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**Rank-preserving additions for topological vector bundles,  
after a construction of Horrocks**

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# Rank-preserving additions for topological vector bundles, after a construction of Horrocks

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We produce group structures on certain sets of topological vector bundles of fixed rank. In particular, we put a group structure on complex rank 2 bundles on  $\mathbb{C}P^3$  with fixed first Chern class. We show that this binary operation coincides with a construction on locally free sheaves due to Horrocks, provided the latter is defined. Using similar ideas, we give group structures on certain sets of rank 3 bundles on  $\mathbb{C}P^5$ .

These groups arise from the study of relative infinite loop space structures on truncated diagrams. Specifically, we show that the  $(2n-2)$ -truncation of an  $n$ -connective map  $X \rightarrow Y$  with a section is a highly structured group object over the  $(2n-2)$ -truncation of  $Y$ . Applying these results to classifying spaces yields the group structures of interest.

55P99, 55R25, 55R35; 55P47, 55Q05

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## 1 Introduction

Despite the importance of vector bundles in geometry and topology, there are few explicit methods to produce them. On complex projective spaces, the simplest complex bundles to write down are sums of line bundles. Indecomposable bundles are difficult to describe explicitly, but there are some famous examples: one is that of the Horrocks–Mumford bundle of rank 2 on  $\mathbb{C}P^4$  [7]; another is the Horrocks bundle of rank 3 on  $\mathbb{C}P^5$  [6].

In [5], Horrocks takes another approach and constructs new algebraic vector bundles from given ones using a modified extension group procedure. Horrocks’ construction takes as input complex rank 2 algebraic

bundles  $V$  and  $W$  on  $\mathbb{C}P^3$ , which must have the same first Chern class and satisfy some technical hypotheses, and outputs another complex rank 2 algebraic vector bundle with the same first Chern class. We will write  $V +_H W$  for the output bundle, although the construction does not define a group structure.

Atiyah and Rees show that Horrocks' construction produces essentially all topological equivalence classes of complex rank 2 bundles on  $\mathbb{C}P^3$  from the simplest ones:

**Theorem 1.1** [2, Theorem 1.1] *Any complex rank 2 topological vector bundle on  $\mathbb{C}P^3$  can be obtained from a sum of line bundles by iteratively*

- *tensoring by a line bundle, and*
- *applying Horrocks' construction.*

**Remark 1.2** This result shows that every topological equivalence class of complex rank 2 vector bundles on  $\mathbb{C}P^3$  admits an algebraic representative.

The algebra involved in Horrocks' construction is quite specialized and does not directly generalize to produce algebraic vector bundles of other ranks or on other spaces. However, we explore Horrocks' construction from a homotopical vantage point and show that, from this perspective, it does generalize.

**Proposition 1.3** (topological Horrocks addition) *Fix an integer  $a_1 \in \mathbb{Z}$ . Let  $\mathcal{G}_{a_1}$  denote the set of topological isomorphism classes of complex rank 2 topological bundles on  $\mathbb{C}P^3$  with first Chern class equal to  $a_1$ .*

- (i)  *$\mathcal{G}_{a_1}$  carries an abelian group structure  $+_{a_1}$ , via an explicit construction on classifying spaces.*
- (ii) *The identity is  $L \oplus \underline{\mathbb{C}}$ , where  $L$  is the complex line bundle determined by  $c_1(L) = a_1$  and  $\underline{\mathbb{C}}$  is the trivial complex line bundle.*
- (iii) *The second Chern class defines a homomorphism  $c_2: \mathcal{G}_{a_1} \rightarrow H^4(\mathbb{C}P^3; \mathbb{Z})$ .*

**Remark 1.4** The existence of a group structure on the underlying set of  $\mathcal{G}_{a_1}$  can also be deduced by identifying elements of  $\mathcal{G}_{a_1}$  with twisted symplectic  $K$ -theory classes. It is not hard to show that this group structure agrees with ours, but our description provides a framework for comparison with Horrocks' construction and for generalization.

We show  $+_{a_1}$  addition agrees with Horrocks' construction as far as possible:

**Theorem 1.5** *Suppose that  $V$  and  $W$  are two rank 2 algebraic vector bundles such that  $c_1(V) = c_1(W) = a_1$  and such that the Horrocks sum  $V +_H W$  is defined. Then*

$$V +_H W \simeq V +_{a_1} W$$

*as topological vector bundles.*

The proof of Theorem 1.5 is conceptually simple. In [2], Atiyah and Rees show that a topological rank 2 bundle on  $\mathbb{C}P^3$  is determined by its first and second Chern classes together with a  $\mathbb{Z}/2$ -valued invariant called  $\alpha$ . So, to prove that  $V +_H W$  and  $V +_{a_1} W$  are topologically isomorphic, it suffices to show their Chern classes and  $\alpha$ -invariants agree.

The groups of Proposition 1.3 generalize as follows:

**Theorem 1.6** (topological Horrocks addition for rank 3 bundles) *Fix a complex rank 2 topological bundle  $V_0$  on  $\mathbb{C}P^5$ . Let  $\mathcal{G}_{V_0}$  denote the set of topological equivalence classes of complex rank 3 topological bundles on  $\mathbb{C}P^5$  with the same first and second Chern classes as  $V_0$ .*

- (i)  $\mathcal{G}_{V_0}$  carries an abelian group structure, via an explicit construction on classifying spaces.
- (ii) The identity is  $V_0 \oplus \mathbb{C}$ .
- (iii) The third Chern class defines a homomorphism  $c_3 : \mathcal{G}_{V_0} \rightarrow H^6(\mathbb{C}P^5; \mathbb{Z})$ .

**Remark 1.7** The underlying set of  $\mathcal{G}_{V_0}$  depends only on a choice of first and second Chern classes of  $V_0$ . However, we need an auxiliary rank 2 bundle to define the group operation.

### 1.1 Methods

Proposition 1.3 and Theorem 1.6 follow from a more general study of relative infinite loop space structures on truncated diagrams. We remind the reader of the following terminology:

**Definition 1.8** Let  $\text{Top}$  denote the category of topological spaces.

- (a) The  $m$ -truncation functor  $\tau_m : \text{Top} \rightarrow \text{Top}_{\leq m}$  is reflection onto the full subcategory  $\text{Top}_{\leq m}$  of spaces  $X$  with  $\pi_i X = 0$  for  $i > m$ . The functor  $\tau_m$  is left adjoint to the inclusion  $\text{Top}_{\leq m} \hookrightarrow \text{Top}$ .
- (b) A map  $f : X \rightarrow Y$  in  $\text{Top}$  is  $n$ -connective if the homotopy fiber of  $f$  has first nonzero homotopy group in degree  $n$ .
- (c) In any category, given objects  $X$  and  $Y$  and a morphism  $s : Y \rightarrow X$ , let  $X \cup_Y X$  denote the pushout of the diagram

$$\begin{array}{ccc} Y & \xrightarrow{s} & X \\ \downarrow s & & \\ X & & \end{array}$$

Let  $1 : X \rightarrow X$  denote the identity. The map  $\text{fold} : X \cup_Y X \rightarrow X$  is the dotted arrow making the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{s} & X \\ \downarrow s & & \downarrow 1 \\ X & \longrightarrow & X \cup_Y X \\ & \searrow 1 & \dashrightarrow \text{fold} \\ & & X \end{array}$$

In the homotopy category of spaces, our key technical result can be stated as follows:

**Proposition 1.9** *Let  $f : X \rightarrow Y$  be a map of pointed, simply connected spaces, with section  $s : Y \rightarrow X$ . Suppose that  $f$  is  $n$ -connective and let  $m := 2n - 2$ . Let  $\tau_m$  denote the  $m$ -truncation functor. Then*

$$\begin{array}{ccc} \tau_m X & \xrightarrow{\tau_m f} & \tau_m Y \\ & \swarrow \tau_m s & \end{array}$$

*is a group object in the homotopy category of  $m$ -truncated spaces over  $\tau_m Y$ , with binary operation given by truncating the diagram*

$$X \times_Y X \xleftarrow{(1 \cup s f) \times (s f \cup 1)} X \cup_Y X \xrightarrow{\text{fold}} X,$$

*where the dotted arrow exists only after truncating.*

Using the skeleton–truncation adjunction on the homotopy category of spaces, we obtain:

**Corollary 1.10** *Let  $f : X \rightarrow Y, s : Y \rightarrow X, n$  and  $m$  be as in Proposition 1.9. Let  $C$  be an  $m$ -skeletal space and fix  $c : C \rightarrow Y$ . Let  $[C, X]_c$  denote homotopy classes of maps  $g : C \rightarrow X$  making*

(1-1) 
$$\begin{array}{ccc} C & \xrightarrow{g} & X \\ & \searrow c & \downarrow f \\ & & Y \end{array}$$

*homotopy commutative. Then  $[C, X]_c$  is a group with identity  $s \circ c$ .*

**Remark 1.11** This corollary can be viewed as a relative version of Borsuk’s classical construction, as given in Spanier [13]. More recently, such ideas have been used by Asok and Fasel [1], for example.

It is rather tedious to prove Proposition 1.9 in a point–set way. Instead, we prove an  $(\infty, 1)$ -categorical refinement. As a consequence of  $n$ -connectivity, the fibers of  $\tau_m f : \tau_m X \rightarrow \tau_m Y$  are infinite loop spaces and the automorphisms of the fibers induced by paths in the base are infinite loop maps; this data should assemble to make  $\tau_m X \rightarrow \tau_m Y$  an infinite loop space over  $\tau_m Y$ . The machinery of an  $(\infty, 1)$ -categorical Grothendieck construction, for example straightening and unstraightening as in Lurie [9], allows us to make this precise.

