Power sum elements in the G_2 skein algebra

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We study the skein algebras of surfaces associated to the exceptional Lie group G_2, using Kuperberg webs. We identify two 2-variable polynomials, $P_n(x, y)$ and $Q_n(x, y)$, and use threading operations along knots to construct a family of central elements in the G_2 skein algebra of a surface, $\mathcal{S}_q^{G_2}(\Sigma)$, when the quantum parameter q is a $2n^{\text{th}}$ root of unity. We verify these elements are central using elementary skein-theoretic arguments. We also obtain a result about the uniqueness of the so-called transparent polynomials P_n and Q_n. Our methods involve a detailed study of the skein modules of the annulus and the twice-marked annulus.

57K31; 17B37

1 Introduction

The subject of skein theory lies at the intersection of low-dimensional topology and the representation theory of quantum groups. In skein theory, one works with linear relations involving pictures of skeins, which are hybrid objects that can be readily interpreted either in the language of topology or in the language of algebra. The simplest examples of these skein relations are the Kauffman bracket skein relations:

$$\diagup\!\!\!\!\diagdown = q\,\big)\,\big(\, + q^{-1}\,\overset{\smile}{\frown}, \quad \bigcirc = -q^2 - q^{-2}.$$

On one hand, these relations can be used to define link invariants, like the Jones polynomial, or to construct the 3-manifold invariants of Witten, Reshetikhin, and Turaev; see Lickorish [18, Chapter 13]. On the other hand, these relations faithfully describe the representation theory of quantum SL_2; see eg Kuperberg [15].

We are interested in the skein theory associated to the rank-2 exceptional Lie group, G_2. In this setting, skeins are certain graphs called *webs*. The skein theory for G_2 webs, and for the cases of other rank-2 Lie groups, was developed in the seminal work of Kuperberg [14; 15]. Webs for higher-rank type A_n were introduced by Murakami, Ohtsuki, and Yamada [23] for the purpose of a graphical definition of the colored HOMFLY polynomial. Later, it was shown by Cautis, Kamnitzer, and Morrison [6] that the higher-rank webs and their skein relations provide a full description of the representation category of

quantum SL_d. Finding explicit skein relations which faithfully describe the representation categories of Lie groups for all Dynkin types is an active area of research; see Bodish, Elias, Rose, and Tatham [2] and Savage and Westbury [28] for recent examples. Many of these skein theories are first developed in the setting where q is a generic parameter. One can also study whether the same skein relations still faithfully describe, in some sense, the representation category when q is root of unity; see Bodish [1] and Elias [7] for instance.

The setting of our work here is of G_2 webs on surfaces, with particular focus on the case when q is a root of unity. Two webs on a surface can be multiplied by stacking web diagrams on the surface, yielding an algebra structure. These skein algebras of surfaces are rich algebras related to both quantum groups and character varieties. Skein algebras are generally noncommutative, and their centers depend on the parameter q. Our main goal is to use the skein relations of Kuperberg to find central elements in the G_2 skein algebras.

1.1 Power sum elements

There is substantial literature on the subject of the SL_2 case of Kauffman bracket skein algebras of surfaces. In the root of unity case, surprising central elements were discovered in the work of Bonahon and Wong [5]. These central elements arise from a topological operation of threading of certain polynomials along knots. The polynomials in question are the Chebyshev polynomials $T_n \in \mathbb{Z}[x]$ which satisfy the following fundamental property regarding the trace of a power of a matrix $A \in SL_2$:

$$T_n(\operatorname{tr}(A)) = \operatorname{tr}(A^n).$$

Since the trace of a power of a matrix is the power sum of the eigenvalues of the matrix, the polynomials T_n can be referred to as power sum polynomials. These central elements obtained by threading power sum polynomials along knots may be referred to as power sum elements in the skein algebra. We are interested in finding analogous elements for the case of G_2.

The Chebyshev elements at roots of unity play a fundamental role in studying the representation theory of SL_2 skein algebras; see Bonahon and Wong [5], Frohman, Kania-Bartoszynska, and Lê [10], and Ganev, Jordan, and Safronov [12]. For closed surfaces, the Chebyshev elements generate the whole center of the skein algebra [10]. We expect that the power sum elements found in this paper will play a similar role for the case of G_2.

Power sum elements have also been studied in the setting of a generic parameter q, and they satisfy product-to-sum formulas which allow for an explicit presentation of the skein algebra of the torus. They have been studied for each of the settings of the SL_2, HOMFLYPT, and Kauffman skein theories; see Frohman and Gelca [9], Morton and Samuelson [22], and Morton, Pokorny, and Samuelson [21]. For the SL_2 case, Chebyshev elements are also involved in the construction of positive bases of skein algebras of surfaces, a construction which is important in the program of categorification of surface skein algebras and also is related to the theory of cluster algebras; see Mandel and Qin [19], Queffelec [26], and Thurston [30].

1.2 Main results

We study the G_2 skein algebra, $\mathcal{S}_q^{G_2}(\Sigma)$, of a surface Σ using the webs of Kuperberg. For our ring of coefficients, we use a commutative integral domain $\mathcal{R}$ containing a distinguished invertible element $q \in \mathcal{R}$. Due to denominators appearing in the defining skein relations, we further assume that $[12]^{-1} \in \mathcal{R}$, where the quantum integer $[k]$ denotes the Laurent polynomial defined by $[k] = (q^k - q^{-k})/(q - q^{-1})$.

In Section 4.2, we construct 2-variable polynomials $P_n(x, y)$ and $Q_n(x, y)$ in $\mathcal{R}[x, y]$ that are related to taking traces of powers of matrices in the two fundamental matrix representations of the Lie group G_2. We hence call elements obtained by threading P_n or Q_n along knots *power sum elements* in the skein algebra. These elements satisfy a transparency property described in Section 3.3, and the polynomials P_n and Q_n provide examples of so-called *transparent polynomials*.

Theorem 1.1 (Theorem 5.7) *If $q^{2n} = 1$ then the element obtained by threading the polynomial P_n or the polynomial Q_n along a knot diagram on a surface Σ is a central element of the G_2 skein algebra of the surface, $\mathcal{S}_q^{G_2}(\Sigma)$.*

We also obtain a result addressing the uniqueness of those polynomials which yield central elements by threadings.

Theorem 1.2 (Theorem 5.9) *Suppose that q^2 is a primitive root of unity of order n coprime to 3. Then any transparent polynomial $S(x, y)$ is generated by P_n and Q_n. In other words, $S(x, y)$ must be an element of the polynomial ring $\mathcal{R}[P_n, Q_n]$.*

Along the way, we study the confluence theory of the G_2 skein relations (Theorem 2.4) to describe bases of G_2 skein modules of surfaces. This had already been studied by Kuperberg [15] and Sikora and Westbury [29] using a smaller set of skein relations than the set that we use. We have incorporated the relations from the more recent works of Bodish and Wu [3] and Sakamoto and Yonezawa [27]. See Remark 2.5 comparing the two potential sets of skein relations.

The computations involved in establishing our main theorems take place inside of an algebra associated to the annulus $\mathcal{A}$, and an algebra associated to the twice-marked annulus $\mathcal{A}_{1,1}$. These computations are inspired by the earlier work of Lê in the SL_2 setting [16]. Using our constructed bases, we show that $\mathcal{A} \cong \mathcal{R}[x, y]$ and that $\mathcal{A}_{1,1}$ is a commutative algebra, obtaining an explicit presentation for it in Corollary 3.10. These are essential ingredients for us in connecting the theory of commutative polynomials to the skein theory of G_2 webs.

A similar approach in constructing transparent polynomials using the explicit skein relations for the setting of SL_d webs was carried out by Bonahon and the third author in [4]. Informally, the strategy common between these papers is to attempt to match up certain elements in the commutative ring $\mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$ with certain diagrammatic elements in $\mathcal{A}_{1,1}$. In the SL_d case, this strategy goes exactly as planned. When the analogous approach was tried in this project for G_2 webs, it produced a surprising error term. In Section 5.2 we discuss this error term and explain how to navigate around it.

Our approach here is elementary, and uses the explicit skein relations of Kuperberg. It would be quite interesting to extend this work to find a skein-theoretic proof that the threading operation respects the Kuperberg web relations and describes an algebra homomorphism $\mathcal{S}_1^{G_2}(\Sigma) \to \mathcal{S}_q^{G_2}(\Sigma)$ when q is an appropriate root of unity. There is an alternative approach to establishing the existence of these central elements, using the theory of quantum groups. It involves the Frobenius map of Lusztig at roots of unity; see Ganev, Jordan, and Safronov [12]. Kuperberg [15] showed that the G_2 webs and skein relations faithfully describe a full subcategory of the category of representations associated to the quantum group G_2 when the parameter q is generic. It is predicted that the same still holds when q is specialized to most roots of unity; see Bodish and Wu [3] and Morrison [20]. Such a result should provide an avenue for the translation of results from the algebraic setting of quantum groups at roots of unity to the skein-theoretic setting of Kuperberg webs. We expect that elements obtained by threading our polynomials P_n and Q_n along knots form a generating set for the image of the Frobenius map described in [12].

1.3 Multivariate Chebyshev polynomials

In the case of SL_2, there are two families of single-variable polynomials T_n and S_n, which satisfy a linear relationship $T_n = S_n - S_{n-2}$ and play fundamental roles in the theory of SL_2 and its representation ring. The polynomials P_n and Q_n that we study here are 2-variable generalizations of the Chebyshev polynomial of the first kind, T_n, which satisfies the power sum property $T_n(\mathrm{tr}(A)) = \mathrm{tr}(A^n)$ for a matrix A in SL_2. The closely related Chebyshev polynomial of the second kind, S_n, satisfies the equation $S_n([V_1]) = [V_n]$ in the representation ring of SL_2, which expresses the representation V_n of highest weight n as a polynomial in the vector representation V_1. The polynomial S_n can be obtained diagrammatically by taking the Jones–Wenzl idempotent in the Temperley–Lieb algebra, $\mathrm{JW}_n \in \mathrm{TL}_n$, and taking its annular closure, considered as an element of the SL_2 skein algebra of the annulus $\mathcal{S}_q^{\mathrm{SL}_2}(\mathbb{A}) \cong \mathcal{R}[x]$; see Lickorish [18, Chapter 13].

Nesterenko, Patera, and Tereszkiewicz [24] give recursive definitions for two families of polynomials, which they call C-polynomials and S-polynomials, generalizing the Chebyshev polynomials of the first and second kinds for compact Lie groups. In the case of G_2, their polynomials $C_{(a,b)}$ are parametrized by dominant integral weights. We believe our polynomials P_n and Q_n are the same as their polynomials $C_{(n,0)}$ and $C_{(0,n)}$, although a change of variables is required. The basic variables of [24] correspond to the so-called C-functions of the two fundamental representations, while our basic variables are the characters of the two fundamental representations.

In [3], for each dominant integral weight (a, b), Bodish and Wu construct the G_2 analogue of a Jones–Wenzl idempotent as a linear combination of G_2 webs. If we take the annular closure of an idempotent, we obtain an element of $\mathcal{S}_q^{G_2}(\mathbb{A})$, which is identified with a polynomial in $\mathcal{R}[x, y]$ by our Proposition 3.2. We expect that under this identification, the idempotent associated to the weight (a, b) yields the polynomial $S(a, b)$ described in [24]. Results analogous to this have been obtained in the case of SL_3 by Ohtsuki and Yamada [25] en route to a skein-theoretic construction of the Witten–Reshetikhin–Turaev 3-manifold

invariant. Those results were further studied by Yuasa [31] for the purpose of categorizing the SL$_3$ analogue of Chebyshev polynomials of the second kind.

