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Surgery sequences and self-similarity of the Mandelbrot set

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We introduce an analog in the context of rational maps of the idea of *hyperbolic Dehn surgery* from the theory of Kleinian groups. A *surgery sequence* is a sequence of postcritically finite maps limiting (in a precise manner) to a postcritically finite map with at least one strictly preperiodic critical orbit. As an application of this idea we give a new and elementary proof of Tan Lei's theorem on the asymptotic self-similarity of Julia sets and the Mandelbrot set at Misiurewicz points.

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For Tan Lei

1 Introduction

Sullivan's dictionary is a framework that seeks to unify two subjects at the heart of one-dimensional holomorphic dynamics: Kleinian groups and the iteration of rational maps. Sullivan introduced the idea of this dictionary in [4], and supplied a number of key entries. It is the purpose of this paper to propose a new entry for this dictionary — between *hyperbolic Dehn surgery* for cusped hyperbolic 3-manifolds and *surgery sequences* of rational maps — and to use this analogy to give a new, elementary proof of Tan Lei's famous theorem on the asymptotic self-similarity of Julia sets and the Mandelbrot set at Misiurewicz points.

Sullivan's original dictionary contains twenty-four entries; the last entry is a correspondence between cocompact Kleinian groups and postcritically finite rational maps. One might reasonably broaden this correspondence from the class of cocompact Kleinian groups to the finite-covolume groups. The idea of this dictionary entry is that in either case, topological data (an irreducible compact 3-manifold with torus boundary components; an equivalence class of postcritically finite branched self-covering maps of the 2-sphere) may be 'geometrized' by a rigid holomorphic dynamical system, unless a purely topological obstruction exists.

One reason to include finite-covolume Kleinian groups in the picture is that these groups arise as *limits* of the cocompact ones. A rank-2 parabolic subgroup H of a Kleinian group G corresponds to a *toral cusp* in the 3-manifold quotient M. Dehn filling a cusp of M with a long slope gives rise to a new 3-manifold M', which is the quotient of $\mathbb{H}^3$ by a new Kleinian group G', which has a rank-1 loxodromic subgroup H' 'in place' of the rank-2 parabolic subgroup H of G. For suitable choice of filling slope, the geometry and topology of M' (resp. the dynamics of G' on the Riemann sphere) will approximate arbitrarily closely the geometry and topology of M away from the cusp (resp. the dynamics of G away from the fixed points of the conjugates of H).

Now let's move to the other side of the dictionary. What is the analog of Dehn filling in the world of rational maps? For a postcritically finite rational map f there is no really good analog of the quotient 3-manifold M on the Kleinian group side, and any sort of approximation must take place in the dynamics on the Riemann sphere. We propose the following informal analogy; for a precise definition see Definition 2.1. We start with the data of a postcritically finite map f with at least one critical point c whose forward orbit contains a repelling cycle O disjoint from c. The cycle O is the analog of the 'cusp' which is to be deformed. We must also make a choice of an infinite backward orbit T of c (called the 'tail') accumulating only on O. A *surgery sequence* is then a sequence of postcritically finite maps f_n so that $f_n \to f$ as maps, and so that the postcritical set of f_n converges (in the Hausdorff topology) to the *union* of the postcritical set of f with T.

Different choices of tail T accumulating on O give rise to different surgery sequences for a fixed f; we may describe the structure of the set of all such surgery sequences, and this lets us recover the (asymptotic) geometry of the Julia set $J(f)$ of f near the orbit O, and in the special case that f has degree 2 and corresponds to a Misiurewicz point in the Mandelbrot set, the (asymptotic) geometry of the Mandelbrot set near f.

2 Surgery sequences

2.1 Definition

Let f be a rational map. We denote the *critical set* of f (ie the set of critical points of f) by $C(f)$. For each $c \in C(f)$ let $P(c) := \bigcup_{n>0} f^n(c)$ denote the forward orbit of c, and let $P(f) := \bigcup_{c\in C(f)} P(c)$ denote the *postcritical set*. The map f is *postcritically finite* (pcf) if $P(f)$ is finite.