**Proposition 1.12** *Let  $f : X \rightarrow Y$  be a map of pointed simply connected spaces with section  $s : Y \rightarrow X$ . Suppose  $f$  has homotopy fiber with first nonzero homotopy group in degree  $n$  and let  $m := 2n - 2$ . The functor of infinity categories*

$$\text{St}_*(\tau_m f) : \tau_m Y \rightarrow \text{Top}_*,$$

*given by applying pointed straightening  $\text{St}_*$  to the  $m$ -truncation of  $f$ , factors as*

$$\begin{array}{ccc} \tau_m Y & \xrightarrow{\text{St}_*(\tau_m f)} & \text{Top}_* \\ \exists ! \downarrow & \nearrow \Omega^\infty & \\ \text{Sp} & & \end{array}$$

where  $\mathrm{Sp}$  denotes the category of spectra. In particular,  $\mathrm{St}_*(\tau_m f)$  takes values in infinite loop spaces and infinite loop maps.

By applying the inverse of  $\mathrm{St}_*$  to  $\mathrm{St}_*(\tau_m f)$ , we recover the binary operation given in Proposition 1.9 on the homotopy category. Moreover, Proposition 1.12 shows that the group object

$$\tau_m X \xrightarrow{\tau_m f} \tau_m Y$$

$\swarrow \tau_m s$

as in Proposition 1.9 is a grouplike  $\mathbb{E}_\infty$ -space over  $\tau_m Y$ , rather than just an  $H$ -group object.

## Paper outline

Section 2 serves as a setup for the rest of the paper, including both technical results and key examples. In Section 2.1, we give proofs of Corollary 1.10 and Propositions 1.9 and 1.12. Section 2.1 relies on  $(\infty, 1)$ -categorical machinery, so we summarize the key facts before our proofs. The geometrically inclined reader may choose to take the proofs of Corollary 1.10 and Proposition 1.9 for granted and begin their perusal with Section 2.2.

Section 2.2 includes the key examples of group structures on sets of vector bundles, which are the focus of the rest of the paper. These include the groups  $\mathcal{G}_{a_1}$  of Proposition 1.3 and  $\mathcal{G}_{V_0}$  of Theorem 1.6. The proof of Theorem 1.5 uses certain elementary comparisons between different groups, which we include in Section 2.3.

In Section 3, we focus on rank 2 bundles on  $\mathbb{C}P^3$ . Section 3.1 summarizes the Atiyah–Rees  $\alpha$ -invariant, topological classification of rank 2 bundles on  $\mathbb{C}P^3$ , and Horrocks’ construction. In Section 3.2, we complete the proof of Proposition 1.3 by showing that  $c_2$  is a group homomorphism. Section 3.3 includes the proof of Theorem 1.5.

Section 4 focuses on group structures on isomorphism classes of rank 3 bundles on  $\mathbb{C}P^5$ . In Section 4.1 we prove part (iii) of Theorem 1.6. We study group structures on rank 3 bundles in more detail in Section 4.2. We show that, in some of the groups  $\mathcal{G}_{V_0}$  from Theorem 1.6, indecomposable bundles are generated by sums of line bundles under the addition operation in the group. We view this as a proof of concept: our addition operation produces interesting bundles from simple ones. We end Section 4.2 with naturally arising questions about these groups of unstable vector bundles.

We also include an appendix, which supplies details of the pointed,  $(\infty, 1)$ -categorical Grothendieck construction for spaces. This material is used in Section 2.1, although the section can be read without the appendix. This material is not new, but our exposition consolidates the necessary background and provides a reference to more comprehensive literature.

## Conventions

- “Vector bundle” refers to a complex, topological vector bundle unless otherwise stated. “Rank” refers to complex rank.
- We write  $\text{Top}$  for the category of spaces and  $\text{Sp}$  for the category of spectra. We write  $\text{Top}_*$  for pointed topological spaces.
- Limits and colimits are implicitly homotopy limits and colimits.
- We write  $\mathcal{O}(k)$  for the vector bundle on  $\mathbb{C}P^n$  with first Chern class equal to  $k$ .
- Given a classifying space  $BU(n)$ , we write  $\gamma_n$  for the universal bundle on it.
- We write  $\underline{\mathbb{C}}$  for the trivial rank 1 vector bundle on any space.
- Given an  $(\infty, 1)$ -category  $C$  and objects  $X, Y$  in  $C$ , we write  $\text{Maps}_C(X, Y)$  for the space of maps from  $X$  to  $Y$ .
- The  $m$ -truncation functor  $\tau_m: \text{Top} \rightarrow \text{Top}_{\leq m}$  is reflection onto the full subcategory  $\text{Top}_{\leq m}$  of spaces with  $X$  with  $\pi_i X = 0$  for  $i > m$ , ie the left adjoint to the inclusion  $\text{Top}_{\leq m} \hookrightarrow \text{Top}$ .
- Given spaces  $X$  and  $Y$ , we write  $[X, Y]$  for homotopy classes of maps with domain  $X$  and codomain  $Y$ .
- A space  $Y$  is said to be  $m$ -skeletal if, for all  $X \in \text{Top}$ ,  $[Y, X] \xrightarrow{\cong} [Y, \tau_m X]$  via postcomposition with the  $m$ -truncation map  $X \rightarrow \tau_m X$ . Examples of  $m$ -skeletal spaces include CW complexes with cells of dimension at most  $m$  and  $m$ -dimensional manifolds.
- Fix  $f: X \rightarrow Y$  in  $\text{Top}$ . Given  $Y \in s$  and  $c: C \rightarrow Y$ , let  $[C, X]_c$  denote homotopy classes of maps  $g: C \rightarrow X$  such that  $f \circ g = c$ .
- We say a map  $f: X \rightarrow Y$  is  $n$ -connective if the homotopy fiber of  $f$  has first nonzero homotopy group in degree  $n$ .

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## 2 Background and setup

This section has two goals: to prove necessary technical results about constructing group objects and to introduce key examples of rank-preserving group structures on sets of vector bundles. These examples will be the focus of the rest of the paper; the reader primarily interested in vector bundles may safely begin with Section 2.2, referring to Section 2.1 as needed.

In Section 2.1, we prove Proposition 1.12 first and then deduce Proposition 1.9 and Corollary 1.10. Much of this relies on aspects of the Grothendieck construction, or straightening and unstraightening, as described in detail by Lurie [9, 2.2 and 3.2] and summarized in the appendix.

Section 2.2 applies Corollary 1.10 to maps from  $\mathbb{C}P^3$  (respectively  $\mathbb{C}P^5$ ) into diagrams involving classifying spaces of rank 2 (respectively rank 3) bundles, yielding group structures on collections of rank 2 bundles on  $\mathbb{C}P^3$  (respectively rank 3 bundles on  $\mathbb{C}P^5$ ). We also show that our methods give multiple a priori distinct group structures on the same set of bundles.

In Section 2.3, we investigate the relationship between two different examples from the previous section. The results in this subsection are needed for the proof of Theorem 1.5.

### 2.1 Technical preliminaries

In this subsection we prove Corollary 1.10 and Propositions 1.9 and 1.12, starting with the latter. We also record several other facts about the additive structures in question, which we will use later to study specific examples.

**Construction 2.1** For  $Y$  a space, let  $(\text{Top}/Y)_*$  denote  $(\infty, 1)$ -category of pointed objects in the  $(\infty, 1)$ -category of spaces over  $Y$ , ie spaces over  $Y$  together with a choice of section. Let  $(\text{Top}_*)^Y$  denote the  $(\infty, 1)$ -category of functors from the Poincaré infinity groupoid of  $Y$  to  $\text{Top}_*$ . The following facts are justified in the appendix:

- (1) Given a map of spaces  $f: X \rightarrow Y$  with section  $s: Y \rightarrow X$ , we can naturally associate a functor  $\text{St}_*(f): Y \rightarrow \text{Top}_*$ , the *pointed straightening of  $f$* . Heuristically,  $\text{St}_*(f)$  takes a point  $p$  in  $Y$  to the homotopy fiber  $f^{-1}(p)$  pointed by  $s(p)$ .
- (2) The pointed straightening functor  $\text{St}_*$  participates in an equivalence of  $(\infty, 1)$ -categories

$$\begin{array}{ccc}
 & \text{St}_* & \\
 & \curvearrowright & \\
 (\text{Top}/Y)_* & & (\text{Top}_*)^Y \\
 & \curvearrowleft & \\
 & \text{Un}_* &
 \end{array}$$

(See Corollary A.7.)

- (3) There is a functor  $\text{forget}: (\text{Top}/Y)_* \rightarrow \text{Top}/Y$  given on objects by  $(Y \rightarrow x) \mapsto x$  and a free basepoint functor  $+: \text{Top}/Y \rightarrow (\text{Top}/Y)_*$  given on objects by  $x \mapsto (Y \rightarrow Y \sqcup x)$ , participating in an adjunction of  $(\infty, 1)$ -categories

$$\begin{array}{ccc} & \text{forget} & \\ & \curvearrowright & \\ \text{Top}/Y & \top & (\text{Top}/Y)_* \\ & \curvearrowleft & \\ & + & \end{array}$$

(See Lemma A.8.)

**Proof of Proposition 1.12** For nonnegative integers  $a \leq b$ , let  $\text{Sp}_{[a,b]}$  denote the full subcategory of spectra with homotopy groups zero outside the range  $[a, b]$ . Similarly, let  $\text{Top}_{*,[a,b]}$  denote the full subcategory of pointed spaces with homotopy groups zero outside the range  $[a, b]$  (ie  $a$ -connective,  $b$ -truncated spaces).

