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Algebraic models for equivariant rational homotopy theory were developed by Triantafillou and Scull for finite group actions and S^1 action, respectively. They showed that given a diagram of rational cohomology algebras from the orbit category of a group G, there is a unique minimal system of DGAs representing a unique G-rational homotopy type that is weakly equivalent to it. However, there can be several equivariant rational homotopy types with the same diagram of cohomology algebras. Halperin, Stasheff, and others studied the problem of classifying rational homotopy types up to cohomology in the nonequivariant case. In this article, we consider this question in the equivariant case. For the case $G = C_p$, for prime p, under suitable conditions, we are able to determine the equivariant rational homotopy types with isomorphic diagram of cohomology algebras in terms of nonequivariant data. We give explicit examples to demonstrate how these theorems can be applied to classify equivariant rational homotopy types with isomorphic cohomology.

55P62, 55P91; 16E45, 18G10

1 Introduction

Any two simply connected spaces are said to have the same *rational homotopy type* if there is a zigzag of morphisms between them, each inducing isomorphism on their rational cohomology. Quillen and Sullivan give algebraic models, namely, differential graded Lie algebras [8] and differential graded commutative algebras, written as DGAs in short (minimal algebras, [14, Section 2]) respectively, describing simply connected spaces up to their rational homotopy type.

We say two G-simply connected spaces (i.e. G-space X whose fixed point spaces, X^H, are simply connected for all subgroups H of G) have the same *G-rational homotopy type* if there is a zigzag of G-maps, each inducing an isomorphism on the rational cohomology of the fixed point of the spaces under every subgroup of G. Triantafillou [16] (for finite G) and Scull [11] (for $G = S^1$) define algebraic models describing G-simply connected spaces up to the same G-rational homotopy type. These models lie in the subcategory of injective objects (Definition 2.2) of the category of functors from the orbit category of G, $\mathcal{O}_G$ (Definition 2.1), to the category of cohomologically 1-connected DGAs. We refer to functors from $\mathcal{O}_G$ to the category of cohomologically 1-connected DGAs (graded algebras/vector spaces) as a *diagram of DGAs* (graded algebras/vector spaces) over $\mathcal{O}_G$ and as a *system of DGAs* over $\mathcal{O}_G$ when it is injective.

In [16, Theorem 6.2; 11, Theorem 6.28], the authors show that there is a one-to-one correspondence between G-simply connected spaces up to the G-rational homotopy type and isomorphism classes of minimal system of DGAs (Definition 2.14) over $\mathcal{O}_G$.

A natural question is to ask for all minimal algebras and, therefore, rational homotopy types with isomorphic (rational) cohomology algebras. Given a graded algebra A^*, define the moduli set of all minimal algebras (up to isomorphism) with cohomology A^*,

$$\mathcal{M}_{A^*} := \{\text{rational homotopy type of } X \mid H^*(X;\mathbb{Q}) \cong A^*\}.$$

In the nonequivariant context, the set $\mathcal{M}_{A^*}$ has been studied by several authors including [5; 7; 10; 12; 13] with different viewpoints. Lupton [7] shows that for any positive integer n there is a graded algebra A^* such that the cardinality of $\mathcal{M}_{A^*}$ is n.

Lemaire and Sigrist [6] produce an infinite family of distinct rational homotopy types with the same cohomology algebra and rational homotopy Lie algebra. Halperin and Stasheff [5], study $\mathcal{M}_{A^*}$ by considering the set of perturbations of a bigraded model constructed from A^*. In particular, they show that for $A^* = H^*((S^2 \vee S^2) \times S^3;\mathbb{Q})$, the set $\mathcal{M}_{A^*}$ consists of two points. Shiga and Yamaguchi [13] study the set $\mathcal{M}_{A^*}$ by constructing a correspondence between $\mathcal{M}_{A^*}$ and rational points of Grassmann manifolds modulo an equivalence relation generated by the group of automorphisms of DGAs [13, Theorem 2.1, Corollary 2.3].

In this article, we study the equivariant analogue of $\mathcal{M}_{A^*}$, defined as the moduli set of all minimal systems of DGAs (up to isomorphism) with cohomology diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_G$,

$$\mathcal{M}^G_{\mathcal{A}^*} := \{\mathcal{M} \mid \mathcal{M} \text{ is a minimal system of DGAs over } \mathcal{O}_G \text{ and } H^*(\mathcal{M};\underline{\mathbb{Q}}) \cong \mathcal{A}^*\},$$

where $\underline{\mathbb{Q}}$ is the constant coefficient system defined by $\underline{\mathbb{Q}}(G/H) := \mathbb{Q}$, for every subgroup H of G. A minimal system of DGAs over $\mathcal{O}_G$ which determines the G-rational homotopy type of a G-simply connected space is obtained by taking elementary extensions (Definition 2.11) inductively. An elementary extension is the equivariant analogue of the Hirsch extension used to construct minimal algebras [4, Chapter 16.2]. Unlike the nonequivariant case, the generators added at n^{th} stage extension in the construction of $\mathcal{M}$, which we denote by $\mathcal{M}_n$, can have degree greater than n.

The construction of an elementary extension of a system of DGAs $\mathcal{U}$ over $\mathcal{O}_G$ depends on the following data: a diagram of vector spaces $\underline{V}$ over $\mathcal{O}_G$ of degree n and an element $[\alpha] \in H^n(\mathcal{U};\underline{V})$, and the extension is denoted by $\mathcal{U}^{\alpha}(\underline{V})$. Any two nonisomorphic minimal systems of DGAs, with an isomorphic diagram of cohomology algebras, differ at some n^{th} stage.

A necessary condition for isomorphic elementary extensions over the same system of DGAs over $\mathcal{O}_G$, is given by Scull [11, Proposition 11.52].

In Section 3, we define *condition C_n* on a system of DGAs which ensures that two elementary extensions over the same system of DGAs are nonisomorphic, as proved in Proposition 3.11. This improves our understanding of $\mathcal{M}^G_{\mathcal{A}^*}$ and more specifically gives a method to construct minimal systems of DGAs that are not quasi-isomorphic but have the isomorphic cohomology diagrams. For instance, we show that $\mathcal{M}^G_{\mathcal{A}^*}$ can have more than one point in Example 3.13.

In order to study $\mathcal{M}^G_{\mathcal{A}^*}$, for a cohomology diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_G$ which can be considered as a diagram of DGAs with zero differential, it is imperative to understand its minimal model, that is, a minimal system of DGAs $\mathcal{M}$ over $\mathcal{O}_G$ with a morphism $\rho\colon \mathcal{M} \to \mathcal{A}^*$ inducing isomorphism in cohomology. However, the cohomology diagram of a given G-space is not always injective, as can be seen from examples (Example 4.2). In such cases, we need to consider the injective envelope of the diagram of DGAs (Theorem 2.5), whose differential need not be zero, making the ensuing computations more complex. In Proposition 4.1, we give a simple condition to verify when a diagram of DGAs over an orbit category of C_p, for prime p, is injective.

Moreover, we observe that at each stage of the construction of a minimal system if the *associated* diagram of vector spaces (Definition 3.9) involved in the elementary extension is injective, then the computations simplify. We therefore, consider the following question:

Question 1.1 *What conditions on an injective cohomology diagram of graded algebras ensure that the associated diagram of vector spaces added for each elementary extension are injective*?

We answer this question when $G = C_p$, where p is prime. In Proposition 4.5, we show that if the cohomology diagram $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$ has the property that the structure map $\mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is a retract (Definition 2.21) and the minimal model is $\mathcal{M}$, then the associated diagram of vector spaces at every stage of the construction of $\mathcal{M}$ is injective. In this case, the equivariant minimal model $\mathcal{M}$ of $\mathcal{A}^*$ is levelwise minimal, i.e. $\mathcal{M}(G/H)$ is a minimal model for $\mathcal{A}^*(G/H)$, reducing the problem to the nonequivariant case.

The diagram of graded algebras, $\mathcal{A}^*$ over $\mathcal{O}_G$, is said to be *equivariantly k-intrinsically formal* if there exists a k-isomorphism (Definition 2.20) between any two minimal systems in the equivariant moduli set $\mathcal{M}^G_{\mathcal{A}^*}$. Further, if this set $\mathcal{M}^G_{\mathcal{A}^*}$ is a singleton, then we say $\mathcal{A}^*$ is *equivariantly intrinsically formal*.

Define a system of DGAs to be *equivariantly formal* if its weak equivalence class (Definition 2.10) can be completely determined by its cohomology diagram. A G-space is equivariantly formal if the minimal system of DGAs corresponding to it is equivariantly formal. With this set up, we consider the following question:

Question 1.2 *Can we compute the cardinality of G-rational homotopy types with isomorphic cohomology diagrams over $\mathcal{O}_G$ or say when the cohomology diagram is equivariantly intrinsically formal*?

We address this question for $G = C_p$, for p prime, in Section 5. We extend the results of [13], for systems of DGAs over $\mathcal{O}_{C_p}$ in Theorem 5.4. In Theorem 5.6, under suitable conditions, we determine the cardinality of a subclass of $\mathcal{M}^{C_p}_{\mathcal{A}^*}$ in terms of the nonequivariant set $\mathcal{M}_{\mathcal{A}^*(C_p/e)}$. In Corollary 5.7, we give a sufficient condition for a diagram of graded algebras to be equivariantly intrinsically formal, that is, the equivariant moduli set corresponding to $\mathcal{A}^*$ is a point. More precisely, we prove the following.

Theorem 5.6 and Corollary 5.7 *Let $\mathcal{A}^*$ be a diagram of graded algebras over $\mathbb{O}_{C_p}$ such that its structure map is a retract with minimal model $\mathcal{M}$. Assume that $\mathcal{A}^*$ is equivariantly $(n-1)$-intrinsically formal and $\mathcal{M}(C_p/C_p)$ does not have elements of degree $\geq n$. Then the following statements are true.*

1. *The set of isomorphism classes of minimal systems containing $\mathcal{M}_{n-1}$ is determined by the moduli set corresponding to $\mathcal{A}^*(C_p/e)$.*
2. *If $\mathcal{A}^*(C_p/e)$ does not have elements of degree $> n+1$, then the cardinality of the equivariant moduli set corresponding to $\mathcal{A}^*$ coincides with that of the moduli set corresponding to $\mathcal{A}^*(C_p/e)$.*
3. *If the minimal models of $\mathcal{M}(C_p/e)$ and $H^*(\mathcal{M}(C_p/e))$ are isomorphic, then $\mathcal{A}^*$ is equivariantly intrinsically formal. In particular, $\mathcal{M}$ is equivariantly formal.*

This allows us to produce examples of equivariantly formal C_p-spaces; for instance, see Example 6.1. As a further application of Theorem 5.6 and Corollary 5.7, in Example 6.2 we demonstrate how to prove that a given diagram of DGAs is equivariantly n-intrinsically formal for a particular n. Further, using the work of Shiga and Yamaguchi [5] for the nonequivariant case, we are able to compute the cardinality of $\mathcal{M}_{\mathcal{A}^*}^{C_p}$ for a given diagram of graded algebras $\mathcal{A}^*$ over $\mathbb{O}_{C_p}$ in Examples 6.3 and 6.5.

In the forthcoming work [9; 15], we extend these results for C_{p^n}- and C_{pq}-diagrams of graded algebras.

2 Background

In this article, we work with DGAs over $\mathbb{Q}$ and assume G to be a finite group.

Definition 2.1 Given a group G, the category of canonical orbits is the category whose objects are G-sets G/H and morphisms are G maps between them. We denote this category by $\mathbb{O}_G$.

