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# Pullbacks of metric bundles and Cannon–Thurston maps

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Metric (graph) bundles were defined by Mj and Sardar (Geom. Funct. Anal. 22 (2012) 1636–1707). In this paper, we introduce the notion of morphisms and pullbacks of metric (graph) bundles. Given a metric (graph) bundle $X$ over $B$ where $X$ and all the fibers are uniformly (Gromov) hyperbolic and nonelementary, and a Lipschitz quasiisometric embedding $i: A \to B$, we show that the pullback $i^*X$ is hyperbolic and the map $i^*: i^*X \to X$ admits a continuous boundary extension, ie the Cannon–Thurston (CT) map $\partial i^*: \partial(i^*X) \to \partial X$. As an application of our theorem, we show that given a short exact sequence of nonelementary hyperbolic groups $1 \to N \to G \xrightarrow{\pi} Q \to 1$ and a finitely generated quasiisometrically embedded subgroup $Q_1 < Q$, $G_1 := \pi^{-1}(Q_1)$ is hyperbolic and the inclusion $G_1 \to G$ admits the CT map $\partial G_1 \to \partial G$. We then derive several interesting properties of the CT map.

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## 1 Introduction

Given a hyperbolic group $G$ and a hyperbolic subgroup $H$ a natural question to ask is if the inclusion $H \to G$ always extends continuously to $\partial H \to \partial G$ (see [3, Q 1.19]). This question was posed by Mahan Mitra (Mj) motivated by the seminal article of Cannon and Thurston [7]. In [7] the authors found the first instance of this phenomenon where $H$ is not quasiisometrically embedded in $G$. It follows from their work that if $G = \pi_1(M)$ where $M$ is a closed hyperbolic 3-manifold fibering over a circle and $H = \pi_1(S)$ with $S$ (an orientable closed surface of genus at least 2) being the fiber, then the boundary extension $\partial H \to \partial G$ exists. More generally, one may ask for a pair of (Gromov) hyperbolic metric spaces

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$Y \subset X$ if there is a continuous extension of the inclusion $Y \to X$ to $\partial Y \to \partial X$. Such an extension is by definition unique (see Definition 2.47) when it exists and is popularly known as the *Cannon–Thurston map* or "*CT map*" for short in geometric group theory. The above question of Mahan Mitra (Mj) has motivated numerous works. The reader is referred to [22] for a detailed history of the problem. Although the general question for groups has been answered in the negative recently by Baker and Riley [2], there are many interesting questions to be answered in this context. In this paper, we pick up the following:

**Question** Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ is a short exact sequence of hyperbolic groups. Suppose $Q_1 < Q$ is quasiisometrically embedded and $G_1 = \pi^{-1}(Q_1)$. Then does the inclusion $G_1 < G$ admit the CT map?

It follows by the results of Mj and Sardar [24] that $G_1$ is hyperbolic (see [24, Remark 4.4]), and so the question makes sense. In this paper, we answer the above question affirmatively. However, we reformulate this question in terms of metric (graph) bundles as defined in [24] (see Section 3) and obtain the following more general result. One is referred to Lemma 2.41 and the discussion following it for the definition of barycenter map. Coarsely surjective maps are introduced in Definition 2.1(3).

**Theorem 5.2** *Suppose $\pi : X \to B$ is a metric (graph) bundle such that*

(1) *$X$ is hyperbolic and*

(2) *all the fibers are uniformly hyperbolic and nonelementary, ie there are $\delta \geq 0$ and $R \geq 0$ such that any fiber $F$ is $\delta$-hyperbolic and the barycenter map $\partial_s^3 F \to F$ is $R$-coarsely surjective.*

*Suppose $i : A \to B$ is a Lipschitz, quasiisometric embedding and $\pi_Y : Y \to A$ is the pullback bundle under $i$ (see Definition 3.18). Then $i^* : Y \to X$ admits the CT map.*

There are two main sources of examples of metric graph bundles mentioned in this paper where the above theorem can be applied. The first one is that of short exact sequences of groups.

**Theorem 6.1** *Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ is a short exact sequence of hyperbolic groups. Suppose $Q_1 < Q$ is quasiisometrically embedded and $G_1 = \pi^{-1}(Q_1)$. Then $G_1$ is a hyperbolic group and the inclusion $G_1 < G$ admits the CT map.*

We note that special cases of Theorems 5.2 and 6.1, namely when $A$ is a point and $Q_1 = (1)$, respectively, were already known. See [20, Theorem 4.3; 24, Theorem 5.3]. Another context where Theorem 5.2 applies is that of complexes of hyperbolic groups. We refer to Section 3.3.2 for relevant definitions.

Suppose $\mathcal{Y}$ is a finite simplicial complex and $\mathbb{G}(\mathcal{Y})$ is a developable complex of nonelementary hyperbolic groups over $\mathcal{Y}$. Suppose that for all face $\sigma$ of $\mathcal{Y}$, $G_\sigma$ is a nonelementary hyperbolic group and for any two faces $\sigma \subset \tau$ the corresponding homomorphism $G_\tau \to G_\sigma$ is an isomorphism onto a finite index subgroup of $G_\sigma$. Suppose that the fundamental group of the complex of groups, $G$ say, is hyperbolic. Suppose we have a *good* subcomplex $\mathcal{Y}_1 \subset \mathcal{Y}$ ie one for which the following two conditions are satisfied.

(1) The natural homomorphism $\pi_1(\mathcal{G}, \mathcal{Y}_1) \to \pi_1(\mathcal{G}, \mathcal{Y})$ is injective.

Let $G_1 = \pi_1(\mathcal{G}, \mathcal{Y}_1)$. Suppose $G_1$ and $G$ are both endowed with word metrics with respect to some finite generating sets. Let $\widehat{G}$ and $\widehat{G}_1$ be the coned off spaces à la Farb [10], obtained by coning off all the face groups in $G$ and $G_1$ respectively.

(2) Then the induced map $\widehat{G}_1 \to \widehat{G}$ of the coned off spaces is a quasiisometric embedding. With these hypotheses we have:

**Theorem 6.2** *The group $G_1$ is hyperbolic and the inclusion $G_1 \to G$ admits the CT map.*

Particularly interesting cases to which the above theorem applies are obtained in [18; 12]. There graphs of groups are considered where all the vertex and edge groups are either surface groups [18] or free groups of rank $\geq 3$ [12], respectively.

Next, we explore properties of the Cannon–Thurston map $\partial Y \to \partial X$ proved in Theorem 5.2. Suppose $F$ is a fiber of the bundle $Y$ over $A$. Then there is a CT map for the inclusions $i_{F,X}\colon F \to X$ and $i_{F,Y}\colon F \to Y$, and the map $i^*\colon Y \to X$. Since $\partial i_{F,X} = \partial i^* \circ \partial i_{F,Y}$, if $\alpha, \beta \in \partial F$ are identified under $\partial i_{F,X}$ then under $\partial i^*$ the points $\partial i_{F,Y}(\alpha)$ and $\partial i_{F,Y}(\beta)$ are identified too. It turns out that a sort of "converse" of this is also true.

**Theorem 6.25** *Suppose we have the hypotheses of Theorem 5.2 and also that the fibers of the bundle are proper metric spaces. Suppose $\gamma$ is a (quasi)geodesic line in $Y$ such that $\gamma(\infty)$ and $\gamma(-\infty)$ are identified by the CT map $\partial i^*\colon \partial Y \to \partial X$. Then $\pi_Y(\gamma)$ is bounded. In particular, given any fiber $F$ of the metric bundle, $\gamma$ is at a finite Hausdorff distance from a quasigeodesic line of $F$.*

On the other hand as an immediate application of Theorem 6.25 (in fact, see Theorem 6.26 and Proposition 6.6) we get the following:

**Theorem** *Suppose we have the hypotheses of Theorem 5.2 and also that the fibers of the bundle are proper metric spaces. Let $F$ be the fiber over a point $b \in A$. Then the CT map $\partial i_{F,X}\colon \partial F \to \partial X$ is surjective if and only if the CT maps $\partial i_{F,Y_\xi}\colon \partial F \to \partial Y_\xi$ are surjective for all $\xi \in \partial B$, where $Y_\xi$ is the pullback of a (quasi)geodesic ray in $B$ asymptotic to $\xi$.*

*In particular, $\partial i_{F,Y}\colon \partial F \to \partial Y$ is surjective if $\partial i_{F,X}\colon \partial F \to \partial X$ is surjective.*

Following Mitra [19] we define the *Cannon–Thurston lamination* $\partial^{(2)}_X(F)$ to be

$$\{(z_1, z_2) \in \partial F \times \partial F : z_1 \neq z_2,\ \partial i_{F,X}(z_1) = \partial i_{F,X}(z_2)\}$$

and following Bowditch [5, Section 2.3] we define for any point $\xi \in \partial B$ a subset of this lamination denoted by $\partial^{(2)}_{\xi,X}(F)$ or simply $\partial^{(2)}_{\xi}(F)$ when $X$ is understood, where $(z_1, z_2) \in \partial^{(2)}_{\xi,X}(F)$ if and only if $\partial i_{F,X}(z_1) = \partial i_{F,X}(z_2) = \tilde{\gamma}(\infty)$, where $\tilde{\gamma}$ is a quasiisometric lift in $X$ of a (quasi)geodesic ray $\gamma$ in $B$ converging to $\xi$. If $(z_1, z_2) \in \partial^{(2)}_{\xi,X}(F)$ and $\alpha$ is a (quasi)geodesic line in $F$ connecting $z_1, z_2$, then $\alpha$ is referred to be a *leaf* of the lamination $\partial^{(2)}_{\xi,X}(F)$. Leaves are assumed to be uniform quasigeodesics in the following theorem using Proposition 2.37.

**Theorem** (properties of $\partial^{(2)}_X(F)$; see Lemmas 6.17–6.24) (1) $\partial^{(2)}_X(F)=\coprod_{\xi\in\partial B}\partial^{(2)}_{\xi,X}(F)$.

(2) *$\partial^{(2)}_X(F)$ and $\partial^{(2)}_{\xi,X}(F)$ are all closed subsets of $\partial^{(2)}F$, where $\partial^{(2)}F=\{(z_1,z_2)\in\partial F\times\partial F : z_1\neq z_2\}$.*

(3) *The leaves of $\partial^{(2)}_{\xi_1,X}(F)$, $\partial^{(2)}_{\xi_2,X}(F)$ are coarsely transverse to each other for all $\xi_1\neq\xi_2\in\partial B$: given $\xi_1\neq\xi_2\in\partial B$ and $D>0$ there exists $R>0$ such that if $\gamma_i$ is leaf of $\partial^{(2)}_{\xi_i,X}(F)$, $i=1,2$ then $\gamma_1\cap N_D(\gamma_2)$ has diameter less than $R$.*

(4) *If $\xi_n\to\xi$ in $\partial B$ and $\alpha_n$ is a leaf of $\partial^{(2)}_{\xi_n,X}(F)$ for all $n\in\mathbb{N}$ which converge to a geodesic line $\alpha$ then $\alpha$ is a leaf of $\partial^{(2)}_{\xi,X}(F)$.*

(5) *$\partial^{(2)}_{\xi,X}(F)=\partial^{(2)}_{\xi,Y}(F)$ for all $\xi\in\partial A$ if we have the hypothesis of Theorem 5.2.*

Finally, we also prove the following interesting property of the CT lamination.

**Theorem 6.30** *Suppose $X$ is a metric (graph) bundle over $B$ satisfying the hypotheses of Theorem 5.2 such that $X$ is a proper metric space. Let $F=F_b$, where $b\in B$. Suppose $\partial F$ is not homeomorphic to a dendrite and also the CT map $\partial F\to\partial X$ is surjective.*

*Then for all $\xi\in\partial B$ we have $\partial^{(2)}_{\xi,X}(F)\neq\varnothing$.*

This applies in particular to the examples of short exact sequence of hyperbolic groups and the complexes of hyperbolic groups mentioned in Theorems 6.1 and 6.2 above.

**Outline of the paper** In Section 2 we recall basic hyperbolic geometry, Cannon–Thurston maps, etc. In Section 3 we recall the basics of metric (graph) bundles and we introduce morphisms of bundles, pullbacks. Here we prove the existence of pullbacks under suitable assumptions. In Section 4 we mainly recall the machinery of [24] and we prove a few elementary results. Section 5 is devoted to the proof of the main theorem. In Section 6 we derive applications of the main result and we mention some related results.

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## 2 Hyperbolic metric spaces

In this section, we remark on the notation and convention to be followed in the rest of the paper and we put together basic definitions and results about hyperbolic metric spaces. We begin with some basic notions from large scale geometry. Most of these are quite standard, eg see [13; 14]. We have used [24] where all the basic notions can be quickly found in one place.

**Notation, convention and some metric space notions** One is referred to [6, Chapters I.1, I.3] for the definitions and basic facts about geodesic metric spaces, metric graphs and length spaces.

(0) For any set $A$, $\mathrm{Id}_A$ will denote the *identity* map $A \to A$. If $A \subset B$ then we denote by $i_{A,B}\colon A \to B$ the inclusion map of $A$ into $B$.

(1) If $x \in X$ and $A \subset X$ then $d(x, A)$ will denote $\inf\{d(x, y) : y \in A\}$ and will be referred to as the *distance of $x$ from $A$*. For $D \geq 0$ and $A \subset X$, $N_D(A) := \{x \in X : d(x, a) \leq D$ for some $a \in A\}$ will be called the *$D$-neighborhood* of $A$ in $X$. For $A, B \subset X$ we shall denote by $d(A, B)$ the quantity $\inf\{d(x, B) : x \in A\}$ and by $Hd(A, B)$ the quantity $\inf\{D > 0 : A \subset N_D(B),\ B \subset N_D(A)\}$ and will refer to it as the *Hausdorff distance* of $A, B$.

(2) If $X$ is a length space we consider only subspaces $Y \subset X$ such that the induced length metric on $Y$ takes values in $[0, \infty)$, or equivalently for any pair of points in $Y$ there is a rectifiable path in $X$ joining them which is contained in $Y$. We shall refer to such subsets as *rectifiably path connected*. If $\gamma$ is a rectifiable path in $X$ then $l(\gamma)$ will denote the length of $\gamma$.

(3) All graphs are connected for us. If $X$ is a metric graph then $\mathcal{V}(X)$ will denote the set of vertices of $X$. Generally, we shall write $x \in X$ to mean $x \in \mathcal{V}(X)$. In metric graphs (see [6, Chapter I.1]) all the edges are assumed to have length 1. In a graph $X$ the paths are assumed to be a sequence of vertices. In other words, these are maps $I \cap \mathbb{Z} \to X$, where $I$ is a closed interval in $\mathbb{R}$ with end points in $\mathbb{Z} \cup \{\pm\infty\}$. We shall informally write this as $\alpha\colon I \to X$ and sometimes refer to it as a *dotted path* for emphasis. Length of such a path $\alpha\colon I \to X$ is defined to be $l(\alpha) = \sum d(\alpha(i), \alpha(i+1))$, where the sum is taken over all $i \in \mathbb{Z}$ such that $i, i+1 \in I$. If $\alpha\colon [0, n] \to X$ and $\beta\colon [0, m] \to X$ are two paths with $\alpha(n) = \beta(0)$, then their concatenation $\alpha * \beta$ will be the path $[0, m+n] \to X$ defined by $\alpha * \beta(i) = \alpha(i)$ if $i \in [0, n]$ and $\alpha * \beta(j) = \beta(j - n)$ if $j \in [n, m+n]$.

(4) If $X$ is a geodesic metric space and $x, y \in X$ then we shall use $[x, y]_X$ or simply $[x, y]$ to denote a geodesic segment joining $x$ to $y$. This applies in particular to metric graphs. For $x, y, z \in X$ we shall denote by $\Delta xyz$ some geodesic triangle with vertices $x, y, z$.

(5) If $X$ is any metric space then for all $A \subset X$, $\mathrm{diam}(A)$ will denote the diameter of $A$.

## 2.1 Basic notions from large scale geometry

Suppose $X, Y$ are any two metric spaces and $k \geq 1$, $\epsilon \geq 0$, $\epsilon' \geq 0$.

**Definition 2.1** [24, Definition 1.1.1] (1) A map $\phi\colon X \to Y$ is said to be *metrically proper* if there is an increasing function $f\colon [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} f(t) = \infty$ such that for any $x, y \in X$ and $R \in [0, \infty)$, $d_Y(\phi(x), \phi(y)) \leq R$ implies $d_X(x, y) \leq f(R)$. In this case we say that $\phi$ is proper *as measured by $f$*.

(2) A subset $A$ of a metric space $X$ is said to be *$r$-dense* in $X$ for some $r \geq 0$ if $N_r(A) = X$.

(3) Suppose $A$ is a set. A map $\phi\colon A \to Y$ is said to be *$\epsilon$-coarsely surjective* if $\phi(A)$ is $\epsilon$-dense in $Y$. We will say that it is coarsely surjective if it is $\epsilon$-coarsely surjective for some $\epsilon \geq 0$.

(4) A map $\phi\colon X \to Y$ is said to be *coarsely $(\epsilon,\epsilon')$-Lipschitz* if for every $x_1, x_2 \in X$, we have $d(\phi(x_1),\phi(x_2)) \leq \epsilon d(x_1,x_2) + \epsilon'$. A coarsely $(\epsilon,\epsilon)$-Lipschitz map will be simply called a *coarsely $\epsilon$-Lipschitz* map. A map $\phi$ is *coarsely Lipschitz* if it is coarsely $\epsilon$-Lipschitz for some $\epsilon \geq 0$.

(5) (i) A map $\phi\colon X \to Y$ is said to be a *$(k,\epsilon)$-quasiisometric embedding* if for every $x_1, x_2 \in X$, one has

$$-\epsilon + d(x_1,x_2)/k \leq d(\phi(x_1),\phi(x_2)) \leq \epsilon + k d(x_1,x_2).$$

A map $\phi\colon X \to Y$ will simply be referred to as a quasiisometric embedding if it is a $(k,\epsilon)$-quasiisometric embedding for some $k \geq 1$, $\epsilon \geq 0$. A $(k,k)$-quasiisometric embedding will be referred to as a *$k$-quasiisometric embedding*.

(ii) A map $\phi\colon X \to Y$ is a *$(k,\epsilon)$-quasiisometry* (resp. *$k$-quasiisometry*) if it is a $(k,\epsilon)$-quasiisometric embedding (resp. $k$-quasiisometric embedding) and moreover, it is $D$-coarsely surjective for some $D \geq 0$.

(iii) A *$(k,\epsilon)$-quasigeodesic* (resp. a *$k$-quasigeodesic*) in a metric space $X$ is a $(k,\epsilon)$-quasiisometric embedding (resp. a $k$-quasiisometric embedding) $\gamma\colon I \to X$, where $I \subseteq \mathbb{R}$ is an interval.

We recall that a $(1,0)$-quasigeodesic is called a *geodesic*.

If $I = [0,\infty)$, then $\gamma$ will be called a *quasigeodesic ray*. If $I = \mathbb{R}$, then we call it a *quasigeodesic line*. One similarly defines a *geodesic ray* and a *geodesic line*. We refer to the constant(s) $k$ (and $\epsilon$) as *quasigeodesic constant(s)*.

Quasigeodesics in a metric graph $X$ will be maps $I \cap \mathbb{Z} \to X$, informally written as $I \to X$ where $I$ is a closed interval in $\mathbb{R}$.

(6) Suppose $\phi, \phi'\colon X \to Y$ are two maps and $\epsilon \geq 0$.

(i) We define $d(\phi,\phi')$ to be the quantity $\sup\{d_Y(\phi(x),\phi'(x)) : x \in X\}$ provided the supremum exists in $\mathbb{R}$; otherwise we write $d(\phi,\phi') = \infty$.

(ii) A map $\psi\colon Y \to X$ is called an $\epsilon$-coarse left (right) inverse of $\phi$ if $d(\psi \circ \phi, \mathrm{Id}_X) \leq \epsilon$ (resp. $d(\phi \circ \psi, \mathrm{Id}_Y) \leq \epsilon$).

If $\psi$ is both an $\epsilon$-coarse left and right inverse then it is simply called an *$\epsilon$-coarse inverse* of $\phi$.

(7) Suppose $S$ is any set. A map $f\colon S \to X$ satisfying some properties $\mathcal{P}_1, \dots, \mathcal{P}_k$ will be called *coarsely unique* if for any other map $g\colon S \to X$ with properties $\mathcal{P}_1, \dots, \mathcal{P}_k$ there is a constant $D$ such that $d(f,g) \leq D$.

The definition (7) above is taken from [24]. See the definition following Lemma 2.9 there. In places where this definition will be used the properties may not be explicitly stated but they will be clear from the context. If $S$ is finite then we talk about a finite subset of $X$ to be coarsely unique, eg see the remark following Lemma 2.56.

**Remark on terminology** (1) All the above definitions are about certain properties of maps and in each case some parameters are involved.

- (i) When the parameters are not important or they are clear from the context then we say that the map has the particular property without explicit mention of the parameters, eg "$\phi\colon X \to Y$ is metrically proper" if $\phi$ is metrically proper as measured by some function.
- (ii) When we have a set of pairs of metric spaces and a map between each pair possessing the same property with the same parameters then we say that the set of maps "uniformly" have the property, eg *uniformly metrically proper*, *uniformly coarsely Lipschitz*, *uniform qi embeddings*, *uniform approximate nearest point projection* etc.

(2) We often refer to a quasiisometric embedding as "qi embedding" and a quasiisometry as "qi".

The following gives a characterization of quasiisometry to be used in the discussion on metric bundles.

**Lemma 2.2** [24, Lemma 1.1] (1) *For every $K_1, K_2 \geq 1$ and $D \geq 0$ there are $K_{2.2} = K_{2.2}(K_1, K_2, D)$, such that the following holds*:
*A $K_1$-coarsely Lipschitz map with a $K_2$-coarsely Lipschitz, $D$-coarse inverse is a $K_{2.2}$-quasiisometry.*

(2) *Given $K \geq 1$, $\epsilon \geq 0$ and $R \geq 0$ there are constants $C_{2.2} = C_{2.2}(K, \epsilon, R)$ and $D_{2.2} = D_{2.2}(K, \epsilon, R)$ such that the following holds*:
*Suppose $X, Y$ are any two metric spaces and $f\colon X \to Y$ is a $(K, \epsilon)$-quasiisometry which is $R$-coarsely surjective. Then there is a $(K_{2.2}, C_{2.2})$-quasiisometric $D_{2.2}$-coarse inverse of $f$.*

The following lemmas follow from simple calculations and hence we omit their proofs.

**Lemma 2.3** (1) *Suppose we have a sequence of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ where $f, g$ are coarsely $L_1$-Lipschitz and $L_2$-Lipschitz, respectively. Then $g \circ f$ is coarsely $(L_1L_2, L_1L_2{+}L_2)$-Lipschitz.*

(2) *Suppose $f\colon X \to Y$ is a $(K_1, \epsilon_1)$-qi embedding and $g\colon Y \to Z$ is a $(K_2, \epsilon_2)$-qi embedding. Then $g \circ f\colon X \to Z$ is a $(K_1K_2, K_2\epsilon_1{+}\epsilon_2)$-qi embedding.*

*Moreover, if $f$ is $D_1$-coarsely surjective and $g$ is $D_2$-coarsely surjective then $g \circ f$ is $(K_2D_1 + \epsilon_2 + D_2)$-coarsely surjective.*

*In particular, the composition of finitely many quasiisometries is a quasiisometry.*

**Lemma 2.4** *Suppose $X'$ is any connected graph and $r > 0$. Suppose $X$ is another graph obtained from $X'$ by introducing some new edges to $X'$ where $e = [v, w]$ is an edge in $X$ but not in $X'$ implies $d_{X'}(v, w) \leq r$. Then the inclusion map $X' \to X$ is a quasiisometry.*

The following lemma appears in [17, Section 1.5] in a somewhat different form. We include a proof for the sake of completeness.

**Lemma 2.5** *Let $X$ be any metric space, $x, y \in X$, $\gamma$ be a (dotted) $k$-quasigeodesic joining $x, y$ and $\alpha\colon I \to X$ is a (dotted) coarsely $L$-Lipschitz path joining $x, y$. Suppose moreover, $\alpha$ is a proper embedding as measured by a function $f\colon [0,\infty) \to [0,\infty)$ and that $Hd(\alpha, \gamma) \leq D$ for some $D \geq 0$. Then $\alpha$ is (dotted) $K_{2.5} = K_{2.5}(k, f, D, L)$-quasigeodesic in $X$.*

**Proof** Suppose $\gamma$ is defined on an interval $J$. Let $a, b \in I$. Then we have

$$d(\alpha(a), \alpha(b)) \leq L|a-b| + L \longrightarrow (1)$$

since $\alpha$ is coarsely $L$-Lipschitz. Now let $a', b' \in J$ be such that $d(\alpha(a), \gamma(a') \leq D$ and $d(\alpha(b), \gamma(b')) \leq D$. Let $R = d(\alpha(a), \alpha(b))$. Then by triangle inequality $d(\gamma(a'), \gamma(b')) \leq 2D + R$. Since $\gamma$ is a $k$-quasigeodesic we have $-k + |a'-b'|/k \leq d(\gamma(a'), \gamma(b')) \leq 2D + R$. Hence, $|a'-b'| \leq k(2D+R) + k^2$. Without loss of generality suppose $a' \leq b'$. Consider the sequence of points $a'_0 = a', a'_1, \dots, a'_n = b'$ in $J$ such that $a'_{i+1} = 1 + a'_i$ for $0 \leq i \leq n-2$ and $a'_n - a'_{n-1} \leq 1$. We note that $n \leq 1 + k(2D+R) + k^2$. Let $a_i \in I$ be such that $d(\gamma(a'_i), \alpha(a_i)) \leq D$, $0 \leq i \leq n$, where $a_0 = a$, $a_n = b$. Once again by the triangle inequality we have

$$d(\alpha(a_i), \alpha(a_{i+1})) \leq 2D + d(\gamma(a'_i), \gamma(a'_{i+1})) \leq 2D + 2k$$

for $0 \leq i \leq n-1$ since $\gamma$ is a $k$-quasigeodesic. This implies $|a_i - a_{i+1}| \leq f(2D+2k)$ since $\alpha$ is a proper embedding as measured by $f$. Hence,

$$|a-b| \leq \sum_{i=0}^{n-1} |a_i - a_{i+1}| \leq nf(2D+k) \leq (1 + k(2D+R) + k^2) f(2D+2k).$$

Thus we have

$$-\frac{1+2kD+k^2}{k} + \frac{1}{kf(2D+2k)}|a-b| \leq R = d(\alpha(a), \alpha(b)) \longrightarrow (2).$$

Hence, by (1) and (2) we can take

$$K_{2.5} = 1 + 2D + k + kf(2D+2k) + L. \qquad \square$$

The following lemma is implicit in the proof of [24, Proposition 2.10]. The proof of this lemma being immediate we omit it.

**Lemma 2.6** *Suppose $X$ is a length space and $Y$ is any metric space. Let $f:X\to Y$ be any map. Then $f$ is coarsely $C$-Lipschitz for some $C\geq 0$ if for all $x_1,x_2\in X$, $d_X(x_1,x_2)\leq 1$ implies $d_Y(f(x_1),f(x_2))\leq C$.*

**Remark** We spend quite some time restating some results proved in [24] in the generality of length spaces since the main result in our paper is about length spaces. For instance (1) the existence of pullback of metric bundles to be defined below is unclear within the category of geodesic metric spaces; and (2) we observe that for the definition of Cannon–Thurston maps the assumption of (Gromov) hyperbolic *geodesic* metric spaces is rather restrictive and unnecessary.

In a length metric space geodesics may not exist joining a pair of points. However, we still have the following.

**Lemma 2.7** *Suppose $X$ is a length space.*

1. *Given any $\epsilon>0$, any pair of points of $X$ can be joined by a continuous, rectifiable, arc length parametrized path which is a $(1,\epsilon)$-quasigeodesic.*
2. *Any pair of points of $X$ can be joined by a dotted 1-quasigeodesic.*

**Metric graph approximation to a length space** Given any length space $X$, we define a metric graph $Y$ as follows. We take the vertex set $V(Y)=X$. We join $x,y\in X$ by an edge (of length 1) if and only if $d_X(x,y)\leq 1$. We let $\psi_X: X\to \mathcal{V}(Y)\subset Y$ be the identity map. Let $\phi_X: Y\to X$ be defined to be the inverse of $\psi_X$ on $\mathcal{V}(Y)$ and for any point $y$ in the interior of an edge $e$ of $Y$ we define $\phi_X(y)$ to be one of the end points of the edge $e$. The following hold.

**Lemma 2.8** [17, Lemma 1.32] (1) *$Y$ is a (connected) metric graph.*

(2) *The maps $\psi_X$ and $\phi_X|_{\mathcal{V}(Y)}$ are coarsely 1-surjective, $(1,1)$-quasiisometries.*

(3) *The map $\phi_X$ is a $(1,3)$-quasiisometry and it is a 1-coarse inverse of $\psi_X$.*

**Remark** We shall refer to the space $Y$ constructed in the proof of the above lemma as the (*canonical*) *metric graph approximation* to X. We also preserve the notation $\psi_X$ and $\phi_X$ to be used in this context only.

**Definition 2.9** (Gromov inner product) Let $X$ be any metric space and let $p,x,y\in X$. Then the *Gromov inner product of $x,y$ with respect to $p$* is defined to be the number $\frac{1}{2}(d(p,x)+d(p,y)-d(x,y))$. It is denoted by $(x.y)_p$.

**Lemma 2.10** *Suppose $X$ is a length space and $x_1,x_2,x_3\in X$. Let $\gamma_{ij}$, $i<j$, $1\leq i,j\leq 3$ denote $(1,1)$-quasigeodesics joining the respective pairs of points $x_i,x_j$. Suppose there are points $w_1\in\gamma_{23}$, $w_2\in\gamma_{13}$ and $w_3\in\gamma_{12}$ such that $d(w_1,w_i)\leq R$ for some $R\geq 0$, $i=2,3$. Then $|(x_2.x_3)_{x_1}-d(x_1,w_1)|\leq 3+2R$.*

**Proof** By triangle inequality we have $|d(x_2,w_1)-d(x_2,w_2)|\le R$, $|d(x_3,w_1)-d(x_3,w_2)|\le R$, $|d(x_1,w_1)-d(x_1,w_i)|\le R$, $i=2,3$. Since the $\gamma_{ij}$'s are $(1,1)$-quasigeodesics it is easy to see that

$$d(x_1,w_3)+d(w_3,x_2)\le d(x_1,x_2)+3,\quad d(x_1,w_2)+d(w_2,x_3)\le d(x_1,x_3)+3,$$
$$d(x_2,w_1)+d(w_1,x_3)\le d(x_2,x_3)+3.$$

It then follows by a simple calculation that

$$2d(x_1,w_1)-6-4R\le d(x_1,x_2)+d(x_1,x_3)-d(x_2,x_3)\le 2d(x_1,w_1)+3+4R.$$

Hence, we have $|(x_2.x_3)_{x_1}-d(x_1,w_1)|\le 3+2R$. □

**Definition 2.11** (1) Suppose $X$ is a length space and $Y_1,Y_2,Z$ are nonempty subsets of $X$. We say that $Z$ *coarsely disconnects* $Y_1,Y_2$ in $X$ if (i) $Y_i\setminus Z\neq\varnothing$, $i=1,2$ and (ii) for all $K\ge 1$ there is $R\ge 0$ such that the following holds: for any $y_i\in Y_i$, $i=1,2$ and any $K$-quasigeodesic $\gamma$ in $X$ joining $y_1,y_2$ we have $\gamma\cap N_R(Z)\neq\varnothing$.

(2) Suppose $Y,Z\subset X$, $Y_1,Y_2\subset Y$. We say that $Z$ *coarsely bisects* $Y$ into $Y_1,Y_2$ in $X$ if $Y=Y_1\cup Y_2$ and $Z$ coarsely disconnects $Y_1,Y_2$ in $X$.

(3) Suppose $\{X_i\}$ is a collection of length spaces and there are nonempty sets $Y_i,Z_i\subset X_i$, $Y_i^+,Y_i^-\subset Y_i$ such that $Y_i=Y_i^+\cup Y_i^-$, $Y_i^+\setminus Z_i\neq\varnothing$, and $Y_i^-\setminus Z_i\neq\varnothing$ for all $i$. We say that $Z_i$'s *uniformly coarsely bisect* $Y_i$'s into $Y_i^+$'s, and $Y_i^-$'s if for all $K\ge 1$ there is $R=R(K)\ge 0$ with the following property: for any $i$, and for any $x_i^+\in Y_i^+$, $x_i^-\in Y_i^-$ and any $K$-quasigeodesic $\gamma_i\subset X_i$ joining $x_i^\pm$ we have $N_R(Z_i)\cap\gamma_i\neq\varnothing$.

We note that the first part of the above definition implies $Y_1\cap Y_2\subset N_{R(1)}(Z)$. Moreover one would like to impose the condition that $Y_i\setminus Z$ are of infinite diameter. Keeping the application we have in mind we do not assume that.

**Definition 2.12** (approximate nearest point projection) (1) Suppose $X$ is any metric space, $A\subset X$, and $x\in X$. Given $\epsilon\ge 0$ and $y\in A$ we say that $y$ is an *$\epsilon$-approximate nearest point projection* of $x$ on $A$ if for all $z\in A$ we have $d(x,y)\le d(x,z)+\epsilon$.

(2) Suppose $X$ is any metric space, $A\subset X$ and $\epsilon\ge 0$. An *$\epsilon$-approximate nearest point projection map* $f\colon X\to A$ is a map such that $f(a)=a$ for all $a\in A$ and $f(x)$ is an $\epsilon$-approximate nearest point projection of $x$ on $A$ for all $x\in X\setminus A$.

For $\epsilon=0$ an $\epsilon$-approximate nearest point projection is simply referred to as a *nearest point projection*. A nearest point projection map from $X$ onto a subset $A$ will be denoted by $P_{A,X}\colon X\to A$ or simply $P_A\colon X\to A$ when there is no possibility of confusion.

We note that given a metric space $X$ and $A\subset X$ a nearest point projection map $X\to A$ may not be defined in general but an $\epsilon$-approximate nearest point projection map $X\to A$ exists by axiom of choice for all $\epsilon>0$.

**Lemma 2.13** *Suppose $X$ is a metric space and $A \subset X$. Suppose $y \in A$ is an $\epsilon$-approximate nearest point projection of $x \in X$. Suppose $\alpha\colon I \to X$ is a $(1,1)$-quasigeodesic joining $x, y$. Then $y$ is an $(\epsilon+3)$-approximate nearest point of $x'$ on $A$ for all $x' \in \alpha$.*

**Proof** Suppose $z \in A$ is any point. Then we know that $d(x,y) \leq d(x,z)+\epsilon$. Since $\alpha$ is a $(1,1)$-quasigeodesic it is easy to see that $d(x,x')+d(x',y) \leq d(x,y)+3$. Hence, $d(x,x')+d(x',y) \leq d(x,z)+3+\epsilon$ which in turn implies that $d(x',y) \leq d(x,z)-d(x,x')+3+\epsilon \leq d(x',z)+\epsilon+3$. Hence, $y$ is an $(\epsilon+3)$-approximate nearest point projection of $x'$ on $A$. □

**Corollary 2.14** *Suppose $X$ is any metric space and $x, y, z \in X$. Suppose $\alpha$, $\beta$ are $(1,1)$-quasigeodesics joining $x, y$ and $y, z$, respectively. If $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\beta$ then $\alpha * \beta$ is $(3, 3+\epsilon)$-quasigeodesic.*

**Proof** Let $x' \in \alpha$ and $y' \in \beta$. Let $\beta'$ denote the segment of $\beta$ from $y$ to $y'$. Then $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\beta'$ too. Hence, by the previous lemma $y$ is an $(\epsilon+3)$-approximate nearest point projection of $x'$ on $\beta'$. Without loss of generality, suppose $\alpha(a)=x'$, $\alpha(a+m)=y$, $\beta(0)=y$, and $\beta(n)=y'$. Now, $d(x',y) \leq d(x',y')+\epsilon+3$. Hence $d(y,y') \leq d(x',y')+d(x',y) \leq 2d(x',y')+\epsilon+3$. Since $\alpha, \beta$ are both $(1,1)$-quasigeodesics it follows that $m-1 \leq d(x',y) \leq d(x',y')+\epsilon+3$ and $n-1 \leq d(y,y') \leq 2d(x',y')+\epsilon+3$. Adding these we get $m+n-2 \leq 3d(x',y')+2\epsilon+6$. On the other hand, $d(x',y') \leq d(x',y)+d(y,y') \leq m+n+2$. Putting everything together we get

$$\tfrac{1}{3}(m+n)-\tfrac{1}{3}(2\epsilon+8) \leq d(x',y') \leq (m+n)+2$$

from which the corollary follows immediately. □

## 2.2 Rips hyperbolicity vs Gromov hyperbolicity

This subsection gives a quick introduction to some basic notions and results about hyperbolic metric spaces. One is referred to [1; 13; 14] for more details. The following definition of hyperbolic metric spaces is due to E Rips and hence we refer to this as the Rips hyperbolicity.

**Definition 2.15** (1) Suppose $\Delta x_1x_2x_3$ is a geodesic triangle in a metric space $X$ and $\delta \geq 0$, $K \geq 0$. We say that the triangle $\Delta x_1x_2x_3$ is *$\delta$-slim* if any side of the triangle is contained in the $\delta$-neighborhood of the union of the remaining two sides.

(2) Let $\delta \geq 0$ and $X$ be a geodesic metric space. We say that $X$ is *$\delta$-hyperbolic* (in the sense of Rips) if all geodesic triangles in $X$ are $\delta$-slim.

A geodesic metric space is said to be (*Rips*) *hyperbolic* if it is $\delta$-hyperbolic in the sense of Rips for some $\delta \geq 0$.

However, in this paper we need to deal with length spaces a lot which a priori need not be geodesic. The following definition is more relevant in that case.

**Definition 2.16** (Gromov hyperbolicity) Suppose $X$ is any metric space, not necessarily geodesic and $\delta \geq 0$.

1. Let $p \in X$. We say that the Gromov inner product on $X$ with respect to $p$, ie the map $X \times X \to \mathbb{R}$ defined by $(x, y) \mapsto (x.y)_p$, is $\delta$-*hyperbolic* if
$$(x.y)_p \geq \min\{(x.z)_p, (y.z)_p\} - \delta$$
for all $x, y, z \in X$.
2. The metric space $X$ is called $\delta$-*hyperbolic* in the sense of Gromov if the Gromov inner product on $X$ is $\delta$-hyperbolic with respect to any point of $X$.

A metric space is called (*Gromov*) *hyperbolic* if it is $\delta$-hyperbolic in the sense of Gromov for some $\delta \geq 0$.

However, it is a standard fact that for geodesic metric spaces the two concepts are equivalent. See [14, Section 6.3C], or [6, Proposition 1.22, Chapter III.H] for instance. In this subsection we observe an analog of Rips hyperbolicity in Gromov hyperbolic length spaces using the next two lemmas.

The following lemma is a crucial property of Rips hyperbolic metric spaces.

**Lemma 2.17** (stability of quasigeodesics in a Rips hyperbolic space [13]) *For all* $\delta \geq 0$ *and* $k \geq 1$, $\epsilon \geq 0$ *there is a constant* $D_{2.17} = D_{2.17}(\delta, k, \epsilon)$ *such that the following holds*:

*Suppose* $Y$ *is a geodesic metric space* $\delta$*-hyperbolic in the sense of Rips. Then the Hausdorff distance between a geodesic and a* $(k, \epsilon)$*-quasigeodesic joining the same pair of end points is less than or equal to* $D_{2.17}$.

One is referred to [25, Theorems 3.18, 3.20] for a proof of the following lemma.

**Lemma 2.18** *Suppose* $X$ *is a metric space which is* $\delta$*-hyperbolic in the sense of Gromov. If* $f: X \to Y$ *is a* $R$*-coarsely surjective*, $(1, C)$*-quasiisometry then* $Y$ *is* $D = D_{2.18}(\delta, R, C)$*-hyperbolic in the sense of Gromov.*

Using metric graph approximations to length spaces (Lemma 2.8) and the fact that for geodesic metric spaces Gromov hyperbolicity implies Rips hyperbolicity we obtain the following three corollaries.

**Corollary 2.19** (stability of quasigeodesics in a Gromov hyperbolic space) *Given* $\delta \geq 0$, $k \geq 1$, $\epsilon \geq 0$ *there is* $D = D_{2.19}(\delta, k, \epsilon)$ *such that the following holds*:

*Suppose* $X$ *is metric space which is* $\delta$*-hyperbolic in the sense of Gromov. Then given* $(k, \epsilon)$*-quasigeodesics* $\gamma_i$, $i = 1, 2$ *with the same end points we have* $Hd(\gamma_1, \gamma_2) \leq D$.

**Corollary 2.20** (analog of Rips hyperbolicity for length spaces) *Suppose $X$ is a length space. If $X$ is $\delta$-hyperbolic in the sense of Gromov then for all $K \geq 1$, $\epsilon \geq 0$ all $(K, \epsilon)$-quasigeodesic triangles in $X$ are $D_{2.20} = D_{2.20}(\delta, K, \epsilon)$-slim.*

*Conversely if all $(K, \epsilon)$-quasigeodesic triangles in $X$ are $R$-slim for some $R \geq 0$ and for some sufficiently large $K, \epsilon$ then $X$ is $\lambda_{2.20} = \lambda_{2.20}(R, K, \epsilon)$-hyperbolic in the sense of Gromov.*

Slimness of triangles immediately implies slimness of polygons:

**Corollary 2.21** (slimness of polygons) *Suppose that $X$ is a length space. If $X$ is $\delta$-hyperbolic in the sense of Gromov then for all $K \geq 1$, $\epsilon \geq 0$ all $(K, \epsilon)$-quasigeodesic $n$-gons in $X$ are $(n-2)D_{2.20} = (n-2)D_{2.20}(\delta, K, \epsilon)$-slim.*

**Convention 2.22** For the rest of the paper a $\delta$-hyperbolic (or simply hyperbolic) space will refer either to (1) a $\delta$-hyperbolic (resp. hyperbolic) space in the sense of Rips if it is a geodesic metric space or (2) a $\delta$-hyperbolic (resp. hyperbolic) space in the sense of Gromov if it is not a geodesic metric space. However, in this case the space will be assumed to be a length space. The constant $\delta$ will be referred to as the *hyperbolicity constant* for the space involved.

## 2.3 Quasiconvex subspaces of hyperbolic spaces

**Definition 2.23** Let $X$ be a hyperbolic geodesic metric space and let $A \subseteq X$. For $K \geq 0$, we say that $A$ is *$K$-quasiconvex* in $X$ if any geodesic with end points in $A$ is contained in $N_K(A)$.

If $X$ is a Gromov hyperbolic length space and $A \subset X$ then we will say that $A$ is *$K$-quasiconvex* if any $(1, 1)$-quasigeodesic joining a pair of points of $A$ is contained in $N_K(A)$.

A subset $A \subset X$ is said to be *quasiconvex* if it is $K$-quasiconvex for some $K \geq 0$.

The following lemma relates quasiconvexity with qi embedding. It is straightforward and is proved in the context of geodesic metric spaces in [17, Chapter 1, Section 1.11]. Hence we skip the proof.

**Lemma 2.24** (1) *Given $\delta \geq 0$ and $k \geq 0$ there are constants $D = D(\delta, k)$ and $K = K(\delta, k)$ such that the following holds*:
*Suppose $X$ is a $\delta$-hyperbolic metric space and $A \subset X$ is $k$-quasiconvex. Then $N_D(A)$ is path connected and with respect to the induced path metric on $N_D(A)$ from $X$ the inclusion map $N_D(A) \to X$ is a $K$-qi embedding.*

(2) *Suppose $X$ is a hyperbolic metric space and $Y$ is a quasiconvex subset. Suppose $Y$ is path connected and with respect to the induced path metric on $Y$ from $X$ the inclusion map $Y \to X$ is metrically proper. Then the inclusion map is a qi embedding.*

In this subsection, in a Gromov hyperbolic setting, we prove a number of results about quasiconvex sets analogous to those in [24, Section 1.2] which were proved in a Rips hyperbolic setting. The importance of the following lemma for this paper can be hardly exaggerated.

**Lemma 2.25** (projection on a quasiconvex set) *Let $X$ be a $\delta$-hyperbolic metric space, $U \subset X$ a $K$-quasiconvex set and $\epsilon \geq 0$. Suppose $y \in U$ is an $\epsilon$-approximate nearest point projection of a point $x \in X$ on $U$. Let $z \in U$. Suppose $\alpha$ is a (dotted) $k$-quasigeodesic joining $x$ to $y$ and $\beta$ is a (dotted) $k$-quasigeodesic joining $y$ to $z$. Then $\alpha * \beta$ is a (dotted) $K_{2.25} = K_{2.25}(\delta, K, k, \epsilon)$-quasigeodesic in $X$.*

*In particular, if $\gamma$ is $k$-quasigeodesic joining $x, z$ then $y$ is contained in the $D_{2.25}(\delta, K, k, \epsilon)$-neighborhood of $\gamma$.*

**Proof** Without loss of generality we shall assume that $X$ is a $\delta$-hyperbolic length space. Suppose $\beta_1$ is a $(1,1)$-quasigeodesic in $X$ joining $y, z$. Since $U$ is $K$-quasiconvex it is clear that $y$ is an $(\epsilon+K)$-approximate nearest point projection of $x$ on $\beta_1$. Hence, if $\alpha_1$ is a $(1,1)$-quasigeodesic joining $x, y$ then $\alpha_1 * \beta_1$ is a $(3, 3+\epsilon+K)$-quasigeodesic in $X$ by Corollary 2.14. By stability of quasigeodesics $Hd(\alpha, \alpha_1) \leq D_{2.19}(\delta, k, \epsilon)$, and $Hd(\beta, \beta_1) \leq D_{2.19}(\delta, k, \epsilon)$. Hence, $Hd(\alpha*\beta, \alpha_1*\beta_1) \leq D_{2.19}(\delta, k, \epsilon)$. By Lemma 2.5 it is enough to show now that $\gamma = \alpha * \beta$ is uniformly properly embedded. Let $\gamma_1 = \alpha_1 * \beta_1$ and $R = D_{2.19}(\delta, k, \epsilon)$. Suppose $\alpha\colon [0,l] \to X$ with $\alpha(0) = x$, $\alpha(l) = y$ and $\beta\colon [0,m] \to X$ with $\beta(0) = y$, $\beta(m) = z$. Let $s \leq t \in [0, l+m]$ and $d(\gamma(s), \gamma(t)) \leq D$ for some $D \geq 0$. We need to find a constant $D_1$ such that $t-s \leq D_1$, where $D_1$ depends on $\delta, k, K$ and $D$ only. However, if $s, t \in [0,l]$ or $s, t \in [l, l+m]$ then we have $-k + (t-s)/k \leq D$ since both $\alpha, \beta$ are $k$-quasigeodesics. Hence, in that case $t - s \leq k^2 + kD$.

