On contact mapping classes of prequantizations

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We present examples of prequantizations over integral symplectic manifolds which admit infinitely many smoothly trivial contact mapping classes. These classes are given by the connected components of the strict contactomorphism group which project to the identity component of the symplectomorphism group of the base manifold. Along the way, we study the lifting problem of symplectomorphisms of the base manifold to strict contactomorphisms of the prequantization.

1 Introduction

One of the fundamental questions in contact topology is to understand the group $\mathrm{Cont}(V, \xi)$ of coorientation preserving contactomorphisms of a given cooriented contact structure ξ on a closed manifold V. Understanding $\mathrm{Cont}(V, \xi)$ requires studying its identity component $\mathrm{Cont}^0(V, \xi)$ and the contact mapping class group $\pi_0(\mathrm{Cont}(V, \xi))$. The aim of this paper is to study $\pi_0(\mathrm{Cont}(V, \xi))$ for a particular class of contact manifolds called prequantizations. These are total spaces of principal circle bundles $V \to M$, with a connection 1-form λ which is a contact form whose Reeb vector field generates the circle action and, therefore, the curvature form of λ is an integral symplectic form ω on M.

Concerning the contact mapping class group, it is interesting to study the homomorphism

$$\mathcal{J} \colon \pi_0(\mathrm{Cont}(V, \xi)) \to \pi_0(\mathrm{Diff}(V))$$

where $\mathrm{Diff}(V)$ is the group of diffeomorphisms of V which preserve the orientation determined by the coorientation of ξ. Our main results, which rely solely on algebraic topological methods, provide a large class of examples of prequantizations (V, λ) for which the kernel of $\mathcal{J}$ contains an infinite subgroup of $\pi_0(\mathrm{Cont}(V, \xi))$. This subgroup is given as follows. Given a prequantization $(V, \lambda) \to (M, \omega)$, we get a natural homomorphism

$$\tag{1} \pi_0(\mathrm{SCont}(V, \lambda)) \to \pi_0(\mathrm{Symp}(M, \omega))$$

where $\mathrm{SCont}(V, \lambda)$ is the group of strict contactomorphisms, that is, contactomorphisms which preserve the contact form λ, and $\mathrm{Symp}(M, \omega)$ is the group of symplectomorphisms of (M, ω). It turns out that

the kernel of the above homomorphism is isomorphic to $H_{dR}^1(M, \mathbb{Z})/\Gamma_\omega$, where Γ_ω is the flux group of (M, ω); see Theorem 2.10. Then we show that for a large class of integral symplectic manifolds for which Γ_ω turns out to be trivial, the composition

$$\mathcal{I}: H_{dR}^1(M, \mathbb{Z}) \to \pi_0(\mathrm{SCont}(V, \lambda)) \to \pi_0(\mathrm{Cont}(V, \xi))$$

is injective. More precisely, we prove the following.

Theorem 1.1 *Let (M, ω) be a closed integral symplectic manifold such that either $[\omega]$ vanishes on $\pi_2(M)$ or (M, ω) is monotone on $\pi_2(M)$ with a nonvanishing monotonicity constant. Suppose, in addition, that any abelian subgroup of $\pi_1(M)$ has rank at most one. If $(V, \lambda) \to (M, \omega)$ is a prequantization, then the homomorphism $\mathcal{I}$ is injective.*

The idea of the proof is as follows. For a given class $\alpha \in H_{dR}^1(M, \mathbb{Z})$, the associated strict contactomorphism L_α acts on a loop δ in the conformally symplectic frame bundle $\mathrm{CSp}(\xi)$, up to free homotopy, by concatenation with an iterate of the *Reeb frame loop*, which is defined by picking a symplectic frame of a contact hyperplane and pushing it forward by the Reeb flow along a circle fiber. The amount of iteration of the Reeb frame loop is $\langle [\alpha], [\bar{\delta}] \rangle$, where $\bar{\delta}$ is the projection of δ to M. Then, we associate a Maslov-type index to the Reeb frame loop which we use to show that the resulting loop $L_\alpha \circ \delta$ is not freely homotopic to δ whenever $\langle [\alpha], [\bar{\delta}] \rangle \neq 0$. Using this Maslov-type index, we show the following statement:

Proposition 1.2 *Let $(V, \lambda) \to (M, \omega)$ be a prequantization. Suppose that either $[\omega]$ vanishes on $\pi_2(M)$ or (M, ω) is monotone on $\pi_2(M)$ with nonvanishing monotonicity constant. Then the Reeb frame loop is of infinite order in $\pi_1(\mathrm{CSp}(\xi))$. In particular, the Reeb flow, seen as a loop in $\mathrm{Cont}(V, \xi)$, is of infinite order in $\pi_1(\mathrm{Cont}(V, \xi))$.*

We note that the Maslov index that we associate to the Reeb frame loop can be seen as a reformulation of the first Maslov index, which is a homomorphism from $\pi_1(\mathrm{Cont}(V, \xi))$ to $\mathbb{Z}$, of Casals, Ginzburg and Presas [7], applied to the Reeb loop (see also [8; 9]). A similar result was also obtained by Albers, Shelukhin, and Zapolsky, as an application (among other applications) of some spectral invariants which they associate to contactomorphisms of prequantization bundles (see [1] for more details).

Theorem 1.3 *Let (Σ_g, ω_0) be a surface of genus $g \geq 2$ with an integral symplectic form and ω_{FS} be the Fubini–Study form on $\mathbb{C}P^n$, which is normalized so that the standard $\mathbb{C}P^1 \subset \mathbb{C}P^n$ has unit area. Consider a prequantization $(V, \lambda) \to (\mathbb{C}P^n \times \Sigma_g, \omega_{\mathrm{FS}} \oplus \omega_0)$. Then the homomorphism*

$$\mathcal{I}: H_{dR}^1(\Sigma_g, \mathbb{Z}) \to \pi_0(\mathrm{Cont}(V, \xi))$$

is injective. Moreover, $\ker(\mathcal{J})$ contains the image of $\mathcal{I}$ for n odd, and contains $\mathcal{I}(2H_{dR}^1(\Sigma_g, \mathbb{Z}))$ for n even.

More generally, we describe a setting for which the composition $\mathcal{J} \circ \mathcal{I}$ fails to be injective; see Proposition 3.12. In principle, this provides examples other than the ones given in the above theorem.

In what follows we give an account of previous results in the direction of this work. The first result we want to mention is given by Giroux [16], which says that $\mathcal{I}$ is injective for $V = \mathbb{T}^3$ and ξ the standard tight contact structure (see also [13, Proposition 2]). The first example, according to our knowledge, of a closed contact manifold for which the kernel of $\mathcal{I}$ is nontrivial is attributed to Chekanov; see Eliashberg and Fraser [12] and Vogel [25]. The statement here is that for any overtwisted contact structure ξ on $\mathbb{S}^3$, $\pi_0(\mathrm{Cont}(\mathbb{S}^3, \xi))$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ depending on the Hopf invariant. Later in [11], Dymara introduced an invariant for overtwisted contact structures on $\mathbb{S}^3$, which enabled her to show that a homomorphism from $\pi_0(\mathrm{Cont}(\mathbb{S}^3, \xi))$ to $\mathbb{Z}_2$ that she constructed is surjective for every overtwisted ξ.

The first example where the contact structure is tight and also the kernel of $\mathcal{I}$ is infinite was given by Ding and Geiges in [10], following an observation made by Gompf in [18]. Here the contact manifold is $\mathbb{S}^1 \times \mathbb{S}^2$ with the standard tight contact structure. Later in [17], Giroux and Massot showed that the kernel of $\mathcal{I}$ is $\mathbb{Z}_d$ for V the d-fold fiber cover of the unit cotangent bundle of a suitable surface of genus at least 2. In [21], Massot and Niederkrüger constructed hypertight contact manifolds in every odd dimension for which the kernel of $\mathcal{I}$ contains nontrivial but finite-order elements. Lanzat and Zapolsky [20] then constructed an embedding of certain (braid) subgroups G of the symplectic mapping class group of the A_m-Milnor fiber, which is noncompact and exact, to the contact mapping class group of the associated contactization and showed that G is contained in the kernel of $\mathcal{I}$.

The last two works we want to mention are due to Gironella. In [14] he constructed examples of overtwisted contact manifolds, in every odd dimension, admitting smoothly trivial finite order contact mapping classes. Lastly, in [15] Gironella generalized the constructions in [10] to provide examples of tight contact manifolds in all odd dimensions, which admit smoothly trivial contact mapping classes of infinite order.

Questions and further discussion

Given a prequantization bundle (V, λ) over an integral symplectic manifold (M, ω), the kernel of the homomorphism (1) is naturally identified with $H^1_{dR}(M, \mathbb{Z})/\Gamma_\omega$ — see Theorem 2.10 — and one obtains a homomorphism

$$\mathcal{I} \colon H^1_{dR}(M, \mathbb{Z})/\Gamma_\omega \to \pi_0(\mathrm{Cont}(V, \xi)).$$

We do not know any example for which the homomorphism $\mathcal{I}$ fails to be injective.

