exactly as in the proof of [24, Theorem 10]. Suppose that we fix a diagram of L, and let $\mu_1, \dots, \mu_s$ denote the meridional generators of the corresponding Wirtinger presentation. If we have a meridian-traceless, nonabelian representation

$$\rho\colon \pi_1(S^3 \setminus L) \to D \subset \mathrm{SU}(2)$$

such that $\rho(\mu_r) = e^{i\theta_r} j$ for each $r = 1, \dots, s$, then the relations in the Wirtinger presentation are equivalent to a system of linear equations, one for each of $r-1$ crossings of the diagram, which at a given crossing take the form

$$\theta_a - 2\theta_b + \theta_c \equiv 0 \pmod{2\pi}$$

if the strand corresponding to μ_b crosses over the strands for μ_a and μ_c. We can conjugate ρ by $e^{-i\theta_s/2} \in D$ to replace a solution $(\theta_1, \dots, \theta_{s-1}, \theta_s)$ with $(\theta_1 - \theta_s, \dots, \theta_{s-1} - \theta_s, 0)$, using the relation

$$e^{-i\theta_s/2} \cdot e^{i\alpha} j \cdot (e^{-i\theta_s/2})^{-1} = e^{i(\alpha-\theta_s)} j.$$

Thus up to conjugacy we are free to take $\theta_s = 0$, and then nonabelian representations will correspond to solutions $(\theta_1, \dots, \theta_{s-1})$ that are not identically zero modulo π.

The $(s-1) \times (s-1)$ integer matrix A describing this system is obtained from an Alexander matrix by setting $t = -1$, so it has determinant

$$\det(A) = \pm \Delta_L(-1) = \pm \det(L).$$

In particular, if we write $\Delta = \det(L)$ for readability then any nonzero solution satisfies

$$(\theta_1, \dots, \theta_{s-1}) \in \frac{1}{\det(A)} (2\pi\mathbb{Z})^{s-1} = \frac{2\pi}{\Delta} \mathbb{Z}^{s-1},$$

and in this case the image $\rho(\mu_r)$ of each meridian lies in the set

(7-1) $$\{e^{2\pi i m/\Delta} \mid 0 \le m < \Delta\} \cup \{e^{2\pi i m/\Delta} \cdot j \mid 0 \le m < \Delta\}.$$

Since $\Delta = \det(L)$ is even, this set is closed under multiplication; hence it is a binary dihedral subgroup of D, with order $2\det(L)$. Then $\operatorname{Im}(\rho)$ is a subgroup of this, so its order divides $2\det(L)$, as promised. □

Remark 7.4 If L were a knot, then the set (7-1) would not be closed under multiplication, because it does not contain -1 if Δ is odd. But adjoining -1 turns it into a subgroup, so in this case we would conclude that $\operatorname{Im}(\rho)$ has order dividing $4\det(L)$ instead.

Proposition 7.5 *Let $L \subset S^3$ be an* SU(2)*-simple link of determinant* 2*. Then L is a Hopf link.*

Proof Suppose that L is not a Hopf link. Xie and Zhang [45, Theorem 1.5] proved that the following are equivalent:

- L is not an unknot, a Hopf link, or a connected sum of Hopf links.
- There is an irreducible representation
$$\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2)$$
  sending every meridian of L to a traceless element.

Now we know by Lemma 7.1 that L must be a two-component link, and we have assumed that it is not a Hopf link, so such a representation ρ must exist, with nonabelian image.

Suppose in addition that after possibly replacing ρ with a conjugate, the image $\operatorname{Im}(\rho)$ lies in the binary dihedral group D. Lemma 7.2 says that ρ sends every meridian of L to an element of the form $e^{i\theta} j$, so by Lemma 7.3 we know that $\operatorname{Im}(\rho)$ is a finite group of order dividing $2\det(L) = 4$. But any such group must be abelian, so we have a contradiction. Thus ρ cannot be conjugate to a binary dihedral representation, and this proves that L is not SU(2)-simple. □

We are now ready to prove Theorem 1.5, asserting that if Y has Heegaard genus 2 and $H_1(Y)$ has order 2 then there is a nonabelian representation $\pi_1(Y) \to \mathrm{SO}(3)$. To do so, we recall how to construct a map

$$B\colon \left\{ \begin{matrix} \text{meridian-traceless representations} \\ \rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2) \end{matrix} \right\} \to \mathrm{Hom}\big(\pi_1(\Sigma_2(L)), \mathrm{SO}(3)\big). \tag{7-2}$$

We have a short exact sequence

$$1 \to \pi_1(X_2) \to \pi_1(S^3 \setminus L) \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

where the map to $\mathbb{Z}/2\mathbb{Z}$ sends the meridian μ_i of each component L_i (say $i = 1, \dots, r$) to 1, and where X_2 is the corresponding double cover of the complement of L. Then we obtain $Y \cong \Sigma_2(L)$ by Dehn filling X_2 along each lift of μ_i^2, so that

$$\pi_1(Y) \cong \frac{\pi_1(X_2)}{\langle\langle \mu_1^2, \mu_2^2, \dots, \mu_r^2 \rangle\rangle}.$$

Now given a meridian-traceless ρ, we know that $\rho(\mu_i^2) = -1$ for each i, so

$$\mathrm{ad}\,\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SO}(3)$$

sends each μ_i^2 to the identity; thus its restriction to $\pi_1(X_2)$ induces a well-defined

$$\tilde{\rho}\colon \pi_1(Y) \to \mathrm{SO}(3).$$

The map (7-2) is defined by $B(\rho) = \tilde{\rho}$.

The following is part of the proof of [46, Proposition 3.1].

Lemma 7.6 [46] *Suppose that* $\det(L) \neq 0$, *and let* $\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2)$ *be a meridian-traceless representation with nonabelian image. If* $B(\rho)$ *has abelian image, then* ρ *is conjugate to a representation with image in the binary dihedral subgroup*

$$D = \{e^{i\theta}\} \cup \{e^{i\theta} j\} \subset \mathrm{SU}(2).$$

The condition $\det(L) \neq 0$ is needed to ensure that $\mathrm{Im}(B(\rho))$ is a *finite* subgroup of SO(3).

Proof of Theorem 1.5 Just as in the proof of Proposition 2.1, we can appeal to Birman and Hilden [7, Theorem 1] to write

$$Y \cong \Sigma_2(L)$$

for some 3-bridge link L in S^3, with determinant $\det(L) = |H_1(Y;\mathbb{Z})| = 2$ and hence (by Lemma 7.1) with two components. If L were the Hopf link then $Y \cong \Sigma_2(L)$ would be $\mathbb{RP}^3$, whose Heegaard genus is only 1, so now Proposition 7.5 says that L is not SU(2)-simple. In particular, there is a meridian-traceless, nonabelian representation

$$\rho\colon \pi_1(S^3 \setminus L) \to \mathrm{SU}(2)$$

whose image is not conjugate to a subgroup of the binary dihedral group $D \subset \mathrm{SU}(2)$.

We now use ρ to construct a representation $\tilde{\rho} = B(\rho)\colon \pi_1(Y) \to \mathrm{SO}(3)$ as in (7-2). Since ρ does not have binary dihedral image, Lemma 7.6 says that $\tilde{\rho}$ cannot have abelian image and so this proves that Y is not SO(3)-abelian. □

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