Suppose $s \in [0,l)$ and $t \in (l,m]$. In this case $\gamma(s) = \alpha(s)$, $\gamma(t) = \beta(t-l)$. Let $x' \in \alpha_1$, $y' \in \beta_1$ be such that $d(x', \gamma(s)) \leq R$ and $d(y', \gamma(t)) \leq R$. Then $d(x', y') \leq 2R + D$. Suppose $\gamma_1(s') = x'$, $\gamma_1(t') = y'$, $\gamma_1(u) = y$, where $s' \leq u \leq t'$. Since $\gamma_1$ is a $(3, 3+\epsilon+K)$-quasigeodesic we have $|s'-t'| \leq 3(3+\epsilon+K) + 3d(x', y') \leq 3(3+\epsilon+K) + 3(2R+D)$. It follows that $|s'-u|$ and $|u-t'|$ are both at most $3(3+\epsilon+K) + 3(2R+D) = 9 + 3\epsilon + 3K + 6R + 3D$. Hence, $d(x', y), d(y, y')$ are both at most $3(9+3\epsilon+3K+6R+3D) + 3 + \epsilon + K = 30 + 10\epsilon + 10K + 18R + 9D = D'$, say. Hence, $d(\gamma(s), y)$, $d(y, \gamma(t))$ are both at most $R + D'$. Since $\alpha, \beta$ are $k$-quasigeodesics it follows that $l - s$ and $t - l$ are both at most $k^2 + k(R+D')$. Hence, $t - s \leq 2(k^2 + k(R+D'))$. Hence, we can take $D_1 = 2k^2 + 2kR + 2kD'$. This completes the proof of the existence of $K_{2.25}$.

Clearly one can set $D_{2.25}(\delta, K, k, \epsilon) = D_{2.19}(\delta, K_{2.25}(\delta, K, k, k), K_{2.25}(\delta, K, k, k))$. □

**Corollary 2.26** *Suppose $X$ is a $\delta$-hyperbolic metric space and $\alpha$ is a $k$-quasigeodesic in $X$ with an end point $y$. Suppose $x \in X$ and $y$ is an $\epsilon$-approximate nearest point projection of $x$ on $\alpha$. Suppose $\beta$ is a $k$-quasigeodesic joining $x$ to $y$. Then $\beta * \alpha$ is a $K_{2.26}(\delta, k, \epsilon)$-quasigeodesic.*

**Proof** We briefly indicate the proof. One first notes by stability of quasigeodesics that images of uniform quasigeodesics are uniformly quasiconvex. Then one applies the preceding lemma. □

The following corollary easily follows from Lemmas 2.25 and 2.13. For instance, the proof is similar to that of [24, Lemma 1.32].

**Corollary 2.27** (projection on nested quasiconvex sets) *Suppose $X$ is a $\delta$-hyperbolic metric space and $V \subset U$ are two $K$-quasiconvex subsets of $X$. Suppose $x \in X$ and $x_1 \in U$, $x_2 \in V$ are $\epsilon$-approximate nearest point projection of $x$ on $U$ and $V$, respectively. Suppose $x_3$ is an $\epsilon$-approximate nearest point projection of $x_1$ on $V$. Then $d(x_2, x_3) \leq D_{2.27}(\delta, K, \epsilon)$.*

*In particular, for any two $\epsilon$-approximate nearest point projections $x_1, x_2$ of $x$ on $U$ we have*

$$d(x_1, x_2) \leq D_{2.27}(\delta, K, \epsilon).$$

**Corollary 2.28** *Given $\delta \geq 0$, $K \geq 0$, $\epsilon \geq 0$ there are constants $L = L_{2.28}(\delta, K, \epsilon)$, $D = D_{2.28}(\delta, K, \epsilon)$ and $R = R_{2.28}(\delta, K, \epsilon)$ such that the following hold:*

(1) *Suppose $X$ is a $\delta$-hyperbolic metric space and $U$ is a $K$-quasiconvex subset of $X$. Then for all $\epsilon \geq 0$ any $\epsilon$-approximate nearest point projection map $P\colon X \to U$ is coarsely $L$-Lipschitz.*

(2) *Suppose $V$ is another $K$-quasiconvex subset of $X$ and $v_1, v_2 \in V$ and $u_i = P(v_i)$, $i = 1, 2$. If $d(u_1, u_2) \geq D$ then $u_1, u_2 \in N_R(V)$.*

*In particular, if the diameter of $P(V)$ is at least $D$ then $d(U, V) \leq R$.*

**Proof** (1) Suppose $x, y \in X$ with $d(x, y) \leq 1$. Then $P(x)$ is an $(\epsilon{+}1)$-approximate nearest point projection of $y$ on $U$. Hence, by Corollary 2.27 we have $d(P(x), P(y)) \leq D_{2.27}(\delta, K, \epsilon + 1)$ and we may take $L_{2.28}(\delta, K, \epsilon) = D_{2.27}(\delta, K, \epsilon + 1)$ by Lemma 2.6.

(2) Consider the quadrilateral formed by $(1,1)$-quasigeodesics joining the pairs $(u_1, u_2)$, $(u_2, v_2)$, $(v_2, v_1)$ and $(v_1, u_1)$. This is $2D_{2.20}(\delta, 1, 1)$-slim by Corollary 2.21. Let $\delta' = 2D_{2.20}(\delta, 1, 1)$. Suppose no point of the side $v_1v_2$ is contained in a $\delta'$-neighborhood of the side $u_1u_2$. Then there are two points say $x_1, x_2 \in v_1v_2$ such that $x_i \in N_{\delta'}(u_iv_i)$, $i = 1, 2$ and $d(x_1, x_2) \leq 2$. Hence there are points $y_i \in u_iv_i$, $i = 1, 2$ such that $d(y_1, y_2) \leq 2 + 2\delta'$. However, $u_i$ is an $(\epsilon{+}3)$-approximate nearest point projection of $y_i$ on $U$ by Lemma 2.13. Hence, by the first part of Corollary 2.28 we have $d(u_1, u_2) \leq L_{2.28}(\delta, K, \epsilon + 3) + (2 + 2\delta')L_{2.28}(\delta, K, \epsilon + 3)$. Hence, if the diameter of $P(V)$ is bigger than $D = L_{2.28}(\delta, K, \epsilon + 3) + (2 + 2\delta')L_{2.28}(\delta, K, \epsilon + 3)$ then there is a point $x \in v_1v_2$ and $y \in u_1u_2$ such that $d(x, y) \leq \delta'$. Since $U$ is $K$-quasiconvex we have thus $x \in N_{K+\delta'}(U)$. Thus we may choose $R = K + \delta'$. □

The second part of the above corollary is implied in Lemma 1.35 of [24] too. The next lemma roughly says that the nearest point projection of a quasigeodesic on a quasiconvex set is close to a quasigeodesic.

**Lemma 2.29** *Given $K \ge 0$, $R \ge 0$, $\delta \ge 0$ there is a constant $D = D_{2.29}(R, K, \delta)$ such that the following holds:*

*Suppose $X$ is a $\delta$-hyperbolic metric space and $A$ is a $K$-quasiconvex subset of $X$. Suppose $x, y \in X$ and $\bar{x}, \bar{y} \in A$, respectively, are their 1-approximate nearest point projections on $A$. Let $[x, y], [\bar{x}, \bar{y}]$ denote 1-quasigeodesics in $X$ joining $x, y$ and $\bar{x}, \bar{y}$, respectively. Suppose $z \in [x, y]$ and $\bar{z}$ is a 1-approximate nearest point projection of $z$ on $A$ and $d(z, \bar{z}) \le R$. Then $d(z, [\bar{x}, \bar{y}]) \le D$.*

**Proof** By Corollary 2.21, quadrilaterals in $X$ formed by 1-quasigeodesics are $2D_{2.20}(\delta, 1, 1)$-slim. Hence, there is $z' \in [x, \bar{x}] \cup [\bar{x}, \bar{y}] \cup [y, \bar{y}]$ such that $d(z, z') \le 2D_{2.20}(\delta, 1, 1)$. If $z' \in [\bar{x}, \bar{y}]$ then we are done. Suppose not. Without loss of generality let us assume that $z' \in [x, \bar{x}]$. Then $d(z', A) \le d(z, z') + d(z, A) \le 2D_{2.20}(\delta, 1, 1) + R$. Since $\bar{x}$ is a 1-approximate nearest point projection of $x$ on $A$, $\bar{x}$ is a 4-approximate nearest point projection of $z'$ on $A$ by Lemma 2.13. Hence, by Corollary 2.28, $d(\bar{x}, \bar{z}) \le L_{2.28}(\delta, K, 4) d(z', z) \le L_{2.28}(\delta, K, 4)(2D_{2.20}(\delta, 1, 1) + R)$. But $d(z, \bar{z}) \le R$. Hence,

$$d(z, \bar{x}) \le R + L_{2.28}(\delta, K, 4)(2D_{2.20}(\delta, 1, 1) + R).$$

Thus we can take $D_{2.29}(R, K, \delta) = \max\{2D_{2.20}(\delta, 1, 1),\ R + L_{2.28}(\delta, K, 4)(2D_{2.20}(\delta, 1, 1) + R)\}$. □

The following lemma asserts that quasiconvexity and nearest point projections are preserved under qi embeddings.

**Lemma 2.30** *Suppose $X$ is a $\delta$-hyperbolic metric graph and $Y \subset X$ is a connected subgraph such that the inclusion $(Y, d_Y) \to (X, d_X)$ is a $k$-qi embedding. Suppose $A \subset Y$ is $K$-quasiconvex in $Y$. Then the following hold:*

1. *$A$ is $K_{2.30}(\delta, k, K)$-quasiconvex in $X$.*
2. *For any $x \in Y$ if $x_1, x_2 \in A$ are the nearest point projections of $x$ on $A$ in $Y$ and $X$, respectively, then $d_Y(x_1, x_2) \le D_{2.30}(\delta, k, K)$.*

**Proof** (1) Suppose $x, y \in A$ and let $\alpha, \beta$ be geodesics joining $x, y$ in $Y$ and $X$, respectively. Since $Y$ is $k$-qi embedded, $\alpha$ is a $(k, k)$-quasigeodesic in $X$ by Lemma 2.3. Hence, by stability of quasigeodesics $Hd(\alpha, \beta) \le D_{2.17}(\delta, k, k)$. However, $A$ being $K$-quasiconvex in $Y$, $\alpha \subset N_K(A)$ in $Y$ and hence in $X$ as well. Thus $\beta \subset N_{K + D_{2.17}(\delta, k, k)}(A)$ in $X$. Hence, we can take $K_{2.30}(\delta, k, K) = K + D_{2.17}(\delta, k, k)$.

(2) Suppose $K_1 = K_{2.30}(\delta, k, K)$. Then $x_2 \in N_D([x, x_1]_X)$ in $X$, where $D = D_{2.25}(\delta, K_1, 1, 1)$. We have $Hd([x, x_1]_Y, [x, x_1]_X) \le D_{2.17}(\delta, k, k)$ by stability of quasigeodesics. Thus there is a point $x_2' \in [x, x_1]_Y$ such that $d_X(x_2, x_2') \le D + D_{2.17}(\delta, k, k) = D_1$, say. Then $d_Y(x_2, x_2') \le k(D_1 + k)$ since $Y$ is $k$-qi embedded in $X$. Since $x_1$ is a nearest point projection of $x$ on $A$ in $Y$, it is also a nearest point projection of $x_2'$ on $A$ in $Y$. Hence, $d_Y(x_2', x_1) \le d_Y(x_2', x_2) \le k(D_1 + k)$. Hence, $d_Y(x_1, x_2) \le 2k(D_1 + k)$ by triangle inequality. Thus we can take $D_{2.30}(\delta, k, K) = 2k(D_1 + k)$. □

**Definition 2.31** Suppose $X$ is a $\delta$-hyperbolic metric space and $A, B$ are two quasiconvex subsets. Let $R > 0$. We say that $A, B$ are *mutually $R$-cobounded, or simply $R$-cobounded*, if the set of all 1-approximate nearest point projections of the points of $A$ on $B$ has a diameter at most $R$ and vice versa.

When the constant $R$ is understood or is not important we just say that $A, B$ are cobounded.

The following corollary is an immediate consequence of Corollary 2.28(2).

**Corollary 2.32** [24, Lemma 1.35] *Given $\delta \geq 0, k \geq 0$ there are constants $D = D_{2.32}(\delta, k)$ and $R = R_{2.32}(\delta, k)$ such that the following holds*:

*Suppose $X$ is a $\delta$-hyperbolic metric space and $A, B \subset X$ are two $k$-quasiconvex subsets. If $d(A, B) \geq D$ then $A, B$ are mutually $R$-cobounded.*

The following proposition and its proof are motivated by an analogous result due to Hamenstädt [16, Lemma 3.5]. See also [24, Corollary 1.52]. Before we state the proposition let us explain the set-up.

(P0) Suppose $X$ is a $\delta$-hyperbolic metric graph and $Y \subset X$ is a $K$-quasiconvex subgraph, for some $\delta \geq 0$, $K \geq 0$. Suppose $I$ is an interval in $\mathbb{R}$ with end points in $\mathbb{Z} \cup \{\infty, -\infty\}$ and $\Pi\colon Y \to I$ is a map such that $I \cap \mathbb{Z} \subset \Pi(Y)$. Let $Y_i := \Pi^{-1}(i)$ for all $i \in I \cap \mathbb{Z}$ and $Y_{ij} = \Pi^{-1}([i, j])$ for all $i, j \in I \cap \mathbb{Z}$ with $i < j$ such that the following hold:

(P1) All the sets $Y_i$ and $Y_{ij}$, $i, j \in I$, $i < j$ are $K$-quasiconvex in $X$.

(P2) $Y_i$ uniformly coarsely bisects $Y$ into $Y_i^- := \Pi^{-1}((-\infty, i] \cap I)$ and $Y_i^+ := \Pi^{-1}([i, \infty) \cap I)$ for all $i \in I$. Let $R \geq 0$ be such that any geodesic in $Y$ joining $Y_i^+$ and $Y_i^-$ passes through $N_R(Y_i)$ for all $i \in I \cap \mathbb{Z}$.

(P3) $d(Y_{ii+1}, Y_{jj+1}) > 2K + 1$ for all $i, j \in I$ if $j + 1 \in I$ and $i + 1 < j$.

(P4) There is $D \geq 0$ such that the sets $Y_i$ and $Y_j$ are $D$-cobounded in $X$ for all $i, j \in I \cap \mathbb{Z}$ with $i < j$ unless $j = i + 1$ and $i, j$ are the end points of $I$.

The proposition below is about a description of uniform quasigeodesics in $X$ joining points of $Y$.

**Proposition 2.33** *Given $\delta \geq 0$, $K \geq 0$, $D \geq 0$, $\lambda \geq 1$, $\epsilon \geq 1$ and $R \geq 0$ there are*

$$\lambda' = \lambda_{2.33}(\delta, K, D, \lambda, \epsilon, R) \geq 1 \quad \text{and} \quad \mu_{2.33} = \mu_{2.33}(\delta, K, D, \epsilon, R) \geq 0$$

*such that the following holds.*

*Suppose we have the aforementioned hypotheses* (P0)–(P4). *Suppose $m, n \in I \cap \mathbb{Z}$ and $y \in Y_m$, $y' \in Y_n$. Suppose $y_i \in Y$, $m \leq i \leq n$ are defined as follows*: *$y_m = y$, $y_{i+1}$ is an $\epsilon$-approximate nearest point projection of $y_i$ on $Y_{i+1}$ for $m \leq i \leq n-1$. Suppose $\alpha_i \subset Y_{ii+1}$ is a $\lambda$-quasigeodesic in $X$ joining $y_i$ and $y_{i+1}$, $m \leq i \leq n-1$ and $\beta$ is a $\lambda$-quasigeodesic joining $y_n$ and $y'$.*

*Then the concatenation of the all the $\alpha_i$'s and $\beta$ is a $\lambda'$-quasigeodesic in $X$ joining $y, y'$. Moreover, each $y_i$ is an $\mu_{2.33}$-approximate nearest point projection of $y$ on $Y_i$ for $m + 2 \leq i \leq n$.*

**Proof** The proof is broken into the following three claims. In course of the proof we shall denote the concatenation of the $\alpha_i$'s and $\beta$ by $\alpha$.

**Claim 1** *Suppose $x \in Y_i^-$ for some $i$. Let $\bar{x}$ be an $\epsilon$-approximate nearest point projection of $x$ on $Y_i$. Then $\bar{x}$ is an $\epsilon'$-approximate nearest point projection of $x$ on $Y_i^+$ where $\epsilon'$ depends only on $\epsilon$ and the parameters $\delta$, $D$, $K$ and $R$.*

**Proof** Suppose $x'$ is a 1-approximate nearest point projection of $x$ on $Y_i^+$. Since $Y_i^+$ is $K$-quasiconvex, $[x, x'] * [x', \bar{x}]$ is a $K_{2.25}(\delta, K, 1, 1)$-quasigeodesic by Lemma 2.25. Let $k_1 = K_{2.25}(\delta, K, 1, 1)$. Then by stability of quasigeodesics there is a point $z \in [x, \bar{x}]$ such that $d(x', z) \le D_{2.17}(\delta, k_1) = D_1$, say. We claim that $z$ is uniformly close to $Y_i$. Since $Y_i^-$ is $K$-quasiconvex there is a point $w \in Y_i^-$ such that $d(z, w) \le K$. It follows that $d(w, x') \le D_1 + K$. By (P2), there is a point $z_1 \in [w, x']$ such that $d(z_1, Y_i) \le R$. Since $d(z_1, w) \le d(w, x') \le D_1 + K$ and $d(w, z) \le K$, it follows by the triangle inequality that $d(z, Y_i) \le 2K + D_1 + R$. Now, by Lemma 2.13 $\bar{x}$ is an $(\epsilon{+}3)$-approximate nearest point projection of $z$ on $Y_i$. Hence, $d(x', \bar{x}) \le d(x', z) + d(z, \bar{x}) \le D_1 + \epsilon + 3 + d(z, Y_i)$. It follows that $\epsilon' = 3 + \epsilon + 2K + 2D_1 + R$ works. ◁

**Note** *We shall use $D_1$ again in the proof of Claim 3 to denote the same constant as in the proof of Claim 1 above.*

**Claim 2** *Next we claim that for all $m + 2 \le i \le n - 1$ there is uniformly bounded set $A_i \subset Y_i$ such that $\epsilon$-nearest point projection of any point of $Y_j^-$, $j < i$ on $Y_i$ is contained in $A_i$.*

**Proof** Consider any $Y_i$, $m + 2 \le i \le n - 1$. Let $B_i \subset Y_i$ be the set of all 1-approximate nearest point projections of points of $Y_{i-1}$ on $Y_i$ in $X$. Then the diameter of $B_i$ is at most $D$ by (P4). Suppose $x \in Y_j^-$, $j < i$. Let $x_1, x_2$ be $\epsilon$-approximate nearest point projections of $x$ on $Y_{i-1}$ and $Y_i$, respectively. Let $x_3$ be an $\epsilon$-nearest point projection of $x_1$ on $Y_i$. Now, by Step 1 $x_1$ is an $\epsilon'$-approximate nearest point projection of $x$ on $Y_{i-1}^+$ and $x_2$, $x_3$ are $\epsilon'$-approximate nearest point projection of $x$ and $x_1$, respectively, on $Y_i^+$. Therefore, by the first part of Corollary 2.27 we have $d(x_2, x_3) \le D_{2.27}(\delta, K, \epsilon')$. However, if $x_1' \in B_i$ is a 1-approximate nearest point projection of $x_1$ on $Y_i$ then by the second part of Corollary 2.27 we have $d(x_3, B_i) \le d(x_3, x_1') \le D_{2.27}(\delta, K, \epsilon)$ since $\epsilon \ge 1$. Hence, $d(x_2, B_i) \le 2D_{2.27}(\delta, K, \epsilon)$. Therefore, we can take $A_i = N_{2D_{2.27}(\delta, K, \epsilon)}(B_i) \cap Y_i$. ◁

Let $r = \sup_{m+2 \le i \le n-1}\{\mathrm{diam}(A_i)\}$. We note that $r \le D + 2D_{2.27}(\delta, K, \epsilon)$.

**Claim 3** *Finally we claim that* (1) *$\alpha$ is contained in a uniformly small neighborhood of a geodesic joining $y, y'$ and* (2) *$\alpha$ is uniformly properly embedded in $X$.*

We note that the proposition follows from Claim 3 using Lemma 2.5.

**Proof of Claim 3** Suppose $x, x' \in \alpha$, $\Pi(x) < \Pi(x')$. Choose smallest $k, l$ such that $x \in \alpha \cap Y_{kk+1}$, $x' \in \alpha \cap Y_{ll+1}$, where $m \le k \le l \le n$. Let $\gamma$ be a geodesic in $X$ joining $x, x'$.

(1) It is enough to show that the segment of $\alpha$ joining $x$ to $x'$ is contained in a uniformly small neighborhood of $\gamma$. Hence, without loss of generality $k<l$. Due to Corollary 2.21 it is enough to prove that the points $y_i,\ k+1\le i\le l-1$ are contained in a uniformly small neighborhood of $\gamma$ in order to show that the segment of $\alpha$ joining $x$ to $x'$ is contained in a uniformly small neighborhood of $\gamma$. (We note that the path $\alpha_{n-1}*\beta$ is a $D_{2.25}(\delta,K,\lambda,\epsilon)$-quasigeodesic joining $y_{n-1}$ and $y'$.) For this first we note that $x$ is on $\alpha_k$. Let $\gamma_k$ be a geodesic joining $y_k, y_{k+1}$. Then by stability of quasigeodesics there is a point $x_1\in\gamma_k$ such that $d(x_1,x)\le D_{2.19}(\delta,\lambda,\lambda)$. Since $y_{k+1}$ is an $\epsilon$-approximate nearest point projection of $y_k$ on $Y_{k+1}$, by Lemma 2.13 $y_{k+1}$ is an $(\epsilon{+}3)$-approximate nearest point projection of $x_1$ on $Y_{k+1}$. Hence, $y_{k+1}$ is an $(\epsilon{+}3{+}D_{2.19}(\delta,\lambda,\lambda))$-approximate nearest point projection of $x_1$ on $Y_{k+1}$. Let $\epsilon_1=\epsilon+3+D_{2.19}(\delta,\lambda,\lambda)$. By Step 1 $y_{k+1}$ is an $\epsilon_1'$-nearest point projection of $x$ on $Y^+_{k+1}$, where $\epsilon_1'=3+\epsilon_1+2D_1++2K+R$. Now the concatenation of a geodesic joining $y_{k+1}$ to $x'$ with the segment of $\alpha$ from $x$ to $y_{k+1}$ is a uniform quasigeodesic by Lemma 2.25. Thus by Corollary 2.19 $y_{k+1}$ is uniformly close to $\gamma$. On the other hand by Step 2 $y_i$ is an $(\epsilon{+}r)$-approximate nearest point projection of $x$ on $Y_i$ and hence an $(\epsilon{+}r)'$-approximate nearest point projection on $Y_i^+$ for all $k+2\le i\le l-1$. Hence, again by Lemma 2.25 and Corollary 2.19 $y_i$ is within a uniformly small neighborhood of $\gamma$. This proves (1).

(2) Suppose $L=\sup\{d(y_i,\gamma):k+1\le i\le l-1\}$. Suppose $x,x'\in\alpha$ as above with $d(x,x')\le N$. Once again, without loss of generality $k<l$. We claim that $l\le k+N$. To see this consider two adjacent vertices $v_i,v_{i+1}$ on $\gamma$. If $v_i\in N_K(Y_{ss+1})$ and $v_{i+1}\in N_K(Y_{tt+1})$ with $s<t$ then by (P3) we have $t=s+1$. The claim follows from this. Suppose $\alpha(s_k)=x,\ \alpha(s_i)=y_i$ for $k+1\le i\le l-1$ and $\alpha(s_l)=x'$. We note that $d(\alpha(s_i),\alpha(s_{i+1}))\le N+2L$ for $k\le i\le l-1$. Since $l-k\le N$ and since the segments of $\alpha$ joining $\alpha(s_i),\alpha(s_{i+1}),\ k\le i\le l-1$ are uniform quasigeodesics, we are done. ◁

For the second part of the proposition we have already noticed that $y_i$ is an $(\epsilon{+}r)$-approximate nearest point projection of any point $Y_j^-$, in particular of $y$, on $Y_i$ for all $j<i,\ m+2\le i\le n-1$. On the other hand, $y_{n-1}$ is an $(\epsilon+r)'=(\epsilon+r+3+2D_1+2K+R)$-approximate nearest point projection of $y$ on $Y^+_{n-1}$. Hence, by Corollary 2.27 if $y_n'$ is a 1-approximate point projection of $y$ on $Y_n\subset Y^+_{n-1}$ then $d(y_n',y_n)\le D_{2.27}(\delta,K,(\epsilon+r)')$. Thus $y_n$ is an $(1+D_{2.27}(\delta,K,(\epsilon+r)'))$-approximate nearest point projection of $y$ on $Y_n$. □

**Lemma 2.34** *Given $\delta\ge 0,\ k\ge 1,\ \epsilon\ge 0$, there is a constant $D=D_{2.34}(\delta,k,\epsilon)$ such that the following is true*:

*Suppose $X$ is a $\delta$-hyperbolic metric space. Suppose $x_1,x_2,p\in X$ and $\alpha$ is a $(k,\epsilon)$-quasigeodesic in $X$ joining $x_1,x_2$. Then $|(x_1.x_2)_p-d(p,\alpha)|\le D$.*

**Proof** Without loss of generality, we shall assume that $X$ is a length space $\delta$-hyperbolic in the sense of Gromov. Let $w\in\alpha$ be a 1-approximate nearest point projection of $p$ on $\alpha$. Let $\beta_1,\beta_2$ be $(1,1)$-quasigeodesics joining the pairs of points $(x_1,p),(x_2,p)$, respectively. Let $\gamma$ be a $(1,1)$-quasigeodesic joining $p, w$ and let $\alpha'$ be a $(1,1)$-quasigeodesic joining $x_1, x_2$. Let $C = D_{2.19}(\delta, k, \epsilon+1)$. Now, by Corollary 2.19 $Hd(\alpha, \alpha') \le C$ and $\alpha$ is $C$-quasiconvex. Let $\alpha_1$ be the portion of $\alpha$ from $x_1$ to $w$ and let $\alpha_2$ be the portion of $\alpha$ from $w$ to $x_2$. Then $\alpha_1 * \gamma$, $\alpha_2 * \gamma$ are $K = K_{2.25}(\delta, C, k+\epsilon, k+\epsilon)$-quasigeodesics. Hence by Corollary 2.19 $Hd(\beta_i, \alpha_i * \gamma) \le D_{2.19}(\delta, K, K)$. Let $w_i \in \beta_i$ be such that $d(w, w_i) \le D_{2.19}(\delta, K, K)$. Since $Hd(\alpha, \alpha') \le C$, there is a point $w' \in \alpha'$ such that $d(w, w') \le C$. Hence, $d(w', w_i) \le C + D_{2.19}(\delta, K, K) = R$, say. Now by Lemma 2.10 $|(x_1.x_2)_p - d(p, w')| \le 3 + 2R$. It follows that $|(x_1.x_2)_p - d(p, w)| \le 3 + 2R + C$. Since $w$ is a 1-approximate nearest point projection of $p$ on $\alpha$ we have for all $z \in \alpha$, $d(p, w) \le d(p, z) + 1$. Thus $|d(p, \alpha) - d(p, w)| \le 1$. Hence, $|(x_1.x_2)_p - d(p, \alpha)| \le 4 + 2R + C$. □

## 2.4 Boundaries of hyperbolic spaces and CT maps

Given a hyperbolic metric space, there are the following three standard ways to define a boundary. Some of the results in this subsection are mentioned without proof. One may refer to [1; 6] and for details.

**Definition 2.35** (1) **Geodesic boundary** Suppose $X$ is a (geodesic) hyperbolic metric space. Let $\mathcal{G}$ denote the set of all geodesic rays in $X$. The *geodesic boundary* $\partial X$ of $X$ is defined to be $\mathcal{G}/\sim$, where $\sim$ is the equivalence relation on $\mathcal{G}$ defined by setting $\alpha \sim \beta$ if and only if $Hd(\alpha, \beta) < \infty$.

(2) **Quasigeodesic boundary** Suppose $X$ is a hyperbolic metric space in the sense of Gromov. Let $\mathcal{Q}$ be the set of all quasigeodesic rays in $X$. Then the *quasigeodesic boundary* $\partial_q X$ is defined to be $\mathcal{Q}/\sim$, where $\sim$ is defined as above.

(3) **Gromov boundary or sequential boundary** Suppose $X$ is a hyperbolic metric space in the sense of Gromov and $p \in X$. Let $\mathcal{S}$ be the set of all sequences $\{x_n\}$ in $X$ such that $\lim_{i,j\to\infty}(x_i.x_j)_p = \infty$. All such sequences are said to converge to infinity. On $\mathcal{S}$ we define an equivalence relation where $\{x_n\} \sim \{y_n\}$ if and only if $\lim_{i,j\to\infty}(x_i.y_j)_p = \infty$ for some (any) base point $p \in X$. The *Gromov boundary or the sequential boundary* $\partial_s X$ of $X$, as a set, is defined to be $\mathcal{S}/\sim$.

**Notation and convention** (1) The equivalence class of a geodesic ray or a quasigeodesic ray $\alpha$ in $\partial X$ or $\partial_q X$ is denoted by $\alpha(\infty)$. It is customary to fix a base point and require that all the rays start from there to define $\partial X$ and $\partial_q X$ but it is not essential.

(2) If $\alpha$ is a (quasi)geodesic ray with $\alpha(0) = x$, $\alpha(\infty) = \xi$ then we say that $\alpha$ joins $x$ to $\xi$. We use $[x, \xi)$ to denote any (quasi)geodesic ray joining $x$ to $\xi$ when the parametrization of the (quasi)geodesic ray is not important or is understood.

(3) If $\alpha$ is a quasigeodesic line with $\alpha(\infty) = \xi_1$, $\alpha(-\infty) = \xi_2 \in \partial_q X$ then we say that $\alpha$ joins $\xi_1, \xi_2$. We denote by $(\xi_1, \xi_2)$ any quasigeodesic line joining $\xi_1, \xi_2$ when the parameters of the quasigeodesic are understood.

(4) If $\xi = [\{x_n\}] \in \partial_s X$ then we write $x_n \to \xi$ or $\xi = \lim_{n\to\infty} x_n$ and say that the sequence $\{x_n\}$ converges to $\xi$.

(5) We shall denote by $\widehat{X}$ the set $X \cup \partial_s X$.

The following lemma and proposition summarizes all the basic properties of the boundary of hyperbolic spaces that we will need in this paper.

**Lemma 2.36** [9, Theorem 11.108] *Let $X$, $Y$ be hyperbolic metric spaces.*

1. *Given a qi embedding $\phi\colon X \to Y$ we have an injective map $\partial\phi\colon \partial_s X \to \partial_s Y$.*
2. (i) *If $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ are qi embeddings then $\partial(\psi \circ \phi) = \partial\psi \circ \partial\phi$.*
   (ii) *$\partial(\mathrm{Id}_X)$ is the identity map on $\partial_s X$.*
   (iii) *A qi induces a bijective boundary map.*

The following proposition relates the three definitions of boundaries.

**Proposition 2.37** (1) *For any metric space $X$ the inclusion $\mathcal{G} \to \mathcal{Q}$ induces an injective map $\partial X \to \partial_q X$.*

(2) *Given a quasigeodesic ray $\alpha$, $\lim_{n\to\infty} \alpha(n)$ is well defined and $\alpha \sim \beta$ implies $\lim_{n\to\infty} \alpha(n) = \lim_{n\to\infty} \beta(n)$. This induces an injective map $\partial_q X \to \partial_s X$.*

(3) *If $X$ is a proper geodesic hyperbolic metric space then the map $\partial X \to \partial_q X$ is a bijection.*

(4) *The map $\partial_q X \to \partial_s X$ is a bijection for all Gromov hyperbolic length spaces.*

*In fact, given $\delta \geq 0$ there is a constant $k_{2.37} = k_{2.37}(\delta)$ such that given any $\delta$-hyperbolic length space $X$, any pair of points $x, y \in \widehat{X}$ can be joined by a $k_{2.37}$-quasigeodesic.*

**Proof** Properties (1), (2), (3) are standard. See [6, Chapter III.H], for instance. Property (4) is proved for geodesic metric spaces in Section 2 of [24]. See Lemma 2.4 there. The same result for a general length space then is a simple consequence of the existence of a metric graph approximation of a length space and the preceding lemma. □

**Lemma 2.38** (ideal triangles are slim) *Suppose $X$ is a $\delta$-hyperbolic metric space in the sense of Rips or Gromov. Suppose $x, y, z \in \widehat{X}$ and we have three $k$-quasigeodesics joining each pair of points from $\{x, y, z\}$. Then the triangle is $R = R_{2.38}(\delta, k)$-slim.*

*In particular, if $\gamma_1, \gamma_2$ are two $k$-quasigeodesic rays with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(\infty) = \gamma_2(\infty)$ then $Hd(\gamma_1, \gamma_2) \leq R$.*

The proof of the above lemma is pretty standard and hence we omit it. However, slimness of ideal triangles immediately implies slimness of ideal polygons:

**Corollary 2.39** (ideal polygons are slim) *Suppose $X$ is a $\delta$-hyperbolic metric space in the sense of Rips or Gromov. Suppose $x_1, x_2, \dots, x_n \in \widehat{X}$ are $n$ points and we have $n$ $k$-quasigeodesics joining pairs of points $(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ and $(x_n, x_1)$. Then this $n$-gon is $R = R_{2.39}(\delta, k, n)$-slim, ie every side is contained in $R$-neighborhood of the union of the remaining $n-1$ sides.*

The following lemma gives a geometric interpretation for sequential boundary in terms of quasigeodesics.

**Lemma 2.40** *Let $x \in X$ be any point. Suppose $\{x_n\}$ is any sequence of points in $X$ and $\beta_{m,n}$ is a $k$-quasigeodesic joining $x_m$ to $x_n$ for all $m, n \in \mathbb{N}$. Suppose $\alpha_n$ is a $k$-quasigeodesic joining $x$ to $x_n$. Then*:

1. *$\{x_n\} \in \mathcal{S}$ if and only if $\lim_{m,n\to\infty} d(x, \beta_{m,n}) = \infty$ if and only if there is a constant $D$ such that for all $M > 0$ there is $N > 0$ with $Hd\big(\alpha_m \cap B(x; M), \alpha_n \cap B(x; M)\big) \le D$ for all $m, n \ge N$.*
2. *Suppose moreover $\xi \in \partial_s X$ and $\gamma_n$ is a $k$-quasigeodesic in $X$ joining $x_n$ to $\xi$ for all $n \in \mathbb{N}$ and $\alpha$ is a $k$-quasigeodesic joining $x$ to $\xi$.*

*Then $x_n \to \xi$ if and only if $d(x, \gamma_n) \to \infty$ if and only if there is constant $D > 0$ such that for all $M > 0$ there is $N > 0$ with $Hd\big(\alpha \cap B(x; M), \alpha_n \cap B(x; M)\big) \le D$ for all $n \ge N$.*

We skip the proof of this lemma. In fact, the first statement of the lemma is an easy consequence of Lemma 2.34 and stability of quasigeodesics. The second statement is a simple consequence of Lemma 2.34, stability of quasigeodesics and Lemma 2.38.

The following lemma is proved in Section 2 of [24] (see Lemmas 2.7 and 2.9 there) for hyperbolic geodesic metric spaces. The same statements are true for length spaces too. To prove it for length spaces one just takes a metric graph approximation. Since the proof is straightforward we omit it.

**Lemma 2.41** (barycenters of ideal triangles) *Given $\delta \ge 0$ there is $r_0 \ge 0$ such that for any $\delta$-hyperbolic length space $X$, any three distinct points $x, y, z \in \widehat{X}$ and any three $k_{2.37}(\delta)$-quasigeodesics joining $x, y, z$ in pairs there is a point $x_0 \in X$ such that $N_{r_0}(x_0)$ intersects all the three quasigeodesics.*

*We refer to a point with this property to be a* **barycenter** *of the ideal triangle $\Delta xyz$. There is a constant $L_0$ such that if $x_0, x_1$ are two barycenters of $\Delta xyz$ then $d(x_0, x_1) \le L_0$.*

Thus we have a coarsely well-defined map $\partial_s^3 X \to X$. We shall refer to this map as the *barycenter map*. It is a standard fact that for a nonelementary hyperbolic group $G$ if $X$ is a Cayley graph of $G$ then the barycenter map $\partial_s^3 X \to X$ is coarsely surjective and vice versa. If $X$ is a hyperbolic metric space such that the barycenter map for $X$ is coarsely surjective then $X$ will be called a *nonelementary* hyperbolic space. In Sections 4 and 5 we deal with spaces with this property.

The following lemma is clear. For instance, we can apply the proof of [24, Lemma 2.9].

**Lemma 2.42** *Barycenter maps being coarsely surjective is a qi invariant property among hyperbolic length spaces.*

### 2.4.1 Topology on $\partial_s X$ and Cannon–Thurston maps

**Definition 2.43** (1) If $\{\xi_n\}$ is a sequence of points in $\partial_s X$, we say that $\{\xi_n\}$ converges to $\xi \in \partial_s X$ if the following holds: Suppose $\xi_n = [\{x^n_k\}_k]$ and $\xi = [\{x_k\}]$. Then $\lim_{n\to\infty}(\liminf_{i,j\to\infty}(x_i.x^n_j)_p) = \infty$.

(2) A subset $A \subset \partial_s X$ is said to be closed if for any sequence $\{\xi_n\}$ in $A$, $\xi_n \to \xi$ implies $\xi \in A$.

The definition of convergence that we have stated here is equivalent to the one stated in [1]. Moreover, that the convergence mentioned above is well-defined follows from [1] and hence we skip it. The next two lemmas give a geometric meaning of the convergence.

**Lemma 2.44** *Given $k \geq 1$ and $\delta \geq 0$, there are constants $D = D_{2.44}(k,\delta)$, $L = L_{2.44}(k,\delta)$ and $r = r_{2.44}(k,\delta)$ with the following properties:*

*Suppose $\alpha, \beta$ are two $k$-quasigeodesic rays starting from a point $x \in X$ such that $\alpha(\infty) \neq \beta(\infty)$ and $\gamma$ is a $k$-quasigeodesic line joining $\alpha(\infty)$ and $\beta(\infty)$. Then the following hold:*

(1) *There exists $N \in \mathbb{N}$ such that $|(\alpha(m).\beta(n))_x - d(x,\gamma)| \leq D$ for all $m,n \geq N$.*
*In particular, $|\liminf_{m,n\to\infty}(\alpha(m).\beta(n))_x - d(x,\gamma)| \leq D$.*

(2) *Suppose $R = d(x,\gamma)$ then $Hd(\alpha \cap B(x;R-r), \beta \cap B(x;R-r)) \leq L$.*

**Proof** (1) Since $\alpha(\infty) \neq \beta(\infty)$ by Lemma 2.38 there is $N \in \mathbb{N}$ such that for all $m,n \geq N$, $\alpha(m) \in N_{R_{2.38}}(\gamma)$ and $\beta(n) \in N_{R_{2.38}}(\gamma)$. Let $x_m, y_n \in \gamma$ be such that

$$d(x_m,\alpha(m)) \leq R_{2.38} \quad\text{and}\quad d(y_n,\beta(n)) \leq R_{2.38}.$$

Then by joining $x_m, \alpha(m)$ and $y_n, \beta(n)$ and applying Corollary 2.21 we see that Hausdorff distance between any $(1,1)$-quasigeodesic joining $\alpha(m), \beta(n)$, say $c_{m,n}$ and the portion of $\gamma$ between $x_m, y_n$ is at most $R_{2.38} + 2D_{2.20}(\delta,k,k)$. It is clear that for large enough $N$, $d(x,\gamma)$ is the same as the distance of $x$ and the segment of $\gamma$ between $x_m, y_n$ if $m,n \geq N$. Thus for such $m,n$ we have $|d(x,c_{m,n}) - d(x,\gamma)| \leq R_{2.38} + 2D_{2.20}(\delta,k,k)$. But by Lemma 2.34, $|(\alpha(m).\beta(n))_x - d(x,c_{m,n}| \leq D_{2.34}(\delta,k,k)$. Hence, $|(\alpha(m).\beta(n))_x - d(x,\gamma)| \leq R_{2.38} + 2D_{2.20}(\delta,k,k) + D_{2.34}(\delta,k,k)$ for all large $m,n$.

(2) To see this we take a 1-approximate nearest point projection, say $z$, of $x$ on $\gamma$. Let $xz$ denote a 1-quasigeodesic joining $x,z$. Then by Corollary 2.26 concatenation of $xz$ and the portions of $\gamma$ joining $z$ to $\gamma(\pm\infty)$ respectively are both $K_{2.26}(\delta,k,k)$-quasigeodesics. Call them $\alpha'$ and $\beta'$, respectively. Note that $\alpha(\infty) = \alpha'(\infty)$ and $\beta(\infty) = \beta'(\infty)$. Let $K = \max\{k, K_{2.26}(\delta,k,\epsilon)\}$. Then by the last part of Lemma 2.38 it follows that $z \in N_r(\alpha) \cap N_r(\beta)$ where $r = R_{2.38}(\delta,K)$. Suppose $x' \in \alpha$, $y' \in \beta$ are such that $d(z,x') \leq r$ and $d(y',z) \leq r$. By Corollary 2.20 the Hausdorff distance between $xz$ and the portions of $\alpha$ from $x$ to $x'$ and the portion of $\beta$ from $x$ to $y'$ are each at most $D_{2.20}(\delta,k,k) + r$. Thus these segments of $\alpha$ and $\beta$ are at a Hausdorff distance at most $L = 2D_{2.20}(\delta,k,k) + 2r$ from each other. This completes the proof. □

**Lemma 2.45** *Let $x \in X$ be any point. Suppose $\{\xi_n\}$ is any sequence of points in $\partial_s X$. Suppose $\beta_{m,n}$ is a k-quasigeodesic line joining $\xi_m$ to $\xi_n$ for all $m, n \in \mathbb{N}$ and $\alpha_n$ is a k-quasigeodesic ray joining $x$ to $\xi_n$ for all $n \in \mathbb{N}$. Then*:

1. $\lim_{m,n\to\infty} d(x, \beta_{m,n}) = \infty$ *if and only if there is a constant $D = D(k, \delta)$ such that for all $M > 0$ there is $N > 0$ with $Hd\big(\alpha_m \cap B(x; M), \alpha_n \cap B(x; M)\big) \le D$ for all $m, n \ge N$ and in this case $\{\xi_n\}$ converges to some point of $\partial_s X$.*
2. *Suppose moreover $\xi \in \partial_s X$, $\gamma_n$ is a k-quasigeodesic ray in $X$ joining $\xi_n$ to $\xi$ for all $n$, and $\alpha$ is a k-quasigeodesic ray joining $x$ to $\xi$. Then $\xi_n \to \xi$ if and only if $d(x, \gamma_n) \to \infty$ if and only if there is constant $D' = D'(k, \delta)$ such that for all $M > 0$ there is $N > 0$ with*
$$Hd\big(\alpha \cap B(x; M), \alpha_n \cap B(x; M)\big) \le D$$
*for all $n \ge N$. In this case* $\lim_{m,n\to\infty} d(x, \beta_{m,n}) = \infty$.

**Proof** (1) The "if and only if" part is an immediate consequence of Lemma 2.44. We prove the last part. Let $n_i$ be an increasing sequence in $\mathbb{N}$ such that for all $m, n \ge n_i$ we have
$$Hd(\alpha_m \cap B(x; i), \alpha_n \cap B(x; i)) \le D.$$
Let $y_i$ be a point of $\alpha_{n_i} \cap B(x; i)$ such that $d(x, y_i) + 1 \ge \sup\{d(x, y) : x \in \alpha_{n_i} \cap B(x; i)\}$. We claim that $y_i$ converges to a point of $\partial_s X$. Clearly $d(x, y_i) \to \infty$. Given $i \le j \in \mathbb{N}$ we have $d(y_i, \alpha_n) \le D$ and $d(y_j, \alpha_n) \le D$ for all $n \ge n_j$. By slimness of polygons we see that any $(1, 1)$-quasigeodesic joining $y_i, y_j$ is uniformly close to $\alpha_n$. It follows that $\lim_{i,j\to\infty}(y_i . y_j)_x = \infty$. Let $\xi = [\{y_n\}]$. It is clear that $\xi_n \to \xi$.

