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Rank-preserving additions for topological vector bundles, after a construction of Horrocks

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We produce group structures on certain sets of topological vector bundles of fixed rank. In particular, we put a group structure on complex rank 2 bundles on $\mathbb{C}P^3$ with fixed first Chern class. We show that this binary operation coincides with a construction on locally free sheaves due to Horrocks, provided the latter is defined. Using similar ideas, we give group structures on certain sets of rank 3 bundles on $\mathbb{C}P^5$.

These groups arise from the study of relative infinite loop space structures on truncated diagrams. Specifically, we show that the $(2n-2)$-truncation of an n-connective map $X \to Y$ with a section is a highly structured group object over the $(2n-2)$-truncation of Y. Applying these results to classifying spaces yields the group structures of interest.

55P99, 55R25, 55R35; 55P47, 55Q05

1 Introduction

Despite the importance of vector bundles in geometry and topology, there are few explicit methods to produce them. On complex projective spaces, the simplest complex bundles to write down are sums of line bundles. Indecomposable bundles are difficult to describe explicitly, but there are some famous examples: one is that of the Horrocks–Mumford bundle of rank 2 on $\mathbb{C}P^4$ [7]; another is the Horrocks bundle of rank 3 on $\mathbb{C}P^5$ [6].

In [5], Horrocks takes another approach and constructs new algebraic vector bundles from given ones using a modified extension group procedure. Horrocks' construction takes as input complex rank 2 algebraic bundles V and W on $\mathbb{C}P^3$, which must have the same first Chern class and satisfy some technical hypotheses, and outputs another complex rank 2 algebraic vector bundle with the same first Chern class. We will write $V +_H W$ for the output bundle, although the construction does not define a group structure.

Atiyah and Rees show that Horrocks' construction produces essentially all topological equivalence classes of complex rank 2 bundles on $\mathbb{C}P^3$ from the simplest ones:

Theorem 1.1 [2, Theorem 1.1] *Any complex rank 2 topological vector bundle on $\mathbb{C}P^3$ can be obtained from a sum of line bundles by iteratively*

- *tensoring by a line bundle, and*
- *applying Horrocks' construction.*

Remark 1.2 This result shows that every topological equivalence class of complex rank 2 vector bundles on $\mathbb{C}P^3$ admits an algebraic representative.

The algebra involved in Horrocks' construction is quite specialized and does not directly generalize to produce algebraic vector bundles of other ranks or on other spaces. However, we explore Horrocks' construction from a homotopical vantage point and show that, from this perspective, it does generalize.

Proposition 1.3 (topological Horrocks addition) *Fix an integer $a_1 \in \mathbb{Z}$. Let $\mathscr{G}_{a_1}$ denote the set of topological isomorphism classes of complex rank 2 topological bundles on $\mathbb{C}P^3$ with first Chern class equal to a_1.*

(i) *$\mathscr{G}_{a_1}$ carries an abelian group structure $+_{a_1}$, via an explicit construction on classifying spaces.*

(ii) *The identity is $L \oplus \underline{\mathbb{C}}$, where L is the complex line bundle determined by $c_1(L) = a_1$ and $\underline{\mathbb{C}}$ is the trivial complex line bundle.*

(iii) *The second Chern class defines a homomorphism $c_2 \colon \mathscr{G}_{a_1} \to H^4(\mathbb{C}P^3; \mathbb{Z})$.*

Remark 1.4 The existence of a group structure on the underlying set of $\mathscr{G}_{a_1}$ can also be deduced by identifying elements of $\mathscr{G}_{a_1}$ with twisted symplectic K-theory classes. It is not hard to show that this group structure agrees with ours, but our description provides a framework for comparison with Horrocks' construction and for generalization.

We show $+_{a_1}$ addition agrees with Horrocks' construction as far as possible:

Theorem 1.5 *Suppose that V and W are two rank 2 algebraic vector bundles such that $c_1(V) = c_1(W) = a_1$ and such that the Horrocks sum $V +_H W$ is defined. Then*

$$V +_H W \simeq V +_{a_1} W$$

as topological vector bundles.

The proof of Theorem 1.5 is conceptually simple. In [2], Atiyah and Rees show that a topological rank 2 bundle on $\mathbb{C}P^3$ is determined by its first and second Chern classes together with a $\mathbb{Z}/2$-valued invariant called α. So, to prove that $V +_H W$ and $V +_{a_1} W$ are topologically isomorphic, it suffices to show their Chern classes and α-invariants agree.

The groups of Proposition 1.3 generalize as follows:

Theorem 1.6 (topological Horrocks addition for rank 3 bundles) *Fix a complex rank 2 topological bundle V_0 on $\mathbb{C}P^5$. Let $\mathcal{G}_{V_0}$ denote the set of topological equivalence classes of complex rank 3 topological bundles on $\mathbb{C}P^5$ with the same first and second Chern classes as V_0.*

(i) *$\mathcal{G}_{V_0}$ carries an abelian group structure, via an explicit construction on classifying spaces.*
(ii) *The identity is $V_0 \oplus \underline{\mathbb{C}}$.*
(iii) *The third Chern class defines a homomorphism $c_3\colon \mathcal{G}_{V_0} \to H^6(\mathbb{C}P^5; \mathbb{Z})$.*

Remark 1.7 The underlying set of $\mathcal{G}_{V_0}$ depends only on a choice of first and second Chern classes of V_0. However, we need an auxiliary rank 2 bundle to define the group operation.

1.1 Methods

Proposition 1.3 and Theorem 1.6 follow from a more general study of relative infinite loop space structures on truncated diagrams. We remind the reader of the following terminology:

Definition 1.8 Let Top denote the category of topological spaces.

(a) The m-truncation functor $\tau_m\colon \mathrm{Top} \to \mathrm{Top}_{\le m}$ is reflection onto the full subcategory $\mathrm{Top}_{\le m}$ of spaces X with $\pi_i X = 0$ for $i > m$. The functor τ_m is left adjoint to the inclusion $\mathrm{Top}_{\le m} \hookrightarrow \mathrm{Top}$.
(b) A map $f\colon X \to Y$ in Top is n-connective if the homotopy fiber of f has first nonzero homotopy group in degree n.
(c) In any category, given objects X and Y and a morphism $s\colon Y \to X$, let $X \cup_Y X$ denote the pushout of the diagram

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s"] & X \\ X & \end{tikzcd}$$

Let $1\colon X \to X$ denote the identity. The map $\mathrm{fold}\colon X \cup_Y X \to X$ is the dotted arrow making the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s"] & X \arrow[d] \arrow[ddr, bend left, "1"] & \\ X \arrow[r] \arrow[drr, bend right, "1"] & X \cup_Y X \arrow[dr, dashed, "\mathrm{fold}"] & \\ & & X \end{tikzcd}$$

In the homotopy category of spaces, our key technical result can be stated as follows:

Proposition 1.9 *Let $f: X \to Y$ be a map of pointed, simply connected spaces, with section $s: Y \to X$. Suppose that f is n-connective and let $m := 2n-2$. Let τ_m denote the m-truncation functor. Then*

$$\begin{tikzcd} \tau_m X \arrow[r, "\tau_m f"] & \tau_m Y \arrow[l, bend left, dashed, "\tau_m s"] \end{tikzcd}$$

is a group object in the homotopy category of m-truncated spaces over $\tau_m Y$, with binary operation given by truncating the diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, bend right, dashed] & X \cup_Y X \arrow[l, "(1 \cup sf)\times(sf\cup 1)"'] \arrow[r, "\mathrm{fold}"] & X, \end{tikzcd}$$

where the dotted arrow exists only after truncating.

Using the skeleton–truncation adjunction on the homotopy category of spaces, we obtain:

Corollary 1.10 *Let $f: X \to Y$, $s: Y \to X$, n and m be as in Proposition 1.9. Let C be an m-skeletal space and fix $c: C \to Y$. Let $[C, X]_c$ denote homotopy classes of maps $g: C \to X$ making*

$$\begin{tikzcd} C \arrow[r, dashed, "g"] \arrow[dr, "c"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \tag{1-1}$$

homotopy commutative. Then $[C, X]_c$ is a group with identity $s \circ c$.

Remark 1.11 This corollary can be viewed as a relative version of Borsuk's classical construction, as given in Spanier [13]. More recently, such ideas have been used by Asok and Fasel [1], for example.

It is rather tedious to prove Proposition 1.9 in a point–set way. Instead, we prove an $(\infty,1)$-categorical refinement. As a consequence of n-connectivity, the fibers of $\tau_m f: \tau_m X \to \tau_m Y$ are infinite loop spaces and the automorphisms of the fibers induced by paths in the base are infinite loop maps; this data should assemble to make $\tau_m X \to \tau_m Y$ an infinite loop space over $\tau_m Y$. The machinery of an $(\infty,1)$-categorical Grothendieck construction, for example straightening and unstraightening as in Lurie [9], allows us to make this precise.

Proposition 1.12 *Let $f: X \to Y$ be a map of pointed simply connected spaces with section $s: Y \to X$. Suppose f has homotopy fiber with first nonzero homotopy group in degree n and let $m := 2n-2$. The functor of infinity categories*

$$\mathrm{St}_*(\tau_m f): \tau_m Y \to \mathrm{Top}_*,$$

given by applying pointed straightening St_* to the m-truncation of f, factors as

$$\begin{tikzcd} \tau_m Y \arrow[r, "\mathrm{St}_*(\tau_m f)"] \arrow[d, dashed, "\exists"'] & \mathrm{Top}_* \\ \mathrm{Sp} \arrow[ur, "\Omega^\infty"'] \end{tikzcd}$$

where Sp *denotes the category of spectra. In particular,* $\mathrm{St}_*(\tau_m f)$ *takes values in infinite loop spaces and infinite loop maps.*

By applying the inverse of St_* to $\mathrm{St}_*(\tau_m f)$, we recover the binary operation given in Proposition 1.9 on the homotopy category. Moreover, Proposition 1.12 shows that the group object

$$\begin{tikzcd} \tau_m X \arrow[r, "\tau_m f"] & \tau_m Y \arrow[l, dashed, bend left, "\tau_m s"] \end{tikzcd}$$

as in Proposition 1.9 is a grouplike $\mathbb{E}_\infty$-space over $\tau_m Y$, rather than just an H-group object.