Definition 2.2 An object I in a category $\mathcal{C}$ is said to be *injective* if for every injective morphism $f: X \to Y$ and every morphism $g: X \to I$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ I & & \end{array}$$

there exists a morphism $h: Y \to I$ such that $h \circ f = g$.

A *diagram of DGAs* is a covariant functor from the orbit category $\mathbb{O}_G$ to the category of cohomologically 1-connected DGAs. If this functor is injective, then we refer to it as a *system of DGAs* in line with [16]. We will denote the category of systems of DGAs by $\mathrm{DGA}^{\mathbb{O}_G}$.

By forgetting the differential in a diagram of DGAs we get a diagram of rational vector spaces, also known as a *dual rational coefficient system*. The category of dual rational coefficient systems will be denoted by Vec^*_G. A *rational coefficient system* is a contravariant functor from $\mathbb{O}_G$ to the category of rational vector spaces. We denote the category of rational coefficient systems by Vec_G.

Let X be a G-space such that X^H is nonempty and simply connected for all $H \le G$. Then the corresponding diagram of cohomology algebra of X with differential 0 is 1-connected (i.e. $H^1(X^H;\mathbb{Q}) = 0$ for every subgroup H of G). This need not be an injective dual coefficient system. However, every dual coefficient system has an injective envelope.

We now describe [11, Proposition 7.34], the embedding of a given coefficient system $\mathcal{A}$ into its injective envelope $\mathcal{I}$.

Definition 2.3 We define

$$V_H := \bigcap_{H \subset K} \ker \mathcal{A}(\hat{e}_{H,K}), \tag{1}$$

where $\hat{e}_{H,K} : G/H \to G/K$ is the projection and $\mathcal{A}(\hat{e}_{H,K})$ is the induced structure map on the functor $\mathcal{A}$. Note that V_G is defined to be $\mathcal{A}(G/G)$. Let $\mathcal{I} = \bigoplus_H \underline{V}^*_H$, where

$$\underline{V}^*_H(G/K) := \mathrm{Hom}_{\mathbb{Q}(N(H)/H)}((G/H)^K, V_H). \tag{2}$$

There is an injective morphism $\mathcal{A} \to \mathcal{I}$ extending the natural inclusions of $\bigcap_{H \subset K} \ker \mathcal{A}(\hat{e}_{H,K})$.

Proposition 2.4 [16, Section 4] *A dual coefficient system $\mathcal{A}$ is injective if and only if it is of the form $\mathcal{A} = \bigoplus_H \underline{V}^*_H$ for some collection of $\mathbb{Q}(N(H)/H)$-modules V_H and*

$$\underline{V}^*_H(G/K) = \mathrm{Hom}_{\mathbb{Q}(N(H)/H)}((G/H)^K, V_H).$$

Given a diagram of DGAs, forgetting the differential will give a dual rational coefficient system whose injective envelope is a diagram of DGAs, with 0 differential. However, the map into the injective envelope of dual rational coefficient system will not be a quasi-isomorphism in general.

Fine and Triantafillou [3] prove the existence of *injective envelope* for a diagram of DGAs.

Theorem 2.5 [3, Theorem 1] *For a diagram of DGAs $\mathscr{A}$ over $\mathbb{O}_G$, where G is finite group, there is an injective system of DGAs $\mathscr{I}(\mathscr{A})$, called the injective envelope of $\mathscr{A}$, along with an inclusion $i : \mathscr{A} \to \mathscr{I}(\mathscr{A})$ which is a quasi-isomorphism.*

We now describe their construction.

Definition 2.6 Let G be a group and $H \le G$. Let A_H be a DGA over $\mathbb{Q}$ such that $N(H)/H$ acts on it by DGA automorphisms. The *associated system of DGAs $\mathscr{A}_H$*, of the DGA A_H, is a system of DGAs defined as follows. Let V_H be a copy of A_H considered as a graded $\mathbb{Q}(N(H)/H)$-module by forgetting the differential and let $\underline{V}^*_H$ be the induced injective diagram of vector spaces Definition 2.3. Let $s\underline{V}^*_H$ be copy of $\underline{V}^*_H$ with a shift of degree by $+1$. We denote by $\bigwedge_H$ the system of acyclic DGAs generated by $\underline{V}^*_H \oplus s\underline{V}^*_H$, where $d(\underline{V}^*_H) = s\underline{V}^*_H$. Now we define the associated system $\mathscr{A}_H$ by

$$\mathscr{A}_H(G/K) = \begin{cases} \bigwedge_H(G/K) & \text{for } (K) < (H), \\ \mathrm{Hom}_{\mathbb{Q}(N(H)/H)}(\mathbb{Q}(G/H)^K, A_H) & \text{for } (K) \neq (H), \end{cases}$$

where (H) is the conjugacy class of H in G. The value of this functor on morphism is obvious.

Definition 2.7 Let $\mathscr{A}$ be a system of DGAs and let A_H be the subalgebra of $\mathscr{A}(G/H)$ which is equal to $\bigcap_{H \subset H'} \ker \mathscr{A}_{H,H'}$, where $\mathscr{A}_{H,H'}$ is the morphism induced by the projection $G/H \to G/H'$. Let $\mathscr{A}_H$ be the associated system to A_H. The enlargement of $\mathscr{A}$ at H is the system of DGAs $\mathscr{I}_H(\mathscr{A})$ defined by

$$\mathscr{I}_H(\mathscr{A}) = \begin{cases} \mathscr{A}(G/K) \otimes \mathscr{A}_H(G/K) & \text{for } K < H, \\ \mathscr{A}(G/K) & \text{otherwise,} \end{cases}$$

where $K < H$ means that K is a proper subgroup of a conjugate of H. The value of the functor $\mathscr{I}_H(\mathscr{A})$ on morphisms is the obvious one, namely, they are equal to the old morphisms when restricted to the subsystems $\mathscr{A}$ and $\mathscr{A}_H$ respectively.

The injective envelope of a system of DGAs $\mathscr{A}$ is constructed by the following steps. First, we consider enlargement at G, namely, $\mathscr{I}_G(\mathscr{A})$ of the given system. Next we consider a maximal subgroup H of G, and take the enlargement of $\mathscr{I}_G(\mathscr{A})$ at H' — that is, we construct $\mathscr{I}_{H'}(\mathscr{I}_G(\mathscr{A}))$ — where H' is some conjugate of H. We repeat this process until we reach the trivial subgroup. For details, see [3].

Remark 2.8 Note in the construction of the injective envelope, we add new elements and kill their cohomology class by adding their suspension. So if we start with an injective diagram of graded algebras, thought of as a diagram of DGAs by considering the differential zero, then it is injective as a diagram of DGAs.

We now want to define weak equivalences on the category $\mathscr{O}_G$[DGA]. Before that, we define the notion of homotopy on systems of DGAs.

Definition 2.9 Given a system of DGAs $\mathscr{U}$, define $\mathscr{U}(t, dt)$ as the DGA diagram

$$\mathscr{U}(t, dt)(G/H) = \mathscr{U}(G/H) \otimes_{\mathbb{Q}} \mathbb{Q}(t, dt),$$

where $\mathbb{Q}(t, dt)$ are free with t in degree 0 and dt in degree 1. Two morphisms of DGAs $f, g\colon \mathscr{U}_1 \to \mathscr{U}_2$ are said to be homotopic if there exists a DGA morphism $H\colon \mathscr{U}_1 \to \mathscr{U}_2 \otimes \mathbb{Q}(t, dt)$ such that $p_0 H = f$ and $p_1 H = g$ where $p_i\colon \mathscr{U}_2 \otimes \mathbb{Q}(t, dt) \to \mathscr{U}_2$ are defined as $p_i(t) = i$ for $i = 0, 1$ and $p_i(dt) = 0$.

This does not define an equivalence relation on $\mathrm{DGA}^{\mathscr{O}_G}$. However, it does give an equivalence relation on *minimal* (Definition 2.14) systems of DGAs, which we discuss later in this section. We define a coarser relation on $\mathrm{DGA}^{\mathscr{O}_G}$. Given two diagrams of DGAs, $\mathscr{U}$ and $\mathscr{B}$, if there is a morphism f in $\mathscr{U} \to \mathscr{B}$ or

$\mathscr{B} \to \mathscr{U}$ inducing a cohomology isomorphism at each level (at G/H for all $H \le G$) then f is said to be a quasi-isomorphism. The equivalence relation generated by quasi-isomorphisms is defined as a weak equivalence of systems of DGAs. Using the notion of injective envelopes, we can define weak equivalence on the category of diagrams of DGAs

Definition 2.10 Let $\mathcal{U}$ and $\mathcal{V}$ be two DGA diagrams over $\mathcal{O}_G$. We say $\mathcal{U}$ and $\mathcal{V}$ are *weakly equivalent* if there is a weak equivalence between their injective envelopes.

Recall that associated with any G-space X, there is the system of DGAs given by the de Rham–Alexander–Spanier algebra $\mathscr{E}(X)(G/H) := \mathcal{A}(X^H)$ for every $H \le G$. Triantafillou [16, Theorem 1.5] proves that there is a bijective correspondence between the G-space X (with every fixed point set simply connected) and the minimal system of DGAs $\mathscr{M}_X$ of $\mathscr{E}(X)$.

Scull generalizes these ideas to spaces with an S^1 action. In [11, Section 21], Scull shows that, unlike the nonequivariant case, the notion of minimality in the equivariant case arising from filtration via minimal extensions of systems of DGAs does not satisfy the decomposability condition.

Note that homotopy defines an equivalence relation on morphisms from $\mathscr{M} \to \mathscr{B}$ for any system of DGAs $\mathscr{B}$, whenever $\mathscr{M}$ is a minimal system [11, Proposition 3.5]. Further, given a quasi-isomorphism $\rho\colon \mathscr{U} \to \mathscr{B}$ of a system of DGAs and a morphism $f : \mathscr{M} \to \mathscr{B}$ from a minimal system $\mathscr{M}$, there is a lift $g : \mathscr{M} \to \mathscr{U}$ such that $\rho g \simeq f$ [11, Proposition 3.6].

The results [16, Theorem 1.5; 11, Theorem 4.13] show that the category of G-spaces (whose fixed points sets are simply connected) up to rational homotopy equivalences is equivalent to the category of minimal systems of 1-connected DGAs modulo homotopy equivalences.

In order to give the construction of a minimal model of a system of DGAs we first define elementary extensions.

Definition 2.11 Given a system of DGAs $\mathcal{U}$, a diagram of vector spaces $\underline{V}$ assigned to be of degree n, and a map $\alpha\colon \underline{V} \to \underline{Z}^{n+1}(\mathcal{U})$ (here $\underline{Z}(\mathcal{U})$ denotes the kernel of $\mathcal{U}$), the *elementary extension* of $\mathcal{U}$ with respect to α and $\underline{V}$, denoted by $\mathcal{U}^{\alpha}(\underline{V})$, is constructed as follows.

Let $\underline{V} \to \underline{V}_0 \xrightarrow{w_0} \underline{V}_1 \xrightarrow{w_1} \underline{V}_2 \to \cdots$ be a minimal injective resolution of $\underline{V}$ constructed by taking $\underline{V}_i$ to be the injective embedding of coker w_{i-1}, which is of finite length.