(2) Both if and only if statements are immediate from Lemma 2.44. The last part follows from slimness of ideal triangle since $d(x, \gamma_n) \to \infty$. □

**Corollary 2.46** *Suppose $\{x_n\}$ is a sequence of points in $\hat{X}$ such that $\{x_n\} \subset X$ or $\{x_n\} \subset \partial_s X$. Suppose $x_n \to \xi \in \partial_s X$ and $\gamma_n$ is a k-quasigeodesic joining $x_n$ to $\xi$ for each $n$. Let $y_n \in \gamma_n$ such that $d(x, y_n) \to \infty$. Then* $\lim_{n\to\infty} y_n = \xi$.

**Definition 2.47** (Cannon–Thurston map [21]) If $f : Y \to X$ is any map of hyperbolic metric spaces then we say that *the Cannon–Thurston* or *the CT map* exists for $f$ or that $f$ admits the CT map if $f$ gives rise to a continuous map $\partial f : \partial Y_s \to \partial X_s$ in the following sense:

Given any $\xi \in \partial_s Y$ and any sequence of points $\{y_n\}$ in $Y$ converging to $\xi$, the sequence $\{f(y_n)\}$ converges to a definite point of $\partial_s X$ independent of the $\{y_n\}$ and the resulting map $\partial f : \partial_s Y \to \partial_s X$ is continuous.

Generally, one assumes that the map $f$ is a proper embedding but for the sake of the definition it is unnecessary. We note that the CT map is unique when it exists. The following lemma gives a sufficient condition for the existence of CT maps.

**Lemma 2.48** (Mitra's criterion [21, Lemma 2.1]) *Suppose $X$, $Y$ are geodesic hyperbolic metric spaces and $f: Y \to X$ is a metrically proper map. Then $f$ admits the CT map if the following holds*:

($*$) *Let $y_0 \in Y$. There exists a function $\tau: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with the property that $\tau(n) \to \infty$ as $n \to \infty$ such that for all geodesic segments $[y_1, y_2]_Y$ in $Y$ lying outside the $n$-ball around $y_0 \in Y$, any geodesic segment $[f(y_1), f(y_2)]_X$ in $X$ joining the pair of points $f(y_1), f(y_2)$ lies outside the $\tau(n)$-ball around $f(y_0) \in X$.*

**Remark** (1) The main set of examples where Lemma 2.48 applies comes from taking $Y$ to be a rectifiably path connected subspace of a hyperbolic space $X$ with induced length metric and the map $f$ is assumed to be the inclusion map. One also considers the orbit map $G \to X$ where $G$ is a hyperbolic group acting properly by isometries on a hyperbolic metric space $X$. In these examples, the map $f$ is coarsely Lipschitz as well as metrically proper. The proof of the lemma by Mitra also assumes that $X$, $Y$ are proper geodesic metric spaces and Mitra considered the geodesic boundaries. However, these conditions are not necessary as the following lemma and examples show.

(2) The proof of Lemma 2.48 by Mitra only checks that the map is a well-defined extension of $f$ rather than it is continuous. However, with very little effort the condition ($*$) can be shown to be sufficient for the well-definedness as well as the continuity of the CT map.

(3) One can easily check that the condition ($*$) is also necessary provided $X, Y$ are proper hyperbolic spaces and $f$ is coarsely Lipschitz and metrically proper.

The following lemma is the main tool for the proof of our theorem of Cannon–Thurston map. We shall refer to this as Mitra's lemma.

**Lemma 2.49** *Suppose $X, Y$ are length spaces hyperbolic in the sense of Gromov*, *and $f: Y \to X$ is any map. Let $p \in Y$.*

($**$) *Suppose for all $N > 0$ there is $M = M(N) > 0$ such that $N \to \infty$ implies $M \to \infty$ with the following property*: *for any $y_1, y_2 \in Y$, any $(1,1)$-quasigeodesic $\alpha$ in $Y$ joining $y_1, y_2$ and any $(1,1)$-quasigeodesic $\beta$ in $X$ joining $f(y_1), f(y_2)$, $B(p, N) \cap \alpha = \varnothing$ implies $B(f(p), M) \cap \beta = \varnothing$.*

*Then the CT map exists for $f: Y \to X$.*

**Proof** Suppose $\{y_n\}$ is any sequence in $Y$. Suppose $\alpha_{i,j}$ is a $(1,1)$-quasigeodesic in $Y$ joining $y_i, y_j$ and suppose $\gamma_{i,j}$ is a $(1,1)$-quasigeodesic in $X$ joining $f(y_i), f(y_j)$. Then by Lemma 2.34 $\lim_{i,j\to\infty}(y_i.y_j)_p = \infty$ if and only if $\lim_{i,j\to\infty} d(p, \alpha_{i,j}) = \infty$ and $\lim_{i,j\to\infty}(f(y_i).f(y_j))_{f(p)} = \infty$ if and only if $\lim_{i,j\to\infty} d_X(f(p), \gamma_{i,j}) = \infty$. On the other hand, by ($**$), $\lim_{i,j\to\infty} d(p, \alpha_{i,j}) = \infty$ implies $\lim_{i,j\to\infty} d_X(f(p), \gamma_{i,j}) = \infty$. Thus $\{y_n\}$ converges to a point of $\partial_s Y$ implies $\{f(y_n)\}$ converges to a point of $\partial_s X$. The same argument shows that if $\{y_n\}$ and $\{z_n\}$ are two sequences in $Y$ representing the same point of $\partial_s Y$ then $\{f(y_n)\}$ and $\{f(z_n)\}$ also represent the same point of $\partial_s X$. Thus we have a well-defined map $\partial f\colon \partial_s Y \to \partial_s X$.

Now we prove the continuity of the map. We need to show that if $\xi_n \to \xi$ in $\partial_s Y$ then $\partial f(\xi_n) \to \partial f(\xi)$. Suppose $\xi_n$ is represented by the class of $\{y^n_k\}_k$ and $\xi$ is the equivalence class of $\{y_k\}$. Then

$$\lim_{n\to\infty} (\liminf_{i,j\to\infty} (y^n_i . y_j)_p) = \infty.$$

By Lemma 2.34 then we have

$$\lim_{n\to\infty} (\liminf_{i,j\to\infty} d(p, \alpha^n_{i,j}) = \infty$$

for any $(1,\epsilon)$-quasigeodesic $\alpha^n_{i,j}$ in $Y$ joining $y^n_i$ and $y_j$. By $(*)$ then we have

$$\lim_{n\to\infty} (\liminf_{i,j\to\infty} d(f(p), \gamma^n_{i,j}) = \infty,$$

where $\gamma^n_{i,j}$ is any $(1,\epsilon)$-quasigeodesic in $X$ joining $f(y^n_i)$, $f(y_j)$. This in turn implies by Lemma 2.34 that

$$\lim_{n\to\infty} \bigl(\liminf_{i,j\to\infty} (f(y^n_i).f(y_j))_{f(p)}\bigr) = \infty.$$

Therefore, $\partial f(\xi_n) \to \partial f(\xi)$ as was required. □

**Examples and remarks** (1) Suppose $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the function $f(x) = e^x - 1$. Then $f$ is not coarsely Lipschitz but $f$ admits the CT map.

(2) One can easily cook up an example along the line of the above example where metric properness is also violated but the CT map exists as we see in the example below. We will see another interesting example in Corollary 6.10.

(3) The condition $(*)$ in the above lemma is also not necessary in general for the existence of the CT map. Here is an example in which both metric properness and $(*)$ fail to hold but nevertheless the CT map exists. Suppose $X$ is a tree built in two steps. First we have a star, ie a tree with one central vertex on which end points of finite intervals are glued where the lengths of the intervals are unbounded. Then two distinct rays are glued to each vertex of the star other than the central vertex. Suppose $Y$ is obtained by collapsing the central star in $X$ to a point and $f$ is the quotient map. Then clearly the CT map exists but $(*)$ is violated.

The following lemma is very standard and hence we skip mentioning its proof.

**Lemma 2.50** (functoriality of CT maps) (1) *Suppose $X, Y, Z$ are hyperbolic metric spaces and $f\colon X \to Y$ and $g\colon Y \to Z$ admit the CT maps. Then so does $g \circ f$ and $\partial(g \circ f) = \partial g \circ \partial f$.*

(2) *If $i\colon X \to X$ is the identity map then it admits the CT map $\partial i$ which is the identity map on $\partial_s X$.*

(3) *If two maps $f, h\colon X \to Y$ are at a finite distance admitting the CT maps then they induce the same CT map.*

(4) *Suppose* $f\colon X \to Y$ *is a qi embedding of hyperbolic length spaces. Then* $f$ *admits the CT map* $\partial f\colon \partial_s X \to \partial_s Y$ *which is a homeomorphism onto the image.*

*If* $f$ *is a quasiisometry then* $\partial f$ *is a homeomorphism. In particular, the action by left multiplication of a hyperbolic group* $G$ *on itself induces an action of* $G$ *on* $\partial G$ *by homeomorphisms.*

### 2.4.2 Limit sets

**Definition 2.51** Suppose $X$ is a hyperbolic metric space and $A \subset X$. Then the *limit set* of $A$ in $X$ is the set $\Lambda_X(A) = \{\lim_{n\to\infty} a_n \in \partial_s X : \{a_n\}$ is a sequence in $A\}$.

When $X$ is understood then the limit set of $A \subset X$ will be denoted simply by $\Lambda(A)$. In this subsection, we collect some basic results on limit sets that we need in Section 6 of the paper. In each case, we briefly indicate the proofs for the sake of completeness. The following is straightforward.

**Lemma 2.52** *Suppose* $X$ *is a hyperbolic metric space and* $A, B \subset X$ *with* $Hd(A, B) < \infty$*. Then* $\Lambda(A) = \Lambda(B)$*.*

**Lemma 2.53** *Suppose* $X$ *is a hyperbolic metric space and* $Y \subset X$*. Suppose* $Z \subset Y$ *coarsely bisects* $Y$ *in* $X$ *into* $Y_1, Y_2$ *where* $Z \subset Y_1 \cap Y_2$*. Then* $\Lambda(Y_1) \cap \Lambda(Y_2) = \Lambda(Z)$*.*

**Proof** This is a straightforward consequence of Lemma 2.34. □

**Lemma 2.54** *Suppose* $X$ *is a* $\delta$*-hyperbolic metric space and* $A \subset X$ *is* $\lambda$*-quasiconvex. Suppose* $\xi \in \Lambda(A)$ *and* $\gamma$ *is a* $K$*-quasigeodesic ray converging to* $\xi$*. Then there are* $N \in \mathbb{N}$ *and* $D = D_{2.54}(\delta, \lambda, K) > 0$ *such that* $\gamma(n) \in N_D(A)$ *for all* $n \geq N$*.*

**Proof** Rather than explicitly computing the constants we indicate how to obtain them. Suppose $\{x_n\}$ is a sequence in $A$ such that $x_n \to \xi$. Let $y_1 \in \gamma$ be a 1-approximate nearest point projection of $x_1$ on $\gamma$. Let $\alpha_1$ denote a $(1,1)$-quasigeodesic joining $x_1, y_1$. Then the concatenation, say $\gamma_1$, of $\alpha_1$ and the segment of $\gamma$ from $y_1$ to $\xi$ is a uniform quasigeodesic by Corollary 2.26. For all $m > 1$, let $y_m$ denote a 1-approximate nearest point projection of $x_m$ on $\gamma_1$. Then $y_m$ is contained in $\gamma_1$ for all large $m$. However, once again by Corollary 2.26 the concatenation of the portion of $\gamma_1$ between $x_1, y_m$ and a 1-quasigeodesic joining $x_m, y_m$ is a uniform quasigeodesic. Now it follows by stability of quasigeodesics that the segment of $\gamma_1$ between $y_1, y_m$ is contained in a uniformly small neighborhood of $A$ since $A$ is quasiconvex. □

**Lemma 2.55** *Suppose* $X, Y$ *are hyperbolic metric spaces, and* $f\colon Y \to X$ *is any metrically proper map. Suppose that the CT map exists for* $f$*. Then we have* $\Lambda(f(Y)) = \partial f(\partial Y)$ *in each of the following cases:*

(1) $Y$ *is a proper metric space.*

(2) $f$ *is a qi embedding.*

**Proof** (1) It is clear that $\partial f(\partial Y) \subset \Lambda(f(Y))$. Suppose $y_n$ is any sequence such that $f(y_n) \to \xi$ for some $\xi \in \partial_s X$. Since $f$ is proper $\{y_n\}$ is an unbounded sequence. Since $Y$ is a proper length space it is a geodesic metric space by Hopf–Rinow theorem (see [6], Proposition 3.7, Chapter I.3). Now it is a standard fact that any unbounded sequence in a proper geodesic metric space has a subsequence converging to a point of the Gromov boundary of the space. Since $Y$ is proper, we have a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \eta$ for some $\eta \in \partial_s Y$. It is clear that $\partial f(\eta) = \xi$. Hence $\Lambda(f(Y)) \subset \partial f(\partial Y)$.

(2) Let $y \in Y$ and $x = f(y)$. Suppose $\{y_n\}$ is a sequence of points in $Y$ such that

$$\lim_{m,n\to\infty} (f(y_m).f(y_n))_x = \infty \quad \text{and} \quad \eta = [\{f(y_m)\}].$$

Then by Lemma 2.34 for any 1-quasigeodesic $\beta_{m,n}$ in $X$ joining $f(y_m), f(y_n)$ for all $m, n \in \mathbb{N}$, we have $\lim_{m,n\to\infty} d_X(x, \beta_{m,n}) = \infty$. Since $f$ is a qi embedding if $\alpha_{m,n}$ is a 1-quasigeodesic in $Y$ joining $y_m, y_n$ for all $m, n \in \mathbb{N}$ then $f(\alpha_{m,n})$ are uniform quasigeodesics in $X$. Hence, by stability of quasigeodesics in $X$ we have $Hd(f(\alpha_{m,n}), \beta_{m,n}) < D$ for some constant $D \geq 0$. Thus $\lim_{m,n\to\infty} d_X(x, f(\alpha_{m,n})) = \infty$. Since $f$ is a qi embedding and $x = f(y)$ it follows that $\lim_{m,n\to\infty} d_Y(y, \alpha_{m,n}) = \infty$. Therefore, $\lim_{m,n\to\infty}(y_m.y_n)_y = \infty$ again by Lemma 2.34. Hence, if $\xi = [\{y_n\}]$ then $\partial f(\xi) = \eta$. □

**Lemma 2.56** (projection of boundary points on quasiconvex sets) *Given $\delta \geq 0$ and $k \geq 0$ there is a constant $R = R_{2.56}(\delta, k)$ such that the following holds:*

*Suppose $X$ is a $\delta$-hyperbolic metric space, $A \subset X$ is $k$-quasiconvex and $\xi \in \partial X \setminus \Lambda(A)$. Then there is a point $x \in A$ with the following property: Suppose $\{x_n\}$ is any sequence where $x_n \to \xi$. Then there is an $N > 0$ such that for all $n \geq N$ we have $P_A(x_n) \in A \cap B(x, R)$.*

**Proof** Suppose $\{x_n\}, \{y_n\}$ are two sequences in $X$ such that $x_n \to \xi$ and $y_n \to \xi$. Let $\alpha_{m,n}$ be a 1-quasigeodesic in $X$ joining $x_m, y_n$ for all $m, n \in \mathbb{N}$. Let $P_A \colon X \to A$ be a 1-approximate nearest point projection on $A$.

**Claim** *There is a constant $R_0 > 0$ depending only on $\delta$ and $k$ and there is $N > 0$ such that*

$$\operatorname{diam}(P_A(\alpha_{m,n})) \leq R_0 \quad \textit{for all } m, n \geq N.$$

We first note that $\lim_{m,n\to\infty} d(A, \alpha_{m,n}) = \infty$. In fact, if this is not the case then there is $r > 0$ such that for all $N > 0$ there are $m_N, n_N \geq N$ with $d(A, \alpha_{m_N,n_N}) \leq r$. In that case let $a_N \in A$ be such that $d(a_N, \alpha_{m_N,n_N}) \leq r$. It is then clear that $a_N \to \xi$ by Lemma 2.40(1), contradicting the hypothesis that $\xi \notin \Lambda(A)$. By stability of quasigeodesics, any 1-quasigeodesic is uniformly quasiconvex in $X$ and $A$ is given to be $k$-quasiconvex. Hence, by Corollary 2.32 there are constants $D_0, R_0$ such that $d(A, \alpha_{m,n}) > D_0$ implies that $\operatorname{diam}(P_A(\alpha_{m,n})) \leq R_0$. Since, $\lim_{m,n\to\infty} d(A, \alpha_{m,n}) = \infty$ there is $N > 0$ such that $d(A, \alpha_{m,n}) > D_0$ for all $m, n \geq N$. This proves the existence of $N$ and $R_0$.

Now, by specializing the claim to the case $\{x_n\} = \{y_n\}$ we have $N_0 > 0$ such that if $\beta_{m,n}$ is a 1-quasigeodesic joining $x_m, x_n$ then $\operatorname{diam}(P_A(\beta_{m,n})) \leq R_0$ for all $m, n \geq N_0$. Let $x = P_A(x_{N_0})$. Now, given any sequence $\{x'_n\}$ in $X$ with $x'_n \to \xi$ by the claim there is $M > 0$ such that for all $m, n \geq M$, $d(P_A(x_m), P_A(x'_n)) \leq R_0$. Hence, if $N = \max\{N_0, M\}$ then

$$d(x, P_A(x'_n)) \leq d(x, P_A(x_N)) + d(P_A(x_N), P_A(x'_n)) \leq 2R_0.$$

Thus we can take $R = 2R_0$. □

Since the point $x \in A$ in the above lemma is coarsely unique we shall call any such point to be *the nearest point projection* of $\xi$ on $A$ and we shall denote it by $P_A(\xi)$.

## 3 Metric bundles

In this section, we recall necessary definitions and some elementary properties of the primary objects of study in this paper namely, metric bundles and metric graph bundles from [24]. We make a minor modification (see Definition 3.2) to the definition of a metric bundle but use the same definition of metric graph bundles as in [24].

### 3.1 Basic definitions and properties

**Definition 3.1** (metric bundles [24, Definition 1.2]) Suppose $(X, d)$ and $(B, d_B)$ are geodesic metric spaces; let $c \geq 1$ and let $\eta\colon [0,\infty) \to [0,\infty)$ be a function. We say that $X$ is an *$(\eta, c)$-metric bundle* over $B$ if there is a surjective 1-Lipschitz map $\pi\colon X \to B$ such that the following conditions hold:

(1) For each point $z \in B$, $F_z := \pi^{-1}(z)$ is a geodesic metric space with respect to the path metric $d_z$ induced from $X$. The inclusion maps $i\colon (F_z, d_z) \to X$ are uniformly metrically proper as measured by $\eta$.

(2) Suppose $z_1, z_2 \in B$, $d_B(z_1, z_2) \leq 1$ and let $\gamma$ be a geodesic in $B$ joining them. Then for any point $z \in \gamma$ and $x \in F_z$ there is a path $\tilde{\gamma}\colon [0,1] \to \pi^{-1}(\gamma) \subset X$ of length at most $c$ such that $\tilde{\gamma}(0) \in F_{z_1}$, $\tilde{\gamma}(1) \in F_{z_2}$ and $x \in \tilde{\gamma}$.

If $X$ is a metric bundle over $B$ in the above sense then we shall refer to it as a *geodesic metric bundle* in this paper. However, the above definition seems a little restrictive. Therefore, we propose the following.

**Definition 3.2** (length metric bundles) Suppose $(X, d)$ and $(B, d_B)$ are *length spaces*, $c \geq 1$ and we have a function $\eta\colon [0,\infty) \to [0,\infty)$. We say that $X$ is an *$(\eta, c)$-length metric bundle* over $B$ if there is a surjective 1-Lipschitz map $\pi\colon X \to B$ such that the following conditions hold:

(1) For each point $z \in B$, $F_z := \pi^{-1}(z)$ is a length space with respect to the path metric $d_z$ induced from $X$. The inclusion maps $i\colon (F_z, d_z) \to X$ are uniformly metrically proper as measured by $\eta$.

(2) Suppose $z_1, z_2 \in B$, and let $\gamma$ be a *path* of length at most 1 in $B$ joining them. Then for any point $z \in \gamma$ and $x \in F_z$ there is a path $\tilde{\gamma}\colon [0,1] \to \pi^{-1}(\gamma) \subset X$ of length at most $c$ such that $\tilde{\gamma}(0) \in F_{z_1}$, $\tilde{\gamma}(1) \in F_{z_2}$ and $x \in \tilde{\gamma}$.

Given length spaces $X$ and $B$ we will say that $X$ is a *length metric bundle* over $B$ if $X$ is an $(\eta, c)$-length metric bundle over $B$ in the above sense for some function $\eta\colon \mathbb{R}^+ \to \mathbb{R}^+$ and some constant $c \geq 1$.

**Convention 3.3** From now on whenever we speak of a metric bundle we mean a length metric bundle.

**Definition 3.4** (metric graph bundles [24, Definition 1.5]) Suppose $X$ and $B$ are metric graphs. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a function. We say that $X$ is an *$\eta$-metric graph bundle* over $B$ if there exists a surjective simplicial map $\pi : X \to B$ such that:

(1) For each $b \in \mathcal{V}(B)$, $F_b := \pi^{-1}(b)$ is a connected subgraph of $X$ and the inclusion maps $i : F_b \to X$ are uniformly metrically proper as measured by $\eta$ for the path metrics $d_b$ induced on $F_b$.

(2) Suppose $b_1, b_2 \in \mathcal{V}(B)$ are adjacent vertices. Then each vertex $x_1$ of $F_{b_1}$ is connected by an edge with a vertex in $F_{b_2}$.

**Remark** Since the map $\pi$ is simplicial it follows that it is 1-Lipschitz.

For a metric (graph) bundle the spaces $(F_z, d_z)$, $z \in B$ will be referred to as *fibers* and the $d_z$-distance between two points in $F_z$ will be referred to as their *fiber distance*. A geodesic in $F_z$ will be called a *fiber geodesic*. The spaces $X$ and $B$ will be referred to as the *total space* and the *base space* of the bundle respectively. By a statement of the form "$X$ is a metric bundle (resp. metric graph bundle)" we will mean that it is the total space of a metric bundle (resp. metric graph bundle).

Most of the results proved for geodesic metric bundles in [24] have their analogs for length metric bundles. We explicitly prove this phenomenon or provide sufficient arguments for all the results needed for our purpose.

**Convention 3.5** Very often in a lemma, proposition, corollary, or theorem we shall omit explicit mention of some of the parameters on which a constant may depend if the parameters are understood.

**Definition 3.6** Suppose $\pi : X \to B$ is a metric (graph) bundle.

(1) Suppose $A \subset B$ and $k \geq 1$. A *$k$-qi section* over $A$ is a $k$-qi embedding $s\colon A \to X$ (resp. $s\colon \mathcal{V}(A) \to X$) such that $\pi \circ s = \mathrm{Id}_A$ (resp. $\pi \circ s = \mathrm{Id}_{\mathcal{V}(A)}$), where $A$ has the restricted metric from $B$ and $\mathrm{Id}_A$ (resp. $\mathrm{Id}_{\mathcal{V}(A)}$) denotes the identity map on $A \to A$ (resp. $\mathcal{V}(A) \to \mathcal{V}(A)$).

(2) Given any metric space (resp. graph) $Z$ and any qi embedding $f\colon Z \to B$ (resp. $f : \mathcal{V}(Z) \to \mathcal{V}(B)$) a *$k$-qi lift* of $f$ is a $k$-qi embedding $\tilde{f}\colon Z \to X$ (resp. $\tilde{f}\colon \mathcal{V}(Z) \to \mathcal{V}(X)$) such that $\pi \circ \tilde{f} = f$.

**Convention 3.7** (1) Most of the time we shall refer to the image of a qi section (or a qi lift) to be the qi section (resp. the qi lift).

(2) Suppose $\gamma\colon I \to B$ is a (quasi)geodesic and $\tilde{\gamma}$ is a qi lift of $\gamma$. Let $b = \gamma(t)$ for some $t \in I$. Then we will denote $\tilde{\gamma}(t)$ by $\tilde{\gamma}(b)$ also.

(3) In the context of a metric graph bundle $(X, B, \pi)$, when we talk about a point in $X$, $B$ or a fiber, we mean that the point is a vertex in the corresponding space.

The following lemma is immediate from the definition of a metric (graph) bundle. Hence we briefly indicate its proof.

**Lemma 3.8** (path lifting lemma) *Suppose $\pi\colon X \to B$ is an $(\eta, c)$-metric bundle or an $\eta$-metric graph bundle.*

1. *Suppose $b_1, b_2 \in B$. Suppose $\gamma\colon [0, L] \to B$ is a continuous, rectifiable, arc length parametrized path (resp. an edge path) in $B$ joining $b_1$ to $b_2$. Given any $x \in F_{b_1}$ there is a path $\tilde{\gamma}$ in $\pi^{-1}(\gamma)$ such that $l(\tilde{\gamma}) \le (L+1)c$ (resp. $l(\tilde{\gamma}) = L$) joining $x$ to some point of $F_{b_2}$.*
   *In particular, when $X$ is a metric graph bundle over $B$, any geodesic $\gamma$ of $B$ can be lifted to a geodesic starting from any given point of $\pi^{-1}(\gamma)$.*
2. *For any $k \ge 1$ and $\epsilon \ge 0$, any* **dotted** *$(k, \epsilon)$-quasigeodesic $\beta\colon [m, n] \to B$ has a lift $\tilde{\beta}$ starting from any point of $F_{\beta(m)}$ such that the following hold, where we assume $c = 1$ for metric graph bundles. For all $i, j \in [m, n]$ we have*
$$-\epsilon + \frac{1}{k}|i-j| \le d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \le c \cdot (k+\epsilon+1)|i-j|.$$
   *In particular it is a $c \cdot (k+\epsilon+1)$-qi lift of $\beta$. Also we have*
$$l(\tilde{\beta}) \le ck(k+\epsilon+1)(\epsilon + d_B(b_1, b_2)).$$

**Proof** (1) We fix a sequence of points $0 = t_0, t_1, \dots, t_n = L$ in $[0, L]$ such that $l(\gamma|_{[t_i, t_{i+1}]}) = 1$ for $0 \le i < n-1$ and $l(\gamma|_{[t_{n-1}, t_n]}) \le 1$ for the metric bundle case. For the metric graph bundle $\gamma(t_i)$ are the consecutive vertices on $\gamma$, $0 \le i \le L = n$. Now given any $x =: x_0 \in F_{t_0}$ we can inductively construct a sequence of points $x_i \in F_{t_i}$, $0 \le i \le n$ and a sequence of paths $\alpha_i$ of length at most $c$ (resp. an edge) joining $x_i$ to $x_{i+1}$ for $0 \le i \le n-1$. Concatenation of these paths gives a candidate for $\tilde{\gamma}$.

The second statement for metric graph bundles follow because $\pi\colon X \to B$ is a 1-Lipschitz map.

(2) We construct a lift $\tilde{\beta}$ of $\beta$ starting from any point $x \in F_{\beta(m)}$ inductively as follows. We know that $d_B(\beta(i), \beta(i+1)) \le k + \epsilon$. Let $\beta_i$ be a path in $B$ joining $\beta(i)$ to $\beta(i+1)$ which is of length at most $k+\epsilon+1$ for $m \le i \le n-1$. We can then find a sequence of paths of length at most $(k+\epsilon+1) \cdot c$ in $\pi^{-1}(\beta_i)$ (where $c = 1$ for metric graph bundle) $m \le i \le n-1$ using the first part of the lemma such that

$\beta_m$ starts at $x$ and $\beta_{i+1}$ starts at the end point of $\beta_i$ for $m+1 \le i \le n-1$. Let $x_i$ be the starting point of $\beta_i$ for $m \le i \le n-1$ and let $x_n$ be the end point of $\beta_{n-1}$. Then we define $\tilde{\beta}$ by setting $\tilde{\beta}(i) = x_i$, $m \le i \le n$.

Clearly $d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \le c \cdot (k+\epsilon+1)|i-j|$. Also,

$$d_B(\pi \circ \tilde{\beta}(i), \pi \circ \tilde{\beta}(j)) = d_B(\beta(i), \beta(j)) \le d_X(\tilde{\beta}(i), \tilde{\beta}(j))$$

since $\pi$ is 1-Lipschitz. Since $\beta$ is a dotted $(k,\epsilon)$ quasigeodesic, we have $-\epsilon + \frac{1}{k}|i-j| \le d_B(\beta(i), \beta(j))$. This proves that

$$-\epsilon + \frac{1}{k}|i-j| \le d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \le c \cdot (k+\epsilon+1)|i-j|.$$

For the last part of (2) we see that

$$l(\tilde{\beta}) = \sum_{i=m}^{n-1} d_X(\tilde{\beta}(i), \tilde{\beta}(i+1)) \le \sum_{i=m}^{n-1} c \cdot (k+\epsilon+1) = (n-m)c \cdot (k+\epsilon+1).$$

On the other hand since $\beta$ is a $(k,\epsilon)$-quasigeodesic we have $-\epsilon + \frac{1}{k}(n-m) \le d_B(b_1, b_2)$. The conclusion immediately follows from these two inequalities. □

The following corollary follows from the proof of Proposition 2.10 of [24]. We include it for the sake of completeness.

**Corollary 3.9** *Given any metric (graph) bundle $\pi\colon X \to B$ and $b_1, b_2 \in B$ we can define a map $\phi\colon F_{b_1} \to F_{b_2}$ such that $d_X(x, \phi(x)) \le 3c + 3c d_B(b_1, b_2)$ (resp. $d(x, \phi(x)) = d_B(b_1, b_2)$) for all $x \in F_{b_1}$.*

**Proof** The statement about the metric graph bundle is trivially true by Lemma 3.8(1). For the metric bundle case, fix a dotted 1-quasigeodesic $\gamma$ joining $b_1$ to $b_2$. Then for all $x \in F_{b_1}$ fix for once and all a dotted lift $\tilde{\gamma}$ as constructed in the proof of Lemma 3.8 which starts from $x$ and set $\phi(x) = \tilde{\gamma}(b_2)$. The statement then follows from Lemma 3.8(2). □

**Remark** For all $b_1, b_2 \in B$ any map $f\colon F_{b_1} \to F_{b_2}$ such that $d_X(x, f(x)) \le D$ for some constant $D$ independent of $x$ will be referred to as a *fiber identification map*.

The proof of the first part of the following lemma is immediate from Corollary 3.9 whereas the next two parts essentially follow from the proof of Proposition 2.10 of [24]. Hence we skip the proofs.

**Lemma 3.10** *Suppose $\pi\colon X \to B$ is an $(\eta, c)$-metric bundle or an $\eta$-metric graph bundle and $R \ge 0$. Suppose $b_1, b_2 \in B$. The we have the following.*

(1) $Hd(F_{b_1}, F_{b_2}) \le 3c + 3c d_B(b_1, b_2)$ (resp. $Hd(F_{b_1}, F_{b_2}) = d_B(b_1, b_2)$).

(2) *Suppose* $\phi_{b_1b_2}\colon F_{b_1} \to F_{b_2}$ *is a map such that for all* $x \in F_{b_1}$, $d(x, \phi_{b_1b_2}(x)) \le R$ *for all* $x \in F_{b_1}$. *Then* $\phi_{b_1b_2}$ *is a* $K_{3.10} = K_{3.10}(R)$*-quasiisometry which is* $D_{3.10}$*-surjective.*

(3) *If* $\psi_{b_1b_2}\colon F_{b_1} \to F_{b_2}$ *is any other map such that* $d(x, \psi_{b_1b_2}(x)) \le R'$ *for all* $x \in F_{b_1}$ *then* $d(\phi_{b_1b_2}, \psi_{b_1b_2}) \le \eta(R + R')$.
*In particular, the maps* $\phi_{b_1b_2}$ *are coarsely unique* (*see Definition 2.1*(7)).

In this lemma, we deliberately suppress the dependence of $K_{3.10}$ on the parameter(s) of the bundle.

**Corollary 3.11** *Suppose* $\pi\colon X \to B$ *is a metric* (*graph*) *bundle and* $b_1, b_2 \in B$ (*resp.* $b_1, b_2 \in \mathcal{V}(B)$) *such that* $d_B(b_1, b_2) \le R$. *Suppose* $\phi_{b_1b_2}\colon F_{b_1} \to F_{b_2}$ *is a fiber identification map as constructed in the proof of Corollary 3.9. Then* $\phi_{b_1b_2}$ *is a* $K_{3.11} = K_{3.11}(R)$*-quasiisometry.*

**Proof** By Corollary 3.9 $d_X(x, \phi_{b_1b_2}(x)) \le 3c + 3cd_B(b_1, b_2) \le 3c + 3cR$ for all $x \in F_{b_1}$ (resp. $d_X(x, \phi_{b_1b_2}(x)) = d_B(b_1, b_2) \le R$ for all $x \in \mathcal{V}(B)$). Hence by Lemma 3.10(2) $\phi_{b_1b_2}$ is $K_{3.11} = K_{3.10}(3c{+}3cR)$-qi for the metric bundle and $K_{3.11} = K_{3.10}(R)$-qi for the metric graph bundle case. □

The following corollary is proved as a simple consequence of Lemma 3.10 and Corollary 3.9. (See Corollaries 1.14 and 1.16 of [24].) Therefore, we skip the proof of it.

**Corollary 3.12** (bounded flaring condition) *For all* $k \in \mathbb{R}$, $k \ge 1$ *there is a function* $\mu_k\colon \mathbb{N} \to \mathbb{N}$ *such that the following holds*:

*Suppose* $\pi\colon X \to B$ *is an* $(\eta, c)$*-metric bundle or an* $\eta$*-metric graph bundle. Let* $\gamma \subset B$ *be a dotted* $(1,1)$*-quasigeodesic* (*resp. a geodesic*) *joining* $b_1, b_2 \in B$, *and let* $\tilde\gamma_1$, $\tilde\gamma_2$ *be two k-qi lifts of* $\gamma$ *in* $X$. *Suppose* $\tilde\gamma_i(b_1) = x_i \in F_{b_1}$ *and* $\tilde\gamma_i(b_2) = y_i \in F_{b_2}$, $i = 1, 2$.

*Then*

$$d_{b_2}(y_1, y_2) \le \mu_k(N) \max\{d_{b_1}(x_1, x_2), 1\}.$$

*if* $d_B(b_1, b_2) \le N$.

In the rest of the paper, we will summarize the conclusion of Corollary 3.12 by saying that a metric (graph) bundle satisfies the *bounded flaring condition*.

**Remark** (metric bundles in the literature) Metric (graph) bundles appear in several places in other people's work. In [5, Section 2.1] Bowditch defines *stacks of* (*hyperbolic*) *spaces* which can easily be shown to be quasiisometric to metric graph bundles over an interval in $\mathbb{R}$. Conversely, a metric (graph) bundle whose base is an interval in $\mathbb{R}$ is clearly a stack of spaces as per [5, Section 2.1]. In [26] Whyte defines *coarse bundles* which are also quasiisometric to metric graph bundles but with additional restrictions.

## 3.2 Some natural constructions of metric bundles

In this section, we discuss a few general constructions that produce metric (graph) bundles.

**Definition 3.13** (1) **Metric bundle morphisms** Suppose $(X_i, B_i, \pi_i)$, $i = 1, 2$ are metric bundles. A morphism from $(X_1, B_1, \pi_1)$ to $(X_2, B_2, \pi_2)$ (or simply from $X_1$ to $X_2$ when there is no possibility of confusion) consists of a pair of coarsely $L$-Lipschitz maps $f: X_1 \to X_2$ and $g: B_1 \to B_2$ for some $L \geq 0$ such that $\pi_2 \circ f = g \circ \pi_1$, ie this diagram is commutative:

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[r, "g"'] & B_2
\end{tikzcd}$$

(2) **Metric graph bundle morphisms** Suppose $(X_i, B_i, \pi_i)$, $i = 1, 2$ are metric graph bundles. A morphism from $(X_1, B_1, \pi_1)$ to $(X_2, B_2, \pi_2)$ (or simply from $X_1$ to $X_2$ when there is no possibility of confusion) consists of a pair of coarsely $L$-Lipschitz maps $f: \mathcal{V}(X_1) \to \mathcal{V}(X_2)$ and $g: \mathcal{V}(B_1) \to \mathcal{V}(B_2)$ for some $L \geq 0$ such that $\pi_2 \circ f = g \circ \pi_1$.

(3) **Isomorphisms** A morphism $(f, g)$ from a metric (graph) bundle $(X_1, B_1, \pi_1)$ to a metric (graph) bundle $(X_2, B_2, \pi_2)$ is called an isomorphism if there is a morphism $(f', g')$ from $(X_2, B_2, \pi_2)$ to $(X_1, B_1, \pi_1)$ such that $f'$ is a coarse inverse of $f$ and $g'$ is a coarse inverse of $g$.

We note that for any morphism $(f, g)$ from a metric (graph) bundle $(X_1, B_1, \pi_1)$ to a metric (graph) bundle $(X_2, B_2, \pi_2)$ we have $f(\pi_1^{-1}(b)) \subset \pi_2^{-1}(g(b))$ for all $b \in B_1$. We will denote by $f_b: \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ the restriction of $f$ to $\pi_1^{-1}(b)$ for all $b \in B_1$. We shall refer to these maps as the *fiber maps* of the morphisms. We also note that in the case of metric graph bundles coarse Lipschitzness is equivalent to Lipschitzness.

**Lemma 3.14** *Given $k \geq 1$, $K \geq 1$ and $L \geq 0$ there are constants $L_{3.14}, K_{3.14}$ such that the following holds.*

*Suppose $(f, g)$ is a morphism of metric (graph) bundles as in the definition above. Then the following hold:*

1. *For all $b \in B_1$ the map $f_b: \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ is coarsely $L_{3.14}$-Lipschitz with respect to the induced length metric on the fibers.*
2. *Suppose $\gamma: I \to B_1$ is a dotted $(1, 1)$-quasigeodesic (or simply a geodesic in the case of a metric graph bundle) and suppose $\tilde{\gamma}$ is a $k$-qi lift of $\gamma$. If $g$ is a $K$-qi embedding then $f \circ \tilde{\gamma}$ is a $K_{3.14} = K_{3.14}(k, K, L)$-qi lift of $g \circ \gamma$.*

**Proof** We shall check the lemma only for the metric bundle case because for metric graph bundles the proofs are similar and in fact easier.

Suppose $\pi_i \colon X_i \to B_i$, $i = 1, 2$ are $(\eta_i, c_i)$-metric bundles.

(1) Let $b \in B_1$ and $x, y \in \pi_1^{-1}(b)$ be such that $d_b(x, y) \le 1$. Since $f$ is coarsely $L$-Lipschitz, $d_{X_2}(f(x), f(y)) \le L + L d_{X_1}(x, y) \le L + L d_b(x, y) \le 2L$. Now, the fibers of $\pi_2$ are uniformly properly embedded as measured by $\eta_2$. Hence, $d_{g(b)}(f(x), f(y)) \le \eta_2(2L)$. Therefore, by Lemma 2.6 the fiber map $f_b \colon \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ is $\eta_2(2L)$-coarsely Lipschitz. Hence, $L_{3.14} = \eta_2(2L)$ will do.

(2) Let $\gamma_2 = g \circ \gamma$ and $\tilde{\gamma}_2 = f \circ \tilde{\gamma}$. Then clearly, $\pi_2 \circ \tilde{\gamma}_2 = \gamma_2$ whence $\tilde{\gamma}_2$ is a lift of $\gamma_2$. By Lemma 2.3(1) $\tilde{\gamma}_2 = f \circ \tilde{\gamma}$ is coarsely $(kL, kL+L)$-Lipschitz. Hence, for all $s, t \in I$ we have

$$d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \le kL|s-t| + (kL + L).$$

On the other hand, for $s, t \in I$ we have

$$d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \ge d_{B_2}(\pi_2 \circ \tilde{\gamma}_2(s), \pi_2 \circ \tilde{\gamma}_2(t)) = d_{B_2}(\gamma_2(s), \gamma_2(t)).$$

However, by Lemma 2.3(2) $\gamma_2 = g \circ \gamma$ is a $(K, 2K)$-qi embedding. Hence, we have

$$d_{X_2}(\tilde{\gamma}_2(s), \tilde{\gamma}_2(t)) \ge d_{B_2}(\gamma_2(s), \gamma_2(t)) \ge -2K + \frac{1}{K}|s-t|.$$

Therefore, it follows that $\tilde{\gamma}_2$ is a $K_{3.14} = \max\{2K, kL + L\}$-qi lift of $\gamma_2$. □

The following theorem characterizes isomorphisms of metric (graph) bundles.

**Theorem 3.15** *If $(f, g)$ is an isomorphism of metric (graph) bundles as in the above definition then the maps $f, g$ are quasiisometries and all the fiber maps are uniform quasiisometries.*

*Conversely, if the map $g$ is a qi and the fiber maps are uniform qi then $(f, g)$ is an isomorphism.*

**Proof** We shall prove the theorem in the case of a metric bundle only. The proof in the case of a metric graph bundle is very similar and hence we skip it.

If $(f, g)$ is an isomorphism then $f, g$ are qi by Lemma 2.2(1). We need to show that the fiber maps are quasiisometries.

Suppose $(f', g')$ is a coarse inverse of $(f, g)$ such that

$$d_{X_2}(f \circ f'(x_2), x_2) \le R \quad \text{and} \quad d_{X_1}(f' \circ f(x_1), x_1) \le R$$

for all $x_1 \in X_1$ and $x_2 \in X_2$. It follows that for all $b_1 \in B_1, b_2 \in B_2$ we have $d_{B_1}(b_1, g' \circ g(b_1)) \le R$ and $d_{B_2}(b_2, g \circ g'(b_2)) \le R$ since the maps $\pi_1, \pi_2$ are 1-Lipschitz. Suppose $f', g'$ are coarsely $L'$-Lipschitz. Let $L_1 = \eta_2(2L)$ and $L_2 = \eta_1(2L')$. Then for all $u \in B_1$, $f_u \colon \pi_1^{-1}(u) \to \pi_2^{-1}(g(u))$ is coarsely $L_1$-Lipschitz and for all $v \in B_2$, $f'_v \colon \pi_2^{-1}(v) \to \pi_1^{-1}(g'(v))$ is coarsely $L_2$-Lipschitz by Lemma 3.14(1).

Let $b \in B_1$. To show that $f_b \colon \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ is a uniform quasiisometry, it is enough by Lemma 2.2(1) to find a uniformly coarsely Lipschitz map $\pi_2^{-1}(g(b)) \to \pi_1^{-1}(b)$ which is uniform coarse inverse of $f_b$. We already know that $f'_{g(b)}$ is $L_2$-coarsely Lipschitz. Let $b_1 = g' \circ g(b)$. We also noted that $d_{B_1}(b, b_1) \le R$. Hence, it follows by Corollaries 3.9 and 3.11 that we have a $K_{3.10}(R)$-qi $\phi_{b_1 b} \colon \pi_1^{-1}(b_1) \to \pi_1^{-1}(b)$ such that $d_{X_1}(x, \phi_{b_1 b}(x)) \le 3c_1 + 3c_1 R$ for all $x \in \pi_1^{-1}(b_1)$. Let $h = \phi_{b_1 b} \circ f'_{g(b)}$. We claim that $h$ is a uniformly coarsely Lipschitz, uniform coarse inverse of $f_b$. Since $f'_{g(b)}$ is $L_2$-coarsely Lipschitz and clearly $\phi_{b_1 b}$ is $K_{3.10}(R)$-coarsely Lipschitz, it follows by Lemma 2.3(1) that $h$ is $(L_2 K_{3.10}(R) + K_{3.10}(R))$-coarsely Lipschitz.

Moreover, for all $x \in \pi_1^{-1}(b)$ we have

$$d_{X_1}(x, h \circ f_b(x)) \le d_{X_1}(x, f'_{g(b)} \circ f_b(x)) + d_{X_1}(f'_{g(b)} \circ f_b(x), h \circ f_b(x)) \le R + 3c_1 + 3c_1 R.$$

Hence, $d_b(x, h \circ f_b(x)) \le \eta_1(R + 3c_1 + 3c_1 R)$. Let $y \in \pi_2^{-1}(g(b))$. Then

$$\begin{aligned} d_{X_2}(y, f_b \circ h(y)) = d_{X_2}(y, f \circ \phi_{b_1 b} \circ f'(y)) &\le d_{X_2}(y, f \circ f'(y)) + d_{X_2}(f \circ f'(y), f \circ \phi_{b_1 b} \circ f'(y)) \\ &\le R + L(3c_1 + 3c_1 R) + L, \end{aligned}$$

since $d_{X_1}(f'(y), \phi_{b_1 b} \circ f'(y)) \le 3c_1 + 3c_1 R$. Hence, $d_{g(b)}(y, f_b \circ h(y)) \le \eta_2(R + L(3c_1 + 3c_1 R) + L)$. Hence by Lemma 2.2(1), $f_b$ is a uniform qi.

Conversely, suppose all the fiber maps of the morphism $(f, g)$ are $(\lambda, \epsilon)$-qi which are $R$-coarsely surjective and $g$ is a $(\lambda_1, \epsilon_1)$-qi which is $R_1$-surjective. Let $g'$ be a coarsely $(K, C)$-quasiisometric, $D$-coarse inverse of $g$ where $K = K_{2.2}(\lambda_1, \epsilon_1, R_1)$, $C = C_{2.2}(\lambda_1, \epsilon_1, R_1)$ and $D = D_{2.2}(\lambda_1, \epsilon_1, R_1)$. For all $u \in B_1$, let $\bar{f}_u$ be a $D_1$-coarse inverse of $f_u \colon F_u \to F_{g(u)}$. We will define a map $f' \colon X_2 \to X_1$ such that $(f', g')$ is morphism from $X_2$ to $X_1$ and $f'$ is a coarse inverse of $f$ as follows.