Paper outline

Section 2 serves as a setup for the rest of the paper, including both technical results and key examples. In Section 2.1, we give proofs of Corollary 1.10 and Propositions 1.9 and 1.12. Section 2.1 relies on $(\infty, 1)$-categorical machinery, so we summarize the key facts before our proofs. The geometrically inclined reader may choose to take the proofs of Corollary 1.10 and Proposition 1.9 for granted and begin their perusal with Section 2.2.

Section 2.2 includes the key examples of group structures on sets of vector bundles, which are the focus of the rest of the paper. These include the groups $\mathcal{G}_{a_1}$ of Proposition 1.3 and $\mathcal{G}_{V_0}$ of Theorem 1.6. The proof of Theorem 1.5 uses certain elementary comparisons between different groups, which we include in Section 2.3.

In Section 3, we focus on rank 2 bundles on $\mathbb{C}P^3$. Section 3.1 summarizes the Atiyah–Rees α-invariant, topological classification of rank 2 bundles on $\mathbb{C}P^3$, and Horrocks' construction. In Section 3.2, we complete the proof of Proposition 1.3 by showing that c_2 is a group homomorphism. Section 3.3 includes the proof of Theorem 1.5.

Section 4 focuses on group structures on isomorphism classes of rank 3 bundles on $\mathbb{C}P^5$. In Section 4.1 we prove part (iii) of Theorem 1.6. We study group structures on rank 3 bundles in more detail in Section 4.2. We show that, in some of the groups $\mathcal{G}_{V_0}$ from Theorem 1.6, indecomposable bundles are generated by sums of line bundles under the addition operation in the group. We view this as a proof of concept: our addition operation produces interesting bundles from simple ones. We end Section 4.2 with naturally arising questions about these groups of unstable vector bundles.

We also include an appendix, which supplies details of the pointed, $(\infty, 1)$-categorical Grothendieck construction for spaces. This material is used in Section 2.1, although the section can be read without the appendix. This material is not new, but our exposition consolidates the necessary background and provides a reference to more comprehensive literature.

Conventions

- "Vector bundle" refers to a complex, topological vector bundle unless otherwise stated. "Rank" refers to complex rank.
- We write Top for the category of spaces and Sp for the category of spectra. We write Top_* for pointed topological spaces.
- Limits and colimits are implicitly homotopy limits and colimits.
- We write $\mathbb{O}(k)$ for the vector bundle on $\mathbb{C}P^n$ with first Chern class equal to k.
- Given a classifying space $BU(n)$, we write γ_n for the universal bundle on it.
- We write $\underline{\mathbb{C}}$ for the trivial rank 1 vector bundle on any space.
- Given an $(\infty, 1)$-category C and objects X, Y in C, we write $\mathrm{Maps}_C(X, Y)$ for the space of maps from X to Y.
- The m-truncation functor $\tau_m\colon \mathrm{Top} \to \mathrm{Top}_{\leq m}$ is reflection onto the full subcategory $\mathrm{Top}_{\leq m}$ of spaces with X with $\pi_i X = 0$ for $i > m$, ie the left adjoint to the inclusion $\mathrm{Top}_{\leq m} \hookrightarrow \mathrm{Top}$.
- Given spaces X and Y, we write $[X, Y]$ for homotopy classes of maps with domain X and codomain Y.
- A space Y is said to be m-skeletal if, for all $X \in \mathrm{Top}$, $[Y, X] \xrightarrow{\cong} [Y, \tau_m X]$ via postcomposition with the m-truncation map $X \to \tau_m X$. Examples of m-skeletal spaces include CW complexes with cells of dimension at most m and m-dimensional manifolds.
- Fix $f\colon X \to Y$ in Top. Given $Y \in s$ and $c\colon C \to Y$, let $[C, X]_c$ denote homotopy classes of maps $g\colon C \to X$ such that $f \circ g = c$.
- We say a map $f\colon X \to Y$ is n-connective if the homotopy fiber of f has first nonzero homotopy group in degree n.

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2 Background and setup

This section has two goals: to prove necessary technical results about constructing group objects and to introduce key examples of rank-preserving group structures on sets of vector bundles. These examples will be the focus of the rest of the paper; the reader primarily interested in vector bundles may safely begin with Section 2.2, referring to Section 2.1 as needed.

In Section 2.1, we prove Proposition 1.12 first and then deduce Proposition 1.9 and Corollary 1.10. Much of this relies on aspects of the Grothendieck construction, or straightening and unstraightening, as described in detail by Lurie [9, 2.2 and 3.2] and summarized in the appendix.

Section 2.2 applies Corollary 1.10 to maps from $\mathbb{C}P^3$ (respectively $\mathbb{C}P^5$) into diagrams involving classifying spaces of rank 2 (respectively rank 3) bundles, yielding group structures on collections of rank 2 bundles on $\mathbb{C}P^3$ (respectively rank 3 bundles on $\mathbb{C}P^5$). We also show that our methods give multiple a priori distinct group structures on the same set of bundles.

In Section 2.3, we investigate the relationship between two different examples from the previous section. The results in this subsection are needed for the proof of Theorem 1.5.

2.1 Technical preliminaries

In this subsection we prove Corollary 1.10 and Propositions 1.9 and 1.12, starting with the latter. We also record several other facts about the additive structures in question, which we will use later to study specific examples.

Construction 2.1 For Y a space, let $(\mathrm{Top}_{/Y})_*$ denote $(\infty,1)$-category of pointed objects in the $(\infty,1)$-category of spaces over Y, ie spaces over Y together with a choice of section. Let $(\mathrm{Top}_*)^Y$ denote the $(\infty,1)$-category of functors from the Poincaré infinity groupoid of Y to Top_*. The following facts are justified in the appendix:

(1) Given a map of spaces $f\colon X \to Y$ with section $s\colon Y \to X$, we can naturally associate a functor $\mathrm{St}_*(f)\colon Y \to \mathrm{Top}_*$, the *pointed straightening of f*. Heuristically, $\mathrm{St}_*(f)$ takes a point p in Y to the homotopy fiber $f^{-1}(p)$ pointed by $s(p)$.

(2) The pointed straightening functor St_* participates in an equivalence of $(\infty,1)$-categories

$$\begin{tikzcd} (\mathrm{Top}_{/Y})_* \arrow[r, bend left, "\mathrm{St}_*"] & (\mathrm{Top}_*)^Y \arrow[l, bend left, "\mathrm{Un}_*"] \end{tikzcd}.$$

(See Corollary A.7.)

(3) There is a functor forget: $(\mathrm{Top}_{/Y})_* \to \mathrm{Top}_{/Y}$ given on objects by $(Y \to x) \mapsto x$ and a free basepoint functor $+\colon \mathrm{Top}_{/Y} \to (\mathrm{Top}_{/Y})_*$ given on objects by $x \mapsto (Y \to Y \sqcup x)$, participating in an adjunction of $(\infty,1)$-categories

$$\begin{tikzcd} \mathrm{Top}_{/Y} \arrow[r, bend right, "+"'] \arrow[r, phantom, "\top"] & (\mathrm{Top}_{/Y})_* . \arrow[l, bend right, "\mathrm{forget}"'] \end{tikzcd}$$

(See Lemma A.8.)

Proof of Proposition 1.12 For nonnegative integers $a \le b$, let $\mathrm{Sp}_{[a,b]}$ denote the full subcategory of spectra with homotopy groups zero outside the range $[a,b]$. Similarly, let $\mathrm{Top}_{*,[a,b]}$ denote the full subcategory of pointed spaces with homotopy groups zero outside the range $[a,b]$ (ie a-connective, b-truncated spaces).

Observe that $\mathrm{St}_*(\tau_m f)$ takes values in $\mathrm{Top}_{*,[n,2n-2]}$, the subcategory of pointed spaces that are n-connective and $(2n-2)$-truncated. By [10, Theorem 5.1.2],

$$\Omega^\infty\colon \mathrm{Sp}_{[n,2n-2]} \to \mathrm{Top}_{*,[n,2n-2]} \tag{2-1}$$

is an equivalence of $(\infty,1)$-categories. So the factorization is automatic and there exists some functor of $(\infty,1)$-categories $A\colon Y \to \mathrm{Sp}$ such that there is an equivalence

$$\Omega^\infty A \simeq \mathrm{St}_*(\tau_m f) \tag{2-2}$$

in the $(\infty,1)$-category of such functors. This factorization is given by applying the inverse of Ω^∞ as in (2-1) to $\mathrm{St}_*(\tau_m f)$ and is therefore functorial. □

Corollary 2.2 *With setup as in Proposition 1.12, the diagram*

$$\begin{tikzcd} \tau_m X \arrow[r, "\tau_m f"] & \tau_m Y \arrow[l, bend left, dashed, "\tau_m s"] \end{tikzcd}$$

is an group object in the category of m-truncated spaces over $\tau_m Y$. The binary operation is given by applying Un_* to the operation on the group object $\mathrm{St}_*(\tau_m f)$.