Let f be a pcf map with $c \in C(f)$ and let $O \subset P(c) - C(f)$ be a periodic orbit with multiplier $\mu := \prod_{x\in O} f'(x)$. Let's suppose O is a repelling orbit, ie $|\mu| > 1$.

Since O is repelling, there is a neighborhood U of O and a unique branch g of f^{-1} with $g\colon U \to U$ so that for every point $x \in U$ the sequence $g^n(x)$ accumulates on O. A *tail* T for c, O is an infinite sequence c_{-n} for $n \in \mathbb{Z}^+$ for which

(1) $f(c_{-1}) = c$;

(2) $f(c_{-n}) = c_{1-n}$ for $n > 1$; and

(3) for n sufficiently large, $c_{-n} \in U$ and $g(c_{-n}) = c_{-1-n}$.

Definition 2.1 Let f be pcf. Let c be critical for f and let $O \subset P(c) - C(f)$ be a repelling periodic orbit for f. A *surgery sequence* for c, O is a sequence of pcf maps f_n for which there is a tail T for c, O such that

(1) the f_n converge to f as rational maps;

(2) the cardinality of $P(f_n)$ is $A + |O|n$ for some integer A; and

(3) the sets $P(f_n)$ converge in the Hausdorff topology to $T \cup P(f)$.

The idea of a surgery sequence is to approximate the 'orbit' $T \cup c \cup P(c)$ of f by some finite periodic critical orbit of f_n.

We shall consider two surgery sequences f_n, g_n to be *isomorphic* either

(1) if they are conjugate by a (convergent) family of Möbius transformations ϕ_n — ie $f_n = \phi_n g_n \phi_n^{-1}$ with $\phi_n \to \mathrm{id}$; or

(2) if there is an integer m with $f_n = g_{n+m}$ for all n (where defined).

In practice the first kind of ambiguity is eliminated by normalizing our surgery sequences somehow, either by fixing a small number of specific critical points and their images, or by fixing some of the coefficients of the rational maps f_n.

2.2 An example

We consider the simplest nontrivial example, a surgery sequence for the pcf map $f : z \to z^2 - 2$. Since f is a polynomial, the critical point ∞ is completely invariant. The only other critical point is 0, whose orbit is $0 \to -2 \to 2$ thereafter left fixed by f. Let O consist solely of the point $p = 2$; the multiplier μ equals 4 so this is indeed a repelling fixed point; and as tail T we choose the sequence $c_{-1} = \sqrt{2}$ and $c_{-n} = \sqrt{c_{1-n} + 2}$, where by convention, we take the 'square root' symbol to denote the unique nonnegative real square root of a nonnegative real number. Viète's formula [6] implies that

$$\lim_{n\to\infty} (c_{-n} - p) \cdot \mu^n = \lim_{n\to\infty} (c_{-n} - 2) \cdot 4^n = -\tfrac{1}{4}\pi^2 \sim -2.4675011$$

See Figure 1.

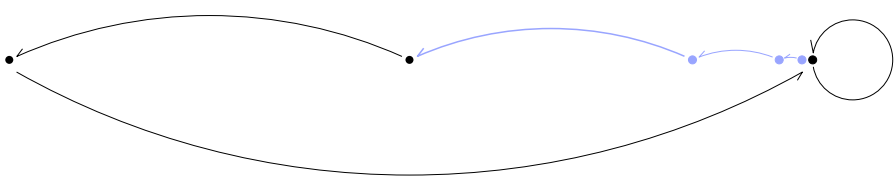

Figure 1: The forward orbit $P(0)$ for the map $f : z \to z^2 - 2$ is in black, and the tail T is in blue (gray) (only three elements of T are shown in the figure).

We normalize the associated surgery sequence in the form $f_n\colon z \to z^2 + v_n$ where each v_n is real and $v_n \to v := -2$ from above, and so that the critical point 0 is periodic for f_n with period $n+2$ consisting of real numbers satisfying

$$-2 < v_n = f_n(0) < 0 = f_n^{n+2}(0) < f_n^{n+1}(0) < \cdots < f_n^2(0) < 2$$

For instance, f_1 has critical orbit

$$0 \to v_1 \to v_1^2 + v_1 \to v_1^4 + 2v_1^3 + v_1^2 + v_1 = 0$$

so that $v_1 \sim -1.7549$. Likewise we can compute $v_2 \sim -1.9408$, $v_3 \sim -1.9854$ and so on. The first three values for $(v_n - v)\cdot\mu^n$ are approximately 0.9805, 0.9472, 0.9328; other approximate values are tabulated here:

v_n	$(v_n - v)\cdot\mu^n$
−1.754877666	0.980489335
−1.940799806	0.947203095
−1.985424253	0.932847805
−1.996376137	0.927708745
−1.999095682	0.926021297
−1.999774048	0.925496551