Acknowledgments

2 G_2 skein relations and reduction rules

In this section, we give the defining skein relations for G_2 skein algebras of surfaces. Let Σ be an oriented surface of finite type. Let $\mathcal{R}$ denote a commutative integral domain containing an invertible element $q \in \mathcal{R}$ such that $[12]^{-1} \in \mathcal{R}$, where the quantum integer $[k]$ denotes the Laurent polynomial $[k] = (q^k - q^{-k})/(q - q^{-1})$.

2.1 G_2 skein algebras of surfaces

An abstract G_2 web is a trivalent graph where each edge is either a single-strand or double-strand,

$$\Big|\, , \quad \Big\| \, .$$
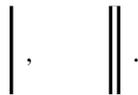
and each vertex is one of the following two types:

$$\curlywedge \, , \quad \text{⅄⅃} \, .$$

We also allow vertexless loops of either strand type. The graph need not be connected. We will study G_2 webs on surfaces subject to certain skein relations of Kuperberg. Linear combinations of webs provide the structure of an $\mathcal{R}$-module called a skein module, and it is equipped with a natural algebra structure. We now provide the necessary definitions.

Definition 2.1 A *web W* on an oriented surface Σ is a diagram on Σ which is an embedding of a G_2 web except for transverse double-points, which are labeled as overcrossings or undercrossings. Two webs W_1 and W_2 are isotopic on Σ if there is an isotopy of Σ taking W_1 to W_2.

Definition 2.2 Let Σ be a smooth oriented surface of finite type. We define the G_2 *skein module* of Σ, denoted by $\mathcal{S}_q^{G_2}(\Sigma)$, to be the $\mathcal{R}$-module spanned by isotopy classes of webs on Σ subject to the skein relations (1)–(12) listed later in this section.

We now describe the algebra structure for the G_2 *skein algebra* of a surface Σ, also denoted by $\mathcal{S}_q^{G_2}(\Sigma)$.

Definition 2.3 The module $\mathcal{S}_q^{G_2}(\Sigma)$ has an algebra structure arising from the bilinear extension of the following product of webs. If W_1 and W_2 are webs, then the product $W_1 \cdot W_2$ is obtained by isotoping the diagrams W_1 and W_2 on Σ so that they only intersect at transverse double points and then taking the union $W_1 \cup W_2$. At each intersection of W_1 with W_2, the crossing data is assigned so that W_1 is drawn above W_2 using overcrossings.

The product operation on webs is well defined with respect to isotopy because the defining skein relations imply the following types of Reidemeister moves, which hold for all strand types and vertex types:

$$\asymp = \backsimeq, \quad \widetilde{\asymp} = \mid\mid, \quad \widetilde{\bigtimes} = \bigvee, \quad \bigwedge = \bigvee.$$

In fact, there is an equivalent definition of the skein algebra of a surface, using the language of thickened surfaces and ribbon graphs, which satisfies the same Reidemeister moves as above. More generally, one may define skein modules spanned by ribbon graphs in oriented 3-manifolds. We focus on the more 2-dimensional perspective of webs on surfaces since that is the setting in which our computations take place.

The following skein relations serve two purposes. The first is that they are the defining relations for the skein modules. The second is that each relation can be interpreted as a reduction rule which replaces a web by a linear combination of simpler webs. We will soon explain how this leads to a basis for skein modules of surfaces, by using the method of confluence theory described in [29].

We use the following rules to remove crossings and internal double-edges:

(1) $\quad \bigtimes = \frac{1}{[2]}\left(q^3\right)\left(+q^{-3}\smile + q\rangle\langle + q^{-1}\supset\subset\right),$

(2) $\quad \bigtimes = \frac{1}{[2]}\left(q^3\rangle\langle + q^{-3}\bigtimes + \frac{1}{[2]}\boxminus\right),$

(3) $\quad \bigtimes = \frac{1}{[2]}\left(q^{-3}\rangle\langle + q^3\bigtimes + \frac{1}{[2]}\boxminus\right),$

(4) $\quad \bigtimes = q^6\right)\left(+q^{-6}\overset{\smile}{\smile} + \frac{2}{[2]^2}\boxminus + \frac{q^3}{[2]^3}\boxminus + \frac{q^{-3}}{[2]^3}\boxminus,$

(5) $\quad \boxtimes = -\right)\left(+\frac{[4][6]}{[2][12]}\overset{\smile}{\frown} + \rangle\langle + \frac{1}{[3]}\boxtimes.$

We call (5) the "double-edge removal" relation. Once we have applied the crossing removals and double-edge removals to a web diagram on a surface, we are left with webs which might have some polygonal faces bounded by single edges. The following defining reduction rules are used to remove small faces:

(6)
$$\bigcirc = \frac{[2][7][12]}{[4][6]}, \quad \circledcirc = \frac{[7][8][15]}{[3][4][5]},$$

(7)
$$\text{⎰} = 0, \quad \text{⎱} = 0,$$

(8)
$$\text{⎶} = -\frac{[3][8]}{[4]}\Big|, \quad \text{⎶} = -[2]^2\Big\|, \quad \text{⎶} = 0,$$

(9)
$$\triangle = \frac{[6]}{[2]}\curlywedge, \quad \triangle = -[3]\curlywedge, \quad \triangle = 0,$$

(10)
$$\sqcap\!\!\sqcup = [3]\big(\,\big)\,\big(+\smile\frown\big) - \frac{[4]}{[2]}\big(\rangle\!\!\langle + \rangle\!\langle\big),$$

(11)
$$\sqcap\!\!\sqcup = \succ\!\!\prec + \rangle\!\!\langle,$$

(12)
$$\overset{\star}{\Longleftrightarrow} = \Big(\curlyvee\!\curlywedge + \curlywedge\!\curlyvee + \curlyvee\!\curlyvee + \curlyvee\!\curlyvee + \curlyvee\!\curlywedge\Big) - \Big(\frown\!\curlyvee + \curlyvee\!\frown + \curlyvee\!\frown + \frown\!\curlyvee + \curlyvee\!\curlyvee\Big).$$

All of these defining relations are local, in the sense that they hold in any disk on the surface. Some of the defining relations follow from the other defining relations. See [3; 27] for a smaller set of defining relations. We have listed the larger set because these are needed to form a set of relations which are confluent, in the following sense.

We can view each relation (1)–(12) as a reduction rule for rewriting a web as a linear combination of webs which are simpler when considering the triple of numbers (crossings, vertices, connected components), considered lexicographically. This allows us to find a spanning set of webs for $\mathcal{S}_q^{G_2}(\Sigma)$ consisting of the set of webs which cannot be reduced further. In order for this spanning set to be a basis, [29] explains that we must check that every overlap of the reduction rules can be resolved consistently. To this end, in addition to the defining relations, we need the following reduction rules. Even though these apply in annular regions instead of disk-shaped regions, they follow from the previous local rules. In the following pictures, the dot represents a region on the surface which may be noncontractible.

(13)
$$\circledcirc\!\!- = \circledcirc\!\!\cdot - [2]^2 \circledcirc\!\!\cdot + \frac{[8]}{[4]}\bigcirc\!\!\cdot - [7]\;\cdot\;,$$

(14)
$$\circledcirc\!\!- = \circledcirc\!\!\cdot - \frac{[6]}{[2][3]}\bigcirc\!\!\cdot.$$

The relation (13) is derived by considering an overlap of the middle relation of (8) and (5). The relation (14) is derived by considering an overlap of the last relation of (9) and of (5). We refer to the region bounded by the web in the left side of (13) as a *self-folded square*, and the region bounded by the web in the left side of (14) a *self-folded pentagon*.

By considering an overlap of the relation (14) with itself, we find we need the equivalence rule (15). This allows us to permute any two simple loops on the surface which bound an embedded annulus on the surface:

(15)
$$\textcircled{\odot} = \textcircled{\odot}$$

We do not consider it as a reduction rule, but we use it as an equivalence relation placed on the set of *irreducible diagrams*, meaning those diagrams which do not contain any of the subdiagrams appearing in the left sides of relations (1)–(14). We identify any two diagrams which are related by a sequence of moves of the form (15).

2.2 Bases of skein modules

When we apply a reduction rule, we replace a web with a linear combination of simpler descendant diagrams. Our relations are confluent if, whenever two reduction rules are applicable to a web, the element obtained by applying the first reduction rule and the element obtained by applying the second can both be reduced to the same element, up to surface isotopy and moves of the form (15). A set of reduction rules is terminal if no diagram has an infinite chain of descendants. If the defining set of relations for a module consists of a set of reduction rules which is terminal and confluent, then the irreducible diagrams form a linearly independent spanning set for the module. We encourage the reader unfamiliar with confluence theory (also called rewriting theory or the diamond lemma) to consult [29] for the description of this procedure in the context of skein theory. Other similar examples of the process of checking the confluence of skein relations can be found in [11; 13; 17].

Theorem 2.4 *The above defining skein relations (1)–(14) are terminal and confluent. Thus a basis for $\mathcal{S}_q^{G_2}(\Sigma)$ consists of irreducible webs to which none of relations (1)–(14) apply. Any two irreducible webs representing the same basis element are related by an isotopy of the surface along with a sequence of loop switch moves (15).*

Proof Each reduction rule (1)–(14) strictly decreases the numbers of the following, in lexicographic order: (crossings, vertices, connected components). Thus the set of reduction rules is terminal. Then the proof follows from applying [29, Theorem 2.3], by checking that the relations are locally confluent for all nontrivial overlaps of the reduction rules.   □

In Proposition 2.8, we give a characterization of irreducible webs by analyzing their faces.

Remark 2.5 Our Theorem 2.4 fills in a small gap in the literature by listing the full set of reduction rules required to describe a basis of the G_2 skein module when both single-strands and double-strands are used. Although relations (5) and (6) appear in [15; 29], the other reduction rules involving double-strands were left out. They are required to complete the relations. For example, without the relation

$$\varphi = -[2]^2 \|,$$

both Schur's lemma and the Reidemeister 2 move fail to hold. Hence the basis claimed in [29, Theorem 7.2] regarding G_2 is not a basis for the version of the skein module we study here. The more recent works [3; 27] do include all of the necessary defining relations. Our analysis of the overlaps of the relations led to the reduction rule (13), which we have not previously found in the literature.

On the other hand, if one considers only webs with single-strands and uses relations only concerning single-strands, then the relations appearing in [15] are complete, and from [29, Theorem 7.2] a basis of that version of the skein module can be deduced by leaving out the double-edge removal relation.

We will also consider skein modules of surfaces with marked points on the boundary. Here, however, the only marked surface we explicitly deal with is the twice-marked annulus $\mathcal{A}_{1,1}$ as described in Section 3.2.

Definition 2.6 Let $M = p_1, \ldots, p_k \in \partial \Sigma$ be a finite set of marked points on the boundary of Σ. The skein module of the marked surface $\mathcal{S}_q^{G_2}(\Sigma, M)$ is the $\mathcal{R}$-module spanned by webs W having single-strand endpoints such that $\partial W = M$. Webs are considered up to isotopies of Σ which fix the boundary components.

Our Theorem 2.4 also applies when (Σ, M) is a surface with marked boundary. Following Kuperberg [14], we can identify irreducible webs by analyzing contributions of web faces to the Euler characteristic of the surface.