Construct a commutative diagram

$$\begin{array}{ccccccccc}
\underline{V} & \longrightarrow & \underline{V}_0 & \xrightarrow{w_0} & \underline{V}_1 & \xrightarrow{w_1} & \underline{V}_2 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \\
\underline{Z}^{n+1}(\mathcal{U}) & \longrightarrow & \mathcal{U}^{n+1} & \xrightarrow[d]{} & \mathcal{U}^{n+2} & \xrightarrow[d]{} & \mathcal{U}^{n+3} & \longrightarrow & \cdots
\end{array}$$

The maps α_i are constructed inductively by first noting that $d\alpha_i w_{i-1} = dd\alpha_{i-1} = 0$, so $d\alpha_i|_{\operatorname{Im} w_{i-1}} = 0$ and then by the injectivity of $\mathcal{U}$ we get a commutative diagram

$$\begin{array}{ccc} \underline{V}_i/\operatorname{Im} w_{i-1} & \xhookrightarrow{\rho^*} & \underline{V}_{i+1} \\ {\scriptstyle d\alpha_i}\big\downarrow & \swarrow{\scriptstyle \alpha_{i+1}} & \\ \mathcal{U}^{n+i+1} & & \end{array}$$

Define $\mathcal{U}^\alpha(\underline{V}) := \mathcal{U} \otimes \big(\bigotimes_i \mathbb{Q}(\underline{V}_i)\big)$, where $\mathbb{Q}(\underline{V}_i)$ is the free graded commutative algebra generated at G/H by the vector space $\underline{V}_i(G/H)$ in degree $n+i$; the differential is defined on $\mathcal{U}$ by the original differential on $\mathcal{U}$, and on the generators of $\underline{V}_i$ by $d = (-1)^i\alpha_i + w_i$. Since $\underline{V}_i$ is injective for all i by construction, as a vector space the system is the tensor product of injectives and hence injective. Thus, $\mathcal{U}^\alpha(\underline{V})$ is a new system of DGAs.

Remark 2.12 We use $\mathcal{U}^\alpha(\underline{V})$ to denote the elementary extension of $\mathcal{U}$ by a diagram of vector spaces $\underline{V}$ with respect to the map α. We denote the elementary extension of $\mathcal{U}$ by the diagram of vector spaces $\underline{V}$ by $\mathcal{U}(\underline{V})$ if we do not want to focus on the map α.

The following result shows when two elementary extensions are isomorphic.

Proposition 2.13 [11, Lemma 11.53] *Suppose $f : \mathcal{U}^\alpha(\underline{V}) \to \mathcal{U}^{\alpha'}(\underline{V}')$ is a map between two degree n elementary extensions of $\mathcal{U}$ with the following properties*:

(1) *f restricts to an isomorphism of $\mathcal{U}$.*

(2) *On $\underline{V}$, $f(x) = g(x) + a(x)$, where $g : \underline{V} \to \underline{V}'$ is an isomorphism and $a(x) \in \mathcal{U}$.*

Then f is an isomorphism.

A minimal system of DGAs is defined as follows.

Definition 2.14 A system of DGAs $\mathcal{M}$ is minimal if $\mathcal{M} = \bigcup_n \mathcal{M}_n$, where $\mathcal{M}_0 = \mathcal{M}_1 = \underline{\mathbb{Q}}$ and $\mathcal{M}_n = \mathcal{M}_{n-1}(\underline{V})$ is the elementary extension for some diagram of vector spaces $\underline{V}$ of degree $\geq n$.

Theorem 2.15 [11, Theorem 3.8] *If $f : \mathcal{M} \to \mathcal{N}$ be a quasi-isomorphism between minimal systems of DGAs, then $f \simeq g$, when g is an isomorphism.*

Thus, if we have two minimal systems $\mathcal{M}$ and $\mathcal{N}$, and quasi-isomorphisms $\rho_1 : \mathcal{M} \to \mathcal{U}$ and $\rho_2 : \mathcal{N} \to \mathcal{U}$ by the lifting property of maps from minimal systems to systems of DGAs, we get a map $f : \mathcal{M} \to \mathcal{N}$ which is a quasi-isomorphism. By Theorem 2.15, we get $f \simeq g$ where g is an isomorphism. Now we define the following.

Definition 2.16 If $\mathcal{M}$ is a minimal system and $\rho : \mathcal{M} \to \mathfrak{U}$ is a quasi-isomorphism, we say that $\mathcal{M}$ is a minimal model of $\mathfrak{U}$.

Maps between two minimal systems of DGAs are much *nicer*, in the sense that they are always homotopy equivalent to a levelwise map of extensions. We will make use of this fact later.

Lemma 2.17 [11, Lemma 13.57] *Any morphism $f : \mathcal{M} \to \mathcal{N}$ between minimal systems of DGAs is homotopic to a morphism g which maps $\mathcal{M}_n$ to $\mathcal{N}_n$ for all n.*

Remark 2.18 Given a morphism g as in Lemma 2.17, by Theorem 2.15 we get that g is an isomorphism. Then Proposition 2.13 implies that if $\mathcal{M}_n = \mathcal{M}_{n-1}(\underline{V})$ and $\mathcal{N}_n = \mathcal{N}_{n-1}(\underline{V}')$ then $\underline{V} \cong \underline{V}'$ and for any $x \in \underline{V}$, $g(x) = x + b$ for some $b \in \mathcal{N}_{n-1}$.

Observe that any minimal system is cohomologically 1-connected, that is, it satisfies $\underline{H}^0(\mathcal{M}) = \mathbb{Q}$ and $\underline{H}^1(\mathcal{M}) = 0$. It can be shown that being cohomologically 1-connected is sufficient for a diagram of DGAs to have a *minimal model*.

Theorem 2.19 [11, Theorem 3.11] *If $\mathcal{U}$ is a system of DGAs which is cohomologically 1-connected, then there exists a minimal model of $\mathcal{U}$, i.e. a minimal system $\mathcal{M}$ and a quasi-isomorphism $\rho : \mathcal{M} \to \mathcal{U}$.*

Note that this construction ensures that ρ restricted to $\mathcal{M}_n \to \mathcal{U}$ is an *n-isomorphism*.

Definition 2.20 We say a morphism $f : \mathcal{U} \to \mathcal{B}$ between two systems of DGAs is an n-isomorphism if $f^* : H^*(\mathcal{U}) \to H^*(\mathcal{B})$ is isomorphism up to degree n and monomorphism at degree $(n+1)$.

In the nonequivariant setup, we define the following.

Definition 2.21 (retract) Given a DGA A and a sub-DGA B of A, we say B is a retract of A if there is a DGA-morphism $r : A \to B$ such that $r \circ i = \mathrm{id}_B$. Here $i : B \to A$ is the inclusion morphism and the morphism r is called the retraction.

3 Equivariantly intrinsically formal graded algebras

A cohomologically 1-connected DGA is said to be *formal* if its minimal model is weakly equivalent to the minimal model of its cohomology algebra. Recall that a *minimal algebra m* is a free graded algebra which can be written as an increasing union of m_i's where $m_0 = m_1 = \mathbb{Q}$, and $m_{k-1} \to m_k$ is a Hirsch extension for every k [4, Theorem 10.3]. A *minimal model* of a 1-connected DGA u, is a pair (m, ρ), where m is a minimal algebra and $\rho : m \to u$ is a quasi-isomorphism.

A simply connected space is said to be *formal* if the corresponding minimal model is formal. The rational homotopy groups of formal spaces can be computed from its cohomology algebra and by rational Postnikov tower [1, Theorem 3.3]. A graded algebra A^* is said to be k-intrinsically formal if for any minimal algebra m with $H^*(m) \cong A^*$ the sub-DGA m_k of m generated by elements degree $\le k$, is unique up to isomorphism. If m_k is unique for every k, the graded algebra A^* becomes intrinsically formal, and any space with cohomology algebra A^* will be formal. In this section, we introduce the notion of formality and intrinsic formality in the equivariant setup with some examples. We first prove the following facts in the nonequivariant case.

Proposition 3.1 *Let m and m' be two minimal algebras with $m_k \cong m'_k$. If there is a morphism $m_{k+1} \to m'_{k+1}$ which is a $(k+1)$-isomorphism, then $m_{k+1} \cong m'_{k+1}$.*

Proof Let $f\colon m_{k+1} \to m'_{k+1}$ be a $(k+1)$-isomorphism, which is an extension of the isomorphism from $m_k \to m'_k$.

Consider the diagram

$$\begin{array}{ccc} m_k \cong m'_k & \longrightarrow & m_{k+1} \\ \downarrow & \overset{g}{\nearrow} & \downarrow f \\ m'_{k+1} & \xrightarrow{\text{id}} & m'_{k+1} \end{array}$$

where the bottom horizontal arrow is the identity. The obstructions to finding a lift $m'_{k+1} \to m_{k+1}$ successively lie in the relative cohomology, [4, Proposition 11.1], $H^{i+1}(m_{k+1}, m'_{k+1}, V^i)$ where V^i are the degree i generators of m'_{k+1}. If the relative cohomology $H^{i+1}(m_{k+1}, m'_{k+1})$ vanishes for $i+1 \le k+2$, then all the obstructions vanish. Note, we only have to consider i up to $k+2$, as m_{k+1} is generated by elements of degrees $\le k+1$.

Consider the long exact sequence in cohomology

$$\cdots \to H^k(m_{k+1}) \to H^k(m'_{k+1}) \to H^{k+1}(m_{k+1}, m'_{k+1}) \to H^{k+1}(m_{k+1}) \to \cdots .$$

Now, from our assumptions on the map f, it follows that $H^{\le k+2}(m_{k+1}, m'_{k+1})$ vanishes. Therefore, we have a lift g such that fg is homotopic to the identity on m'_{k+1}.

The existence of this g implies that f is cohomologically surjective in all degrees. Also, g is cohomologically injective in all degrees. Since f is a cohomological isomorphism in all degrees $\le k+1$, so is g.

Similarly, consider the diagram

$$\begin{array}{ccc} m_k \cong m'_k & \longrightarrow & m'_{k+1} \\ \downarrow & \overset{h}{\nearrow} & \downarrow g \\ m_{k+1} & \xrightarrow{\text{id}} & m_{k+1} \end{array}$$

We get a lift h as before, and conclude that gh is homotopic to the identity. Thus, g is cohomologically surjective in all degrees. Therefore, it is a cohomological isomorphism in all degrees.

Since $fg \simeq \text{id}$, f is also a cohomological isomorphism.

Both m_{k+1} and m'_{k+1} are minimal algebras, and quasi-isomorphism implies isomorphism. Hence $m_{k+1} \cong m'_{k+1}$. □

Corollary 3.2 *Let m and m' be minimal algebras. A k-isomorphism between m_k and m'_k induces a quasi-isomorphism.*

We can talk about formality and intrinsic formality in the equivariant case as follows.

Definition 3.3 We say a system of DGAs, $(\mathscr{A}, d)$ is equivariantly formal if there is a weak equivalence between $(\mathscr{A}, d)$ and the injective envelope of $(H(\mathscr{A}), 0)$. That is, the isomorphism of graded algebras is realized by a zigzag of quasi-isomorphisms of the system of DGAs. A G-space X is said to be equivariantly formal if the minimal system of DGAs representing it is equivariantly formal.

Definition 3.4 (equivariantly intrinsically formal) A diagram of graded algebras $\mathcal{A}$ over $\mathcal{O}_G$, for some finite group G, is called equivariantly k-intrinsically formal (abbreviated to equivariantly k-IF) if for any two minimal systems $\mathcal{M}$ and $\mathcal{M}'$ with $H^*(\mathcal{M}) = \mathcal{A} = H^*(\mathcal{M}')$, the subsystems $\mathcal{M}_k$ and $\mathcal{M}'_k$ have the property that there is a map from $\mathcal{M}_k \to \mathcal{M}'_k$ or $\mathcal{M}'_k \to \mathcal{M}_k$ which is a k-isomorphism. For any two such minimal systems $\mathcal{M}$ and $\mathcal{M}'$, if there is a map $\phi\colon \mathcal{M} \to \mathcal{M}'$ which is a quasi-isomorphism, we say $\mathcal{A}$ is equivariantly intrinsically formal (abbreviated to equivariantly IF).