Proof of Corollary 1.10 Let $c\colon C \to Y$ be a map from an m-skeletal space C to our base space Y, where $m = 2n-2$. Let c_+ denote the object

$$\begin{tikzcd} C \sqcup \tau_m Y \arrow[r, "c"'] & \tau_m Y \arrow[l, bend right, dashed] \end{tikzcd}$$

given by applying $+$ as in Construction 2.1(3) to the object $\tau_m c\colon C \to \tau_m Y$.

Proposition 1.12 implies that the space

$$\mathrm{Maps}_{(\mathrm{Top}_*)^{\tau_m Y}}(\mathrm{St}_*(c_+), \mathrm{St}_*(\tau_m f))$$

possesses the structure of a grouplike $\mathbb{E}_\infty$-space, ie an infinite loop space.

By Construction 2.1(2), we have natural equivalences of grouplike $\mathbb{E}_\infty$-spaces

$$\begin{aligned}\mathrm{Maps}_{(\mathrm{Top})_*^{\tau_m Y}}(\mathrm{St}_*(c_+), \mathrm{St}_*(\tau_{\le m} f)) &\simeq \mathrm{Maps}_{(\mathrm{Top}_{/\tau_m Y})_*}(\mathrm{Un}_*\,\mathrm{St}_*(c_+), \mathrm{Un}_*\,\mathrm{St}_*(\tau_m f))\\ &\simeq \mathrm{Maps}_{(\mathrm{Top}_{/\tau_m Y})_*}(C \sqcup \tau_m Y, \tau_m X)\\ &\simeq \mathrm{Maps}_{\mathrm{Top}_{/\tau_m Y}}(C, \tau_m X),\end{aligned}$$

where the third step is the definition of c_+ and Construction 2.1(3).

This implies

$$\pi_0\,\mathrm{Maps}_{\mathrm{Top}_{/\tau_m Y}}(C, \tau_m X) \simeq \pi_0\,\mathrm{Maps}_{\mathrm{Top}_{/Y}}(\mathrm{sk}^m\, C, X) \simeq \pi_0\,\mathrm{Maps}_{\mathrm{Top}_{/Y}}(C, X) \simeq [C, X]_c,$$

giving the group structure on $[C, X]_c$. Note that the identity of $\pi_0\,\mathrm{Maps}_{\mathcal{S}_{/\tau_m Y}}(C, \tau_m X)$ is obtained by postcomposing $c\colon C \to \tau_m Y$ with $s\colon \tau_m Y \to \tau_m X$. Tracing through the isomorphisms above, this shows that $s \circ c\colon C \to X$ is the identity in $[C, X]_c$. □

We record one additional result that will be useful later:

Lemma 2.3 *Suppose that we have a commutative solid diagram*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"] & & X' \arrow[dl, "f'"'] \\ & Y \arrow[ul, dashed, bend left, "s"] \arrow[ur, dashed, bend right, "s'"'] & \end{tikzcd}$$

where f and f' are both n-connective, and s and s' are sections of f and f', respectively. Let $m = 2n-2$. Then $\mathrm{St}_*(\tau_m h)\colon \mathrm{St}_*(\tau_m f) \to \mathrm{St}_*(\tau_m f')$ is a natural transformation through infinite loop maps.

Proof Note that the functor $\mathrm{St}_*(\tau_m -)\colon (\mathrm{Top}_{/Y})_* \to (\mathrm{Top}_*)^{\tau_m Y}$ factors through $\mathrm{Sp}^{\tau_m Y}$. □

This immediately implies:

Corollary 2.4 *With setup as in Lemma 2.3, let C be an m-skeletal space and let $c\colon C \to Y$ be given. The map $h \circ -\colon [C, X]_c \to [C, X']_c$ is a group homomorphism.*

We now explicitly describe the group operation of Corollary 2.2 in the homotopy category of spaces by unraveling the binary operation on $\mathrm{Un}_*\,\mathrm{St}_*(\tau_m f)$.

Proof of Proposition 1.9 Recall the fold map from Definition 1.8(c). From the proof of Proposition 1.12, equation (2-2) supplied a functor $A\colon \tau_m Y \to \mathrm{Sp}$ such that $\mathrm{St}_*(\tau_m f) = \Omega^\infty A$. The group object structure on A is given by

$$A \times A \simeq A \sqcup A \xrightarrow{\text{fold}} A.$$

Applying Un_*, we have a homotopy commutative diagram

$$\begin{tikzcd}
\tau_m X \times_{\tau_m Y} \tau_m X \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{Un}_* \Omega^\infty(A \times A) & \mathrm{Un}_* \Omega^\infty(A \sqcup A) \arrow[l, "\simeq"'] \arrow[r, "\mathrm{Un}_* \Omega^\infty \mathrm{fold}"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\star"] & \mathrm{Un}_* \Omega^\infty A \arrow[d, "\simeq"] \\
\tau_m(X \times_Y X) & & \tau_m X \cup_{\tau_m Y} \tau_m X \arrow[r, "\mathrm{fold}"] \arrow[d, "\dagger"] \arrow[dr, phantom, "\star\star"] & \tau_m X \arrow[d, "\dagger"] \\
\tau_m(X \cup_Y X) \arrow[u, "\simeq", "\tau_m(sf \times 1 \cup 1 \times sf)"'] \arrow[rr, "*"] & & \tau_m(\tau_m X \cup_{\tau_m Y} \tau_m X) \arrow[r, "\tau_m \mathrm{fold}"] & \tau_m X
\end{tikzcd} \tag{2-3}$$

where the maps labeled † above are the natural map from a space into its truncation and the map labeled $*$ is the τ_m-truncation of the map on pushouts induced by the map of spans

$$\begin{tikzcd}
X \arrow[d] & Y \arrow[l, "s"'] \arrow[r, "s"] \arrow[d] & X \arrow[d] \\
\tau_m X & \tau_m Y \arrow[l, "\tau_m s"'] \arrow[r, "\tau_m s"] & \tau_m X
\end{tikzcd} \tag{2-4}$$

where the vertical arrows are again the natural maps from a space to its truncation.

In (2-3), arrow $\tau_m(sf \times 1 \cup 1 \times sf)$ is an equivalence by the hypothesis that f is n-connective. Inspecting the diagram, it is clear that the two right rectangles labeled $\star$ and $\star\star$ commute up to homotopy, by naturality. One then checks that starting at the bottom left corner, going up and around the large left rectangle produces the map $*$ induced by (2-4).

The composite obtained by starting at the upper left-hand corner of the outer rectangle, going down by two, and then right by two computes the proposed operation on $\tau_m X$ over $\tau_m Y$. On the other hand, starting in the upper left-hand corner but going first right by two and then down by two computes the group object structure obtained in Corollary 2.2. □

From this description of the group structure, we obtain group homomorphisms from homotopy classes of maps over one base to homotopy classes of maps over another base. First note the following elementary lemma:

Lemma 2.5 *Consider a diagram*

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"] & X' \arrow[d, "f'"'] \\
Y \arrow[r, "j"] \arrow[u, bend left, "s"] & Y' \arrow[u, bend right, "s'"']
\end{tikzcd} \tag{2-5}$$

such that $hs \simeq s'j$, $jf \simeq f'h$, and s and s' are sections of f and f', respectively. Such a diagram gives rise to a homotopy commutative diagram

$$\begin{tikzcd}
X \times_Y X \arrow[d] & X \cup_Y X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
X' \times_{Y'} X' & X' \cup_{Y'} X' \arrow[l] \arrow[r] & X'
\end{tikzcd}$$

Corollary 2.6 *Let $h\colon X \to X'$ be as in (2-5). Suppose also that f and f' are n-connective and let $m = 2n-2$. Let C be m-skeletal and let $c\colon C \to Y$ be given. Then postcomposition with h induces a group homomorphism $h \circ -\colon [C, X]_c \to [C, X']_{j \circ c}$.*

2.2 Group structures on vector bundles

The goal of this section is to introduce and explore examples of groups comprising unstable vector bundles. First we note the following general procedure for obtaining group structures:

Example 2.7 Let $g\colon X \to Y$ be a map of connected spaces. Consider the diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "p_1"] & Y \arrow[d, "\tau_k"] \\ X \arrow[r, "\tau_k \circ g"] \arrow[u, bend left, dashed, "s_g"] & \tau_k Y \end{tikzcd}$$

where $Z := \tau_k Y$, p_1 is the natural projection, and the section s_g from X to the pullback is induced by the identity on X and $g\colon X \to Y$. The map p_1 is $k+1$-connective since τ_k is. By Corollary 2.2, $\tau_{2k} f\colon \tau_{2k}(X \times_Z Y) \to \tau_{2k} X$ is a group object in $2k$-truncated spaces over $\tau_{2k} X$. So, for any $2k$-skeletal space C with $c\colon C \to X$, the set $[C, X]_c$ inherits a group structure.

The next two examples are the ones that relate to Horrocks' construction [5].

Example 2.8 The map $\det(\gamma_2)\colon BU(2) \to BU(1)$ is 4-connective and admits a section $(\gamma_1 \otimes \mathcal{O}(-b)) \oplus \mathcal{O}(b)$, where $b \in \mathbb{Z}$ is arbitrary. If we fix $a_1 \in \mathbb{Z}$, this gives a group structure on isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$ with first Chern class equal to a_1. The identity is $\mathcal{O}(a_1 - b) \oplus \mathcal{O}(b)$.