Observe that  $\text{St}_*(\tau_m f)$  takes values in  $\text{Top}_{*,[n,2n-2]}$ , the subcategory of pointed spaces that are  $n$ -connective and  $(2n-2)$ -truncated. By [10, Theorem 5.1.2],

$$(2-1) \quad \Omega^\infty: \text{Sp}_{[n,2n-2]} \rightarrow \text{Top}_{*,[n,2n-2]}$$

is an equivalence of  $(\infty, 1)$ -categories. So the factorization is automatic and there exists some functor of  $(\infty, 1)$ -categories  $A: Y \rightarrow \text{Sp}$  such that there is an equivalence

$$(2-2) \quad \Omega^\infty A \simeq \text{St}_*(\tau_m f)$$

in the  $(\infty, 1)$ -category of such functors. This factorization is given by applying the inverse of  $\Omega^\infty$  as in (2-1) to  $\text{St}_*(\tau_m f)$  and is therefore functorial. □

**Corollary 2.2** *With setup as in Proposition 1.12, the diagram*

$$\begin{array}{ccc} \tau_m X & \xrightarrow{\tau_m f} & \tau_m Y \\ & \swarrow \tau_m s & \\ & & \end{array}$$

*is an group object in the category of  $m$ -truncated spaces over  $\tau_m Y$ . The binary operation is given by applying  $\text{Un}_*$  to the operation on the group object  $\text{St}_*(\tau_m f)$ .*

**Proof of Corollary 1.10** Let  $c: C \rightarrow Y$  be a map from an  $m$ -skeletal space  $C$  to our base space  $Y$ , where  $m = 2n - 2$ . Let  $c_+$  denote the object

$$C \sqcup \tau_m Y \xrightarrow{c} \tau_m Y$$

given by applying  $+$  as in Construction 2.1(3) to the object  $\tau_m c: C \rightarrow \tau_m Y$ .

Proposition 1.12 implies that the space

$$\text{Maps}_{(\text{Top}_*)\tau_m Y}(\text{St}_*(c_+), \text{St}_*(\tau_m f))$$

possesses the structure of a grouplike  $\mathbb{E}_\infty$ -space, ie an infinite loop space.

By Construction 2.1(2), we have natural equivalences of grouplike  $\mathbb{E}_\infty$ -spaces

$$\begin{aligned} \text{Maps}_{(\text{Top})_*}^{\tau_m Y}(\text{St}_*(c_+), \text{St}_*(\tau_{\leq m} f)) &\simeq \text{Maps}_{(\text{Top}/\tau_m Y)_*}(\text{Un}_* \text{St}_*(c_+), \text{Un}_* \text{St}_*(\tau_m f)) \\ &\simeq \text{Maps}_{(\text{Top}/\tau_m Y)_*}(C \sqcup \tau_m Y, \tau_m X) \\ &\simeq \text{Maps}_{\text{Top}/\tau_m Y}(C, \tau_m X), \end{aligned}$$

where the third step is the definition of  $c_+$  and Construction 2.1(3).

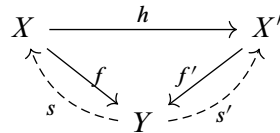
This implies

$$\pi_0 \text{Maps}_{\text{Top}/\tau_m Y}(C, \tau_m X) \simeq \pi_0 \text{Maps}_{\text{Top}/Y}(\text{sk}^m C, X) \simeq \pi_0 \text{Maps}_{\text{Top}/Y}(C, X) \simeq [C, X]_c,$$

giving the group structure on  $[C, X]_c$ . Note that the identity of  $\pi_0 \text{Maps}_{\text{Top}/\tau_m Y}(C, \tau_m X)$  is obtained by postcomposing  $c: C \rightarrow \tau_m Y$  with  $s: \tau_m Y \rightarrow \tau_m X$ . Tracing through the isomorphisms above, this shows that  $s \circ c: C \rightarrow X$  is the identity in  $[C, X]_c$ .  $\square$

We record one additional result that will be useful later:

**Lemma 2.3** *Suppose that we have a commutative solid diagram*



where  $f$  and  $f'$  are both  $n$ -connective, and  $s$  and  $s'$  are sections of  $f$  and  $f'$ , respectively. Let  $m = 2n - 2$ . Then  $\text{St}_*(\tau_m h): \text{St}_*(\tau_m f) \rightarrow \text{St}_*(\tau_m f')$  is a natural transformation through infinite loop maps.

**Proof** Note that the functor  $\text{St}_*(\tau_m -): (\text{Top}/Y)_* \rightarrow (\text{Top}_*)^{\tau_m Y}$  factors through  $\text{Sp}^{\tau_m Y}$ .  $\square$

This immediately implies:

**Corollary 2.4** *With setup as in Lemma 2.3, let  $C$  be an  $m$ -skeletal space and let  $c: C \rightarrow Y$  be given. The map  $h \circ -: [C, X]_c \rightarrow [C, X']_c$  is a group homomorphism.*

We now explicitly describe the group operation of Corollary 2.2 in the homotopy category of spaces by unraveling the binary operation on  $\text{Un}_* \text{St}_*(\tau_m f)$ .

**Proof of Proposition 1.9** Recall the fold map from Definition 1.8(c). From the proof of Proposition 1.12, equation (2-2) supplied a functor  $A: \tau_m Y \rightarrow \text{Sp}$  such that  $\text{St}_*(\tau_m f) = \Omega^\infty A$ . The group object structure on  $A$  is given by

$$A \times A \simeq A \sqcup A \xrightarrow{\text{fold}} A.$$

Applying  $\text{Un}_*$ , we have a homotopy commutative diagram

$$\begin{array}{ccccc}
 \tau_m X \times_{\tau_m Y} \tau_m X & \xrightarrow{\simeq} & \text{Un}_* \Omega^\infty(A \times A) & \xleftarrow{\simeq} & \text{Un}_* \Omega^\infty(A \sqcup A) & \xrightarrow{\text{Un}_* \Omega^\infty \text{ fold}} & \text{Un}_* \Omega^\infty A \\
 \downarrow \simeq & & & & \downarrow \simeq & & \downarrow \simeq \\
 (2-3) \quad \tau_m(X \times_Y X) & & & & \tau_m X \cup_{\tau_m Y} \tau_m X & \xrightarrow{\text{fold}} & \tau_m X \\
 \uparrow \simeq \tau_m(sf \times 1 \cup 1 \times sf) & & & & \downarrow \dagger & & \downarrow \dagger \\
 \tau_m(X \cup_Y X) & \xrightarrow{*} & \tau_m(\tau_m X \cup_{\tau_m Y} \tau_m X) & \xrightarrow{\tau_m \text{ fold}} & \tau_m X & & \tau_m X
 \end{array}$$

where the maps labeled  $\dagger$  above are the natural map from a space into its truncation and the map labeled  $*$  is the  $\tau_m$ -truncation of the map on pushouts induced by the map of spans

$$\begin{array}{ccccc}
 X & \xleftarrow{s} & Y & \xrightarrow{s} & X \\
 \downarrow & & \downarrow & & \downarrow \\
 \tau_m X & \xleftarrow{\tau_m s} & \tau_m Y & \xrightarrow{\tau_m s} & \tau_m X
 \end{array}$$

where the vertical arrows are again the natural maps from a space to its truncation.

In (2-3), arrow  $\tau_m(sf \times 1 \cup 1 \times sf)$  is an equivalence by the hypothesis that  $f$  is  $n$ -connective. Inspecting the diagram, it is clear that the two right rectangles labeled  $\star$  and  $\star\star$  commute up to homotopy, by naturality. One then checks that starting at the bottom left corner, going up and around the large left rectangle produces the map  $*$  induced by (2-4).

The composite obtained by starting at the upper left-hand corner of the outer rectangle, going down by two, and then right by two computes the proposed operation on  $\tau_m X$  over  $\tau_m Y$ . On the other hand, starting in the upper left-hand corner but going first right by two and then down by two computes the group object structure obtained in Corollary 2.2.  $\square$

From this description of the group structure, we obtain group homomorphisms from homotopy classes of maps over one base to homotopy classes of maps over another base. First note the following elementary lemma:

**Lemma 2.5** Consider a diagram

$$(2-5) \quad \begin{array}{ccc}
 X & \xrightarrow{h} & X' \\
 s \left( \begin{array}{c} \uparrow \\ \downarrow f \end{array} \right) & & \left( \begin{array}{c} \uparrow f' \\ \downarrow \end{array} \right) s' \\
 Y & \xrightarrow{j} & Y'
 \end{array}$$

such that  $hs \simeq s'j$ ,  $jf \simeq f'h$ , and  $s$  and  $s'$  are sections of  $f$  and  $f'$ , respectively. Such a diagram gives rise to a homotopy commutative diagram

$$\begin{array}{ccccc}
 X \times_Y X & \longleftarrow & X \cup_Y X & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 X' \times_{Y'} X' & \longleftarrow & X' \cup_{Y'} X' & \longrightarrow & X'
 \end{array}$$

**Corollary 2.6** Let  $h: X \rightarrow X'$  be as in (2-5). Suppose also that  $f$  and  $f'$  are  $n$ -connective and let  $m = 2n - 2$ . Let  $C$  be  $m$ -skeletal and let  $c: C \rightarrow Y$  be given. Then postcomposition with  $h$  induces a group homomorphism  $h \circ -: [C, X]_c \rightarrow [C, X']_{j \circ c}$ .

### 2.2 Group structures on vector bundles

The goal of this section is to introduce and explore examples of groups comprising unstable vector bundles. First we note the following general procedure for obtaining group structures:

**Example 2.7** Let  $g: X \rightarrow Y$  be a map of connected spaces. Consider the diagram

$$\begin{array}{ccc}
 X \times_Z Y & \longrightarrow & Y \\
 \begin{array}{c} \nearrow s_g \\ \downarrow p_1 \\ \searrow \end{array} & & \downarrow \tau_k \\
 X & \xrightarrow{\tau_k \circ g} & \tau_k Y
 \end{array}$$

where  $Z := \tau_k Y$ ,  $p_1$  is the natural projection, and the section  $s_g$  from  $X$  to the pullback is induced by the identity on  $X$  and  $g: X \rightarrow Y$ . The map  $p_1$  is  $k+1$ -connective since  $\tau_k$  is. By Corollary 2.2,  $\tau_{2k} f: \tau_{2k}(X \times_Z Y) \rightarrow \tau_{2k} X$  is a group object in  $2k$ -truncated spaces over  $\tau_{2k} X$ . So, for any  $2k$ -skeletal space  $C$  with  $c: C \rightarrow X$ , the set  $[C, X]_c$  inherits a group structure.