Remark 3.5 (1) The above notion of intrinsic formality is compatible with that of the nonequivariant case by Proposition 3.1.

(2) Any diagram of graded algebras is 2-intrinsically formal. Let $\mathcal{A}$ be a diagram of graded algebras and let $\mathcal{M}$ be any minimal algebra such that $H^*(\mathcal{M}) = \mathcal{A}$. By the definition of minimal algebra $\mathcal{M}_0 = \mathbb{Q} = \mathcal{M}_1$. For any diagram of vector spaces $\underline{V}$, $H^3(\mathcal{M}_1; \underline{V}) = 0$. So in order to get $\mathcal{M}_2 = \mathcal{M}_1^{\gamma}(\underline{V})$, the choice of $\gamma \in H^3(\mathcal{M}_1; \underline{V})$ is unique and $\mathcal{M}_2$ is uniquely determined.
Thus, if $\mathcal{A}$ is equivariantly IF and the diagram of cohomology algebras for a G-space X is isomorphic to $\mathcal{A}$, then X is equivariantly formal.

Proposition 3.6 *Let a cohomology diagram $\mathcal{A}$ be equivariantly n-IF with $\mathcal{A}^{n+1} = \mathcal{A}^{n+2} = 0$. Let $\mathcal{M}$ be the minimal model for $\mathcal{A}$ with $\mathcal{M}_n = \mathcal{M}_{n+1}$. Then $\mathcal{A}$ is equivariantly $(n+1)$-IF.*

Proof First, note that if $\mathcal{M}$ is any minimal system then for any $n \geq 0$ the inclusion $\mathcal{M}_n \to \mathcal{M}$ is an n-isomorphism.

Let $\mathcal{M}'$ be another minimal system with $H^*(\mathcal{M}') = \mathcal{A}$. As $\mathcal{A}$ is equivariantly n-IF, let $\phi_n\colon \mathcal{M}_n \to \mathcal{M}'_n$ be an n-isomorphism.

Since $\mathcal{M}_n = \mathcal{M}_{n+1}$, we rewrite the map as $\phi_n\colon \mathcal{M}_{n+1} \to \mathcal{M}'_{n+1}$.

Note that $H^{n+2}(\mathcal{M}_{n+1}) = H^{n+2}(\mathcal{M}_n) = 0$ as $\mathcal{M}_{n+1} \to \mathcal{M}$ is $(n+1)$-isomorphism and $H^{n+1}(\mathcal{M}) = \mathcal{A}^{n+1} = 0$. We claim that $H^{n+2}(\mathcal{M}'_{n+1}) = 0$. Now if $\alpha \in H^{n+2}(\mathcal{M}'_{n+1})$ is nonzero then as $\mathcal{M}'_{n+1} \to \mathcal{M}'$ is an $(n+1)$-isomorphism by injectivity, we have a nonzero member which is the image of α in $H^{n+2}(\mathcal{M}') \cong \mathcal{A}^{n+2} = 0$ and this is a contradiction. The map $H^{n+2}(\mathcal{M}_{n+1}) \to H^{n+2}(\mathcal{M}'_{n+1})$ induced by ϕ_n is a zero map and hence a monomorphism.

Using a similar argument, we conclude that $H^j(\mathcal{M}_{n+1}) \to H^j(\mathcal{M}'_{n+1})$ is an n-isomorphism for $j \leq n$. Hence, $\mathcal{A}$ is equivariantly $(n+1)$-IF. □

Example 3.7 (1) Consider the space $X = S^n \vee S^n \vee \cdots \vee S^n$, which is $(p+1)$ many copies of spheres of dimension $n \geq 2$. There is an action of C_p on X which permutes the first p copies of S^n and keeps the last copy fixed. Note that the cohomology algebra is equivariantly n-IF as the fixed point sets are $(n-1)$-connected. The cohomology algebra also satisfies the hypothesis of Proposition 3.6 so it is equivariantly $(n+1)$-IF. Since all higher cohomologies are zero, by Proposition 3.6 we conclude that the cohomology algebra is equivariantly IF.

(2) Consider $X = S^n \times \cdots \times S^n$ be product of p-copies of S^n's with p a prime number and $n \geq 3$. Then there is a C_p-action on X by permutation and the fixed point set is homeomorphic to S^n. As both fixed point sets are $(n-1)$ connected, so the space is equivariantly n-IF. As $n \geq 3$, using Proposition 3.6 the cohomology diagram is equivariantly $(n+1)$-IF.

We give a condition under which a map of systems of DGAs can be extended to a map from an elementary extension of the domain.

Proposition 3.8 *Let $f : \mathcal{U} \to \mathcal{B}$ be a map of systems of DGAs and $\mathcal{U}^\alpha(\underline{V})$ be an elementary extension with respect to some α.*

(1) *If $f' : \underline{V} \to \mathcal{B}$ satisfies $f\alpha = df'$, one can extend f to $\mathcal{U} \otimes \mathbb{Q}(\underline{V})$ using f' on $\underline{V}$.*

(2) *The converse is also true. Let $f : \mathcal{U}^\alpha(\underline{V}) \to \mathcal{B}$ be a morphism, i.e. $df = fd$. If $f' = f|_{\underline{V}}$, then f' satisfies $f\alpha = df'$.*

Proof Suppose $f' : \underline{V} \to \mathcal{B}$ satisfies $f\alpha = df'$, we can extend f to $\mathcal{U} \otimes \mathbb{Q}(\underline{V})$ using f' on $\underline{V}$; the condition on f' ensures that this map respects the differential. We now extend the map to $\underline{V}_0$ using the injectivity of $\mathcal{B}$ and define it on the rest of the resolution inductively. Given f' on $\underline{V}_i$, we must define a map $f' : \underline{V}_{i+1} \to \mathcal{B}$ such that $f'd = df'$. To ensure this is satisfied, we consider the differential from $\underline{V}_i$ to $\underline{V}_{i+1}$; defined by $d = (-1)^i \alpha_i + w_i$. We need to find a map f' such that

$$f'((-1)^i \alpha_i + w + i) = (-1)^i f'\alpha_i + f'w_i = df',$$

or equivalently, $(-1)^i f'\alpha_i - df' = f w_i$. Observe that since

$$\begin{aligned}((-1)^i f'\alpha_i - df)w_{i-1} &= (-1)^i f\alpha_i w_{i-1} - df w_{i-1} \\ &= (-1)^i fd\alpha_{i-1} - fdw_{i-1} = fd((-1)^i \alpha_{i-1} - w_{i-1}) = fd(-d) = 0,\end{aligned}$$

the map $(-1)^i f'\alpha - df$ vanishes on $\mathrm{Im}(w_{i-1}) \subset \underline{V}_i$, and we have

$$\begin{array}{ccc} \underline{V}_i / \mathrm{Im}\, w_{i-1} & \hookrightarrow & \underline{V}_{i+1} \\ \downarrow & \swarrow_{f'} & \\ \mathcal{B}^{n+i+1} & & \end{array}$$

where the map $\underline{V}_i/\operatorname{Im} w_{i-1} \to \mathcal{B}^{n+i+1}$ is given by $(-1)^i f\alpha_i - df$. Since $\mathcal{B}$ is injective, we can define f' on V_{i+1}. Continuing in this manner, we extend f' to all generators, and therefore to a DGA map on all of $\mathcal{U}(\underline{V})$.

Conversely, take $f' = f|_{\underline{V}}$. Consider the injective resolution of $\underline{V}$, i.e.

$$\underline{V} \to \underline{V}_0 \to \underline{V}_1 \to \cdots .$$

The map $\underline{V}_0 \xrightarrow{w_0} \underline{V}_1$ when restricted to $\underline{V}(\subset \underline{V}_0)$ is $w_0|_{\underline{V}} = 0$, since $\underline{V}_1$ is the injective envelope of $\operatorname{coker}(w)$. By definition, if we pick an element $x \in \underline{V}$, the derivation d has no horizontal component on $\underline{V}$, that is, $d = \alpha$ on $\underline{V}$. Hence $f\alpha = df'$, on $\underline{V}$, where $f' = f|_{\underline{V}}$:

$$\begin{array}{ccccccccc}
\underline{V}_\alpha & \xrightarrow{w} & \underline{V}_0 & \xrightarrow{w_0} & \underline{V}_1 & \xrightarrow{w_1} & \underline{V}_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \\
\underline{Z}^{n+1}(\mathcal{U}) & \longrightarrow & \mathcal{U}_d^{n+1} & \longrightarrow & \mathcal{U}_d^{n+2} & \longrightarrow & \mathcal{U}_d^{n+3} & \longrightarrow & \cdots
\end{array}$$

□

We now recall the construction of the minimal model of a system of DGAs from [11, Theorem 3.11]. Let $\mathcal{U}$ be a system of DGAs; then inductively we build the minimal model $\mathcal{M} = \bigcup \mathcal{M}_d$, where each $\mathcal{M}_n$ is an elementary extension as in Section 2, so that $\mathcal{M}_n = \mathcal{M}_{n-1}^{\gamma}(\underline{V})$. Here $\gamma \in H^{n+1}(\mathcal{M}_{n-1}; \underline{V})$.

Consider the diagram

$$\begin{array}{ccccc}
 & & \mathcal{M}_{n-1} & & \\
 & & {\scriptstyle \alpha}\downarrow & \searrow{\scriptstyle \rho} & \\
\ker\beta & \xrightarrow[i]{} & \mathcal{M}'_{n-1} & \xrightarrow[\beta]{} & \mathcal{U}
\end{array}$$

Here ρ is an $(n-1)$-isomorphism, α is a quasi-isomorphism and β is surjective.

The system of DGAs $\mathcal{M}'_{n-1} = \mathcal{M}_{n-1} \otimes \bigl(\bigotimes_H \mathbb{Q}\bigl(\underline{U}_H^* \oplus \sum \underline{U}_H^*\bigr)\bigr)$, where U_H and $\underline{U}_H^*$ are from Definition 2.3 and $\sum \underline{U}_H^*$ is the system obtained from $\underline{U}_H^*$ by considering the degree of every element shifted by $+1$. The differential on $\mathcal{M}'_{n-1}$ is defined accordingly. Here, the map

$$\beta|_{\mathcal{M}_{n-1}} = \rho, \qquad \beta|_{\underline{U}} = \mathrm{id}, \qquad \beta\Bigl(\sum x\Bigr) = dx \quad \text{for } x \in \sum \underline{U}.$$

Let $\underline{\mathbb{Q}}$ be the coefficient system defined by $\underline{\mathbb{Q}}(G/H) = \mathbb{Q}$ for every subgroup H of G. Define $R := \ker\beta \oplus \underline{\mathbb{Q}}$, and the diagram of vector spaces $\underline{V} = H^{n+1}(R)$. The map γ is the elementary extension obtained by considering $[\mathrm{id}] \in H^*(\underline{V}; \underline{V})$ and considering its image, under the inclusion $i : \ker\beta \to \mathcal{M}'_{n-1}$, in $H^*(\mathcal{M}'_{n-1}; \underline{V})$. Since α is quasi-isomorphism, there is a preimage $\gamma \in H^*(\mathcal{M}_{n-1}; \underline{V})$, of $i^*[\mathrm{id}]$ such that $\alpha^*[\gamma] = i^*[\mathrm{id}]$.

Let X be a G-space and $\mathcal{A}$ be its cohomology diagram. Let $\mathcal{M}$ be the minimal model of $\mathcal{A}$. Then there exists a quasi-isomorphism $\mathcal{M} \xrightarrow{\rho} \mathcal{A}$. The system of DGAs, $\mathcal{M}$, is a minimal system and by definition we have $\mathcal{M} = \bigcup_{i \ge 0} \mathcal{M}_i$, where $\mathcal{M}_n = \mathcal{M}_{n-1}^{\gamma}(\underline{V})$.