As a special case, setting $b = 0$, we obtain the following:

Example 2.9 The determinant map $\det(\gamma_2)\colon BU(2) \to BU(1)$ is 4-connective and admits a section represented by the bundle $\gamma_1 \oplus \underline{\mathbb{C}}$ on $BU(1)$. If we fix $\mathcal{O}(a_1)\colon \mathbb{C}P^3 \to BU(1)$ for $a_1 \in \mathbb{Z}$, this gives a group structure on isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$ with first Chern class equal to a_1. The identity is $\mathcal{O}(a_1) \oplus \underline{\mathbb{C}}$.

We will explore the previous two examples and how they are related in Section 2.3, so we establish notation for each.

Definition 2.10 Let $+_{a_1,b}$ denote the operation on rank 2 bundles on $\mathbb{C}P^3$ with first Chern class a_1 given in Example 2.8. We write $\mathcal{G}_{a_1,b}$ for this group. In the case $b = 0$ (ie Example 2.9), we write $+_{a_1}$ for the operation and $\mathcal{G}_{a_1}$ for the group.

Construction 2.11 Let $b = -\frac{1}{2}a_1$. Consider the commutative diagram

$$\begin{tikzcd} BU(2) \arrow[r, "(-)\otimes\mathcal{O}(-b)"] \arrow[d, "\det(-)"] & BU(2) \arrow[d, "\det(-)"'] \\ BU(1) \arrow[r, "\mathrm{id}"] \arrow[u, dashed, bend left, "(-)\oplus\underline{\mathbb{C}}"] & BU(1) \arrow[u, dashed, bend right, "(-\otimes\mathcal{O}(-b))\oplus\mathcal{O}(b)"'] \end{tikzcd} \tag{2-6}$$

where for maps we write the functor represented on vector bundles. By Corollary 2.6, (2-6) induces a group isomorphism $\phi\colon \mathcal{G}_{a_1} \to \mathcal{G}_{0,b}$ given on bundles by

$$V \mapsto V \otimes \mathcal{O}(-b).$$

Next, we introduce group structures for isomorphism classes of rank 3 bundles on $\mathbb{C}P^5$, which will be our subject in Section 4.

Example 2.12 Consider the diagram

$$\begin{tikzcd} \widetilde{BU}(3) \arrow[r] \arrow[d, "v"] \arrow[dr, phantom, "\lrcorner"] & BU(3) \arrow[d, "c_1\times c_2"] \\ BU(2) \arrow[r, "c_1\times c_2"] \arrow[u, dashed, bend left, "s"] & K(\mathbb{Z},2)\times K(\mathbb{Z},4) \end{tikzcd} \tag{2-7}$$

where

- $c_1 \times c_2$ is the product of the first Chern class and second Chern class;
- $\widetilde{BU}(3)$ is the indicated homotopy pullback; and
- the dotted arrow s to the pullback is induced by the identity map on $BU(2)$ and by $\gamma_2 \oplus \underline{\mathbb{C}}\colon BU(2) \to BU(3)$.

The map v is 6-connective. By Corollary 1.10, for any fixed $V_0\colon \mathbb{C}P^5 \to BU(2)$, the set of homotopy classes of lifts $[\mathbb{C}P^5, \widetilde{BU}(3)]_{V_0}$ carries a group structure. A priori, an element in $[\mathbb{C}P^5, \widetilde{BU}(3)]_{V_0}$ consists of a homotopy class $\mathbb{C}P^5 \to BU(3)$ and a homotopy witnessing $c_i(V) = c_i(V_0)$ for $i = 1, 2$. Such homotopies form a torsor for $H^1(\mathbb{C}P^5;\mathbb{Z}) \times H^3(\mathbb{C}P^5;\mathbb{Z}) \simeq 0$. Therefore the underlying set of this group is in fact isomorphism classes of rank 3 bundles on $\mathbb{C}P^5$ with the same first and second Chern classes as V_0. The identity is $V_0 \oplus \underline{\mathbb{C}}$.

Example 2.12 is our focus in Section 4, so we introduce the following terminology:

Definition 2.13 Let V_0 be a fixed rank 2 bundle on $\mathbb{C}P^3$. Let $+_{V_0}$ denote the operation on rank 3 bundles on $\mathbb{C}P^5$ with first and second Chern classes equal to those of V_0, as defined by Example 2.12. We write $\mathcal{G}_{V_0}$ for this group.

Convention 2.14 We will refer to the binary operations of Definitions 2.10 and 2.13 as *topological Horrocks additions*.

Remark 2.15 To generalize topological Horrocks addition from rank 2 bundles to rank 3 bundles, one might hope to consider isomorphism classes of rank 3 vector bundles on $\mathbb{C}P^5$ with fixed c_1 and c_2.

However, the additional choice of V_0 is necessary. To apply our setup, we would need a section σ of $c_1 \times c_2 \colon BU(3) \to K(\mathbb{Z},2) \times K(\mathbb{Z},4)$.

In $H^*(BU(3);\mathbb{Z}/3\mathbb{Z})$, $P^1(c_2) = c_1^2 c_2 + c_2^2 - c_1 c_3$. In $H^*(K(\mathbb{Z},2) \times K(\mathbb{Z},4);\mathbb{Z}/3\mathbb{Z})$, $P^1(\iota_4) = 0$. Since $(c_1 \times c_2)^*(\iota_{2i}) = c_i$ for $i = 1, 2$, existence of σ would force $\sigma^*(P^1(c_2)) = P^1(\iota_4)$ and $\iota_2^2 \iota_4 + \iota_4^2 = 0$, a contradiction.

2.3 Properties of $+_{a_1}$ and $+_{a_1,b}$

Recall from Definition 2.10 that $+_{a_1}$ and $+_{a_1,b}$ are binary operations, with different identities, on the set of isomorphism classes of rank 2 vector bundles over $\mathbb{C}P^3$ with first Chern class a_1.

Lemma 2.16 *Let V, W and Z be rank 2 vector bundles on $\mathbb{C}P^3$ with first Chern class a_1. Then*

$$(V +_{a_1} W) +_{a_1,b} Z = V +_{a_1} (W +_{a_1,b} Z).$$

Proof Consider the commutative diagram

$$\begin{tikzcd}
 & BU(1) \arrow[dl, "\gamma_1 \oplus \underline{\mathbb{C}}"'] \arrow[dr, "\gamma_1 \oplus \underline{\mathbb{C}}"] & & BU(1) \arrow[dl, "g'"'] \arrow[dr, "g'"] & \\
BU(2) \arrow[dr, "\det \gamma_2"] & & BU(2) \arrow[dl, "\det \gamma_2"'] \arrow[dr, "\det \gamma_2"] & & BU(2) \arrow[dl, "\det \gamma_2"'] \\
 & BU(1) & & BU(1) &
\end{tikzcd} \tag{2-8}$$

where $g' := \gamma_1 \otimes \mathbb{O}(-b) \oplus \mathbb{O}(b)$. To further simplify notation, let

$$g := \gamma_1 \oplus \underline{\mathbb{C}}, \quad d := \det \gamma_2, \quad B_2 := BU(2), \quad B_1 := BU(1).$$

Diagram (2-8) gives a homotopy commutative diagram

$$\begin{tikzcd}
 & & B_2 & & \\
B_2 \cup_{B_1,g'} B_2 \arrow[urr, "\text{fold}"] & & & & B_2 \cup_{B_1,g} B_2 \arrow[ull, "\text{fold}"'] \\
(B_2 \cup_{B_1,g} B_2) \cup_{B_1,g'} B_2 \arrow[u, "\text{fold} \cup \text{id}"] \arrow[d, "\simeq_{\le 6}"] & & B_2 \cup_g B_2 \cup_{g'} B_2 \arrow[ll, "\simeq"'] \arrow[rr, "\simeq"] \arrow[dd] & & B_2 \cup_{B_1,g} (B_2 \cup_{B_1,g'} B_2) \arrow[u, "\text{id} \cup \text{fold}"'] \arrow[d, "\simeq_{\le 6}"'] \\
(B_2 \times_{B_1,d} B_2) \times_{B_1,d} B_2 & & & & B_2 \times_{B_1,d} (B_2 \times_{B_1,d} B_2) \\
 & & B_2 \times_{B_1,d} B_2 \times_{B_1,d} \times B_2 \arrow[ull, "\simeq"] \arrow[urr, "\simeq"'] \arrow[uuuu, dashed, bend left=90, "x"] \arrow[uuuu, dashed, bend right=90, "y"'] & &
\end{tikzcd}$$

where $\simeq_{\leq 6}$ indicates a map that becomes an equivalence after applying the 6-truncation functor and the dashed arrows x and y are the composites defined after applying τ_6 to the entire diagram. The fact that

$$\pi_0 \operatorname{Maps}_{\mathrm{Top}}(\mathbb{C}P^3, -)(x) \cong \pi_0 \operatorname{Maps}_{\mathrm{Top}}(\mathbb{C}P^3, -)(y)$$

proves the lemma. □

This implies:

Corollary 2.17 *Let V and W be rank 2 vector bundles on $\mathbb{C}P^3$ with $c_1(V) = c_1(W) = a_1$. Then*

$$V +_{a_1,b} W = V +_{a_1} W -_{a_1} (\mathbb{O}(a_1 - b) \oplus \mathbb{O}(b)).$$

Proof This is a special case of a general fact. Suppose that $+$ and $*$ are two abelian group structures on a set S such that, for all $x, y, z \in S$,

$$(x + y) * z = x + (y * z).$$

Let e_+ and e_* be the respective identities. Then

$$(x + e_*) * (y + e_*) = (x + e_*) * (e_* + y) = x + (e_* * e_*) + y = x + y + e_*.$$

Set $x = x + (-e_*)$ and $y = y + (-e_*)$ in the above formula, where $-e_*$ is the additive inverse of e_*. This yields

$$x * y = x - e_* + y - e_* + e_* = x + y - e_*.$$

We get the result by applying this to the set of isomorphism classes of vector bundles with first Chern class a_1, letting $+ = +_{a_1}$ and $* = +_{a_1,b}$, and noting that $e_* = \mathbb{O}(a_1 - b) \oplus \mathbb{O}(b)$. □

3 Comparing group structures on rank 2 bundles

In the previous section, we defined the group structures $\mathcal{G}_{a_1}$ and $\mathcal{G}_{a_1,b}$ on the set of isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$ with first Chern class equal to a_1 (see Examples 2.9 and 2.8 and Definition 2.10). Our project in this section is to relate these group structures to Horrocks' construction and prove Theorem 1.5. This will involve Atiyah and Rees' classification of rank 2 bundles on $\mathbb{C}P^3$, so we begin in Section 3.1 with a review of relevant material from [2], including discussion of the α-invariant. We also recall Horrocks' construction for locally free sheaves and its necessary properties.