We shall see in Section 3 that this sequence converges, and in fact

$$\lim_{n\to\infty}(v_n - v)\cdot\mu^n = \lim_{n\to\infty}(v_n - (-2))\cdot 4^n = \tfrac{3}{32}\pi^2 \sim 0.9252754$$

where $\frac{3}{32}\pi^2 = -\frac{3}{8}\cdot -\frac{1}{4}\pi^2$. We have already seen the factor of $-\frac{1}{4}\pi^2$; the factor of $-\frac{3}{8}$ will be explained in Section 3.

3 Surgery sequences for Misiurewicz points

In this section we completely analyze surgery sequences in the quadratic family: polynomials of the form $z \to z^2 + v$ with $v \in \mathbb{C}$. We use the following standard terminology: the *Mandelbrot set* $\mathcal{M}$ is the set of parameters v for which the critical point 0 of $z \to z^2 + v$ is not in the attracting basin of infinity (some authors call this the 'filled Mandelbrot set'). For a quadratic polynomial $z \to z^2 + v$ the critical point ∞ is periodic, so in order for it to give rise to a surgery sequence the other critical point 0 must be strictly preperiodic. Such a parameter v is called a *Misiurewicz point*.

Let $f\colon z \to z^2 + v$ be Misiurewicz, and let $O \subset P(0) - 0$ be the periodic orbit with period m and multiplier μ. In the sequel we use the following well-known facts about Misiurewicz points:

Proposition 3.1 *Let v be a Misiurewicz point.*

(1) *v is in the Mandelbrot set $\mathcal{M}$.*

(2) *The multiplier μ satisfies $|\mu| > 1$.*

(3) *The Julia set $J(f)$ is a dendrite.*

For a proof see Douady–Hubbard [1].

Let E_μ denote the elliptic curve $\mathbb{C}^*/\langle\mu\rangle$. We warn the reader that throughout this section we set up our notation slightly differently than in Section 2.1. Choose $p = f^k(0) \in O$ for some least $k > 0$ and let U be a connected open neighborhood of $p \in O$ on which $g\colon U \to U$ is an attracting local branch of f^{-m}. Any tail T for f intersects U in a sequence of points which by abuse of notation we denote by c_{-j} for which $g(c_{-j}) = c_{-j-1}$, and thus $c_{-n} \to p$ (this indexing and the definition of g and U differs from Section 2.1 when $m \neq 1$ but agrees when $m = 1$).

A holomorphic map may be linearized (ie holomorphically conjugated to a linear map) near an attracting fixed point; see eg [3]. Thus g is holomorphically conjugate near p to the map $z \to z/\mu$ near 0. In particular, $\lim_{n\to\infty} \mu^n(c_{-n} - p)$ exists. We denote by $x(T)$ the image of this limit in E_μ. Another way to say this is to use the holomorphic conjugacy to identify E_μ with the quotient $(U - p)/\langle g\rangle$ and then the g-orbit $\{c_{-n}\}$ becomes identified with $x(T) \in E_\mu$.

Let $X \subset E_\mu$ denote the closure of the set of points $x(T)$ that arise in this way.