One may perhaps find situations where the image of $\mathcal{I}$ contains elements which are isotopic to the identity in the group of formal contactomorphisms, namely the group of pairs (φ, Φ), where $\varphi \in \mathrm{Diff}(V)$ and Φ is a bundle map of TV covering φ and preserving ξ conformally symplectically. A candidate setting for this is when the Reeb frame loop is contractible, in other words, when M admits an element in $\pi_2(M)$ of unit symplectic area on which $c_1(TM)$ vanishes.

On the other hand, the noninjectivity of $\mathcal{I}$ has the potential to provide examples of strict contactomorphisms that are contact isotopic to the identity but without translated points in the sense of Sandon [23]. The

setting of such an example could be that there is some $\alpha \in \ker \mathscr{I}$ represented by a nonsingular closed 1-form on M such that the time-one-map of its symplectic gradient flow is without fixed points. Recently, Cant [6] provided examples of contactomorphisms on standard contact spheres that are contact isotopic to identity yet without translated points. However there is no known example of a strict contactomorphism with these properties.

Outline of the paper

In Section 2, we discuss generalities about prequantizations, and study the homomorphism (1). In Section 3, we give our main results, namely Theorem 3.11, Proposition 3.12 and Corollary 3.13.

Acknowledgements

2 Generalities on strict contactomorphisms of prequantizations

In this section we fix our setting, recall some generalities on prequantizations and determine the kernel of the homomorphism (1). Most of the results presented in this section can be found in the classical books of Banyaga [2] and Souriau [24]. We try to keep our presentation minimal and we refer to [2; 24] for further details.

Let (M, ω) be a $2n$-dimensional closed integral symplectic manifold, that is, the value of $[\omega]$ over any integral homology 2-cycle is an integer. Then a classical fact due to Kobayashi provides our main object of study. We fix $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ for the rest of this manuscript.

Theorem 2.1 [4; 19] *Let $\pi : V \to M$ be a principal $\mathbb{S}^1$-bundle whose Euler class is a lift of $-[\omega]$. Then there is a contact form λ on the total space V such that*

(p1) *the vector field that generates the $\mathbb{S}^1$-action is the Reeb vector field of λ;*

(p2) $\pi^*\omega = d\lambda$.

We call such a bundle (V, λ), together with a contact form as above, a *prequantization* over (M, ω).

Given a prequantization (V, λ) over (M, ω) we want to understand the induced homomorphism

$$(2) \qquad\qquad \mathrm{SCont}(V, \lambda) \to \mathrm{Symp}(M, \omega), \quad \varphi \mapsto \bar{\varphi},$$

where $\mathrm{SCont}(V, \lambda)$ is the group of strict contactomorphisms of (V, λ), that is, diffeomorphisms of V that preserve λ, and $\mathrm{Symp}(M, \omega)$ is the group of symplectomorphisms of (M, ω). Notice that any strict contactomorphism φ of (V, λ) is in particular an $\mathbb{S}^1$-equivariant diffeomorphism of V and therefore descends to a diffeomorphism $\bar{\varphi}$ of M, which preserves ω due to (p2).

Our main objective is to characterize the image and the kernel of the map (2) and of the induced map

$$(3) \qquad\qquad \pi_0(\mathrm{SCont}(V, \lambda)) \to \pi_0(\mathrm{Symp}(M, \omega))$$

on mapping class groups. The surjectivity of the map (2), and of the map (3), relies on the answer to the question whether a given $\bar{\varphi} \in \mathrm{Symp}(M, \omega)$ can be lifted (maybe after an isotopy) to a map $\varphi \in \mathrm{SCont}(V, \lambda)$.

In order to study this lifting problem, we fix the $\mathbb{S}^1$-bundle $\pi \colon V \to M$ as in Theorem 2.1 and consider the space $\mathcal{P}$ of all contact forms on V that satisfy (p1) and (p2). Notice that $\mathcal{P}$ is an affine space given by

$$(4) \qquad \mathcal{P} = \{\lambda + \beta \mid \beta \in \Omega^1_{\mathrm{cl}}(V) \text{ and } \beta(X) = 0\} = \{\lambda + \pi^*\alpha \mid \alpha \in \Omega^1_{\mathrm{cl}}(M)\}$$

for any fixed $\lambda \in \mathcal{P}$, where Ω^*_{cl} denotes the space of closed forms. Let $\mathrm{Diff}_{\mathbb{S}^1}(V)$ be the group of $\mathbb{S}^1$-equivariant diffeomorphisms of V. We consider the action of the subgroups

$$(5) \qquad\qquad \mathrm{Diff}^\omega_{\mathbb{S}^1}(V) = \{\varphi \in \mathrm{Diff}_{\mathbb{S}^1}(V) \mid \bar{\varphi}^*\omega = \omega\}$$

and

$$(6) \qquad\qquad \mathrm{Diff}^{\mathrm{Id}}_{\mathbb{S}^1}(V) = \{\varphi \in \mathrm{Diff}_{\mathbb{S}^1}(V) \mid \bar{\varphi} = \mathrm{Id}\}.$$

on $\mathcal{P}$ by the pull-back. We note that $\varphi \in \mathrm{Diff}^\omega_{\mathbb{S}^1}(V)$ if and only if $\varphi \in \mathrm{Diff}_{\mathbb{S}^1}(V)$ and $\varphi^*(d\lambda) = d\lambda$ for all $\lambda \in \mathcal{P}$ and the stabilizer of any $\lambda \in \mathcal{P}$ in $\mathrm{Diff}^\omega_{\mathbb{S}^1}(V)$ is precisely $\mathrm{SCont}(V, \lambda)$. On the other hand we have the identification

$$(7) \qquad\qquad \mathrm{Diff}^{\mathrm{Id}}_{\mathbb{S}^1}(V) \cong C^\infty(M, \mathbb{S}^1)$$

since any $\varphi \in \mathrm{Diff}^{\mathrm{Id}}_{\mathbb{S}^1}(V)$ is given by

$$(8) \qquad\qquad \varphi(p) = \phi^X_{f(\pi(p))}(p)$$

for a unique smooth map $f \colon M \to \mathbb{S}^1$, where ϕ^X_t denotes the flow of the vector field X that generates the $\mathbb{S}^1$-action.

Theorem 2.2 *Let (V, λ) be a prequantization over (M, ω) and assume that $H^2(M, \mathbb{Z})$ is torsion-free. Then the homomorphism (3) is surjective.*

Proof We first show that the identity component of $\mathrm{Diff}^\omega_{\mathbb{S}^1}(V)$ acts transitively on $\mathcal{P}$. In fact, given the line segment $\lambda_t = \lambda + t\beta$, $t \in [0, 1]$, joining λ and $\lambda + \beta$, we define the vector field Y on V by the

equations $\lambda(Y) = 0$ and $\beta = -\iota_Y d\lambda$. Note that $\dot\lambda_t = \beta$, $d\lambda_t = d\lambda$ and $\lambda_t(Y) = 0$ for all $t \in [0, 1]$. Now let ϕ_t be the flow of Y. We get

$$\frac{d}{dt}\phi_t^*\lambda_t = \phi_t^*(\dot\lambda_t + \mathcal{L}_Y\lambda_t) = \phi_t^*(\beta + \iota_Y d\lambda_t + d(\lambda_t(Y))) = \phi_t^*(\beta + \iota_Y d\lambda) = 0.$$

Hence $\phi_t^*(\lambda + t\beta) = \lambda$ and $\phi_t \in \mathrm{Diff}_{\mathbb{S}^1}^\omega(V)$ for all $t \in [0, 1]$. In fact $[X, Y] = 0$ since

$$0 = \lambda([X, Y]) = X(\lambda(Y)) - Y(\lambda(X)) - d\lambda(X, Y)$$

and

$$0 = \iota_{[X,Y]}d\lambda = \mathcal{L}_X\iota_Y d\lambda - \iota_Y\mathcal{L}_X d\lambda = -\mathcal{L}_X\beta = 0.$$

Note that any $\varphi \in \mathrm{Symp}(M, \omega)$ lifts to some $\widetilde\varphi \in \mathrm{Diff}_{\mathbb{S}^1}^\omega(V)$ since φ pulls back the circle bundle $V \to M$ to a circle bundle on M, which has the same Euler class since $H^2(M, \mathbb{Z})$ is torsion-free. The resulting bundle equivalence provides $\widetilde\varphi$. Then there is some φ in the identity component of $\mathrm{Diff}_{\mathbb{S}^1}^\omega(V)$ such that $\psi^*(\widetilde\varphi^*\lambda) = \lambda$. So $\widetilde\varphi \circ \psi \in \mathrm{SCont}(V, \lambda)$ and $\overline{\widetilde\varphi \circ \psi}$ is isotopic to φ in $\mathrm{Symp}(M, \omega)$.   □

Lemma 2.3 *Let $\lambda \in \mathcal{P}$ and $\varphi \in \mathrm{SCont}(M, \lambda)$ be such that $\overline\varphi = \mathrm{Id}$. Then $\varphi = \phi_t^X$ for some $t \in \mathbb{S}^1$.*

Proof Given $\varphi \in \mathrm{Diff}_{\mathbb{S}^1}^{\mathrm{Id}}(V)$ as in (8), an easy computation shows that

(9) $$\varphi^*\lambda = \lambda + \pi^* df$$

where df is the differential of $f\colon M \to \mathbb{S}^1$ viewed as the 1-form $f^* dt$. Now if φ preserves λ, then f is a constant map and therefore $\varphi = \phi_t^X$ for some $t \in \mathbb{S}^1$.   □

Theorem 2.4 *Let (V, λ) be a prequantization over (M, ω) and assume that $H^2(M, \mathbb{Z})$ is torsion-free. Then the following are equivalent.*

(1) *The action of $\mathrm{Diff}_{\mathbb{S}^1}^{\mathrm{Id}}(V)$ on $\mathcal{P}$ is transitive.*

(2) *The first cohomology group $H_{dR}^1(M, \mathbb{R})$ is trivial.*

(3) *The homomorphism (2) is surjective. In other words, the sequence*

$$0 \to \mathbb{S}^1 \to \mathrm{SCont}(V, \lambda) \to \mathrm{Symp}(M, \omega) \to 0$$

   is exact.