For all $u \in B_2$ we define $f'_u \colon F_u \to F_{g'(u)}$ as the composition $\bar{f}_{g'(u)} \circ \phi_{ug(g'(u))}$ where $\phi_{ug(g'(u))}$ is a fiber identification map as constructed in the proof of Corollary 3.9. Collectively this defines $f'$. Now we shall check that $f'$ satisfies the desired properties.

(i) We first check that $(f', g')$ is a morphism. It is clear from the definition that $\pi_1 \circ f' = g' \circ \pi_2$. Hence we will be done by showing that $f'$ is coarsely Lipschitz. By Lemma 2.6 it is enough to show that for all $u_2, v_2 \in B_2$ and $x \in F_{u_2}$, $y \in F_{v_2}$ with $d_{X_2}(x, y) \le 1$, $d_{X_1}(f'(x), f'(y))$ is uniformly small. We note that $d_{B_2}(u_2, v_2) \le 1$. Let $u_1 = g'(u_2)$ and $v_1 = g'(v_2)$. Then $d_{B_1}(u_1, v_1) \le K + C$, $d_{B_2}(u_2, g(u_1)) \le D$ and $d_{B_2}(v_2, g(v_1)) \le D$. This means $d_{X_2}(x, \phi_{u_2 g(u_1)}(x)) \le 3Dc_2 + 3c_2$ and $d_{X_2}(y, \phi_{v_2 g(v_1)}) \le 3Dc_2 + 3c_2$ by Lemma 3.8 and Corollary 3.9. Hence,

$$d_{X_2}(\phi_{u_2 g(u_1)}(x), \phi_{v_2 g(v_1)}(y)) \le 1 + 6c_2 + 6Dc_2.$$

Let $x_2 = \phi_{u_2 g(u_1)}(x)$, $y_2 = \phi_{v_2 g(v_1)}(y)$, $x_1 = f'(x_2) = \bar{f}_{g(u_1)}(x_2)$ and $y_1 = f'(y_2) = \bar{f}_{g(v_1)}(y_2)$. Therefore, $d_{X_2}(x_2, y_2) \le 1 + 6c_2 + 6Dc_2 = R_2$, say and we want to show that $d_{X_1}(x_1, y_1)$ is uniformly small. Let $x'_2 = f(x_1) = f_{u_1}(x_1)$, $y'_2 = f(y_1) = f_{v_1}(y_1)$. Then $d_{X_2}(x_2, x'_2) \le D_1$ and $d_{X_2}(y_2, y'_2) \le D_1$.

Hence, $d_{X_2}(x'_2, y'_2) \le R_2 + 2D_1$. Since $d_{B_1}(u_1, v_1) \le K + C$ there is a point $y'_1 \in F_{u_1}$ such that $d_{X_1}(x_1, y'_1) \le (K+C)c_1 + c_1$. Hence, $d_{X_2}(x'_2, f(y'_1)) \le ((K+C)c_1 + c_1).L + L$. Hence,

$$d_{X_2}(f(y'_1), y'_2) \le d_{X_2}(f(y'_1), x'_2) + d_{X_2}(x'_2, y'_2) \le ((K+C)c_1 + c_1).L + L + 2D_1 + R_2.$$

This implies that $d_{v_2}(f(y'_1), f(y_1)) \le \eta_2(((K+C)c_1 + c_1).L + L + 2D_1 + R_2) = D_2$, say. Since $f_{v_1}$ is a $(\lambda, \epsilon)$-qi we have $-\epsilon + \frac{1}{\lambda} d_{v_1}(y_1, y'_1) \le D_2$. Hence, $d_{v_1}(y_1, y'_1) \le (\epsilon + D_2)\lambda$. Thus,

$$d_{X_1}(x_1, y_1) \le d_{X_1}(x_1, y'_1) + d_{X_1}(y'_1, y_1) \le (K+C)c_1 + c_1 + (\epsilon + D_2)\lambda.$$

(ii) We already know that $g'$ is a coarse inverse of $g$. Hence we will be done by checking that $f'$ is a coarse inverse of $f$. We will check only that $d(f' \circ f, \mathrm{Id}_{X_1}) < \infty$ leaving the proof of $d(f \circ f', \mathrm{Id}_{X_2}) < \infty$ for the reader. Suppose $b \in B_1$ and $x \in \pi_1^{-1}(b)$. Then

$$f'(f(x)) = \bar{f}_{g' \circ g(b)} \circ \phi_{g(b) g \circ g'(g(b))} \circ f_b(x).$$

We want to show that $d_{X_1}(x, f'(f(x)))$ is uniformly small. Let $h = f_{g' \circ g(b)} \circ \bar{f}_{g' \circ g(b)}$. Then

$$\begin{aligned} &d_{X_2}\big(f(x), f\big(f'(f(x))\big)\big) \\ &\qquad = d_{X_2}(f_b(x), h \circ \phi_{g(b) g \circ g'(g(b))} \circ f_b(x)) \\ &\qquad \le d_{X_2}\big(f_b(x), \phi_{g(b) g \circ g'(g(b))}(f_b(x))\big) + d_{X_2}\big(\phi_{g(b) g \circ g'(g(b))}(f_b(x)), h \circ \phi_{g(b) g \circ g'(g(b))} \circ f_b(x)\big). \end{aligned}$$

Now since, $d(g \circ g', \mathrm{Id}_{B_2}) \le D$,

$$d_{X_2}\big(f_b(x), \phi_{g(b) g \circ g'(g(b))}(f_b(x))\big) \le 3Dc_2 + 3c_2.$$

Since $d(h, \mathrm{Id}_{F_{g(g'(g(b)))}}) \le D_1$ we have $d_{X_2}\big(\phi_{g(b) g \circ g'(g(b))}(f_b(x)), h \circ \phi_{g(b) g \circ g'(g(b))} \circ f_b(x)\big) \le D_1$. Thus $d_{X_2}\big(f(x), f\big(f'(f(x))\big)\big) \le 3Dc_2 + 3c_2 + D_1$. Hence, it is enough to show that $f$ is a proper embedding. Here is how this is proved. Suppose $b, b' \in B$, $x \in \pi_1^{-1}(b)$ and $x' \in \pi_1^{-1}(b')$. Suppose $d_{X_2}(f(x), f(x')) \le N$ for some $N \ge 0$. This implies $d_{B_2}(g(b), g(b')) = d_{B_2}(\pi_2 \circ f(x), \pi_2 \circ f(x')) \le N$. Since $g$ is a $(\lambda_1, \epsilon_1)$-qi we have $-\epsilon_1 + d_{B_1}(b, b')/\lambda_1 \le N$, ie $d_{B_1}(b, b') \le (N + \epsilon_1)\lambda_1 = N_1$, say. Hence by Corollary 3.9 there is a point $x'' \in \pi_1^{-1}(b')$ such that $d_{X_1}(x, x'') \le 3N_1c_1 + 3c_1$. Since $f$ is coarsely $L$-Lipschitz we have $d_{X_2}(f(x), f(x'')) \le L(3N_1c_1 + 3c_1) + L$. It follows that

$$d(f(x'), f(x'')) \le d(f(x'), f(x)) + d(f(x), f(x'')) \le N + L(3N_1c_1 + 3c_1) + L = N_2,$$

say. Hence, $d_{g(b')}(f(x'), f(x'')) \le \eta_2(N_2)$. Since $f_{b'}$ is a $(\lambda, \epsilon)$-qi we have $d_{X_1}(x', x'') \le d_{b'}(x', x'') \le \lambda(\epsilon + \eta_2(N_2))$. Hence, $d_{X_1}(x, x') \le d_{X_1}(x, x'') + d_{X_1}(x', x'') \le 3N_1c_1 + 3c_1 + \lambda(\epsilon + \eta_2(N_2))$. □

**Definition 3.16** (subbundle) Suppose $(X_i, B, \pi_i)$, $i = 1, 2$ are metric (graph) bundles with the same base space $B$. We say that $(X_1, B, \pi_1)$ is subbundle of $(X_2, B, \pi_2)$ or simply $X_1$ is a subbundle of $X_2$ if there is a metric (graph) bundle morphism $(f, g)$ from $(X_1, B, \pi_1)$ to $(X_2, B, \pi_2)$ such that all the fiber maps $f_b$, $b \in B$ are uniform qi embeddings and $g$ is the identity map on $B$ (resp. on $\mathcal{V}(B)$).

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "f^Y"] \arrow[rd, dashed, "f'"] \arrow[rdd, bend right, "\pi_2"'] & & \\
 & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X \arrow[d, "\pi"] \\
 & B_1 \arrow[r, "g"'] & B
\end{tikzcd}$$

Figure 1

The most important example of a subbundle that concerns us is that of ladders which we discuss in a later section. The following gives another way to construct a metric (graph) bundle. We omit the proof since it is immediate.

**Lemma 3.17** (restriction bundle) *Suppose $\pi\colon X \to B$ is a metric (graph) bundle and $A \subset B$ is a connected subset such that any pair of points in $A$ can be joined by a path of finite length in $A$ (resp. $A$ is a connected subgraph). Then the restriction of $\pi$ to $Y = \pi^{-1}(A)$ gives a metric (graph) bundle with the same parameters as that of $\pi\colon X \to B$ where $A$ and $Y$ are given the induced length metrics from $B$ and $X$, respectively.*

*Moreover, if $f\colon Y \to X$ and $g\colon A \to B$ are the inclusion maps then $(f, g)\colon (Y, A) \to (X, B)$ is a morphism of metric (graph) bundles.*

**Definition 3.18** (1) **Pullback of a metric bundle** Given a metric bundle $(X, B, \pi)$ and a coarsely Lipschitz map $g\colon B_1 \to B$ a pullback of $(X, B, \pi)$ under $g$ is a metric bundle $(X_1, B_1, \pi_1)$ together with a morphism $(f\colon X_1 \to X,\ g\colon B_1 \to B)$ such that the following universal property holds: Suppose $\pi_2\colon Y \to B_1$ is another metric bundle and $(f^Y, g)$ is a morphism from $Y$ to $X$. Then there is a coarsely unique morphism $(f', \mathrm{Id}_{B_1})$ from $Y$ to $X_1$ making the diagram of Figure 1 commutative.

(2) **Pullback of a metric graph bundle** In the case of a metric graph bundle, the diagram is replaced by one where we have the vertex sets instead of the whole spaces.

The following lemma follows by a standard argument.

**Lemma 3.19** *Suppose we have a metric bundle $(X, B, \pi)$ and a coarsely Lipschitz map $g\colon B_1 \to B$ for which there are two pullbacks ie metric bundles $(X_i, B_1, \pi_i)$ together with morphisms*

$$(f_i\colon X_i \to X,\ g\colon B_1 \to B), \quad i = 1, 2$$

*satisfying the universal property of Definition 3.18. Then there is a coarsely unique metric (graph) bundle isomorphism from $X_1$ to $X_2$.*

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "f^Y"] \arrow[rd, dashed, "f'"] \arrow[rdd, bend right, "\pi_2"'] & & \\
& X_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & X \\
& B_1 &
\end{tikzcd}$$

Figure 2

With the above lemma in mind, in the context of Definition 3.18, we say that $f\colon X_1 \to X$ is the pullback of $X$ under $g\colon B_1 \to B$ or simply $X_1$ is *the* pullback of $X$ under $g$ when all the other maps are understood.

**Lemma 3.20** *Suppose we are given $L \geq 0$ and functions $\phi_1, \phi_2\colon [0,\infty) \to [0,\infty)$ and the commutative diagram of maps between metric spaces shown in Figure 2 satisfies these three properties*:

1. *All the maps* (*except possibly $f'$*) *are coarsely $L$-Lipschitz.*
2. *If $d_{B_1}(b,b') \leq N$ then $Hd(\pi_1^{-1}(b), \pi_1^{-1}(b')) \leq \phi_1(N)$ for all $b, b' \in B_1$ and $N \in [0,\infty)$.*
3. *The restrictions of $f$ on the fibers of $\pi_1$ are uniformly properly embedded as measured by $\phi_2$.*

*Then there is a function $\phi\colon [0,\infty) \to [0,\infty)$ such that $d_Y(y,y') \leq R$ implies $d_{X_1}(f'(y), f'(y')) \leq \phi(R)$ for all $y', y \in Y$ and $R \in [0,\infty)$. In particular, if $Y$ is a length space or the vertex set of a connected metric graph with restricted metric then $f'$ is coarsely $\phi(1)$-Lipschitz.*

*Moreover, $f'$ is coarsely unique, ie there is a constant $D > 0$ such that if $f''\colon Y \to X_1$ is another map making the above diagram commutative then $d(f', f'') \leq D$.*

**Proof** Suppose $y, y' \in Y$ with $d_Y(y,y') \leq R$. Let $x = f'(y), x' = f'(y')$. Then $d_{B_1}(\pi_1(x), \pi_1(x')) = d_{B_1}(\pi_2(y), \pi_2(y')) \leq LR + L$. Let $b = \pi_2(y), b' = \pi_2(y')$. Then

$$Hd(\pi_1^{-1}(b), \pi_1^{-1}(b')) \leq \phi_1(LR + L) = R_1,$$

say. Let $x_1' \in \pi_1^{-1}(b')$ be such that $d_{X_1}(x, x_1') \leq R_1$. Then $d_X(f(x), f(x_1')) \leq LR_1 + L$. On the other hand $d_X(f(x), f(x')) = d_X(f^Y(y), f^Y(y')) \leq LR + L$. By triangle inequality, we have $d_X(f(x'), f(x_1')) \leq LR + L + LR_1 + L = 2L + RL + R_1 L$. Hence, by the hypothesis (3) of the lemma $d_{X_1}(x', x_1') \leq \phi_2(2L + RL + R_1 L)$. Thus

$$d_{X_1}(x, x') \leq d_{X_1}(x, x_1') + d_{X_1}(x', x_1') \leq R_1 + \phi_2(2L + RL + R_1 L).$$

Hence, we may choose $\phi(t) = \phi_1(Lt + L) + \phi_2(2L + tL + L\phi_1(Lt + L))$.

If $Y$ is a length space or the vertex set of a connected metric graph, it follows by Lemma 2.6 that $f'$ is coarsely $\phi(1)$-Lipschitz.

Lastly, suppose $f'': Y \to X_1$ is another map making the diagram commutative. In particular we have $f^Y = f \circ f' = f \circ f''$. Hence for all $y \in Y$ we have $f(f'(y)) = f(f''(y))$. Since $\pi_1(f'(y)) = \pi_1(f''(y)) = \pi_2(y)$ by the hypothesis (3) of the lemma it follows that $d_{X_1}(f'(y), f''(y)) \le \phi_2(0)$. Hence $d(f', f'') \le \phi_2(0)$. □

**Remark** We note that the condition (2) of the lemma above holds when $\pi_1: X_1 \to B_1$ is a metric (graph) bundle.

**Proposition 3.21** (pullbacks of metric bundles) *Suppose $(X, B, \pi)$ is a metric bundle and $g: B_1 \to B$ is a Lipschitz map. Then there is a pullback.*

*More precisely the following hold: Suppose $X_1$ is the set theoretic pullback with the induced length metric from $X \times B_1$ and let $\pi_1: X_1 \to B_1$ be the projection on the second coordinate and let $f: X_1 \to X$ be the projection on the first coordinate. Then:*

1. *$\pi_1: X_1 \to B_1$ is metric bundle and $f$ is a coarsely Lipschitz map so that $(f, g)$ is a morphism from $X_1$ to $X$.*
2. *$f: X_1 \to X$ is the metric bundle pullback of $X$ under $g$.*
3. *All the fiber maps $f_b: \pi_1^{-1}(b) \to \pi^{-1}(g(b))$, $b \in B_1$ are isometries with respect to induced length metrics from $X_1$ and $X$, respectively.*

**Proof** By definition $X_1 = \{(x,t) \in X \times B_1 : g(t) = \pi(x)\}$. We put on it the induced length metric from $X \times B_1$. Let $\pi_1: X_1 \to B_1$ be the restriction of the projection map $X \times B_1 \to B_1$ to $X_1$. We first show that $X_1$ is a length space. Suppose $g$ is $L$-Lipschitz. Let $(x,s), (y,t) \in X_1$. Let $\alpha$ be a rectifiable path joining $s, t$ in $B_1$. Then $g \circ \alpha$ is a rectifiable path in $B$ of length at most $l(\alpha)L$. By Lemma 3.8 and Corollary 3.9 this path can be lifted to a rectifiable path in $X$ starting from $x$ and ending at some point say $z$ in $F_t$ such that the length of the path is at most $3c + 3cLl(\alpha)$. By construction this lift is contained in $X_1$. Finally we can join $(y,t), (z,t)$ by a rectifiable path in $F_t$. This show that $(x,s)$ and $(y,t)$ can be joined in $X_1$ by a rectifiable path. This proves that $X_1$ is a length space. Now, since $\pi_1^{-1}(t) = \pi^{-1}(g(t))$ is uniformly properly embedded in $X$ for all $t \in B_1$ and $X$ is properly embedded in $X \times B_1$, $\pi_1^{-1}(t)$ is uniformly properly embedded in $X_1$ for all $t \in B_1$. The same argument also shows that any path in $B_1$ of length at most 1 can be lifted to a path of length at most $3c + 3cL$ verifying the condition (2) of metric bundles. Hence $(X_1, B_1, \pi_1)$ is a metric bundle. Let $f: X_1 \to X$ be the restriction of the projection map $X \times B_1 \to X$ to $X_1$. Clearly $f: X_1 \to X$ is a morphism of metric bundles. Finally, we check the universal property. If there is a metric bundle $\pi_2: Y \to B_1$ and a morphism $(f^Y, g)$ from $Y$ to $X$ then there is a map $f': Y \to X_1$ making the diagram 1 commutative since we are working with the set theoretic pullback. That $f'$ is a coarsely unique, coarsely Lipschitz map now follows from Lemma 3.20. In fact, condition (2) of that lemma follows from Lemma 3.10(1) since $\pi_1: X_1 \to B_1$ is a metric bundle and (3) follows because fibers of metric bundles are uniformly properly embedded and in this case the restriction of $f$, $\pi_1^{-1}(b) \to \pi^{-1}(g(b)) \subset X$ is an isometry with respect to the induced path metric on $\pi_1^{-1}(b)$ and $\pi^{-1}(g(b))$ for all $b \in B_1$. □

$$\begin{tikzcd}
X_2 \arrow[drr, bend left, "f_2"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "\pi_2"'] & & \\
& X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X \arrow[d, "\pi"] \\
& B_1 \arrow[r, "g"'] & B
\end{tikzcd}$$

Figure 3

**Corollary 3.22** *Suppose* $(X, B, \pi)$ *is a metric bundle and* $g\colon B_1 \to B$ *is a Lipschitz map. Suppose* $\pi_2\colon X_2 \to B_1$ *is an arbitrary metric bundle and* $(f_2\colon X_2 \to X, g)$ *is a morphism of metric bundles. If* $X_2$ *is the pullback of* $X$ *under* $g$ *and* $f_2\colon X_2 \to X$ *is the pullback map then for all* $b \in B_1$ *the fiber map* $(f_2)_b\colon \pi_2^{-1}(b) \to \pi^{-1}(g(b))$ *is a uniform quasiisometry with respect to the induced length metrics on the fibers of* $\pi_2$ *and* $\pi$*, respectively.*

**Proof** Suppose $X_1$ is the pullback of $X$ under $g$ as constructed in the proof of Proposition 3.21. Then the fiber maps $f_b\colon \pi_1^{-1}(b) \to \pi^{-1}(g(b))$ are isometries with respect to the induced metrics on the fibers of $\pi_1$ and $\pi$, respectively. On the other hand by Lemma 3.19 there is a coarsely unique metric bundle isomorphism $(h, \mathrm{Id})$ from $X_2$ to $X_1$ making the diagram of Figure 3 commutative.

Now, by Theorem 3.15 the fiber maps $h_b\colon \pi_2^{-1}(b) \to \pi_1^{-1}(b)$ are uniform quasiisometries with respect to the induced length metrics on the fibers of $\pi_2$ and $\pi_1$, respectively. Since $(f_2)_b = f_b \circ h_b$ for all $b \in B_1$ are done by Lemma 2.3(2). □

**Example** Suppose $(X, B, \pi)$ is a metric bundle and $B_1 \subset B$ which is path connected and such that with respect to the path metric induced from $B$, $B_1$ is a length space. Let $X_1 = \pi^{-1}(B_1)$ be endowed with the induced path metric from $X$. Let $\pi_1\colon X_1 \to B_1$ be the restriction of $\pi$ to $X_1$. Let $g\colon B_1 \to B$ and $f\colon X_1 \to X$ be the inclusion maps. It is clear that $(X_1, B_1, \pi_1)$ is a metric bundle and also that $X_1$ is the pullback of $g$.

**Remark** The notion of morphisms of metric bundles was implicit in the work of Whyte [26]. Along the line of [26], one can define a more general notion of metric bundles by relaxing the hypothesis of length spaces. In that category of spaces, pullbacks should exist under any coarsely Lipschitz maps. However, we do not delve into it here.

**Proposition 3.23** (pullbacks for metric graph bundles) *Suppose* $(X, B, \pi)$ *is an* $\eta$*-metric graph bundle,* $B_1$ *is a metric graph and* $g\colon \mathcal{V}(B_1) \to \mathcal{V}(B)$ *is a coarsely* $L$*-Lipschitz map for some constant* $L \geq 1$*. Then there is a pullback* $\pi_1\colon X_1 \to B_1$ *of* $g$ *such that all the fiber maps* $f_b\colon \pi_1^{-1}(b) \to \pi^{-1}(g(b))$*,* $b \in \mathcal{V}(B_1)$ *are isometries with respect to induced length metrics from* $X_1$ *and* $X$*, respectively.*

$$\begin{tikzcd}
\mathcal{V}(Y) \arrow[dr, dashed, "f'"'] \arrow[drr, bend left, "f^Y"] \arrow[ddr, bend right, "\pi_2"'] & & \\
& \mathcal{V}(X_1) \arrow[r, "f"] \arrow[d, "\pi_1"'] & \mathcal{V}(X) \arrow[d, "\pi"] \\
& \mathcal{V}(B_1) \arrow[r, "g"'] & \mathcal{V}(B)
\end{tikzcd}$$

Figure 4

**Proof** The proof is a little long. Hence we break this into steps for the sake of clarity.

**Step 1** (*construction of* $X_1$ *and* $\pi_1\colon X_1 \to B_1$ *and* $f\colon \mathcal{V}(X_1) \to \mathcal{V}(X)$) We first construct a metric graph $X_1$, a candidate for the total space of the bundle. The vertex set of $X_1$ is the disjoint union of the vertex sets of $\pi^{-1}(g(b))$, $b \in \mathcal{V}(B_1)$. There are two types of edges. First of all for all $b \in \mathcal{V}(B_1)$, we take all the edges appearing in $\pi^{-1}(g(b))$. In other words, the full subgraph $\pi^{-1}(g(b))$ is contained in $X_1$. Let us denote that by $F_b$. For all adjacent vertices $s, t \in B_1$ we introduce some other edges with one end point in $F_s$ and the other in $F_t$. We note that $F_s, F_t \subset X_1$ are identical copies of $F_{g(s)}$ and $F_{g(t)}$, respectively. Let $f_s\colon F_s \to F_{g(s)}$ denote this identification. Let $e$ be an edge joining $s, t$ and let $\alpha$ be a geodesic in $B$ joining $g(s), g(t)$. Now for each $x \in F_s$ we lift the path $\alpha$ starting from $f_s(x)$ isometrically by Lemma 3.8(1) to say $\tilde{\alpha}$. For each such lift we join $x$ by an edge to $y \in V(F_t)$ if and only if $f_t(y) = \tilde{\alpha}(g(t))$. This completes the construction of $X_1$. We note that $d_B(g(s), g(t)) \le 2L$ and hence $l(\tilde{\alpha}) \le 2L$ too. Now we define $f\colon \mathcal{V}(X_1) \to \mathcal{V}(X)$ by setting $f(x) = f_{\pi_1(x)}(x)$ for all $x \in \mathcal{V}(X_1)$. It is clear that this map is $2L$-Lipschitz.

**Step 2** ($\pi_1\colon X_1 \to B_1$ *is a metric graph bundle and* $(f, g)$ *is a morphism*) We need to verify that the fibers are uniformly properly embedded in $X_1$ so that $X_1$ is a metric graph bundle. Suppose $x, y \in F_s$ and $d_{X_1}(x, y) \le D$. Let $\alpha$ be a (dotted) geodesic in $X_1$ joining $x, y$. Then $f \circ \alpha$ is a (dotted) path of length at most $2LD$. Thus $d_X(f(x), f(y)) \le 2LD$. Since $X$ is an $\eta$-metric graph bundle $d_{g(s)}(f(x), f(y)) \le \eta(2LD)$. Since $f$ is an isometry when restricted to $F_s$ we have $d_s(x, y) \le \eta(2LD)$. This proves that $X_1$ is a metric graph bundle over $B_1$.

On the other hand, $f$ is $2L$-Lipschitz by step 1 and $g$ is coarsely $L$-Lipschitz by hypothesis. It is also clear that $\pi \circ f = g \circ \pi_1$ by the definition of $f$. Thus $(f, g)$ is a morphism of metric graph bundles from $X_1$ to $X$.

**Step 3** ($X_1$ *is a pullback*) Now we check that $X_1$ is a pullback of $X$ under $g$. Suppose $\pi_2\colon Y \to B_1$ is a metric graph bundle and $(f^Y, g)$ is a morphism of metric graph bundles from $Y$ to $X$ where $f^Y$ is coarsely $L_1$-Lipschitz We need to find a coarsely unique, coarsely Lipschitz map $f'\colon \mathcal{V}(Y) \to \mathcal{V}(X_1)$ such that $(f', \mathrm{Id})$ is a morphism from $Y$ to $X_1$ and the whole diagram of Figure 4 is commutative, where $\mathrm{Id}\colon \mathcal{V}(B_1) \to \mathcal{V}(B_1)$ is the identity map.

**The map $f'$** For all $s \in \mathcal{V}(B_1)$ we define $f'$ on $\mathcal{V}(\pi_2^{-1}(s))$ as the composition $f_s^{-1} \circ f_s^Y$. Collectively these maps define $f'$. It is clear that $f'$ makes the whole diagram above commutative.

The rest of the argument follows from Lemma 3.20. In fact, condition (2) of that lemma follows from Lemma 3.10(1) since $\pi_1 : X_1 \to B_1$ is a metric graph bundle and (3) follows because fibers of metric graph bundles are uniformly properly embedded and in this case the restriction of $f$, $\pi_1^{-1}(b) \to \pi^{-1}(g(b)) \subset X$ is an isometry with respect to the induced path metric on $\pi_1^{-1}(b)$ and $\pi^{-1}(g(b))$ for all $b \in \mathcal{V}(B_1)$. □

The corollary below follows immediately from the proof of the above proposition.

**Corollary 3.24** *Suppose $\pi : X \to B$ is a metric graph bundle. Suppose $A$ is a connected subgraph of $B$. Let $g : A \to B$ denote the inclusion map. Let $X_A = \pi^{-1}(A)$, $\pi_A$ be the restriction of $\pi$ and let $f : X_A \to X$ denote the inclusion map. Then $X_A$ is the pullback of $X$ under $g$.*

The proof of the following corollary is similar to that of Corollary 3.22 and hence we omit the proof.

**Corollary 3.25** *Suppose $(X, B, \pi)$ is a metric graph bundle and $g : \mathcal{V}(B_1) \to \mathcal{V}(B)$ is a coarsely Lipschitz map. Suppose $\pi_2 : X_2 \to B_1$ is an arbitrary metric graph bundle and $(f_2 : \mathcal{V}(X_2) \to \mathcal{V}(X), g)$ is a morphism of metric graph bundles. If $X_2$ is the pullback of $X$ under $g$ and $f_2 : \mathcal{V}(X_2) \to \mathcal{V}(X)$ is the pullback map then for all $b \in \mathcal{V}(B_1)$ the fiber map $(f_2)_b : \mathcal{V}(\pi_2^{-1}(b)) \to \mathcal{V}(\pi^{-1}(g(b)))$ is a uniform quasiisometry with respect to the induced length metrics on the fibers of $\pi_2$ and $\pi$, respectively.*

## 3.3 Some examples

In this section we discuss in detail two main sources of examples for metric graph bundles to which the main theorem of this paper will be applied.

**3.3.1 Short exact sequence of groups** Given a short exact sequence of finitely generated groups

$$1 \to N \to G \xrightarrow{\pi} Q \to 1$$

we have a naturally associated metric graph bundle. This is the main motivating example of metric graph bundles. We recall the definition from [24, Example 1.8] with a minor modification.

Suppose $H < Q$ is a finitely generated subgroup. Let $G_1 = \pi^{-1}(H)$. We fix a generating set $S_N$ of $N$, a generating set $S \supseteq S_N$ of $G$ such that $S$ contains a generating set $S_1$ of $G_1$, $S_N \subset S_1$ and $N \cap S = S_N$. Let $S_Q = \pi(S) \setminus \{1\}$ and $S_H = \pi(S_1) \setminus \{1\}$. Then we have a metric graph bundle $\pi : \Gamma(G, S) \to \Gamma(Q, S_Q)$. Clearly $\Gamma(H, S_H)$ is a subgraph of $\Gamma(Q, S_Q)$ and $\Gamma(G_1, S_1) = \pi^{-1}(\Gamma(H, S_H))$. Hence, by Corollary 3.24 it follows that $\Gamma(G_1, S_1)$ is the pullback of $\Gamma(G, S)$ under the inclusion $\Gamma(H, S_H) \hookrightarrow \Gamma(Q, S_Q)$.

**3.3.2 Complexes of groups** For this example, we refer to [15]. Suppose $\mathcal{Y}$ is a finite simplicial complex and $\boldsymbol{G}(\mathcal{Y})$ is a developable complex of groups defined over $\mathcal{Y}$. (See [15, Definition 2.2].) For any face $\sigma$ of $\mathcal{Y}$, let $K_\sigma$ be a $K(G_\sigma, 1)$-space. Then by [15, Theorem 3.4.1] there is a complex of spaces $p\colon \mathcal{X} \to \mathcal{Y}$ (compare with *good* complexes of spaces due to Corson [8]) which is a cellular aspherical realization (see [15, Definition 3.3.4]) of the complex of groups $\boldsymbol{G}(\mathcal{Y})$ such that inverse image under $p$ of the barycenter of each face $\sigma$ is $K_\sigma$. It follows from the construction of $\mathcal{X}$ that there is a continuous section $s$ of $p\colon \mathcal{X} \to \mathcal{Y}$ over the 1-skeleton $\mathcal{Y}^{(1)}$ of $\mathcal{Y}$. We fix a maximal tree of $s(\mathcal{Y}^{(1)})$ and a base vertex $v_0 \in \mathcal{Y}^{(0)}$ in it. Let $G = \pi_1(\mathcal{X}, s(v_0))$. Thus for any $v \in \mathcal{Y}^{(0)}$ we have a natural injective homomorphism $\pi_1(X_v, s(v)) \to G$. We identify the image of the same with $G_v$. Next following Corson [8] we take the universal cover $\pi_{\mathcal{X}}\colon \widetilde{\mathcal{X}} \to \mathcal{X}$. We put a CW complex structure on $\widetilde{\mathcal{X}}$ in the standard way so that $\pi_{\mathcal{X}}$ is a cellular map. Then for all $y \in \mathcal{Y}$, we collapse each connected component of $(p \circ \pi_{\mathcal{X}})^{-1}(y)$ to a point. Suppose $\mathcal{B}$ is the quotient complex thus obtained and let $q\colon \widetilde{\mathcal{X}} \to \mathcal{B}$ be the quotient map. Then we note that there is a cellular map $\bar{\pi}_{\mathcal{X}}\colon \mathcal{B} \to \mathcal{Y}$ making the following diagram commutative:

$$\begin{tikzcd}
\widetilde{\mathcal{X}} \arrow[r, "q"] \arrow[d, "\pi_{\mathcal{X}}"'] & \mathcal{B} \arrow[d, "\bar{\pi}_{\mathcal{X}}"] \\
\mathcal{X} \arrow[r, "p"'] & \mathcal{Y}
\end{tikzcd}$$

Now for our purpose, we shall also assume that all the face groups $G_\sigma$ are finitely generated, the 0-skeleton of each $K_\sigma$ is a point $x_\sigma$, the 1-skeleton is a wedge of finitely many circles and the developable complex of groups satisfies the qi condition as defined below.

**Definition 3.26** Suppose we have a developable complex of groups $(\mathcal{G}, \mathcal{Y})$.

(1) We say that it satisfies the *qi condition* if for any faces $\sigma \subset \tau$ of $\mathcal{Y}$ the corresponding homomorphism $G_\tau \to G_\sigma$ is an isomorphism onto a finite index subgroup of $G_\sigma$.

(2) If all the face groups of $G_\sigma$ satisfies a group theoretic property $\mathcal{P}$ then we shall say that $(\mathcal{G}, \mathcal{Y})$ is a developable complex of groups with property $\mathcal{P}$.

For instance, we shall work in Section 6 with the developable complexes of nonelementary hyperbolic groups.

However, we now aim to associate to the complex of groups a metric graph bundle as follows. Let $X' = (p \circ \pi_{\mathcal{X}})^{-1}(\mathcal{Y}^{(1)})^{(1)}$ and $B = \bar{\pi}_{\mathcal{X}}^{-1}(\mathcal{Y}^{(1)})^{(1)}$, where we denote by $\mathcal{Z}^{(1)}$ the 1-skeleton of any CW complex $\mathcal{Z}$. Now we construct a metric graph bundle $\pi\colon X \to B$ as follows. For all $v \in \mathcal{B}^{(0)}$ let $F_v := q^{-1}(v)^{(1)}$. Suppose $v, w \in \mathcal{B}^{(0)}$ are connected by an edge $e$. We look at the subcomplex $\widetilde{\mathcal{X}}_{[v,w]} = q^{-1}([v, w])$. Let $v_0 = \bar{\pi}_{\mathcal{X}}(v)$, $w_0 = \bar{\pi}_{\mathcal{X}}(w)$ and $e_0 = \bar{\pi}_{\mathcal{X}}(e)$. Then $\widetilde{\mathcal{X}}_{[v,w]} \subset \pi_{\mathcal{X}}^{-1}(p^{-1}([v_0, w_0])$. However, we recall from Haefliger [15] how $p^{-1}([v_0, w_0]) \subset \mathcal{X}$ is built from the spaces the $K_{v_0}$, $K_{w_0}$

and $K_{e_0}$. There are injective homomorphisms $G_{e_0} \to G_{v_0}$ and $G_{e_0} \to G_{w_0}$. We choose cellular maps $f_0 : K_{e_0} \to K_{v_0}$, and $f_1 : K_{e_0} \to K_{w_0}$ such that the induced maps in the fundamental groups are those group homomorphisms. Then one glues $K_{e_0} \times [0,1]$ to $K_{v_0} \bigsqcup K_{w_0}$ by gluing $K_{e_0} \times \{0\}$ to $K_{v_0}$ and $K_{e_0} \times \{1\}$ to $K_{w_0}$ using the maps $f_0, f_1$, respectively. Let $m_0$ be the midpoint of $x_{e_0} \times [0,1] \subset p^{-1}([v_0, w_0])$ and let $m \in e$ be the midpoint of $e$. Then through any $a \in q^{-1}(m)^{(0)}$ we lift $x_{e_0} \times [0,1]$. The lift is a 1-cell joining $a_v \in q^{-1}(v)^{(0)}$ to $a_w \in q^{-1}(w)^{(0)}$. Let us denote the map $a \mapsto a_v$ by $f_{e,v}$ and the map $a \mapsto a_w$ by $f_{e,w}$.

**Lemma 3.27** (1) *The map $f_{e,v} : q^{-1}(m)^{(0)} \to q^{-1}(v)^{(0)}$ is uniformly coarsely surjective with respect to the graph metric on $q^{-1}(m)^{(0)}$, $q^{-1}(v)^{(0)}$ coming from $q^{-1}(m)^{(1)}$, $q^{-1}(v)^{(1)}$, respectively.*

(2) *A similar statement holds for $f_{e,w}$.*

**Proof** We will only prove (1) as the proof of (2) is similar. The group $G_v < G$ is isomorphic to $G_{v_0}$ and $q^{-1}(v)$ is a universal cover of $K_{v_0}$ since the complex of groups is developable. The groups $G_v$ acts properly discontinuously with quotient $K_{v_0}$. Since the action is cellular the action of $G_v$ on $q^{-1}(v)^{(1)}$ is simply transitive. Similarly the action of $G_m$ is simply transitive on $q^{-1}(m)^{(1)}$. We note that $G_m < G_v$ and the map $f_{e,v}$ is equivariant. It is also clear that $[G_v : G_m] = [G_{v_0} : G_{e_0}]$. Finally we note that $q^{-1}(v)^{(1)}$ is naturally isometric to a Cayley graph of $G_v$ when $q^{-1}(v)^{(1)}$ is given graph metric where each edge has length 1. The lemma is immediate from this. □

Let $R > 0$ be such that $f_{e,v}$ is coarsely $R$-surjective for all 0-cell $v$ and 1-cell $e$ of $\mathcal{B}$ where $e$ is incident on $v$. Then we construct a graph $X$ from $X'$ by introducing new edges as follows. Given $v, w \in \mathcal{B}^{(0)}$ connected by an edge $e$ we join all $x \in q^{-1}(v)$ to $y \in q^{-1}(w)$ by an edge if there is $a \in q^{-1}(m_e)^{(0)}$ such that $d(x, f_{e,v}(a)) \le R$ and $d(y, f_{e,w}(a)) \le R$, where the distances are taken in the respective 1-skeletons of $q^{-1}(v)$ and $q^{-1}(w)$.

**Proposition 3.28** *Suppose we identify $G$ as the group of deck transformation on the covering map $\pi_{\mathcal{X}} : \widetilde{\mathcal{X}} \to \mathcal{X}$. Then we have the following:*

1. *$G$ acts on $X$ and on $B$ through simplicial maps. The map $q$ is $G$-equivariant.*
2. *The $G$-action is proper and cofinite on $X$ but it is only cofinite on $B$. Also $B/G$ is isomorphic to $\mathcal{Y}^{(1)}$.*
3. *For all $v \in \mathcal{Y}^{(0)}$ and $\tilde{v} \in \bar{\pi}_{\mathcal{X}}^{-1}(v)$, $G_{\tilde{v}}$ is a conjugate of $G_v$ in $G$.*
4. *The action of $G_{\tilde{v}}$ on $X_{\tilde{v}} = q^{-1}(\tilde{v})$ is proper and cocompact. In fact the action on $V(X_{\tilde{v}})$ is transitive and on $E(X_{\tilde{v}})$ is cofinite. In particular if the $G_v$ is hyperbolic for all $v \in \mathcal{Y}^{(0)}$ then for all $v \in \mathcal{Y}^{(0)}$ and $\tilde{v} \in \bar{\pi}_{\mathcal{X}}^{-1}(v)$, $X_{\tilde{v}}$ is uniformly hyperbolic.*
5. *$\pi : X \to B$ is a metric graph bundle.*

**Proof** The group $G$ acts through deck transformations of the covering map $\pi_{\mathcal{X}} : \widetilde{\mathcal{X}} \to \mathcal{X}$. Hence it follows that $G$ permutes the connected components of $(p \circ \pi_{\mathcal{X}})^{-1}(y)$ for all $y \in \mathcal{Y}$. The action is also simplicial.

Hence, (1) follows from this. For (2) we note that the action of $G$ on $X'$ is proper and cofinite. On the other hand, the inclusion map $X' \to X$ is a $G$-equivariant quasiisometry by Lemma 2.4. Hence the $G$-action on $X$ is proper and cofinite. Clearly, $B/G$ is isomorphic to $\mathcal{Y}^{(1)}$ whence the $G$-action on $B$ is cofinite. Property (3) is a consequence of a basic covering space argument using the $G$-equivariance of the map $q$. In (4) the properness follows from the properness of the action of $G$ on $X'$. Cocompactness is due to the fact that $X'/G$ is finite. The second part also follows from the nature of $K(G_v, 1)$ used to construct $\mathcal{X}$, where $v = \bar{\pi}_{\mathcal{X}}(\tilde{v})$. The last part follows from the second by Milnor–Schwarz lemma. What remains is to prove (5). For all $\tilde{v} \in V(B)$, let $X_{\tilde{v}} = \pi^{-1}(\tilde{v})$. Since $B/G$ is finite and the map $q$ is $G$-equivariant the $X_{\tilde{v}}$'s are uniformly properly embedded in $X'$ if and only if for all $w \in V(B/G)$ there is one $\tilde{w} \in \bar{\pi}_{\mathcal{X}}^{-1}(w)$ such that $X_{\tilde{w}}$ is uniformly properly embedded in $X'$. However, each inclusion $X_{\tilde{v}} \to X'$ is $G_{\tilde{v}}$-equivariant, the $G_{\tilde{v}}$ action on $X_{\tilde{v}}$ is proper and cocompact and $G_{\tilde{v}}$ is a finitely generated subgroup of $G$. Since each finitely generated subgroup of a finitely generated group is uniformly properly embedded it follows that $X_{\tilde{v}}$ is properly embedded in $X'$. Since $X'$ is quasiisometric to $X$, it follows that the $X_{\tilde{v}}$'s are properly embedded in $X$. This verifies property (1) of metric graph bundles. Property (2) follows from Lemma 3.27 and the construction of the new edges. □

**Subcomplexes of groups** In the above set-up we now assume further that we have a connected subcomplex $\mathcal{Y}_1 \subset \mathcal{Y}$. Let $\mathcal{X}_1 = p^{-1}(\mathcal{Y}_1)$. We shall assume that the base point $x_0 \in \mathcal{X}$ is contained in $\mathcal{X}_1$ and a maximal tree of $s(\mathcal{Y}_1^{(1)})$ is chosen so that it is contained in the chosen maximal tree of $s(\mathcal{Y}^{(1)})$. Suppose the inclusion $\mathcal{X}_1 \to \mathcal{X}$ is $\pi_1$-injective. Then the restriction $\boldsymbol{G}(\mathcal{Y}_1)$ of $\boldsymbol{G}(\mathcal{Y})$ to $\mathcal{Y}_1$ is a developable complex of groups by [6, Corollary 2.15]. Let $G_1 = \pi_1(\mathcal{X}_1, x_0)$. However, $\mathcal{X}_1 \to \mathcal{Y}_1$ is a complex of spaces which is a cellular aspherical realization of the complex of groups $\boldsymbol{G}(\mathcal{Y}_1)$. Hence, we can build a metric graph bundle $\pi_1 \colon X_1 \to B_1$ as described in Proposition 3.28.

In fact fixing a point $\tilde{x}_0 \in \pi_{\mathcal{X}}^{-1}(x_0)$ we may identify $G$ as the group of deck transformations on $\widetilde{\mathcal{X}}$. Then $G_1$ stabilizes the connected component of $\pi^{-1}(\mathcal{X}_1)$ containing $\tilde{x}_0$. Since $\mathcal{X}_1 \to \mathcal{X}$ is $\pi_1$-injective this connected component, say $\widetilde{\mathcal{X}}_1$, is a universal cover of $\mathcal{X}_1$. We set $B_1 = q(\widetilde{\mathcal{X}}_1) \cap B$ and $X_1 = \pi^{-1}(B_1)$. The following proposition records these in a nutshell.

**Proposition 3.29** *Suppose $\mathcal{Y}$ is a finite connected simplicial complex and $\boldsymbol{G}(\mathcal{Y})$ is a developable complex of groups with qi condition and with fundamental group $G$ and suppose $\mathcal{Y}_1$ is a connected subcomplex of $\mathcal{Y}$. Suppose $G_1$ is the fundamental group of $\boldsymbol{G}(\mathcal{Y}_1)$. Suppose the inclusion $\boldsymbol{G}(\mathcal{Y}_1) \to \boldsymbol{G}(\mathcal{Y})$ induces injective homomorphism $G_1 \to G$.*

*Then there is a metric graph bundle $\pi \colon X \to B$, a connected subgraph $B_1 \subset B$ such that the following hold:*

(1) *$G$ acts on $X$ and on $B$ through simplicial maps. The map $\pi$ is $G$-equivariant. The action is proper and cofinite on $X$ but it is only cofinite on $B$. Also, there is a simplicial $G$-equivariant map $B \to \mathcal{Y}^{(1)}$ with trivial action on $\mathcal{Y}^{(1)}$ inducing an isomorphism of graphs $B/G \to \mathcal{Y}^{(1)}$. The group*

*$G_b < G$ is a conjugate of $G_{\bar{b}}$ in $G$, where $\bar{b}$ is the image of $b$ under the map $B \to \mathcal{Y}^{(1)}$. Also the $G_b$-action on $F_b$ is proper and cofinite for all $b \in V(B)$.*

(2) *Let $X_1 = \pi^{-1}(B_1)$. Then $G_1$ stabilizes $X_1$ and the $G_1$-action on $X_1$ is proper and cofinite. Also the restriction of the map $B/G \to \mathcal{Y}^{(1)}$ to $B_1/G_1$ is an isomorphism of graphs $B_1/G_1 \to \mathcal{Y}_1^{(1)}$.*

Later on we shall work with rather special subcomplexes of groups as defined below.