Definition 2.7 (Euler measure) Let W be a crossingless web on a surface Σ of finite type. We call a connected component of $\Sigma \setminus W$ a *face* of W. Each face U of W is bounded by some number of edges and vertices. The edges could be edges (or loops) of W or could be boundary edges (or boundary circles) of Σ, if W has endpoints on the boundary. If every edge bounding U is an edge of W, then U is an *interior face* of W. If at least one edge bounding U is a boundary edge of Σ, then we call U a *boundary face* of W. If the interior of the face U is homeomorphic to an open disk, then we call the face an *n-gon*, where n is the number of edges bounding the face, counted with multiplicity, so that an edge is counted twice if the face U appears on both sides of the edge. For example, the left sides of relations (10) and (13) depict faces which are both 4-gons.

For each face U of W, we define its *Euler measure* $\mu(U)$ to be

$$\mu(U) = \tfrac{1}{3}V + \tfrac{1}{2}V_\partial - \tfrac{1}{2}E - E_\partial + \chi(U),$$

where V and V_∂ are the number of interior vertices and boundary vertices of W, E is the number of edges of U which belong to W, and E_∂ is the number of edges of U which belong to $\partial \Sigma$. Vertices and edges are both counted with multiplicity. The quantity $\chi(U)$ is the ordinary Euler characteristic of the face U considered as a submanifold of the surface Σ. Loops of W and boundary circles of $\partial \Sigma$ do not contribute to the Euler measure.

A classical fact is that the Euler measure satisfies the following fundamental property. If the faces of W are labeled by $U_1, \ldots, U_k$, then we have

$$\sum_{i=1}^{k} \mu(U_i) = \chi(\Sigma).$$

If U_n is an interior n-gon, then $\mu(U_n) = \frac{1}{3}n - \frac{1}{2}n + 1 = 1 - \frac{1}{6}n$. If U is a face that is not simply connected, then $\mu(U) \leq 0$. From this discussion we obtain the following characterization of terminal elements:

Proposition 2.8 *A web W on Σ is irreducible if and only if it contains no crossings, no interior double-edges, and if every interior face U of W satisfies $\mu(U) \leq 0$.*

Proof It follows from the discussion preceding Proposition 2.8 that the property of a web W having no interior face U of positive Euler measure is equivalent to W having no interior face an n-gon for $n \leq 5$. By analyzing our reduction rules (1)–(14), we will show that this property holds. Let W be an irreducible web diagram. By rules (1)–(5), W has no crossings and no interior double-edges. Furthermore, by Definition 2.6, W has no double-edge ending on the boundary of Σ, and so every interior vertex of W is of the form $\curlywedge$. The remaining reduction rules eliminate all interior n-gon faces for $n \leq 5$. These eliminated n-gons include self-folded ones by observing the following. Consider a sequence S of adjacent edges (considered with multiplicity) which bound an n-gon face U of W. Since each vertex of W is trivalent, any two instances of the same edge of W cannot be adjacent to each other in the sequence S. This implies that self-folded n-gon faces do not exist for $n \leq 3$ and that any self-folded interior 4-gon and interior 5-gon are eliminated by rules (13) and (14), respectively. $\square$

Kuperberg used the notion of Euler measure to prove the following:

Proposition 2.9 [14] *If Σ is a disk (with no marked points) or is a sphere, then $\mathcal{S}_q^{G_2}(\Sigma) \cong \mathcal{R}$.*

Proof It suffices to show that our basis of $\mathcal{S}_q^{G_2}(\Sigma)$ consists of only the empty diagram. Suppose that W is a basis web for $\mathcal{S}_q^{G_2}(\Sigma)$. Then W contains no internal double-edges, no crossings, and no n-gons for $n \leq 5$. Since Σ is a sphere or is a disk, the Euler characteristic of Σ is positive. If W is nonempty, then it must have a face of positive Euler measure. However, a face of positive Euler measure must be an n-gon with $n \leq 5$. Since W has no such face, W must be empty. $\square$

3 Annuli

We now study some algebras associated to skein modules of the annulus. In this section, we study these algebras from a purely skein-theoretic perspective. Later on, we will return to these algebras to perform computations with power sum polynomials.

3.1 The annulus and the algebra $\mathcal{A}$

Let $\mathbb{A} = S^1 \times [0, 1]$ denote the annulus.

Definition 3.1 Define $\mathcal{A}$ to be the $\mathcal{R}$-algebra spanned by closed webs in $\mathbb{A}$. If $W_1, W_2 \in \mathcal{A}$ are two webs, we define their product, $W_1 W_2$ to be the web obtained by isotoping W_1 so it is contained in $S^1 \times \left(\frac{1}{2}, 1\right]$, isotoping W_2 so it is contained in $S^1 \times \left[0, \frac{1}{2}\right)$, and taking the union of the webs. This product agrees with that of $\mathcal{S}_q^{G_2}(\mathbb{A})$.

The multiplicative unit $1 \in \mathcal{A}$ is the empty diagram. We define $x, y \in \mathcal{A}$ to be the following distinguished elements:

$$x = \left(\begin{array}{c} \text{(loop with small circle)} \end{array} \right), \quad y = \left(\begin{array}{c} \text{(double loop with small circle)} \end{array} \right).$$

Proposition 3.2 *The algebra $\mathcal{A}$ associated to the annulus is isomorphic to $\mathcal{R}[x, y]$ and has a basis of diagrams consisting of concentric loops around the core of the annulus.*

Proof We first describe the basis of $\mathcal{A}$. Let W be a basis web. Thus W is a crossingless web which contains no internal double-edge, and has no interior n-gon face for $n \leq 5$. Any boundary face of W is not simply connected and must have nonpositive Euler measure. Consequently, W has no face of positive Euler measure. If W has a trivalent vertex, then as W must have a nonsimply connected face, W would have a face of negative Euler measure. Therefore, as the Euler characteristic of $\mathbb{A}$ is zero, W cannot contain a trivalent vertex. Thus W consists of concentric loops around the core of the annulus, and the description of the basis is as claimed.

Now we describe the algebra structure of $\mathcal{A}$. Due to the equivalence rule (15), the two kinds of loops in $\mathcal{A}$ commute. Consequently, there is a well-defined algebra homomorphism $\mathcal{R}[x, y] \to \mathcal{A}$ sending x to an essential single-strand loop and sending y to an essential double-strand loop. This map sends the standard basis of monomials of $\mathcal{R}[x, y]$ injectively onto the basis we have just described for $\mathcal{A}$. Thus this algebra map is an isomorphism $\mathcal{R}[x, y] \cong \mathcal{A}$. $\qquad \square$

3.2 The twice-marked annulus and the algebra $\mathcal{A}_{1,1}$

Consider the annulus $\mathbb{A} = S^1 \times [0, 1]$ and a fixed point $p \in S^1$. Let $p_0 = (p, 0)$ and $p_1 = (p, 1)$ be two marked points on $\partial \mathbb{A}$. We consider webs with boundary equal to $M = \{p_0, p_1\}$.
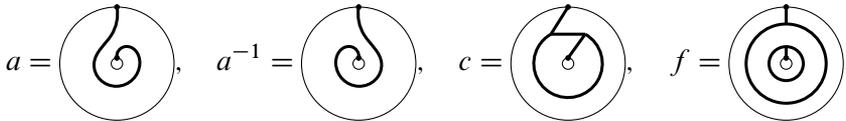
Definition 3.3 Define $\mathcal{A}_{1,1}$ to be the skein module $\mathcal{S}_q^{G_2}(\mathbb{A}, \{p_0, p_1\})$ of the marked annulus spanned by webs with one single-strand endpoint at p_0 on the interior boundary of the annulus and one single-strand endpoint at p_1 on the exterior boundary of the annulus. The algebra structure for $\mathcal{A}_{1,1}$ is given by concatenating annuli outward, as described for $\mathcal{A}$. Note that this product is not the same as a skein algebra product.

Remark 3.4 In the literature, $\mathcal{A}_{1,1}$ is also known as an endomorphism space in the *annular representation category*, *annular spider*, *skein category of the annulus*, or as a corner of the *tube algebra*. The product structure is to be thought of as the composition of morphisms. We will later show that $\mathcal{A}_{1,1}$ is commutative (Corollary 3.9).

The multiplicative unit of $\mathcal{A}_{1,1}$ is the following diagram:

$$1 = \left(\vphantom{\bigcirc} \right).$$
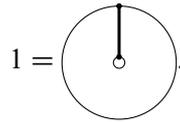
We identify the following distinguished elements $a, a^{-1}, c, f \in \mathcal{A}_{1,1}$:

$$a = \left(\vphantom{\bigcirc}\right), \quad a^{-1} = \left(\vphantom{\bigcirc}\right), \quad c = \left(\vphantom{\bigcirc}\right), \quad f = \left(\vphantom{\bigcirc}\right).$$

In addition to the fundamental elements $a, a^{-1}, 1, c, f \in \mathcal{A}_{1,1}$ listed, we also need to provide notation for the elements $f_{i,j} \in \mathcal{A}_{1,1}$ which consist of the diagram f along with i concentric single-strand loops and j concentric double-strand loops inside of the open annular region bounded by the diagram of f.

For example, the following depicts the element $f_{1,1}$:

$$f_{1,1} = \left(\vphantom{\bigcirc}\right).$$
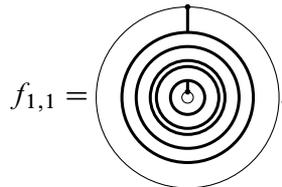
Using these elements, we can describe a basis for $\mathcal{A}_{1,1}$ over $\mathcal{R}$ (Proposition 3.6) and later a presentation for $\mathcal{A}_{1,1}$ as an $\mathcal{R}$-algebra (Corollary 3.10).

We first record some topological facts about diagrams in $\mathcal{A}_{1,1}$. Recall that the mapping class group of the annulus, $\mathrm{MCG}(\mathbb{A})$, is the group of isotopy classes of orientation-preserving diffeomorphisms of $\mathbb{A}$ which fix the boundary. It is a classical fact (see [8]) that $\mathrm{MCG}(\mathbb{A}) \cong \mathbb{Z}$ and is generated by a Dehn twist of the annulus. The annulus is equipped with a natural orientation-reversing diffeomorphism $\mathbb{A} \to \mathbb{A}$, called the mirror image, which reverses the orientation of the S^1 factor of $S^1 \times [0, 1]$. From these facts we observe that if a web $W_1 \in \mathcal{A}_{1,1}$ is taken to a web $W_2 \in \mathcal{A}_{1,1}$ by an orientation-preserving diffeomorphism, then $W_1 = a^k W_2$ for some integer power k of the twist diagram a. Similarly, if W_1 is taken to W_2 by an orientation-reversing diffeomorphism, then W_1 and the mirror image of W_2 are related by a power of a. We use these observations to record the following:

Lemma 3.5 *Suppose $W_1 \in \mathcal{A}_{1,1}$ is a web and $W_2 \in \mathcal{A}_{1,1}$ is one of $\{1, c^j, f_{ij}\}$. If W_1 and W_2 are diffeomorphic then there exists some $k \in \mathbb{Z}$ such that*

$$W_1 = a^k W_2.$$

Moreover, if $W_2 = f_{ij}$, then we may use $k = 0$.