Proposition 3.2 (X is Julia) *With notation as above, the set X is equal to $(J(f) \cap (U - p))/\langle g\rangle$ under the identification of $(U - p)/\langle g\rangle$ with E_μ.*

Proof The Julia set $J(f)$ is f-invariant and therefore also g-invariant (where defined). Let $Y \subset E_\mu$ denote the image of $(J(f) \cap (U - p))/\langle g\rangle$ under the identification of $(U - p)/\langle g\rangle$ with E_μ. Since $J(f)$ is g-invariant and closed, it follows that Y is closed. Since O is a repelling periodic orbit for f contained in $P(0)$, it follows that $0 \in J(f)$ and therefore also $T \subset J(f)$ for every T. Thus $x(T) \in Y$, so that $X \subset Y$. Conversely, the complete backward orbit of every element of $J(f)$ is dense in $J(f)$, so that every point of $J(f)$ may be approximated by some element of some tail. Thus $Y \subset X$ so $X = Y$. □

Let us now construct the surgery sequence associated to a tail T and a Misiurewicz point v. Repelling periodic orbits are structurally stable, so that for all $w \in \mathbb{C}$ sufficiently close to v there is a unique repelling point $p(w)$ of period m for $f_w\colon z \to z^2 + w$ close to p; furthermore, $p(w)$ depends holomorphically on w with $p(v) = p$. For $|w - v|$ sufficiently small, there is a unique local branch g_w of f_w^{-m} with $g_w\colon U \to U$ fixing $p(w)$.

In a similar manner we may define $c_{-j}(w)$ to be the preimage of 0 under a suitable power of f_w close to c_{-j}, so that each $c_{-j}(w)$ depends holomorphically on w, and $g_w(c_{-j}(w)) = c_{-1-j}(w)$ and therefore $c_{-n}(w) \to p(w)$.

On the other hand, we may define $q_w := f_w^k(0)$, so that $p_v = q_v = p$. Define $\nu := d/dw|_{w=v}(q_w - p_w)$.

Lemma 3.3 *With notation as above, $\nu \neq 0$.*

Proof This is the only nonelementary point in the paper; it follows from Thurston's theorem [2] on the uniqueness of pcf maps of a given topological type. Let us see how.

Thurston's theorem [2] gives necessary and sufficient conditions (not relevant here) that a critically finite branched map from S^2 to S^2 is equivalent to a rational map, and furthermore that such a rational map is unique (up to holomorphic conjugacy) provided the map has 'hyperbolic orbifold'. This condition is rather technical to state completely, but we remark that it is satisfied automatically when $|P(f)| > 4$ and thus in our context there are only finitely many exceptional cases where it does not hold where $v \neq 0$ may be checked (numerically) by hand.

Since q_w and p_w both depend holomorphically on w, we may write $q_w = p_w + h(w - v)$ for some holomorphic function h with $h(0) = 0$. The first observation is that h is not identically zero. For, if it were, the f_w would all be topologically equivalent pcf maps, and therefore (by Thurston's theorem) holomorphically conjugate. But distinct elements of the quadratic family are never holomorphically conjugate; the first claim follows.

If $h(z)$ is not identically zero then we can write $h(z) = \alpha z^k + O(z^{k+1})$ for some $\alpha \neq 0$. If $k > 1$ then there is a real $\epsilon > 0$ so that if S is the circle $|z| = \epsilon$, the image of S under h has winding number k around 0. Choose n sufficiently large so that for all $w \in S + v$, the difference $|c_{-n}(w) - p(w)|$ is small compared to the minimum of $|h|$ on S. Then as $w - v$ winds around S the difference $q_w - c_{-n}(w)$ also winds k times around 0, and therefore there are k distinct values of w near v for which $q_w = c_{-n}(w)$. But then for each of these w the map f_w is pcf (indeed 0 is periodic) and furthermore these maps are all topologically equivalent, violating Thurston's theorem.

It follows that $k = 1$, so that $h'(0) := v$ is nonzero, as claimed. □

The winding number argument of Lemma 3.3 actually shows for all sufficiently large n that there is a unique v_n near v with $q_{v_n} = c_{-n}(v_n)$. Evidently the f_{v_n} are the surgery sequence associated to the given tail T.

Proposition 3.4 *With notation as above,* $\lim_{n\to\infty}(v_n - v)\mu^n$ *exists, and its image in* E_μ *is equal to* $v^{-1}x(T)$.