**Definition 3.30** Suppose $\mathcal{Y}$ is a finite connected simplicial complex and $(\mathcal{G}, \mathcal{Y})$ is a developable complexes of groups with qi condition over $\mathcal{Y}$. We shall call a connected subcomplex $\mathcal{Y}_1 \subset \mathcal{Y}$ a *good* subcomplex if the following hold:

(1) The induced natural homomorphism $\pi_1(\mathcal{G}, \mathcal{Y}_1) \to \pi_1(\mathcal{G}, \mathcal{Y})$ is injective. Suppose the image is $G_1$.

(2) If $\pi : X \to B$ is a metric graph bundle obtained as in Proposition 3.28 from $(\mathcal{G}, \mathcal{Y})$ and $B_1 \subset B$ is as in Proposition 3.29. Then the inclusion $B_1 \subset B$ is a qi embedding.

We note that $X$ is quasiisometric to $G$ and $X_1$ is quasiisometric to $G_1$. Thus it follows that $B$ is quasiisometric to the "coned-off" space à la Farb [10] obtained from $G$ by coning off the cosets of the various face groups of $(\mathcal{G}, \mathcal{Y})$. Similarly $B_1$ is obtained by coning off various cosets of the face groups of $(\mathcal{G}, \mathcal{Y}_1)$. Thus condition (2) of the above definition is intrinsic and independent of the particular metric graph bundle obtained from $(\mathcal{G}, \mathcal{Y})$.

# 4 Geometry of metric bundles

In this section, we recall some results from [24] and also add a few of our own which are going to be useful for the proof of our main theorem in the next section. Especially some of the results which were stated for geodesic metric spaces in [24] but whose proofs require little adjustments to hold true for length spaces are mentioned here.

## 4.1 Metric graph bundles arising from metric bundles

An analog of the following result is proved in [24, Lemmas 1.17–1.21]. We give an independent and relatively simpler proof here. We also construct an approximating metric graph bundle morphism starting with a given metric bundle morphism. However, one disadvantage of our construction is that the metric graphs so obtained are never proper.

**Proposition 4.1** *Suppose $\pi' : X' \to B'$ is an $(\eta, c)$-metric bundle. Then there is a metric graph bundle $\pi : X \to B$ along with quasiisometries $\psi_B : B' \to B$ and $\psi_X : X' \to X$ such that*

(1) *$\pi \circ \psi_X = \psi_B \circ \pi'$ and*

(2) *for all $b \in B'$ the map $\psi_X$ restricted to $\pi'^{-1}(b)$ is a $(1,1)$-quasiisometry onto $\pi^{-1}(\psi_B(b))$.*

*Moreover, the maps $\psi_X, \psi_B$ have coarse inverses $\phi_X, \phi_B$, respectively, making the diagram of Figure 5 commutative.*

$$\begin{tikzcd}
X' \arrow[r, shift left, "\psi_X"] \arrow[d, "\pi'"'] & X \arrow[l, dashed, shift left, "\phi_X"] \arrow[d, "\pi"] \\
B' \arrow[r, shift left, "\psi_B"] & B \arrow[l, dashed, shift left, "\phi_B"]
\end{tikzcd}$$

Figure 5

**Proof** (1) For the proof we use the construction of Lemma 2.8. We shall briefly recall the construction of the spaces. We define $\mathcal{V}(B) = B'$ and $s, t \in \mathcal{V}(B)$ are connected by an edge if and only if $s \neq t$ and $d_{B'}(s,t) \leq 1$. This defines the graph. We also have a natural map $\psi_B \colon B' \to B$ which is just the inclusion map when $B'$ is identified with the vertex set of $B$. To define $X$, we take $\mathcal{V}(X) = X'$. Edges are of two types.

**Type 1 edges** For all $s \in B'$, $x, y \in \pi'^{-1}(s)$ are connected by an edge if and only if $d_s(x,y) \leq 1$.

**Type 2 edges** If $s \neq t \in B'$, $x \in \pi'^{-1}(s)$ and $y \in \pi'^{-1}(t)$ then $x, y$ are connected by an edge if and only if $d_{B'}(s,t) \leq 1$ and $d_{X'}(x,y) \leq c$.

The map $\psi_X \colon X' \to X$ is defined as before to be the inclusion map. By Lemma 2.8 $\psi_B$ is a qi. We also note that $\pi \circ \psi_X = \psi_B \circ \pi'$. We need to verify that $\psi_X$ is a qi. For that, it is enough to produce Lipschitz coarse inverses $\phi_X$, $\phi_B$ as claimed in the second part of the proposition and then apply Lemma 2.2 since it is clear that $\psi_X$ is 1-Lipschitz. We first choose a coarse inverse $\phi_B$ of $\psi_B$ as follows. On $\mathcal{V}(B)$ it is simply the identity map. The interior of each edge is then sent to one of its end points. The map $\phi_X$ on $\mathcal{V}(X)$ is also defined as the identity map. The interior of a type 1 edge is sent to one of its end points. Then interior of each type 2 edge $e = [x,y]$ is sent to one of the end points $x$ or $y$ according as the edge $\pi(e)$ is mapped by $\phi_B$ to $\pi(x)$ or $\pi(y)$, respectively. It follows that the diagram of Figure 5 commutes. We just need to check that $\phi_X$ is coarsely Lipschitz, since $\phi_B, \phi_X$ are inverses of $\psi_B, \psi_X$, respectively on a 1-dense subset, they will be coarse inverse automatically. However, by Lemma 2.6 it is enough to show that edges are mapped to small diameter sets. This is again clear. In fact, the image of an edge has diameter at most $c$. This proves the first part of the proposition.

(2) This is immediate from the definition of $\psi_X$ and the construction in Lemma 2.8.

(3) Finally, we need to check that $(X, B, \pi)$ is a metric graph bundle. Let $s \in B$ and $x, y \in \pi^{-1}(s)$ such that $d_X(x,y) \leq M$ for some $M > 0$. Since $\phi_X$ is a quasiisometry, $d_{X'}(x,y) \leq M'$, where $M' > 0$ depends on $M$ and $\phi_X$. Since $\pi'^{-1}(\phi_B(s))$ is properly embedded in $X'$ as measured by $\eta$, we have $d_{\phi_B(s)}(x,y) \leq \eta(M')$. Now, using the above fact that $\pi'^{-1}(\phi_B(s))$ is $(1,1)$-quasiisometric to $\pi^{-1}(s)$, we have $d_s(x,y) \leq \eta(M') + 1$. Hence, $\pi^{-1}(s)$ is uniformly properly embedded in $X$. Next we check the condition (2) of Definition 3.4. Suppose $s, t \in \mathcal{V}(B)$ are adjacent vertices. Then, $d_{B'}(s,t) \leq 1$. Let $\alpha$ be a path in $B'$ joining $s, t$ with $l_{B'}(\alpha) \leq 1$. Then, for any $x \in \pi'^{-1}(s)$, $\alpha$ can be lifted to a path of length at most $c$, joining $x$ to some $y \in \pi'^{-1}(t)$. Then there exists an edge joining $x$ and $y$ in $X$, which is a lift of the edge joining $s$ and $t$ in $B$. □

**Remark** We shall refer to the metric graph bundle $X$ obtained from $X'$ as the *canonical metric graph bundle* associated to the bundle $X$. Since we are working with length metric spaces some of the machinery of [24] may not apply directly. Proposition 4.1 then comes to the rescue. We sometimes modify our definitions suitably to make things work. Consequently, all the results proved for metric graph bundles have their close analogs in metric bundles. We shall make this precise for instance in Proposition 4.3 and Definition 4.5.

**Approximating a metric bundle morphism** Suppose $\pi' \colon X' \to B'$ is a metric bundle and $g \colon A' \to B'$ is a Lipschitz map. Suppose $Y'$ is the pullback of the bundle under the map $g$ as constructed in the proof of Proposition 3.21, ie $Y'$ is also the set theoretic pullback. Let $g^*\pi' \colon Y' \to A'$ be the corresponding bundle projection map and $f \colon Y' \to X'$ be the pullback map. Suppose we use the recipe of the above proposition to construct metric graph bundles $\pi_X : X \to B$, $\pi_Y : Y \to A$ with quasiisometries $\psi_A \colon A' \to A$, $\psi_B \colon B' \to B$, $\psi_Y \colon Y' \to Y$ and $\psi_X \colon X' \to X$ such that $\pi_Y \circ \psi_Y = \psi_A \circ g^*\pi'$ and $\pi_X \circ \psi_X = \psi_B \circ \pi'$.

Suppose $\phi_X, \phi_B, \phi_Y, \phi_A$ are the coarse inverses (as constructed in the proposition above) of $\psi_X$, $\psi_B$, $\psi_Y$, and $\psi_A$, respectively. We then have a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, dashed, shift right, "\phi_Y"'] \arrow[d, "\pi_Y"'] & Y' \arrow[l, shift right, "\psi_Y"'] \arrow[r, "f"] \arrow[d, "g^*\pi'"'] & X' \arrow[r, shift left, "\psi_X"] \arrow[d, "\pi'"'] & X \arrow[l, dashed, shift left, "\phi_X"] \arrow[d, "\pi_X"'] \\
A \arrow[r, dashed, shift left, "\phi_A"] & A' \arrow[l, shift left, "\psi_A"] \arrow[r, "g"'] & B' \arrow[r, shift right, "\psi_B"'] & B \arrow[l, dashed, shift right, "\phi_B"']
\end{tikzcd}$$

Let $\bar{f}, \bar{g}$ denote the restrictions of $\psi_X \circ f \circ \phi_Y$ and $\psi_B \circ g \circ \phi_A$ on the vertex sets of $Y$ and $A$, respectively.

**Proposition 4.2** (1) *The pair of maps $(\bar{f}, \bar{g})$ gives a morphism of metric graph bundles from $Y$ to $X$. Moreover, if $Y'$ is the pullback of $X'$ under $g$ and $f$ is the pullback map then $Y$ is the pullback of $X$ under $\bar{g}$ and $\bar{f}$ is the pullback map.*

(2) *If $X'$, $Y'$ are hyperbolic, then $f$ admits the CT map if and only if $\bar{f}$ does also.*

**Proof** (1) Since all the maps in consideration, ie $\psi_X, f, \phi_Y, \psi_B, g, \phi_A$ are coarsely Lipschitz the maps $\bar{f}, \bar{g}$ are also coarsely Lipschitz by Lemma 2.3(1). It also follows that $\pi_X \circ \bar{f} = \bar{g} \circ \pi_Y$. Thus $(\bar{f}, \bar{g})$ is a morphism.

Suppose $Y'$ is a the pullback of $X'$ under $g$. To show that $Y$ is the pullback of $X$ we need to verify the universal property. Suppose $\pi_1 \colon Y_1 \to A$ is any metric bundle and $f_1 \colon \mathcal{V}(Y_1) \to \mathcal{V}(X)$ is a coarsely Lipschitz map such that the pair $(f_1, \bar{g})$ is a morphism of metric graph bundles from $Y_1$ to $X$. We note that $\pi' \circ (\phi_X \circ f_1) = g \circ (\phi_A \circ \pi_1)$. Since $Y'$ is a set theoretic pullback there is a unique map $f_2 \colon \mathcal{V}(Y_1) \to Y'$ making the whole diagram of Figure 6 commutative.

$$\begin{tikzcd}
Y_1 \arrow[dr, "f_2"] \arrow[drrr, bend left, "f_1"] \arrow[dd, bend right, "\pi_1"'] & & & \\
Y \arrow[r, dashed, shift right, "\phi_Y"'] \arrow[d, "\pi_Y"] & Y' \arrow[l, shift right, "\psi_Y"'] \arrow[r, "f"] \arrow[d, "g^*\pi'"] & X' \arrow[r, shift left, "\psi_X"] \arrow[d, "\pi'"] & X \arrow[l, dashed, shift left, "\phi_X"] \arrow[d, "\pi_X"] \\
A \arrow[r, dashed, shift left, "\phi_A"] & A' \arrow[l, shift left, "\psi_A"] \arrow[r, "g"'] & B' \arrow[r, shift right, "\psi_B"'] & B \arrow[l, dashed, shift right, "\phi_B"']
\end{tikzcd}$$

Figure 6

Now, by Lemma 2.3(1) the maps $\phi_X \circ f_1$ and $\phi_A \circ \pi_1$ are coarsely Lipschitz. Hence, it follows by Lemma 3.20 and the subsequent Remark on page 2706 that the map $f_2$ is coarsely Lipschitz. Let $h = \psi_Y \circ f_2$. Then $h$ is coarsely Lipschitz by Lemma 2.3(1) and we have $\bar{f} \circ h = f_1$ and $\pi_Y \circ h = \pi_1$. Hence, $(h, \mathrm{Id}_A)$ is a morphism from $Y_1$ to $Y$. Finally coarse uniqueness of $h$ follows from Lemma 3.20.

(2) This is a simple application of Lemma 2.50. □

## 4.2 Metric bundles with hyperbolic fibers

For the rest of this section we shall assume that all our metric (graph) bundles $\pi\colon X \to B$ have the following property:

(†) Each of the fibers $F_b$, $b \in B$ (resp. $b \in \mathcal{V}(B)$) is a $\delta'$-hyperbolic metric space with respect to the path metric $d_b$ induced from $X$.

We will refer to this by saying that the metric (graph) bundle has uniformly hyperbolic fibers. Moreover, the following property is crucial for the existence of (global) qi sections:

(††) There is $N \geq 0$ such that for all $b \in B$ the barycenter map $\phi_b\colon \partial^3 F_b \to F_b$ is coarsely $N$-surjective. (Recall that barycenter maps were defined right after Lemma 2.41.)

**Proposition 4.3** (global qi sections for metric (graph) bundles [24, Propositions 2.10, 2.12]) *For all $\delta', c \geq 0$, $N \geq 0$ and $\eta\colon [0,\infty) \to [0,\infty)$ there exists $K_0 = K_0(c, \eta, \delta', N)$ such that the following holds.*

*Suppose $p\colon X' \to B'$ is an $(\eta, c)$-metric bundle or an $\eta$-metric graph bundle satisfying (†) and (††). Then there is a $K_0$-qi section over $B'$ through each point of $X'$ (where we assume $c = 1$ for the metric graph bundle).*

**Convention 4.4** (1) With the notation of Proposition 4.1, we note that for any qi section $\Sigma$ in $X$ over $B$, $\phi_X(\Sigma) = \Sigma$ since $\phi_X$ is the identity map when restricted to $\mathcal{V}(X)$. We shall refer to it as a qi section of the metric graph bundle *transported* to the metric bundle.

(2) Whenever we talk about a $K$-qi section in a metric bundle we shall mean that it is the transport of a $K$-qi section contained in the associated canonical metric graph bundle.

**Definition 4.5** [24, Definition 2.13] Suppose $\Sigma_1$ and $\Sigma_2$ are two $K$-qi sections of the metric graph bundle $X$. For each $b \in \mathcal{V}(B)$ we join the points $\Sigma_1 \cap F_b$, $\Sigma_2 \cap F_b$ by a geodesic in $F_b$. We denote the union of these geodesics by $\mathbb{L}(\Sigma_1, \Sigma_2)$, and call it a *K-ladder* (formed by the sections $\Sigma_1$ and $\Sigma_2$).

For a metric bundle by a ladder, we will mean one transported from the canonical metric graph bundle associated to it (by the canonical map $\phi_X$ as in Proposition 4.1.)

The following are the most crucial properties of a ladder summarized from [24].

**Proposition 4.6** *Given $K \geq 0$, $\delta \geq 0$ there are $C = C_{4.6}(K) \geq 0$, $R = R_{4.6}(K) \geq 0$ and $K_{4.6}(\delta, K) \geq 0$ such that the following holds*:

*Suppose $\pi : X \to B$ is an $\eta$-metric graph bundle satisfying* (†). *Suppose $\Sigma_1, \Sigma_2$ are two $K$-qi sections in $X$ and $\mathbb{L} = \mathbb{L}(\Sigma_1, \Sigma_2)$ is the ladder formed by them. Then the following hold*:

(1) ***Ladders are coarse Lipschitz retracts*** *There is a coarsely $C$-Lipschitz retraction $\pi_{\mathbb{L}} : X \to \mathbb{L}$ defined as follows*:

*For all $x \in X$ we define $\pi_{\mathbb{L}}(x)$ to be a nearest point projection of $x$ in $F_{\pi(x)}$ on $\mathbb{L} \cap F_{\pi(x)}$.*

(2) *Given a $k$-qi section $\gamma$ in $X$ over a geodesic in $B$, $\pi_{\mathbb{L}}(\gamma)$ is a $(C+2kC)$-qi section in $X$ contained in $\mathbb{L}$ over the same geodesic in $B$.*

(3) ***QI sections in ladders*** *If $X$ also satisfies* (††) *then through any point of $\mathbb{L}$ there is $(1+2K)C$-qi section contained in $\mathbb{L}$.*

(4) ***Quasiconvexity of ladders*** *The $R$-neighborhood of $\mathbb{L}$ is* (i) *connected and* (ii) *uniformly qi embedded in $X$.*

*In particular if $X$ is $\delta$-hyperbolic then $\mathbb{L}$ is $K_{4.6}(\delta, K)$-quasiconvex in $X$.*

**Proof** (1) This is stated as Theorem 3.2 in [24].

(2), (3) These are immediate from (1) or one can refer to Lemma 3.1 of [24].

(4) This is proved in Lemma 3.6 in [24] assuming (††). However, we briefly indicate the argument here without assuming (††).

(4)(i) Suppose $b, b' \in B$, $d_B(b, b') = 1$. Let $x \in \mathbb{L} \cap F_b$. Then there is a point $x' \in F_{b'}$ such that $d(x, x') = 1$. Hence, $d(\pi_{\mathbb{L}}(x), \pi_{\mathbb{L}}(x')) = d(x, \pi_{\mathbb{L}}(x')) \leq 2C$. If we define $R = 2C$ then clearly the $R$-neighborhood of $\mathbb{L}$ is connected.

(4)(ii) We first claim that the $N_R(\mathbb{L}) = Y$ say, is also properly embedded in $X$. Suppose $x', y' \in Y$ with $d_X(x', y') \leq N$. Let $x, y \in \mathbb{L}$ be such that $d(x, x') \leq R$ and $d(y, y') \leq R$. Then $d(x, y) \leq 2R + N$. Hence, $d_B(\pi(x), \pi(y)) \leq 2R + N$. Let $\alpha$ be a geodesic in $B$ joining $\pi(x), \pi(y)$. Then by Lemma 3.8

there is a geodesic lift $\tilde{\alpha}$ of $\alpha$ starting from $x$. It follows that for all adjacent vertices $b_1, b_2 \in \alpha$ we have $d(\pi_{\mathbb{L}}(\tilde{\alpha})(b_1), \pi_{\mathbb{L}}(\tilde{\alpha})(b_2)) \leq 2C$. Hence, the length of $\pi_{\mathbb{L}}(\tilde{\alpha})$ is at most $2C(2R+N)$. Hence,

$$d\big(y, \pi_{\mathbb{L}}\big(\tilde{\alpha}(\pi(y))\big)\big) \leq d(x,y) + d\big(x, \pi_{\mathbb{L}}\big(\tilde{\alpha}(\pi(y))\big)\big) \leq 2R+N+l(\pi_{\mathbb{L}}(\tilde{\alpha})) \leq 2R+N+2C(2R+N).$$

Hence, $d_{\pi(y)}\big(y, \pi_{\mathbb{L}}\big(\tilde{\alpha}(\pi(y))\big)\big) \leq \eta(2R+N+4CR+2CN)$. Since $\pi_{\mathbb{L}}(\tilde{\alpha}) \subset Y$,

$$d_Y(x,y) \leq d_{\pi(y)}\big(y, \pi_{\mathbb{L}}\big(\tilde{\alpha}(\pi(y))\big)\big) + l(\pi_{\mathbb{L}}(\tilde{\alpha})) \leq \eta(2R+N+4CR+2CN)+4CR+2CN.$$

Hence, $d_Y(x', y') \leq 4CR+2CN+\eta(2R+N+4CR+2CN)$.

Finally we prove the qi embedding. Let $f(N) = \eta(2R+N+4CR+2CN)+4CR+2CN$ for all $N \in \mathbb{N}$. Given $x, y \in \mathbb{L}$, $d_X(x,y) = n$ and a geodesic $\gamma: [0,n] \to X$ joining them. By the proof of (4)(i) we have $d_Y(\pi_{\mathbb{L}}(\gamma(i)), \pi_{\mathbb{L}}(\gamma(i+1))) \leq f(2C)$ for all $0 \leq i \leq n-1$, whence $d_{\mathbb{L}}(x,y) \leq nf(2C) = f(2C)d_X(x,y)$. Clearly $d_X(x,y) \leq d_{\mathbb{L}}(x,y)$. This proves the qi embedded part.

It follows that for all $x, y \in \mathbb{L}$ a geodesic joining $x, y$ in $Y$ is a $(f(2C), 0)$-quasigeodesic in $X$. Since $X$ is $\delta$-hyperbolic stability of quasigeodesics implies that $\mathbb{L}$ is uniformly quasiconvex. In fact, we can take $K_{4.6}(\delta, K) = R + D_{2.19}(\delta, f(2C), 0)$. □

**Remark** Part (3) and (4) are clearly also true for metric bundles which satisfy the properties (†) and (††).

The following corollary is immediate.

**Corollary 4.7** (ladders form subbundles) *Suppose $\pi: X \to B$ is an $\eta$-metric graph bundle satisfying (†) and (††). Let $C$, $R$ be as in the previous proposition. Suppose $\mathbb{L} = \mathbb{L}(\Sigma_1, \Sigma_2)$ is a $K$-ladder. Consider the metric graph $Z$ obtained from $\mathbb{L}$ by introducing some extra edges as follows: Suppose $b, b' \in B$ are adjacent vertices then for all $x \in \mathbb{L} \cap F_b$, $x' \in \mathbb{L} \cap F_{b'}$ we join $x, x'$ by an edge if and only if $d_X(x,x') \leq C + 2KC$. Let $\pi_Z: Z \to B$ be the simplicial map such that $\pi = \pi_Z$ on $\mathcal{V}(Z)$ and the extra edges are mapped isometrically to edges of $B$.*

*Then $Z$ is a metric graph bundle and the natural map $Z \to X$ gives a subbundle of $X$ which is also a (uniform) qi onto $N_R(\mathbb{L})$ and hence a (uniform) qi embedding in $X$.*

In the next section of the paper, we will exclusively deal with bundles $\pi: X \to B$ which are hyperbolic satisfying (†) and (††) and we will need to understand geodesics in $X$. Since ladders are quasiconvex we look for quasigeodesics contained in ladders. The lemma below is the last technical piece of information needed for that purpose. However, we need the following definitions for stating the lemma.

**Definition 4.8** Suppose $X$ is a metric graph bundle over $B$ and suppose $\Sigma_1, \Sigma_2$ are any two qi sections.

(1) **Neck of ladders** [24, Definition 2.16] Suppose $R \geq 0$. Then the set

$$U_R(\Sigma_1, \Sigma_2) = \{b \in B : d_b(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b) \leq R\}$$

is called the *$R$-neck* of the ladder $\mathbb{L}(\Sigma_1, \Sigma_2)$.

For a metric bundle the $R$-neck of a ladder will be defined to be the one transported from the canonical metric graph bundle associated to it, ie the image under $\phi_B$.

(2) **Girth of ladders** [24, Definition 2.15] The quantity $\min\{d_b(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b) : b \in B\}$ is called the *girth* of the ladder $\mathbb{L}(\Sigma_1, \Sigma_2)$ and it will be denoted by $d_h(\Sigma_1, \Sigma_2)$.

Motivated by the hallway flaring condition of Bestvina and Feighn [4], flaring conditions for metric (graph) bundles were defined in [24, Definition 1.12]. Below we slightly modify those definitions to suit to our context and to add a little more clarity.

**Definition 4.9** (flaring for metric graph bundles [24, Definition 1.12]) (1) Let $k \geq 1$ be a constant. We say that a metric graph bundle $\pi : X \to B$ satisfies a *flaring condition for k-qi lifts* if there exist constants $\nu > 1$, and $n, M \in \mathbb{N}$ such that the following holds:

Suppose $\gamma : [-n, n] \to B$ is any geodesic, and suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are any two $k$-qi lifts of $\gamma$ in $X$. If $d_{\gamma(0)}(\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \geq M$, then we have

$$\nu . d_{\gamma(0)}(\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \leq \max\bigl\{d_{\gamma(n)}(\tilde{\gamma}_1(n), \tilde{\gamma}_2(n)),\ d_{\gamma(-n)}(\tilde{\gamma}_1(-n), \tilde{\gamma}_2(-n))\bigr\}.$$

(2) We say that the metric graph bundle $\pi : X \to B$ satisfies a *flaring condition* if it satisfies a flaring condition for $k$-qi lifts for all $k \geq 1$.

We note that the assertion that a metric graph bundle "satisfies a flaring condition" means that for any $k \geq 1$ there are three constants $\nu > 1$, and $n, M \in \mathbb{N}$ (depending on $k$) with the said property in Definition 4.9(1). However, when we wish to emphasize the dependence of these three numbers on $k$, we say that the metric bundle satisfies a $(\nu_k, M_k, n_k)$*-flaring condition.* This property is independent of the hypotheses about metric graph bundles and the conditions (†) and (††) mentioned in the beginning of this subsection.

**Definition 4.10** (flaring for metric bundles) We shall say that a metric bundle $\pi : X \to B$ satisfies a $(\nu_k, M_k, n_k)$*-flaring* condition if the canonical metric graph bundle associated to it satisfies a $(\nu_k, M_k, n_k)$-flaring condition.

**Remark** (1) Since the base for a metric bundle need not be a geodesic metric space, it is not reasonable to use [24, Definition 1.12] of flaring for metric bundles. However, one can formulate analogous flaring of qi sections over uniform quasigeodesics in the base and then show that this is indeed equivalent to Definition 4.10. Since this discussion is not directly related to the rest of the paper we move it to the end of the paper and we include it as an appendix. See Lemmas A.5 and A.6.

(2) This definition of flaring for metric bundles is equivalent to [24, Definition 1.12] in the case of geodesic metric bundles. In fact it follows from Lemmas A.5 and A.6 that a geodesic metric bundle satisfies flaring as per [24, Definition 1.12] if and only if the canonical metric graph bundle associated to it also satisfies flaring.

The following lemma will be crucial for the next section of the paper.

**Lemma 4.11** (quasiconvexity of necks of ladders [24, Lemma 2.18]) *Let $X$ be an $\eta$-metric graph bundle over $B$ satisfying* $(v_k, M_k, n_k)$***-flaring condition*** *for all $k \geq 1$. Then for all $c_1 \geq 1$ and $R > 1$ there are constants $D_{4.11} = D_{4.11}(c_1, R)$ and $K_{4.11} = K_{4.11}(c_1)$ such that the following holds:*

*Suppose $\Sigma_1, \Sigma_2$ are two $c_1$-qi sections of $B$ in $X$ and let $L \geq \max\{M_{c_1}, d_h(\Sigma_1, \Sigma_2)\}$.*

1. *Let $\gamma : [t_0, t_1] \to B$ be a geodesic, $t_0, t_1 \in \mathbb{Z}$, such that*
   - (a) *$d_{\gamma(t_0)}(\Sigma_1 \cap F_{\gamma(t_0)}, \Sigma_2 \cap F_{\gamma(t_0)}) = LR$,*
   - (b) *$\gamma(t_1) \in U_L := U_L(\Sigma_1, \Sigma_2)$ but for all $t \in [t_0, t_1) \cap \mathbb{Z}$, $\gamma(t) \notin U_L$.*

   *Then the length of $\gamma$ is at most $D_{4.11}(c_1, R)$.*
2. *For any $b_1, b_2 \in U_L$ and any geodesic $[b_1, b_2]$ joining them in $B$, we have $[b_1, b_2] \subset N_{K_{4.11}}(U_L)$. In particular, if $B$ is hyperbolic then $U_L$ is $K_{4.11}$-quasiconvex in $B$.*
3. *If $d_h(\Sigma_1, \Sigma_2) \geq M_{c_1}$ then the diameter of the set $U_L$ is at most $D'_{4.11} = D'_{4.11}(c_1, L)$.*

Part (2) of the above lemma is slightly different from that of [24, Lemma 2.18] but the proof there actually showed this. However, ladders with short necks to which Lemma 4.11 applies are given a special name:

**Definition 4.12** (small girth ladders) Given two $K$-qi sections $\Sigma_1, \Sigma_2$ in a metric graph bundle satisfying a flaring condition the ladder $\mathbb{L}(\Sigma_1, \Sigma_2)$ is called a *small girth ladder* if $U_L(\Sigma_1, \Sigma_2) \neq \varnothing$, where $L = M_K$.

**Remark** Suppose $X' \to B'$ is a metric bundle and $X \to B$ is the canonical metric graph bundle associated to it. Suppose a flaring condition holds for $X$. This is the case for instance when $X$ or equivalently $X'$ is hyperbolic. In such a case, a small girth ladder in $X'$ for us will be, by definition, the transport of a small girth ladder from $X$ under $\phi_X$ (as in Proposition 4.1).

We end this section with two simple lemmas. We note that flaring condition is not needed for these to hold.

**Lemma 4.13** *Given $D \geq 0$, $K \geq 1$ there is $R = R_{4.13}(D, K)$ such that the following holds.*

*Suppose $\Sigma$ is $K$-qi section in $X$ and $x \in X$. Let $b = \pi(x)$. Then $d(x, \Sigma) \geq D$ if $d_b(x, \Sigma \cap F_b) \geq R$.*

**Proof** Suppose $y \in \Sigma$ a nearest point from $x$. Let $\alpha \subset \Sigma$ be the lift of a geodesic $[b, \pi(y)]$ joining $b$ to $\pi(y)$ joining $y$ to $\Sigma \cap F_b$. We note that $d_B(b, \pi(y)) \leq d(x, y)$. Hence, $d(y, \alpha(b)) \leq Kd(x, y) + K$. Therefore, $d(x, \alpha(b)) \leq d(x, y) + d(y, \alpha(b)) \leq (K + 1)d(x, y) + K$. This implies $d(x, y) \geq \frac{1}{K+1} d(x, \alpha(b))$ since all distances are integers in this case. Now fibers of $X$ are properly embedded as measured by $\eta$. Thus if $d_b(x, \alpha(b)) \geq \eta((K + 1)D)$ then $d(x, y) \geq D$. Hence, we can take $R = \eta(KD + D)$. □

The corollary below gives a relation between the girth of a ladder $\mathbb{L}(\Sigma_1, \Sigma_2)$ and $d(\Sigma_1, \Sigma_2)$.

**Corollary 4.14** *Given $D \geq 0$, $K \geq 1$ there is an $R = R_{4.14}(D, K)$ such that the following holds:*

*Suppose $\Sigma_1, \Sigma_2$ are two $K$-qi sections in $X$. Then $d(\Sigma_1, \Sigma_2) \geq D$ if $U_R(\Sigma_1, \Sigma_2) = \varnothing$.*

The next lemma is a generalization of Lemma 4.13. Nevertheless we keep both of them since they are used many times in the next section.

**Lemma 4.15** *Given $K, D$ there is $R = R_{4.15}(K, D)$ such that the following holds.*

*Suppose $\Sigma_1, \Sigma_2$ are two $K$-qi sections in $X$ and $\mathbb{L} = \mathbb{L}(\Sigma_1, \Sigma_2)$. Suppose $x \in X$ and $\pi(x) = b$. Then $d(x, \mathbb{L}) \geq D$ if $d_b(x, \mathbb{L} \cap F_b) \geq R$.*

**Proof** Suppose $y \in \mathbb{L}$ is a nearest point from $x$. Let $\alpha$ be a geodesic lift of any geodesic $[b, \pi(y)]$ joining $b$ to $\pi(y)$ such that $\alpha$ joins $y$ to $F_b$. Now $\pi_{\mathbb{L}}(\alpha)$ is a $2C$-qi lift of $[b, \pi(y)]$ where $C = C_{4.6}(K)$. Thus $d(y, \pi_{\mathbb{L}}(\alpha)(b)) \leq 2Cd_B(b, \pi(y) + 2C \leq 2Cd(x, y) + 2C$. Hence,

$$d(x, \mathbb{L} \cap F_b) \leq d(x, y) + d(y, \pi_{\mathbb{L}}(\alpha)(b)) \leq (2C + 1)d(x, y) + 2C.$$

Therefore, $d(x, y) \geq \dfrac{1}{2C+1} d(x, \mathbb{L} \cap F_b)$. Hence, we can take $R = \eta((2C + 1)D)$. □

# 5 Cannon–Thurston maps for pullback bundles

In this section, we prove the main result of the paper. Here is the set-up. From now on we suppose that $\pi\colon X \to B$ is an $(\eta, c)$-metric bundle or an $\eta$-metric graph bundle satisfying the following hypotheses:

(H1) $B$ is a $\delta_0$-hyperbolic metric space.

(H2) Each of the fibers $F_b$, $b \in B$ is a $\delta_0$-hyperbolic metric space with respect to the path metric induced from $X$.

(H3) The barycenter maps $\partial^3 F_b \to F_b$, $b \in B$ (resp. $b \in \mathcal{V}(B)$) are $N_0$-coarsely surjective for some constant $N_0$.

(H4) The $(v_k, M_k, n_k)$-flaring condition is satisfied for all $k \geq 1$.

The following theorem is the main result of [24]:

**Theorem 5.1** [24, Theorem 4.3 and Proposition 5.8] *If $\pi\colon X \to B$ is a geodesic metric bundle or a metric graph bundle satisfying* (H1)–(H3) *then $X$ is a hyperbolic metric space if and only if $X$ satisfies a flaring condition.*

## 5.1 Proof of the main theorem

We are now ready to state and prove the main theorem of the paper.

**Theorem 5.2** (main theorem) *Suppose $\pi\colon X \to B$ is a metric (graph) bundle satisfying the hypotheses* (H1)–(H4)*. Suppose $g\colon A \to B$ is a Lipschitz $k$-qi embedding and suppose $p\colon Y \to A$ is the pullback bundle. Let $f\colon Y \to X$ be the pullback map.*

*Then $Y$ is a hyperbolic metric space and the CT map exists for $f\colon Y \to X$.*

**Proof** We first note that $X$ is hyperbolic. This follows from Theorem 5.1 if $X$ is a metric graph bundle (or a geodesic metric bundle). If $X$ is a (length) metric bundle, one may first pass to the canonical metric graph bundle associated to it, and then verify the hypotheses of Theorem 5.1 for it. In fact, if any metric bundle satisfies (H1), (H2), and (H3) then the canonical metric graph bundle associated to it also has these properties with possibly different values of the respective parameters. Flaring condition (H4) follows from Definition 4.10. It then follows that the metric graph bundle is hyperbolic. Consequently, $X$ is hyperbolic by Proposition 4.1. We shall assume that $X$ is $\delta$-hyperbolic. We begin with the following reductions:

(1) *It is enough to prove the theorem only for metric graph bundles.*

Indeed this follows from Proposition 4.2(2). So for the rest of the proof we shall assume that $\pi\colon X \to B$ is a metric graph bundle satisfying (H1)–(H4).

Since we work with graphs from now, for the rest of the section by hyperbolicity we shall mean Rips hyperbolicity.

(2) *We may moreover assume that $A$ is a connected subgraph, $g\colon A \to B$ is the inclusion map and $Y$ is the restriction bundle for that inclusion. In particular, $f\colon Y \to X$ is the inclusion map and $Y = \pi^{-1}(A)$.*

Since $g\colon A \to B$ is a $k$-qi embedding and $B$ is $\delta_0$-hyperbolic, $g(A)$ is $D_{2.19}(\delta_0, k, k)$-quasiconvex in $B$. Let $A'$ be the $D_{2.19}(\delta_0, k, k)$-neighborhood of $g(A)$ in $B$. Then clearly $A'$ is connected subgraph of $B$ and $g\colon A \to A'$ is a quasiisometry with respect to the induced path metric on $A'$ from $B$. Clearly $A'$ is $(1, 4D_{2.19}(\delta_0, k, k))$-qi embedded. Let $\pi'\colon X' = \pi^{-1}(A') \to A'$ be the restriction of $\pi$ on $X'$. Then $\pi'\colon X' \to A'$ is a metric graph bundle by Lemma 3.17. Also, we note that $(f, g)\colon Y \to X'$ is a morphism of metric graph bundles. By Corollary 3.25 the fiber maps of the morphism $f\colon Y \to X'$ are uniform quasiisometries and hence by Theorem 3.15 we see that $f\colon Y \to X'$ is an isomorphism of metric graph bundles. Since (Rips) hyperbolicity of graphs is a qi invariant, we are reduced to proving hyperbolicity of $X'$ and also by Lemma 2.50(1) we are reduced to proving the existence of the CT map for the inclusion $X' \to X$.

**Hyperbolicity of $Y$** $Y$ is hyperbolic by Remark 4.4 of [24]. In fact, by Theorem 5.1 it is enough to check that flaring holds for the bundle $Y \to A$. This is a consequence of flaring of the bundle $\pi\colon X \to B$ and bounded flaring.

**Remark** (1) The sole purpose of (H3) is to have global uniform qi sections through every point of $X$ which is guaranteed by Proposition 4.3. For the rest of this section, we shall also assume:

(H3$'$) Through any point of $X$ there is a global $K_0$-qi section.

(2) Clearly $Y$ is an $\eta$-metric graph bundle over $A$ satisfying (H2) and (H3). We shall assume that $A$ is $\delta_0'$-hyperbolic. We shall also assume the bundle $Y$ satisfies a $(\nu_k', M_k', n_k')$-flaring condition for all $k \geq 1$.

**Existence of CT map** *Outline of the proof*: To prove the existence of the CT map we use Lemma 2.49. The different steps used in the proof are as follows. (1) Given $y, y' \in Y$ first we define a uniform quasigeodesic $c(y, y')$ in $X$ joining $y, y'$. This is extracted from [24]. (2) In the next step we modify $c(y, y')$ to obtain a path $\bar{c}(y, y')$ in $Y$. (3) We then check that these paths are uniform quasigeodesics in $Y$. (4) Finally we verify the condition of Lemma 2.49 for the paths $c(y, y')$ and $\bar{c}(y, y')$. Since $X, Y$ are hyperbolic metric spaces, stability of quasigeodesics and Lemma 2.49 finishes the proof. To maintain modularity of the arguments we state intermediate observations as lemma, proposition etc.

**Remark** Although we assumed that $y, y' \in Y$ as is necessary for our proof, $c(y, y')$ as defined below is a uniform quasigeodesic for all $y, y' \in X$ as it will follow from the proof.

However, we would like to note that description of uniform quasigeodesics in a metric graph bundle with the above properties (H1)–(H4) is already contained in [24], eg see Propositions 3.4 and 3.14 of [24]. We make it more explicit with the help of Proposition 2.33.

**Step 1** (*descriptions of the uniform quasigeodesic* $c(y, y')$) The description of the paths and the proof that they are uniform quasigeodesics in $X$ is broken up into three further substeps.

**Step 1(a)** (*choosing a ladder containing* $y, y'$) We begin by choosing any two $K_0$-qi sections $\Sigma, \Sigma'$ in $X$ containing $y, y'$, respectively. Let $\mathbb{L}(\Sigma, \Sigma')$ be the ladder formed by them. Throughout Step 1 we shall work with these qi sections and ladder. The path $c(y, y')$ that we shall construct in Step 1(c) will be contained in this ladder.

**Step 1(b)** (*decomposition of the ladder into small girth ladders*) We next choose finitely many qi sections in $\mathbb{L}(\Sigma, \Sigma')$ after [24, Proposition 3.14] in a way suitable for using Proposition 2.33. This requires a little preparation. We start with the following.

**Lemma 5.3** *For all* $K \geq 1$ *there is* $D_{5.3}(K)$ *such that the following holds in* $X$.

*Suppose* $\Sigma_1, \Sigma_2$ *are two* $K$*-qi sections and* $d_h(\Sigma_1, \Sigma_2) \geq M_K$. *Then* $\Sigma_1, \Sigma_2$ *are* $D_{5.3}(K)$*-cobounded.*

**Proof** We note that $\Sigma_1, \Sigma_2$ are $K' = D_{2.19}(\delta, K, K)$-quasiconvex in $X$. Suppose $P\colon X \to \Sigma_1$ is a 1-approximate nearest point projection map and the diameter of $P(\Sigma_2)$ is bigger than $D = D_{2.28}(\delta, K', 1)$. Then $d(\Sigma_1, \Sigma_2) \leq R = R_{2.28}(\delta, K', 1)$. If $x \in \Sigma_2$ such that $d(x, \Sigma_1) \leq R$ and $b = \pi(x)$ then

$$d_b(x, \Sigma_1 \cap F_b) \leq R_{4.13}(R, K') = \bar{R},$$

say. Hence, $\pi(P(\Sigma_2)) \subset U_{\bar{R}}(\Sigma_1, \Sigma_2)$. However, by Lemma 4.11 the diameter of $U_{\bar{R}}(\Sigma_1, \Sigma_2)$ is at most $D'_{4.11}(K', \bar{R})$. It follows that the diameter of $P(\Sigma_2)$ is at most $K + KD'_{4.11}(K', \bar{R})$. Hence we may choose $D_{5.3}(K) = \max\{D_{2.28}(\delta, K', 1),\ K + KD'_{4.11}(K', \bar{R})\}$. □

**Lemma 5.4** *Suppose* $\Sigma_1, \Sigma_2$ *are two* $K$*-qi sections and* $\Sigma \subset \mathbb{L}(\Sigma_1, \Sigma_2)$ *is* $K$*-qi section. Then* $\Sigma$ *coarsely uniformly bisects* $\mathbb{L}(\Sigma_1, \Sigma_2)$ *into the subladders* $\mathbb{L}(\Sigma_1, \Sigma)$ *and* $\mathbb{L}(\Sigma, \Sigma_2)$.

**Proof** First of all any ladder formed by $K$-qi sections is $K_{4.6}(\delta, K)$-quasiconvex. Let $K' = K_{4.6}(\delta, K)$. Let $k \geq 1$, and $x_i \in \Sigma_i$, $i = 1, 2$ be any points. Let $\gamma_{x_1x_2} \colon I \to X$ be a $k$-quasigeodesic joining them where $I$ is an interval. Then there are points $t_1, t_2 \in I$ with $|t_1 - t_2| \leq 1$ such that $\gamma_{x_1x_2}(t_1) \in N_{K'}(\mathbb{L}(\Sigma_1, \Sigma))$ and $\gamma_{x_1x_2}(t_2) \in N_{K'}(\mathbb{L}(\Sigma, \Sigma_2))$. Let $y_1 \in \mathbb{L}(\Sigma_1, \Sigma)$ and $y_2 \in \mathbb{L}(\Sigma, \Sigma_2)$ be such that $d(y_i, \gamma_{x_1x_2}(t_i)) \leq K'$, $i = 1, 2$. We note that $d(\gamma_{x_1x_2}(t_1), \gamma_{x_1x_2}(t_2)) \leq 2k$. Hence, $d(y_1, y_2) \leq 2K' + 2k$. Let $b = \pi(y_1)$. Then $d_b(y_1, \mathbb{L}(\Sigma, \Sigma_2) \cap F_b) \leq R_{4.15}(K, 2K' + 2k)$. This implies $d_b(y_1, \Sigma \cap F_b) \leq R_{4.15}(K, 2K' + 2k)$. Thus $d(\gamma_{x_1x_2}(t_1), \Sigma) \leq K' + R_{4.15}(K, 2K' + 2k)$. This proves the lemma. □

**Lemma 5.5** *If $\mathcal{Q}$ is a $K$-qi section in $X$ then $\mathcal{Q} \cap Y$ is a $K_{5.5}(K)$-qi section of $A$ in $Y$.*

**Proof** Suppose $s \colon B \to X$ is the $K$-qi embedding such that $s(B) = \mathcal{Q}$. Let $s$ also denote the restriction on $A$. Since the bundle map $Y \to A$ is 1-Lipschitz we have $d_A(u, v) \leq d_Y(s(u), s(v))$ for all $u, v \in A$. Thus it is enough to show that $s \colon A \to Y$ is uniformly coarsely Lipschitz. Suppose $u, v \in A$ are adjacent vertices. Then $d_X(s(u), s(v)) \leq 2K$. Now, there is a vertex $x \in F_v$ adjacent to $s(u) \in F_u$. Hence, $d_X(s(v), x) \leq 1 + 2K$. Therefore, $d_v(s(v), x) \leq \eta(1 + 2K)$. Hence, $d_Y(s(u), s(v)) \leq 1 + \eta(1 + 2K)$. It follows that for all $u, v \in A$ we have $d_Y(s(u), s(v)) \leq (1 + \eta(1 + 2K)) d_A(u, v)$. Hence, we can take $K_{5.5}(K) = 1 + \eta(1 + 2K)$. □

The following corollary is proved exactly as Lemma 5.3. Hence we omit the proof.

**Corollary 5.6** *For all $K \geq 1$ there is $D_{5.6}(K) \geq 0$ such that the following holds.*

*Suppose $\Sigma_1, \Sigma_2$ are two $K$-qi sections in $X$ and $d_h(\Sigma_1, \Sigma_2) \geq M_K$. Then $\Sigma_1 \cap Y, \Sigma_2 \cap Y$ are $D_{5.6}(K)$-cobounded in $Y$.*

Before describing the decomposition of ladders the following conclusions and notation on qi sections and ladders will be useful to record.

**Convention 5.7** (C0) We recall that $A$ is $k$-qi embedded in $B$. We let $k_0 = D_{2.17}(\delta_0, k, k)$ so that $A$ is $k_0$-quasiconvex in $B$. Finally we assume that $Y$ is $\delta'$ hyperbolic.

(C1) Let $K_{i+1} = (1 + 2K_0) C_{4.6}(K_i)$ for all $i \in \mathbb{N}$ where $K_0$ is as in (H3′). Therefore, through any point of a $K_i$-ladder in $X$, there is a $K_{i+1}$-qi section contained in the ladder. Let $K_i' = K_{5.5}(K_i)$.