Proof We recall that every class in $H_{dR}^1(M, \mathbb{Z})$ is represented by the differential of a circle valued function. Combining this fact with (9) gives the noncanonical identification

$$\mathcal{P}/\mathrm{Diff}_{\mathbb{S}^1}^{\mathrm{Id}}(V) \simeq H_{dR}^1(M, \mathbb{R})/H_{dR}^1(M, \mathbb{Z}),$$

which shows the equivalence of (1) and (2).

Now assume that $\mathrm{Diff}_{\mathbb{S}^1}^{\mathrm{Id}}(V)$ acts transitively on $\mathcal{P}$ and let $\varphi \in \mathrm{Symp}(M, \omega)$. Then φ lifts to some $\widetilde\varphi \in \mathrm{Diff}_{\mathbb{S}^1}^\omega(V)$ as in the proof of Theorem 2.2. We have some $\psi \in \mathrm{Diff}_{\mathbb{S}^1}^{\mathrm{Id}}(V)$ such that $\psi^*(\widetilde\varphi^*\lambda) = \lambda$. Hence $\widetilde\varphi \circ \psi$ is in $\mathrm{SCont}(V, \lambda)$ and lifts φ. Together with Lemma 2.3 we get the exact sequence above. For the converse statement, we take $\lambda' \in \mathcal{P}$ and by the proof of Theorem 2.2, there is some φ in the

identity component of $\mathrm{Diff}^{\omega}_{\mathbb{S}^1}(V)$ such that $\varphi^*\lambda' = \lambda$ and $\overline{\varphi} \in \mathrm{Symp}(M,\omega)$. By assumption there is a lift $\widetilde{\varphi} \in \mathrm{SCont}(V,\lambda)$ of $\overline{\varphi}$. Then we get $\varphi \circ \widetilde{\varphi}^{-1} \in \mathrm{Diff}^{\mathrm{Id}}_{\mathbb{S}^1}(V)$ and $(\varphi \circ \widetilde{\varphi}^{-1})^*\lambda' = \lambda$. $\qquad\square$

Remark 2.5 The assumption that $H^2(M,\mathbb{Z})$ is torsion-free is essential for Theorems 2.2 and 2.4. It is possible to construct an integral symplectic manifold (M,ω) with nontrivial torsion in $H^2(M,\mathbb{Z})$ and a symplectomorphism that does not preserve a class $e \in H^2(M,\mathbb{Z})$ that lifts $-[\omega]$. Such a symplectomorphism cannot be lifted even to an $\mathbb{S}^1$-equivariant map of the circle bundle with Euler class e.

Given a prequantization $(V,\lambda) \to (M,\omega)$, we now want to understand the kernel of the map (3). To this end we define the subgroups

$$\mathscr{L} := \{\varphi \in \mathrm{Symp}(M,\omega) \mid \varphi \text{ lifts to } \mathrm{SCont}(V,\lambda)\}$$

and

$$\mathscr{H} := \mathscr{L} \cap \mathrm{Symp}^0(M,\omega)$$

where $\mathrm{Symp}^0(M,\omega)$ is the identity component.

By Lemma 2.3, the map (2) now fits into the exact sequence

$$0 \to \mathbb{S}^1 \to \mathrm{SCont}(V,\lambda) \to \mathscr{L} \to 0$$

which defines a Serre fibration. In fact, it is easy to see that once restricted to the identity components, the above sequence reads

$$0 \to \mathbb{S}^1 \to \mathrm{SCont}^0(V,\lambda) \to \mathrm{Ham}(M,\omega) \to 0$$

where $\mathrm{Ham}(M,\omega)$ is the group of Hamiltonian diffeomorphisms. Then the homotopy lifting property for 0-cells follows from the nontrivial fact that any path of Hamiltonian diffeomorphism starting at Id is a Hamiltonian isotopy. For the details, we refer to [3].

An immediate consequence of the above lemma is the following exact sequence of mapping class groups:

$$0 \to \pi_0(\mathrm{SCont}(V,\lambda)) \to \pi_0(\mathscr{L}) \to 0.$$

On the other hand if $H^2(M,\mathbb{Z})$ is torsion-free we have the exact sequence

$$0 \to \pi_0(\mathscr{H}) \to \pi_0(\mathscr{L}) \to \pi_0(\mathrm{Symp}(M,\omega)) \to 0.$$

It turns out that the elements of $\mathscr{L}$ are characterized by their effect on holonomy and this effect is explicitly determined for the elements of $\mathscr{H}$. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth curve and $p \in \pi^{-1}(\gamma(0))$. Then there exists a unique curve $\hat{\gamma}\colon [0,1] \to V$ such that

$$\hat{\gamma}(0) = p, \quad \pi \circ \hat{\gamma} = \gamma, \quad \lambda(\dot{\hat{\gamma}}) = 0.$$

We call $\hat{\gamma}$, suppressing its initial condition, *the horizontal lift* of γ. If γ is a loop, that is $\gamma(0) = \gamma(1)$, then the *holonomy* $\mathrm{Hol}(\gamma) \in \mathbb{S}^1$ of γ is defined by

$$\phi^X_{\mathrm{Hol}(\gamma)}(\hat{\gamma}(1)) = \hat{\gamma}(0).$$

We note that the holonomy does not depend on $\tilde{\gamma}(0)$ and it is invariant under positive reparameterizations of γ. Moreover, $\operatorname{Hol}(\gamma^{-1}) = -\operatorname{Hol}(\gamma)$ where γ^{-1} is some (hence any) negative reparametrization of γ. Now given a diffeomorphism φ of M one considers its effect on holonomy, namely the map

$$(10) \qquad E_\varphi \colon \operatorname{Loop}(M) \to \mathbb{S}^1, \quad \gamma \mapsto \operatorname{Hol}(\varphi \circ \gamma) - \operatorname{Hol}(\gamma),$$

where $\operatorname{Loop}(M)$ is the space of piecewise smooth maps from $\mathbb{S}^1$ to M. The following fact, which we spell out in accordance with our purposes, is due to Kostant and says that the holonomy effect determines whether a symplectomorphism φ lifts to $\operatorname{SCont}(V, \lambda)$.

Theorem 2.6 (Kostant [22, Proposition 3.3]) *A symplectomorphism $\varphi \in \operatorname{Symp}(M, \omega)$ admits a lift to $\operatorname{SCont}(V, \lambda)$ if and only if $E_\varphi = 0$.*

Proof We fix a point $x \in M$ and put $y = \varphi(x)$. After trivializing the circle bundle near x and y, we fix a map $\eta \colon \pi^{-1}(x) \to \pi^{-1}(y)$, which reads as a rotation with respect to any other choice of such trivializations. Now given $p \in V$ we take a path γ such that $\gamma(0) = \pi(p)$ and $\gamma(1) = x$ and we define $\tilde{\varphi}(p)$ by the equation

$$\widehat{\varphi \circ \gamma}(1) = \eta(\hat{\gamma}(1))$$

where $\widehat{\varphi \circ \gamma}$ is the horizontal lift of $\varphi \circ \gamma$ with $\widehat{\varphi \circ \gamma}(0) = \tilde{\varphi}(p)$ and $\hat{\gamma}$ is the horizontal lift of γ with $\hat{\gamma}(0) = p$.

Note that $\tilde{\varphi}$ is well defined if and only if $E_\varphi = 0$. Moreover $\tilde{\varphi}$, whenever is well defined, is an $\mathbb{S}^1$-equivariant lift of φ, which maps horizontal lifts of curves in M to horizontal lifts of their images under φ. It is not difficult to show that this property together with the fact that φ is symplectic implies that $\tilde{\varphi}$ is in $\operatorname{SCont}(V, \lambda)$. $\qquad \square$

Given $\varphi \in \operatorname{Symp}^0(M, \omega)$, it is easy to understand its effect on holonomy. We begin with the following observation.