The next two examples are the ones that relate to Horrocks' construction [5].

**Example 2.8** The map  $\det(\gamma_2): BU(2) \rightarrow BU(1)$  is 4-connective and admits a section  $(\gamma_1 \otimes \mathbb{C}(-b)) \oplus \mathbb{C}(b)$ , where  $b \in \mathbb{Z}$  is arbitrary. If we fix  $a_1 \in \mathbb{Z}$ , this gives a group structure on isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$  with first Chern class equal to  $a_1$ . The identity is  $\mathbb{C}(a_1 - b) \oplus \mathbb{C}(b)$ .

As a special case, setting  $b = 0$ , we obtain the following:

**Example 2.9** The determinant map  $\det(\gamma_2): BU(2) \rightarrow BU(1)$  is 4-connective and admits a section represented by the bundle  $\gamma_1 \oplus \underline{\mathbb{C}}$  on  $BU(1)$ . If we fix  $\mathbb{C}(a_1): \mathbb{C}P^3 \rightarrow BU(1)$  for  $a_1 \in \mathbb{Z}$ , this gives a group structure on isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$  with first Chern class equal to  $a_1$ . The identity is  $\mathbb{C}(a_1) \oplus \underline{\mathbb{C}}$ .

We will explore the previous two examples and how they are related in Section 2.3, so we establish notation for each.

**Definition 2.10** Let  $+_{a_1, b}$  denote the operation on rank 2 bundles on  $\mathbb{C}P^3$  with first Chern class  $a_1$  given in Example 2.8. We write  $\mathcal{G}_{a_1, b}$  for this group. In the case  $b = 0$  (ie Example 2.9), we write  $+_{a_1}$  for the operation and  $\mathcal{G}_{a_1}$  for the group.

**Construction 2.11** Let  $b = -\frac{1}{2}a_1$ . Consider the commutative diagram

$$(2-6) \quad \begin{array}{ccc} BU(2) & \xrightarrow{(-)\otimes\mathbb{C}(-b)} & BU(2) \\ (-)\oplus\mathbb{C} \downarrow \uparrow \det(-) & & \det(-) \downarrow \uparrow (-)\otimes\mathbb{C}(-b)\oplus\mathbb{C}(b) \\ BU(1) & \xrightarrow{\text{id}} & BU(1) \end{array}$$

where for maps we write the functor represented on vector bundles. By Corollary 2.6, (2-6) induces a group isomorphism  $\phi: \mathcal{G}_{a_1} \rightarrow \mathcal{G}_{0,b}$  given on bundles by

$$V \mapsto V \otimes \mathbb{C}(-b).$$

Next, we introduce group structures for isomorphism classes of rank 3 bundles on  $\mathbb{C}P^5$ , which will be our subject in Section 4.

**Example 2.12** Consider the diagram

$$(2-7) \quad \begin{array}{ccc} \widetilde{BU}(3) & \longrightarrow & BU(3) \\ s \downarrow \uparrow v & \lrcorner & \downarrow c_1 \times c_2 \\ BU(2) & \xrightarrow{c_1 \times c_2} & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \end{array}$$

where

- $c_1 \times c_2$  is the product of the first Chern class and second Chern class;
- $\widetilde{BU}(3)$  is the indicated homotopy pullback; and
- the dotted arrow  $s$  to the pullback is induced by the identity map on  $BU(2)$  and by  $\gamma_2 \oplus \mathbb{C}: BU(2) \rightarrow BU(3)$ .

The map  $v$  is 6-connective. By Corollary 1.10, for any fixed  $V_0: \mathbb{C}P^5 \rightarrow BU(2)$ , the set of homotopy classes of lifts  $[\mathbb{C}P^5, \widetilde{BU}(3)]_{V_0}$  carries a group structure. A priori, an element in  $[\mathbb{C}P^5, \widetilde{BU}(3)]_{V_0}$  consists of a homotopy class  $\mathbb{C}P^5 \rightarrow BU(3)$  and a homotopy witnessing  $c_i(V) = c_i(V_0)$  for  $i = 1, 2$ . Such homotopies form a torsor for  $H^1(\mathbb{C}P^5; \mathbb{Z}) \times H^3(\mathbb{C}P^5; \mathbb{Z}) \simeq 0$ . Therefore the underlying set of this group is in fact isomorphism classes of rank 3 bundles on  $\mathbb{C}P^5$  with the same first and second Chern classes as  $V_0$ . The identity is  $V_0 \oplus \mathbb{C}$ .

Example 2.12 is our focus in Section 4, so we introduce the following terminology:

**Definition 2.13** Let  $V_0$  be a fixed rank 2 bundle on  $\mathbb{C}P^3$ . Let  $+_{V_0}$  denote the operation on rank 3 bundles on  $\mathbb{C}P^5$  with first and second Chern classes equal to those of  $V_0$ , as defined by Example 2.12. We write  $\mathcal{G}_{V_0}$  for this group.

**Convention 2.14** We will refer to the binary operations of Definitions 2.10 and 2.13 as *topological Horrocks additions*.

**Remark 2.15** To generalize topological Horrocks addition from rank 2 bundles to rank 3 bundles, one might hope to consider isomorphism classes of rank 3 vector bundles on  $\mathbb{C}P^5$  with fixed  $c_1$  and  $c_2$ .

However, the additional choice of  $V_0$  is necessary. To apply our setup, we would need a section  $\sigma$  of  $c_1 \times c_2: BU(3) \rightarrow K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)$ .

In  $H^*(BU(3); \mathbb{Z}/3\mathbb{Z})$ ,  $P^1(c_2) = c_1^2 c_2 + c_2^2 - c_1 c_3$ . In  $H^*(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4); \mathbb{Z}/3\mathbb{Z})$ ,  $P^1(\iota_4) = 0$ . Since  $(c_1 \times c_2)^*(\iota_{2i}) = c_i$  for  $i = 1, 2$ , existence of  $\sigma$  would force  $\sigma^*(P^1(c_2)) = P^1(\iota_4)$  and  $\iota_2^2 \iota_4 + \iota_4^2 = 0$ , a contradiction.

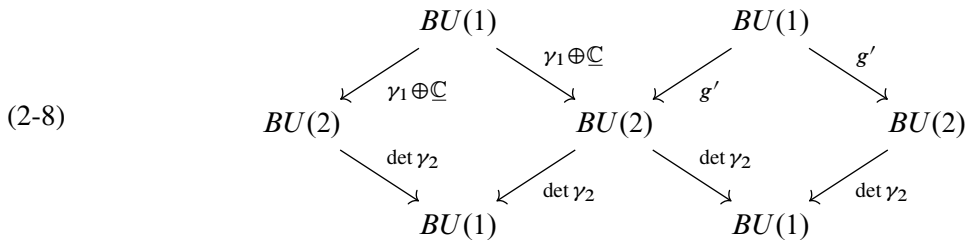
### 2.3 Properties of $+_{a_1}$ and $+_{a_1,b}$

Recall from Definition 2.10 that  $+_{a_1}$  and  $+_{a_1,b}$  are binary operations, with different identities, on the set of isomorphism classes of rank 2 vector bundles over  $\mathbb{C}P^3$  with first Chern class  $a_1$ .

**Lemma 2.16** *Let  $V, W$  and  $Z$  be rank 2 vector bundles on  $\mathbb{C}P^3$  with first Chern class  $a_1$ . Then*

$$(V +_{a_1} W) +_{a_1,b} Z = V +_{a_1} (W +_{a_1,b} Z).$$

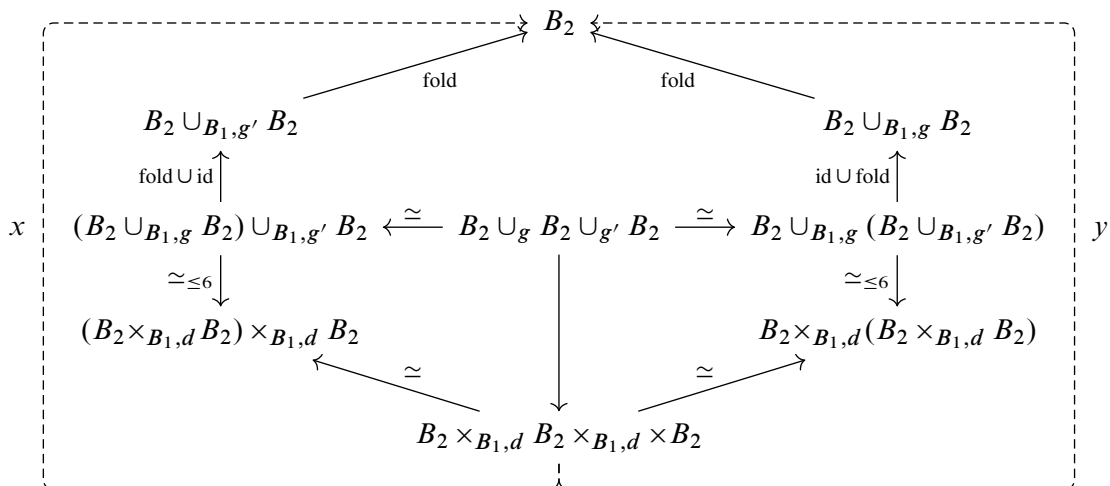
**Proof** Consider the commutative diagram



where  $g' := \gamma_1 \otimes \mathbb{C}(-b) \oplus \mathbb{C}(b)$ . To further simplify notation, let

$$g := \gamma_1 \oplus \mathbb{C}, \quad d := \det \gamma_2, \quad B_2 := BU(2), \quad B_1 := BU(1).$$