Proof The webs 1 and f_{ij} are invariant under the mirror-image operation, so the statement follows in these cases from the previous discussion. Moreover, it can be seen diagrammatically that $a^k f_{ij} = f_{ij}$ for all k. Now consider the case $W_2 = c^j$. It is an easy observation that the mirror image of the diagram c is

equal to $a^{-1}c$. From this, and the fact that the mirror image operation induces an algebra homomorphism $\mathcal{A}_{1,1} \to \mathcal{A}_{1,1}$, it follows that the mirror image of c^j is equal to $a^{-j}c^j$. Thus W_1 is related by a power of a to the diagram $a^{-j}c^j$, as required. $\qquad\square$

Proposition 3.6 *The algebra $\mathcal{A}_{1,1}$ associated to the twice-marked annulus has a basis of irreducible webs constructed in [Theorem 2.4](#), and consists of the set of webs $\{a^i c^j \mid i \in \mathbb{Z},\ j \in \mathbb{Z}_{\geq 0}\} \cup \{f_{i,j} \mid i, j \in \mathbb{Z}_{\geq 0}\}$.*

The proof of [Proposition 3.6](#) follows from Lemmas [3.7](#) and [3.8](#).

Lemma 3.7 *The elements of $\{a^i c^j \mid i \in \mathbb{Z},\ j \in \mathbb{Z}_{\geq 0}\} \cup \{f_{i,j} \mid i, j \in \mathbb{Z}_{\geq 0}\}$ are distinct and each element of the set is some basis element of $\mathcal{A}_{1,1}$.*

Proof   Since the elements of $\{a^i c^j\} \cup \{f_{i,j}\}$ are all irreducible, they are basis elements. To show they are distinct, we must show that they are not related to each other by sequences of isotopies and moves of the form [(15)](#). Since isotopies and loop switch moves [(15)](#) preserve the number of single-strand and double-strand loops, the set $\{f_{i,j}\}$ consists of distinct elements. Since every element of $\{f_{i,j}\}$ is disconnected while every element of $\{a^i c^j\}$ is connected, $\{f_{i,j}\} \cap \{a^i c^j\} = \varnothing$.

We now show the elements of $\{a^i c^j\}$ are distinct. If $j \neq j'$, then $a^i c^j \neq a^{i'} c^{j'}$ for any i and i', since the two diagrams have different numbers of vertices. If $i \neq i'$, then a^i is not isotopic to $a^{i'}$ since the diagrams have different winding numbers. Now suppose that $a^i c^j$ is isotopic to $a^{i'} c^j$. Within the underlying web of c^j, there is a path from p_0 to p_1 which is isotopic rel endpoints to the identity element $1 \in \mathcal{A}_{1,1}$. Concatenating this path with a^i yields a path γ in $a^i c^j$ isotopic rel endpoints to a^i. On the other hand, if we concatenate the same path in c^j with $a^{i'}$, we find a path γ' in $a^{i'} c^j$ isotopic rel endpoints to $a^{i'}$. An isotopy of $\mathbb{A}_{1,1}$ taking $a^{i'} c^j$ to $a^i c^j$ must take γ to γ', so we must have $i = i'$. Therefore elements of the set $\{a^i c^j\}$ are distinct. $\qquad\square$

Lemma 3.8 *If W is a basis web of $\mathcal{A}_{1,1}$, then W must be an element of*

$$\{a^i c^j \mid i \in \mathbb{Z},\ j \in \mathbb{Z}_{\geq 0}\} \cup \{f_{i,j} \mid i, j \in \mathbb{Z}_{\geq 0}\}.$$

Proof   Let W be a basis web. We consider two cases. If p_0 and p_1 belong to different connected components of W, we claim that $W = f_{i,j}$ for some i and j. Let W_0 be the connected component of W containing the endpoint p_0 and let W_1 be the connected component of W containing the endpoint p_1. It will suffice to show the equality of webs $W_0 \cup W_1 = f$. Since W_0 has an endpoint at p_0 but not at p_1, the web W_0 must contain a trivalent vertex. Consider the face of $\mathbb{A} \setminus W_0$ containing p_1. It is not simply connected, and so must have negative Euler measure. Thus some other face U_0 of W_0 must have positive Euler measure. Since W_0 is irreducible, this face U_0 must be adjacent to the boundary edge of $\mathbb{A}$ containing p_0. We now analyze U_0. It is connected and simply connected, but not a sphere. Thus U_0 is topologically a disk and $\chi(U_0) = 1$. Furthermore, exactly one edge of U_0 is a boundary edge of $\mathbb{A}$, and exactly two vertices (both being p_0, counted with multiplicity) of U_0 are on the boundary of $\mathbb{A}$. Since the edge of W_0 emerging from p_0 is seen to have U_0 on both of its sides, U_0 is a self-folded n-gon. So

far, we have deduced that $n - 1$ of the edges of U_0 are interior edges and $n - 2$ of the vertices of U_0 are interior vertices. We then compute its Euler measure via Definition 2.7:

$$\mu(U_0) = \tfrac{1}{3}(n-1) - \tfrac{1}{2}(n-2) + \tfrac{1}{2}(2) - 1 + 1 = \tfrac{1}{6}(-n+5).$$

Since $\mu(U_0) > 0$, we have that $n < 5$. Since self-folded n-gons for $n < 4$ do not exist for G_2 webs, we conclude that $n = 4$ and U_0 is a self-folded square. Hence W_0 is of the form claimed. The same argument shows that W_1 is of the form claimed, and we must have the equality of diagrams $W_0 \cup W_1 = f$. Since the region between W_0 and W_1 is an annulus, any irreducible diagram in that region must be a collection of concentric loops by Proposition 3.2, so W must be equal to $f_{i,j}$ for some i and j.

Next, if W contains a path from p_0 to p_1, we claim $W = a^i c^j$ for some $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 0}$. By Lemma 3.5 it suffices to show that W is diffeomorphic to c^j for some $j \geq 0$. Let W' be the connected component of W containing p_0 and p_1. If W' contains no *interior* trivalent vertex, then W' is diffeomorphic to $1 = c^0$. Otherwise, W' contains a trivalent vertex and, like above, we consider the boundary face U' of W' adjacent to p_0. If the Euler measure of U' is negative, then the same analysis as in the first case shows that the boundary face of W adjacent to p_1 would be a self-folded square. However, this case is excluded since we are assuming p_0 and p_1 are connected by a path in W. Therefore $\mu(U') \geq 0$ and U' is a self-folded boundary n-gon. By the above computation of Euler measure, it must be either a self-folded square or a self-folded pentagon. The case of U' being a self-folded square is again excluded since p_0 is connected to p_1. Therefore the boundary face U' must be a self-folded pentagon. Similarly, the boundary face adjacent to p_1 must be a self-folded pentagon as well. Since these boundary faces account for an Euler measure of zero, any other face of W' must also have Euler measure zero. Therefore all other faces of W' must be hexagons. By inspection, they must be self-folded hexagons and so the diagram W' must be diffeomorphic to some power of c. Since all faces of W' are topologically disks, any other component of W would be contained in the complement of W' and would thus be reducible by Proposition 2.9. So $W = W' = a^i c^j$, as claimed. $\square$

Corollary 3.9 *The algebra $\mathcal{A}_{1,1}$ is commutative.*

Proof By the characterization of basis elements of $\mathcal{A}_{1,1}$ in Proposition 3.6, one only needs to check that the elements a, c, and $f_{i,j}$ commute with each other. Some of these are checked by simple isotopies while others are checked using short computations similar to the ones in the proof of the next corollary. $\square$

Corollary 3.10 *The algebra $\mathcal{A}_{1,1}$ has the following presentation as an $\mathcal{R}$-algebra. It is the quotient of the algebra generated by the commuting variables a, a^{-1}, c, and f_{ij} for $i, j \geq 0$, subject to the relations*

$$a f_{i,j} = f_{i,j}, \quad a^{-1} f_{i,j} = f_{i,j}, \quad c f_{i,j} = f_{i+1,j} - \frac{[6]}{[2][3]} f_{i,j},$$

$$f_{i,j} f_{k,l} = f_{i+k+2,j+l} - [2]^2 f_{i+k,j+l+1} + \frac{[8]}{[4]} f_{i+k+1,j+l} - [7] f_{i+k,j+l}.$$

Proof We first suppose the relations above hold. Since each element of our basis for $\mathcal{A}_{1,1}$ can be written as a product of these proposed generators, our list is a generating set for $\mathcal{A}_{1,1}$. On the other hand,

any element written as a linear combination of products of these generators can be reduced to a linear combination of basis elements using only the relations listed in the proposition. Therefore, if the relations hold, then they are a full set of relations for $\mathcal{A}_{1,1}$.

We now check this. The relation $a f_{i,j} = f_{i,j} = a^{-1} f_{i,j}$ follows from observing a simple isotopy. The relation which rewrites $f_{i,j} f_{k,l}$ follows from the computation $f^2 = f_{2,0} - [2]^2 f_{0,1} + ([8]/[4]) f_{1,0} - [7] f$, using the self-folded square relation (13). The relation which rewrites $c f_{i,j}$ follows from the computation $cf = f_{1,0} - ([6]/([2][3])) f$, using the self-folded pentagon relation (14). $\qquad\square$

The following will be used later on to identify a nicely behaved subalgebra of $\mathcal{A}_{1,1}$ with a subalgebra of the ring $\mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$.

Corollary 3.11 *The Laurent polynomial ring $\mathcal{R}[a^{\pm 1}, c]$ is a subalgebra of $\mathcal{A}_{1,1}$.*

Remark 3.12 The algebra $\mathcal{A}_{1,1}$ actually admits a finite set of generators $\{a, a^{-1}, c, f\}$. Indeed, in the next section we introduce elements $x^\star$ and $y^\star$, and it can be seen that $f_{i,j} = (x^\star)^i (y^\star)^j f$. Then in the proof of Proposition 5.2, it is shown that $x^\star$ and $y^\star$ are generated by a, a^{-1}, c and f. However, it is more complicated to describe the algebra using the finite list of generators.

3.3 Transparency and actions of $\mathcal{A}$ on $\mathcal{A}_{1,1}$

We now describe two actions of $\mathcal{A}$ on $\mathcal{A}_{1,1}$ and use them to define transparent polynomials, analogous to the notion defined in [16].

Definition 3.13 We define an algebra homomorphism denoted by $(-)^\star \colon \mathcal{A} \to \mathcal{A}_{1,1}$ such that if $D \in \mathcal{A}$ is a diagram, the element $D^\star \in \mathcal{A}_{1,1}$ is the diagram of D placed over the identity strand $1 \in \mathcal{A}_{1,1}$. It can be checked diagrammatically that this is a well-defined algebra homomorphism.

Similarly, we define $(-)_\star \colon \mathcal{A} \to \mathcal{A}_{1,1}$ to be the algebra homomorphism such that if $D \in \mathcal{A}$ is a diagram, the element $D_\star \in \mathcal{A}_{1,1}$ is the diagram of D placed below the identity strand $1 \in \mathcal{A}_{1,1}$.

Recall that the algebra $\mathcal{A}$ is isomorphic to $\mathcal{R}[x, y]$. Below we show the images of the homomorphisms $(-)^\star$ and $(-)_\star$ on the generators x, y of $\mathcal{A}$:
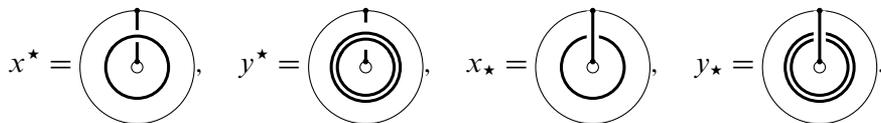
The following definition uses the fact that $(-)^\star$ and $(-)_\star$ are algebra homomorphisms:

Definition 3.14 A polynomial $S(x, y) \in \mathcal{A}$ is said to be *transparent* if either of the following equivalent conditions hold:

(i) $S(x, y)^\star = S(x, y)_\star \in \mathcal{A}_{1,1}$.