Lemma 2.7 *Let $\gamma_1, \gamma_2, \dots, \gamma_k \colon \mathbb{S}^1 \to M$ be a family of smooth loops. Suppose there is an oriented surface Σ with boundary $\partial\Sigma = S_1 \cup S_2 \cup \dots \cup S_k$ consisting of k circles and a smooth map $\sigma \colon \Sigma \to M$ extending each γ_j, that is, for each j, once S_j is oriented as the boundary of Σ and identified with $\mathbb{S}^1$, $\sigma|_{S_j} = \gamma_j$. Then we have*

$$\sum_{j=1}^{k} \operatorname{Hol}(\gamma_j) = \int_\Sigma \sigma^* \omega \quad \mod \mathbb{Z}.$$

Proof We first note that for any smooth $\gamma \colon \mathbb{S}^1 \to M$,

$$\operatorname{Hol}(\gamma) = \int_{\tilde{\gamma}} \lambda \quad \mod \mathbb{Z}$$

where $\tilde{\gamma}$ is any loop that lifts γ. In fact, given a lift $\tilde{\gamma} \colon \mathbb{S}^1 \to V$, we consider the horizontal lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = \tilde{\gamma}(0)$ and define a smooth map $\tau \colon [0, 1] \to \mathbb{S}^1$ such that

$$\phi_{\tau(s)}^X(\hat{\gamma}(s)) = \tilde{\gamma}(s).$$

Note that $\tau(1) = \mathrm{Hol}(\gamma)$ modulo $\mathbb{Z}$. We lift τ to a map into $\mathbb{R}$, still denoted by τ, such that $\tau(0) = 0$ and consider the smooth map

$$\eta\colon \Delta \to V, \quad (s,t) \mapsto \phi_t^X(\hat{\gamma}(s))$$

on the domain $\Delta := \{(s,t) \in \mathbb{R}^2 \mid 0 \le t \le \tau(s),\, s \in [0,1]\}$. Note that $\eta^* d\lambda = 0$. By Stokes' theorem,

$$0 = \int_\Delta \eta^* d\lambda = \int_{\hat{\gamma}} \lambda + \tau(1) - \int_{\tilde{\gamma}} \lambda$$

and the claim follows.

Now let $\sigma\colon \Sigma \to M$ be as in the statement. Since Σ has the homotopy type of a graph, there exists a lift $\tilde{\sigma}\colon \Sigma \to V$ of φ. By the above observation we get

$$\int_\Sigma \sigma^* \omega = \int_\Sigma \tilde{\sigma}^* d\lambda = \int_{\partial \Sigma} \tilde{\sigma}^* \lambda = \sum_{j=1}^k \mathrm{Hol}(\gamma_j) \quad \mathrm{mod}\ \mathbb{Z}. \qquad \square$$

An immediate consequence of this lemma is the following.

Proposition 2.8 *For any $\varphi \in \mathrm{Symp}(M,\omega)$, the map* (10) *descends to a homomorphism*

$$E_\varphi\colon H_1(M,\mathbb{Z}) \to \mathbb{S}^1, \quad [\gamma] \mapsto E_\varphi(\gamma).$$

Moreover the map

(11) $$\mathrm{Symp}^0(M,\omega) \to \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{S}^1), \quad \varphi \mapsto E_\varphi$$

is a homomorphism.

Proof Given smooth loops γ_1 and γ_2 representing the same class in $H_1(M,\mathbb{Z})$, there is an oriented surface Σ with two oriented boundary components S_1 and S_2 and a smooth map $\sigma\colon \Sigma \to M$ such that $\sigma|_{S_1} = \gamma_1$ and $\sigma|_{S_2} = \gamma_2^{-1}$. We get

$$\mathrm{Hol}(\gamma_1) + \mathrm{Hol}(\gamma_2^{-1}) = \mathrm{Hol}(\gamma_1) - \mathrm{Hol}(\gamma_2) = \int_\Sigma \sigma^* \omega$$

and therefore

$$E_\varphi(\gamma_1) - E_\varphi(\gamma_2) = \int_\Sigma (\varphi \circ \sigma)^* \omega - \int_\Sigma \sigma^* \omega = \int_\Sigma \sigma^* (\varphi^* \omega) - \int_\Sigma \sigma^* \omega = 0.$$

Now given $\varphi_1, \varphi_2 \in \mathrm{Symp}^0(M,\omega)$ and $[\gamma] \in H_1(M,\mathbb{Z})$, we have $[\varphi_2 \circ \gamma] = [\gamma]$ in $H_1(M,\mathbb{Z})$ and

$$E_{\varphi_1 \circ \varphi_2}(\gamma) = E_{\varphi_1}(\varphi_2 \circ \gamma) + E_{\varphi_2}(\gamma) = E_{\varphi_1}(\gamma) + E_{\varphi_2}(\gamma). \qquad \square$$

Notice that $\mathcal{H}$ is precisely the kernel of the homomorphism (11). Our next objective is to relate the homomorphism (11) to the flux homomorphism

(12) $$\mathrm{Flux}\colon \mathrm{Symp}^0(M,\omega) \to H^1_{dR}(M,\mathbb{R})/\Gamma_\omega.$$

Recall that if $(\varphi_t)_{t\in[0,1]}$ is a symplectic isotopy of a closed symplectic manifold (M, ω), not necessarily integral, generated by the time dependent symplectic vector field X_t, the flux of $(\varphi_t)_{t\in[0,1]}$ is defined by

$$\mathrm{Flux}(\varphi_t) = \int_0^1 [\iota_{X_t}\omega]dt \in H^1_{dR}(M, \mathbb{R}).$$

It turns out that $\mathrm{Flux}(\varphi_t)$ stays invariant under any homotopy of the path $(\varphi_t)_{t\in[0,1]}$ in $\mathrm{Symp}^0(M, \omega)$, which fixes the end points Id and φ_1 and consequently defines a homomorphism

$$\mathrm{Flux}\colon \widetilde{\mathrm{Symp}^0}(M, \omega) \to H^1_{dR}(M, \mathbb{R})$$

where $\widetilde{\mathrm{Symp}^0}(M, \omega)$ is the universal cover of $\mathrm{Symp}^0(M, \omega)$, seen as the space of homotopy classes, in the above sense, of paths in $\mathrm{Symp}^0(M, \omega)$ starting form the identity. Viewing $\pi_1(\mathrm{Symp}^0(M, \omega), \mathrm{Id})$ as a subgroup of $\widetilde{\mathrm{Symp}^0}(M, \omega)$, one defines the *flux group*

$$\Gamma_\omega := \mathrm{Flux}\big(\pi_1(\mathrm{Symp}^0(M, \omega), \mathrm{Id})\big) \subseteq H^1_{dR}(M, \mathbb{R})$$

and obtains the homomorphism (12). It turns out that $\mathrm{Flux}(\varphi_t)$ vanishes if and only if (φ_t) is homotopic to a Hamiltonian isotopy. Hence the kernel of (12) is precisely the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms.

The following observation relates Flux to the homomorphism (11) if ω is integral.

Lemma 2.9 *Let (φ_t) be a symplectic isotopy starting at Id. Then for any smooth loop $\gamma\colon \mathbb{S}^1 \to M$, we have*

$$\int_\gamma \mathrm{Flux}(\varphi_t) = E_{\varphi_1}(\gamma) \mod \mathbb{Z}.$$

Proof Given γ and φ_t as above, one defines the smooth map $\sigma\colon [0, 1]\times \mathbb{S}^1 \to M$ by $\sigma(s, t) = \varphi_t(\gamma(s))$. An easy computation shows that

$$(13)\qquad\qquad \int_{[\gamma]} \mathrm{Flux}(\varphi_t) = \int_{[0,1]\times\mathbb{S}^1} \sigma^*\omega$$

and from Lemma 2.7 it follows that

$$\int_{[0,1]\times\mathbb{S}^1} \sigma^*\omega = \mathrm{Hol}(\gamma^{-1}) + \mathrm{Hol}(\varphi_1\circ\gamma) = E_{\varphi_1}(\gamma) \mod \mathbb{Z}. \qquad \square$$

We have the following characterization of the mapping class group $\pi_0(\mathscr{H})$.

Theorem 2.10 *We have the following canonical isomorphisms of groups*:

$$(14)\qquad\qquad \pi_0(\mathscr{H}) = \mathscr{H}/\mathrm{Ham}(M, \omega) \cong H^1_{dR}(M, \mathbb{Z})/\Gamma_\omega.$$

Proof The statement follows immediately from Lemma 2.9 and the fact that Γ_ω is contained in $H^1_{dR}(M, \mathbb{Z})$ as ω is integral. $\qquad\square$

3 $H^1_{dR}(M, \mathbb{Z})$ in $\pi_0(\mathrm{Cont}(V, \xi))$ and in $\pi_0(\mathrm{Diff}(V))$

The aim of this section is to provide a class of examples of prequantizations $(V, \lambda) \to (M, \omega)$ such that $\pi_0(\mathscr{H})$ is an infinite subgroup of $\pi_0(\mathrm{SCont}(V, \lambda))$ and the homomorphism

$$(15) \qquad\qquad \pi_0(\mathscr{H}) \to \pi_0(\mathrm{Cont}(V, \xi))$$

is injective, while the homomorphism

$$(16) \qquad\qquad \pi_0(\mathscr{H}) \to \pi_0(\mathrm{Diff}(V))$$

is trivial. Here ξ stands for the kernel of λ, $\mathrm{Cont}(V, \xi)$ is the group of coorientation (given by λ) preserving contactomorphisms of (V, ξ) and $\mathrm{Diff}(V)$ is the group of orientation (determined by λ) preserving diffeomorphisms of V.