Diagram (2-8) gives a homotopy commutative diagram



where  $\simeq_{\leq 6}$  indicates a map that becomes an equivalence after applying the 6-truncation functor and the dashed arrows  $x$  and  $y$  are the composites defined after applying  $\tau_6$  to the entire diagram. The fact that

$$\pi_0 \text{Maps}_{\text{Top}}(\mathbb{C}P^3, -)(x) \cong \pi_0 \text{Maps}_{\text{Top}}(\mathbb{C}P^3, -)(y)$$

proves the lemma. □

This implies:

**Corollary 2.17** *Let  $V$  and  $W$  be rank 2 vector bundles on  $\mathbb{C}P^3$  with  $c_1(V) = c_1(W) = a_1$ . Then*

$$V +_{a_1, b} W = V +_{a_1} W -_{a_1} (\mathcal{O}(a_1 - b) \oplus \mathcal{O}(b)).$$

**Proof** This is a special case of a general fact. Suppose that  $+$  and  $*$  are two abelian group structures on a set  $S$  such that, for all  $x, y, z \in S$ ,

$$(x + y) * z = x + (y * z).$$

Let  $e_+$  and  $e_*$  be the respective identities. Then

$$(x + e_*) * (y + e_*) = (x + e_*) * (e_* + y) = x + (e_* * e_*) + y = x + y + e_*.$$

Set  $x = x + (-e_*)$  and  $y = y + (-e_*)$  in the above formula, where  $-e_*$  is the additive inverse of  $e_*$ . This yields

$$x * y = x - e_* + y - e_* + e_* = x + y - e_*.$$

We get the result by applying this to the set of isomorphism classes of vector bundles with first Chern class  $a_1$ , letting  $+$  be  $+_{a_1}$  and  $*$  be  $+_{a_1, b}$ , and noting that  $e_* = \mathcal{O}(a_1 - b) \oplus \mathcal{O}(b)$ . □

### 3 Comparing group structures on rank 2 bundles

In the previous section, we defined the group structures  $\mathcal{G}_{a_1}$  and  $\mathcal{G}_{a_1, b}$  on the set of isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$  with first Chern class equal to  $a_1$  (see Examples 2.9 and 2.8 and Definition 2.10). Our project in this section is to relate these group structures to Horrocks' construction and prove Theorem 1.5. This will involve Atiyah and Rees' classification of rank 2 bundles on  $\mathbb{C}P^3$ , so we begin in Section 3.1 with a review of relevant material from [2], including discussion of the  $\alpha$ -invariant. We also recall Horrocks' construction for locally free sheaves and its necessary properties.

In Section 3.2, we prove that  $c_2: \mathcal{G}_{a_1} \rightarrow H^2(\mathbb{C}P^3; \mathbb{Z})$  is a group homomorphism, completing the proof of Proposition 1.3. Verifying additivity of  $c_2$  involves only the definition of the group structure and functoriality results about the group structures of interest, as given in Corollary 2.4. We also show that the  $\alpha$ -invariant defines a group homomorphism  $\alpha: \mathcal{G}_0 \rightarrow \mathbb{Z}/2$ .

In Section 3.3, we show that topological Horrocks addition and Horrocks' construction produce the same underlying topological isomorphism class, when both are defined (Theorem 1.5). We do this by verifying that both constructions have the same effect on complete invariants of isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$ .

### 3.1 The classification of rank 2 bundles on $\mathbb{C}P^3$ and Horrocks' construction

In [2], Atiyah and Rees study complex rank 2 topological vector bundles on  $\mathbb{C}P^3$ . They define a  $\mathbb{Z}/2$ -valued invariant  $\alpha$  for bundles with even first Chern class.

**Construction 3.1** (the  $\alpha$ -invariant [2]) Let  $V$  be rank 2 bundle on  $\mathbb{C}P^3$  with  $c_1 = 0$ . Such bundles are classified by  $BSU(2)$ . The accidental isomorphism  $BSU(2) \simeq BSp(1)$  composed with stabilization gives

$$\tilde{\alpha}: BSU(2) \rightarrow BSp \simeq \Omega^\infty \Sigma^4 KO,$$

where  $KO$  denotes the real  $K$ -theory spectrum. Thus we have a class  $\tilde{\alpha} \in KO^4(BSU(2))$  and we define

$$\alpha_0(V) := p_* V^*(\tilde{\alpha}),$$

where  $p_*$  is the  $KO$ -theory pushforward for the spin manifold  $\mathbb{C}P^3$ .

Atiyah and Rees extend  $\alpha$  to all bundles with  $c_1(V) \equiv 0 \pmod{2}$  by letting

$$(3-1) \quad \alpha(V) := \alpha_0(V \otimes \mathbb{O}(-\frac{1}{2}c_1(V))).$$

Atiyah and Rees give various formulas for computing the  $\alpha$ -invariant of special bundles, eg:

**Proposition 3.2** [2, Theorem 7.2] For  $V$  a rank 2 bundle on  $\mathbb{C}P^3$  with  $c_1(V) \equiv 0 \pmod{2}$ , let

$$\Delta(V) := \frac{1}{4}(c_1^2 - 4c_2).$$

If  $V$  extends to  $\mathbb{C}P^4$ , then  $\alpha(V) \equiv \frac{1}{12}\Delta(\Delta - 1) \pmod{2}$ .

The utility of the  $\alpha$ -invariant is as follows:

**Theorem 3.3** [2, Theorems 1.1, 2.8 and 3.3] Given  $a_1, a_2 \in \mathbb{Z}$  with  $a_1 a_2 \equiv 0 \pmod{2}$ , the number of isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$  with  $i^{\text{th}}$  Chern class  $a_i$  is

- equal to 2 if  $a_1 \equiv 0 \pmod{2}$ , and
- equal to 1 otherwise.

In the first case, a rank 2 vector bundle on  $\mathbb{C}P^3$  is determined by  $c_1, c_2$  and  $\alpha$ .

**Remark 3.4** The condition  $a_1 a_2 \equiv 0 \pmod{2}$  is necessary and sufficient for two integers to be the Chern classes of complex rank 2 topological bundles on  $\mathbb{C}P^3$ .

We now review Horrocks' construction for rank 2 algebraic vector bundles.

**Construction 3.5** (Horrocks' construction [5]) Let  $V_1$  and  $V_2$  be rank 2 locally free sheaves on  $\mathbb{C}P^3$ . Suppose that

- we have isomorphisms  $\wedge^2 V_1 \simeq \wedge^2 V_2 \simeq \mathbb{O}(a)$  for some  $a \in \mathbb{Z}$ ;
- we have regular<sup>1</sup> sections  $s_i: \mathbb{O} \rightarrow V_i^*$ ; and
- the sheaves  $\mathcal{R}_i := \text{coker}(s_i^*: \mathcal{F}_i \rightarrow \mathbb{O})$  have disjoint supports.

<sup>1</sup>A section  $s$  of  $V$  is regular if its vanishing locus is of codimension equal to the rank of  $V$ .

For each  $i = 1, 2$ , the Koszul complex relative to  $s_i$  has the form  $0 \rightarrow \mathcal{O}(a) \rightarrow V_i \rightarrow \mathbb{C}$  and is exact since  $s_i$  is regular. By definition, we have exact sequences

$$0 \rightarrow \mathcal{O}(a) \rightarrow V_i \rightarrow \mathbb{C} \rightarrow \mathcal{R}_i \rightarrow 0.$$

The sheaf  $V_1 +_H V_2$  is defined by the diagram

$$(3-2) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(a) \oplus \mathcal{O}(a) & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & \mathcal{R}_1 \oplus \mathcal{R}_2 & \longrightarrow & 0 \\ & & \uparrow \simeq & & \uparrow \lrcorner & & \uparrow \Delta & & \uparrow \simeq & & \\ 0 & \longrightarrow & \mathcal{O}(a) \oplus \mathcal{O}(a) & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{R}_1 \oplus \mathcal{R}_2 & \longrightarrow & 0 \\ & & \downarrow \nabla & & \downarrow \lrcorner & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{O}(a) & \longrightarrow & V_1 +_H V_2 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{R}_1 \oplus \mathcal{R}_2 & \longrightarrow & 0 \end{array}$$

where  $\mathcal{W}$  is the indicated pullback along the diagonal  $\Delta$  and  $V_1 +_H V_2$  is the pushout along the fold map  $\nabla$ . The middle and bottom rows in (3-2) are both exact and  $V_1 +_H V_2$  is locally free of rank 2 [5, Theorem 1]. The bottom exact sequence of (3-2) implies that

$$c_1(V_1 +_H V_2) = c_1(V_1) = c_1(V_2) = a.$$

Atiyah and Rees and Horrocks study the effect of  $+_H$  on algebraic invariants:

**Theorem 3.6** [5, Theorem 1; 2, Corollary 5.7] *Let  $V_1$  and  $V_2$  be rank 2 algebraic bundles on  $\mathbb{C}P^3$  with  $c_1(V_1) = c_1(V_2) = -m$ , with  $m \geq 0$ .<sup>2</sup> Then*

$$c_2(V_1 +_H V_2) = c_2(V_1) + c_2(V_2).$$

Furthermore, suppose that  $m = 2n$  with  $n \geq 0$ .