(ii) $S(x^\star, y^\star) = S(x_\star, y_\star) \in \mathcal{A}_{1,1}$.

For example, the empty diagram in $\mathcal{A}$ is a transparent element regardless of the choice of $q \in \mathcal{R}$. If $q = 1$ then every element of $\mathcal{A}$ is transparent. When q is chosen to be a root of unity, we will find more interesting elements which are transparent.

We now briefly explain the process which takes a transparent polynomial and constructs central elements in the skein algebra $\mathcal{S}_q^{G_2}(\Sigma)$ of a surface Σ.

Definition 3.15 Let a knot diagram K on Σ represent a framed knot in the thickened surface $\Sigma \times [0, 1]$. Given a monomial $x^i y^j \in \mathcal{R}[x, y]$, the *threading* of $x^i y^j$ along K is denoted by $K^{[x^i y^j]}$. It is obtained by taking $i + j$ parallel copies of K, in the direction of the framing, such that i of these copies are colored by a single-strand and j of these copies are colored by a double-strand. We can extend this definition linearly to obtain the threading $K^{[S(x,y)]}$ of a polynomial $S(x, y) \in \mathcal{R}[x, y]$ along K.

Lemma 3.16 *If an element $S(x, y) \in \mathcal{A}$ is transparent, then the threading $K^{[S]}$ of S along any knot K is a central element of $\mathcal{S}_q^{G_2}(\Sigma)$.*

Proof Let K be a knot diagram on Σ, representing a framed knot which is an embedded annulus in the thickened surface given by an orientation-preserving embedding $k : \mathbb{A} \hookrightarrow \Sigma \times [0, 1]$. Let W be an arbitrary web on Σ. Consider the element $KW \in \mathcal{S}_q^{G_2}(\Sigma)$ which is represented by a diagram with finitely many transverse crossings of K and W, all of which involve K crossing over an edge of W. Consider $K^{[S]}$, which is the image of the embedding k applied to the element $S(x, y)$ on $\mathbb{A}$. To show that $K^{[S]}$ is central, we argue that in each spot that $K^{[S]}$ crosses over a strand of W, the overcrossings can be changed to undercrossings, which implies that $K^{[S]}W = WK^{[S]}$. Each crossing change involving a single-strand of W can be viewed as the image of k applied to the transparency property for $S(x, y)$. Since the skein relations are preserved under the embedding k, these crossing changes are valid in $\mathcal{S}_q^{G_2}(\Sigma)$.

To finish the proof of the lemma, it suffices that W can be replaced by an equivalent web so that all of the crossings of K and W involve single-strands of W. This follows by using the second relation of (8),

$$\left\| \; = -\frac{1}{[2]^2} \right\rangle\!\!\!\langle \, ,$$

which shows that a small interval of a double-strand can be replaced by two single-strands. □

Threadings of polynomials can be defined similarly for links, and threadings of transparent polynomials along links yield central elements in the skein algebra as well.

4 Power sum polynomials for G_2

We now develop the ingredients to construct transparent polynomials.

4.1 Polynomials in $\mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$

Consider the 2-variable Laurent polynomial algebra $E = \mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$. Later, we will relate certain elements of this ring with elements in the algebra $\mathcal{A}_{1,1}$.

Definition 4.1 To a monomial $\lambda_1^i \lambda_2^j \in E$ we associate a degree d_1 and a bidegree d_2 given by

$$d_1(\lambda_1^i \lambda_2^j) = i + j \in \mathbb{Z}, \quad d_2(\lambda_1^i \lambda_2^j) = (i, j) \in \mathbb{Z}^2.$$

We note that E has a natural basis consisting of monomials of distinct bidegree:

$$\{\lambda_1^i \lambda_2^j \mid i, j \in \mathbb{Z}\}.$$

We can place a linear order on our monomials by using the lexicographic order and declaring $d_2(\lambda_1^i \lambda_2^j) > d_2(\lambda_1^{i'}, \lambda_2^{j'})$ if either $i > i'$, or $i = i'$ and $j > j'$.

Definition 4.2 This linear order on monomials allows us to extend d_1 and d_2 to all of E by the following. If $p = \sum a_{ij} \lambda_1^i \lambda_2^j$ is a finite $\mathcal{R}$-linear combination of basis monomials, we define the (bi)degree of p to be the (bi)degree of the top monomial of p:

$$d_1(p) = \max\{d_1(\lambda_1^i \lambda_2^j) \mid a_{ij} \neq 0\}, \quad d_2(p) = \max\{d_2(\lambda_1^i \lambda_2^j) \mid a_{ij} \neq 0\}.$$

Since $\mathcal{R}$ is an integral domain, the degrees satisfy the following property:

Proposition 4.3 *If $p_1, p_2 \in E$, then*

$$d_1(p_1 p_2) = d_1(p_1) + d_1(p_2), \quad d_2(p_1 p_2) = d_2(p_1) + d_2(p_2).$$

Consider the two elements $\lambda_1 + \lambda_2$ and $\lambda_1 \lambda_2 \in E$. For $i \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$ a monomial, $(\lambda_1 + \lambda_2)^i (\lambda_1 \lambda_2)^j$ is a monic element in E of bidegree $(i + j, j)$. Thus these monomials are linearly independent over $\mathcal{R}$. Equivalently, $\lambda_1 + \lambda_2$ and $\lambda_1 \lambda_2$ are algebraically independent over $\mathcal{R}$. These expressions generate a subalgebra $E^{\text{Sym}} \subseteq E$ of symmetric Laurent polynomials:

$$E^{\text{Sym}} = \mathcal{R}[\lambda_1 + \lambda_2, (\lambda_1 \lambda_2)^{\pm 1}].$$

Consequently, a basis of E^{Sym} is given by

$$\{(\lambda_1 + \lambda_2)^i (\lambda_1 \lambda_2)^j \mid i \in \mathbb{Z}_{\geq 0} \text{ and } j \in \mathbb{Z}\}.$$

We are interested in two distinguished elements $\boldsymbol{x}, \boldsymbol{y} \in E^{\text{Sym}}$ given by

$$\boldsymbol{x} = \lambda_1 + \lambda_2 + \lambda_1 \lambda_2 + 1 + (\lambda_1 \lambda_2)^{-1} + \lambda_2^{-1} + \lambda_1^{-1},$$

$$\boldsymbol{y} = \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 + \lambda_1 \lambda_2 + \lambda_1 + \lambda_2 + \lambda_1 \lambda_2^{-1} + 1$$
$$+ 1 + \lambda_1^{-1} \lambda_2 + \lambda_2^{-1} + \lambda_1^{-1} + (\lambda_1 \lambda_2)^{-1} + (\lambda_1 \lambda_2^2)^{-1} + (\lambda_1^2 \lambda_2)^{-1}.$$

The expression defining the element x models the trace of a matrix in the 7-dimensional fundamental representation of G_2, while y models the same for the 14-dimensional fundamental representation. When viewing the exponent (a, b) of a monomial $\lambda_1^a \lambda_2^b$ as a G_2 weight, elements x and y can be viewed as the characters of the first and second fundamental representations, recording the dimensions of weight spaces of the representations.

We have written x as a sum of 7 terms and y as a sum of 14 terms. We remark that $y = e_2 - x$, where e_2 is the second elementary symmetric sum (see Definition 4.5 below) of the terms of x.

Proposition 4.4   *The elements x and y are algebraically independent over $\mathcal{R}$.*

Proof   We view each monomial $x^i y^j$ as a monic element of $E = \mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$ and compute its bidegree $d_2(x^i y^j) = d_2(x^i) + d_2(y^j) = i d_2(x) + j d_2(y) = (i, i) + (2j, j) = (i + 2j, i + j)$. We see the set of monomials $\{x^i y^j\}_{i,j \geq 0}$ consists of monic elements of E of distinct bidegrees. Therefore the monomials are linearly independent over $\mathcal{R}$, and x and y are algebraically independent. $\square$

More generally, we will be using the elements of E^{Sym}:

$$x^{(i)} = \lambda_1^i + \lambda_2^i + (\lambda_1 \lambda_2)^i + 1 + (\lambda_1 \lambda_2)^{-i} + \lambda_2^{-i} + \lambda_1^{-i},$$

$$\begin{aligned} y^{(i)} = {} & (\lambda_1^2 \lambda_2)^i + (\lambda_1 \lambda_2^2)^i + (\lambda_1 \lambda_2)^i + \lambda_1^i + \lambda_2^i + (\lambda_1 \lambda_2^{-1})^i + 1 \\ & + 1 + (\lambda_1^{-1} \lambda_2)^i + \lambda_2^{-i} + \lambda_1^{-i} + (\lambda_1 \lambda_2)^{-i} + (\lambda_1 \lambda_2^2)^{-i} + (\lambda_1^2 \lambda_2)^{-i}. \end{aligned}$$

Using this notation, $x^{(1)} = x$ and $y^{(1)} = y$. We remark that $x^{(i)}$ gives the ith power sum (Definition 4.5) of the summands of x, and similarly for y.

4.2   Polynomials in $\mathcal{R}[x, y]$

Consider the polynomial ring in two variables $\mathcal{R}[x, y]$. The main goal of this section is to provide recursive definitions of two families of polynomials $P_i(x, y)$ and $Q_i(x, y)$ which will allow us to express $x^{(i)}$ and $y^{(i)}$ as polynomials of x and y defined in the previous section. We will study the polynomials from a purely algebraic perspective before we return to studying the diagrammatically defined algebra $\mathcal{A}$, which was shown in Proposition 3.2 to be isomorphic to $\mathcal{R}[x, y]$.