We begin with an explicit description of the inverse of the isomorphism (14). Let α be a closed integral 1-form on M given by $\alpha = df$ for some $f \in C^\infty(M, \mathbb{S}^1)$. We define the symplectic gradient X_α of α via $-\alpha = \iota_{X_\alpha}\omega$ and we let $(\overline{\varphi}_t)$ denote its flow. Then the time-one-map $\overline{\varphi}_1$ belongs to $\mathscr{H}$ with a lift

$$(17) \qquad\qquad L_\alpha : V \to V, \quad L_\alpha = \phi^X_{f \circ \pi} \circ \varphi_1$$

in $\mathrm{SCont}(V, \lambda)$, where φ_1 is the time-one-map of the flow (φ_t) of the horizontal lift X_h of X_α and the map $\phi^X_{f \circ \pi}$ is given by (8). Indeed as we saw in the proof of Theorem 2.2,

$$(18) \qquad\qquad (\varphi_t)^*\lambda = \lambda - t\pi^*\alpha.$$

Together with (9) we see that L_α preserves λ. We also note that the maps $\phi^X_{f \circ \pi}$ and φ_t commute for all $t \in \mathbb{R}$. Indeed as in the proof of Theorem 2.2, the vector fields X and X_h commute and the map $f \circ \pi$ is invariant under the flow of X_h. We have the following observation on the action of L_α on loops in V.

Lemma 3.1 *Let* $\delta : \mathbb{S}^1 \to V$ *be a loop. Then* $L_\alpha \circ \delta$ *is freely homotopic to the loop* $\delta * \gamma_V^k$ *where* $\gamma_V : \mathbb{S}^1 \to V$ *denotes the parameterized Reeb orbit starting at the point* $\delta(0)$ *and* $k = \int_\delta \pi^*\alpha$.

Proof   Note that the loop $L_\alpha \circ \delta$ is freely homotopic to the loop $\phi^X_{f \circ \pi} \circ \delta$ via the isotopy (φ_t) so it is enough to show that $\delta * \gamma_V^k$ is freely homotopic to the loop $\phi^X_{f \circ \pi} \circ \delta$. To this end we consider the pull-back bundle $\overline{\delta}^*V \to \mathbb{S}^1$, where $\overline{\delta} = \pi \circ \delta$ and trivialize it via the section δ. After identifying the total space with $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, where the bundle projection π corresponds to the projection to the first circle, the loop $\delta * \gamma_V^k$ reads $(\mathbb{S}^1 \times \{0\}) * (\{0\} \times \mathbb{S}^1)^k$ and the loop $\phi^X_{f \circ \pi} \circ \delta$ reads $t \mapsto (t, f(t))$, where $f : \mathbb{S}^1 \to \mathbb{S}^1$ is of degree k. It is clear that these loops are homotopic in $\mathbb{T}^2$. $\square$

As a consequence of the lemma we get the following class of prequantizations where the map (15) is injective due to topological reasons; more precisely, due to the fact that (16) is injective.

Proposition 3.2   *Let* (M, ω) *be a closed integral symplectic manifold such that*

(1) $[\omega]$ *vanishes on* $\pi_2(M)$;

(2) *any abelian subgroup of* $\pi_1(M)$ *has rank at most one.*

Let $(V, \lambda) \to (M, \omega)$ be a prequantization. Then for any closed integral 1-form α on M, L_α is isotopic to Id in $\mathrm{Diff}(V)$ if and only if $[\alpha]$ is trivial in $H^1_{dR}(M, \mathbb{Z})$.

Lemma 3.3 *Let H be an abelian group of rank at most one. Then every $\mathbb{Z}$-central extension of H is trivial. More precisely, if G is a group with a central subgroup C isomorphic to $\mathbb{Z}$ and G/C is isomorphic to H then G is isomorphic to $H \times \mathbb{Z}$. In particular, G is abelian.*

Proof The commutator subgroup $[G, G]$ is contained in $C = \mathbb{Z}$. The map $G \times G \to \mathbb{Z}$ that sends a couple (a, b) to the commutator $aba^{-1}b^{-1}$ factors to a skew-symmetric bilinear map $B \colon H \times H \to \mathbb{Z}$. Since H has rank at most one, it follows that B is identically zero. $\qquad\square$

Proof of Proposition 3.2 The homotopy exact sequence associated to the bundle $V \to M$ reads

$$(19) \qquad \cdots \to \pi_2(V) \to \pi_2(M) \to \pi_1(\mathbb{S}^1) \to \pi_1(V) \to \pi_1(M) \to 0$$

where the connecting homomorphism is given by integrating ω over spheres in M. By the first assumption, the fiber γ_V is of infinite order in $\pi_1(V)$. We note also that the image of $\pi_1(\mathbb{S}^1)$ is always central in $\pi_1(V)$, that is γ_V commutes with every element of $\pi_1(V)$. In fact, given any loop η in V (having the same base point with γ_V) and its projection $\bar\eta$ in M, γ_V and η can be seen as loops in the total space of the pull-back bundle $\bar\eta^* V \to \mathbb{S}^1$, which is diffeomorphic to $\mathbb{T}^2$ and therefore has abelian fundamental group.

Now if L_α is isotopic to Id and δ is a loop in V, then by Lemma 3.1, δ is freely homotopic to $\delta * \gamma_V^k$. Then there is a loop β in V such that $\beta * \delta * \beta^{-1} = \delta * \gamma_V^k$ in $\pi_1(V)$. We write $\bar\delta = \pi \circ \delta$ and $\bar\beta = \pi \circ \beta$. Then $\bar\delta$ commutes with $\bar\beta$ in $\pi_1(M)$ and by assumption $\bar\delta$ and $\bar\beta$ are contained in a subgroup H of rank at most one. Then, by Lemma 3.3, the subgroup $\pi_*^{-1}(H)$ is abelian. As δ and β are contained in $\pi_*^{-1}(H)$, we conclude that γ_V^k is contractible and therefore $k = \int_{\bar\delta} \alpha = 0$. Since δ is arbitrary we get $[\alpha] = 0$. $\quad\square$

Remark 3.4 The above statement shows that the flux group Γ_ω is trivial in this case and $\pi_0(\mathscr{H})$, being isomorphic to $H^1_{dR}(M, \mathbb{Z})$, injects into $\pi_0(\mathrm{Diff}(V))$.

Example 3.5 A typical example for the above proposition is an oriented surface of genus at least two.

Our next objective is to use Proposition 3.2 as a springboard to describe prequantizations for which the homomorphism (15) is injective for contact topological reasons. We want to replace the role of the Reeb orbit with the loop induced by the Reeb flow in the bundle $\mathrm{CSp}(\xi) \to V$ of conformally symplectic frames of ξ.

The bundle $\mathrm{CSp}(\xi) \to V$ is a principal $\mathrm{CSp}(2n, \mathbb{R})$-bundle, where $\mathrm{CSp}(2n, \mathbb{R})$ denotes the group of conformally symplectic matrices. As the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ is a deformation retract of $\mathrm{CSp}(2n, \mathbb{R})$, it is natural to consider the isomorphism, called the *Maslov index*, given by

$$\mathrm{Index} \colon \pi_1(\mathrm{CSp}(2n, \mathbb{R})) = \pi_1(\mathrm{Sp}(2n, \mathbb{R})) \to \pi_1(\mathrm{U}(n)) \to \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

where the first arrow is given by the polar decomposition and the second arrow is induced by the determinant map. We have the homotopy exact sequence

$$(20) \qquad \cdots \to \pi_2(V) \to \pi_1(\mathrm{CSp}(2n, \mathbb{R})) \to \pi_1(\mathrm{CSp}(\xi)) \to \pi_1(V) \to 0$$

and it is easy to see that the connecting homomorphism is given by integrating the first Chern class $c_1(\xi)$ over spheres in V. Here we view ξ as a complex vector bundle of rank n, by choosing a complex structure on ξ compatible with $d\lambda$. We note that the space of such complex structures is nonempty and contractible. In the same vein, $c_1(TM)$ is well defined and we have $c_1(\xi) = \pi^* c_1(TM)$. Combining this with (20), we conclude that the homomorphism

$$(21) \qquad \pi_1(\mathrm{CSp}(2n, \mathbb{R})) \to \pi_1(\mathrm{CSp}(\xi))$$

is injective if and only if $c_1(TM)$ vanishes on every sphere of vanishing symplectic area in M. Note that this is equivalent to assuming that (M, ω) is *monotone on* $\pi_2(M)$, that is, there is some $\kappa \in \mathbb{R}$ (in fact $\kappa \in \mathbb{Q}$), which we call the *monotonicity constant*, such that $c_1(TM) = \kappa\omega$ on $\pi_2(M)$. We also remark that the image of $\pi_1(\mathrm{CSp}(2n, \mathbb{R}))$ in $\pi_1(\mathrm{CSp}(\xi))$ is always central. This follows from the argument given in the proof of Proposition 3.2 for the bundle $V \to M$ and the fact that $\pi_1(\mathrm{CSp}(2n, \mathbb{R}))$ is abelian.