In Section 3.2, we prove that $c_2 \colon \mathcal{G}_{a_1} \to H^2(\mathbb{C}P^3; \mathbb{Z})$ is a group homomorphism, completing the proof of Proposition 1.3. Verifying additivity of c_2 involves only the definition of the group structure and functoriality results about the group structures of interest, as given in Corollary 2.4. We also show that the α-invariant defines a group homomorphism $\alpha \colon \mathcal{G}_0 \to \mathbb{Z}/2$.

In Section 3.3, we show that topological Horrocks addition and Horrocks' construction produce the same underlying topological isomorphism class, when both are defined (Theorem 1.5). We do this by verifying that both constructions have the same effect on complete invariants of isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$.

3.1 The classification of rank 2 bundles on $\mathbb{C}P^3$ and Horrocks' construction

In [2], Atiyah and Rees study complex rank 2 topological vector bundles on $\mathbb{C}P^3$. They define a $\mathbb{Z}/2$-valued invariant α for bundles with even first Chern class.

Construction 3.1 (the α-invariant [2]) Let V be rank 2 bundle on $\mathbb{C}P^3$ with $c_1 = 0$. Such bundles are classified by $B\mathrm{SU}(2)$. The accidental isomorphism $B\mathrm{SU}(2) \simeq B\mathrm{Sp}(1)$ composed with stabilization gives

$$\widetilde{\alpha}\colon B\mathrm{SU}(2) \to B\mathrm{Sp} \simeq \Omega^\infty \Sigma^4 KO,$$

where KO denotes the real K-theory spectrum. Thus we have a class $\widetilde{\alpha} \in KO^4(B\mathrm{SU}(2))$ and we define

$$\alpha_0(V) := p_* V^*(\widetilde{\alpha}),$$

where p_* is the KO-theory pushforward for the spin manifold $\mathbb{C}P^3$.

Atiyah and Rees extend α to all bundles with $c_1(V) \equiv 0 \pmod 2$ by letting

$$\alpha(V) := \alpha_0\big(V \otimes \mathbb{O}\big(-\tfrac{1}{2}c_1(V)\big)\big). \tag{3-1}$$

Atiyah and Rees give various formulas for computing the α-invariant of special bundles, eg:

Proposition 3.2 [2, Theorem 7.2] *For V a rank 2 bundle on $\mathbb{C}P^3$ with $c_1(V) \equiv 0 \pmod 2$, let*

$$\Delta(V) := \tfrac{1}{4}(c_1^2 - 4c_2).$$

If V extends to $\mathbb{C}P^4$, then $\alpha(V) \equiv \frac{1}{12}\Delta(\Delta - 1) \pmod 2$.

The utility of the α-invariant is as follows:

Theorem 3.3 [2, Theorems 1.1, 2.8 and 3.3] *Given $a_1, a_2 \in \mathbb{Z}$ with $a_1 a_2 \equiv 0 \pmod 2$, the number of isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$ with i^{th} Chern class a_i is*

- *equal to 2 if $a_1 \equiv 0 \pmod 2$, and*
- *equal to 1 otherwise.*

In the first case, a rank 2 vector bundle on $\mathbb{C}P^3$ is determined by c_1, c_2 and α.

Remark 3.4 The condition $a_1 a_2 \equiv 0 \pmod 2$ is necessary and sufficient for two integers to be the Chern classes of complex rank 2 topological bundles on $\mathbb{C}P^3$.

We now review Horrocks' construction for rank 2 algebraic vector bundles.

Construction 3.5 (Horrocks' construction [5]) Let V_1 and V_2 be rank 2 locally free sheaves on $\mathbb{C}P^3$. Suppose that

- we have isomorphisms $\wedge^2 V_1 \simeq \wedge^2 V_2 \simeq \mathbb{O}(a)$ for some $a \in \mathbb{Z}$;
- we have regular[1] sections $s_i : \mathbb{O} \to V_i^*$; and
- the sheaves $\mathcal{R}_i := \mathrm{coker}(s_i^* : \mathcal{F}_i \to \mathbb{O})$ have disjoint supports.

[1]A section s of V is regular if its vanishing locus is of codimension equal to the rank of V.

For each $i = 1, 2$, the Koszul complex relative to s_i has the form $0 \to \mathcal{O}(a) \to V_i \to \mathcal{O}$ and is exact since s_i is regular. By definition, we have exact sequences

$$0 \to \mathcal{O}(a) \to V_i \to \mathcal{O} \to \mathcal{R}_i \to 0.$$

The sheaf $V_1 +_H V_2$ is defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] & O(a) \oplus \mathcal{O}(a) \arrow[r] & V_1 \oplus V_2 \arrow[r] & \mathcal{O} \oplus \mathcal{O} \arrow[r] & \mathcal{R}_1 \oplus \mathcal{R}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(a) \oplus \mathcal{O}(a) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\nabla"'] & \mathcal{W} \arrow[r] \arrow[u] \arrow[d] \arrow[ur, phantom, "\urcorner"] & \mathcal{O} \arrow[r] \arrow[u, "\Delta"'] \arrow[d, "\simeq"'] & \mathcal{R}_1 \oplus \mathcal{R}_2 \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & O(a) \arrow[r] \arrow[ul, phantom, "\ulcorner"] & V_1 +_H V_2 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{R}_1 \oplus \mathcal{R}_2 \arrow[r] & 0
\end{tikzcd} \tag{3-2}$$

where $\mathcal{W}$ is the indicated pullback along the diagonal Δ and $V_1 +_H V_2$ is the pushout along the fold map ∇. The middle and bottom rows in (3-2) are both exact and $V_1 +_H V_2$ is locally free of rank 2 [5, Theorem 1]. The bottom exact sequence of (3-2) implies that

$$c_1(V_1 +_H V_2) = c_1(V_1) = c_1(V_2) = a.$$

Atiyah and Rees and Horrocks study the effect of $+_H$ on algebraic invariants:

Theorem 3.6 [5, Theorem 1; 2, Corollary 5.7] *Let V_1 and V_2 be rank 2 algebraic bundles on $\mathbb{C}P^3$ with $c_1(V_1) = c_1(V_2) = -m$, with $m \geq 0$.*[2] *Then*

$$c_2(V_1 +_H V_2) = c_2(V_1) + c_2(V_2).$$

Furthermore, suppose that $m = 2n$ with $n \geq 0$.

- *If n is odd or $n \equiv 0 \pmod 4$, then $\alpha(V_2 +_H V_2) = \alpha(V_1) + \alpha(V_2) \in \mathbb{Z}/2\mathbb{Z}$.*
- *If $n \equiv 2 \pmod 4$, then $\alpha(V_1 +_H V_2) = \alpha(V_1) + \alpha(V_2) + 1$.*

3.2 Proof of Proposition 1.3

In the previous section, we defined the group structures $\mathcal{G}_{a_1}$ on the set of isomorphism classes of rank 2 bundles on $\mathbb{C}P^3$ with first Chern class equal to a_1 (see Example 2.9 and Definition 2.10). This gives the group structure required by part (i) of Proposition 1.3.

For part (ii), note that, for any $f\colon X \to Y$ with section $s\colon X \to Y$ satisfying the hypotheses of Proposition 1.12, and any $c\colon C \to Y$ with C an m-skeletal space, the identity element is given by $s \circ c$. In this case, $s \circ c \simeq \mathcal{O}(a_1) \oplus \underline{\mathbb{C}}$.

Part (iii) is a consequence of the following result:

[2] This is necessary for V_i to admit regular sections.

Proposition 3.7 *Let $V, W\colon \mathbb{C}P^3 \to BU(2)$ be two bundles with $\det V \simeq \det W \simeq \mathcal{O}(a_1)$. Let $+_{a_1}$ denote topological Horrocks addition as in Definition 2.10. Then*

- *for any $a \in \mathbb{Z}$, the second Chern class c_2 is a homomorphism for $+_{a_1}$; and*
- *if $a_1 = 0$, then α is a homomorphism for $+_0$.*

Proof Let $K(\mathbb{Z},4)$ denote the Eilenberg–Mac Lane space that is determined by $\pi_4(K(\mathbb{Z},4)) = \mathbb{Z}$ and $\pi_i(K(\mathbb{Z},4)) = 0$ for $i \neq 4$. Note that the diagram

$$\begin{tikzcd} BU(2) \arrow[r, "c_2"] \arrow[d, "\det"] & K(\mathbb{Z},4) \arrow[d] \\ BU(1) \arrow[r, "0"] \arrow[u, dashed, bend left, "s"] & * \arrow[u, dashed, bend right] \end{tikzcd}$$

The solid diagram is homotopy commutative and the subdiagram with only dashed vertical arrows is homotopy commutative. By Corollary 2.6, we get a group homomorphism $c_2\colon [\mathbb{C}P^3, BU(2)]_c \to H^4(\mathbb{C}P^3, \mathbb{Z})$.