Define the polynomials $P_k(x, y)$ and $Q_k(x, y)$ in $\mathcal{R}[x, y]$ by the recursive relations

$$P_k(x, y) = \begin{cases} 7 & k = 0, \\ \sum_{i=1}^{k-1} (-1)^{i-1} e_i P_{k-i}(x, y) + (-1)^{k-1} k e_k & 0 < k < 7, \\ \sum_{i=1}^{7} (-1)^{i-1} e_i P_{k-i}(x, y) & k \geq 7, \end{cases}$$

$$Q_k(x, y) = \begin{cases} 14 & k = 0, \\ \sum_{i=1}^{k-1} (-1)^{i-1} f_i Q_{k-i}(x, y) + (-1)^{k-1} k f_k & 0 < k < 14, \\ \sum_{i=1}^{14} (-1)^{i-1} f_i Q_{k-i}(x, y) & k \geq 14, \end{cases}$$

where the coefficients $e_i := e_i(x, y)$ and $f_i := f_i(x, y)$ are given by

$$
\begin{aligned}
e_0 &= 1, & f_0 &= 1, \\
e_1 &= x, & f_1 &= y, \\
e_2 &= x + y, & f_2 &= x^3 - x^2 - 2xy - x, \\
e_3 &= x^2 - y, & f_3 &= x^4 - x^3 - 3x^2 y - x^2 + 2y^2 + x + y, \\
e_k &= e_{7-k} \quad \text{if } 0 \le k \le 7, & f_4 &= x^3 y - x^3 - x^2 y - 2xy^2 + x^2 + xy - y^2 + x + y,
\end{aligned}
$$

and

$$
\begin{aligned}
f_5 &= x^5 - 2x^4 - 5x^3 y + 3x^2 y + 6xy^2 + y^3 + 2x^2 + 5xy + 2y^2 - x, \\
f_6 &= x^4 - 3x^3 y + x^2 y^2 - x^2 y + 4xy^2 - 2x^2 + 3xy + 2y^2 + y, \\
f_7 &= -2x^5 + 4x^4 + 6x^3 y + 2x^2 y^2 + 2x^3 - 4x^2 y - 8xy^2 - 2y^3 - 6x^2 - 6xy - 6y^2 + 2, \\
f_k &= f_{14-k} \quad \text{if } 0 \le k \le 14.
\end{aligned}
$$

The first few terms of each sequence of polynomials are as follows:

$$
\begin{aligned}
P_0(x, y) &= 7, & Q_0(x, y) &= 14, \\
P_1(x, y) &= x, & Q_1(x, y) &= y, \\
P_2(x, y) &= x^2 - 2x - 2y, & Q_2(x, y) &= y^2 - 2x^3 + 2x^2 + 4xy + 2x.
\end{aligned}
$$

Definition 4.5 (elementary symmetric sums and power sums) Given terms $t_1, \ldots, t_s$, the ith *elementary symmetric sum* of the terms is defined to be

$$
\sum_{1 \le j_1 < j_2 < \cdots < j_i \le s} t_{j_1} t_{j_2} \cdots t_{j_i},
$$

while the ith *power sum* of the terms is defined to be

$$
\sum_{j=1}^{s} t_j^i.
$$

The following proposition relates the coefficients defining P_n and Q_n to elementary symmetric sums of the summands of x and y. It can be verified by direct computation, which is made easier by use of mathematical software.

Proposition 4.6 *For each $0 \le i \le 7$, the quantity $e_i(x^{(1)}, y^{(1)})$ is equal to the ith elementary symmetric sum of the terms of $x^{(1)}$. Similarly, for each $0 \le i \le 14$, the quantity $f_i(x^{(1)}, y^{(1)})$ is equal to the ith elementary symmetric sum of the terms of $y^{(1)}$.*

We can now relate our polynomials P_k and Q_k to the kth power sums of the terms of x and y.

Proposition 4.7 *For each $k \geq 0$,*

$$P_k(\pmb{x}^{(1)}, \pmb{y}^{(1)}) = \pmb{x}^{(k)}, \quad Q_k(\pmb{x}^{(1)}, \pmb{y}^{(1)}) = \pmb{y}^{(k)}.$$

More generally, for each $k \geq 0$ and for each $i \geq 0$,

$$P_k(\pmb{x}^{(i)}, \pmb{y}^{(i)}) = \pmb{x}^{(ik)}, \quad Q_k(\pmb{x}^{(i)}, \pmb{y}^{(i)}) = \pmb{y}^{(ik)}.$$

Proof We consider P_k. The case of Q_k is handled similarly.

By [Proposition 4.6](#), the quantity $e_l(\pmb{x}^{(1)}, \pmb{y}^{(1)})$ is equal to the lth elementary symmetric sum of the terms of $\pmb{x}^{(1)}$. Let A be a 7-by-7 diagonal matrix such that each diagonal entry A_{jj} is the jth summand of $\pmb{x}^{(1)}$. Each e_l is a coefficient of the characteristic polynomial of A. The Cayley–Hamilton theorem applied to the matrix A^k then gives

$$P_k(\pmb{x}^{(1)}, \pmb{y}^{(1)}) = \operatorname{tr} A^k = \pmb{x}^{(k)}.$$

Next, consider the algebra endomorphism $\phi_i \colon E \to E$ defined on the generators by $\phi_i(\lambda_1) = \lambda_1^i$ and $\phi_i(\lambda_2) = \lambda_2^i$. Then $\phi_i(\pmb{x}^{(1)}) = \pmb{x}^{(i)}$ and $\phi_i(\pmb{y}^{(1)}) = \pmb{y}^{(i)}$.

We then use ϕ_i and the first part of the proposition to see that

$$P_k(\pmb{x}^{(i)}, \pmb{y}^{(i)}) = P_k(\phi_i(\pmb{x}^{(1)}), \phi_i(\pmb{y}^{(1)})) = \phi_i(P_k(\pmb{x}^{(1)}, \pmb{y}^{(1)})) = \phi_i(\pmb{x}^{(k)}) = \pmb{x}^{(ik)}. \qquad \square$$

Corollary 4.8 *For each $k \geq 0$ and for each $i \geq 0$ the following hold in $\mathcal{R}[x, y]$:*

$$P_k(P_i(x, y), Q_i(x, y)) = P_{ik}(x, y), \quad Q_k(P_i(x, y), Q_i(x, y)) = Q_{ik}(x, y).$$

Proof Since $\pmb{x}^{(1)}$ and $\pmb{y}^{(1)}$ are algebraically independent, there is an injective algebra homomorphism $\psi \colon \mathcal{R}[x, y] \to E$ defined on generators by $x \mapsto \pmb{x}^{(1)}$ and $y \mapsto \pmb{y}^{(1)}$. Applying ψ to the left side of the first equation in the proposition yields

$$\psi(P_k(P_i(x, y), Q_i(x, y))) = P_k(P_i(\pmb{x}^{(1)}, \pmb{y}^{(1)}), Q_i(\pmb{x}^{(1)}, \pmb{y}^{(1)})) = P_k(\pmb{x}^{(i)}, \pmb{y}^{(i)}) = P_{ki}(\pmb{x}^{(1)}, \pmb{y}^{(1)}).$$

Applying ψ to the right side of the first equation in the proposition yields the same answer. The injectivity of ψ then proves the proposition for P_k. The case of Q_k is proven similarly. $\qquad \square$

Definition 4.9 For a monomial $x^i y^j \in \mathcal{R}[x, y]$, we define its bidegree to be

$$D_2(x^i y^j) = (i + 2j, i + j) \in \mathbb{Z}_{\geq 0}.$$

Using a lexicographic ordering of bidegrees, we extend D_2 to $\mathcal{R}[x, y]$ by defining for an element $p = \sum a_{ij} x^i y^j \in \mathcal{R}[x, y]$ its bidegree

$$D_2(p) = \max\{D_2(x^i y^j) \mid a_{ij} \neq 0\}.$$

The bidegree satisfies the following compatibility with multiplication in $\mathcal{R}[x, y]$.

Proposition 4.10 *For any two polynomials $S(x, y), T(x, y) \in \mathcal{R}(x, y)$ we have*

$$D_2(S(x, y)T(x, y)) = D_2(S(x, y)) + D_2(T(x, y)).$$

The following propositions provide support for the unusual choice of the bidegree D_2:

Proposition 4.11 *For $k \geq 1$, the polynomials $P_k(x, y)$ and $Q_k(x, y)$ are monic polynomials in $\mathcal{R}[x, y]$. Their bidegrees are $D_2(P_k) = (k, k)$ and $D_2(Q_k) = (2k, k)$.*

Proof By inspecting the coefficients e_i and f_i in the recursive definitions of P_k and Q_k we observe inductively that $P_k = xP_{k-1} + $ (lower terms) and $Q_k = yQ_{k-1} + $ (lower terms). Thus $D_2(P_k) = D_2(x) + D_2(P_{k-1}) = (1, 1) + D_2(P_{k-1})$ and $D_2(Q_k) = D_2(y) + D_2(Q_{k-1}) = (2, 1) + D_2(Q_{k-1})$. The result follows by induction. $\qquad \square$

Thus P_0 and Q_0 are the only nonmonic polynomials in our sequences of polynomials. In the following proposition we renormalize P_0 and Q_0 so that $P_0(x, y) = 1 = Q_0(x, y)$ (but the other polynomials stay unchanged).

Proposition 4.12 *The set of products of polynomials*

$$\{P_k(x, y)Q_l(x, y)\}_{k,l \geq 0}$$

forms a basis for the 2-variable polynomial ring $\mathcal{R}[x, y]$.

Proof We observe that each $P_k Q_l$ is a monic polynomial of $\mathcal{R}[x, y]$ of distinct bidegree $D_2(P_k Q_l) = D_2(P_k) + D_2(Q_l) = (k, k) + (2l, l) = (k + 2l, k + l)$. The standard basis $\{x^k y^l\}$ of $\mathcal{R}[x, y]$ also consists of monic polynomials of distinct bidegrees $D_2(x^k y^l) = (k + 2l, k + l)$. Thus there is a unitriangular change of basis from the standard basis $\{x^k y^l\}_{k,l \geq 0}$ to $\{P_k(x, y)Q_l(x, y)\}_{k,l \geq 0}$. $\qquad \square$

Proposition 4.13 *For $i, j \geq 0$ we have*

$$D_2(x^i y^j) = d_2(\mathbf{x}^{(i)} \mathbf{y}^{(j)}).$$

Proof This follows from the compatibility of D_2 and d_2 with multiplication and the following observation:

$$D_2(x^i) = (i, i) = d_2(\mathbf{x}^{(i)}), \quad D_2(y^j) = (2j, j) = d_2(\mathbf{y}^{(j)}). \qquad \square$$

5 Transparent elements

In this section we connect the algebra E^{Sym} to the algebra $\mathcal{A}_{1,1}$ and use the relationship to show that the polynomials P_n and Q_n are transparent when $q^{2n} = 1$. As a reminder, throughout the paper we have assumed that our ring of coefficients $\mathcal{R}$ contains $[12]^{-1}$, which gives some restriction on $\mathcal{R}$ and q.

5.1 The maps $F^\star$ and $F_\star$

We next define the maps $F^\star$ and $F_\star$, which are the key ingredients in connecting our algebra $E^{\mathrm{Sym}} = \mathcal{R}[\lambda_1 + \lambda_2, (\lambda_1\lambda_2)^{\pm 1}]$ and our diagrammatic algebra $\mathcal{A}_{1,1}$.

Definition 5.1 Below we define algebra maps $F^\star, F_\star \colon E^{\mathrm{Sym}} \to \mathcal{A}_{1,1}$ on generators:

$$F^\star(\lambda_1\lambda_2) = q^2 a, \qquad\qquad F_\star(\lambda_1\lambda_2) = q^{-2} a,$$

$$F^\star(\lambda_1 + \lambda_2) = \frac{q}{[2]}(c - a - 1), \quad F_\star(\lambda_1 + \lambda_2) = \frac{q^{-1}}{[2]}(c - a - 1).$$

In particular, the map $F_\star$ is obtained from the map $F^\star$ by replacing q with q^{-1}. The following proposition provides the precise relationship between our distinguished elements of E^{Sym} and our distinguished elements of $\mathcal{A}_{1,1}$:

Proposition 5.2 $F^\star$ and $F_\star \colon E^{\mathrm{Sym}} \to \mathcal{A}_{1,1}$ *are well-defined algebra homomorphisms and satisfy*

$$F^\star(x^{(1)}) = x^\star, \quad F^\star(y^{(1)}) = \bar{y} := y^\star + \frac{1}{[2]^2} f, \quad F_\star(x^{(1)}) = x_\star, \quad F_\star(y^{(1)}) = \underline{y} := y_\star + \frac{1}{[2]^2} f.$$

Proof We will prove the statement for $F^\star$. The statement for $F_\star$ is proved by replacing q with q^{-1}. As $\lambda_1 + \lambda_2$ and $(\lambda_1\lambda_2)^{\pm 1}$ generate $E^{\mathrm{Sym}} = \mathcal{R}[\lambda_1 + \lambda_2, (\lambda_1\lambda_2)^{\pm 1}]$, we see that $F^\star$ is a well-defined algebra homomorphism by noting that $q^2 a$ is an invertible element of $\mathcal{A}_{1,1}$ and that $\mathcal{A}_{1,1}$ is a commutative algebra.