Definition 3.9 Let $\mathcal{A}$ be an injective cohomology diagram and $(\mathcal{M}, \rho)$ be its minimal model. Let $\mathcal{M}_n = \mathcal{M}^{\gamma}_{n-1}(\underline{V})$, be the n^{th} stage construction of $\mathcal{M}$, which is obtained by taking elementary extension of $\mathcal{M}_{n-1}$ with the injective resolution of $\underline{V}$. We refer to $\underline{V}$ as the *n^{th} stage associated diagram of vector spaces of $\mathcal{A}$.*

Let $\mathcal{U}$ be a system of DGAs over $\mathcal{O}_G$ and $\mathcal{M}$ be the minimal model of $\mathcal{U}$. Let the restriction at the $(n-1)^{\text{st}}$ level, $\rho\colon \mathcal{M}_{n-1} = \mathcal{N} \to \mathcal{U}$ be such that ρ is an $(n-1)$-isomorphism, $\rho^*[\gamma] = 0$, for $\gamma \in H^{n+1}(\mathcal{N}; \underline{V})$ and $\mathcal{N}^{\gamma}(\underline{V})$ is the n^{th} stage of $\mathcal{M}$. That is, $\rho\colon \mathcal{M}_n = \mathcal{N}^{\gamma}(\underline{V}) \to \mathcal{U}$ is an *n-isomorphism*, a cohomology isomorphism up to degree n and monomorphism at degree $n+1$.

Let $\gamma' \in H^{n+1}(\mathcal{N}; \underline{V})$. We say $\mathcal{N}^{\gamma'}(\underline{V})$ satisfies *condition C_n* with respect to $\mathcal{N}^{\gamma}(\underline{V})$ if the following is true:

Condition $\mathbf{C}_n$ With the above assumptions, there exists $\kappa \in \operatorname{Aut}(\mathcal{N})$ and $g \in \operatorname{Aut}(\underline{V})$ such that $\kappa\gamma = d_{\gamma'}(g + \beta)$, where $\beta\colon \underline{V} \to \mathcal{N}$.

Remark 3.10 Note that if we choose $\sigma = \rho^*$ and $\gamma = \gamma'$ then $\rho^*[\gamma] = 0$. Also, there always exists $\kappa = \mathrm{id} \in \operatorname{Aut}(\mathcal{N})$ and $g = \mathrm{id} \in \operatorname{Aut}(\underline{V})$ so that $\kappa \circ \mathrm{id} = \mathrm{id} \circ \kappa$ (considering $\beta = 0$).

Proposition 3.11 *Let $\mathcal{M}$ be a minimal system with $\mathcal{M}_{n-1} = \mathcal{N}$ and $\mathcal{M}_n = \mathcal{N}^{\gamma}(\underline{V})$. Assume $\gamma' \in H^{n+1}(\mathcal{N}; \underline{V})$. Then the following statements are equivalent*:

(1) *$\mathcal{N}^{\gamma'}(\underline{V})$ satisfies condition C_n.*

(2) *$\mathcal{N}^{\gamma}(\underline{V})$ and $\mathcal{N}^{\gamma'}(\underline{V})$ are isomorphic.*

Proof (2) $\Longrightarrow$ (1) If $f\colon \mathcal{N}^{\gamma}(\underline{V}) \to \mathcal{N}^{\gamma'}(\underline{V})$ is an isomorphism then by Remark 2.18 and Lemma 2.17, we may assume that f takes $(\mathcal{N}^{\gamma}(\underline{V}))(n)$ to $(\mathcal{N}^{\gamma'}(\underline{V}))(n)$; that is, f is a levelwise isomorphism. Since isomorphisms are quasi-isomorphisms, and a quasi-isomorphism between minimal systems is homotopic to an isomorphism which is level invariant by Remark 2.18, we can assume, $f\colon \mathcal{N} \to \mathcal{N}$ is an isomorphism. For $x \in \underline{V}$ we have $f(x) = x + b(x)$, where $b\colon \underline{V} \to \mathcal{N}$.

As, $fd = df$, we have for $x \in \underline{V}$, $f\gamma(x) = d_{\gamma'} f(x) = d_{\gamma'}(x + b) = d_{\gamma'}(\mathrm{id}(x) + b)$ (here the first equality comes from the converse part of Proposition 3.8). So the maps $f\colon \mathcal{N} \to \mathcal{N}$ and $\mathrm{id}\colon \underline{V} \to \underline{V}$ give that condition C_n is satisfied.

(1) $\Longrightarrow$ (2) If there exists $\kappa \in \operatorname{Aut}(\mathcal{N})$ and $g \in \operatorname{Aut}(\underline{V})$ such that $\kappa\gamma = d_{\gamma'}(g+\beta)$, then by Proposition 3.8 this implies that κ extends to a map $\tilde{\kappa}\colon \mathcal{N}^{\gamma}(\underline{V}) \to \mathcal{N}^{\gamma'}(\underline{V})$. By Proposition 2.13, $\tilde{\kappa}$ is isomorphism. □

Example 3.12 We define a $G = C_2$ action on $X = S^3 \vee S^3 \vee S^5$, where C_2 acts on $(S^3 \vee S^3)$ by switching copies and acts on S^5 trivially.

The fixed point set $X^G = S^5$. We denote the cohomology diagram of X by $\mathcal{A}$ and, the generators of the cohomology by x, y, and z. Note that $\deg(x) = 3 = \deg(y)$ and $\deg(z) = 5$.

Then $\mathcal{A}(e) = \mathbb{Q}(x, y, z)/D$ and $\mathcal{A}(G) = \mathbb{Q}(z)/E$, where $D = \langle x^2, z^2, xy, xz, yz\rangle$ and $E = \langle z^2\rangle$.

We will compute the minimal system for X under G action up to the 6^{th} stage. We get that

$$\begin{aligned}
&\mathcal{M}_0 = \mathbb{Q} = \mathcal{M}_1 = \mathcal{M}_2,\\
&\mathcal{M}_3 = \begin{cases} \wedge(a_3, b_3) & \text{at } G/e,\\ \mathbb{Q} & \text{at } G/G,\end{cases}\\
&\mathcal{M}_4 \cong \mathcal{M}_3,\\
&\mathcal{M}_5 = \begin{cases} \wedge(a_3, b_3, c_5, d_5) & \text{at } G/e,\\ \wedge(c_5) & \text{at } G/G.\end{cases}
\end{aligned}$$

The differential is $d(a_3) = 0 = d(b_3)$, $d(d_5) = ab$, and $d(c_5) = 0$, so

$$\mathcal{M}_6 \cong \mathcal{M}_5.$$

At each stage, $H^{n+2}(\mathcal{M}_n; \underline{V})$ is zero for $n \le 5$; thus, condition C_5 is satisfied trivially.

Later in Examples 6.1 and 6.4, we show that this diagram of graded algebras is equivariantly 5-IF.

Example 3.13 Consider $X = (S^3 \vee S^3) \times S^5$ with a diagonal action of $G = C_2$, where C_2 acts on $(S^3 \vee S^3)$ by switching copies and acts on S^5 trivially. Then $X^G \cong S^5$. We denote the cohomology diagram by $\mathcal{A}$, and the generators of the cohomology by x, y and z. Note that, $\deg(x) = 3 = \deg(y)$ and $\deg(z) = 5$, so

$$\mathcal{A} := \begin{cases} \wedge(x, y)/\langle xy\rangle \otimes \wedge(z) & \text{at } G/e,\\ \wedge(z) & \text{at } G/G.\end{cases}$$

At the 7^{th} stage, we have

$$\mathcal{M}_7 = \mathcal{M}_6^{\gamma_1}(\underline{V}) \begin{cases} \wedge(a_3, b_3, c_5, d_5, e_7, f_7) & \text{at } G/e,\\ \wedge(c_5) & \text{at } G/G,\end{cases}$$

with $d(a) = d(b) = d(c) = 0$, $d(d_5) = ab$, $d(e_7) = ad$, $d(f_7) = bd$, and

$$\mathcal{M}'_7 = \mathcal{M}_6^{\gamma_2}(\underline{V}) \begin{cases} \wedge(a_3, b_3, c_5, d_5, e'_7, f'_7) & \text{at } G/e,\\ \wedge(c_5) & \text{at } G/G,\end{cases}$$

with $d(a) = d(b) = d(c) = 0$, $d(d_5) = ab$, $d(e'_7) = ad + ac$ and $d(f'_7) = bd$. Clearly, $\mathcal{M}_7$ and $\mathcal{M}'_7$ are not quasi-isomorphic as a system of DGAs by Proposition 3.11.

Let γ, γ' and $\underline{V}$ be as in Proposition 3.11. We say $\mathcal{N}^{\gamma'}(\underline{V})$ satisfies *condition PH_n* if the following holds.

Condition $\mathbf{PH}_n$ (1) $\mathcal{N}^{\gamma'}(\underline{V})$ does not satisfy condition C_n with respect to $\mathcal{N}^{\gamma}(\underline{V})$.

(2) There exists a map $\sigma_n^*\colon H^*(\mathcal{N}^{\gamma'}(\underline{V})) \to \mathcal{U}$ such that σ_n^i is isomorphism for $i \le n$ and monomorphism for $i = n+1$, and

$$\begin{array}{ccc} H^*(\mathcal{N}) & \xrightarrow{\rho^*} & \mathcal{U}\\ {\scriptstyle i^*}\downarrow & \nearrow{\scriptstyle \sigma_n^*} & \\ H^*(\mathcal{N}^{\gamma'}(\underline{V})) & & \end{array}$$

commutes for $* \le n-1$.

Inductively, if the condition PH_n holds for all $n \geq 1$, we say that the *condition plural homotopy type* is satisfied. In this case we will get a new minimal system $\mathcal{M}_\infty = \mathcal{N} \cup \mathcal{N}^{\gamma'}(\underline{V}) \cup (\mathcal{N}^{\gamma'}(\underline{V}))_\delta(\underline{W}) \cup \cdots$, and a map $\sigma_\infty \colon H^*(\mathcal{M}_\infty) \to \mathcal{U}$ which is an isomorphism. Thus, we get more than one nonisomorphic minimal algebra with the same cohomology algebra.

Remark 3.14 (1) It is difficult to check condition C_n for each n, as it involves the computation of elementary extension at each level.

(2) Note, this condition allows us to find plural homotopy types. But when this condition is not satisfied, it is not clear whether the given diagram of graded algebras will be equivariantly intrinsically formal or not. We now provide a few examples where the condition plural homotopy type is not satisfied. We prove later in the article that they are equivariantly intrinsically formal.

We give a couple of examples of equivariantly intrinsically formal and equivariantly n-intrinsically formal diagrams of graded cohomology algebras.

Example 3.15 Let n be an odd integer ≥ 3 and p be a prime. Let $X = S^n \times \cdots \times S^n$ be the product of p copies of S^n with the $G = C_p$ action $t(a_1, a_2, \ldots, a_p) = (a_2, a_3, \ldots, a_{p-1}, a_1)$ where t is a generator of C_p. The fixed points of X under G, $X^G \cong S^n$. Thus, the cohomology diagram of X, which we denote by $\mathcal{A}$, is given by

$$\mathcal{A} = \begin{cases} \mathbb{Q}_3[x_1, x_2, \ldots, x_p]/\langle x_i^2 \mid i = 1, 2, \ldots, p\rangle & \text{at } G/e, \\ \mathbb{Q}_3[y]/\langle y^3\rangle & \text{at } G/G, \end{cases}$$

where y corresponds to the generator of the cohomology algebra for X^G. We want to compute the minimal model $\mathcal{M}$ for this cohomology diagram.