(C2) We let $\lambda_i = \max\{D_{2.19}(\delta, K_i, K_i), K_{4.6}(\delta, K_i), D_{2.19}(\delta', K_i', K_i'), K_{4.6}(\delta', K_i')\}$ so that any $K_i$-qi section $\mathcal{Q} \subset X$ and any ladder $\mathbb{L} \subset X$ formed by two $K_i$-qi sections in $X$ are $\lambda_i$-quasiconvex in $X$ and moreover $\mathcal{Q} \cap Y$ and $\mathbb{L} \cap Y$ are $\lambda_i$-quasiconvex in $Y$.

(C3) If $\Sigma_1, \Sigma_2$ are two $K_i$-qi sections in $X$ and $d_h(\Sigma_1, \Sigma_2) \geq M_{K_i}$ then they are $D_i$-cobounded in $X$, as are $\Sigma_1 \cap Y$, $\Sigma_2 \cap Y$ in $Y$, where $D_i = \max\{D_{5.3}(K_i), D_{5.6}(K_i)\}$.

(C4) For each pair of $K_i$-qi sections $\Sigma_1, \Sigma_2$ in $X$ that satisfies

$$d_h(\Sigma_1, \Sigma_2) > r_i = \max\{R_{4.14}(2\lambda_i + 1, K_i), R_{4.14}(2\lambda_i + 1, K_i')\}$$

we have $d_X(\Sigma_1, \Sigma_2) > 2\lambda_i + 1$ and $d_Y(\Sigma_1 \cap Y, \Sigma_2 \cap Y) > 2\lambda_i + 1$.

The following proposition is extracted from Proposition 3.14 of [24]. The various parts of this proposition are contained in the different steps of the proof of [24, Proposition 3.14].

Let us fix a point $b_0 \in A$ once and for all. Suppose $\alpha\colon [0, l] \to F_{b_0} \cap \mathbb{L}(\Sigma, \Sigma')$ is an isometry such that $\alpha(0) = \Sigma \cap F_{b_0}$ and $\Sigma' \cap F_{b_0} = \alpha(l)$.

**Proposition 5.8** (see [24, Corollary 3.13 and Proposition 3.14]) *There is a constant $L_0$ such that for all $L \geq L_0$ there is a partition $0 = t_0 < t_1 < \cdots < t_n = l$ of $[0, l]$ and $K_1$-qi sections $\Sigma_i$ passing through $\alpha(t_i)$, $0 \leq i \leq n$ inside $\mathbb{L}(\Sigma, \Sigma')$ such that the following hold*:

(1) $\Sigma_0 = \Sigma$, $\Sigma_n = \Sigma'$.

(2) *For* $0 \leq i \leq n-2$, $\Sigma_{i+1} \subset \mathbb{L}(\Sigma_i, \Sigma')$.

(3) *For* $0 \leq i \leq n-2$ *either* (I) $d_h(\Sigma_i, \Sigma_{i+1}) = L$, *or* (II) $d_h(\Sigma_i, \Sigma_{i+1}) > L$ *and there is a* $K_2$-*qi section* $\Sigma_i'$ *through* $\alpha(t_{i+1} - 1)$ *inside* $\mathbb{L}(\Sigma_i, \Sigma_{i+1})$ *such that* $d_h(\Sigma_i, \Sigma_i') < C + CL$, *where* $C = C_{4.6}(K_1)$.

(4) $d_h(\Sigma_{n-1}, \Sigma_n) \leq L$.

However, we will need a slightly different decomposition of $\mathbb{L}(\Sigma, \Sigma')$ than what is described here. It is derived as the following corollary to Proposition 5.8.

**Convention 5.9** We shall fix $L = L_0 + M_{K_3} + r_3$ and denote it by $R_0$ for the rest of the paper. Also we shall define $R_1 = C + CR_0$, where $C = C_{4.6}(K_1)$. Thus we have the following.

**Corollary 5.10** (decomposition of $\mathbb{L}(\Sigma, \Sigma')$) *There is a partition $0 = t_0 < t_1 < \cdots < t_n = l$ of $[0, l]$ and $K_1$-qi sections $\Sigma_i$ passing through $\alpha(t_i)$, $0 \leq i \leq n$ inside $\mathbb{L}(\Sigma, \Sigma')$ such that the following hold*:

(1) $\Sigma_0 = \Sigma$, $\Sigma_n = \Sigma'$.

(2) *For* $0 \leq i \leq n-2$, $\Sigma_{i+1} \subset \mathbb{L}(\Sigma_i, \Sigma')$.

(3) *For* $0 \leq i \leq n-2$ *either* (I) $d_h(\Sigma_i, \Sigma_{i+1}) = R_0$, *or* (II) $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$ *and there is a* $K_2$-*qi section* $\Sigma_i'$ *through* $\alpha(t_{i+1} - 1)$ *inside* $\mathbb{L}(\Sigma_i, \Sigma_{i+1})$ *such that* $d_h(\Sigma_i, \Sigma_i') < R_1$.
*In either case* $d_X(\Sigma_i, \Sigma_{i+1}) > 2\lambda_1 + 1$ *and* $\Sigma_i, \Sigma_{i+1}$ *are* $D_1$-*cobounded in* $X$.

(4) $d_h(\Sigma_{n-1}, \Sigma_n) \leq R_0$.

We note that the second part of (3) follows from (C1), (C2), (C3) above. However, a subladder $\mathbb{L}(\Sigma_i, \Sigma_{i+1})$ of $\mathbb{L}(\Sigma, \Sigma')$ will be referred to as a *type* (I) *subladder* or a *type* (II) *subladder* according as $d_h(\Sigma_i, \Sigma_{i+1}) = R_0$ or $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$ respectively.

**Remark** (1) We note that by the choice of $R_0, R_1$ it follows that $d_Y(\Sigma_i \cap Y, \Sigma_{i+1} \cap Y) > 2\lambda_1 + 1$ and $\Sigma_i \cap Y, \Sigma_{i+1} \cap Y$ are $D_1$-cobounded in $Y$ for $0 \le i \le n-2$.

(2) *We shall use* $\Sigma_i$ *to mean qi sections in* $\mathbb{L}(\Sigma, \Sigma')$ *exactly as in the corollary above for the rest of this section.*

(3) Finally we note that $\Sigma_n, \Sigma_{n-1}$ need not be cobounded in general and the same remark applies to $\Sigma_n \cap Y,\ \Sigma_{n-1} \cap Y$.

**Lemma 5.11** *Let* $\Pi : \mathbb{L}(\Sigma, \Sigma') \to [0, n]$ *be any map that sends* $\Sigma_i$ *to* $i \in [0, n] \cap \mathbb{Z}$ *and sends any point of* $\mathbb{L}(\Sigma_i, \Sigma_{i+1}) \setminus \{\Sigma_i \cup \Sigma_{i+1}\}$ *to a point in* $(i, i+1)$. *Then the hypotheses of Proposition 2.33 are verified for both* $\Pi$ *and its restriction* $\mathbb{L}(\Sigma, \Sigma') \cap Y \to [0, n]$.

**Proof** For both $\Pi$ and its restriction to $\mathbb{L}(\Sigma, \Sigma') \cap Y$, (P0) and (P1) follow from (C2), (P2) follows from Lemma 5.4, and (P3) follows from (C4). For $\Pi$ (P4) follows from (C3) and for the restriction of $\Pi$ to $\mathbb{L}(\Sigma, \Sigma') \cap Y$ from (1) of the Remark. □

**Step 1(c)** (*joining* $y, y'$ *inside* $\mathbb{L}(\Sigma, \Sigma')$) We now inductively define a finite sequence of points $y_i \in \Sigma_i$, $0 \le i \le n+1$ with $y_0 = y,\ y_{n+1} = y'$ such that each $y_i$, $1 \le i \le n$, is a uniform approximate nearest point projection of $y_{i-1}$ on $\Sigma_i$ in $X$. We also define uniform quasigeodesics $\gamma_i$ in $X$ joining $y_i, y_{i+1}$. The concatenation of these $\gamma_i$'s then forms a uniform quasigeodesic in $X$ joining $y, y'$ by Proposition 2.33 and Lemma 5.11.

*We define* $\gamma_n$ *to be the lift of* $[\pi(y_n), \pi(y_{n+1})]$ *in* $\Sigma'$.

Suppose $y_0, \dots, y_i$ and $\gamma_0, \dots, \gamma_{i-1}$ are already constructed, $0 \le i \le n-2$. We next explain how to define $y_{i+1}$ and $\gamma_i$.

**Case I** Suppose $\mathbb{L}_i = \mathbb{L}(\Sigma_i, \Sigma_{i+1})$ is of type (I) ie $d_h(\Sigma_i, \Sigma_{i+1}) = R_0$ or $i = n-1$. Then, $U_{R_0}(\Sigma_i, \Sigma_{i+1})$ is nonempty. Let $u_i$ be a nearest point projection of $\pi(y_i)$ on $U_{R_0}(\Sigma_i, \Sigma_{i+1})$. We define $y_{i+1} = \Sigma_{i+1} \cap F_{u_i}$. Let $\alpha_i$ be the lift of $[\pi(y_i), u_i]$ in $\Sigma_i$, and let $\sigma_i$ be the subsegment of $F_{u_i} \cap \mathbb{L}_i$ joining $\alpha_i(u_i)$ and $y_{i+1}$. We define $\gamma_i$ to be the concatenation of $\alpha_i$ and $\sigma_i$. Then clearly $\gamma_i$ is a $(K_1 + R_0)$-quasigeodesic in $X$. That $y_{i+1}$ is a uniform approximate nearest point projection of $y_i$ on $\Sigma_{i+1}$ follows from the following lemma.

**Lemma 5.12** *Given* $K \ge 1$ *and* $R \ge M_K$ *there are constants* $\epsilon_{5.12}(K, R)$ *and* $\epsilon'_{5.12}(K, R)$ *such that the following holds.*

*Suppose* $\mathcal{Q}_1, \mathcal{Q}_2$ *are two* $K$*-qi sections and* $d_h(\mathcal{Q}_1, \mathcal{Q}_2) \le R$. *Let* $x \in \mathcal{Q}_1$ *and let* $U = U_R(\mathcal{Q}_1, \mathcal{Q}_2)$. *Suppose* $b$ *is a nearest point projection of* $\pi(x)$ *on* $U$. *Then* $\mathcal{Q}_2 \cap F_b$ *is* $\epsilon_{5.12}(K, R)$*-approximate nearest point projection of* $x$ *on* $\mathcal{Q}_2$.

*If* $d_h(\mathcal{Q}_1, \mathcal{Q}_2) \ge M_K$ *then for any* $b' \in U$ *the point* $\mathcal{Q}_2 \cap F_{b'}$ *is an* $\epsilon'_{5.12}(K, R)$*-approximate nearest point projection of any point of* $\mathcal{Q}_1$ *on* $\mathcal{Q}_2$.

This lemma follows from Corollary 1.40 and Proposition 3.4 of [24] given that ladders are quasiconvex. However, we give an independent proof using the hyperbolicity of $X$.

**Proof** Suppose $\bar{x}$ is a nearest point projection of $x$ on $\mathcal{Q}_2$ and let $x' = \mathcal{Q}_2 \cap F_b$. Let $\gamma_{xx'}$ be the concatenation of the lift in $\mathcal{Q}_1$ of any geodesic in $B$ joining $\pi(x)$ to $b$ and any geodesic in $F_b$ joining $\mathcal{Q}_1 \cap F_b$ to $\mathcal{Q}_2 \cap F_b$. Clearly it is a $(K+R)$-quasigeodesic in $X$. Also by Lemma 2.25 the concatenation of any 1-quasigeodesics joining $x, \bar{x}$ and $\bar{x}, x'$ is a $K_{2.25}(\delta, K, 1, 0)$-quasigeodesic. Hence, by stability of quasigeodesics we have $\bar{x} \in N_D(\gamma_i)$ where $D = D_{2.19}(\delta, K', K')$ and $K' = \max\{K + R, K_{2.25}(\delta, K, 1, 0)\}$. This implies there is a point $z \in \gamma_{xx'}$ such that $d(z, \bar{x}) \le D$. If $z \in F_b \cap \gamma_{xx'}$ then $d(\bar{x}, x') \le D + R$ and hence $x'$ is a $(D+R)$-approximate nearest point projection of $x$ on $\mathcal{Q}_2$.

Suppose $z \in \mathcal{Q}_1 \cap \gamma_{xx'}$. Then $d_{\pi(z)}(z, \mathcal{Q}_2 \cap F_{\pi(z)}) \le R_{4.13}(D, K)$. Hence, by Lemma 4.11 we have $d_B(\pi(z), b) \le D_{4.11}(K, R')$, where $R' = R_{4.13}(D, K)/R$. Therefore,

$$d(\bar{x}, x') \le d(\bar{x}, z) + d(z, \mathcal{Q}_1 \cap F_b) + d(\mathcal{Q}_1 \cap F_b, x') \le D + (K + KD_{4.11}(K, R')) + R.$$

Hence in this case $x'$ is a $(D+K+KD_{4.11}(K, R')+R)$-approximate nearest point projection of $x$ on $\mathcal{Q}_2$. We may set $\epsilon_{5.12}(K, R) = D + K + KD_{4.11}(K, R') + R$.

For the last part, we note that the diameter of $U$ is at most $D'_{4.11}(K, R)$. Thus clearly $\epsilon'_{5.12}(K, R) = \epsilon_{5.12}(K, R) + K + KD'_{4.11}(K, R)$ works. ☐

**Case II** Suppose $\mathbb{L}_i = \mathbb{L}(\Sigma_i, \Sigma_{i+1})$ is of type (II), ie $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$. In this case there exists a $K_2$-qi section $\Sigma'_i$ inside $\mathbb{L}_i = \mathbb{L}(\Sigma_i, \Sigma_{i+1})$ passing through $\alpha(t_{i+1} - 1)$ such that $d_h(\Sigma_i, \Sigma'_i) \le R_1$. We thus use Case (I) twice as follows. First we project $y_i$ on $\Sigma'_i$. Suppose the projection is $y'_i$. Then we project $y'_i$ on $\Sigma_{i+1}$ which we call $y_{i+1}$ and so on. Here are the details involved.

Let $v_i$ be a nearest point projection of $\pi(y_i)$ on $U_{R_1}(\Sigma_i, \Sigma'_i)$ and let $w_i$ be a nearest point projection $v_i$ on $U_{R_1}(\Sigma'_i, \Sigma_{i+1})$. Then $y_{i+1} = \Sigma_{i+1} \cap F_{w_i}$. In this case we let $\alpha_i$ denote the lift of $[\pi(y_i), v_i]$ in $\Sigma_i$ and let $\beta_i$ denote the lift of $[v_i, w_i]$ in $\Sigma'_i$. Then $\gamma_i$ is the concatenation of the paths $\alpha_i$, $[\Sigma_i \cap F_{v_i}, \Sigma'_i \cap F_{v_i}]_{v_i}$, $\beta_i$ and $[\Sigma'_i \cap F_{w_i}, \Sigma_{i+1} \cap F_{w_i}]_{w_i}$. That $y_{i+1}$ is a uniform approximate nearest point projection of $y_i$ on $\Sigma_{i+1}$ and that $\gamma_i$ is a uniform quasigeodesic follow immediately from Lemma 5.12 and the last part of Proposition 2.33.

**Remark** We note that $\mathbb{L}(\Sigma, \Sigma') \cap Y$ is a ladder in $Y$ formed by the qi sections $\Sigma \cap Y$ and $\Sigma' \cap Y$ defined over $A$. However, in this case the subladders $\mathbb{L}(\Sigma_i, \Sigma_{i+1}) \cap Y$ may not be of type (I) or (II). Therefore, we cannot directly use the above procedure to construct a uniform quasigeodesic in $Y$ joining $y, y'$.

**Step 2** (*modification of the path* $c(y, y')$) In this step we shall construct a path $\bar{c}(y, y')$ in $Y$ joining $y$ and $y'$ by modifying $c(y, y')$. For $0 \le i \le n$, let $b_i$ be a nearest point projection of $\pi(y_i)$ on $A$ and let $\bar{y}_i = F_{b_i} \cap \Sigma_i$. We define a path $\bar{\gamma}_i \subset Y$ joining the points $\bar{y}_i, \bar{y}_{i+1}$ for $0 \le i \le n$. Finally the path $\bar{c}(y, y')$ is defined to be the concatenation of these paths. *The path* $\bar{\gamma}_n$ *is the lift of* $[\pi(y_{n+1}), \pi(\bar{y}_n)]_A$ *in* $\Sigma' \cap Y$. The definition of $\bar{\gamma}_i$, for $0 \le i \le n-1$, depends on the type of the subladder $\mathbb{L}_i = \mathbb{L}(\Sigma_i, \Sigma_{i+1})$ given by Corollary 5.10(3).

**Case 2(I)** Suppose $\mathbb{L}_i$ is of type (I) or $i = n-1$. Let $\bar{\alpha}_i$ denote the lift of $[b_i, b_{i+1}]_A$ in $\Sigma_i$ starting at $\bar{y}_i$. The path $\bar{\gamma}_i$ is defined to be the concatenation of $\bar{\alpha}_i$ and the fiber geodesic $F_{b_{i+1}} \cap \mathbb{L}(\Sigma_i, \Sigma_{i+1})$.

**Case 2(II)** Suppose $\mathbb{L}_i$ is of type (II). In this case, we apply Case 2(I) to each of the subladders $\mathbb{L}(\Sigma_i, \Sigma'_i)$ and $\mathbb{L}(\Sigma'_i, \Sigma_{i+1})$. Let $y'_i$ be as defined in Step 1(c). Let $b'_i \in A$ be a nearest point projection $\pi(y'_i)$ on $A$ and $\bar{y}'_i = \pi^{-1}(b'_i) \cap \Sigma'_i$. Next we connect $\bar{y}_i, \bar{y}'_i$ and $\bar{y}'_i, \bar{y}_{i+1}$ as in Case 2(I) inside the ladders $\mathbb{L}(\Sigma_i \cap Y, \Sigma'_i \cap Y)$ and $\mathbb{L}(\Sigma'_i \cap Y, \Sigma_{i+1} \cap Y)$ respectively. We shall denote by $\bar{\alpha}_i$ and $\bar{\beta}_i$ the lift of $[b_i, b'_i]_A$ in $\Sigma_i \cap Y$ and $[b'_i, b_{i+1}]_A$ in $\Sigma'_i \cap Y$ respectively. The concatenation of the paths $\bar{\alpha}_i$, $[\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}]_{b'_i} \subset \mathbb{L}(\Sigma, \Sigma')$, $\bar{\beta}_i$ and $[\Sigma'_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}}]_{b_{i+1}} \subset \mathbb{L}(\Sigma, \Sigma')$ is defined to be $\bar{\gamma}_i$.

**Step 3** (*proving that* $\bar{c}(y, y')$ *is a uniform quasigeodesic in* $Y$) To show that $\bar{c}(y, y')$ is a quasigeodesic it is enough, by Proposition 2.33, to show that the paths $\bar{\gamma}_i$ are all uniform quasigeodesics in $Y$ and that for $0 \le i \le n-1$, $\bar{y}_{i+1}$ is an approximate nearest point projection of $\bar{y}_i$ in $\Sigma_{i+1} \cap Y$. The proof of this is broken into three cases depending on the type of the ladder $\mathbb{L}_i$. We start with the following lemma as a preparation for the proof.

The lemma below is true for any metric bundle that satisfies the hypotheses (H1)–(H4) and (H3′), although we are stating it for $X$ only. For instance, it is true for $Y$ too.

**Lemma 5.13** *Suppose* $b \in B$, $x, y \in F_b$. *Suppose for all* $K \ge K_0$ *and* $R \ge M_K$ *there is a constant* $D = D(K, R) \ge 0$ *such that for all* $x', y' \in [x, y]_b$ *and any two* $K$*-qi sections* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *in* $X$ *passing through* $x', y'$, *respectively, either* $U_R(\mathcal{Q}_1, \mathcal{Q}_2) = \varnothing$ *or* $d_B(b, U_R(\mathcal{Q}_1, \mathcal{Q}_2)) \le D$. *Then the following hold*:

1. $[x, y]_b$ *is a* $\lambda_{5.13}$*-quasigeodesic in* $X$, *where* $\lambda_{5.13}$ *depends on the function* $D$ (*and the parameters of the metric bundle*).
2. *If* $\mathcal{Q}$ *and* $\mathcal{Q}'$ *are two* $K$*-qi sections passing through* $x, y$ *respectively then* $x$ *is a uniform approximate nearest point projection of* $y$ *on* $\mathcal{Q}$ *and* $y$ *is a uniform approximate nearest point projection of* $x$ *on* $\mathcal{Q}'$.

**Proof** (1) Since the arc length parametrization of $[x,y]_b$ is a uniform proper embedding, by Lemma 2.5 it is enough to show that $[x,y]_b$ is uniformly close to a geodesic in $X$ joining $x,y$.

**Claim** *Suppose $\Sigma_x, \Sigma_y$ are two $K_0$-qi sections passing through $x,y$ respectively. Given any $z \in [x,y]_b$ and any $K_1$-qi section $\Sigma_z$ passing through $z$ contained in the ladder $\mathbb{L}(\Sigma_x,\Sigma_y)$ the nearest point projection of $x$ on $\Sigma_z$ is uniformly close to $z$.*

We note that once the claim is proved then applying Proposition 2.33 to the ladder $\mathbb{L}(\Sigma_x,\Sigma_y) = \mathbb{L}(\Sigma_x,\Sigma_z) \cup \mathbb{L}(\Sigma_z,\Sigma_y)$ it follows that $z$ is uniformly close to a geodesic joining $x,y$. From this (1) follows immediately.

**Proof of Claim** First suppose $U_{M_{K_1}}(\Sigma_x,\Sigma_z) \neq \varnothing$. Then we can find a uniform approximate nearest point projection of $x$ on $\Sigma_z$ using Step 1(c), Case I and Lemma 5.12 above which is uniformly close to $z$ by hypothesis. ◁

Now suppose $U_{M_{K_1}}(\Sigma_x,\Sigma_z) = \varnothing$. Let $\alpha_{zx} : [0,l] \to F_b$ be the unit speed parametrization of the geodesic $\mathbb{L}(\Sigma_x,\Sigma_z) \cap F_b$ joining $z$ to $x$. By Corollary 5.10 there is a $K_2$-qi section $\Sigma_{z'}$ contained in the ladder $\mathbb{L}(\Sigma_x,\Sigma_z)$ passing through $z' = \alpha_{zx}(t)$ for some $t \in [0,l]$ such that $\mathbb{L}(\Sigma_z,\Sigma_{z'})$ is a $K_2$-ladder of type (I) or (II). Let $x'$ be a nearest point projection of $x$ on $\Sigma_{z'}$. By the last part of Proposition 2.33 applied to $\mathbb{L}(\Sigma_x,\Sigma_z)$, it is enough to find a uniform approximate nearest point projection of $x'$ on $\Sigma_z$ which is also uniformly close to $z$. However, in this case $\Sigma_z, \Sigma_{z'}$ are $D_2$-cobounded. Hence it is enough to find a uniform approximate nearest point projection of $z'$ on $\Sigma_z$ which is uniformly close to $z$. The proof of this is broken into two cases as follows.

(I) Suppose $d_h(\Sigma_z,\Sigma_{z'}) = R_0$. By the last part of Lemma 5.12 if $v \in U_{R_0}(\Sigma_z,\Sigma_{z'})$ then $F_v \cap \Sigma_z$ is a uniform approximate nearest point projection of any point of $\Sigma_{z'}$. Since $d_A(b,v)$ is uniformly small by hypothesis, $d(z, F_v \cap \Sigma_z)$ is also uniformly small.

(II) Suppose $d_h(\Sigma_z,\Sigma_{z'}) > R_0$. Then there is a $K_3$-qi section $\Sigma_{z''}$ in $\mathbb{L}(\Sigma_z,\Sigma_{z'})$ passing through $z'' = \alpha_{zx}(t-1)$ such that $U_{R_0}(\Sigma_z,\Sigma_{z''}) \neq \varnothing$. Let $v'$ be a nearest point projection of $b$ on $U_{R_0}(\Sigma_z,\Sigma_{z''})$. Then by hypothesis $d(b,v')$ is uniformly small whence $d(z, F_{v'} \cap \Sigma_z)$ is uniformly small. Also by Lemma 5.12 the point $\Sigma_z \cap F_{v'}$ is a uniform approximate nearest point projection of $z''$ on $\Sigma_z$. It follows that $z$ is a uniform approximate nearest point projection of $z''$ on $\Sigma_z$. Finally, since $d(z',z'') \le 1$, $z$ is a uniform approximate nearest point projection of $z'$.

(2) We shall prove only the first statement since the proof of the second would be an exact copy. Suppose $x_1 \in Q$ is a nearest point projection of $y$ on $Q$. Consider the $K$-qi section over $[b, \pi(x_1)]$ contained in $Q$. This is a $K$-quasigeodesic of $X$ joining $x, x_1$. Since $Q$ is a $K$-qi section, by stability of quasigeodesics it is $D_{2.17}(\delta,K,K)$-quasiconvex in $X$. Hence by Lemma 2.25 the concatenation of this quasigeodesic with a geodesic in $X$ joining $y$ to $x_1$ is a $K_{2.25}(\delta,\widetilde{K},K,0)$-quasigeodesic where $\widetilde{K} = D_{2.17}(\delta,K,K)$. Let $k' = \max\{\widetilde{K}, \lambda_{5.13}\}$. Since $[x,y]_b$ is a $\lambda_{5.13}$-quasigeodesic, by stability of quasigeodesics we have

$x_1 \in N_{2D'}([x,y]_b)$, where $D' = D_{2.17}(\delta, k', k')$. Suppose $z \in [x,y]_b$ be such that $d(x_1, z) \le 2D'$. Then $d_B(\pi(x_1), \pi(z)) = d_B(\pi(x_1), b) \le 2D'$. Hence, $d(x, x_1) \le K + 2D'K$. Thus $x$ is a $(K+2D'K)$-approximate nearest point projection of $y$ on $Q$. □

**Remark** The proof of the first part of the above lemma uses the hypothesis for $K \le K_3$ only whereas the proof of the second part follows directly from the statement of the first part and is independent of the hypotheses of the lemma.

The following lemma is actually a trivial consequence of flaring (Lemma 4.13) and it is going to be used in the next two lemmas following it.

**Lemma 5.14** *Given $R \ge 0$, $K, K' \ge 1$ and $R' \ge M_{K'}$, there is a constant $R_{5.14}(R, R', K, K')$ and $D_{5.14}(R, R', K, K')$ such that the following holds.*

*Suppose $u \in B$ and $P_A(u) = b$, where $P_A \colon B \to A$ is a nearest point projection map. Suppose $x, y \in F_b$ and let $\gamma_x, \gamma_y$ be two $K$-qi sections over $[u,b]$. Let $Q_1, Q_2$ be two $K'$-qi sections over $A$ in $Y$ and $U = U_{R'}(Q_1, Q_2)$. If $d_u(\gamma_x(u), \gamma_y(u)) \le R$ and $U \ne \varnothing$, then $d_b(x,y) \le R_{5.14}(R, R', K, K')$ and $d_A(b, U) \le D_{5.14}(R, R', K, K')$.*

**Proof** Suppose $U \ne \varnothing$ and $d_u(\gamma_x(u), \gamma_y(y)) \le R$. Let $b' \in U_{M_{K'}}(Q_1, Q_2)$ be any point and let $[b,b']$ denote a geodesic in $A$ joining $b, b'$. Then the concatenation $[u,b] * [b,b']$ is a $K_{2.25}(\delta_0, k_0, k, 0)$-quasigeodesic in $B$ by Lemma 2.25 since $A$ is $k$-qi embedded and $k_0$-quasiconvex. Concatenation of $\gamma_x, \gamma_y$ with the qi sections over $[b,b']$ contained in $Q_1, Q_2$ respectively defines $\max\{K, K'\}$-qi sections over $[u,b] * [b,b']$ passing through $x, y$, respectively. Let $k' = K_{2.25}(\delta_0, k_0, k, 0)$ and $k'' = \max\{K, K'\}$. Then by Lemma 2.3 these qi sections are $(k'k'', k''k'+k'')$-quasigeodesics in $X$. Since $X$ is $\delta$-hyperbolic and $d(\gamma_x(u), \gamma_y(u)) \le R$ and $d(Q_1 \cap F_{b'}, Q_2 \cap F_{b'}) \le R'$, by Corollary 2.21 $x$ is contained in the $D' := (R+R'+2D_{2.20}(\delta, k'k'', k'k''+k''))$-neighborhood of the qi section over $[u,b]*[b,b']$ passing through $y$. Applying Lemma 4.13 to the restriction bundles over $[u,b]$ and $[b,b']$ we have $d_b(x,y) \le R'_1$, where $R'_1 = R_{4.13}(D', K)$. Hence, we can take $R_{5.14}(R, R', K, K') = R'_1$. Finally by Lemma 4.11 $d_A(b, U) \le D_{4.11}(K', R'_1/M_{K'})$. This completes the proof by taking $D_{5.14}(R, R', K, K') = D_{4.11}(K', R'_1/M_{K'})$. □

We recall that the paths $\bar{c}(y, y')$ were constructed from $c(y, y')$ by replacing parts of $c(y, y')$ by some fiber geodesic segments. The main aim of the following three lemmas is to proving that these fiber geodesic segments are uniform quasigeodesics in $Y$. Depending on how the corresponding subladders of $X$ intersect $Y$ we have three scenarios and hence we divided the proof into three lemmas.

**Lemma 5.15** *Given $K \ge K_0$ and $R \ge M_K$, there are constants $K_{5.15} = K_{5.15}(K, R)$, $\epsilon_{5.15} = \epsilon_{5.15}(K, R)$ and $D_{5.15} = D_{5.15}(K, R)$ such that the following holds.*

*Suppose $Q, Q'$ are two $K$-qi sections in $X$ and $d_h(Q, Q') \le R$ in $X$. Let $U = U_R(Q, Q')$. Suppose $d_h(Q \cap Y, Q' \cap Y) \ge R$ in $Y$. Then the following hold:*

(1) *The projection of* $U$ *on* $A$ *is of diameter at most* $D_{5.15}$.

(2) *For any* $b \in P_A(U)$, $F_b \cap \mathbb{L}(Q, Q')$ *is a* $K_{5.15}$*-quasigeodesic in* $Y$; *moreover*, $F_b \cap Q$ *is an* $\epsilon_{5.15}$*-approximate nearest point projection of any point of* $Q'$ *on* $Q$ *and vice versa.*

**Proof** (1) We know that $A$ is $k_0$-quasiconvex in $B$. By Lemma 4.11 $U$ is $K_{4.11}(K)$-quasiconvex in $B$. Let $\lambda' = \max\{k_0, K_{4.11}(K)\}$. Suppose $P_A\colon B \to A$ is a nearest point projection map and $a, a' \in P_A(U)$ with $d_B(a, a') \geq D_{2.28}(\delta, \lambda', 0)$. Then there are $u, u' \in U$ such that $d_B(a, u) \leq R_{2.28}(\delta, \lambda', 0)$ and $d_B(a', u') \leq R_{2.28}(\delta, \lambda', 0)$. Let $D = R_{2.28}(\delta, \lambda', 0)$. We know $d_u(Q \cap F_u, Q' \cap F_u) \leq R$. Hence by the bounded flaring condition we have $d_a(Q \cap F_a, Q' \cap F_a) \leq \mu_K(D)R$. Similarly

$$d_{a'}(Q \cap F_{a'}, Q' \cap F_{a'}) \leq \mu_K(D)R.$$

Let $R_1 = \mu_K(D)R$. Thus, $a, a' \in U_{R_1}(Q \cap Y, Q' \cap Y)$. Since $R_1 \geq M_K$, by Lemma 4.11 we have $\operatorname{diam}(U_{R_1}(Q \cap Y, Q' \cap Y)) \leq D_{4.11}(K, R_1)$. This proves (1). In fact, we can take $D_{5.15} = \max\{D_{2.28}(\delta, \lambda', 0), D_{4.11}(K, R_1)\}$.

(2) We derive this from Lemma 5.13 as follows. Let $u \in U$ be such that $P_A(u) = b$ and let $x, y \in F_b \cap \mathbb{L}(Q, Q')$. Suppose $Q_1, Q'_1$ are two $K'$-qi sections in $Y$ passing through $x, y$, respectively and $U' = U_{M_{K'}}(Q_1, Q'_1)$. Suppose $U' \neq \varnothing$. Consider the restriction $Z$ of the bundle $X$ on $[u, b] \subset B$. In this bundle $Q \cap Z$, $Q' \cap Z$ are $K$-qi sections. By Proposition 4.6(3) there are $(1+2K_0)C_{4.6}(K)$-qi sections over $ub$ contained in the ladder $\mathbb{L}(Q \cap Z, Q' \cap Z)$ passing through $x, y$. Call them $\gamma_x, \gamma_y$, respectively. We note that $d(\gamma_x(u), \gamma_y(u)) \leq R$. Now applying Lemma 5.14 we know that $d_B(b, U')$ is uniformly small. This verifies the hypothesis of Lemma 5.13. Thus $Q \cap F_b$ is a uniform approximate nearest point projection of $Q' \cap F_b$ on $Q$. Since $d_h(Q \cap Y, Q' \cap Y) \geq R \geq M_K$ the qi sections $Q \cap Y, Q' \cap Y$ are uniformly cobounded by Lemma 5.3. This shows that $Q \cap F_b$ is a uniform approximate nearest point projection of any point of $Q'$ on $Q$. That $Q' \cap F_b$ is a uniform approximate nearest point projection of any point of $Q$ on $Q'$ is similar and hence we skip it. □

**Lemma 5.16** *Given* $D \geq 0$, $K \geq K_0$ *and* $R \geq M_K$ *there are constants* $K_{5.16} = K_{5.16}(D, K, R)$ $\epsilon_{5.16} = \epsilon_{5.16}(D, K, R)$ *and* $D_{5.16} = D_{5.16}(D, K, R)$ *such that the following holds.*

*Suppose* $Q, Q'$ *are two* $K$*-qi sections in* $X$ *and* $d_h(Q, Q') \leq R$ *in* $X$. *Let* $U = U_R(Q, Q')$. *Suppose* $U \neq \varnothing$ *and* $\operatorname{diam}(U) \leq D$. *Then the following hold*:

(1) $\operatorname{diam}(P_A(U)) \leq D_{5.16}$.

(2) *For any* $b \in P_A(U)$, $F_b \cap \mathbb{L}(Q, Q')$ *is a* $K_{5.16}$*-quasigeodesic in* $Y$.

(3) $F_b \cap Q$ *is an* $\epsilon_{5.16}$*-approximate nearest point projection of any point of* $Q'$ *on* $Q$ *and vice versa.*

**Proof** (1) Since $B$ is $\delta_0$-hyperbolic and $A$ is $k_0$-quasiconvex in $B$ any nearest point projection map $P_A\colon B \to A$ is coarsely $L := L_{2.28}(\delta_0, k_0, 0)$-Lipschitz. Hence, $\operatorname{diam}(P_A(U)) \leq L + DL$.

(2), (3) We can derive these from Lemma 5.13 and the hypotheses of Lemma 5.13 can be verified using Lemma 5.14. The proof is an exact copy of the proof of Lemma 5.15(2),(3). Hence we omit it. The only part that requires explanation is why $\mathcal{Q} \cap Y$, $\mathcal{Q}' \cap Y$ are uniformly cobounded in $Y$. If $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) > R$ then we are done by Lemma 5.3. Suppose this is not the case. Then by the hypothesis $\operatorname{diam}(U_R(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y)) \le k(k + D)$ since $A$ is $k$-qi embedded in $B$. Then we are done by the first part of Lemma 5.12. □

**Lemma 5.17** *Given $K \ge K_0$ and $R \ge M_K$ there is a constant $D_{5.17} = D_{5.17}(K, R)$ such that the following holds.*

*Suppose $\mathcal{Q}, \mathcal{Q}'$ are two $K$-qi sections in $X$ and $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) \le R$. Let $U = U_R(\mathcal{Q}, \mathcal{Q}')$. Then for any $b \in P_A(U)$, $d_b(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \le D_{5.17}$.*

**Proof** Suppose $u \in U$ and $P_A(u) = b$. If $u \in A$ then $b = u$ and $d_b(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \le R$. Suppose $u \notin A$. We note that $U \cap Y = U(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) \neq \varnothing$. Let $v \in U(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y)$. Then by Lemma 2.25 $[u, b] * [b, v]$ is a $K_{2.25}(\delta_0, k_0, 1, 0)$-quasigeodesic in $B$. Since $U$ is $K_{4.11}(K)$-quasiconvex in $B$. Let $k' = K_{2.25}(\delta_0, k_0, 1, 0)$. Hence, by Lemma 2.17, $b \in N_D(U)$, where $D = D_{2.17}(\delta_0, k', k') + K_{4.11}(K)$. Finally by the bounded flaring $d_b(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \le R \max\{1, \mu_K(D)\}$. Hence we can take $D_{5.17} = R \max\{1, \mu_K(D)\}$. □

Finally, we are ready to finish the proof of Step 3.

**Lemma 5.18** *For $0 \le i \le n - 1$ we have the following.*

1. *$\bar{y}_{i+1}$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma_{i+1} \cap Y$.*
2. *$\bar{\gamma}_i$ is a uniform quasigeodesic in $Y$.*

**Proof** The proof is broken into three cases depending on the type of $\mathbb{L}_i$.

**Case 1** (*$i \le n - 2$ and $\mathbb{L}_i$ is of type* (I)) By Lemma 4.11 $U_{R_0}(\Sigma_i, \Sigma_{i+1})$ has uniformly small diameter. Hence by Lemma 5.16(2), $[\Sigma_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}}]_{b_{i+1}}$ is a uniform quasigeodesic in $Y$. By the part (3) of the same lemma $\Sigma_{i+1} \cap F_{b_{i+1}}$ is a uniform approximate nearest point projection of $\Sigma_i \cap F_{b_{i+1}}$ on $\Sigma_{i+1} \cap Y$ and $\Sigma_i \cap F_{b_{i+1}}$ is a uniform approximate nearest point projection of $\Sigma_{i+1} \cap F_{b_{i+1}}$ on $\Sigma_i \cap Y$ in $Y$. Hence the second part of the lemma follows, in this case, by Lemma 2.25.

**Case 2** (*$i \le n - 2$ and $\mathbb{L}_i$ is of type* (II)) Suppose $\mathbb{L}_i$ is a ladder of type (II). In this case, it is enough, by Proposition 2.33, to show the following two statements ($2'$) and ($2''$):

($2'$) *$\bar{y}'_i$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma'_i \cap Y$ in $Y$ and the concatenation of $\bar{\alpha}_i$ and the fiber geodesic $[\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}]_{b'_i}$ is a uniform quasigeodesic in $Y$.*

We know that $d_h(\Sigma_i, \Sigma'_i) \le R_1$. Depending on the nature of $d_h(\Sigma_i \cap Y, \Sigma'_i \cap Y)$ the proof of $(2')$ is broken into the following two cases.

**Case (2′)(i)** Suppose $d_h(\Sigma_i \cap Y, \Sigma'_i \cap Y) \le R_1$. In this case $d_{b'_i}(\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i})$ is uniformly small by Lemma 5.17. By Lemma 5.12 if $b''_i$ is a nearest point projection of $\pi(\bar{y}_i)$ on $U_{R_1}(\Sigma_i \cap Y, \Sigma'_i \cap Y)$ then $F_{b''_i} \cap \Sigma'_i$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma'_i \cap Y$ in $Y$. Thus it is enough to show that $d_B(b''_i, b'_i)$ uniformly bounded to prove that $\bar{y}'_i$ is a uniform approximate nearest point projection of $\bar{y}_i$ on $\Sigma'_i \cap Y$ in $Y$. Then since $\Sigma_i \cap Y$ is $K'_1$-qi section in $Y$ and $d_{b'_i}(\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i})$ is uniformly small it will follow that the concatenation of $\bar{\alpha}_i$ and the fiber geodesic $[\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}]_{b'_i}$ is a uniform quasigeodesic in $Y$.

That $d_B(b''_i, b'_i)$ uniformly bounded is proved as follows. Let $U = U_{R_1}(\Sigma_i, \Sigma'_i)$ and $V = U \cap A = U_{R_1}(\Sigma_i \cap Y, \Sigma'_i \cap Y)$. Since $B$ is $\delta_0$-hyperbolic, $A$ is $k$-qi embedded in $B$ and $V$ is $\lambda_2$-quasiconvex in $A$, $V$ is $K_{2.30}(\delta_0, k, \lambda_2)$-quasiconvex in $B$. Let $k' = \max\{\lambda_2, k_0, K_{4.11}(K_2), K_{2.30}(\delta_0, k, \lambda_2)\}$. Then $A, U, V$ are all $k'$-quasiconvex in $B$. By the definitions of $y_i$'s we know that $\pi(y'_i)$ is the nearest point projection of $\pi(y_i)$ on $U$. Let $\bar{b}'_i$ be a nearest point projection of $\pi(y'_i)$ on $V$. Also $b'_i = \pi(\bar{y}'_i)$ is the nearest point projection of $\pi(y'_i)$ on $A$. On the other hand, $b_i = \pi(\bar{y}_i)$ is a nearest point projection of $\pi(y_i)$ on $A$ and $b''_i$ is the nearest point projection of $b_i$ on $V$. Therefore, $d_B(b''_i, \bar{b}'_i) \le 2D_{2.27}(\delta_0, k', 0)$ by Corollary 2.27.

Now, by Lemma 5.17 $d_{b'_i}(\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}) \le D_{5.17}(K_2, R_1)$. Hence, by Lemma 4.11 $d_A(b'_i, V) \le D_{4.11}(K_2, D_{5.17}(K_2, R_1)/R_1) = D_1$, say. Let $v \in V$ be such that $d_A(b'_i, v) \le D_1$. Then $d_B(b'_i, v) \le kD_1 + k$. Hence,

$$Hd([\pi(y'_i), b'_i]_B, [\pi(y'_i), v]_B) \le \delta_0 + k + kD_1.$$

However, the concatenation $[\pi(y'_i), \bar{b}'_i]_B * [\bar{b}'_i, v]_B$ is a $K_{2.25}(\delta_0, k', 1, 0)$-quasigeodesic. Hence, there is a point $w \in [\pi(y'_i), v]_B$ such that $d_B(w, \bar{b}'_i) \le D_{2.17}\big(\delta_0, K_{2.25}(\delta_0, k', 1, 0), K_{2.25}(\delta_0, k', 1, 0)\big) = D_2$, say. Thus there is a point $w' \in [\pi(y'_i), b'_i]$ such that $d_B(w', \bar{b}'_i) \le D_2 + \delta_0 + k + kD_1 = D_3$, say. But $b'_i$ is a nearest point projection of $\pi(y'_i)$ on $A$ and $\bar{b}'_i \in V \subset A$. Thus $d_B(w', b'_i) \le D_3$. Thus $d_B(\bar{b}'_i, b'_i) \le 2D_3$. Hence, $d_B(b'_i, b''_i) \le d_B(b''_i, \bar{b}'_i) + d_B(\bar{b}'_i, b'_i) \le 2D_{2.27}(\delta_0, k', 0) + 2D_3$.

**Case (2′)(ii)** Suppose $d_h(\Sigma_i \cap Y, \Sigma'_i \cap Y) \ge R_1$. In this case Lemmas 5.15 and 2.25 do the job.

$(2'')$ *$\bar{y}_{i+1}$ is a uniform approximate nearest point projection of $\bar{y}'_i$ on $\Sigma_{i+1} \cap Y$ in $Y$ and the concatenation of $\bar{\beta}$ and the fiber geodesic $[\Sigma'_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}}]_{b_{i+1}}$ is a uniform quasigeodesic joining $\bar{y}'_i$ to $\bar{y}_{i+1}$ in $Y$.*

In this case $d_h(\Sigma'_i \cap Y, \Sigma_{i+1} \cap Y) \le 1$ hence we are done as in Case (2′)(i).

**Case 3** $(i = n-1)$ The proof of this case is also analogous to that of the proof of Case (2′)(i) since $d_h(\Sigma_{n-1}, \Sigma_n) \le R_0$. □

**Remark** The conclusion of Lemma 5.16 is subsumed by Lemmas 5.15 and 5.17. But we still keep Lemma 5.16 for the sake of ease of explanation.

Thus by Lemmas 5.11 and 5.18, we have proved the following.