Our aim now is to study the loop in $\mathrm{CSp}(\xi)$ that is associated to the Reeb flow and to determine sufficient conditions such that it is of infinite order in $\pi_1(\mathrm{CSp}(\xi))$. Let $p \in V$ and $\mathscr{F}_p$ be a symplectic frame of ξ_p. We define the *Reeb frame loop* in the frame bundle by

$$(22) \qquad \Gamma_V \colon \mathbb{S}^1 \to \mathrm{CSp}(\xi), \quad t \mapsto (\phi_t^X)_* \mathscr{F}_p,$$

which defines an element in $\pi_1(\mathrm{CSp}(\xi), \mathscr{F}_p)$. Next we assume that the Reeb orbit γ_V is contractible. Let $j \colon \mathbb{D} \to V$ be a capping disc, that is $j = \gamma_V$ on $\partial\mathbb{D}$. Let $\mathscr{F}$ be a symplectic framing of $j^*\xi$. Writing Γ_V with respect to the framing $\mathscr{F}|_{\partial\mathbb{D}}$, we get the loop

$$\Gamma_V^j \colon \mathbb{S}^1 \to \mathrm{CSp}(2n, \mathbb{R})$$

to which we can associate its Maslov index.

Lemma 3.6 *Let* $\bar{j} \colon S^2 := \mathbb{D}/\partial\mathbb{D} \to M$ *be the sphere given by* $\bar{j} = \pi \circ j$. *Then we have*

$$\mathrm{Index}(\Gamma_V^j) = -\langle c_1(\bar{j}^*TM), S^2 \rangle = -\langle c_1(TM), \bar{j}(S^2) \rangle.$$

In particular, $\mathrm{Index}(\Gamma_V^j)$ *does not depend on* $\mathscr{F}$. *Moreover,* $\mathrm{Index}(\Gamma_V^j)$ *is independent of the capping disc if and only if* (M, ω) *is monotone on* $\pi_2(M)$.

Proof Let $x \in S^2$ denote the image of $\partial\mathbb{D}$ in the quotient. We take a collar neighborhood U of $\partial\mathbb{D}$ in $\mathbb{D}$ and let $\overline{U}$ be the corresponding disc neighborhood of $x \in S^2$. We pick a symplectic framing $\overline{\mathscr{G}}$ of $\bar{j}^*TM|_{\overline{U}}$, say extending $\pi_*\mathscr{F}_p$, and lift it to a Reeb invariant framing $\mathscr{G}$ of $j^*\xi|_U$ extending $\mathscr{F}_p$. Note that this is possible since $\overline{U}$ is contractible. On the other hand, $\mathscr{F}$ descends to a symplectic framing $\overline{\mathscr{F}}$ of

$\bar{j}^*TM|_{S^2\setminus\{x\}}$. Viewing $c_1(\bar{j}^*TM)$ as the homotopy class of the clutching function that is used to glue the trivial bundles $(\bar{j}^*TM|_{S^2\setminus\overline{U}}, \overline{\mathcal{F}})$ and $(\bar{j}^*TM|_{\overline{U}}, \overline{\mathcal{G}})$ along $\partial\overline{U}$, we get

$$\langle c_1(\bar{j}^*TM), S^2\rangle = \text{Index}(\overline{\mathcal{F}}|_{\partial\overline{U}}) = -\text{Index}(\overline{\mathcal{G}}|_{\partial\overline{U}})$$

where we view $\overline{\mathcal{F}}|_{\partial\overline{U}}$ (resp. $\overline{\mathcal{G}}|_{\partial\overline{U}}$) as a loop in $\text{CSp}(2n, \mathbb{R})$ using the framing $\overline{\mathcal{G}}|_{\partial\overline{U}}$ (resp. $\overline{\mathcal{F}}|_{\partial\overline{U}}$). Using the symplectic isomorphism π_*, we get

$$\langle c_1(\bar{j}^*TM), S^2\rangle = -\text{Index}(\mathcal{G}|_\eta)$$

where η denotes the preimage of $\partial\overline{U}$ in $\mathbb{D}$ and the loop $\mathcal{G}|_\eta$ is obtained via the framing $\mathcal{F}$. Since $\mathcal{G}|_\eta$ is homotopic to the loop $\mathcal{G}|_{\partial\mathbb{D}}$, which is defined via the framing $\mathcal{F}$, and since Γ_V^j reads as the constant loop with respect to the framing $\mathcal{G}|_{\partial\mathbb{D}}$, we get

$$\langle c_1(\bar{j}^*TM), S^2\rangle = -\text{Index}(\mathcal{G}|_{\partial\mathbb{D}}) = -\text{Index}(\Gamma_V^j),$$

where the left hand side is clearly independent of the framing $\mathcal{F}$.

Recall that (M, ω) is monotone on $\pi_2(M)$ if and only if $c_1(\xi)$ vanishes on $\pi_2(V)$. Now given two capping discs $j_1, j_2 \colon \mathbb{D} \to V$, we glue them together, say after reversing the orientation of the second, to get a sphere $j \colon S^2 \to V$. We have

$$\text{Index}(\Gamma_V^{j_1}) - \text{Index}(\Gamma_V^{j_2}) = \langle c_1(j^*\xi), S^2\rangle.$$

Hence $\text{Index}(\Gamma_V^{j_1}) = \text{Index}(\Gamma_V^{j_2})$ if $c_1(\xi)$ vanishes on $\pi_2(V)$. On the other hand, given a sphere S in V, one can choose some capping disc $j_1 \colon \mathbb{D} \to V$ with $j_1(0) \in S$ and define the capping disc j_2 to be the concatenation of j_1 and S. Repeating the above construction for these two capping discs leads to a sphere j in the homotopy class of S. Then the above formula says that $c_1(\xi)$ vanishes on S if the index is independent of the capping discs. □

Remark 3.7 The above construction of the Maslov index can also be carried out in the case where γ_V is null homologous. In this case the capping disc is replaced by a capping surface $j \colon \Sigma \to V$ and the relative index is given by evaluating $c_1(TM)$ over the closed surface $\bar{j} \colon \overline{\Sigma} := \Sigma/\partial\Sigma \to M$.

An immediate consequence of the above lemma is the following.

Proposition 3.8 Let $(V, \lambda) \to (M, \omega)$ be a prequantization. Suppose that either $[\omega]$ vanishes on $\pi_2(M)$ or (M, ω) is monotone on $\pi_2(M)$ with nonvanishing monotonicity constant. Then the Reeb frame loop Γ_V is of infinite order in $\pi_1(\text{CSp}(\xi))$.

Proof Notice that if $[\omega]$ is trivial on $\pi_2(M)$ then the Reeb orbit is of infinite order in $\pi_1(V)$. Since Γ_V projects to γ_V the statement follows.

Now assume that there is sphere in M with unit symplectic area. Then γ_V is contractible. Given a capping disc $j \colon \mathbb{D} \to V$ with $p = j(1)$, with a framing $\mathcal{F}$ of $j^*\xi$ extending $\mathcal{F}_p$, the loop Γ_V based at

$\mathscr{F}_p$ is homotopic to a loop in $\mathrm{CSp}(\xi_p)$ based at $\mathscr{F}_p$, whose homotopy class is given by $\mathrm{Index}(\Gamma_V^j)$ after $(\mathrm{CSp}(\xi_p), \mathscr{F}_p)$ is identified by $(\mathrm{CSp}(2n, \mathbb{R}), \mathrm{Id})$. By monotonicity, the homomorphism (21) is injective. Hence Γ_V is of infinite order in $\pi_1(\mathrm{CSp}(\xi), \mathscr{F}_p)$ if and only if $\mathrm{Index}(\Gamma_V^j)$ is nonzero.

On the other hand,

$$1 = \int_{\gamma_V} \lambda = \int_{\mathbb{D}} j^* d\lambda = \int_{S^2} \bar{j}^* \omega.$$

Since the monotonicity constant is assumed to be nonzero, $\mathrm{Index}(\Gamma_V^j)$ does not vanish.