On further calculation we get

$$\mathcal{M}_0 = \cdots = \mathcal{M}_{n-1} = \underline{\mathbb{Q}}$$

and

$$\mathcal{M}_n = \begin{cases} \wedge(a_{12}, a_{23}, \ldots, a_{(p-1)p}, b) & \text{at } G/e, \\ \wedge(b) & \text{at } G/G, \end{cases}$$

with zero differential. Here the process ends at the n^{th} stage, since we get a quasi-isomorphism from $\mathcal{M}_n \to \mathcal{A}$. So the minimal system is obtained at the n^{th} stage.

From the calculation, we see that

$$H^{r+1}(\mathcal{M}_{r-1}) = 0$$

for every r, and we conclude that condition PH_r is not satisfied for any r. Also, if $\mathcal{M}'$ is any minimal algebra with $\underline{H}^*(\mathcal{M}') = \mathcal{A}$, then $\mathcal{M}'$ consists of at least as many generators as $\mathcal{M}_n$. Thus, one can define an n-isomorphism via inclusion from $\mathcal{M}_n \to \mathcal{M}'$. Later in Example 6.1(1) we will show that the cohomology diagram is equivariantly IF.

Example 3.16 Let n be an even integer ≥ 2 and p be a prime. Let $X = S^n \times \cdots \times S^n$ be the product of p copies of S^n with $G = C_p$ action given by $t(a_1, a_2, \dots, a_p) = (a_2, a_3, \dots, a_{p-1}, a_1)$ where t is a generator of C_p. The fixed points of X under G is $X^G \cong S^n$. Thus, the cohomology diagram of X, which we denote by $\mathcal{A}$, is given by

$$\mathcal{A} = \begin{cases} \mathbb{Q}_n[x_1, x_2, \dots, x_p]/\langle x_i^2 \mid i = 1, 2, \dots, p\rangle & \text{at } G/e, \\ \mathbb{Q}_n[y]/\langle y^2 \rangle & \text{at } G/G, \end{cases}$$

where y corresponds to the generator of the cohomology algebra for X^G. We want to compute the minimal model for the cohomology diagram.

Putting all the stages together, we get the following:

$$\mathcal{M}_0 = \underline{\mathbb{Q}} = \mathcal{M}_1 = \cdots = \mathcal{M}_{n-1},$$

$$\mathcal{M}_n = \begin{cases} \wedge(a_{12}, a_{23}, \dots, a_{(p-1)p}, b) & \text{at } G/e, \\ \wedge(b) & \text{at } G/G. \end{cases}$$

Since the both G/e and G/G levels of the spaces are $(n-1)$-connected, the cohomology diagram is equivariantly $(n-1)$-IF.

Next, we claim that the cohomology diagram is equivariantly n-IF. If we consider any minimal system $\mathcal{M}'$ with $H^*(\mathcal{M}') = \mathcal{A}$ then $\mathcal{M}'_{n-1} = \mathcal{M}_{n-1}$. Now $\mathcal{M}'_n = \mathcal{M}_{n-1}(\underline{V})$. Then we claim that there is a map $\mathcal{M}_n \to \mathcal{M}'_n$ that is an n-isomorphism.

First, note that $\mathcal{M}(G/e)$ and $\mathcal{M}(G/G)$ are (nonequivariantly) minimal algebras. For any other minimal system $\mathcal{M}'$ and, for any $H \leq G$, $\mathcal{M}'_n(G/H)$ has at least as many generators as that in $\mathcal{M}_n(G/H)$. Since the cohomology diagrams of both $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, the generators which contribute to the nonzero cohomology classes of $\mathcal{M}_n$ mapping to the generators of $\mathcal{M}'_n$ which are nontrivial classes defines an inclusion map $\mathcal{M}_n \to \mathcal{M}'_n$. This map is an n-isomorphism by construction.

Also,

$$\mathcal{M}_n = \cdots = \mathcal{M}_{2n-2}.$$

The map $\mathcal{M}_{2n-2} = \mathcal{M}_n \to \mathcal{M}'_n \to \mathcal{M}'_{2n-2}$ is a $(2n-2)$-isomorphism since $\mathcal{A}^i = 0$, for $n+1 \leq i \leq 2n-1$;

$$\mathcal{M}_{2n-1} = \begin{cases} \wedge(a_{12}, a_{23}, \dots, a_{(p-1)p}, c_{12}, \dots, c_{(p-1)p}, b, b') & \text{at } G/e, \\ \wedge(b, b') & \text{at } G/G, \end{cases}$$

with $d(c_{ij}) = a_{ij}^2$ for all i, j and $d(b') = b^2$.

Given that $\mathcal{M}_{2n-1}(G/H)$ are minimal algebras for all $H < G$, using the earlier argument (the way we show that $\mathcal{A}$ is equivariantly n-IF) we can show that $\mathcal{A}$ is equivariantly $(2n-1)$-IF.

Later in Example 6.1(2) we will show that the product of even spheres under the above action is equivariantly formal.

4 Injectivity of the associated diagram of vector spaces

In the nonequivariant case, various authors [12, Theorem 3.2; 5, Example 6.5; 13] give different methods to determine the plural homotopy types of a given graded algebra. These results often use the fact that at the n^{th} extension stage, we are adding generators only in degree n. This is not true in the equivariant case. However, if the associated diagram of graded vector spaces at the n^{th} stage is injective then, the generators added are only in degree n.

In this section, we restrict our group to $G = C_p$, for prime p, and describe conditions under which a diagram of DGAs or a diagram of vector spaces is injective.

Proposition 4.1 *Let $G = C_p$, where p is a prime. Let $\mathscr{A} \in \mathrm{DGA}^{\mathcal{O}_G}$. Then $\mathscr{A}$ as an element of Vec^*_G is injective if and only if the map $\mathscr{A}(\hat{e}_{e,G})\colon \mathscr{A}(G/e) \to \mathscr{A}(G/G)$ is surjective.*

Proof The injective envelope for $\mathscr{A}$ is given by $\mathscr{I}(\mathscr{A}) = \mathcal{I}^*_G \oplus \mathcal{I}^*_e$, where $\mathcal{I}^*_e$ and $\mathcal{I}^*_G$ are systems corresponding to the vector spaces $I_e = \ker \mathscr{A}(\hat{e}_{e,G})$ and $I_G = \mathscr{A}(G/G)$ respectively. Given that the map $\mathscr{A}(\hat{e}_{e,G})$ is surjective, we have a short exact sequence

$$0 \to \ker \mathscr{A}(\hat{e}_{e,G}) \to \mathscr{A}(G/e) \to \mathscr{A}(G/G) \to 0, \tag{3}$$

which splits as $\mathbb{Q}$-vector spaces. So $\mathscr{A}(G/e) = \ker \mathscr{A}(\hat{e}_{e,G}) \oplus \mathscr{A}(G/G)$.

Also,

$$\mathcal{I}^*_G(G/G) = \mathrm{Hom}_{\mathbb{Q}(G/G)}(\mathbb{Q}(G/G)^G, I_G) \cong I_G, \tag{4}$$

$$\mathcal{I}^*_G(G/e) = \mathrm{Hom}_{\mathbb{Q}(e)}(\mathbb{Q}(G/G)^e, I_G) \cong I_G, \tag{5}$$

$$\mathcal{I}^*_e(G/G) = \mathrm{Hom}_{\mathbb{Q}(G)}(\mathbb{Q}(G/e)^G, I_e) = 0, \tag{6}$$

$$\mathcal{I}^*_e(G/e) = \mathrm{Hom}_{\mathbb{Q}(G/e)}(\mathbb{Q}(G/e)^e, I_e) \cong I_e. \tag{7}$$

Since $\mathscr{I}(A) = \mathcal{I}^*_G \oplus \mathcal{I}^*_e$ we get that

$$\mathscr{I}(\mathscr{A})(G/e) = \mathcal{I}^*_G(G/e) \oplus \mathcal{I}^*_e(G/e) = \ker \mathscr{A}(\hat{e}_{e,G}) \oplus \mathscr{A}(G/G) = \mathscr{A}(G/e),$$
$$\mathscr{I}(\mathscr{A})(G/G) = \mathcal{I}^*_G(G/G) \oplus \mathcal{I}^*_e(G/G) = I_G = \mathscr{A}(G/G).$$

Therefore, the injective envelope of $\mathscr{A}$ is itself implying that $\mathscr{A}$ is injective.

Conversely, if $\mathscr{A}$ is injective, then $\mathscr{A} \cong \mathcal{I}^*_G \oplus \mathcal{I}^*_e$. Then using (3)–(7) we get that the map $\mathscr{A}(\hat{e}_{e,G})$ is the projection

$$\mathscr{A}(G/G) \oplus \ker \mathscr{A}(\hat{e}_{e,G}) = \mathcal{I}^*_G(G/e) \oplus \mathcal{I}^*_e(G/e) \to \mathcal{I}^*_G(G/G) \oplus \mathcal{I}^*_e(G/G) = \mathscr{A}(G/G),$$

and hence is surjective. □

Example 4.2 Consider the G-space $X = S^3$, where $G = C_2$ acts on S^3 by reflection, which fixes the equator sphere S^2. So here $G = C_2$, $X^G = S^2$ and $X^e = S^3$. The corresponding cohomology diagram is given by $H^*(X; \underline{\mathbb{Q}})$, which is not injective. This follows from Proposition 4.1.

Note that if a cohomology diagram $\mathcal{A}$ is injective and for each n, and the associated diagram of vector spaces for $\mathcal{A}$ is injective, then at elementary extension we only add elements of degree n to $\mathcal{M}_{n-1}(G/H)$ to obtain $\mathcal{M}_n(G/H)$, for all $H \leq G$. Then by [1, Lemma 3.2] the differential will be levelwise decomposable. Also, the map $\rho(G/H)\colon(\mathcal{M}(G/H), d) \to (\mathcal{A}(G/H), 0)$ is a quasi-isomorphism and surjective, since the differential in $\mathcal{A}(G/H)$ is 0. Thus, by the lifting lemma [2, Lemma 12.4], there is a map $\alpha\colon N_H \to \mathcal{M}(G/H)$. Since α_H is a quasi-isomorphism between two minimal algebras, it is an isomorphism. In view of this, we have the following proposition.

Proposition 4.3 *Let $\mathcal{A}$ be a cohomology diagram over $\mathcal{O}_G$, which is injective, and for each n, the associated diagram of vector spaces of $\mathcal{A}$ is injective. If (N_H, ρ_H) is the minimal model for $\mathcal{A}(G/H)$ where $\rho_H\colon N_H \to \mathcal{A}(G/H)$ is a quasi-isomorphism and $\mathcal{M}$ be a minimal system for $\mathcal{A}$, then $\mathcal{M}(G/H) \cong N_H$.*

Remark 4.4 Let us consider the above proposition when $G = C_p$ for p prime and (N_e, ρ_e) and (N_G, ρ_G) are minimal models for $\mathcal{A}(G/e)$ and, $\mathcal{A}(G/G)$ respectively. If there exists a map $\theta\colon N_e \to N_G$ such that the following diagram commutes:

$$\begin{array}{ccccc} N_e & \xrightarrow{\alpha_e} & \mathcal{M}(G/e) & \xrightarrow{\rho_e} & \mathcal{A}(G/e) \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \mathcal{M}(\hat{e}_{e,G})} & & \downarrow{\scriptstyle \mathcal{A}(\hat{e}_{e,G})} \\ N_G & \xrightarrow[\alpha_G]{} & \mathcal{M}(G/G) & \xrightarrow[\rho_G]{} & \mathcal{A}(G/G) \end{array}$$

Then the minimal system for the diagram of graded algebras $\mathcal{A}$ can be given by

$$\mathcal{M}(G/e) = N_e, \quad \mathcal{M}(G/G) = N_G.$$

We have the following result.