- If  $n$  is odd or  $n \equiv 0 \pmod{4}$ , then  $\alpha(V_2 +_H V_2) = \alpha(V_1) + \alpha(V_2) \in \mathbb{Z}/2\mathbb{Z}$ .
- If  $n \equiv 2 \pmod{4}$ , then  $\alpha(V_1 +_H V_2) = \alpha(V_1) + \alpha(V_2) + 1$ .

### 3.2 Proof of Proposition 1.3

In the previous section, we defined the group structures  $\mathcal{G}_{a_1}$  on the set of isomorphism classes of rank 2 bundles on  $\mathbb{C}P^3$  with first Chern class equal to  $a_1$  (see Example 2.9 and Definition 2.10). This gives the group structure required by part (i) of Proposition 1.3.

For part (ii), note that, for any  $f: X \rightarrow Y$  with section  $s: X \rightarrow Y$  satisfying the hypotheses of Proposition 1.12, and any  $c: C \rightarrow Y$  with  $C$  an  $m$ -skeletal space, the identity element is given by  $s \circ c$ . In this case,  $s \circ c \simeq \mathcal{O}(a_1) \oplus \underline{\mathbb{C}}$ .

Part (iii) is a consequence of the following result:

<sup>2</sup>This is necessary for  $V_i$  to admit regular sections.



Item (i) follows from additivity of  $c_2$  for both operations, by Theorem 3.6 and Proposition 3.7.

Checking (ii) is complicated by the fact that the  $\alpha$ -invariant does not play well with  $+_{a_1}$  when  $a_1 \neq 0$ . We bootstrap from the case  $a_1 = 0$  to obtain a formula for  $\alpha(V +_{a_1} W)$  in terms of  $\alpha(V)$  and  $\alpha(W)$ . This involves the study of the groups  $\mathcal{G}_{0,b}$  for  $b$  nonzero (see Example 2.8).

Let  $a_1 \in \mathbb{Z}$  and suppose that  $V$  and  $W$  are rank 2 bundles on  $\mathbb{C}P^3$  with  $c_1(V) = c_1(W) = a_1$ . By Proposition 1.3(iii),  $c_2(V +_{a_1} W) = c_2(V) + c_2(W) = c_2(V +_H W)$ . In the case  $a_1 \equiv 1 \pmod{2}$ , Theorem 3.3 implies  $V +_{a_1} W \simeq V +_H W$  and we are done.

Now suppose  $a_1 \equiv 0 \pmod{2}$ . Recall Construction 2.11 and the associated group isomorphism  $\phi: \mathcal{G}_{a_1} \rightarrow \mathcal{G}_{0,b}$ . Using the definition of  $\alpha$  (see (3-1)), Proposition 3.7 and Corollary 2.17, we have

$$\begin{aligned} \alpha(V +_{a_1} W) &= \alpha(\phi(V +_{a_1} W)) \\ &= \alpha(\phi(V) +_{0,b} \phi(W)) \\ &= \alpha(\phi(V) +_0 \phi(W) -_0 \mathbb{O}(-b) \oplus \mathbb{O}(b)) \\ &= \alpha(V) + \alpha(W) - \alpha(\mathbb{O}(-b) \oplus \mathbb{O}(b)). \end{aligned}$$

To compute  $\alpha(\mathbb{O}(b) \oplus \mathbb{O}(-b))$ , we use Proposition 3.2. Since  $\mathbb{O}(-b) \oplus \mathbb{O}(b)$  extends to  $\mathbb{C}P^4$  and  $\Delta(\mathbb{O}(b) \oplus \mathbb{O}(-b)) = b^2$ , we see that

$$\alpha(\mathbb{O}(-b) \oplus \mathbb{O}(b)) \equiv \frac{1}{12} b^2 (b^2 - 1) \equiv \frac{1}{4} b^2 (b + 1)(b - 1) \pmod{2}.$$

So we must determine when  $b^2(b + 1)(b - 1)$  is divisible by 8.

- If  $b \equiv 0 \pmod{4}$ , then 16 divides  $b^2(b + 1)(b - 1)$ , so  $\alpha(\mathbb{O}(b) \oplus \mathbb{O}(-b)) = 0$ .
- If  $b$  is odd, then both  $b + 1$  and  $b - 1$  are even and one of  $b + 1$  or  $b - 1$  is divisible by 4, so  $\alpha(\mathbb{O}(b) \oplus \mathbb{O}(-b)) = 0$ .
- If  $b \equiv 2 \pmod{4}$ , then both  $b + 1$  and  $b - 1$  are odd and  $b^2$  is divisible by 4 but not 8, so  $\alpha(\mathbb{O}(b) \oplus \mathbb{O}(-b)) = 1$ .

Thus,

$$\alpha(V +_a W) = \alpha(V) + \alpha(W) + \epsilon(a), \quad \text{where } \epsilon(a) := \begin{cases} 0 & \text{if } a \not\equiv 4 \pmod{8}, \\ 1 & \text{if } a \equiv 4 \pmod{8}. \end{cases}$$

Comparing this with Theorem 3.6 yields the conclusion  $\alpha(V +_{a_1} W) = \alpha(V +_H W)$ . □

While this completes the argument that algebraic and topological Horrocks addition produce the same underlying topological isomorphism class, our method is indirect. The reader might hope for a more geometric relationship between the constructions.

**Problem 3.8** Is there an explicit, bundle-theoretic comparison between topological Horrocks addition  $+_{a_1}$  and Horrocks’ original construction for locally free sheaves?

### 4 Group structures on rank 3 bundles on $\mathbb{C}P^5$

We now explore additive structures on rank 3 bundles on  $\mathbb{C}P^5$ . Recall that, given a fixed rank 2 bundle  $V_0$  on  $\mathbb{C}P^5$ , Example 2.12 gives a group  $\mathcal{G}_{V_0}$  of isomorphism classes of rank 3 vector bundles with the same first and second Chern classes as  $V_0$ . The identity is  $V_0 \oplus \underline{\mathbb{C}}$ . Our primary interest here is to better understand the groups  $\mathcal{G}_{V_0}$  and their properties.

We complete the proof of Theorem 1.6 in Section 4.1 by showing that  $c_3$  is a group homomorphism. In Section 4.2, we explore the structure of  $\mathcal{G}_{V_0}$  for  $V_0 = \mathcal{O}(a) \oplus \mathcal{O}(b)$ . We show that  $+_{V_0}$  allows for the construction of interesting bundles from simple ones, as did Horrocks’ construction on rank 2 bundles on  $\mathbb{C}P^3$  (see Theorem 1.1).

**Proposition 4.1** *For infinitely many isomorphism classes of rank 2 bundles  $V_0 := \mathcal{O}(a) \oplus \mathcal{O}(b)$  on  $\mathbb{C}P^5$ , there exists a bundle  $W = \mathcal{O}(x) \oplus \mathcal{O}(y) \oplus \mathcal{O}(z)$  with  $W \in \mathcal{G}_{V_0}$  such that the subgroup generated by  $W$  contains bundles that are not sums of line bundles. Moreover, the subgroup generated by  $W$  is of finite index in  $\mathcal{G}_{V_0}$ .*

The proof of this result is elementary, involving only the study of possible Chern classes of sums of line bundles and additivity of  $c_3$ .

#### 4.1 The third Chern class and the structure of $\mathcal{G}_{V_0}$ for rank 3 bundles

We first prove the additivity of  $+_{V_0}$  for  $c_3$  and complete a theorem stated in the introduction.

**Proof of Theorem 1.6** We have already established the group structure in Example 2.12. The identity element is  $V_0 \oplus \underline{\mathbb{C}}$  by Corollary 1.10.

To show that the third Chern class  $c_3$  is a homomorphism, consider the diagram

$$\begin{array}{ccc}
 \widetilde{BU}(3) & \xrightarrow{c_3} & K(\mathbb{Z}, 6) \\
 \begin{array}{c} \uparrow \text{---} \downarrow \\ s \text{---} \downarrow \end{array} & & \begin{array}{c} \downarrow \text{---} \uparrow \\ \downarrow \end{array} \\
 BU(2) & \xrightarrow{0} & *
 \end{array}$$

where the solid diagram is homotopy commutative and the subdiagram with only dashed vertical arrows is homotopy commutative. By Corollary 2.6, we get a group homomorphism  $c_3: [\mathbb{C}P^5, BU(3)]_c \rightarrow H^6(\mathbb{C}P^5, \mathbb{Z})$ . □

In [11], we prove that rank 3 bundles on  $\mathbb{C}P^5$  are determined by Chern classes unless  $c_1(V) \equiv c_2(V) \equiv 0 \pmod{3}$ , in which case there are three pairwise nonisomorphic bundles with the same Chern classes as  $V$ . This proves the following result:

**Corollary 4.2** *The kernel of  $c_3: \mathcal{G}_{V_0} \rightarrow \mathbb{Z}$  is*

- (1) *trivial if  $c_1(V_0) \not\equiv 0 \pmod{3}$  or  $c_2(V_0) \not\equiv 0 \pmod{3}$ , and*
- (2) *isomorphic to  $\mathbb{Z}/3$  if  $c_1(V_0) \equiv c_2(V_0) \equiv 0 \pmod{3}$ .*

**Remark 4.3** The image of  $c_3$  is an infinite subgroup of  $\mathbb{Z}$ , since the conditions for three integers to be the Chern classes of a rank 3 bundle on  $\mathbb{C}P^5$  are a finite number of congruences (see [11, Lemma 2.16]). In the first case in Corollary 4.2,  $\mathcal{G}_{V_0}$  is abstractly isomorphic to  $\mathbb{Z}$ ; in the second,  $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$ .