For the second item, consider the diagram

$$\begin{tikzcd} BU(2) \arrow[d, "c_1"] & BSU(2) \arrow[l, "\iota"'] \arrow[r, "\alpha"] \arrow[d] & \Omega^\infty \Sigma^4 KO \arrow[d] \\ K(\mathbb{Z},2) \arrow[u, dashed, bend left] & * \arrow[l] \arrow[r] \arrow[u, dashed, bend left] & * \arrow[u, dashed, bend left] \end{tikzcd} \tag{3-3}$$

where the solid diagram is homotopy commutative and the subdiagram with only dashed vertical arrows is homotopy commutative. All vertical homotopy fibers are 4-connective, so we can apply Corollary 1.10 to all vertical diagrams to obtain group structures on homotopy classes of maps from $\mathbb{C}P^3$ into each diagram. By Corollary 2.6, we get group homomorphisms

$$[\mathbb{C}P^3, BU(2)]_{c_1=0} \xleftarrow{\iota\circ(-)} [\mathbb{C}P^3, BSU(2)] \xrightarrow{\alpha\circ(-)} [\mathbb{C}P^3, \Omega^\infty\Sigma^4 KO] \simeq KO^4(\mathbb{C}P^3). \tag{3-4}$$

The map $\iota \circ -$ is an isomorphism. For $V\colon \mathbb{C}P^3 \to BU(2)$ with $c_1(V) = 0$,

$$p_*\big(\alpha \circ \big((\iota \circ (-))^{-1}(V)\big)\big) = \alpha(V).$$

Since pushforward on cohomology is a group homomorphism, we conclude that α is additive for $+_0$, as was to be shown. □

3.3 Proof of Theorem 1.5

To show that $V +_{a_1} W \simeq V +_H W$ when both $+_{a_1}$ and $+_H$ are defined, we check that $V +_{a_1} W$ and $V +_H W$ have the same Chern classes and α-invariant. Both $+_{a_1}$ and $+_H$ fix the first Chern class, so it suffices to show that

(i) $c_2(V +_{a_1} W) = c_2(V +_H W)$, and

(ii) $\alpha(V +_{a_1} W) = \alpha(V +_H W)$.

Item (i) follows from additivity of c_2 for both operations, by Theorem 3.6 and Proposition 3.7.

Checking (ii) is complicated by the fact that the α-invariant does not play well with $+_{a_1}$ when $a_1 \neq 0$. We bootstrap from the case $a_1 = 0$ to obtain a formula for $\alpha(V +_{a_1} W)$ in terms of $\alpha(V)$ and $\alpha(W)$. This involves the study of the groups $\mathcal{G}_{0,b}$ for b nonzero (see Example 2.8).

Let $a_1 \in \mathbb{Z}$ and suppose that V and W are rank 2 bundles on $\mathbb{C}P^3$ with $c_1(V) = c_1(W) = a_1$. By Proposition 1.3(iii), $c_2(V +_{a_1} W) = c_2(V) + c_2(W) = c_2(V +_H W)$. In the case $a_1 \equiv 1 \pmod 2$, Theorem 3.3 implies $V +_{a_1} W \simeq V +_H W$ and we are done.

Now suppose $a_1 \equiv 0 \pmod 2$. Recall Construction 2.11 and the associated group isomorphism $\phi\colon \mathcal{G}_{a_1} \to \mathcal{G}_{0,b}$. Using the definition of α (see (3-1)), Proposition 3.7 and Corollary 2.17, we have

$$\begin{aligned}\alpha(V +_{a_1} W) &= \alpha(\phi(V +_{a_1} W))\\ &= \alpha(\phi(V) +_{0,b} \phi(W))\\ &= \alpha(\phi(V) +_0 \phi(W) -_0 \mathcal{O}(-b) \oplus \mathcal{O}(b))\\ &= \alpha(V) + \alpha(W) - \alpha(\mathcal{O}(-b) \oplus \mathcal{O}(b)).\end{aligned}$$

To compute $\alpha(\mathcal{O}(b) \oplus \mathcal{O}(-b))$, we use Proposition 3.2. Since $\mathcal{O}(-b) \oplus \mathcal{O}(b)$ extends to $\mathbb{C}P^4$ and $\Delta(\mathcal{O}(b) \oplus \mathcal{O}(-b)) = b^2$, we see that

$$\alpha(\mathcal{O}(-b) \oplus \mathcal{O}(b)) \equiv \tfrac{1}{12} b^2(b^2-1) \equiv \tfrac{1}{4} b^2(b+1)(b-1) \pmod 2.$$

So we must determine when $b^2(b+1)(b-1)$ is divisible by 8.

- If $b \equiv 0 \pmod 4$, then 16 divides $b^2(b+1)(b-1)$, so $\alpha(\mathcal{O}(b) \oplus \mathcal{O}(-b)) = 0$.
- If b is odd, then both $b+1$ and $b-1$ are even and one of $b+1$ or $b-1$ is divisible by 4, so $\alpha(\mathcal{O}(b) \oplus \mathcal{O}(-b)) = 0$.
- If $b \equiv 2 \pmod 4$, then both $b+1$ and $b-1$ are odd and b^2 is divisible by 4 but not 8, so $\alpha(\mathcal{O}(b) \oplus \mathcal{O}(-b)) = 1$.

Thus,

$$\alpha(V +_a W) = \alpha(V) + \alpha(W) + \epsilon(a), \qquad \text{where} \qquad \epsilon(a) := \begin{cases} 0 & \text{if } a \not\equiv 4 \pmod 8, \\ 1 & \text{if } a \equiv 4 \pmod 8. \end{cases}$$

Comparing this with Theorem 3.6 yields the conclusion $\alpha(V +_{a_1} W) = \alpha(V +_H W)$. □

While this completes the argument that algebraic and topological Horrocks addition produce the same underlying topological isomorphism class, our method is indirect. The reader might hope for a more geometric relationship between the constructions.

Problem 3.8 Is there an explicit, bundle-theoretic comparison between topological Horrocks addition $+_{a_1}$ and Horrocks' original construction for locally free sheaves?

4 Group structures on rank 3 bundles on $\mathbb{C}P^5$

We now explore additive structures on rank 3 bundles on $\mathbb{C}P^5$. Recall that, given a fixed rank 2 bundle V_0 on $\mathbb{C}P^5$, Example 2.12 gives a group $\mathscr{G}_{V_0}$ of isomorphism classes of rank 3 vector bundles with the same first and second Chern classes as V_0. The identity is $V_0 \oplus \underline{\mathbb{C}}$. Our primary interest here is to better understand the groups $\mathscr{G}_{V_0}$ and their properties.

We complete the proof of Theorem 1.6 in Section 4.1 by showing that c_3 is a group homomorphism. In Section 4.2, we explore the structure of $\mathscr{G}_{V_0}$ for $V_0 = \mathbb{O}(a) \oplus \mathbb{O}(b)$. We show that $+_{V_0}$ allows for the construction of interesting bundles from simple ones, as did Horrocks' construction on rank 2 bundles on $\mathbb{C}P^3$ (see Theorem 1.1).

Proposition 4.1 *For infinitely many isomorphism classes of rank 2 bundles $V_0 := \mathbb{O}(a) \oplus \mathbb{O}(b)$ on $\mathbb{C}P^5$, there exists a bundle $W = \mathbb{O}(x) \oplus \mathbb{O}(y) \oplus \mathbb{O}(z)$ with $W \in \mathscr{G}_{V_0}$ such that the subgroup generated by W contains bundles that are not sums of line bundles. Moreover, the subgroup generated by W is of finite index in $\mathscr{G}_{V_0}$.*

The proof of this result is elementary, involving only the study of possible Chern classes of sums of line bundles and additivity of c_3.

4.1 The third Chern class and the structure of $\mathscr{G}_{V_0}$ for rank 3 bundles

We first prove the additivity of $+_{V_0}$ for c_3 and complete a theorem stated in the introduction.

Proof of Theorem 1.6 We have already established the group structure in Example 2.12. The identity element is $V_0 \oplus \underline{\mathbb{C}}$ by Corollary 1.10.

To show that the third Chern class c_3 is a homomorphism, consider the diagram

$$\begin{tikzcd}
\widetilde{BU}(3) \arrow[r, "c_3"] \arrow[d, shift left] & K(\mathbb{Z},6) \arrow[d, shift right] \\
BU(2) \arrow[r, "0"] \arrow[u, dashed, bend left, "s"] & * \arrow[u, dashed, bend right]
\end{tikzcd}$$

where the solid diagram is homotopy commutative and the subdiagram with only dashed vertical arrows is homotopy commutative. By Corollary 2.6, we get a group homomorphism $c_3 \colon [\mathbb{C}P^3, BU(3)]_c \to H^6(\mathbb{C}P^5, \mathbb{Z})$. $\square$

In [11], we prove that rank 3 bundles on $\mathbb{C}P^5$ are determined by Chern classes unless $c_1(V) \equiv c_2(V) \equiv 0 \pmod 3$, in which case there are three pairwise nonisomorphic bundles with the same Chern classes as V. This proves the following result:

Corollary 4.2 *The kernel of* $c_3\colon \mathcal{G}_{V_0} \to \mathbb{Z}$ *is*

(1) *trivial if* $c_1(V_0) \not\equiv 0 \pmod 3$ *or* $c_2(V_0) \not\equiv 0 \pmod 3$, *and*

(2) *isomorphic to* $\mathbb{Z}/3$ *if* $c_1(V_0) \equiv c_2(V_0) \equiv 0 \pmod 3$.