Proof Fix some small positive $\epsilon > 0$ and fix some large j so that $c_{-j}(w)$ is contained in U for $|w - v| < \epsilon$. If ϵ is small enough, then for each w with $|w - v| < \epsilon$ the multiplier $\mu(w)$ of f_w^m at $p(w)$ satisfies $|\mu(w)| > 1$ and there are a family of maps $\phi_w : U \to \mathbb{C}$ (depending holomorphically on w) with $\phi_w(p(w)) = 0$ and $\phi_w(c_{-j}(w)) = 1$ conjugating $g_w : U \to U$ to the map $z \to \mu(w)^{-1}z$ on $\phi_w(U)$. We may suppose for concreteness that the disk of radius 2 about 0 is contained in all $\phi_w(U)$ for $|w - v| < \epsilon$.

For $z := w - v$ with $|z| < \epsilon$ let $f(z) = \phi_w(q(w))$. Then $f(0) = 0$ and $\rho := f'(0) = v\phi_v'(p)$ and we may write $f(z) = z\rho/(1 + zb_1 + z^2b_2 + \cdots)$ and $h(z) := \mu(w)^{-1} = \mu^{-1}(1 + za_1 + z^2a_2 + \cdots)$ for power series uniformly convergent on $|z| < \epsilon$. With this notation, $v_n = v + z_n$ where z_n is the solution to $f(z_n) = h(z_n)^n$.

An elementary estimate (Lemma 3.5) shows that $\lim_{n\to\infty} z_n\mu^n = \rho^{-1}$. But $\rho^{-1} = v^{-1}(\phi_v'(p))^{-1}$ and if we choose j large enough so that $|c_{-j} - p|$ is small, then $(\phi_v'(p))^{-1}$ is approximately equal to $(c_{-j} - p)^{-1}$. The claim follows, modulo the proof of Lemma 3.5. □

We now prove the desired estimate, completing the proof of Proposition 3.4.

Lemma 3.5 *Let $\mu, \rho \in \mathbb{C}$ with $0 < |\mu^{-1}| < 1$ and $\rho \neq 0$. Let $h(z) := \mu^{-1}(1 + za_1 + z^2a_2 + \cdots)$ and $f(z) := z\rho/(1 + zb_1 + z^2b_2 + \cdots)$ be holomorphic in some open neighborhood of 0. Then for $n \gg 1$ there is a unique $z_n \in \mathbb{C}$ with $|z_n| \ll 1$ such that $f(z_n) = h(z_n)^n$, and furthermore $\lim_{n\to\infty} z_n\mu^n = \rho^{-1}$.*

Proof By definition z_n is the solution to

$$z\rho = \mu^{-n}(1 + za_1 + z^2a_2 + \cdots)^n(1 + zb_1 + z^2b_2 + \cdots) := \mu^{-n}\Big(1 + \sum_{j>0} \kappa_{j,n} z^j\Big)$$

for suitable coefficients $\kappa_{j,n}$ depending on n. Since $f(z)$ and $h(z)$ are holomorphic in an open neighborhood of 0, there are positive real constants α and β so that $|a_k| < \alpha^k$ and $|b_k| < \beta^k$ for all k, and therefore we may estimate $|\kappa_{j,n}| \leq \frac{(n+1+j)!}{n!j!}\sigma^j$ where $\sigma = \max(\alpha, \beta)$.

Set $\zeta_1 := \mu^{-n}\rho^{-1}$ and recursively define

$$\zeta_{k+1} := \mu^{-n}\rho^{-1}\Big(1 + \sum_{j>0} \kappa_{j,n}\zeta_k^j\Big)$$

Then it follows by induction for n sufficiently large that there are constants $C_1 > 0$ and $0 < C_2 < 1$ independent of n and k so that

$$|\zeta_{k+1} - \zeta_k| < C_1\mu^{n(k+1)C_2}$$

(in fact we can take C_2 to be any fixed positive number < 1 at the cost of adjusting C_1). In particular, for each fixed n, the ζ_k converge at a geometric rate to z_n, and by inspection as $n \to \infty$ we have $z_n\mu^n \to \rho^{-1}$. □

Example 3.6 Let's return to the example we worked out in Section 2.2. The fixed points of the quadratic map $f_w : z \to z^2 + w$ are the roots of $z^2 - z + w$, which are $\frac{1}{2}\big(1 \pm \sqrt{1-4w}\big)$. Thus for our family where $v = -2$ and $|w - v|$ is small, the root $p(w)$ is equal to $\frac{1}{2}\big(1 + \sqrt{1-4w}\big)$ and $p'(-2) = -\frac{1}{3}$. On the other hand, $q(w) = w^2 + w$, so that $q'(-2) = -3$. Thus $\nu := q'(-2) - p'(-2) = -3 + \frac{1}{3} = -\frac{8}{3}$.