In [11], we define a  $\mathbb{Z}/3$ -valued invariant  $\rho$  for rank 3 bundles  $V$  on  $\mathbb{C}P^5$  such that  $c_1(V) \equiv c_2(V) \equiv 0 \pmod{3}$ . We prove that  $c_1, c_2, c_3$  and  $\rho$  are complete invariants of isomorphism classes of complex rank 3 topological bundles on  $\mathbb{C}P^5$ . Additivity of  $c_3$  leads to the following:

**Problem 4.4** For general  $V_0$  with first and second Chern classes divisible by 3, what is a formula for  $\rho(V +_{V_0} W)$  in terms of  $\rho(V), \rho(W)$  and  $\rho(V_0)$ ?

Note that a necessary condition for  $\rho$  to be a group homomorphism is that  $\rho(V_0 \oplus \underline{\mathbb{C}}) = 0$ .

### 4.2 Constructing rank 3 bundles on $\mathbb{C}P^5$ from sums of line bundles

We begin this subsection by showing that there exist infinitely many isomorphism classes  $V_0 = \mathbb{O}(a) \oplus \mathbb{O}(b)$  on  $\mathbb{C}P^3$  such that  $\mathcal{G}_{V_0}$  contains another sum of line bundles that is not the group identity. This is a preliminary to prove Proposition 4.1. First, we simplify notation.

**Definition 4.5** Given  $n_1, \dots, n_k \in \mathbb{Z}$ , let  $\mathbb{O}(n_1, \dots, n_k) := \mathbb{O}(n_1) \oplus \dots \oplus \mathbb{O}(n_k)$ .

An element  $\mathbb{O}(x, y, z) \in \mathcal{G}_{\mathbb{O}(a,b)}$  is equivalent to an integer solution to the equations

$$(4-1) \quad x + yz = a + b, \quad xy + yz + zx = ab.$$

Note that, since the identity of  $\mathcal{G}_{\mathbb{O}(a,b)}$  is  $\mathbb{O}(a, b) \oplus \underline{\mathbb{C}}$ , a sum of line bundles  $\mathbb{O}(x, y, z) \in \mathcal{G}_{\mathbb{O}(a,b)}$  is a nonidentity element if and only if  $c_3(\mathbb{O}(x, y, z)) = xyz$  is nonzero.

Let  $c := a - x$  and  $d := b - y$ . Substituting in (4-1), we obtain

$$Q := c^2 + d^2 - bd - ac + cd = 0.$$

This is the equation of a quadric hypersurface in  $\mathbb{P}^3_{a,b,c,d}$ , whose rational points can be found via standard methods. For any  $u, l, v, w \in \mathbb{Z}$ , we get solutions to (4-1)

$$\begin{aligned} a &= w(u^2 + v^2 + uv - lv), & x &= w(v^2 + uv - lv), \\ b &= wul, & y &= wu(l - v), \\ & & z &= wu(u + v). \end{aligned}$$

**Remark 4.6** The previous computation shows that there are infinitely many  $\mathcal{G}_{V_0}$ , where  $V_0 = \mathbb{O}(a, b)$ , such that  $W = \mathbb{O}(x, y, z)$  is a nonidentity element of  $\mathcal{G}_{V_0}$ .

**Example 4.7** Consider  $W = \mathcal{O}(2, -1, 2)$  and  $V_0 = \mathcal{O}(3, 0)$ . By construction,  $W \in \mathcal{G}_{V_0}$ . Since  $c_1$  is even and  $c_1$  and  $c_2$  are both divisible by 3, the Schwarzenberger conditions as in [11, Lemma 2.16] imply that there exists a bundle  $W' \in \mathcal{G}_{V_0}$  with  $c_3(W') = a$  if and only if  $a$  is even. By Remark 4.3,  $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$  and, under this identification,  $c_3$  is projection onto the first factor. Since  $c_3(W) = 4$ , this implies that the subgroup generated by  $W$  is of index 6.

**Proof of Proposition 4.1** Let  $V_0 = \mathcal{O}(a, b)$  and  $W = \mathcal{O}(x, y, z)$  be as in Remark 4.6 with  $c_3(W) \neq 0$ . Inductively, let

$$+_{V_0}^n W := W +_{V_0} (+_{V_0}^{n-1} W).$$

Assume for a contradiction that  $+_{V_0}^n W$  is a sum of line bundles for all positive integers  $n$ . Let  $a_1, a_2, a_3 \in \mathbb{Z}$  denote the Chern classes of  $W$ . Since  $+_{V_0}$  preserves  $c_1$  and  $c_2$ , and  $c_3: \mathcal{G}_{V_0} \rightarrow H^6(\mathbb{C}P^5; \mathbb{Z})$  is a group homomorphism,

$$c_1(+_{V_0}^n W) = a_1, \quad c_2(+_{V_0}^n W) = a_2 \quad \text{and} \quad c_3(+_{V_0}^n W) = na_3.$$

Thus, the equations

$$X + Y + Z = a_1, \quad XYZ = na_3, \quad XY + YX + ZX = a_2$$

have a solution  $(X, Y, Z) = (x_n, y_n, z_n)$  for all  $n \in \mathbb{Z}$ . Let  $p$  be a prime such that  $p > |3a_i|$  for  $i = 1, 3$ . Take  $n = p$  in the above equations. Since  $c_3(W) = a_3 \neq 0$ , the third equation implies that  $p$  divides one of  $x_p, y_p$  or  $z_p$ . Without loss of generality,  $p$  divides  $x_p$ . But note that

$$|z_p y_p| \leq |a_3| < \frac{1}{3} p$$

and so  $|z_p| < \frac{1}{3} p$ , and  $|y_p| < \frac{1}{3} p$ . Since  $|x_p|$  is divisible by  $p$ , it is larger than  $|y_p|, |z_p|$  and  $|y_p + z_p|$ . Therefore,

$$|x_p + y_p + z_p| \geq |x_p| - |y_p| - |z_p| \geq \frac{1}{3} p.$$

On the other hand,  $|x_p + y_p + z_p| = |a_1| < \frac{1}{3} p$ , a contradiction.

Now consider the subgroup generated  $W$ . Since  $c_3(W) \neq 0$ ,  $\ker(c_3) \cap \langle W \rangle$  is trivial and we have an exact sequence of groups

$$\ker(c_3) \rightarrow \mathcal{G}_{V_0}/\langle W \rangle \rightarrow \mathbb{Z}/c_3(W).$$

Finally,  $\ker(c_3)$  is finite by Remark 4.3 and  $\mathbb{Z}/c_3(W)$  is finite, so  $\mathcal{G}_{V_0}/H$  is also finite. □

This section gives insight into the groups  $\mathcal{G}_{V_0}$ , but many questions remain.

**Problem 4.8** (1) For groups  $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$ , what are generators for 3-torsion?

(2) Are the Horrocks bundles of rank 3 on  $\mathbb{C}P^5$  elements of some  $\mathcal{G}_{V_0}$ ? If so, what subgroups do they generate?

(3) What can be said about the structure of  $\mathcal{G}_{V_0}$  for  $V_0$  not a sum of line bundles?

## Appendix The pointed Grothendieck construction for spaces

This appendix describes the pointed version straightening and unstraightening for functors to spaces, combining facts from [9]. The particular goal is to make precise the statement that a diagram of spaces

$$(A-1) \quad X \begin{array}{c} \xrightarrow{f} \\ \xleftarrow[s]{\quad} \end{array} Y$$

where  $f$  is  $n$ -connective can equivalently be viewed as a functor from  $Y$  to infinite loop spaces, after  $(2n-2)$ -truncation.

**Notation A.1** We establish terminology and conventions and provide references.

- (1) Let  $\mathbf{sSet}$  denote the category of simplicial sets and  $\mathbf{Cat}_\Delta$  denote the category of simplicially enriched categories.
- (2) By  $(\infty, 1)$ -category, we will mean a simplicial set that satisfies the inner horn condition (a quasi-category [9, 1.1.2.4]).
- (3) Let  $\mathcal{C}$  denote the left adjoint to the homotopy coherent nerve functor  $N : \mathbf{Cat}_\Delta \rightarrow \mathbf{sSet}$  [9, 1.1.5]. There is a Quillen equivalence of simplicial model categories

$$\mathbf{Cat}_\Delta \begin{array}{c} \xleftarrow{\mathcal{C}} \\ \xrightarrow{N} \end{array} \mathbf{sSet}$$

between simplicial sets with the Joyal model structure and simplicially enriched categories with the Bergner model structure (see [9, Section 1.1.5] for discussion, and [4; 3; 8] for a proof).

- (4) Let  $\mathbf{sSet}/_S$  denote the simplicial model category of simplicial sets over  $S$ , endowed with the contravariant model structure. We have an associated simplicial category  $\mathbf{RFib}(S)$  obtained by taking fibrant–cofibrant objects.
- (5) Let  $\mathbf{Kan}$  denote the  $(\infty, 1)$ -category of Kan complexes obtained by taking fibrant–cofibrant objects in the simplicial model category of simplicial sets with the Kan model structure and applying  $N$ . Given a Kan complex  $S$ , let  $\mathbf{Kan}/_S$  denote the  $(\infty, 1)$ -overcategory of Kan complexes over  $S$ .
- (6) Given a  $(\infty, 1)$ -category  $C$  and a simplicial set  $S$ , we write  $\mathbf{Fun}(S, C)$  for the simplicial set of maps from  $S$  to  $C$ . This is an  $(\infty, 1)$ -category since  $C$  is, and models functors from  $S$  to  $C$  [9, 1.2.7.2].
- (7) We ignore set-theoretic issues and refer the concerned reader to [9, 1.2.15].