Remark 4.3 The image of c_3 is an infinite subgroup of $\mathbb{Z}$, since the conditions for three integers to be the Chern classes of a rank 3 bundle on $\mathbb{C}P^5$ are a finite number of congruences (see [11, Lemma 2.16]). In the first case in Corollary 4.2, $\mathcal{G}_{V_0}$ is abstractly isomorphic to $\mathbb{Z}$; in the second, $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$.

In [11], we define a $\mathbb{Z}/3$-valued invariant ρ for rank 3 bundles V on $\mathbb{C}P^5$ such that $c_1(V) \equiv c_2(V) \equiv 0 \pmod 3$. We prove that c_1, c_2, c_3 and ρ are complete invariants of isomorphism classes of complex rank 3 topological bundles on $\mathbb{C}P^5$. Additivity of c_3 leads to the following:

Problem 4.4 For general V_0 with first and second Chern classes divisible by 3, what is a formula for $\rho(V +_{V_0} W)$ in terms of $\rho(V)$, $\rho(W)$ and $\rho(V_0)$?

Note that a necessary condition for ρ to be a group homomorphism is that $\rho(V_0 \oplus \underline{\mathbb{C}}) = 0$.

4.2 Constructing rank 3 bundles on $\mathbb{C}P^5$ from sums of line bundles

We begin this subsection by showing that there exist infinitely many isomorphism classes $V_0 = \mathcal{O}(a) \oplus \mathcal{O}(b)$ on $\mathbb{C}P^3$ such that $\mathcal{G}_{V_0}$ contains another sum of line bundles that is not the group identity. This is a preliminary to prove Proposition 4.1. First, we simplify notation.

Definition 4.5 Given $n_1, \dots, n_k \in \mathbb{Z}$, let $\mathcal{O}(n_1, \dots, n_k) := \mathcal{O}(n_1) \oplus \dots \oplus \mathcal{O}(n_k)$.

An element $\mathcal{O}(x, y, z) \in \mathcal{G}_{\mathcal{O}(a,b)}$ is equivalent to an integer solution to the equations

$$x + yz = a + b, \quad xy + yz + zx = ab. \tag{4-1}$$

Note that, since the identity of $\mathcal{G}_{\mathcal{O}(a,b)}$ is $\mathcal{O}(a, b) \oplus \underline{\mathbb{C}}$, a sum of line bundles $\mathcal{O}(x, y, z) \in \mathcal{G}_{\mathcal{O}(a,b)}$ is a nonidentity element if and only if $c_3(\mathcal{O}(x, y, z)) = xyz$ is nonzero.

Let $c := a - x$ and $d := b - y$. Substituting in (4-1), we obtain

$$Q := c^2 + d^2 - bd - ac + cd = 0.$$

This is the equation of a quadric hypersurface in $\mathbb{P}^3_{a,b,c,d}$, whose rational points can be found via standard methods. For any $u, l, v, w \in \mathbb{Z}$, we get solutions to (4-1)

$$\begin{aligned} a &= w(u^2 + v^2 + uv - lv), & x &= w(v^2 + uv - lv), \\ b &= wul, & y &= wu(l - v), \\ & & z &= wu(u + v). \end{aligned}$$

Remark 4.6 The previous computation shows that there are infinitely many $\mathcal{G}_{V_0}$, where $V_0 = \mathcal{O}(a, b)$, such that $W = \mathcal{O}(x, y, z)$ is a nonidentity element of $\mathcal{G}_{V_0}$.

Example 4.7 Consider $W = \mathcal{O}(2,-1,2)$ and $V_0 = \mathcal{O}(3,0)$. By construction, $W \in \mathcal{G}_{V_0}$. Since c_1 is even and c_1 and c_2 are both divisible by 3, the Schwarzenberger conditions as in [11, Lemma 2.16] imply that there exists a bundle $W' \in \mathcal{G}_{V_0}$ with $c_3(W') = a$ if and only if a is even. By Remark 4.3, $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$ and, under this identification, c_3 is projection onto the first factor. Since $c_3(W) = 4$, this implies that the subgroup generated by W is of index 6.

Proof of Proposition 4.1 Let $V_0 = \mathcal{O}(a,b)$ and $W = \mathcal{O}(x,y,z)$ be as in Remark 4.6 with $c_3(W) \neq 0$. Inductively, let

$$+^n_{V_0} W := W +_{V_0} (+^{n-1}_{V_0} W).$$

Assume for a contradiction that $+^n_{V_0} W$ is a sum of line bundles for all positive integers n. Let $a_1, a_2, a_3 \in \mathbb{Z}$ denote the Chern classes of W. Since $+_{V_0}$ preserves c_1 and c_2, and $c_3 : \mathcal{G}_{V_0} \to H^6(\mathbb{C}P^5; \mathbb{Z})$ is a group homomorphism,

$$c_1(+^n_{V_0} W) = a_1, \quad c_2(+^n_{V_0} W) = a_2 \quad \text{and} \quad c_3(+^n_{V_0} W) = na_3.$$

Thus, the equations

$$X + Y + Z = a_1, \quad XYZ = na_3, \quad XY + YX + ZX = a_2$$

have a solution $(X, Y, Z) = (x_n, y_n, z_n)$ for all $n \in \mathbb{Z}$. Let p be a prime such that $p > |3a_i|$ for $i = 1, 3$. Take $n = p$ in the above equations. Since $c_3(W) = a_3 \neq 0$, the third equation implies that p divides one of x_p, y_p or z_p. Without loss of generality, p divides x_p. But note that

$$|z_p y_p| \le |a_3| < \tfrac{1}{3} p$$

and so $|z_p| < \frac{1}{3} p$, and $|y_p| < \frac{1}{3} p$. Since $|x_p|$ is divisible by p, it is larger than $|y_p|$, $|z_p|$ and $|y_p + z_p|$. Therefore,

$$|x_p + y_p + z_p| \ge |x_p| - |y_p| - |z_p| v \ge \tfrac{1}{3} p.$$

On the other hand, $|x_p + y_p + z_p| = |a_1| < \frac{1}{3} p$, a contradiction.

Now consider the subgroup generated W. Since $c_3(W) \neq 0$, $\ker(c_3) \cap \langle W \rangle$ is trivial and we have an exact sequence of groups

$$\ker(c_3) \to \mathcal{G}_{V_0}/\langle W \rangle \to \mathbb{Z}/c_3(W).$$

Finally, $\ker(c_3)$ is finite by Remark 4.3 and $\mathbb{Z}/c_3(W)$ is finite, so $\mathcal{G}_{V_0}/H$ is also finite. □

This section gives insight into the groups $\mathcal{G}_{V_0}$, but many questions remain.

Problem 4.8 (1) For groups $\mathcal{G}_{V_0} \simeq \mathbb{Z} \oplus \mathbb{Z}/3$, what are generators for 3-torsion?

(2) Are the Horrocks bundles of rank 3 on $\mathbb{C}P^5$ elements of some $\mathcal{G}_{V_0}$? If so, what subgroups do they generate?

(3) What can be said about the structure of $\mathcal{G}_{V_0}$ for V_0 not a sum of line bundles?

Appendix The pointed Grothendieck construction for spaces

This appendix describes the pointed version straightening and unstraightening for functors to spaces, combining facts from [9]. The particular goal is to make precise the statement that a diagram of spaces

$$X \underset{s}{\overset{f}{\rightleftarrows}} Y \tag{A-1}$$

where f is n-connective can equivalently be viewed as a functor from Y to infinite loop spaces, after $(2n-2)$-truncation.

Notation A.1 We establish terminology and conventions and provide references.

1. Let sSet denote the category of simplicial sets and Cat_Δ denote the category of simplicially enriched categories.
2. By $(\infty,1)$-category, we will mean a simplicial set that satisfies the inner horn condition (a quasi-category [9, 1.1.2.4]).
3. Let $\mathfrak{C}$ denote the left adjoint to the homotopy coherent nerve functor $N\colon \mathrm{Cat}_\Delta \to \mathrm{sSet}$ [9, 1.1.5]. There is a Quillen equivalence of simplicial model categories
$$\begin{tikzcd} \mathrm{Cat}_\Delta \arrow[r, bend right, "N"'] & \mathrm{sSet} \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd}$$
between simplicial sets with the Joyal model structure and simplicially enriched categories with the Bergner model structure (see [9, Section 1.1.5] for discussion, and [4; 3; 8] for a proof).
4. Let $\mathrm{sSet}_{/S}$ denote the simplicial model category of simplicial sets over S, endowed with the contravariant model structure. We have an associated simplicial category $\mathrm{RFib}(S)$ obtained by taking fibrant–cofibrant objects.
5. Let Kan denote the $(\infty,1)$-category of Kan complexes obtained by taking fibrant–cofibrant objects in the simplicial model category of simplicial sets with the Kan model structure and applying N. Given a Kan complex S, let $\mathrm{Kan}_{/S}$ denote the $(\infty,1)$-overcategory of Kan complexes over S.
6. Given a $(\infty,1)$-category C and a simplicial set S, we write $\mathrm{Fun}(S,C)$ for the simplicial set of maps from S to C. This is an $(\infty,1)$-category since C is, and models functors from S to C [9, 1.2.7.2].
7. We ignore set-theoretic issues and refer the concerned reader to [9, 1.2.15].