Finally, if γ_V is torsion, say of order d, in $\pi_1(V)$, then we consider the loop Γ_V^d in $\pi_1(\mathrm{CSp}(\xi))$ and apply the Maslov index construction to Γ_V^d. The argument above applies word by word to Γ_V^d and we conclude that Γ_V^d and consequently Γ_V is of infinite order. $\square$

Given $p \in V$ and a symplectic frame $\mathscr{F}_p$ of ξ_p, we define a map

$$\mathrm{Cont}(V, \xi) \to \mathrm{CSp}(\xi), \quad \varphi \mapsto \varphi_* \mathscr{F}_p$$

and get the induced homomorphisms $\pi_1(\mathrm{Cont}(V, \xi), \mathrm{Id}) \to \pi_1(\mathrm{CSp}(\xi), \mathscr{F}_p)$ which maps the homotopy class of the Reeb loop $(\phi_t^X)_{t \in \mathbb{S}^1}$ to the homotopy class of Γ_V. Hence the Reeb loop is of infinite order in $\pi_1(\mathrm{Cont}(V, \xi), \mathrm{Id})$ if Γ_V is of infinite order in $\pi_1(\mathrm{CSp}(\xi), \mathscr{F}_p)$. Hence we get the following corollary.

Corollary 3.9 *Let* $(V, \lambda) \to (M, \omega)$ *be a prequantization. Suppose that either* $[\omega]$ *vanishes on* $\pi_2(M)$ *or* (M, ω) *is monotone on* $\pi_2(M)$ *with nonvanishing monotonicity constant. Then the Reeb loop* $(\phi_t^X)_{t \in \mathbb{S}^1}$ *is of infinite order in* $\pi_1(\mathrm{Cont}(V, \xi))$.

After establishing the setting in which Γ_V is suitable to take the role of γ_V in the proof of Proposition 3.2, we are ready to prove the lemma that is analogous to Lemma 3.1.

Lemma 3.10 *Let* $\hat{\delta} \colon \mathbb{S}^1 \to \mathrm{CSp}(\xi)$ *be a loop with its projection* $\delta \colon \mathbb{S}^1 \to V$. *Then* $(L_\alpha)_* \circ \hat{\delta}$ *is freely homotopic to the loop* $\hat{\delta} * \Gamma_V^k$ *where* $\Gamma_V \colon \mathbb{S}^1 \to \mathrm{CSp}(\xi)$ *is the Reeb frame loop based at* $\hat{\delta}(0)$ *and* $k = \int_\delta \pi^* \alpha$.

Proof Recall that $L_\alpha = \varphi_1 \circ \phi_{f \circ \pi}^X$ where $\alpha = df$ for some $f \colon M \to \mathbb{S}^1$ and φ_1 is the time-one-map of the flow (φ_t) of the horizontal vector field X_h. From (18) it follows that

$$(\varphi_t)_* \colon (\xi_0, d\lambda) \to (\xi_t, d\lambda), \quad t \in [0, 1]$$

is a smooth path of isomorphisms of rotating contact structures

$$\xi_t := \ker(\lambda + (t-1)\pi^* \alpha), \quad t \in [0, 1],$$

seen as symplectic vector bundles. Note that $\xi_1 = \ker \lambda = \xi$ and by (9) we have that

$$(\phi_{f \circ \pi}^X)_* \colon (\xi_1, d\lambda) \to (\xi_0, d\lambda)$$

is also an isomorphism. Notice that the vector field X is the Reeb vector field for every contact form $\lambda + (t-1)\pi^*\alpha$ and by projecting along X, we get a smooth path

$$P_t : (\xi_t, d\lambda) \to (\xi_1, d\lambda)$$

of isomorphisms of symplectic vector bundles. Observe that P_1 is the identity map. Now we define a path of loops

$$\sigma : \mathbb{S}^1 \times [0,1] \to \mathrm{CSp}(\xi), \quad (s,t) \mapsto P_t \circ (\varphi_t)_* \circ (\phi^X_{f \circ \pi})_* \circ \hat{\delta}(s),$$

which defines a free homotopy between $\sigma(\,\cdot\,, 0) = P_0 \circ (\phi^X_{f \circ \pi})_* \circ \hat{\delta}$ and $\sigma(\,\cdot\,, 1) = (L_\alpha)_* \circ \hat{\delta}$.

We claim that $P_0 \circ (\phi^X_{f \circ \pi})_* \circ \hat{\delta}$ is freely homotopic to $\hat{\delta} * \Gamma^k_V$ in $\mathrm{CSp}(\xi)$. To see this, we write $\bar{\delta} := \pi \circ \delta$ and consider the pull-back bundle $\bar{\delta}^* V \to \mathbb{S}^1$. Let $\Delta : \bar{\delta}^* V \to V$ be the bundle map covering $\bar{\delta}$. Next we take a symplectic trivialization of $\bar{\delta}^* TM$ and lift it to an $\mathbb{S}^1$-invariant trivialization of the pull-back bundle $\Delta^* \xi \to \bar{\delta}^* V$. Pairing this trivialization with the trivialization of $\bar{\delta}^* V \to \mathbb{S}^1$ described in the proof of Lemma 3.1, we identify $\mathrm{CSp}(\Delta^* \xi)$ with $\mathbb{T}^2 \times \mathrm{CSp}(2n, \mathbb{R})$ in such a way that $\hat{\delta}$ reads as $(\mathbb{S}^1 \times \{0\})$ in $\mathbb{T}^2$ paired with a loop η in $\mathrm{CSp}(2n, \mathbb{R})$ based at Id, and Γ^k_V reads as $(\{0\} \times \mathbb{S}^1)^k$ in $\mathbb{T}^2$ paired with the constant loop Id in $\mathrm{CSp}(2n, \mathbb{R})$ since $\Delta^* \xi$ is trivialized in an $\mathbb{S}^1$-invariant fashion. Hence $\hat{\delta} * \Gamma^k_V$ reads as $(\mathbb{S}^1 \times \{0\}) * (\{0\} \times \mathbb{S}^1)^k$ paired with η. On the other hand since P_0 is given by the projection along X, for any $s \in \mathbb{S}^1$ and $u \in \hat{\delta}(s)$, we have

$$P_0 \circ (\phi^X_{f \circ \pi})_* u = P_0\big(d(\phi^X_{f(\bar{\delta}(s))})_{\delta(s)}[u] + \alpha_{\bar{\delta}(s)}(\pi_* u)X\big) = d(\phi^X_{f(\bar{\delta}(s))})_{\delta(s)}[u].$$

Again since $\Delta^* \xi$ is trivialized in an $\mathbb{S}^1$-invariant fashion, $P_0 \circ (\phi^X_{f \circ \pi})_* \circ \hat{\delta}$ reads

$$s \mapsto \big((s, f(s)), \eta(s)\big) \in \mathbb{T}^2 \times \mathrm{CSp}(2n, \mathbb{R}),$$

where $f : \mathbb{S}^1 \to \mathbb{S}^1$ is of degree k. Therefore, $P_0 \circ (\phi^X_{f \circ \pi})_* \circ \hat{\delta}$ is freely homotopic to $\hat{\delta} * \Gamma^k_V$ in $\mathbb{T}^2 \times \mathrm{CSp}(2n, \mathbb{R})$. $\square$

Now we are ready to state our first main result.

Theorem 3.11 *Let (M, ω) be a closed integral symplectic manifold such that either $[\omega]$ vanishes on $\pi_2(M)$ or (M, ω) is monotone on $\pi_2(M)$ with a nonvanishing monotonicity constant. Suppose that any abelian subgroup of $\pi_1(M)$ has rank at most one. If $(V, \lambda) \to (M, \omega)$ is a prequantization, then for any closed integral 1-form α on M, L_α is isotopic to* Id *in* $\mathrm{Cont}(V, \xi)$ *if and only if $[\alpha]$ is trivial in $H^1_{dR}(M, \mathbb{Z})$.*

Proof If $[\omega]$ vanishes on $\pi_2(M)$ then this follows from Proposition 3.2. Now, suppose $[\omega]$ does not vanish on $\pi_2(M)$ and (M, ω) is monotone on $\pi_2(M)$ with a nonvanishing monotonicity constant. If L_α is isotopic to Id in $\mathrm{Cont}(V, \xi)$ and $\hat{\delta}$ is a loop in $\mathrm{CSp}(\xi)$, then by Lemma 3.10 $\hat{\delta}$ is freely homotopic to $\hat{\delta} * \Gamma^k_V$. So there is a loop $\hat{\beta}$ in $\mathrm{CSp}(\xi)$ such that $\hat{\beta} * \hat{\delta} * \hat{\beta}^{-1} = \hat{\delta} * \Gamma^k_V$ in $\pi_1(\mathrm{CSp}(\xi))$. Let δ and β be the projections of $\hat{\delta}$ and $\hat{\beta}$ to V respectively. Then $\delta^{-1}\beta\delta\beta^{-1} = \gamma^k_V$ where γ^k_V is the projection of Γ^k_V. But, we can assume γ^k_V is contractible by iterating L_α (or equivalently, by considering a suitable multiple