To check that $F^\star(x^{(1)}) = x^\star$, we compute diagrammatically that

$$= \frac{1}{[2]}(q^3 a + q^{-3} a^{-1} + qc + q^{-1} a^{-1} c).$$

On the other hand, we compute that

$$F^\star(x^{(1)}) = F^\star(\lambda_1 + \lambda_2) + F^\star(\lambda_1\lambda_2) + F^\star((\lambda_1\lambda_2)^{-1}) + F^\star(1) + F^\star((\lambda_1\lambda_2)^{-1}(\lambda_1 + \lambda_2))$$

$$= \frac{q}{[2]}(c - a - 1) + q^2 a + q^{-2} a^{-1} + 1 + q^{-2} a^{-1} \frac{q}{[2]}(c - a - 1)$$

$$= \frac{1}{[2]}(q^3 a + q^{-3} a^{-1} + qc + q^{-1} a^{-1} c).$$

Therefore $F^\star(x^{(1)}) = x^\star$.

Now, to check that $F^\star(y^{(1)}) = \bar{y} = y^\star + (1/[2]^2)f$, we compute diagrammatically that

$$(16) \qquad y^\star = \left(\!\!\includegraphics[diagram]\!\!\right) = \frac{1}{[2]}\left(q^3\left(\!\!\includegraphics[diagram]\!\!\right) + q^{-3}\left(\!\!\includegraphics[diagram]\!\!\right) + \frac{1}{[2]}\left(\!\!\includegraphics[diagram]\!\!\right)\right).$$

Next we apply the double-edge removal relation to all three diagrams and then continue to fully simplify. The following is the calculation for the final diagram of (16):

$$\left(\!\!\includegraphics[diagram]\!\!\right) = -\left(\!\!\includegraphics[diagram]\!\!\right) + \frac{[4][6]}{[2][12]}\left(\!\!\includegraphics[diagram]\!\!\right) + \left(\!\!\includegraphics[diagram]\!\!\right) + \frac{1}{[3]}\left(\!\!\includegraphics[diagram]\!\!\right)$$

$$= -f - \frac{[6][3][8]}{[2][12]}1 + a^{-1}c^2 + \frac{1}{[3]}([3](a+a^{-1}) - \frac{[4]}{[2]}(c+a^{-1}c)).$$

The other two diagrams of the right side of (16) are reduced using the same process and we record the results:

$$\left(\!\!\includegraphics[diagram]\!\!\right) = -a^2 + \frac{[4][6]}{[2][12]}1 + ac + \frac{1}{[3]}c, \qquad \left(\!\!\includegraphics[diagram]\!\!\right) = -a^{-2} + \frac{[4][6]}{[2][12]}1 + a^{-2}c + \frac{1}{[3]}a^{-1}c.$$

Combining these computations and substituting them into (16), we write $y^\star$ in our basis for $\mathcal{A}_{1,1}$ as

$$y^\star = \frac{1}{[2]}\left(-q^3a^2 - q^{-3}a^{-2} + q^3ac + q^{-3}a^{-2}c + \frac{1}{[2]}a^{-1}c^2\right.$$

$$\left. + \frac{1}{[2]}a + \frac{1}{[2]}a^{-1} + \frac{q^2-1}{[2]}c + \frac{q^{-2}-1}{[2]}a^{-1}c - \frac{q^2+q^{-2}}{[2]}1 - \frac{1}{[2]}f\right).$$

On the other hand, we compute

$$F^\star(y^{(1)}) = F^*(\lambda_1 + \lambda_2) + F^*(\lambda_1^{-1} + \lambda_2^{-1}) + F^*(\lambda_1\lambda_2) + F^*(\lambda_1^{-1}\lambda_2^{-1}) + F^*(\lambda_1\lambda_2(\lambda_1+\lambda_2))$$

$$+ F^*(\lambda_1^{-1}\lambda_2^{-1}(\lambda_1^{-1}+\lambda_2^{-1})) + F^*((\lambda_1+\lambda_2)(\lambda_1^{-1}+\lambda_2^{-1}))$$

$$= q\frac{c-a-1}{[2]} + q^{-1}\frac{a^{-1}(c-a-1)}{[2]} + q^2a + q^{-2}a^{-1} + q^2a\left(q\frac{c-a-1}{[2]}\right)$$

$$+ q^{-2}a^{-1}\left(q^{-1}\frac{a^{-1}(c-a-1)}{[2]}\right) + \left(q\frac{c-a-1}{[2]}\right)\left(q^{-1}\frac{a^{-1}(c-a-1)}{[2]}\right)$$

$$= \frac{1}{[2]}\left(-q^3a^2 - q^{-3}a^{-2} + q^3ac + q^{-3}a^{-2}c + \frac{1}{[2]}a^{-1}c^2\right.$$

$$\left. + \frac{1}{[2]}a + \frac{1}{[2]}a^{-1} + \frac{q^2-1}{[2]}c + \frac{q^{-2}-1}{[2]}a^{-1}c - \frac{q^2+q^{-2}}{[2]}1\right).$$

Thus $F^\star(y^{(1)}) = y^\star + (1/[2]^2)f$, as claimed. $\qquad\square$

Proposition 5.3   *The maps $F^\star$ and $F_\star$ each give an isomorphism $E^{\mathrm{Sym}} \cong \mathcal{R}[a^{\pm 1}, c] \subseteq \mathcal{A}_{1,1}$. In particular, $F^\star$ and $F_\star$ are injective.*

Proof   We check the statement for $F^\star$. The other case is similar. By recalling the definition (Definition 5.1) of $F^\star$ on generators of E^{Sym}, we see that the image of $F^\star$ is contained in the subalgebra $\mathcal{R}[a^{\pm 1}, c] \subseteq \mathcal{A}_{1,1}$. Since [2] is invertible in $\mathcal{R}$ we deduce from the definition of $F^\star$ on the generators of E^{Sym} that $a^{\pm 1}$ and c are in the image of $F^\star$. Thus $F^\star$ is surjective.

On the subalgebra $\mathcal{R}[a^{\pm 1}, c]$, for $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 0}$, we can assign the bidegree (j, k) to the monomial $a^k c^j$ and give the set of all bidegrees the lexicographic ordering. We then see that $F^\star$ sends the monomial $(\lambda_1 \lambda_2)^i (\lambda_1 + \lambda_2)^j$ to an element of $\mathcal{R}[a^{\pm 1}, c]$ whose top bidegree is $(j, i + j)$. Thus $F^\star$ sends a spanning set of E^{Sym} to a linearly independent subset of $\mathcal{A}_{1,1}$. Therefore $F^\star$ is injective. $\square$

5.2   The error term

Our original strategy was to attempt to define an isomorphism from a subalgebra of $E = \mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$ to $\mathcal{A}_{1,1}$ whose image on $x^{(1)}$ was $x^\star$ and whose image on $y^{(1)}$ was $y^\star$. We did not find it possible to achieve this goal on the nose since the algebra $\mathcal{A}_{1,1}$ contains zero divisors while E does not. Indeed, as observed earlier, we have the identity $af = a$ and so $(a - 1)f = 0$ while both $a - 1$ and f are nonzero by consideration of our basis for $\mathcal{A}_{1,1}$.

Instead, we found an injective algebra homomorphism $F^\star$ from the subalgebra $E^{\mathrm{Sym}} \subset E$ onto the subalgebra $\mathcal{R}[a^{\pm 1}, c] \subset \mathcal{A}_{1,1}$. We did succeed in defining $F^\star$ so that its image on $x^{(1)}$ is $x^\star$. However, its image on $y^{(1)}$ is not quite $y^\star$. It is off by an error term: $F^\star(y^{(1)}) = y^\star + (1/[2]^2)f$. Below we show that the error term is essentially negligible in our context since the diagram of f allows loops to pass through it.

For example, we observe the identity $y^\star f = y_\star f$ in $\mathcal{A}_{1,1}$:

$$y^\star f = \quad = \quad = \quad = y_\star f.$$

More generally, we will make use of the following observation:

Lemma 5.4   *The following identities hold in $\mathcal{A}_{1,1}$:*

$$x^\star f = x_\star f, \quad y^\star f = y_\star f, \quad \bar{y} f = \underline{y} f.$$

Proof   The first two identities follow diagrammatically by using a 3-dimensional isotopy to pass a loop through the empty annular region in the diagram of f. The third identity follows from the second and from the defining equations $\bar{y} := y^\star + (1/[2]^2)f$ and $\underline{y} := y_\star + (1/[2]^2)f$. $\square$

Proposition 5.5 *An element $S(x, y) \in \mathcal{A}$ is transparent ([Definition 3.14]) if and only if*

$$S(x^\star, \bar{y}) = S(x_\star, \underline{y}) \in \mathcal{A}_{1,1}.$$

Proof It will suffice to show that

$$S(x^\star, y^\star) - S(x_\star, y_\star) = S(x^\star, \bar{y}) - S(x_\star, \underline{y}).$$

Write $S(x, y) \in \mathcal{A}$ as $S(x, y) = \sum a_{ij} x^i y^j$ for $a_{ij} \in \mathcal{R}$.

Then

$$
\begin{aligned}
S(x^\star, y^\star) &= \sum a_{ij}(x^\star)^i (y^\star)^j = \sum a_{ij}(x^\star)^i \left(\bar{y} - \frac{1}{[2]^2} f \right)^j \\
&= \sum a_{ij}(x^\star)^i \left(\sum_{k=0}^{j} \binom{j}{k} (\bar{y})^k \left(-\frac{1}{[2]^2} f \right)^{j-k} \right) \\
&= \sum a_{ij}(x^\star)^i (\bar{y})^j + \sum_{k<j} a_{ij} \binom{j}{k} (x^\star)^i (\bar{y})^k \left(-\frac{1}{[2]^2} f \right)^{j-k} \\
&= S(x^\star, \bar{y}) + \sum_{k<j} a_{ij} \binom{j}{k} (x_\star)^i (\underline{y})^k \left(-\frac{1}{[2]^2} f \right)^{j-k}.
\end{aligned}
$$

The last equality follows from [Lemma 5.4]. Similarly, we compute that

$$S(x_\star, y_\star) = S(x_\star, \underline{y}) + \sum_{k<j} a_{ij} \binom{j}{k} (x_\star)^i (\underline{y})^k \left(-\frac{1}{[2]^2} f \right)^{j-k},$$

and the result follows. □

5.3 The transparent elements

Recall the definition of the degree d_1 from [Definition 4.1].