The fundamental result that we need is the following:

Theorem A.2 (Lurie [9]) *Let S be a Kan complex. There is an equivalence of $(\infty,1)$-categories*

$$\begin{tikzcd} \mathrm{Kan}_{/S} \arrow[r, bend right, "\mathrm{St}"'] & \mathrm{Fun}(S,\mathrm{Kan}). \arrow[l, bend right, "\mathrm{Un}"'] \end{tikzcd} \tag{A-2}$$

Proof By [9, 2.2.3.11], for any simplicial set we have an equivalence of simplicial categories

$$\begin{tikzcd} \mathrm{RFib}(S) \arrow[r, bend right, "\mathrm{St}"'] & (\mathrm{sSet}^{\mathfrak{C}[S]^{\mathrm{op}}})^{\circ}, \arrow[l, bend right, "\mathrm{Un}"'] \end{tikzcd} \tag{A-3}$$

where sSet has the Kan model structure, $\mathrm{sSet}^{\mathfrak{C}[S]^{\mathrm{op}}}$ has the projective model structure, and $(-)^{\circ}$ indicates we take fibrant–cofibrant objects. Since S is a Kan complex, [9, 3.1.5.1(A3)] implies $\mathrm{Kan}_{/S} \simeq N(\mathrm{RFib}(S))$ as $(\infty, 1)$-categories.

By [9, 4.2.4.4], applying the homotopy coherent nerve to the right side of (A-3) recovers $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Kan})$. By [12, Section 57] combined with [12, Proposition 14.14], every Kan complex is equivalent to its opposite as an $(\infty, 1)$-category. □

To obtain a version for spaces over a base with a section, we take pointed objects on either side of Theorem A.2.

Definition A.3 Given an $(\infty, 1)$-category C with final object t [9, 1.2.12], let C_* denote the full $(\infty, 1)$-subcategory of $\mathrm{Fun}(\Delta^1, C)$ spanned by functors that restrict to t on 0. Explicitly, if fib_t denotes the (1-categorical) fiber in sSet at t, then

$$C_* := \mathrm{fib}_t(\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^0, C)),$$

where the map is given by precomposition with $\{0\} \simeq \Delta^0 \hookrightarrow \Delta^1$.

Remark A.4 Given an $(\infty, 1)$-category C with final object t, the infinity category C_* is also modeled by the $(\infty, 1)$-undercategory $C_{t/}$. See [9, 1.2.9] for a discussion of $(\infty, 1)$-undercategories and [9, 4.2.1.5 and 7.2.2.8] for the equivalence.

Since equivalent $(\infty, 1)$-categories support the same category theory, the equivalences St and Un will induce one on pointed objects.

Lemma A.5 *An equivalence of $(\infty, 1)$-categories induces an equivalence upon taking pointed objects.*

Proof This is the dual statement of [9, 1.2.9.3] (dualized, for example, following the method of [9, 1.2.9.5]), specialized to the case of a final object. □

We can apply Lemma A.5 directly to the equivalences of Theorem A.2, but to understand the result we first describe pointed objects on the left side of (A-2).

Lemma A.6 *For any $(\infty, 1)$-category C with a final object t, the $(\infty, 1)$-category $\mathrm{Fun}(S, C)$ has as final object the constant functor at t. There is an equivalence of $(\infty, 1)$-categories $\mathrm{Fun}(S, C)_* \simeq \mathrm{Fun}(S, C_*)$.*

Proof The first statement follows from the dual of [9, 5.1.2.3]. For the second, recall that $\mathrm{Fun}(S, C)_*$ is modeled by the fiber at the final object of the map

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(S, C)) \to \mathrm{Fun}(\Delta^0, \mathrm{Fun}(S, C))$$

given by restricting to $\{0\} \hookrightarrow \Delta^1$. Moreover, there is an adjunction $S \times - \dashv \mathrm{Fun}(S, -)$ as enriched functors $\mathrm{sSet} \to \mathrm{sSet}$, since $\mathrm{Fun}(S, -)$ is an internal hom in the monoidal category $(\mathrm{sSet}, \times, \Delta^0)$. So we get isomorphisms of simplicial sets

$$\begin{aligned}\mathrm{fib}\big(\mathrm{Fun}(\Delta^1, \mathrm{Fun}(S, C)) \to \mathrm{Fun}(\Delta^0, \mathrm{Fun}(S, C))\big) &\simeq \mathrm{fib}(\mathrm{Fun}(\Delta^1 \times S, C) \to \mathrm{Fun}(\Delta^0 \times S, C)) \\ &\simeq \mathrm{fib}\big(\mathrm{Fun}(S, \mathrm{Fun}(\Delta^1, C)) \to \mathrm{Fun}(S, \mathrm{Fun}(\Delta^0, C))\big).\end{aligned}$$

Since right adjoints preserve limits,

$$\mathrm{fib}\big(\mathrm{Fun}(S, \mathrm{Fun}(\Delta^1, C)) \to \mathrm{Fun}(S, \mathrm{Fun}(\Delta^0, C))\big) \simeq \mathrm{Fun}\big(S, \mathrm{fib}(\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^0, C))\big).$$

Tracing through our equivalences, we have shown that $\mathrm{Fun}(S, C)_* \simeq \mathrm{Fun}(S, C_*)$. □

Corollary A.7 *Straightening and unstraightening induce mutually inverse equivalences of* $(\infty, 1)$*-categories* $\mathrm{St}_* \colon (\mathrm{Kan}_{/S})_* \to \mathrm{Fun}(S, \mathrm{Kan}_*)$ *and* $\mathrm{Un}_* \colon \mathrm{Fun}(S, \mathrm{Kan}_*) \to (\mathrm{Kan}_{/S})_*$.

We record one more result about pointed objects in $(\infty, 1)$-categories.

Lemma A.8 *Let* $\mathscr{C}$ *be a* $(\infty, 1)$*-category with a final object* t *and with coproducts. Then there is an adjunction of* $(\infty, 1)$*-categories*

$$\begin{tikzcd} C \arrow[r, bend right, "+"'] \arrow[r, phantom, "\top"] & C_* \arrow[l, bend right, "\text{forget}"'] \end{tikzcd}.$$

Proof We write F for the forgetful functor. This functor is given on objects by $(t \to Y) \mapsto Y$. The functor $+$ is given on objects by $c \mapsto c \sqcup t$, where $\sqcup$ is the $(\infty, 1)$-coproduct in C [9, 4.4.1].

To show $+$ is left adjoint to F, we use [9, 5.2.2.8]: it suffices to provide a unit transformation

$$u \colon \mathrm{Id}_C \to \mathrm{F} \circ +$$

in $\mathrm{Fun}(C, C)$ such that a certain associated composite is an isomorphism in the homotopy category. Let $u_c \colon c \to t \sqcup c$ be the structure map of the coproduct. We must show

$$\mathrm{Maps}_{C_*}(t \to c \sqcup t, t \to y) \xrightarrow{\text{apply F}} \mathrm{Maps}_C(\mathrm{F}(t \to c \sqcup t), \mathrm{F}(t \to y)) \xrightarrow{u_c \circ -} \mathrm{Maps}_C(c, y)$$

is an isomorphism in the homotopy category of spaces for all c and y in C. Combining [9, 5.5.5.12] with [9, Lemma 7.2.2.8 and Proposition 4.2.1.5], we have an equivalence of spaces

$$\mathrm{Maps}_{C_*}(t \to c \sqcup t, t \to y) \simeq \mathrm{hofib}(\mathrm{Maps}_C(c \sqcup t, y) \to \mathrm{Maps}_C(t, y)),$$

where the arrow $\mathrm{Maps}_C(c \sqcup t, y) \to \mathrm{Maps}_C(t, y)$ is induced by precomposing with the given map $t \to c \sqcup t$, and the fiber is taken over $t \to y \in C_*$ viewed as an object in $\mathrm{Maps}_C(t, y)$. Consider the homotopy commutative diagram of spaces

$$\begin{tikzcd}
\mathrm{Maps}_C(c \sqcup t, y) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Maps}_C(t, y) \arrow[d, equal] \\
\mathrm{Maps}_C(c, y) \times \mathrm{Maps}_C(t, y) \arrow[r, "\pi_2"] & \mathrm{Maps}_C(t, y)
\end{tikzcd}$$

This implies

$$\mathrm{Maps}_{C_*}(t \to c \sqcup t, t \to y) \simeq \mathrm{hofib}(\pi_2 \colon \mathrm{Maps}_C(c, y) \times \mathrm{Maps}_C(t, y) \to \mathrm{Maps}_C(t, y)) \simeq \mathrm{Maps}_C(c, y).$$

We have a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Maps}_{C_*}(t \to c \sqcup t, t \to y) \arrow[r, "\simeq"] \arrow[d, "\text{apply F}"'] & Z \arrow[r, "\simeq"] \arrow[d] & \mathrm{Maps}_C(c, y) \arrow[dl, "1 \times (t \to y)"] \arrow[ddl, bend left, "1"] \\
\mathrm{Maps}_C(c \sqcup t, y) \arrow[r, "\simeq"] \arrow[d, "u_c \circ -"'] & \mathrm{Maps}_C(c, y) \times \mathrm{Maps}_C(t, y) \arrow[d, "\pi_1"'] & \\
\mathrm{Maps}_C(c, y) \arrow[r, equal] & \mathrm{Maps}_C(c, y) &
\end{tikzcd}$$

where $Z := \mathrm{fib}\big((\mathrm{Maps}_C(c, y) \times \mathrm{Maps}_C(t, y)) \xrightarrow{\pi_2} \mathrm{Maps}_C(t, y)\big)$. The rightmost map is a weak homotopy equivalence, as are all the horizontal arrows, so the left vertical composite is too. $\square$

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