Every f_w in a surgery family is pcf, and therefore w is in the filled Mandelbrot set. Let v be a Misiurewicz point, and let $\mathcal{M} - v \subset \mathbb{C}$ denote the result of translating the Mandelbrot set $\mathcal{M}$ so that v is moved to the origin. For each n let $\mu^n(\mathcal{M} - v)$ be the subset of $\mathbb{C}$ obtained by multiplying $\mathcal{M} - v$ by μ^n.

Define $\widehat{Z} := \liminf \mu^n(\mathcal{M} - v)$ where the limit is taken in the Hausdorff topology. In other words, a point p is in $\widehat{Z}$ if and only if for every infinite subsequence of dilations $\mu^{n_i}(\mathcal{M} - v)$ there are points $p_i \in \mu^{n_i}(\mathcal{M}_v)$ with $p_i \to p$. Evidently $\widehat{Z}$ is invariant under multiplication by μ, and therefore $\widehat{Z} - 0$ covers a closed subset $Z \subset E_\mu$.

Proposition 3.7 *The translate $\nu^{-1}Z$ in E_μ contains X.*

Proof This is a direct consequence of Propositions 3.4 and 3.2. □

Conversely, we have the following:

Proposition 3.8 *Let v be a Misiurewicz point, and suppose $v_{n_i} \in \mathcal{M}$ is a sequence of points with $v_{n_i} \to v$ so that $(v_{n_i} - v)\mu^{n_i}$ converge to some limit z. Then the image of $\nu^{-1}z$ in E_μ is in X.*

Proof With notation as above, we let O be the periodic orbit of f_v, choose $f^k(0) = p \in O$, and let U be an open neighborhood of p on which $g\colon U \to U$ is an attracting local branch of f^{-m}. Let $V = g(U)$. Since $v_{n_i} \to v$ we have $f^k_{v_{n_i}}(0) \to p$ and therefore there is some least k_i for which $x_i := f^{k_i}_{v_{n_i}}(0) \in V - U$ and $f^j_{v_{n_i}}(0) \in U$ for $k \leq j < k_i$ (in fact, $(k_i - k)/m - n_i$ is a constant). After passing to a subsequence if necessary, we may assume $x_i \to x \in V - U$. Since $v_{n_i} \in \mathcal{M}$ it follows that the forward orbit of x_i under $f_{v_{n_i}}$ is uniformly bounded (independent of i) and therefore the forward orbit of x under $f = f_v$ is uniformly bounded, so that x is in the filled Julia set of f. Since v is Misiurewicz, it follows that x is in the Julia set of f and therefore that the image of x in E_μ is contained in X. As in our previous calculations (ie Lemma 3.5), this image is equal to $\nu^{-1} z$. □

This concludes our proof of Tan Lei's theorem:

Corollary 3.9 (Tan Lei [5, Theorem 5.1]) *Let $v \in \mathcal{M}$ be a Misiurewicz point associated to a quadratic polynomial $f\colon z \to z^2 + v$ with Julia set $J(f)$ and let $p \in P(0)$ be periodic for f with multiplier μ. Let $\nu \neq 0$ be defined as above. Then the Hausdorff limits*

$$\lim_{n\to\infty} \nu^{-1}\mu^n(\mathcal{M} - v) \quad \text{and} \quad \lim_{n\to\infty} \mu^n(J(f) - p)$$

are equal.

Remark 3.10 There is nothing important about the quadratic family so far as Propositions 3.2 or 3.4 are concerned, except that our maps have been constrained to lie in a one (complex) dimensional family.