Proposition 4.5 *Let $G = C_p$, for p prime. If the structure map in the cohomology diagram $\mathcal{A}$, $\mathcal{A}_{e,G}\colon \mathcal{A}(G/e) \to \mathcal{A}(G/G)$, is a retraction of DGAs then the associated diagram of vector spaces is injective. In particular, the minimal model of the cohomology diagram is levelwise minimal.*

Proof Since the structure map $\mathcal{A}_{e,G}\colon \mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is a retraction, there exists $i\colon \mathcal{A}(G/G) \to \mathcal{A}(G/e)$ such that $\mathcal{A}_{e,G} \circ i = \mathrm{id}$. This implies $\mathcal{A}_{e,G}$ is surjective, and it follows that $\mathcal{A}$ is injective diagram of graded algebras. Note that for any minimal system of DGAs $\mathcal{N}$, the DGA $\mathcal{N}(G/G)$ is nonequivariantly minimal by construction. Let $\rho\colon \mathcal{M} \to \mathcal{A}$ be the minimal model and let $\mathcal{M}_{e,G}\colon \mathcal{M}(G/e) \to \mathcal{M}(G/G)$ be the corresponding structure map. We claim that there exists an inclusion map of DGAs $j\colon \mathcal{M}(G/G) \to \mathcal{M}(G/e)$. Since $\mathcal{A}(G/e)$ is a DGA with zero differential, $\rho(G/e)\colon \mathcal{M}(G/e) \to \mathcal{A}(G/e)$ is a surjective quasi-isomorphism, by the lifting lemma, there exists a lift $j\colon \mathcal{M}(G/G) \to \mathcal{M}(G/e)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(G/e) & \xrightarrow{\rho(G/e)} & \mathcal{A}^i(G/e) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{M}(G/G) & \xrightarrow[\rho(G/G)]{} & \mathcal{A}^i(G/G) \end{array}$$

Therefore,

$$\mathcal{A}_{e,G} \circ i \circ \rho(G/G) = \mathcal{A}_{e,G} \circ \rho(G/e) \circ j \implies \rho(G/G) = \rho(G/G) \circ \mathcal{M}_{e,G} \circ j. \tag{8}$$

Since $\rho(G/G)$ is a quasi-isomorphism, $\mathcal{M}_{e,G} \circ j : \mathcal{M}(G/G) \to \mathcal{M}(G/G)$ is a quasi-isomorphism. It then follows that $\mathcal{M}_{e,G} \circ j$ is an isomorphism and therefore $j : \mathcal{M}(G/G) \to \mathcal{M}(G/e)$ is an inclusion.

Next, we show that all the associated systems of vector spaces are injective by induction on n where $\mathcal{M} = \bigcup \mathcal{M}_m$. Recall $\mathcal{M}_n = \mathcal{M}_{n-1}(\underline{V})$, where $\underline{V}$ is $H^{n+1}(\ker(\beta) \oplus \underline{\mathbb{Q}})$ is the associated diagram of vector spaces at the n^{th} stage. Any element of $\underline{V}(G/G)$ looks like the product of the elements of $\mathcal{M}_{n-1}(G/G)$, $\mathcal{A}(G/G)$ and $\sum \mathcal{A}(G/G)$. We study case by case to conclude that $\underline{V}(\hat{e}_{e,G})$ is surjective. Let $[x] \in \underline{V}(G/G)$.

(1) If $x \in \mathcal{M}_{n-1}^{n+1}(G/G)$, i.e. $[x] \in \underline{V}(G/G)$, then $j(x) \in \mathcal{M}_{n-1}(G/e)$.
As $\beta = \rho$ on $\mathcal{M}_{n-1}$, we have $i \circ \rho(G/G)(x) = \rho(G/e) \circ j(x)$. This implies $i \circ \beta(G/G)(x) = \beta(G/e) \circ j(x)$, which implies $j(x) \in \ker \beta(G/e)$. As j is a DGA-map, we get $dj(x) = jd(x)$, which gives $j(x) \in \underline{V}(G/e)$.

(2) If $x \in \sum \mathcal{A}(G/G)$ then by injectivity of $\mathcal{A}$, one gets a preimage in $\sum \mathcal{A}(G/e)$. The differential is zero for elements in $\sum \mathcal{A}$, so we get a preimage in $\underline{V}(G/e)$.

(3) Assume x is the product of elements in $\mathcal{M}_{n-1}$, $\mathcal{A}$ and $\sum \mathcal{A}$. In this case, note that the maps i, j induce a DGA-map $g \colon \mathcal{M}(G/G) \otimes \mathbb{Q}\big(\mathcal{A} \oplus \sum \mathcal{A}\big)(G/G) \to \mathcal{M}(G/e) \otimes \mathbb{Q}\big(\mathcal{A} \oplus \sum \mathcal{A}\big)(G/e)$. If $x = m.a.sb$ where $m \in \mathcal{M}(G/G)$, $a \in \mathcal{A}(G/G)$ and $sb \in \sum \mathcal{A}(G/G)$, with $[x] \in \underline{V}(G/G)$, then we have $[g(m.a.sb)] \in \underline{V}(G/e)$.

Hence, $\underline{\mathrm{V}}$ is injective and it follows that the minimal model $\mathcal{M}$ is levelwise minimal. □

Example 4.6 Consider $X = (S^3 \vee S^3) \times S^5$ with action of G, where $G = C_2$ acts on $(S^3 \vee S^3)$ by switching copies and acts on S^5 trivially. We denote the cohomology diagram by $\mathcal{A}$.

Then $\mathcal{A}(G/e) = \wedge(x, y)/\langle xy \rangle \otimes \wedge(z)$ and $\mathcal{A}(G/G) = \wedge(z)$ where the generators of the cohomology are x, y and z, and $\deg(x) = 3 = \deg(y)$ and $\deg(z) = 5$.

Using Proposition 4.1 we get that the given cohomology diagram is injective. Let $\mathcal{M}$ denote its minimal model.

We have

$$\mathcal{M}_3 = \begin{cases} \wedge(a, b) & \text{at } G/e, \\ \mathbb{Q} & \text{at } G/G. \end{cases}$$

Note that $\rho \colon \mathcal{M}_3 \to \mathcal{A}$ is the extension of the map $\rho \colon \mathcal{M}_2 \to \mathcal{A}$. It is defined by $\rho(a) = x$ and $\rho(b) = y$.

It can be verified that $\mathcal{M}_4 \cong \mathcal{M}_3$, as there are no elements in the cohomology of degree 4.

Computation for $\mathcal{M}_5$: From [11], the construction of minimal model we get

$$\underline{V} := H^6(R) = \begin{cases} \mathbb{Q}_6(sz, ab) & \text{at } G/e, \\ \mathbb{Q}_6(sz) & \text{at } G/G, \end{cases}$$

where $\mathbb{Q}_n(x_i)$ denotes the $\mathbb{Q}$-vector space generated by the elements x_i of degree n.

Using Proposition 4.1, we see that $\underline{V}$ is injective and can compute $\mathcal{M}_5$.

One can show that

$$H^6(\mathcal{M}_4; \underline{V}) \cong \operatorname{Hom}(\underline{V}, H^6(\mathcal{M}_4)) = \mathbb{Q}_\gamma \cong \mathbb{Q}.$$

Here the map $\gamma\colon \underline{V} \to H^6(\mathcal{M}_4)$ takes $ab \to ab$. Thus,

$$\mathcal{M}_5 = \mathcal{M}_4^\gamma(\underline{V}) = \begin{cases} \wedge(a_3, b_3, c_5, d_5) & \text{at } G/e, \\ \wedge(c_5) & \text{at } G/G. \end{cases}$$

The differential is $d(a_3) = 0 = d(b_3)$, $d(d_5) = ab$, $d(c_5) = 0$, and $\rho\colon \mathcal{M}_5 \to \mathcal{A}$ maps $c_5 \to z$ and $d_5 \to 0$.

5 Classifying rational homotopy types with isomorphic cohomology

In this section, we consider the diagram of graded algebras, $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$, where p is a prime number and the structure map $\mathcal{A}_{e,G}$ is a retract. Then by Proposition 4.5, we get that the minimal model for $\mathcal{A}^*$ is levelwise minimal. We denote the minimal model for $\mathcal{A}^*(G/H)$ by N_H. We use techniques from [13] to give an inductive construction for the minimal systems of DGAs with cohomology algebra $\mathcal{A}^*$. We assume $\mathcal{M}$ is the minimal model for $\mathcal{A}^*$, so that $\mathcal{M}(G/H) \cong N_H$, for each subgroup H of G. We cannot directly extend the results of [13] because minimal algebras with cohomology $\mathcal{A}^*(G/H)$, for every $H \leq G$, do not always give rise to a minimal system with cohomology diagram $\mathcal{A}^*$.

The following proposition compares the levelwise weak equivalence of diagrams of DGAs with a weak equivalence of the diagrams of DGAs when they are levelwise minimal in the nonequivariant sense.

Proposition 5.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be two DGA diagrams over $\mathcal{O}_G$, for some finite group G having the same cohomology at each level of subgroups, and $\mathcal{M}(G/H)$ and $\mathcal{N}(G/H)$ are minimal algebras for every subgroup H of G. Let $\mathcal{I}_\mathcal{M}$ and $\mathcal{I}_\mathcal{N}$ denote their injective envelopes. If $\mathcal{I}_\mathcal{M}$ and $\mathcal{I}_\mathcal{N}$ are weakly equivalent then for every subgroup H of G, $\mathcal{M}(G/H)$ and $\mathcal{N}(G/H)$ have the same rational homotopy type.*

Conversely, if there is a DGA diagram map $r\colon \mathcal{M} \to \mathcal{N}$ such that $r(G/H)\colon \mathcal{M}(G/H) \to \mathcal{N}(G/H)$ is a weak equivalence for every subgroup H of G, then $\mathcal{I}_\mathcal{M}$ and $\mathcal{I}_\mathcal{N}$ are weakly equivalent.

Proof Let $\mathcal{I}_\mathcal{M}$ and $\mathcal{I}_\mathcal{N}$ denote the injective envelopes for $\mathcal{M}$ and, $\mathcal{N}$ respectively. Let us assume that there is a weak equivalence $\phi\colon \mathcal{I}_\mathcal{M} \to \mathcal{I}_\mathcal{N}$. Then we have the diagram

$$\begin{array}{ccc} \mathcal{M} & & \mathcal{N} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{I}_\mathcal{M} & \xrightarrow{\phi} & \mathcal{I}_\mathcal{N} \end{array}$$

where i and j are inclusions, which are also quasi-isomorphisms. Thus, we get a weak equivalence between $\mathcal{M}$ and $\mathcal{N}$, and for every subgroup H of G, $\mathcal{M}(G/H)$ and $\mathcal{N}(G/H)$ are quasi-isomorphic as DGAs.

Conversely, if there exists a map $r\colon \mathcal{M} \to \mathcal{N}$ satisfying the hypothesis then consider the composition map

$$\mathcal{M} \xrightarrow{r} \mathcal{N} \to \mathcal{I}_{\mathcal{N}}.$$

By [3, Proposition 8], we get a map $\phi\colon \mathcal{I}_{\mathcal{M}} \to \mathcal{I}_{\mathcal{N}}$ which is the extension of the composition above:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{r} & \mathcal{N} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{I}_{\mathcal{M}} & \underset{\phi}{\dashrightarrow} & \mathcal{I}_{\mathcal{N}} \end{array}$$

Hence, this is a quasi-isomorphism, since other maps in the above diagram are quasi-isomorphisms. □

We now describe the subset $\mathcal{N}^{C_p}_{\mathcal{A}^*}$ of $\mathcal{M}^{C_p}_{\mathcal{A}^*}$, which is defined to be the set of systems of DGAs over $\mathcal{O}_{C_p}$ which have cohomology $\mathcal{A}^*$ up to weak equivalence and are minimal at each level G/H.