**Proposition 5.19** *Let $x, y \in Y$ and let $\Sigma$ and $\Sigma'$ be two $K_0$-qi sections in $X$ through $x$ and $y$, respectively. Let $c(x, y)$ be a uniform quasigeodesic in $X$ joining $x$ and $y$ which is contained in $\mathbb{L}(\Sigma, \Sigma')$ as constructed in Step 1(c). Then the corresponding modified path $\tilde{c}(x, y)$, as constructed in Step 2, is a uniform quasigeodesic in $Y$.*

**Step 4** (*verification of the hypothesis of Lemma 2.49*)

**Lemma 5.20** (proper embedding of the pullback $Y$) *The pullback $Y$ is metrically properly embedded in $X$. In fact, the distortion function for $Y$ is the composition of a linear function with $\eta$, the common distortion function for all the fibers of the bundle $X$.*

**Proof** As was done in the proof of the main theorem, we shall assume that $g$ is the inclusion map and $Y = \pi^{-1}(A)$ and $p$ is the restriction of $\pi$. Let $x, y \in Y$ such that $d_X(x, y) \leq M$. Let $\pi(x) = b_1$ and $\pi(y) = b_2$. Then, $d_B(b_1, b_2) \leq M$ and hence $d_A(b_1, b_2) \leq k + kM$. Let $[b_1, b_2]_A$ be a geodesic joining $b_1$ and $b_2$ in $A$. This is a quasigeodesic in $B$. By Lemma 3.8, there exists an isometric section $\gamma$ over $[b_1, b_2]_A$, through $x$ in $Y$. Clearly, $\gamma$ is a qi lift in $X$, say $k'$-qi lift. We have, $l_X(\gamma) \leq k'(kM + k) + k' =: D(M)$. The concatenation of $\gamma$ and the fiber geodesic $[\gamma \cap F_{b_2}, y]_{F_{b_2}}$ is a path, denoted by $\alpha$, joining $x$ and $y$ in $X$. So,

$$d_X(\gamma \cap F_{b_2}, y) \leq d_X(\gamma \cap F_{b_2}, x) + d_X(x, y) \leq l_X(\gamma) + d_X(x, y) \leq D(M) + M.$$

Now, since $F_{b_2}$ is uniformly properly embedded as measured by $\eta$, we have,

$$d_{b_2}(\gamma \cap F_{b_2}, y) \leq \eta(D(M) + M).$$

Now, $\alpha$ lies in $Y$ and $l_Y(\gamma) \leq kM + k$. Then,

$$d_Y(x, y) \leq l_Y(\alpha) \leq l_Y(\gamma) + d_Y(\gamma \cap F_{b_2}, y) \leq kM + k + d_{b_2}(\tilde{\gamma} \cap F_{b_2}, y).$$

Therefore, $d_Y(x, y) \leq kM + k + \eta(D(M) + M)$. Setting $\eta_0(M) := kM + k + \eta(D(M) + M)$, we have the following: for all $x, y \in Y$, $d(x, y) \leq M$ implies $d_Y(x, y) \leq \eta_0(M)$. □

We recall that we fixed a vertex $b_0 \in A$ to define the paths $c(y, y')$ in the last step. Let $y_0 \in F_{b_0}$. However, the following lemma completes the proof of Theorem 5.2.

**Lemma 5.21** *Given $D > 0$, there is $D_1 > 0$ such that the following holds:*

*If $d_X(y_0, c(y, y')) \leq D$ then $d_Y(y_0, \bar{c}(y, y')) \leq D_1$.*

**Proof** Let $x \in c(y, y')$ be such that $d_X(y_0, x) \leq D$. This implies that $d_B(\pi(x), b_0) \leq D$. We recall that the path $c(y, y')$ is a concatenation of $\gamma_j$, $j = 0, 1, \dots, n$. Suppose $x \in \gamma_i$, $0 \leq i \leq n$. We claim that there is a point of $\overline{\gamma}_i$ uniformly close to $y_0$. Now, $\gamma_i$ is either a lift of geodesic segments of $B$ in a

$K_2$-qi section $\Sigma_i$ or possibly $\Sigma'_i$ or it is the concatenation of such a lift and a fiber geodesic of length at most $R_1$. Let $\mathcal{Q}$ denote the corresponding qi section and suppose $c(y,y')\cap\mathcal{Q}$ joins the points $z\in\mathcal{Q}$ to $w\in\mathcal{Q}$. If $i=n$ then $\gamma_i$ is a qi lift of $[\pi(z),\pi(w)]_B$ in $\mathcal{Q}$ joining $z,w$. Otherwise there is a fiber geodesic $\sigma\subset c(y,y')\cap F_{\pi(w)}$ connecting $\mathcal{Q}$ to the next qi section $\mathcal{Q}'$, say. Then both the points $z$ and $\mathcal{Q}'\cap\sigma$ are one of the $y_i$'s or $y'_j$'s. Let $z'=\mathcal{Q}'\cap\sigma$ and $b'=\pi(z')$. Let $b$ be the nearest point projection of $\pi(x)$ on $A$. It follows that $d_B(\pi(x),b)\le D$.

Suppose $x\in\sigma$. By the definition of $\bar{c}(y,y')$ we have $\mathcal{Q}'\cap F_b\in\bar{c}(y,y')$. However, $d_B(b,b_0)\le d_B(b,\pi(x))+d_B(b_0,\pi(x))\le 2D$. Since $A$ is $k$-qi embedded in $B$ we have $d_A(b,b_0)\le k+2Dk$. Hence, $d_Y(\mathcal{Q}'\cap F_{b_0},\mathcal{Q}'\cap F_b)\le K_2+(k+2Dk).K_2$. On the other hand in this case $\pi(x)=b'$ and $d_{b'}(z',x)\le R_1$. Hence, $d_X(z',y_0)\le R_1+D$. Thus $d_X(y_0,\mathcal{Q}'\cap F_{b_0})\le d_X(y_0,x)+d_X(x,z')+d_X(z',\mathcal{Q}'\cap F_{b_0})\le D+R_1+K_2+DK_2$ since $d_B(b,b_0)\le 2D$. Hence,

$$d_Y(y_0,\mathcal{Q}'\cap F_{b_0})\le d_{b_0}(y_0,\mathcal{Q}'\cap F_{b_0})\le\eta(D+R_1+K_2+DK_2).$$

Thus

$$\begin{aligned}d_Y(y_0,\mathcal{Q}'\cap F_b)&\le d_Y(y_0,\mathcal{Q}'\cap F_{b_0})+d_Y(\mathcal{Q}'\cap F_b,\mathcal{Q}'\cap F_{b_0})\\&\le d_Y(y_0,\mathcal{Q}'\cap F_{b_0})+K_2+K_2d_A(b,b_0)\\&\le\eta(D+R_1+K_2+DK_2)+K_2+(k+2Dk)K_2.\end{aligned}$$

Hence, in this case $d_Y(y_0,\bar{c}(y,y'))\le(1+k+2Dk)K_2+\eta(D+R_1+K_2+DK_2)$.

Otherwise suppose $x$ is contained in the lift of $[\pi(z),\pi(w)]_B$ in $\mathcal{Q}$. We note that $\pi(x)\in[\pi(z),\pi(w)]_B$ and $d_B(\pi(x),A)\le D$. Now $A$ is $k_0$-quasiconvex in $B$. Hence, by Lemma 2.29 we have

$$d_B(\pi(x),[\overline{\pi(z)},\overline{\pi(w)}]_B)\le D_{2.29}(D,k_0,\delta).$$

where $\overline{\pi(z)}$, $\overline{\pi(w)}$ are nearest point projections of $\pi(z),\pi(w)$, respectively, on $A$. Since $A$ is $k$-qi embedded in $B$ by stability of quasigeodesics $Hd([\overline{\pi(z)},\overline{\pi(w)}]_B,[\overline{\pi(z)},\overline{\pi(w)}]_A)\le D_{2.17}(\delta,k,k)$. Hence, $d_B(\pi(x),[\overline{\pi(z)},\overline{\pi(w)}]_A)\le D_{2.29}(D,k_0,\delta)+D_{2.17}(\delta,k,k)$. Let $\alpha$ be the lift of $[\overline{\pi(z)},\overline{\pi(w)}]_A$ in $\mathcal{Q}$. Then $\alpha\subset\bar{c}(y,y)$. On the other hand,

$$d_X(x,\alpha)\le K_2+K_2d_B(\pi(x),[\overline{\pi(z)},\overline{\pi(w)}]_A)\le K_2+K_2(D_{2.29}(D,k_0,\delta)+D_{2.17}(\delta,k,k))=D_1,$$

say. Hence, $d_X(y_0,\alpha)\le d_X(y_0,x)+d_X(x,\alpha)\le D+D_1$. This implies that $d_Y(y_0,\alpha)$ is also bounded by a function of $D$ and the other parameters of the metric graph bundles $X$ and $Y$, by Lemma 5.20. □

## 5.2 An example

For the convenience of the reader, we briefly illustrate a special case of our main theorem where $B=\mathbb{R}$, $A=(-\infty,0]$. This discussion will also be used in the proof of the last proposition of the next section. We shall assume $b_0=0$ here.

As in the proof of Lemma 5.21, suppose $\mathcal{Q},\mathcal{Q}'$ are two qi sections among the various $\Sigma_i,\Sigma'_j$'s and let $w'\in\mathcal{Q}'$, $z,w\in\mathcal{Q}$ be points of $c(y,y')$, where $\pi(w')=\pi(w)$, $d_{\pi(w)}(w,w')\le R_1$ and the concatenation

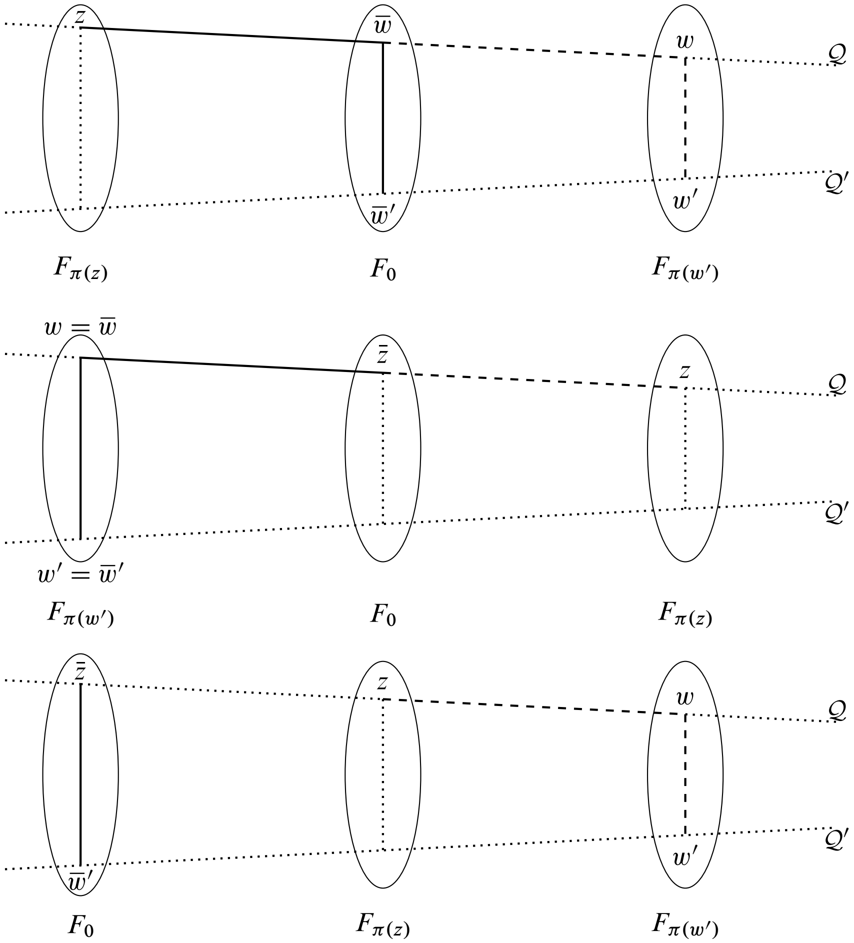

Figure 7: Top: Case 2. Middle: Case 3. Bottom: Case 4. The dashed lines denote the portion of $c(y, y')$, the thick lines denote the portion of $\bar{c}(y, y')$ and dotted lines are portions of the qi sections $\mathcal{Q}, \mathcal{Q}'$.

of the lift say $\alpha$, of $[\pi(z), \pi(w)]$ in $\mathcal{Q}$ and the vertical geodesic segment, say $\sigma$, in $F_{\pi(w)}$ is a part of $c(y, y')$. The following are the possibilities:

**Case 1** If $w', z \in Y \cap c(y, y')$ then $\alpha * \sigma \subset Y$ and it is the corresponding part of $\bar{c}(y, y')$.

**Case 2** $(z \in Y,\ w' \notin Y)$ In this case, the modified segment is formed as the concatenation of subsegment of $\alpha$ joining $z$ to $\mathcal{Q} \cap F_0$ and the fiber geodesic $[\mathcal{Q} \cap F_0, \mathcal{Q}' \cap F_0]_0$. See Figure 7, top.

**Case 3** $(w' \in Y,\ z \notin Y)$ In this case the modified segment is the concatenation of the segment of $\alpha$ from $\mathcal{Q} \cap F_0$ to $w$ and the fiber geodesic segment $\sigma$. See Figure 7, middle.

**Case 4** $(z, w' \notin Y)$ In this case the modified segment is the fiber geodesic $[\mathcal{Q} \cap F_0, \mathcal{Q}' \cap F_0]_0$. See Figure 7, bottom.

# 6 Applications, examples and related results

As the first application of our main theorem, we have the following. Given a short exact sequence of finitely generated groups there is a natural way to associate a metric graph bundle to it as mentioned in Example 1.8 of [24]. See also the example of Section 3.3.1. Having said that, Theorem 5.2 gives the following as an immediate consequence.

**Theorem 6.1** *Suppose* $1 \to N \to G \overset{\pi}{\to} Q \to 1$ *is a short exact sequence of hyperbolic groups where* $N$ *is nonelementary hyperbolic. Suppose* $Q_1$ *is a finitely generated, qi embedded subgroup of* $Q$ *and* $G_1 = \pi^{-1}(Q_1)$. *Then the* $G_1$ *is hyperbolic and the inclusion* $G_1 \to G$ *admits the CT map.*

The next application is in the context of complexes of hyperbolic groups. Suppose $\mathcal{Y}$ is a finite, connected simplicial complex and $\mathbb{G}(\mathcal{Y})$ is a developable complex of nonelementary hyperbolic groups with qi condition defined over $\mathcal{Y}$ (see Section 3.3.2) such that the fundamental group $G$ of the complex of groups is hyperbolic. Suppose we have a good subcomplex $\mathcal{Y}_1 \subset \mathcal{Y}$ and $G_1$ is the image of $\pi_1(\mathcal{G}, \mathcal{Y}_1)$ in $G$ under the natural homomorphism $\pi_1(\mathcal{G}, \mathcal{Y}_1) \to \pi_1(\mathcal{G}, \mathcal{Y})$. Then we have the following pullback diagram as obtained in Proposition 3.29 satisfying the properties of Theorem 5.2:

$$\begin{tikzcd}
X_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & X \arrow[d, "\pi"] \\
B_1 \arrow[r, "i"] & B
\end{tikzcd}$$

Thus we have:

**Theorem 6.2** *The group* $G_1$ *is hyperbolic and the inclusion* $G_1 \to G$ *admits the CT map.*

**Remark** The rest of the paper is devoted to properties of the boundary of metric (graph) bundles and Cannon–Thurston maps. We recall that qi sections, ladders etc for a metric bundle are defined as transport of the same from the canonical metric graph bundle associated to it. *All the results in the rest of the section are meant for metric bundles as well as metric graph bundles. However, using the dictionary provided by Proposition 4.1 it is enough to prove the results only for metric graph bundles. Therefore, we shall state and prove results only for metric graph bundles in what follows starting with the convention below.*

**Convention 6.3** (1) For the rest of the paper we shall assume that $\pi : X \to B$ is a $\delta$-hyperbolic $\eta$-metric graph bundle over $B$ satisfying the hypotheses (H1), (H2), (H3′) and (H4) of Section 5.

(2) By Proposition 2.37 any point of $\partial B$ can be joined to any point of $B \cup \partial B$ and any point of $\partial X$ can be joined to $X \cup \partial X$ by a uniform quasigeodesic ray or line. We shall assume that these are $\kappa_0$-quasigeodesics.

(3) We shall assume that any geodesic in $B$ has a $c$-qi lift in $X$ using the path lifting lemma for metric graph bundles.

(4) We recall that through any point of $X$ there is a $K_0$-qi section over $B$.

## 6.1 Some properties of $\partial X$

**Lemma 6.4** *Suppose* $\alpha, \beta\colon [0,\infty) \to B$ *are two* $k$*-quasigeodesic rays for some* $k \geq 1$ *with* $\alpha(\infty) = \beta(\infty) = \xi$. *Suppose* $\widetilde{\beta}$ *is a* $K$*-qi lift of* $\beta$ *for some* $K \geq 1$. *Then there is a* $K'$*-qi lift* $\widetilde{\alpha}$ *of* $\alpha$ *such that* $\widetilde{\alpha}(\infty) = \widetilde{\beta}(\infty)$, *where* $K'$ *depends on* $k$, $K$, $d_B(\alpha(0), \beta(0))$ *and the various parameters of the metric graph bundle.*

**Proof** Suppose $\alpha, \beta\colon [0,\infty) \to B$ are two $k$-quasigeodesic rays for some $k \geq 1$ with $\alpha(\infty) = \beta(\infty) = \xi$. This means $Hd(\alpha, \beta) < \infty$. Let $R = Hd(\alpha, \beta)$. Then for all $s \in [0,\infty)$ there is $t = t(s) \in [0,\infty)$ such that $d_B(\alpha(s), \beta(t)) \leq R$. Let $\phi_{ts}\colon F_{\beta(t)} \to F_{\alpha(s)}$ be fiber identification maps such that $d_X(x, \phi_{ts}(x)) \leq 3c + 3cR$ for all $x \in F_{\beta(t)}$, $t \in [0,\infty)$, where $c = 1$ for metric graph bundles. (See Lemma 3.10.) Let $\widetilde{\beta}$ be a $K$-qi lift of $\beta$. Now, for all $s \in [0,\infty)$ we define $\widetilde{\alpha}(s) = \phi_{ts}(\widetilde{\beta}(t))$. It is easy to verify that $\widetilde{\alpha}$ thus defined is a uniform qi lift of $\alpha$. Also clearly $\widetilde{\alpha} \subset N_{3c+3cR}(\widetilde{\beta})$. It follows that $\widetilde{\alpha}(\infty) = \widetilde{\beta}(\infty)$ □

**Corollary 6.5** *Let* $\xi \in \partial B$ *and let* $\alpha$ *be a quasigeodesic ray in* $B$ *joining* $b$ *to* $\xi$. *Let*

$$\partial^{\xi}_{\alpha} X := \{\gamma(\infty) : \gamma \text{ is a qi lift of } \alpha\}.$$

*Then* $\partial^{\xi}_{\alpha} X$ *is independent of* $\alpha$; *it is determined by* $\xi$.

Due to the above corollary, we shall use the notation $\partial^{\xi} X$ for all $\xi \in \partial B$ without further explanation. The following proposition is motivated by a similar result proved by Bowditch [5, Proposition 2.3.2].

**Proposition 6.6** *Let* $b \in B$ *be an arbitrary point and* $F = F_b$. *Then we have*

$$\partial X = \Lambda(F) \cup \Big( \coprod_{\xi \in \partial B} \partial^{\xi} X \Big).$$

**Proof** We first fix a point $x \in F$. Let $\gamma$ be a quasigeodesic ray in $X$ starting from $x$. Let $b_n = \pi(\gamma(n))$. Let $\alpha_n$ be a $(1,1)$-quasigeodesic in $B$ joining $b$ to $b_n$. Let $\widetilde{\alpha}_n$ be a $K_0$-qi lift of $\alpha_n$ joining $\gamma(n)$ to $\widetilde{\alpha}_n(b) = x_n \in F$. There are two possibilities.

Suppose $\{x_n\}$ has an unbounded subsequence say $\{x_{n_k}\}$. Then $d(x_{n_k}, x) \to \infty$. We note that the $\widetilde{\alpha}_{n_k}$'s are uniform quasigeodesics in $X$ whose distance from $x$ is going to infinity by Lemma 4.13. Hence, by Lemma 2.34 $x_{n_k} \to \gamma(\infty)$ and thus $\gamma(\infty) \in \Lambda(F)$.

Otherwise, suppose $\{x_n\}$ is a bounded sequence.

**Claim** *In this case* $\pi \circ \gamma$ *is a quasigeodesic ray.*

**Proof of Claim** We note that by stability of quasigeodesics (Corollary 2.19) and slimness of triangles (Corollary 2.20) $Hd(\tilde{\alpha}_n, \gamma|_{[0,n]})$ is uniformly small for all $n$. This implies that $Hd(\alpha_n, (\pi \circ \gamma)|_{[0,n]})$ is uniformly small for all $n$; in particular $d_B(b_m, \alpha_n)$ is uniformly small for all $n \geq m$. Next we note that $d_B(b, b_n) \to \infty$ for otherwise $d(\gamma(n), x)$ will be bounded. Then it follows that $\lim_{m,n\to\infty}(b_m.b_n)_b = \infty$. Let $\xi = \lim_{n\to\infty} b_n$ and let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$. Now, to show that $\pi \circ \gamma$ is a quasigeodesic it is enough to show by Lemma 2.5 that $\pi \circ \gamma$ is (1) uniformly close to $\alpha$ and (2) properly embedded.

(1) Fix an arbitrary $m \in \mathbb{N}$ and consider all $n \geq m$. Since $\lim_{n\to\infty} b_n = \alpha(\infty) = \xi$, by Lemma 2.45(2) for any $\kappa_0$-quasigeodesic ray $\beta_n$ joining $b_n$ to $\xi$ we have $d(b, \beta_n) \to \infty$. Since the triangles with vertices $b_n, b, \xi$ are uniformly slim by Lemma 2.38 and $d_B(b_m, \alpha_n)$ are uniformly small it follows that $b_m$ is uniformly close to $\alpha$. This shows (1).

(2) Since $\pi$ is Lipschitz and $\gamma$ is a quasigeodesic it follows that $\pi \circ \gamma$ is coarsely Lipschitz. Suppose $d_B(b_n, b_m) \leq D$ for some $D \geq 0$ and $m, n \in \mathbb{N}$, $m \leq n$. We claim that $d_X(\gamma(m), \gamma(n))$ is uniformly small. Note that this would then imply that $n - m$ is uniformly small since $\gamma$ is quasigeodesic, and also that $\gamma$ is a qi lift of $\pi \circ \gamma$. We know that $Hd(\tilde{\alpha}_n, \gamma|_{[0,n]}) \leq R$ for some constant $R$ independent of $n$. Hence, $d_X(\gamma(m), \tilde{\alpha}_n) \leq R$. Let $y_{m,n} \in \tilde{\alpha}_n$ be such that $d_X(\gamma(m), y_{m,n}) \leq R$. Since $\pi$ is 1-Lipschitz we have $d_X(b_m, \pi(y_{m,n})) \leq R$. Then $d_B(\pi(y_{m,n}), b_n) \leq d_B(\pi(y_{m,n}), b_m) + d_B(b_m, b_n) \leq R + D$. Since $\tilde{\alpha}_n$ is $K_0$-qi lift of $\alpha_n$ and $\pi \circ \tilde{\alpha}(n) = b_n$ it follows that $d_X(y_{m,n}, \tilde{\alpha}(n)) = d_X(y_{m,n}, \gamma(n)) \leq K_0(R + D) + K_0$. Hence, $d_X(\gamma(m), \gamma(n)) \leq d_X(\gamma(m), y_{m,n}) + d_X(y_{m,n}, \gamma(n)) \leq R + K_0(R + D) + K_0$. Since $\gamma$ is quasigeodesic it follows that $(n - m)$ is uniformly small. This proves (2) and along with this the claim. ◁

It follows that $\gamma(\infty) \in \partial^{\xi} X$.

It remains to check that for all $\xi_1, \xi_2 \in \partial B$, $\partial^{\xi_1} X \cap \partial^{\xi_2} X \neq \varnothing$ implies $\xi_1 = \xi_2$. Suppose $\gamma_i$ is a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi_i$, $i = 1, 2$. Suppose $\tilde{\gamma}_i$ is a qi lift of $\gamma_i$, $i = 1, 2$ such that $\tilde{\gamma}_1(\infty) = \tilde{\gamma}_2(\infty)$, ie $Hd(\tilde{\gamma}_1, \tilde{\gamma}_2) < \infty$. Then $Hd(\gamma_1, \gamma_2) < \infty$ because $\pi \colon X \to B$ is 1-Lipschitz. Thus $\xi_1 = \xi_2$. This finishes the proof. □

**Corollary 6.7** *Suppose $F$ is a bounded metric space. Then $\partial X = \coprod_{\xi \in \partial B} \partial^{\xi} X$.*

For instance suppose $\Sigma_1, \Sigma_2$ are two qi sections and $\mathbb{L} = \mathbb{L}(\Sigma_1, \Sigma_2)$ then by Corollary 4.7 there is a metric graph subbundle $\pi_Z \colon Z \to B$ of $X$ where the bundle map $Z \to X$ is a qi embedding onto a finite neighborhood of $\mathbb{L}$. It follows that $Z$ is hyperbolic and fibers are uniformly quasiisometric to intervals. Therefore, the conclusion of Corollary 6.7 applies to the metric bundle $Z$ too. Hence, informally speaking we have the following.

**Corollary 6.8** *For any ladder $\mathbb{L} = \mathbb{L}(\Sigma_1, \Sigma_2)$ we have*

$$\partial \mathbb{L} = \coprod_{\xi \in \partial B} \partial^{\xi} \mathbb{L}.$$

**Lemma 6.9** *Suppose $b \in B$ and $\alpha_n\colon [0,\infty) \to B$ is a sequence of uniform quasigeodesic rays starting from $b$. Suppose $\tilde{\alpha}_n$ is a uniform qi lift of $\alpha_n$ for all $n$ such that the set $\{\tilde{\alpha}_n(0)\}$ has finite diameter. If $\tilde{\alpha}_n(\infty) \to z \in \partial X$ then $\lim_{n\to\infty} \alpha_n(\infty)$ exists. If $\xi = \lim_{n\to\infty} \alpha_n(\infty)$ and $\alpha\colon [0,\infty) \to B$ is a $\kappa_0$-quasigeodesic ray joining $b$ to $\xi$ then there is a uniform qi lift $\tilde{\alpha}$ of $\alpha$ such that $\tilde{\alpha}(\infty) = z$.*

**Proof** Since $\tilde{\alpha}_n(\infty) \to \xi$ there is a constant $D$ such that for all $M > 0$ there is $N = N(M) > 0$ with $Hd(\tilde{\alpha}_m|_{[0,M]}, \tilde{\alpha}_n|_{[0,M]}) \le D$ for all $m, n \ge N$ by Lemma 2.45(1). It follows that for all $M > 0$, $Hd(\alpha_m|_{[0,M]}, \alpha_n|_{[0,M]}) \le D$ for all $m, n \ge N$. Hence, again by Lemma 2.45(1) $\alpha_n(\infty)$ converges to a point of $\xi \in \partial B$. Let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$. We claim $z \in \partial^{\xi} X$. Given any $t \in [0,\infty)$ by Lemma 2.45(2) there is $N' = N'(t) \in \mathbb{N}$ such that $d(\alpha(t), \alpha_n) \le D'$ for all $n \ge N'$ where $D'$ depends only on $\kappa_0$ and $\delta$. Let $N_0 = \max\{N(t), N'(t)\}$. Let $t'$ be such that $d_X(\alpha(t), \alpha_{N_0}(t')) \le D'$. Define $\tilde{\alpha}(t) = \phi_{uv}(\tilde{\alpha}_{N_0}(t'))$ where $u = \alpha_{N_0}(t')$, $v = \alpha(t)$ and $\phi_{uv}$ is a fiber identification map. It is now easy to check that this defines a qi section over $\alpha$ and $z = \tilde{\alpha}(\infty)$. □

**Corollary 6.10** *If fibers of the metric (graph) bundle are of finite diameter then the map $\partial X = \bigcup_{\xi\in\partial B} \partial^{\xi} X \to \partial B$ defined by sending $\partial^{\xi} X$ to $\xi$ for all $\xi \in \partial B$ is continuous.*

### 6.2 Cannon–Thurston lamination

Suppose $b_0 \in B$ is an arbitrary point and $F = F_{b_0}$. Then we know that the inclusion $i = i_{F,X}\colon F \hookrightarrow X$ admits the CT map $\partial i\colon \partial F \to \partial X$. For any set $S$ we define

$$S^{(2)} = \{(a,b) \in S \times S : a \neq b\}.$$

Now, following Mitra [19] we define the following.

**Definition 6.11** (1) **Cannon–Thurston lamination** Let $\partial^{(2)}_X(F) = \{(\alpha,\beta) \in \partial^{(2)} F : \partial i(\alpha) = \partial i(\beta)\}$.

(2) Suppose $\xi \in \partial B$. Let $\partial^{(2)}_{\xi,X}(F) = \{(\alpha,\beta) \in \partial^{(2)} F : \partial i(\alpha) = \partial i(\beta) \in \partial^{\xi} X\}$. *We shall denote $\partial^{(2)}_{\xi,X}(F)$ simply by $\partial^{(2)}_{\xi}(F)$ when $X$ is understood.*

In this subsection we are going to discuss the various properties of the CT lamination. First we need some definitions. We recall that for all $b, s \in B$ we have the fiber identification map $\phi_{bs}\colon F_b \to F_s$ which is a uniform quasiisometry depending on $d_B(b,s)$. This induces a bijection $\partial\phi_{bs}\colon \partial F_b \to \partial F_s$. Suppose $z \in \partial F_b$. Let $z_s = \partial\phi_{bs}(z)$ for all $s \in B$.

**Convention 6.12** For the rest of the subsection, by "quasigeodesic rays" or "lines", we shall always mean $\kappa_0$-quasigeodesic rays and lines in the fibers of a metric (graph) bundle unless otherwise specified.

**Definition 6.13** (1) **Semi-infinite ladders** Suppose $\Sigma_1$ is a qi section over $B$ in $X$. For all $s \in B$ let $\gamma_s \subset F_s$ be a (uniform) quasigeodesic ray joining $\Sigma_1 \cap F_s$ to $z_s = \partial\phi_{bs}(z)$. The union of all the rays will be denoted by $\mathbb{L}(\Sigma_1; z)$.

*This set is coarsely well-defined by Lemma 2.38. We shall refer to this as the* ***semi-infinite ladder*** *defined by* $\Sigma_1$ *and* $z$.

(2) **Bi-infinite ladders** Suppose $b \in B$ and $z, z' \in \partial F_b$, $z \neq z'$. Now for all $s \in B$ join $z_s = \partial\phi_{bs}(z)$ to $z'_s = \partial\phi_{bs}(z')$ by a (uniform) quasigeodesic line in $F_s$. The union of all these lines will be denoted by $\mathbb{L}(z; z')$.

*As before, this set is coarsely well-defined by Lemma 2.38. We shall refer to this as the* ***bi-infinite ladder*** *defined by* $z$ *and* $z'$.

We shall refer to either of these ladders as an "infinite girth ladder".

**Lemma 6.14** (properties of infinite girth ladders) *Suppose* $\mathbb{L}$ *is an infinite girth ladder.*

1. ***Coarse retract*** *There is a uniformly coarsely Lipschitz retraction* $\pi_{\mathbb{L}}\colon X \to \mathbb{L}$ *such that for all* $b \in B$ *and* $x \in F_b$, $\pi_{\mathbb{L}}(x)$ *is a (uniform approximate) nearest point projection of* $x$ *in* $F_b$ *on* $\mathbb{L} \cap F_b$. *Consequently, infinite girth ladders are uniformly quasiconvex and their uniformly small neighborhoods are qi embedded in* $X$.
2. ***QI sections in ladders*** *Through any point of* $\mathbb{L}$, *there exists a uniform qi section contained in* $\mathbb{L}$.
3. ***QI sections coarsely bisect ladders*** *Any qi section in* $\mathbb{L}$ *coarsely bisects it into two subladders.*

**Proof** We shall briefly indicate the proofs comparing with the proof of the analogous results for finite girth ladders. Property (3) follows exactly as Lemma 5.4. Property (2) is immediate from (1). In fact given $x \in \mathbb{L}$ one takes a $K_0$-qi section $\Sigma$ in $X$ containing $x$ and then $\pi_{\mathbb{L}}(\Sigma)$ is the required qi section. Therefore, we are left with proving (1). This is an exact analog of Proposition 4.6(1). The reader is referred to [19, Theorem 4.6] for supporting arguments. □

**Convention 6.15** All semi-infinite ladders $\mathbb{L}(\Sigma; z)$ are formed by $K_0$-qi section $\Sigma$. We shall assume that through any point of an infinite girth ladder there is a $\bar{K}_0$-qi section contained in the ladder. Also, all infinite girth ladders are assumed to be $\bar{\lambda}_0$-quasiconvex.

**6.2.1 Properties of the CT lamination $\partial^{(2)}_X(F)$** In this subsection, we prove many properties of the CT lamination using coarse bisection of ladders by qi sections. These are motivated by analogous results proved in [5; 19]. For the rest of the subsection, we will use the following set up. Let $b_0 \in B$ and $F = F_{b_0}$. Suppose $(z_1, z_2) \in \partial^{(2)} = \partial^{(2)}_X(F)$ and $\mathbb{L} = \mathbb{L}(z_1; z_2)$. Let $\gamma\colon \mathbb{R} \to F$ be a $\kappa_0$-quasigeodesic line in $F$ joining $z_1$ to $z_2$ such that $\mathrm{Im}(\gamma) = \mathbb{L} \cap F$. Let $i_{F,X}\colon F \to X$ denote the inclusion map and $\partial i_{F,X}\colon \partial F \to \partial X$ denote the CT map.

**Lemma 6.16** *Suppose $\Sigma$ is any qi section contained in $\mathbb{L}$. Then $\partial i_{F,X}(z_i) \in \Lambda(\Sigma)$, $i = 1, 2$.*

**Proof** Let $\Sigma$ be a qi section contained in $\mathbb{L}$. Then $\Sigma$ coarsely separates $\mathbb{L}$ in $X$ into $\mathbb{L}_1 = \mathbb{L}(\Sigma; z_1)$ and $\mathbb{L}_2 = \mathbb{L}(\Sigma; z_2)$. We note that $\partial i_{F,X}(z_1) = \partial i_{F,X}(z_2) \in \Lambda(\mathbb{L}_1) \cap \Lambda(\mathbb{L}_2)$. Hence we are done by Lemma 2.53. □

**Lemma 6.17** *Suppose $(z_1, z_2) \in \partial^{(2)}_X(F)$ and $\mathbb{L} = \mathbb{L}(z_1; z_2)$. There is a unique $\xi \in \partial B$ such that $(z_1, z_2) \in \partial^{(2)}_{\xi,X}(F)$. Moreover, for any $\kappa_0$-quasigeodesic $\beta\colon [0,\infty) \to B$ joining $b_0$ to $\xi$ and any qi section $\Sigma$ contained in $\mathbb{L}$, if $\tilde{\beta}$ is the lift of $\beta$ in $\Sigma$ then $\tilde{\beta}(\infty) = \partial i_{F,X}(z_1) = \partial i_{F,X}(z_2)$.*

*In particular $\partial^{(2)}_X(F) = \coprod_{\xi\in\partial B} \partial^{(2)}_{\xi}(F)$.*

**Proof** Let $\sigma\colon B \to X$ be a qi section with image $\Sigma$ contained in $\mathbb{L}$. By Lemma 6.16 $\partial i_{F,X}(z_1) \in \Lambda(\Sigma)$. But $\Lambda(\Sigma) = \partial\sigma(\partial B)$ by Lemma 2.55. Hence, there is a $\kappa_0$-quasigeodesic ray $\beta\colon [0,\infty) \to B$ such that $\partial\sigma(\beta(\infty)) = \partial i_{F,X}(z_1)$. Let $\xi = \beta(\infty)$. If $\tilde{\beta} = \sigma \circ \beta$ then $\tilde{\beta}$ is a qi lift of $\beta$ and $\partial i_{F,X}(z_1) = \tilde{\beta}(\infty) \in \partial^{\xi} X$. Thus $(z_1, z_2) \in \partial^{(2)}_{\xi,X}(F)$. This shows the existence of $\xi$. Thus we have $\partial^{(2)}_X(F) = \bigcup_{\xi\in\partial B} \partial^{(2)}_{\xi}(F)$. Also for $\xi, \xi' \in \partial B$, $\xi \neq \xi'$ we have $\partial^{\xi_1} X \cap \partial^{\xi_2} X = \varnothing$ by Proposition 6.6 which immediately implies $\partial^{(2)}_{\xi,X}(F) \cap \partial^{(2)}_{\xi',X}(F) = \varnothing$. This shows that the point $\xi$ is independent of the chosen section $\Sigma$ in $\mathbb{L}$. The last part of the lemma is immediate from these observations. □

We next aim to show that the sets $\partial^{(2)}_{\xi,X}(F)$ are closed subsets of $\partial^{(2)}_X(F)$. Let $\beta\colon [0,\infty) \to B$ be a continuous, arc length parametrized $\kappa_0$-quasigeodesic in $B$ with $\beta(0) = b_0$ and $\beta(\infty) = \xi$ as in the proof of Lemma 6.17. Let $A = \beta([0,\infty))$. Let $Y = \pi^{-1}(A)$ be the restriction of the bundle $X$ over $A$. Let $i_{Y,X}\colon Y \to X$, $i_{F,Y}\colon F \to Y$ be inclusion maps.

**Lemma 6.18** *If $(z_1, z_2) \in \partial^{(2)}_{\xi,X}(F)$ then $\partial i_{F,Y}(z_1) = \partial i_{F,Y}(z_2)$, ie $(z_1, z_2) \in \partial^{(2)}_{\xi,Y}(F)$.*

**Proof** Let $\Sigma_n$ be any qi section in $\mathbb{L}$ over $B$ passing through $\gamma(n)$, $n \in \mathbb{Z}$. Then by Lemma 6.17, $\Sigma_m \cap Y$ and $\Sigma_n \cap Y$ are asymptotic for all $m, n \in \mathbb{Z}$ in $X$. Since $Y$ is properly embedded in $X$ by Lemma 5.20 they are still asymptotic in $Y$. Clearly $d_Y(\gamma(0), \Sigma_n \cap Y) \to \infty$ as $n \to \pm\infty$. Thus by Lemma 2.45(1) $\lim_{n\to\pm\infty} \gamma(n) = \tilde{\beta}_0(\infty)$ in $Y$ where $\tilde{\beta}_0$ is the lift of $\beta$ in $\Sigma_0$. This completes the proof. □

**Corollary 6.19** *Let $\tilde{\beta}$ be any qi lift of $\beta$ in $\mathbb{L}$. Then $\tilde{\beta}(\infty) = \partial i_{F,X}(z_1)$. In particular any two qi lifts of $\beta$ in $\mathbb{L}$ are asymptotic.*

**Proof** We know that $\tilde{\beta}$ coarsely separates $\mathbb{L} \cap Y$ into two semi-infinite ladders, $\mathbb{L}^+$ and $\mathbb{L}^-$ in $Y$. It follows that $\Lambda(\mathbb{L}^+) \cap \Lambda(\mathbb{L}^-) = \Lambda(\tilde{\beta}) = \tilde{\beta}(\infty)$. It then follows that the limit of $\gamma(n)$ in $\partial\mathbb{L}$ is $\tilde{\beta}(\infty)$. □

**Corollary 6.20** (1) *$\partial(\mathbb{L} \cap Y)$ is a point.* (2) *$\Lambda_Y(\mathbb{L} \cap Y)$ is a point.* (3) *$\Lambda_X(\mathbb{L} \cap Y)$ is a point.*

**Proof** We know by Lemma 6.14(1) (see also Proposition 4.6(4)) that a small neighborhood, say $\mathbb{L}'_Y = N_R(\mathbb{L} \cap Y)$, of $\mathbb{L} \cap Y$ in $Y$ is qi embedded in $Y$ and hence it is a hyperbolic metric space by its own right. Also, this is a subbundle of $Y$ by Corollary 4.7.

(1) The first part is an informal way of saying that $\partial(\mathbb{L}'_Y)$ is a point. However, this is immediate from Proposition 6.6 and Corollary 6.19.

(2) By Lemma 2.55 $\Lambda_Y(\mathbb{L}'_Y)$ is the image of the CT map for the inclusion $\mathbb{L}'_Y \to Y$ since $\mathbb{L}'_Y$ is qi embedded in $Y$. But $\partial\mathbb{L}'_Y$ is a point by the first part. Thus $\Lambda_Y(\mathbb{L}'_Y)$ is a singleton. Finally, $\Lambda_Y(\mathbb{L}'_Y) = \Lambda_Y(\mathbb{L} \cap Y)$ by Lemma 2.52. Hence we are done.

(3) Lastly, it follows that $\mathbb{L} \cap Y$ is quasiconvex in $X$ too since by Corollary 6.19 $\mathbb{L} \cap Y$ is the union of qi lifts of $\beta$ contained in $\mathbb{L} \cap Y$ all of which converge to the same point of $\partial X$. Hence $\mathbb{L}'_Y$ is also quasiconvex in $X$. Since $Y$ is properly embedded in $X$ by Lemma 5.20 and $\mathbb{L}'_Y$ is qi embedded in $Y$ it follows that $\mathbb{L}'_Y$ is properly embedded in $X$. Thus $\mathbb{L}'_Y$ is qi embedded in $X$ by Lemma 2.24(2). As in (2) we are done by Lemma 2.55. □

**Corollary 6.21**
$$\partial^{(2)}_Y(F) = \partial^{(2)}_{\xi,Y}(F) = \partial^{(2)}_{\xi,X}(F).$$

*In particular, each* $\partial^{(2)}_{\xi,X}(F)$ *is a closed subset of* $\partial^{(2)}F$.

**Proof** The first equality follows from Lemma 6.17 applied to the metric bundle $Y$ over $A$. We will now prove the second one. Since $\partial i_{F,X} = \partial i_{Y,X} \circ \partial i_{F,Y}$, clearly $\partial^{(2)}_{\xi,Y}(F) \subset \partial^{(2)}_{\xi,X}(F)$. The opposite inclusion is an immediate consequence of Lemma 6.18.

Since $\partial i_{F,Y}$ is continuous it follows that $\partial^{(2)}_{\xi,X}(F)$ is a closed subset of $\partial^{(2)}F$. One has to use the standard fact that the Gromov boundaries are Hausdorff spaces. □

The following three results are motivated by similar results proved in [19]. The proof ideas are very similar. However, we get rid of the group actions that were there and in our setting properness is never needed.

**Definition 6.22** Suppose $Z_1, Z_2$ are hyperbolic metric spaces. Suppose $f: Z_1 \to Z_2$ is a metrically proper map that admits the CT map. If $\gamma \subset Z_1$ is a quasigeodesic line such that $\partial f(\gamma(\infty)) = \partial f(\gamma(-\infty))$ then we refer to $\gamma$ as a *leaf* of the CT lamination $\partial^{(2)}_{Z_2}(Z_1)$.

We recall that in our context the quasigeodesic lines are assumed to be $\kappa_0$-quasigeodesic lines.

**Lemma 6.23** *Suppose* $\xi_1 \neq \xi_2 \in \partial B$. *Given* $D > 0$ *there exists* $R = R_{6.23}(D) > 0$ *such that the following holds:*

*Suppose* $\gamma_1$ *is a leaf of* $\partial^{(2)}_{\xi_1,X}(F)$ *and* $\gamma_2$ *is a leaf of* $\partial^{(2)}_{\xi_2,X}(F)$. *Then* $\gamma_1 \cap N_D(\gamma_2)$ *has diameter less than* $R$.

**Proof** Let $\alpha$ be a $\kappa_0$-quasigeodesic line in $B$ joining $\xi_1, \xi_2$. Let $b_0' \in \alpha$ be a nearest point projection of $b_0$ on $\alpha$. Let $c$ be a geodesic in $B$ joining $b_0$ to $b_0'$. Let $\alpha_i$ be the concatenation of $c$ with the portion of $\alpha$ joining $b_0'$ to $\xi_i$, $i = 1, 2$. We note that $\kappa_0$-quasigeodesics in $B$ are $D_{2.17}(\delta_0, \kappa_0, \kappa_0)$-quasiconvex by stability of quasigeodesics. Let $K = D_{2.17}(\delta_0, \kappa_0, \kappa_0)$. Hence, the $\alpha_i$'s are $K_{2.25}(\delta_0, K, \kappa_0, 1)$-quasigeodesics by Lemma 2.25(2). Let $k = K_{2.25}(\delta_0, K, \kappa_0, 1)$.

Next suppose $x_i, x_i' \in \gamma_i$, $i = 1, 2$ are such that $d_F(x_1, x_2) \le D$ and $d_F(x_1', x_2') \le D$. Let $\Sigma_i$, $\Sigma_i'$ be two qi sections in each $\mathbb{L}_i = \mathbb{L}(\gamma_i(\infty), \gamma_i(-\infty))$ passing through $x_i$ and $x_i'$, respectively, $i = 1, 2$. Let $\tilde{\alpha}_i$ and $\tilde{\alpha}_i'$ be lifts of $\alpha_i$ in $\mathbb{L}_i$ through $x_i$ and $x_i'$ respectively for $i = 1, 2$. We now look at the quasigeodesic hexagon in $X$ with vertices $x_i, x_i', \xi_i$, $i = 1, 2$, where the $\tilde{\alpha}_i$'s and $\tilde{\alpha}_i'$'s form four sides and the other two sides are formed by geodesics joining $x_1$ to $x_2$ and $x_1'$ to $x_2'$, respectively. We note that the infinite sides of this polygon are all $(k\bar{K}_0 + k + \bar{K}_0)$-quasigeodesics. Let $\tilde{k} = k\bar{K}_0 + k + \bar{K}_0$. Hence, such a hexagon is $R_{2.39}(\delta, \tilde{k}, 6)$-slim by Corollary 2.39. Let $R_1 = R_{2.39}(\delta, \tilde{k}, 6)$. Let $b_2$ be a point on $\alpha_2$ such that $d_B(b_2, \alpha_1) = D + R_1 + 1 = R$, say and let $y_2 = \tilde{\alpha}_2(b_2)$. Then $y_2 \in N_{R_1}(\tilde{\alpha}_2')$. In particular, $y_2 \in N_R(\Sigma_2')$. Hence, by Lemma 4.13 $d_{b_2}(\Sigma_2 \cap F_{b_2}, \Sigma_2' \cap F_{b_2}) \le R_{4.13}(\bar{K}_0, R)$. It follows by bounded flaring that $d_{b_0}(x_2, x_2') \le \mu_{\tilde{k}}(R_{4.13}(\bar{K}_0, R))$. □

**Lemma 6.24** *If $\xi_n \to \xi$ in $\partial B$, $(z_n, w_n) \in \partial^{(2)}_{\xi_n, X}(F)$ and $(z_n, w_n) \to (z, w) \in \partial^{(2)} F$. Then*

$$(z, w) \in \partial^{(2)}_{\xi, X}(F).$$

**Proof** Since $\partial i_{F,X}(z_n) = \partial i_{F,X}(w_n)$ for all $n$ and $\partial i_{F,X}$ is continuous it follows that $\partial i_{F,X}(z) = \partial i_{F,X}(w)$, whence $(z, w) \in \partial^{(2)}_X F$. Let $[z_n, w_n], [z_n, z], [w_n, w]$ and $[z, w]$ denote $\kappa_0$-quasigeodesic lines in $F$ joining these pairs of points. Let $x \in [z, w] \cap F$ and let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$.