$n\alpha$ of α). In this case, δ commutes with β in $\pi_1(V)$. It follows that δ and β are contained in an abelian subgroup H of rank at most one since the $\mathbb{S}^1$-fiber of V is torsion in $\pi_1(V)$ (by the assumption that $[\omega]$ does not vanish on $\pi_2(M)$) and every abelian subgroup of $\pi_1(M)$ has rank at most one. It follows, by Lemma 3.3, that the subgroup $\pi_*^{-1}(H)$ is an abelian subgroup of $\pi_1(\mathrm{CSp}(\xi))$ since it is a $\mathbb{Z}$-central extension of H (the Reeb frame loop is of infinite order by Proposition 3.8). Hence $\hat{\delta}$ and $\hat{\beta}$ are contained in the abelian subgroup $\pi_*^{-1}(H)$ and therefore Γ_V^k is contractible. But by Proposition 3.8 this is possible only if $k = \int_{\bar{\delta}} \alpha = 0$. Since δ is arbitrary we get $\alpha = 0$ in $H^1_{dR}(M, \mathbb{Z})$. $\qquad\square$

Now let (W, ω_W) be a simply connected integral symplectic manifold. Consider the product symplectic manifold $M := W \times \Sigma_g$, $\omega := \omega_W \oplus \omega_0$, where ω_0 is any integral symplectic form on the surface Σ_g of genus $g \geq 2$ and let $\pi : (V, \lambda) \to (M, \omega)$ be a prequantization. Then V restricted to $W \times \{\mathrm{pt}\}$, denoted by $V|_W$, defines a prequantization of (W, ω_W). Note that $\pi_1(M) = \pi_1(\Sigma_g)$ and Γ_ω is trivial (see, for example, [5, Theorem 3]). Then by Theorem 2.10, the kernel of the homomorphism (which is surjective in this case)

$$\pi_0(\mathrm{SCont}(V, \lambda)) \to \pi_0(\mathrm{Symp}(M, \omega))$$

is given by $H^1_{dR}(M, \mathbb{Z}) = H^1_{dR}(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. So we get the homomorphism

$$\mathscr{I} : H^1_{dR}(\Sigma_g, \mathbb{Z}) \to \pi_0(\mathrm{Cont}(V, \xi))$$

and we have the natural homomorphism $\mathscr{J} : \pi_0(\mathrm{Cont}(V, \xi)) \to \pi_0(\mathrm{Diff}(V))$.

Proposition 3.12 *Suppose that the $\mathbb{S}^1$-action on $V|_W$, given by the Reeb flow, defines a torsion loop in $\pi_1(\mathrm{Diff}_0(V|_W))$ of order k. Then every mapping class in the subgroup $k H^1_{dR}(\Sigma_g, \mathbb{Z}) \subset \pi_0(\mathrm{SCont}(V, \lambda))$ is smoothly trivial.*

Proof Let $\sigma : \mathbb{S}^1 \times [0, \varepsilon] \to \Sigma_g$ be a symplectic embedding of a small cylinder and let $f : [0, \varepsilon] \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be a smooth function that vanishes near 0 and ε. Now, f induces a function $\tilde{f} : \Sigma_g \to \mathbb{S}^1$ as follows: extend f to $\mathbb{S}^1 \times [0, \varepsilon]$ in the obvious way, so it induces a function on the image of σ which can be extended to be identically zero outside. The differential form $d\tilde{f}$ represents an integral cohomology class and, conversely, every integral cohomology is represented by such a function (with a suitable choice of σ and the degree of f). Therefore, every element of $H^1_{dR}(M, \mathbb{Z})$ can be represented as the differential of a function $\tilde{f} \circ p$ where $p : M \to \Sigma_g$ is the projection and $\tilde{f}$ is as above.

Recall that by (17), the lift of $d(\tilde{f} \circ p)$ to $\mathrm{SCont}(V, \lambda)$ is given by $\phi^X_{(\tilde{f} \circ p) \circ \pi} \circ \varphi_1$ where φ_1 is the time-one-map of the flow (φ_t) of the horizontal lift of the symplectic gradient of $d(\tilde{f} \circ p)$. Since φ_1 is isotopic to the identity, one needs only to show that if the degree of f is k (where $f : [0, \varepsilon] \to \mathbb{S}^1$ is seen as a loop) then $\phi^X_{(\tilde{f} \circ p) \circ \pi}$ is isotopic to the identity.

Let $\sigma : \mathbb{S}^1 \times [0, \varepsilon] \to \Sigma_g$ be as above and $f : [0, \varepsilon] \to \mathbb{S}^1$ be such a map of degree k. Denote by C the image of σ. Let $V|_C$ denote the restriction of the $\mathbb{S}^1$-bundle V to $W \times C$ and $Z = V|_W$ denote the restriction of V to $W \times \{\mathrm{pt}\}$. Then the map $p \circ \pi : V|_C \to C$ defines a fiber bundle with fiber Z and the

$\mathbb{S}^1$-action preserves each fiber. Our claim is that $p \circ \pi \colon V|_C \to C$ is equivalent, in an $\mathbb{S}^1$-equivariant way, to $Z \times C \to C$ where Z is seen as the prequantization of (W, ω_W).

We first show that the bundle $p \circ \pi \colon V|_C \to C$ is trivial, in an $\mathbb{S}^1$-equivariant way, when restricted to $\mathbb{S}^1 \times \{0\} \subset C$, where C is identified with $\mathbb{S}^1 \times [0, \varepsilon]$. Fix $y_0 \in W$ and suppose, without loss of generality, that the loop $S_{y_0} := \{y_0\} \times \mathbb{S}^1 \times \{0\}$ has zero holonomy. Then for every $y \in W$, the loop $S_y := \{y\} \times \mathbb{S}^1 \times \{0\}$ also has zero holonomy since for some $\varphi \in \mathrm{Ham}(W, \omega_W)$ with $\varphi(y_0) = y$, the Hamiltonian diffeomorphism $\varphi \times \mathrm{Id}$ of M maps S_{y_0} to S_y. Now, horizontal lifts of the loops S_y for $y \in W$ produce an $\mathbb{S}^1$-equivariant identification of all the fibers $(p \circ \pi)^{-1}(z)$ for $z \in \mathbb{S}^1 \times \{0\}$. Now we can argue in the same fashion and identify all the fibers $(p \circ \pi)^{-1}(z)$, for $z \in \mathbb{S}^1 \times [0, \varepsilon]$, $\mathbb{S}^1$-equivariantly just by considering the foliation of $\mathbb{S}^1 \times [0, \varepsilon]$ by intervals $\{s\} \times [0, \varepsilon]$ and their associated horizontal lifts.

Now, the map $\phi^X_{(\tilde{f} \circ p) \circ \pi}$ reads

$$\phi^X_{(\tilde{f} \circ p) \circ \pi} \colon Z \times C \to Z \times C, \quad (q, (x, y)) \mapsto (f(y) \cdot q, (x, y))$$

where $f(y) \cdot q$ means the action of $f(y) \in \mathbb{S}^1$ on q. So, $\phi^X_{(\tilde{f} \circ p) \circ \pi}$ has the form

$$(q, (x, y)) \mapsto (\phi_y(q), (x, y))$$

where $\phi \colon [0, \varepsilon] \to \mathrm{Diff}^0(Z)$ is a loop based at Id. In our case, ϕ is contractible if f has degree k (since, by assumption, the Reeb flow defines a torsion loop of order k in $\pi_1(\mathrm{Diff}^0(Z))$). This shows that $\phi^X_{(\tilde{f} \circ p) \circ \pi}$ is isotopic to the identity. $\qquad \square$

Corollary 3.13 *Let (Σ_g, ω_0) be a surface of genus $g \geq 2$ with an integral symplectic form and ω_{FS} be the Fubini–Study form on $\mathbb{C}P^n$, which is normalized so that the standard $\mathbb{C}P^1 \subset \mathbb{C}P^n$ has unit area. Consider a prequantization $(V, \lambda) \to (\mathbb{C}P^n \times \Sigma_g, \omega_{\mathrm{FS}} \oplus \omega_0)$. Then the homomorphism*

$$\mathcal{I} \colon H^1_{dR}(\Sigma_g, \mathbb{Z}) \to \pi_0(\mathrm{Cont}(V, \xi)),$$

is injective. Moreover, $\ker(\mathcal{I})$ contains the image of $\mathcal{I}$ for n odd, and contains $\mathcal{I}(2H^1_{dR}(\Sigma_g, \mathbb{Z}))$ for n even.

Proof The injectivity of $\mathcal{I}$ follows from [Theorem 3.11]. Note that the prequantization $V|_{\mathbb{C}P^n}$ is the Hopf bundle $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$ and the Reeb loop is given by the loop $t \mapsto \mathrm{diag}(e^{2\pi i t}, \ldots, e^{2\pi i t})$ in $\mathrm{U}(n+1)$. This defines a loop in $\mathrm{SO}(2n+2)$ which is contractible if n is odd and of order 2 if n is even. The rest is a direct consequence of [Proposition 3.12]. $\qquad \square$

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Fakultät für Mathematik, Ruhr-Universität Bochum
Bochum, Germany

Department Mathematik/Informatik, Abteilung Mathematik, Universität zu Köln
Köln, Germany

souheib.allout@rub.de,   msaglam@math.uni-koeln.de

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