Proposition 5.6 *Suppose $\lambda \in E^{\mathrm{Sym}} \subseteq E = \mathcal{R}[\lambda_1^{\pm 1}, \lambda_2^{\pm 1}]$ is a homogeneous element of degree $d_1(\lambda) = k$. Then $F^\star(\lambda) = q^{2k} F_\star(\lambda)$.*

Proof Since the generators $\lambda_1 + \lambda_2$ and $(\lambda_1 \lambda_2)^{\pm 1}$ are homogenous elements of degree 1 and degree ± 2, respectively, it suffices to check the statement on these generators. We observe that $F^\star(\lambda_1 \lambda_2) = q^2 a = q^4(q^{-2}a) = q^4 F_\star(\lambda_1 \lambda_2)$. Similarly, $F^\star(\lambda_1 + \lambda_2) = (q/[2])(c - a - 1) = q^2(q^{-1}/[2])(c - a - 1) = q^2 F_\star(\lambda_1 + \lambda_2)$. □

Recall the polynomials P_n and Q_n and their power sum properties from [Section 4.2]. We are now able to show that P_n and Q_n are transparent when q is a $2n^{\text{th}}$ root of unity.

Theorem 5.7 If $q \in \mathcal{R}$ satisfies $q^{2n} = 1$, then both the elements $P_n(x, y)$ and $Q_n(x, y)$ in $\mathcal{A}$ are *transparent elements*.

Proof We compute

$$
\begin{aligned}
P_n(x^\star, \bar{y}) = P_n(F^\star(x^{(1)}), F^\star(y^{(1)})) &= F^\star(P_n(x^{(1)}, y^{(1)})) = F^\star(x^{(n)}) \\
&= F^\star(\lambda_1^n + \lambda_2^n) + F^\star(\lambda_1^{-n} + \lambda_2^{-n}) + F^\star((\lambda_1\lambda_2)^n) + F^\star((\lambda_1\lambda_2)^{-n}) + F^\star(1) \\
&= q^{2n} F_\star(\lambda_1^n + \lambda_2^n) + q^{-2n} F_\star(\lambda_1^{-n} + \lambda_2^{-n}) + q^{4n} F_\star((\lambda_1\lambda_2)^n) + q^{-4n} F_\star((\lambda_1\lambda_2)^{-n}) + F_\star(1).
\end{aligned}
$$

Similarly, we compute

$$
P_n(x_\star, \underline{y}) = F_\star(\lambda_1^n + \lambda_2^n) + F_\star(\lambda_1^{-n} + \lambda_2^{-n}) + F_\star((\lambda_1\lambda_2)^n) + F_\star((\lambda_1\lambda_2)^{-n}) + F_\star(1).
$$

Thus when $q^{2n} = 1$ we have $P_n(x^\star, \bar{y}) = P_n(x_\star, \underline{y})$, and P_n is transparent by [Proposition 5.5](). For the case of Q_n we perform a similar computation

$$
\begin{aligned}
Q_n(x^\star, \bar{y}) &= F^\star(y^{(n)}) \\
&= F^\star((\lambda_1^2\lambda_2)^n + (\lambda_1\lambda_2^2)^n) + F^\star((\lambda_1\lambda_2)^n) + F^\star(\lambda_1^n + \lambda_2^n) + F^\star((\lambda_1\lambda_2^{-1})^n + (\lambda_1^{-1}\lambda_2)^n) \\
&\quad + F^\star(1) + F^\star(1) + F^\star(\lambda_2^{-n} + \lambda_1^{-n}) + F^\star((\lambda_1\lambda_2)^{-n}) + F^\star((\lambda_1\lambda_2^2)^{-n} + (\lambda_1^2\lambda_2)^{-n}) \\
&= q^{6n} F_\star((\lambda_1^2\lambda_2)^n + (\lambda_1\lambda_2^2)^n) + q^{4n} F_\star((\lambda_1\lambda_2)^n) + q^{2n} F_\star(\lambda_1^n + \lambda_2^n) + F_\star((\lambda_1\lambda_2^{-1})^n \\
&\quad + (\lambda_1^{-1}\lambda_2)^n) + F_\star(1) + F_\star(1) + q^{-2n} F_\star(\lambda_2^{-n} + \lambda_1^{-n}) + q^{-4n} F_\star((\lambda_1\lambda_2)^{-n}) \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad + q^{-6n} F_\star((\lambda_1\lambda_2^2)^{-n} + (\lambda_1^2\lambda_2)^{-n}),
\end{aligned}
$$

to see that Q_n is transparent when $q^{2n} = 1$. $\qquad\qquad\square$

Notation Inspired by the computations in the previous proof, and to make the proof of the upcoming [Theorem 5.9]() cleaner, we will use the following notation:

$$
\begin{aligned}
\tilde{x}^{(i)} &= q^{2i}(\lambda_1^i + \lambda_2^i) + q^{-2i}(\lambda_1^{-i} + \lambda_2^{-i}) + q^{4i}(\lambda_1\lambda_2)^i + q^{-4i}(\lambda_1\lambda_2)^{-i} + 1, \\
\tilde{y}^{(j)} &= q^{6j}((\lambda_1^2\lambda_2)^j + (\lambda_1\lambda_2^2)^j) + q^{4j}(\lambda_1\lambda_2)^j + q^{2j}(\lambda_1^j + \lambda_2^j) + (\lambda_1\lambda_2^{-1})^j + (\lambda_1^{-1}\lambda_2)^j \\
&\quad + 1 + 1 + q^{-2j}(\lambda_2^{-j} + \lambda_1^{-j}) + q^{-4j}(\lambda_1\lambda_2)^{-j} + q^{-6j}((\lambda_1\lambda_2^2)^{-j} + (\lambda_1^2\lambda_2)^{-j}).
\end{aligned}
$$

By the injectivity of $F_\star$, the elements $\tilde{x}^{(i)}, \tilde{y}^{(j)} \in E^{\mathrm{Sym}}$ are the unique elements satisfying

$$
F^\star(x^{(i)}) = F_\star(\tilde{x}^{(i)}), \quad F^\star(y^{(j)}) = F_\star(\tilde{y}^{(j)}).
$$

We now discuss some terms of $x^{(i)}y^{(j)}$ which behave like leading terms. Recall the bidegree d_2 from [Definition 4.2](). The top two terms of $x^{(i)}y^{(j)}$, with respect to the bidegree d_2, are $\lambda_1^{i+2j}\lambda_2^{i+j}$ and $\lambda_1^{i+2j}\lambda_2^j$. The following lemma shows that the top terms of a monomial $x^{(s)}y^{(t)}$ cannot be canceled out by any terms from a monomial $x^{(i)}y^{(j)}$ of lower bidegree.

Lemma 5.8 *Suppose $d_2(x^{(i)} y^{(j)}) < d_2(x^{(s)} y^{(t)})$ for some $s, t, i, j \in \mathbb{Z}_{\geq 0}$. Then $x^{(i)} y^{(j)}$ contains no term $\lambda_1^{s+2t} \lambda_2^{s+t}$ and no term $\lambda_1^{s+2t} \lambda_2^{t}$.*

Proof Suppose the statement is false. By inspection, the highest power of λ_1 in any of the terms of $x^{(i)} y^{(j)}$ occurs in $\lambda_1^{i+2j} \lambda_2^{i+j}$ and $\lambda_1^{i+2j} \lambda_2^{j}$. By assumption, $i + 2j \leq s + 2t$. Thus, if $x^{(i)} y^{(j)}$ contains such a term, $i + 2j = s + 2t$. Then by assumption $s + t > i + j \geq j$ and so neither the term $\lambda_1^{i+2j} \lambda_2^{i+j}$ nor the term $\lambda_1^{i+2j} \lambda_2^{j}$ can be equal to $\lambda_1^{s+2t} \lambda_2^{s+t}$. For the remaining case, suppose one of the terms is equal to $\lambda_1^{s+2t} \lambda_2^{t}$. If $\lambda_1^{i+2j} \lambda_2^{j} = \lambda_1^{s+2t} \lambda_2^{t}$ then $j = t$ and $i = s$, which contradicts our hypothesis. If $\lambda_1^{i+2j} \lambda_2^{i+j} = \lambda_1^{s+2t} \lambda_2^{t}$ then $i + 2j = s + 2t$ and $i + j = t$. This system implies that $s = -i$, which implies that $i = s = 0$ and $j = t$, and is a contradiction. $\qquad\square$

Theorem 5.9 *If $S(x, y) \in \mathcal{A}$ is a transparent element, then q^2 is a root of unity of some order n. Furthermore, if n is coprime to 3, then $S(x, y) \in \mathcal{R}[P_n, Q_n]$, where $P_n := P_n(x, y)$ and $Q_n := Q_n(x, y)$.*

Proof Suppose $S(x, y) \in \mathcal{A}$ is transparent. By [Proposition 4.12](), $\{P_i Q_j\}_{i,j \geq 0}$ is a basis for $\mathcal{A}$. We write $S(x, y) = \sum a_{ij} P_i Q_j$.

Since $S(x, y)$ is transparent, we have by [Proposition 5.5]() that $S(x^\star, \bar{y}) - S(x_\star, \underline{y}) = 0 \in \mathcal{A}_{1,1}$.

We compute

$$
\begin{aligned}
S(x^\star, \bar{y}) - S(x_\star, \underline{y}) &= \sum a_{ij} (P_i(x^\star, \bar{y}) Q_j(x^\star, \bar{y}) - P_i(x_\star, \underline{y}) Q_j(x_\star, \underline{y})) \\
&= \sum a_{ij} (F^\star(x^{(i)} y^{(j)}) - F_\star(x^{(i)} y^{(j)})) = \sum a_{ij} (F_\star(\tilde{x}^{(i)} \tilde{y}^{(j)}) - F_\star(x^{(i)} y^{(j)})) \\
&= F_\star \left(\sum a_{ij} (\tilde{x}^{(i)} \tilde{y}^{(j)} - x^{(i)} y^{(j)}) \right).
\end{aligned}
$$

Let $P_s Q_t$ be the unique term of $S(x, y)$ of maximal bidegree $(s + 2t, s + t)$.

Using the injectivity of $F_\star$ and considering the leading terms of [Lemma 5.8](), we have

$$
a_{st}(q^{4s+6t} - 1) = 0, \quad a_{st}(q^{2s+6t} - 1) = 0,
$$

where $a_{st}(q^{4s+6t} - 1)$ is the coefficient of $\lambda_1^{s+2t} \lambda_2^{s+t}$ in $\sum a_{ij} (\tilde{x}^{(i)} \tilde{y}^{(j)} - x^{(i)} y^{(j)})$ and $a_{st}(q^{2s+6t} - 1)$ is the coefficient of $\lambda_1^{s+2t} \lambda_2^{t}$.

Since $a_{st} \neq 0$ and $\mathcal{R}$ is an integral domain, q^2 is a root of unity of some order n such that

$$
2s + 3t \equiv 0 \mod n, \quad s + 3t \equiv 0 \mod n.
$$

So $s, 3t \equiv 0 \mod n$. If n is not divisible by 3, then $s, t \equiv 0 \mod n$. In this case, there exist $k_s, k_t \in \mathbb{Z}_{\geq 0}$ such that $s = k_s n$ and $t = k_t n$.

By [Corollary 4.8](),

$$
P_s(x, y) = P_{k_s n}(x, y) = P_{k_s}(P_n(x, y), Q_n(x, y)) \quad \text{and} \quad Q_t(x, y) = Q_{k_t}(P_n(x, y), Q_n(x, y)).
$$

Thus the top term of $S(x, y)$ is in $\mathcal{R}[P_n, Q_n]$. Consequently, $S(x, y) - a_{st} P_s Q_t$ is a transparent element of lower bidegree than $S(x, y)$. A repeated application of this argument shows that $S(x, y) \in \mathcal{R}[P_n, Q_n]$. $\qquad\square$

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