Let f be an arbitrary pcf map with a critical point $c \in C(f)$ and repelling periodic orbit $O \subset P(c) - C(f)$. Let V be a one-dimensional family of nonconjugate rational maps with $f \in V$. For each critical point $c \in C(f)$ and each $g \in V$ near f there is a critical point $c(g) \in C(g)$ close to c; suppose for all $g \in V$ that $P(c')$ is finite for all $c' \in C(g) - c(g)$. Choose $p \in O$ equal to $f^k(c)$ for some least c. For g near f we can define $q(g) := g^k(c(g))$ and $p(g)$ to be the repelling periodic point for g near p, and we may define $\nu := d/dg|_{g=f} q(g) - p(g)$. Then providing $\nu \neq 0$ we may construct a surgery family $f_n \in V$ associated to any tail T for O, c exactly as above, and the analog of Proposition 3.4 holds for $\lim_{n\to\infty} \mu^n(f_n - f)$. The condition $\nu \neq 0$ holds providing the pcf maps obtained by deforming f are holomorphically rigid, which according to Thurston's theorem holds automatically if $|P(f)| > 4$. Presumably the exceptions may be enumerated.

It is worth pointing out that $\nu = 0$ if V is a Lattès family; see eg [3], and indeed such a family evidently does not contain a surgery sequence for any $f \in V$.

4 Another example

The map $z\colon \to z^2 + i$ has $P(0) := \{i, i-1, -i\}$ and we can take $O := \{i-1, -i\}$ and $p = -i \in O$. The multiplier is $\mu := 4(1+i)$. The Julia set J and a blow-up near $-i$ are illustrated in Figure 2, top row. The Mandelbrot set $\mathcal{M}$ and a blow-up near i are illustrated in Figure 2, bottom row, for comparison.

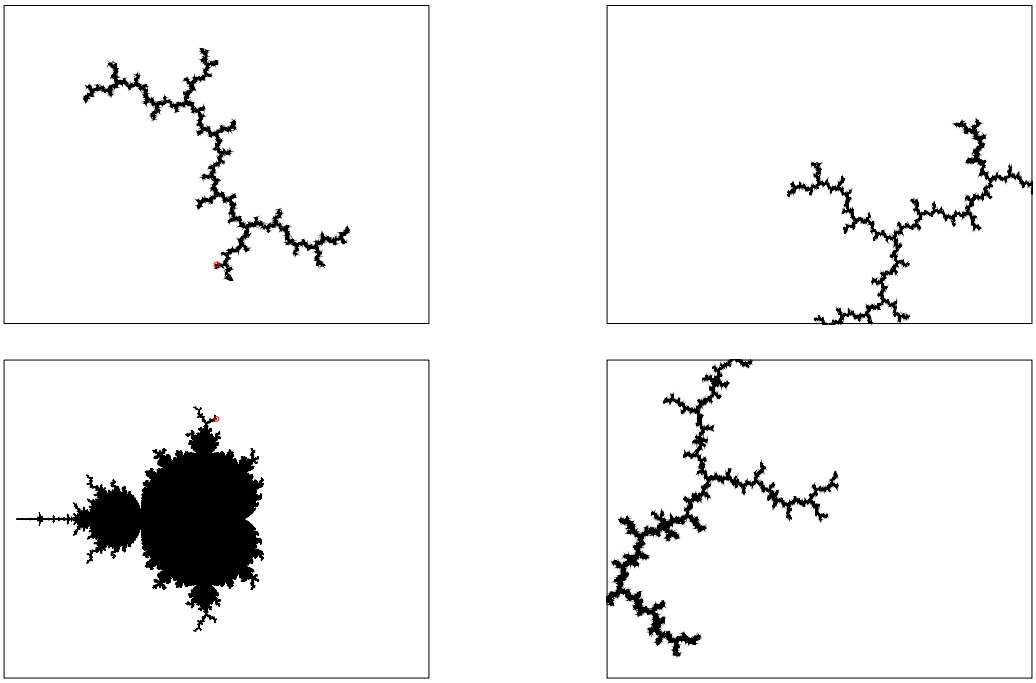

Figure 2: Top row: the Julia set of $z \to z^2 + i$ and a blow-up near $-i$. Bottom row: the Mandelbrot set and a blow-up near i.

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I met Tan Lei in person only once, at the Thurston memorial conference in 2013, although I corresponded with her by email from time to time. I have always loved her theorem on the asymptotic self-similarity of Julia and Mandelbrot sets, and I regret that I cannot share this argument with her (which I believe she would have found amusing) but regret her absence much more.

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