The fundamental result that we need is the following:

**Theorem A.2** (Lurie [9]) *Let  $S$  be a Kan complex. There is an equivalence of  $(\infty, 1)$ -categories*

$$(A-2) \quad \mathbf{Kan}/_S \begin{array}{c} \xleftarrow{\text{Un}} \\ \xrightarrow{\text{St}} \end{array} \mathbf{Fun}(S, \mathbf{Kan}).$$

**Proof** By [9, 2.2.3.11], for any simplicial set we have an equivalence of simplicial categories

$$(A-3) \quad \text{RFib}(S) \begin{array}{c} \xleftarrow{\text{Un}} \\ \text{---} \\ \xrightarrow{\text{St}} \end{array} (\text{sSet}^{\text{e}[S]^{\text{op}}})^{\circ},$$

where  $\text{sSet}$  has the Kan model structure,  $\text{sSet}^{\text{e}[S]^{\text{op}}}$  has the projective model structure, and  $(-)^{\circ}$  indicates we take fibrant–cofibrant objects. Since  $S$  is a Kan complex, [9, 3.1.5.1(A3)] implies  $\text{Kan}_{/S} \simeq N(\text{RFib}(S))$  as  $(\infty, 1)$ -categories.

By [9, 4.2.4.4], applying the homotopy coherent nerve to the right side of (A-3) recovers  $\text{Fun}(S^{\text{op}}, \text{Kan})$ . By [12, Section 57] combined with [12, Proposition 14.14], every Kan complex is equivalent to its opposite as an  $(\infty, 1)$ -category.  $\square$

To obtain a version for spaces over a base with a section, we take pointed objects on either side of Theorem A.2.

**Definition A.3** Given an  $(\infty, 1)$ -category  $C$  with final object  $t$  [9, 1.2.12], let  $C_*$  denote the full  $(\infty, 1)$ -subcategory of  $\text{Fun}(\Delta^1, C)$  spanned by functors that restrict to  $t$  on 0. Explicitly, if  $\text{fib}_t$  denotes the (1-categorical) fiber in  $\text{sSet}$  at  $t$ , then

$$C_* := \text{fib}_t(\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\Delta^0, C)),$$

where the map is given by precomposition with  $\{0\} \simeq \Delta^0 \hookrightarrow \Delta^1$ .

**Remark A.4** Given an  $(\infty, 1)$ -category  $C$  with final object  $t$ , the infinity category  $C_*$  is also modeled by the  $(\infty, 1)$ -undercategory  $C_{t/}$ . See [9, 1.2.9] for a discussion of  $(\infty, 1)$ -undercategories and [9, 4.2.1.5 and 7.2.2.8] for the equivalence.

Since equivalent  $(\infty, 1)$ -categories support the same category theory, the equivalences  $\text{St}$  and  $\text{Un}$  will induce one on pointed objects.

**Lemma A.5** *An equivalence of  $(\infty, 1)$ -categories induces an equivalence upon taking pointed objects.*

**Proof** This is the dual statement of [9, 1.2.9.3] (dualized, for example, following the method of [9, 1.2.9.5]), specialized to the case of a final object.  $\square$

We can apply Lemma A.5 directly to the equivalences of Theorem A.2, but to understand the result we first describe pointed objects on the left side of (A-2).

**Lemma A.6** *For any  $(\infty, 1)$ -category  $C$  with a final object  $t$ , the  $(\infty, 1)$ -category  $\text{Fun}(S, C)$  has as final object the constant functor at  $t$ . There is an equivalence of  $(\infty, 1)$ -categories  $\text{Fun}(S, C)_* \simeq \text{Fun}(S, C_*)$ .*

**Proof** The first statement follows from the dual of [9, 5.1.2.3]. For the second, recall that  $\text{Fun}(S, C)_*$  is modeled by the fiber at the final object of the map

$$\text{Fun}(\Delta^1, \text{Fun}(S, C)) \rightarrow \text{Fun}(\Delta^0, \text{Fun}(S, C))$$

given by restricting to  $\{0\} \hookrightarrow \Delta^1$ . Moreover, there is an adjunction  $S \times - \dashv \text{Fun}(S, -)$  as enriched functors  $\text{sSet} \rightarrow \text{sSet}$ , since  $\text{Fun}(S, -)$  is an internal hom in the monoidal category  $(\text{sSet}, \times, \Delta^0)$ . So we get isomorphisms of simplicial sets

$$\begin{aligned} \text{fib}(\text{Fun}(\Delta^1, \text{Fun}(S, C)) \rightarrow \text{Fun}(\Delta^0, \text{Fun}(S, C))) &\simeq \text{fib}(\text{Fun}(\Delta^1 \times S, C) \rightarrow \text{Fun}(\Delta^0 \times S, C)) \\ &\simeq \text{fib}(\text{Fun}(S, \text{Fun}(\Delta^1, C)) \rightarrow \text{Fun}(S, \text{Fun}(\Delta^0, C))). \end{aligned}$$

Since right adjoints preserve limits,

$$\text{fib}(\text{Fun}(S, \text{Fun}(\Delta^1, C)) \rightarrow \text{Fun}(S, \text{Fun}(\Delta^0, C))) \simeq \text{Fun}(S, \text{fib}(\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\Delta^0, C))).$$

Tracing through our equivalences, we have shown that  $\text{Fun}(S, C)_* \simeq \text{Fun}(S, C_*)$ . □

**Corollary A.7** *Straightening and unstraightening induce mutually inverse equivalences of  $(\infty, 1)$ -categories  $\text{St}_* : (\text{Kan}/S)_* \rightarrow \text{Fun}(S, \text{Kan}_*)$  and  $\text{Un}_* : \text{Fun}(S, \text{Kan}_*) \rightarrow (\text{Kan}/S)_*$ .*

We record one more result about pointed objects in  $(\infty, 1)$ -categories.

**Lemma A.8** *Let  $\mathcal{C}$  be a  $(\infty, 1)$ -category with a final object  $t$  and with coproducts. Then there is an adjunction of  $(\infty, 1)$ -categories*

$$C \begin{array}{c} \xleftarrow{\text{forget}} \\ \top \\ \xrightarrow{+} \end{array} C_*$$

**Proof** We write  $F$  for the forgetful functor. This functor is given on objects by  $(t \rightarrow Y) \mapsto Y$ . The functor  $+$  is given on objects by  $c \mapsto c \sqcup t$ , where  $\sqcup$  is the  $(\infty, 1)$ -coproduct in  $C$  [9, 4.4.1].

To show  $+$  is left adjoint to  $F$ , we use [9, 5.2.2.8]: it suffices to provide a unit transformation

$$u : \text{Id}_C \rightarrow F \circ +$$

in  $\text{Fun}(C, C)$  such that a certain associated composite is an isomorphism in the homotopy category. Let  $u_c : c \rightarrow t \sqcup c$  be the structure map of the coproduct. We must show

$$\text{Maps}_{C_*}(t \rightarrow c \sqcup t, t \rightarrow y) \xrightarrow{\text{apply } F} \text{Maps}_C(F(t \rightarrow c \sqcup t), F(t \rightarrow y)) \xrightarrow{u_{c \circ -}} \text{Maps}_C(c, y)$$

is an isomorphism in the homotopy category of spaces for all  $c$  and  $y$  in  $C$ . Combining [9, 5.5.5.12] with [9, Lemma 7.2.2.8 and Proposition 4.2.1.5], we have an equivalence of spaces

$$\text{Maps}_{C_*}(t \rightarrow c \sqcup t, t \rightarrow y) \simeq \text{hofib}(\text{Maps}_C(c \sqcup t, y) \rightarrow \text{Maps}_C(t, y)),$$

where the arrow  $\text{Maps}_C(c \sqcup t, y) \rightarrow \text{Maps}_C(t, y)$  is induced by precomposing with the given map  $t \rightarrow c \sqcup t$ , and the fiber is taken over  $t \rightarrow y \in C_*$  viewed as an object in  $\text{Maps}_C(t, y)$ . Consider the homotopy

commutative diagram of spaces

$$\begin{array}{ccc} \text{Maps}_{\mathcal{C}}(c \sqcup t, y) & \longrightarrow & \text{Maps}_{\mathcal{C}}(t, y) \\ \downarrow \simeq & & \parallel \\ \text{Maps}_{\mathcal{C}}(c, y) \times \text{Maps}_{\mathcal{C}}(t, y) & \xrightarrow{\pi_2} & \text{Maps}_{\mathcal{C}}(t, y) \end{array}$$

This implies

$$\text{Maps}_{\mathcal{C}_*}(t \rightarrow c \sqcup t, t \rightarrow y) \simeq \text{hofib}(\pi_2: \text{Maps}_{\mathcal{C}}(c, y) \times \text{Maps}_{\mathcal{C}}(t, y) \rightarrow \text{Maps}_{\mathcal{C}}(t, y)) \simeq \text{Maps}_{\mathcal{C}}(c, y).$$

We have a homotopy commutative diagram

$$\begin{array}{ccccc} \text{Maps}_{\mathcal{C}_*}(t \rightarrow c \sqcup t, t \rightarrow y) & \xrightarrow{\simeq} & Z & \xrightarrow{\simeq} & \text{Maps}_{\mathcal{C}}(c, y) \\ \text{apply } F \downarrow & & \downarrow & \swarrow 1 \times (t \rightarrow y) & \\ \text{Maps}_{\mathcal{C}}(c \sqcup t, y) & \xrightarrow{\simeq} & \text{Maps}_{\mathcal{C}}(c, y) \times \text{Maps}_{\mathcal{C}}(t, y) & & \\ u_c \circ - \downarrow & & \pi_1 \downarrow & & \\ \text{Maps}_{\mathcal{C}}(c, y) & \xlongequal{\quad} & \text{Maps}_{\mathcal{C}}(c, y) & \xleftarrow{1} & \end{array}$$

where  $Z := \text{fib}((\text{Maps}_{\mathcal{C}}(c, y) \times \text{Maps}_{\mathcal{C}}(t, y)) \xrightarrow{\pi_2} \text{Maps}_{\mathcal{C}}(t, y))$ . The rightmost map is a weak homotopy equivalence, as are all the horizontal arrows, so the left vertical composite is too. □

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
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