**Claim** *There is a uniform qi lift $\tilde{\alpha}$ of $\alpha$ through $x$ such that $\tilde{\alpha}(\infty) = \partial i_{F,X}(z) = \partial i_{F,X}(w)$.*

**Proof of Claim** Since $z_n \to z$ and $w_n \to w$ by Lemma 2.45(1), we have $d_{b_0}(x, [z_n, z]) \to \infty$ and $d_{b_0}(x, [w_n, w]) \to \infty$. Hence, by Corollary 2.39 there is $N \in \mathbb{N}$ such that $d_{b_0}(x, [z_n, w_n]) \le R = R_{2.39}(\delta_0, \kappa_0, 4)$ for all $n \ge N$. Now, let $x_n \in [z_n, w_n]$ such that $d_{b_0}(x, x_n) \le R$. Let $\alpha_n$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi_n$. Then by Corollary 6.19 we know that there is a uniform qi lift $\tilde{\alpha}_n$ of each $\alpha_n$, $n \ge N$ such that $\tilde{\alpha}_n(0) = x_n$ and $\tilde{\alpha}_n(\infty) = \partial i_{F,X}(z_n)$. Hence, by Lemmas 6.9 and 6.4 there is a qi lift $\tilde{\alpha}$ starting from $x$ such that $\tilde{\alpha}(\infty) = \partial i_{F,X}(z) = \partial i_{F,X}(w)$. ◁

However, this means that $\partial i_{F,X}(z) = \partial i_{F,X}(w) \in \partial^{\xi} X$. Therefore, $(z, w) \in \partial^{(2)}_{\xi, X}(F)$. □

**6.2.2 Leaves of CT laminations for pullback bundles** The following result is motivated by a similar result proved in [17] for trees of hyperbolic spaces which in turn was suggested by Mahan Mj. We gratefully acknowledge the same.

*Suppose we have the hypotheses of Theorem 5.2. We identify $Y$ as a subspace of $X$ and $A$ as a subspace of $B$. Similarly, $\partial A$ is identified as a subset of $\partial B$. With that in mind, we have the following:*

**Theorem 6.25** *Suppose we have a metric graph bundle satisfying the hypotheses of Theorem 5.2 such that the fibers of the bundle are all proper metric spaces. Suppose* $\gamma$ *is a quasigeodesic line in* $Y$ *such that* $(\gamma(\infty), \gamma(-\infty)) \in \partial^{(2)}_X(Y)$. *Let* $F = F_b$ *be any fiber of* $Y$*.*

*Then*:

1. $\gamma(\pm\infty) \in \partial i_{F,Y}(\partial F)$.
2. *There is a point* $\xi \in \partial B \setminus \partial A$ *determined by* $\gamma(\pm\infty)$ *such that if* $z_\pm \in \partial F$ *with* $\partial_{F,Y}(z_\pm) = \gamma(\pm)$ *then* $(z_+, z_-) \in \partial^{(2)}_{\xi,X}(F)$.
3. $\pi(\gamma)$ *is bounded. Moreover,* $\gamma$ *is within a finite Hausdorff distance from a* $\kappa_0$*-quasigeodesic line* $\sigma$ *of* $F$ *so that* $\partial i_{F,Y}(\sigma(\pm\infty)) = \gamma(\pm\infty)$. *Also,* $(\sigma(\infty), \sigma(-\infty)) \in \partial^{(2)}_{\xi,X}(F)$ *for some* $\xi \in \partial B \setminus \partial A$.
4. *If* $b$ *is a nearest point projection of* $\xi$ *on* $A$. *Then* $\sigma$ *(as defined in* (3)*) is a uniform quasigeodesic line in* $Y$*.*

**Proof** We have $\partial Y = \Lambda_Y(F) \cup \big(\bigcup_{\xi\in\partial A} \partial^\xi Y\big)$ by Proposition 6.6. Also since $F$ is a proper metric space, by Lemma 2.55 $\Lambda_Y(F) = \partial i_{F,Y}(\partial F)$. Thus $\partial Y = \partial i_{F,Y}(\partial F) \cup \big(\bigcup_{\xi\in\partial A} \partial^\xi Y\big)$. We shall use the following observation a few times in the proof, which is immediate from the fact that $A$ is qi embedded in $B$:

*Suppose* $\alpha$ *is a quasigeodesic ray in* $A$ *and* $\tilde\alpha$ *is a qi lift of* $\alpha$ *in* $Y$*. Then* $\tilde\alpha$ *is a quasigeodesic ray in* $Y$ *as well as in* $X$*. Also any pair of such rays are asymptotic in* $Y$ *if and only if they are asymptotic in* $X$ *since* $Y$ *is properly embedded in* $X$*.*

(1) The proof of this assertion is by elimination of the possibilities coming from the decomposition $\partial i_{F,Y}(\partial F) \cup \big(\bigcup_{\xi\in\partial A} \partial^\xi Y\big)$ of $\partial Y$.

Suppose $\gamma(\infty) \in \partial^{\xi_1} Y$ and $\gamma(-\infty) \in \partial^{\xi_2} Y$ for some $\xi_1, \xi_2 \in \partial A$. However, this case is not possible due to the above observation.

Suppose $\gamma(\infty) \in \partial^\xi Y$ for some $\xi \in \partial A$ and $\gamma(-\infty) \in \partial i_{F,Y}(\partial F) \setminus \bigcup_{\xi\in\partial A} \partial^\xi Y$ or vice versa. We show below that this case is also not possible.

Let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $A$ joining $b$ to $\xi$ and let $\tilde\alpha$ be a $K_0$-qi lift of $\alpha$ in $Y$ such that $\tilde\alpha(\infty) = \gamma(\infty)$. Also let $\beta$ be a $\kappa_0$-quasigeodesic ray in $F$ such that $\partial i_{F,Y}(\beta(\infty)) = \gamma(-\infty)$. Now, for all $n \in \mathbb{N}$ let $\Sigma_n$ be a $K_0$-qi section in $X$ passing through $\beta(n)$ and let $\mathbb{L}_n = \mathbb{L}(\Sigma_n, \beta(\infty))$. Then $\mathbb{L}_n$ is $\bar\lambda_0$-quasiconvex in $X$. Clearly $\gamma(\infty) = \tilde\alpha(\infty) \in \Lambda_X(\mathbb{L}_n)$. Hence, by Lemma 2.54, $\tilde\alpha$ is asymptotic to $\mathbb{L}_n$. It follows by Proposition 4.6 and Lemma 4.15 that $\pi_{\mathbb{L}_n}(\tilde\alpha)$ is a uniform qi lift of $\alpha$ and it is asymptotic to $\tilde\alpha$. Since $Y$ properly embedded in $X$ by Lemma 5.20, it follows that these qi lifts are asymptotic in $Y$ too. In particular, $\pi_{\mathbb{L}_n}(\tilde\alpha)(\infty) = \gamma(\infty)$. Now, since $d_F(\beta(0), \beta(n)) \to \infty$, by Lemma 4.15 $d_Y(\beta(0), \pi_{\mathbb{L}_n}(\tilde\alpha)) \to \infty$. It follows from Lemma 2.45 that $\lim_{n\to\infty} \beta(n) = \gamma(\infty)$ in $\partial Y$. This gives a contradiction since $\lim_{n\to\infty} \beta(n) = \gamma(-\infty) \neq \gamma(\infty)$.

Therefore, the only possibility is that

$$\gamma(\pm\infty) \in \partial i_{F,Y}(\partial F) \setminus \bigcup_{\xi \in \partial A} \partial^{\xi} Y,$$

proving part (1) of the theorem.

Let $z, z' \in \partial F$ be such that $\partial i_{F,Y}(z) = \gamma(\infty)$ and $\partial i_{F,Y}(z') = \gamma(-\infty)$.

(2) Since $\partial i_{F,X} = \partial i_{Y,X} \circ \partial i_{F,Y}$ by Lemma 2.50(1), we have $(z, z') \in \partial^{(2)}_{X}(F)$ and hence $(z, z') \in \partial^{(2)}_{\xi,X}(F)$ for some $\xi \in \partial B$ by Lemma 6.17. From Corollary 6.21 it follows that $\xi \in \partial B \setminus \partial A$. This proves part (2) of the theorem.

(3) Let $\mathbb{L} = \mathbb{L}(z; z')$ be the bi-infinite ladder in $X$ formed by $z, z'$. Let $\sigma = \mathbb{L} \cap F$ which is an arc length parametrized $\kappa_0$-quasigeodesic line in $F$ joining $z, z'$. Let $\alpha$ be a $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$.

Let $\Sigma_n$ be a $\overline{K}_0$-qi section in $\mathbb{L}$ passing through $\sigma(n)$, $n \in \mathbb{N}$. By Corollary 6.19 qi lifts of $\alpha$ contained in these qi sections are asymptotic. Denote the qi section of $\alpha$ contained in $\Sigma_n$ by $\tilde{\alpha}_n$. We note that these are $k = (\overline{K}_0\kappa_0 + \overline{K}_0 + \kappa_0)$-quasigeodesics by Lemma 2.3(2). Hence, by Lemma 2.38 given $m, n \in \mathbb{N}$ we have $\tilde{\alpha}_n(i) \in N_R(\tilde{\alpha}_{-m})$ (and $\tilde{\alpha}_{-m}(i) \in N_R(\tilde{\alpha}_n)$), where $R = D_{2.38}(\delta, k)$ as long as $\tilde{\alpha}_n(i)$ (resp. $\tilde{\alpha}_{-m}(i)$) is not contained in the $R$-neighborhood of any 1-quasigeodesic joining $\sigma(-m), \sigma(n)$. In particular for such $i$ we have $\tilde{\alpha}_n(i) \in N_R(\Sigma_{-m})$, $\tilde{\alpha}_{-m}(i) \in N_R(\Sigma_n)$. Hence, by Lemma 4.13 we have

$$d_{\alpha(i)}(\tilde{\alpha}_n(i), \tilde{\alpha}_{-m}(i)) \le R_1 = R_{4.13}(R, \overline{K}_0)$$

for all such $i$. Let $R_2 = \max\{R_1, M_{\overline{K}_0}\}$. Thus for all $n \in \mathbb{N}$, $U_n = U_{R_2}(\Sigma_n, \Sigma_{-n}) \neq \varnothing$. Let $b_n \in U_n$ be a nearest point projection of $b$ on $U_n$ and let $b'_n$ be a nearest point projection of $b_n$ on $A$. Then it follows from Lemma 5.18 that the concatenation of the segments of $\tilde{\alpha}_n, \tilde{\alpha}_{-n}$ over the portion of $\alpha$ joining $b, b'_n$ and the fiber geodesic segment $\mathbb{L} \cap F_{b'_n}$ is a uniform quasigeodesic in $Y$ joining $\sigma(\pm n)$. Call it $\gamma'_n$. Since $\lim_{n\to\infty} \sigma(n) \neq \lim_{n\to\infty} \sigma(-n)$ in $Y$ there is a constant $D \ge 0$ such that $d_Y(\sigma(0), \gamma'_n) \le D$ by Lemma 2.34. We claim that this means $d_B(b, b'_n)$ is bounded. In fact $d_Y(\sigma(0), \tilde{\alpha}_{\pm n}) \to \infty$ by Lemma 4.13. Thus for all large $n$ we have $d_Y(\sigma(0), \mathbb{L} \cap F_{b'_n}) \le D$, whence $d_B(b, b'_n) \le D$. It follows from Proposition 4.6(3) that the Hausdorff distance of $\mathbb{L} \cap F_{b'_n}$ and the segment of $\sigma$ between $\sigma(n)$ and $\sigma(-n)$ is at most $(1+2K_0)C_{4.6}(\overline{K}_0)$. Since $\sigma$ is a proper embedding in $Y$ it follows by Lemma 2.5 that $\sigma$ is a uniform quasigeodesic in $Y$ depending on $D$. Let $K \ge 1$ be such that both $\sigma$ and $\gamma$ are $K$-quasigeodesics in $Y$. Then, since $Y$ is $\delta'$-hyperbolic, $Hd(\sigma, \gamma) \le R_{2.39}(\delta', K, 2)$. Thus $\operatorname{diam}(\pi(\gamma)) \le R_{2.39}(\delta', K, 2)$.

We note here that $\operatorname{diam}(\pi(\gamma))$ as well as the quasigeodesic constant of $\sigma$ depends only on $\max\{d_B(b, b'_n)\}$.

(4) We shall use the notation of the proof of (3). Thus we know that there is $D_n \ge 0$ such that for all $i \ge D_n$ we have $d_{\alpha(i)}(\tilde{\alpha}_n(i), \tilde{\alpha}_{-n}(i)) \le R_1$ whence $\alpha(i) \in U_n$ for all $i \ge D_n$. Also we know that the sets $U_n$ are $K_{4.11}(\overline{K}_0)$-quasiconvex in $B$ by Lemma 4.11. Let $t_n \ge \max\{D_n, d_B(b, b_n)\}$. Then $\alpha(t_n) \in U_n$. Thus $[b, b_n]_B * [b_n.\alpha(t_n)]_B$ is a $K_{2.25}(\delta_0, K_{4.11}(\overline{K}_0), \kappa_0, \epsilon)$-quasigeodesic segment. Let

$K' = \max\{\kappa_0, K_{2.25}(\delta_0, K_{4.11}(\bar{K}_0), \kappa_0, \epsilon)\}$. Hence, by stability of quasigeodesics (Lemma 2.17) we get that $b_n \in N_{R'}(\alpha)$ where $R' = D_{2.17}(\delta_0, K', K')$. We also note that $d_B(b, b_n) \to \infty$ by the bounded flaring condition (Corollary 3.12) since $d_b(\tilde{\alpha}_n(0), \tilde{\alpha}_{-n}(0)) \to \infty$. This implies that $b_n \to \xi$. Hence by Lemma 2.56 there exists $N > 0$ such that $d(b'_n, b) \le R_{2.56}(\delta_0, k_0)$ for all $n \ge N$ since $B$ is $\delta_0$-hyperbolic and $A$ is $k_0$-quasiconvex. Hence, we are done by the note left at the end of the proof of (3). □

### Surjectivity of the CT maps

**Theorem 6.26** *Suppose we have the hypotheses of Theorem 5.2 such that the fibers of the bundle are proper metric spaces. Let $F$ be the fiber over a point $b \in A$. Suppose the CT map $\partial i_{F,X} \colon \partial F \to \partial X$ is surjective. Then the CT map $\partial i_{F,Y} \colon \partial F \to \partial Y$ is also surjective.*

*Conversely for any geodesic ray $\alpha \colon [0,\infty) \to B$ with $\alpha(0) = b$, let $Y_\alpha = \pi^{-1}(\alpha)$. If for all $z \in \partial B$ and for some (any) geodesic ray $\alpha$ joining $b$ to $z$ the CT map $\partial_{F,Y_\alpha} \colon \partial F \to \partial Y_\alpha$ is surjective then the CT map $\partial_{F,X} \colon \partial F \to \partial X$ is also surjective.*

**Proof** Let $\xi \in \partial Y$. We want to show that $\xi \in \mathrm{Im}(\partial i_{F,Y})$. Since $\partial i_{F,X} \colon \partial F \to \partial X$ is surjective there exists $z \in \partial F$ such that $\partial i_{F,X}(z) = \partial i_{Y,X}(\xi)$. If $\partial i_{F,Y}(z) = \xi$ we are done. Suppose not. However, $\partial i_{F,X} = \partial i_{Y,X} \circ \partial i_{F,Y}$. Hence, $\partial i_{Y,X}(\partial i_{F,Y}(z)) = \partial i_{Y,X}(\xi)$. Then by Theorem 6.25(3) we are done.

The converse part is a direct consequence of Corollary 6.5 and Proposition 6.6. □

**Corollary 6.27** *Suppose $\pi \colon X \to B$ is a metric (graph) bundle such that $X$, $B$ are hyperbolic and the fibers are all proper, uniformly quasiisometric to the hyperbolic plane $\mathbb{H}^2$. Then for all $b \in B$, the CT map $\partial_{F_b,X} \colon \partial F_b \to \partial X$ is surjective.*

**Proof** This is an immediate consequence of the second part of Theorem 6.26 and the following proposition of Bowditch. □

**Proposition 6.28** [5, Proposition 2.6.1] *Suppose $\pi \colon X \to B$ is a metric (graph) bundle where $B = [0,\infty)$, $X$ is hyperbolic and the fibers are all uniformly quasiisometric to the hyperbolic plane $\mathbb{H}^2$. Then for all $b \in B$, the CT map $\partial_{F_b,X} \colon \partial F_b \to \partial X$ is surjective.*

We would like to remark that Bowditch stated the above proposition in the case that the fibers are all isometric to the hyperbolic plane, but the same proof goes through for fibers uniformly quasiisometric to the hyperbolic plane.

A special case of the following result was proved by E Field [11, Theorem B].

**Theorem 6.29** *Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ is a short exact sequence of infinite hyperbolic groups. Suppose $A \subset Q$ is qi embedded and $Y = \pi^{-1}(A)$. Then the CT map $\partial N \to \partial Y$ is surjective.*

**Proof** Since $N$ is a normal subgroup of the hyperbolic group $G$ it is a standard fact that $\Lambda(N) = \partial G$. Thus by Lemma 2.55 the CT map $\partial N \to \partial G$ is surjective. Now we are done by Theorem 6.26. □

**Fibers of the CT maps**

**Theorem 6.30** *Suppose $X$ is a metric (graph) bundle over $B$ satisfying the hypotheses of Theorem 5.2 such that $X$ is a proper metric space. Let $F = F_b$, where $b \in B$. Suppose $\partial F$ is not homeomorphic to a dendrite and also the CT map $\partial F \to \partial X$ is surjective.*

*Then for all $\xi \in \partial B$ we have $\partial^{(2)}_{\xi,X}(F) \neq \varnothing$.*

**Proof** Suppose $\alpha$ is an arc length parametrized $\kappa_0$-quasigeodesic ray in $B$ joining $b$ to $\xi$. Let $Y = \pi^{-1}(\alpha)$. Since the CT map $\partial F \to \partial X$ is surjective, the map $\partial i_{F,Y} \colon \partial F \to \partial Y$ is also surjective by Theorem 6.26. Now, $\partial^{(2)}_{\xi,X}(F) = \partial^{(2)}_{\xi,Y}(F)$ by Corollary 6.21. Hence, it is enough to show that $\partial^{(2)}_{\xi,Y}(F) \neq \varnothing$. However, $\partial^{(2)}_{\xi,Y}(F) = \varnothing$ if and only if $\partial i_{F,Y}$ is injective. It follows that $\partial^{(2)}_{\xi,Y}(F) = \varnothing$ if and only if $\partial i_{F,Y}$ is bijective. Since $X$ is proper, so are $F$ and $Y$. Hence, $\partial F$ and $\partial Y$ are compact metrizable spaces. (See [6, Chapter III.H, Propositions 3.7 and 3.21] for instance.) Hence, $\partial i_{F,Y}$ is bijective implies $\partial i_{F,Y}$ is a homeomorphism between $\partial F$ and $\partial Y$. Since $\partial F$ is not a dendrite this is impossible due to the following result of Bowditch. Hence, $\partial^{(2)}_{\xi,Y}(F) \neq \varnothing$. □

**Theorem 6.31** [5, Proposition 2.5.2] *Suppose $X$ is hyperbolic metric (graph) bundle over $B = [0, \infty)$ satisfying the hypotheses* (H1)–(H4) *of Section 5. Suppose moreover that $X$ is a proper metric space. Then $\partial X$ is a dendrite.*

We note that a special case of interest of Theorem 6.30 is when the fibers are uniformly quasiisometric to the hyperbolic plane. For instance, we have the following.

**Corollary 6.32** *Suppose we have an exact sequence of infinite hyperbolic groups $1 \to N \to G \to Q \to 1$ where $N$ is either the fundamental group of an orientable closed surface of genus $g \geq 2$ or a free group $F_n$ on $n \geq 3$ generators. Then for all $\xi \in \partial Q$, $\partial^{(2)}_{\xi,G}(N) \neq \varnothing$.*

**Remark** We remark that much stronger results than the above corollary were already proved by Mj and Rafi [23]. For instance, see Theorems 3.12, 5.7 and Proposition 5.8 there.

Another context is that of complexes of groups where Theorem 6.30 can be applied.

**Corollary 6.33** *Suppose $G$ is the fundamental group of a finite developable complexes of nonelementary hyperbolic groups $(\mathcal{G}, \mathcal{Y})$ with qi condition. Suppose $X$ is the metric bundle over $B$ obtained from this data as constructed in Section 3.3.2. Suppose $G$ is hyperbolic.*

*Then for all $\xi \in \partial B$ and any vertex group $G_v$, $v \in V(\mathcal{Y})$ we have $\partial^{(2)}_{\xi,G}(G_v) \neq \varnothing$.*

**Proof** We need to check the hypotheses of Theorem 6.30. It is a standard fact that the boundary of a hyperbolic group is not a dendrite. Since the fibers of the metric bundle under consideration are quasiisometric to nonelementary hyperbolic groups $\partial F$ is not a dendrite for any fiber $F$. We also note that the metric bundle satisfies (H1)–(H4) of Section 5. Finally, $G$ acts on $X$ and $B$ so that the map $\pi\colon X \to B$ is equivariant, the action of $G$ on $X$ is proper and cocompact and on $B$ is cocompact. Thus any orbit map $G \to X$ is a qi by Milnor–Schwarz lemma and therefore induces a homeomorphism $\partial X \to \partial G$.

Now, given any fiber $F$ and $g \in G$, $gF$ is another fiber of the metric bundle. By Lemma 3.10(1) $Hd(F, gF) < \infty$. Hence, by Lemma 2.52 $\Lambda(F) = \Lambda(gF) = g\Lambda(F)$. It is a standard fact that the action of a nonelementary hyperbolic group on its boundary is minimal, ie the only invariant closed subsets are the empty set and the whole set. Hence, it follows that $\Lambda(F) = \partial X$. By Lemma 2.55 we have $\Lambda(F) = \partial i_{F,X}(\partial F)$. Thus the CT map $\partial i_{F,X}\colon \partial F \to \partial X$ is surjective. Finally, clearly $X$ is a proper metric space. Hence, we have $\partial^{(2)}_{\xi,X}(F) \neq \varnothing$ by Theorem 6.30. Finally since $G_v$ acts properly and cocompactly on $X_v$, any orbit map $G_v \to X_v$ is a quasiisometry. Hence, this induces a homeomorphism $\partial G_v \to \partial X_v$. Therefore, taking $F = X_v$ we are done. ☐

**Definition 6.34** Suppose $Z$ is any hyperbolic metric space and $S \subset Z$. Then a point $z \in \Lambda(S) \subset \partial Z$ will be called a *conical limit point* of $S$ if for some (any) quasigeodesic $\gamma$ converging to $z$ in $Z$ there is a constant $D > 0$ such that $N_D(\gamma) \cap S$ is a subset of infinite diameter in $Z$.

**Proposition 6.35** *Suppose we have the hypotheses of Theorem 5.2. Let $\partial i_{Y,X}\colon \partial Y \to \partial X$ be the CT map. If $\xi \in \partial X$ is a conical limit point of $Y$, then $|\partial i_{Y,X}^{-1}(\xi)| = 1$.*

**Proof** Suppose $z \neq z' \in \partial Y$ such that $\partial i_{Y,X}(z) = \partial i_{Y,X}(z') = \xi$. Then by Theorem 6.25 there is $\xi_B \in \partial B \setminus \partial A$ and a qi lift of $\gamma$ of a quasigeodesic ray joining $b$ to $\xi_B$ such that $\xi = \gamma(\infty)$. Since $\xi_B \in \partial B \setminus \partial A$ and $A$ is quasiconvex $\xi_B$ is not a limit point of $A$ in $\partial B$. Thus it is clear that $\xi$ is not a conical limit point of $Y$. This gives a contradiction and proves the proposition. ☐

## 6.3 QI embedding fibers in a product of bundles

The lemma below is the product of answering a question due to Misha Kapovich.

**Lemma 6.36** *Suppose $\pi\colon X \to \mathbb{R}$ is a metric (graph) bundle satisfying the hypotheses of Section 5 and $X^{\pm}$ are the restrictions of it to $[0,\infty)$ and $(-\infty,0]$, respectively. Then the diagonal embedding $f\colon F_0 \to X^+ \times X^-$ is a qi embedding where the latter is given the $l_2$ metric.*

**Proof** Without loss of generality, we assume $(X,d)$ is a metric graph bundle. Let $d_{\pm}$ be the induced length metric on $X^{\pm}$, respectively. Then the $l_2$ metric $d_Y$ on $Y := X^+ \times X^-$ is given by $d_Y((x_1,x_2),(y_1,y_2))^2 = d_+(x_1,y_1)^2 + d_-(x_2,y_2)^2$ for all $x_1, y_1 \in X^+$ and $x_2, y_2 \in X^-$. We note that the inclusion maps $F_0 \to X^{\pm}$ are 1-Lipschitz.

Let $x, y \in F_0$. Then,

$$d_Y(f(x), f(y))^2 = d_Y((x,x),(y,y))^2 = d_+(x,y)^2 + d_-(x,y)^2 \le d_0(x,y)^2 + d_0(x,y)^2 = 2d_0(x,y)^2,$$

which implies that $d_Y(f(x), f(y)) \le \sqrt{2}d_0(x,y)$. A reverse inequality is obtained as follows.

Let $\Sigma, \Sigma'$ be a pair of $K_0$-qi sections in $X$ through $x, y$ respectively. Let $\mathbb{L} = \mathbb{L}(\Sigma, \Sigma')$ be the ladder formed by them. Let $\lambda = \mathbb{L} \cap F_0$. This is a geodesic in $F_0$ joining $x, y$. Now, suppose $c(x,y)$ is a uniform quasigeodesic in $X$ joining $x, y$ constructed as in Section 5 by decomposing $\mathbb{L}$ into subladders using the qi sections $\Sigma_i$'s and $\Sigma'_j$'s. Let $\bar{c}_+ := \bar{c}_+(x,y), \bar{c}_- := \bar{c}_-(x,y)$ be the modified paths joining $x, y$ in $X^+, X^-$, respectively. By our main theorem in Section 5, $\bar{c}_+, \bar{c}_-$ are uniform quasigeodesics in $X^+, X^-$, respectively. Suppose these are $K$-quasigeodesics. As in the discussion at the end of Section 5, suppose $\mathcal{Q}, \mathcal{Q}'$ are consecutive qi sections in the decomposition of $\mathbb{L} = \mathbb{L}(\Sigma, \Sigma')$ and $z, w \in \mathcal{Q}$, $w' \in \mathcal{Q}'$ with $b' = \pi(w) = \pi(w')$ are such that $\mathbb{L}(\mathcal{Q}, \mathcal{Q}') \cap c(y, y')$ is made of the fiber geodesic $[w, w']_{b'}$ and the lift of $[\pi(z), \pi(w)]_B$ in $\mathcal{Q}$. However, if $b' \in [0, \infty)$ then $\lambda \cap \mathbb{L}(\mathcal{Q}, \mathcal{Q}') \subset \bar{c}_-$ and similarly if $b' \in (-\infty, 0]$ then $\lambda \cap \mathbb{L}(\mathcal{Q}, \mathcal{Q}') \subset \bar{c}_+$. Thus $\lambda \subset \bar{c}_+ \cup \bar{c}_-$. Therefore we have,

$$\begin{aligned} d_0(x,y) &\le l_+(\tilde{c}_+) + l_-(\tilde{c}_-) \le Kd_+(x,y) + K + Kd_-(x,y) + K \\ &= K(d_+(x,y) + d_-(x,y)) + 2K \\ &= 2Kd_Y((x,x),(y,y)) + 2K = 2Kd_Y(f(x), f(y)) + 2K. \end{aligned}$$

Thus, $-1 + \frac{1}{2K}d_0(x,y) \le d_Y(f(x), f(y)) \le \sqrt{2}d_0(x,y)$. Hence, $f$ is $(2K, 1)$-qi embedding. □

In the same way, we obtain the following.

**Lemma 6.37** *If $v_0$ is a cut point of $B$ and removing it produces two quasiconvex subsets $A_1, A_2$ and $Y_1, Y_2$ are the restrictions of the bundle to $A_1, A_2$ respectively then the diagonal map $F_{v_0} \to Y_1 \times Y_2$ is a qi embedding.*

**Corollary 6.38** *If $v_0$ is a cut point of $B$ and removing it produces finitely many quasiconvex subsets $A_i$, $1 \le i \le n$ and $Y_i$'s are the restrictions of the bundle to $A_i$'s, respectively, then the diagonal map $F_{v_0} \to \Pi_i Y_i$ is a qi embedding.*

**Remark** In [19] Mitra defined an ending lamination for an exact sequence of groups. Given any point $\xi \in \partial Q$ he defined a lamination $\Lambda_\xi$ and then showed that $\Lambda_\xi = \partial^{(2)}_{\xi,X}(F)$. However, for formulating and proving these sorts of results one needs additional structure on the bundle, eg action of a group on the bundle through morphisms which has uniformly bounded quotients when restricted to the fibers. Results of this type are proved in [23, Section 3]; see also [5, Section 4.4].

# Appendix Flaring in a metric bundle and its canonical metric graph bundle

Suppose $\pi'\colon X' \to B'$ is an $(\eta, c)$-metric bundle and $\pi\colon X \to B$ is the canonical metric graph bundle associated to it. *We shall assume that $B'$ and $B$ are both $\delta$-hyperbolic. However, there will be no assumption about the fibers of the bundles.* We shall freely use the notation from Section 4 of the paper. The purpose of this appendix is to show that a metric bundle satisfies a sort of "generalized flaring property" (see Property (†) below) if and only if the associated canonical metric graph bundle satisfies a flaring condition.

**Note** If $b_0, b_1, \dots, b_n$ are consecutive vertices on a geodesic in $B$ then $\alpha'\colon i \mapsto b_i$ is a dotted $(1,3)$-quasigeodesic of $B'$ by Lemma 2.8. Thus there is a constant $D_0$ such that if $\beta'$ is any $(1,1)$-quasigeodesic in $B'$ joining $b_0, b_n$ then $Hd(\alpha', \beta') \le D_0$. We will preserve $D_0$ to denote this constant for the rest of this section.

Suppose $b \in V(B)$ and $p \in B'$ are such that $d_{B'}(p, b) \le D_0$. Then for any $x \in \pi^{-1}(b)$ we can lift a $(1,1)$-quasigeodesic of $B'$ joining $b$ to $p$ to $X'$ which starts from $x$ and ends at $x'$, say. This way we get a "fiber identification map" $V(\pi^{-1}(b)) \to \pi'^{-1}(p)$. If we denote this map by $f_{bp}$ then we have the following lemma. Since the proof is evident we skip it.

**Lemma A.1** *We have*

$$-C_0 + \frac{1}{C_0} d_b(x, y) \le d'_p(f_{bp}(x), f_{bp}(y)) \le C_0 + C_0 d_b(x, y)$$

*for all $x, y \in \pi^{-1}(b)$ and for some uniform constant $C_0$ where $d_b$ is the fiber distance in $\pi^{-1}(b)$ for the metric graph bundle $X$ and $d'_p$ is the fiber distance in $\pi'^{-1}(p)$ for the metric bundle $X'$.*

Suppose $\alpha$ is a geodesic in $B$ and $\tilde{\alpha}$ is a $C$-qi lift of $\alpha$ in $X$. Let $\alpha'$ be a $(1,1)$-quasigeodesic in $B'$ joining the end points of $\alpha$. Let $\sigma\colon \alpha \to \alpha'$ be any map such that $d_{B'}(b, \sigma(b)) \le D_0$ for all $b \in \alpha$. Let $\tilde{p} = f_{bp}(\tilde{\alpha}(b)) \in \pi'^{-1}(p)$ for all $b \in \alpha$, where $p = \sigma(b)$. Now it is easy to find a uniform qi lift $\tilde{\alpha}'$ of $\alpha'$ such that $\tilde{\alpha}'(\sigma(b)) = \tilde{p}$, where $p = \sigma(b)$ for all $b \in \alpha$. We record this as a lemma.

**Lemma A.2** *There is a constant $C'$ depending on $C$ and a $C'$-qi lift $\tilde{\alpha}'$ of $\alpha'$ such that $\tilde{\alpha}'(\sigma(b)) = \tilde{p}$, where $p = \sigma(b)$ for all $b \in \alpha$.*

The following lemma roughly says that if two qi leaves start flaring in one direction then they keep on flaring in the same direction. The proof follows immediately from the definition of flaring. One may also look up the proof of [24, Lemma 2.17(1)].

**Lemma A.3** (persistence of flaring in graph bundles) *Suppose the metric graph bundle satisfies the $(\nu_k, M_k, n_k)$-flaring condition for all $k \ge 1$. Suppose $\alpha\colon [-m, n] \to B$ is a geodesic where $m \ge n_k$, $n \ge n_k$ and $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are two $k$-qi lifts of $\alpha$ in $X$ with $d_{\alpha(0)}(\tilde{\alpha}_1(0), \tilde{\alpha}_2(0)) \ge M_k$. Suppose*

$$d_{\alpha(sn_k)}(\tilde{\alpha}_1(sn_k), \tilde{\alpha}_2(sn_k)) \ge \nu_k d_{\alpha(0)}(\tilde{\alpha}_1(0), \tilde{\alpha}_2(0)),$$

*where* $s$ *is either* $1$ *or* $-1$*. Let* $t$ *be the largest integer smaller than* $n/n_k$ *or* $m/n_k$ *according as* $s=1$ *or* $-1$*. Then for all integers* $1 \le l \le t$ *we have*

$$d_{\alpha(lsn_k)}(\tilde{\alpha}_1(lsn_k), \tilde{\alpha}_2(lsn_k)) \ge \nu_k^l d_{\alpha(0)}(\tilde{\alpha}_1(0), \tilde{\alpha}_2(0)).$$

The same idea of proof gives the next lemma also. We will need a definition.

**Property** (†) We shall say that the metric bundle $X'$ has Property (†) if for any $k \ge 1$, there exist $\nu_k > 1$ and $n_k, M_k \in \mathbb{N}$ such that the following holds:

Suppose $\alpha' \colon [-n_k, n_k] \to B'$ is a 1-quasigeodesic and $\tilde{\alpha}'_1$ and $\tilde{\alpha}'_2$ are two $k$-qi lifts of $\alpha'$ in $X'$. If $d_{\gamma(0)}(\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \ge M_k$ then we have

$$\nu_k \cdot d_{\alpha'(0)}(\tilde{\alpha}'_1(0), \tilde{\alpha}'_2(0)) \le \max\bigl\{d_{\alpha'(n_k)}(\tilde{\alpha}'_1(n_k), \tilde{\alpha}'_2(n_k)),\ d_{\alpha'(-n_k)}(\tilde{\alpha}'_1(-n_k), \tilde{\alpha}'_2(-n_k))\bigr\}.$$

Note that one could define the flaring condition for a length metric bundle using Property (†).

**Lemma A.4** (persistence of flaring in metric bundles) *Suppose the metric bundle satisfies* (†)*. Let* $k \ge 1$*. Suppose* $\alpha' \colon [-m, n] \to B$ *is a geodesic where* $m \ge n_k$*,* $n \ge n_k$ *and* $\tilde{\alpha}'_1$ *and* $\tilde{\alpha}'_2$ *are two* $k$*-qi lifts of* $\alpha'$ *in* $X$ *with* $d_{\alpha'(0)}(\tilde{\alpha}'_1(0), \tilde{\alpha}'_2(0)) \ge M_k$*. Suppose*

$$d_{\alpha'(sn_k)}(\tilde{\alpha}'_1(sn_k), \tilde{\alpha}'_2(sn_k)) \ge \nu_k d_{\alpha'(0)}(\tilde{\alpha}'_1(0), \tilde{\alpha}'_2(0)),$$

*where* $s$ *is either* $1$ *or* $-1$*. Let* $t$ *be the largest integer smaller than or equal to* $n/n_k$ *or* $m/n_k$ *according as* $s=1$ *or* $-1$*. Then for all integer* $l \le t$ *we have*

$$d_{\alpha'(lsn_k)}(\tilde{\alpha}'_1(lsn_k), \tilde{\alpha}'_2(lsn_k)) \ge \nu_k^l d_{\alpha'(0)}(\tilde{\alpha}'_1(0), \tilde{\alpha}'_2(0)).$$

Following is one of the main results of this appendix.

**Lemma A.5** *Suppose the metric bundle* $X'$ *has Property* (†)*. Then the canonical metric graph bundle* $\pi \colon X \to B$ *associated to* $X'$ *satisfies a* $(\hat{\nu}_k, \widehat{M}_k, \hat{\lambda}_k)$*-flaring condition.*

*In particular if a geodesic metric bundle satisfies a flaring condition* (*see* [24, Definition 1.12]) *then its canonical metric graph bundle satisfies the flaring condition.*

**Proof** Suppose $\alpha \colon [-n, n] \to B$ is a geodesic and $\tilde{\alpha}, \tilde{\tilde{\alpha}}$ are two $k$-qi lifts of $\alpha$ in $X$, where $n \in \mathbb{N}$ and $k \ge 1$. Let $\alpha'$ be a $(1,1)$-quasigeodesic in $B'$ joining $\alpha(n)$ and $\alpha(-n)$. Then there are $k'$-qi lifts $\tilde{\alpha}', \tilde{\tilde{\alpha}}'$ of $\alpha'$, respectively, as in Lemma A.2. We shall choose a parametrization $\alpha' \colon [-m', n'] \to B'$ so that $\alpha'(n') = \alpha(n)$, $\alpha'(-m') = \alpha(-n)$ and $d_{B'}(\alpha(0), \alpha'(0)) \le D_0$. Note that

$$d'_{\alpha'(0)}(\tilde{\alpha}'(0), \tilde{\tilde{\alpha}}'(0)) \ge -C_0 + \frac{1}{C_0} d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0))$$

by Lemma A.1. Hence, if we assume $d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0)) \ge C_0(C_0+M_{k'})$ then $d'_{\alpha'(0)}(\tilde{\alpha}'(0),\tilde{\tilde{\alpha}}'(0)) \ge M_{k'}$. Clearly, if we choose $n$ large enough then we have $n_{k'} < \min\{m',n'\}$. (In the course of the proof we will be more precise.) Without loss of generality we shall assume $\nu_k \cdot d'_{\alpha'(0)}(\tilde{\alpha}'(0),\tilde{\tilde{\alpha}}'(0)) \le d'_{\alpha'(n_{k'})}(\tilde{\alpha}'(n_{k'}),\tilde{\tilde{\alpha}}'(n_{k'}))$. Let $l$ be the greatest integer less than or equal to $n'/n_{k'}$. Then by Lemma A.4

$$\nu_k^l \cdot d'_{\alpha'(0)}(\tilde{\alpha}'(0),\tilde{\tilde{\alpha}}'(0)) \le d'_{\alpha'(ln_{k'})}(\tilde{\alpha}'(ln_{k'}),\tilde{\tilde{\alpha}}'(ln_{k'})). \tag{1}$$

Note that $d_{B'}(\alpha'(ln_{k'}),\alpha'(n')) \le 2n_{k'}$. Let $b'=\alpha'(ln_{k'})$ and $b''=\alpha'(n')$. Then by Corollary 3.11 the fiber identification map $\phi_{b'b''}$ referred to in that corollary is a $K_{3.11}(2n_{k'})$-quasiisometry. Let $K=K_{3.11}(2n_{k'})$. Note that

$$d_{X'}(\tilde{\alpha}'(ln_{k'}),\tilde{\alpha}'(n')) \le k'+2n_{k'}k' \tag{2}$$

since $\tilde{\alpha}'$ is a $k'$-qi section and $d_{B'}(\alpha'(ln_{k'}),\alpha'(n')) \le 2n_{k'}$. Also, by Corollary 3.9 we have

$$d_{X'}\big(\tilde{\alpha}'(ln_{k'}),\phi_{b'b''}(\tilde{\alpha}'(ln_{k'}))\big) \le 3c+6cn_{k'}. \tag{3}$$

Using the inequalities (2) and (3) we have

$$d_{X'}\big(\tilde{\alpha}'(n'),\phi_{b'b''}(\tilde{\alpha}'(ln_{k'}))\big) \le 3c+6cn_{k'}+k'+2n_{k'}k'.$$

Since $X'$ is an $(\eta,c)$-metric bundle we have

$$d'_{\alpha'(n')}\big(\tilde{\alpha}'(n'),\phi_{b'b''}(\tilde{\alpha}'(ln_{k'}))\big) \le \eta(3c+6cn_{k'}+k'+2n_{k'}k').$$

In the same way we have

$$d'_{\alpha'(n')}\big(\tilde{\tilde{\alpha}}'(n'),\phi_{b'b''}(\tilde{\tilde{\alpha}}'(ln_{k'}))\big) \le \eta(3c+6cn_{k'}+k'+2n_{k'}k').$$

Now using the fact that $\phi_{b'b''}$ is a $K$-quasiisometry and letting $R_1=2\eta(3c+6cn_{k'}+k'+2n_{k'}k')$, we have by triangle inequality

$$d'_{\alpha'(ln_{k'})}(\tilde{\alpha}'(ln_{k'}),\tilde{\tilde{\alpha}}'(ln_{k'})) \le (K^2+2R_1K)+Kd'_{\alpha'(n')}(\tilde{\alpha}'(n'),\tilde{\tilde{\alpha}}'(n')). \tag{4}$$

However, by Lemma 2.8 and Proposition 4.1(2) we have

$$d'_{\alpha'(n')}(\tilde{\alpha}'(n'),\tilde{\tilde{\alpha}}'(n')) \le 3+d_{\alpha(n)}(\tilde{\alpha}(n),\tilde{\tilde{\alpha}}(n)). \tag{5}$$

Then it follows from the inequalities (1), (4) and (5) that

$$\nu_k^l \cdot d'_{\alpha'(0)}(\tilde{\alpha}'(0),\tilde{\tilde{\alpha}}'(0)) \le RK+Kd_{\alpha(n)}(\tilde{\alpha}(n),\tilde{\tilde{\alpha}}(n)), \tag{6}$$

where $R=3+K+2R_1$. Finally since $d'_{\alpha'(0)}(\tilde{\alpha}'(0),\tilde{\tilde{\alpha}}'(0)) \ge -C_0+\frac{1}{C_0}d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0))$ using (6) we have

$$\nu_{k'}^l\Big(-C_0+\frac{1}{C_0}d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0))\Big) \le RK+Kd_{\alpha(n)}(\tilde{\alpha}(n),\tilde{\tilde{\alpha}}(n)). \tag{7}$$

Recall that we assumed $d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0)) \ge C_0(C_0+M_{k'})$. Hence,

$$-C_0+\frac{1}{C_0}d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0)) \ge \Big(\frac{1}{C_0}-\frac{1}{C_0+M_{k'}}\Big)d_{\alpha(0)}(\tilde{\alpha}(0),\tilde{\tilde{\alpha}}(0)). \tag{8}$$

Let

$$\lambda = \frac{1}{K}\Big(\frac{1}{C_0} - \frac{1}{C_0 + M_{k'}}\Big).$$

Then we have, using (7) and (8),

$$\nu^l_{k'} \cdot \lambda \cdot d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0)) \le R + d_{\alpha(n)}(\tilde{\alpha}(n), \tilde{\tilde{\alpha}}(n)). \tag{9}$$

It is clear that

$$-R + \nu^l_{k'} \cdot \lambda \cdot d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0)) \ge \tfrac{1}{2}\lambda \nu^l_{k'} d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0))$$

if $d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0)) \ge 2R/(\lambda \nu^l_{k'})$. In particular, since $\nu^l_{k'} > 1$, we have

$$\tfrac{1}{2}\lambda \nu^l_{k'} d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0)) \le d_{\alpha(n)}(\tilde{\alpha}(n), \tilde{\tilde{\alpha}}(n)) \tag{10}$$

using (9) if $d_{\alpha(0)}(\tilde{\alpha}(0), \tilde{\tilde{\alpha}}(0)) \ge 2R/\lambda$. Thus it is enough to choose

$$\widehat{M}_k = \max\Big\{\frac{2R}{\lambda}, C_0(C_0 + M_{k'})\Big\}, \quad \hat{\lambda}_k = 2$$

and to show that if $n$ is sufficiently large then $l$ is so large that $\frac{1}{2}\lambda \nu^l_{k'} \ge 2$ which will give a choice for $\hat{n}_k$. This is easy to verify and hence left to the reader. □

The converse of Lemma A.5 is also true and has an exactly similar proof. However, in this case one uses Lemma A.3 instead of Lemma A.2. We state it without proof to avoid repetition.

**Lemma A.6** *Suppose the metric graph bundle $\pi : X \to B$ satisfies a $(\nu_k, M_k, n_k)$-flaring condition for all $k \ge 1$. Then the metric bundle $\pi' : X' \to B'$ satisfies Property (†) for three functions $\nu'_k, M'_k, n'_k$ of $k$.*

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