Construction 5.2 Let $\mathcal{A}^*$ be a diagram of graded algebras over $\mathcal{O}_G$, where $G = C_p$ so that the structure map $\mathcal{A}_{e,G}\colon \mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is a retract.

We say that minimal algebras p^i_{n-1}, $i = 1, 2$, satisfy the minimal $(n-1)$ property for $\mathcal{A}^*$ if they satisfy the following conditions:

$(1)_{n-1}$ Both the algebras p^i_{n-1}, for $i = 1, 2$, are generated by elements of degree $\le n-1$ and for $i = 1, 2$ they are $\mathbb{Q}[G/e]$- and $\mathbb{Q}[G/G]$-modules respectively.

$(2)_{n-1}$ There is a C_p-DGA-map $\delta\colon p^1_{n-1} \to p^2_{n-1}$ such that $\mathcal{P}_{n-1} = \{p^1_{n-1}, p^2_{n-1}, \delta\}$ defines a diagram of DGAs with a morphism between diagram of graded algebras

$$\sigma^j_{n-1}\colon (H^*(\mathcal{P}_{n-1})(n))^j \to \mathcal{A}^j$$

which is an isomorphism for $j \le n-1$, and a monomorphism for $j = n$, where $((H^*\mathcal{P}_{n-1})(n))$ is subgraded algebra diagram of $H^*(\mathcal{P}_{n-1})$ generated by elements of degree $\le n$.

For instance, let $(\mathcal{M}, \rho)$ be the minimal model of $\mathcal{A}$. By Proposition 4.5 it follows that $\mathcal{M}(G/e)$ and $\mathcal{M}(G/G)$ are minimal models for $\mathcal{A}(G/e)$ and $\mathcal{A}(G/G)$. There are maps

$$\rho(G/e)\colon \mathcal{M}(G/e) \to \mathcal{A}(G/e), \quad \rho(G/G)\colon \mathcal{M}(G/G) \to \mathcal{A}(G/G),$$

which are quasi-isomorphisms. Then the subalgebra $(\mathcal{M}(G/H))_{n-1}$ of $\mathcal{M}(G/H)$ generated by elements of degree less than or equal to $n-1$ satisfies the minimal $(n-1)$ property for $\mathcal{A}^*$ for H equal to $\{e\}$ or G.

We denote by $\underline{K}$ the kernel of the map $\sigma^*_{n-1}\colon H^*((\mathcal{P}_{n-1}(n)) \to \mathcal{A}^*$. Also, we let K^1 and K^2 denote $(\underline{K}(G/e))^{n+1}$ and $(\underline{K}(G/G))^{n+1}$ respectively.

Let p^i_D be the minimal algebras obtained by adding generators to p^i_{n-1}, whose differentials form a basis for K^i for $i = 1, 2$.

Consider the diagram of DGAs $\mathcal{P}_D = \{p_D^1, p_D^2, \theta\}$, where θ is induced from δ in $\mathcal{P}_{n-1}$ and let

$$\sigma_D\colon (H^*(\mathcal{P}_D)(n))^* \to \mathcal{A}^*$$

be the morphism induced by σ_{n-1}.

We set

$$\dim_{\mathbb{Q}} \mathcal{A}(G/H)^{n+1} = u_i, \quad \dim_{\mathbb{Q}} \mathcal{A}(G/H)^{n+1}/(\mathcal{A}(G/H)(n))^{n+1} = s_i,$$

for $(i,H) = (1,e)$ or $(2,G)$ and for $i = 1,2$ we set

$$\dim_{\mathbb{Q}} H^{n+1}(p_D^i) = v_i, \quad \dim_{\mathbb{Q}} \frac{H^{n+1}(p_D^i)}{(H^*(p_D^i)(n))^{n+1}} = t_i.$$

Since the subalgebras generated by elements of degree $< n$ of the diagram of algebras $H^{n+1}(\mathcal{P}_D)$ and $H^{n+1}(\mathcal{P}_{n-1})$ are isomorphic, we have

$$u_i - s_i = v_i - t_i \quad \text{for } i = 1,2.$$

For $i = 1,2$, let l_i be integers satisfying

$$\max(0, t_i - s_i) \le l_i \le t_i \quad \text{for } i = 1,2,$$

and W_i be subspaces of $H^{n+1}(p_D^i)$ such that $\dim W_i = l_i$ for $i = 1,2$ with the condition that

$$W_i \cap (H^*(p_D^i)(n))^{n+1} = \{0\}. \tag{9}$$

Denote the minimal algebras obtained by adding l_i generators (in degree n) to p_D^i, whose differentials span W_i, by p^{i,W_i}.

This implies that $H(p^{i,W_i})(n) = H(p_D^i)(n)$ for $i = 1,2$. We then have graded algebra maps

$$\sigma_{i,D}\colon (H(p^{i,W_i})(n)) \to \mathcal{A}^*(G/H)$$

with

$$H^{n+1}(p^{i,W_i}) \oplus W_i = H^{n+1}(p_D^i) \quad \text{for } i = 1,2,$$

such that

$$\dim_{\mathbb{Q}} \frac{H^{n+1}(p^{i,W_i})}{(H(p^{i,W_i})(n))^{n+1}} = t_i - l_i \le s_i = \dim_{\mathbb{Q}} \frac{\mathcal{A}^{n+1}(G/H)}{(\mathcal{A}(n))^{n+1}(G/H)}$$

for $(i,H) = (1,G)$ or $(2,e)$.

Let p_n^{i,W_i} be the minimal algebras obtained by adding generators (in degree n) with zero differential to p^{i,W_i} so that the cokernel of the map

$$\sigma_{i,D}\colon (H(p^{i,W_i})(n))^n \to \mathcal{A}^n(G/H)$$

becomes trivial for $(i,H) = (1,e)$ or $(2,G)$. We get graded algebra maps

$$\sigma^*_{W_i,n-1}\colon (H(p_n^{i,W_i})(n))^* \to \mathcal{A}^*(G/H)$$

such that for $i = 1,2$, $\sigma^*_{W_i,n-1}$ are isomorphisms for $* \le n$.

We say that the pair (W_1, W_2) satisfies *condition K* if the following holds.

Condition K (a) Given linear monomorphisms

$$\phi_i \colon H^{n+1}(p^{i,W_i})/(H(p^{i,W_i})(n))^{n+1} \to \mathcal{A}^{n+1}(G/H)/(\mathcal{A}(G/H)(n))^{n+1},$$

where $(i, H) = (1, e)$ or $(2, G)$, the maps $\sigma_{W_i,n-1} \oplus \phi_i$ can be extended to graded algebra maps

$$\sigma_{W_i,n} \colon (H(p^{i,W_i})(n+1))^* \to \mathcal{A}^*(G/H),$$

which are $\mathbb{Q}(G/e)$ and $\mathbb{Q}(G/G)$-module maps for $i = 1, 2$ respectively.

(b) There exists morphism of DGAs, $\eta \colon p_n^{1,W_1} \to p_n^{2,W_2}$ such that $\mathcal{P}_n^{(W_1,W_2)} = \{p_n^{1,W_1}, p_n^{2,W_2}, \eta\}$ is a diagram of DGAs and the pair of maps $\{\sigma_{W_1,n}, \sigma_{W_2,n}\}$ induce a morphism

$$\sigma_{(W_1,W_2),n} \colon H^*(\mathcal{P}_n^{(W_1,W_2)}(n+1))^* \to \mathcal{A}^*$$

satisfying conditions $(1)_n$ and $(2)_n$.

Remark 5.3 (1) If there exists W_i's such that $\dim_{\mathbb{Q}} W_i = t_i$, the map η always exists so that condition $(1)_n$ and $(2)_n$ are satisfied. In fact, they will lead us to the minimal model for $\mathcal{A}^*$.

(2) The pairs $(p_n^{1,W_1}, \sigma_{W_1,n})$ and $(p_n^{2,W_2}, \sigma_{W_2,n})$ depend on the subspaces W_i. If there are several choices of W_i, i.e. the choices of l_i are large, there is a higher possibility of finding a minimal system containing p_{n-1}^1 and p_{n-1}^2.

(3) Consider a diagram of graded algebras $\mathcal{A}$ which is equivariantly $(n-1)$ IF with minimal model $\mathcal{M}$ such that $\mathcal{M}_{n-1}(G/e) = p_{n-1}^1$ and $\mathcal{M}_{n-1}(G/G) = p_{n-1}^2$. The map δ always exists when $l_i = t_i$ and gives the minimal model for $(H^*(\mathcal{A}), 0)$. If δ doesn't exist for any $l_i < t_i$ then what we can say is that there is no other minimal system with cohomology $\mathcal{A}$, whose n^{th} stage at G/H, for both $H = e, G$ is a minimal algebra.

Let $\mathrm{Gr}(v_i, l_i)(\mathbb{Q})$ be the set of rational points of the Grassmann manifolds of l_i-dimensional $\mathbb{Q}$-subspaces in the v_i-dimensional spaces $H^{n+1}(p_D^1)$ and $H^{n+1}(p_D^2)$ respectively for $i = 1, 2$. Define

$$\mathcal{M}_{(l_1,l_2)} := \{(W_1, W_2) \in \mathrm{Gr}(v_1, l_1)(\mathbb{Q}) \times \mathrm{Gr}(v_2, l_2)(\mathbb{Q}) \mid W_i \cap (H^*(p_D^i)(n))^{n+1} = \{0\}\},$$
$$\mathcal{O}_{(l_1,l_2)} := \{(W_1, W_2) \in \mathcal{M}_{(l_1,l_2)} \mid (W_1, W_2) \text{ satisfies condition K}\}.$$

Let G_1 and G_2 be the group of DGA automorphisms of p_D^1 and p_D^2 respectively. Then G_1 and G_2 act on $H^{n+1}(p_D)$ and $H^{m+1}(p_D^1)$ respectively, and hence on $\mathrm{Gr}(v_1, l_1)(\mathbb{Q})$ and $\mathrm{Gr}(v_2, l_2)(\mathbb{Q})$. Let $(W_1, W_2) \in \mathcal{O}_{l_1,l_2}$ and (ϕ_1, ϕ_2) be an element of $G_1 \times G_2$.

Then it is easy to see that (ϕ_1, ϕ_2) can be extended to DGA isomorphisms

$$\phi_i \colon p_n^{i,W_i} \to p_n^{i,\phi_i(W_i)}$$

for $i = 1, 2$ respectively.

Conversely, let $(W_1 \oplus W_2)$ and $(W_1' \oplus W_2')$ be (l_1+l_2)-dimensional subspaces of $H^{n+1}(p_D^1) \oplus H^{m+1}(p_D^2)$ such that there are DGA isomorphisms

$$f_1\colon p_n^{1,W_1} \to p_n^{1,W_1'}, \quad f_2\colon p_n^{2,W_2} \to p_n^{2,W_2'}.$$

Then $(f_1|p_D^1, f_2|p_D^2) \in G_1 \times G_2$.

However, it is not always true that such a pair will give rise to a morphism of DGA diagrams over $\mathcal{O}_{C_p}$; that is, for such a pair there may not exist a C_p-DGA map θ such that condition K(b) is satisfied.

Further, for isomorphic pairs (W_1, W_2) and (W_1', W_2') if there are two pairs of maps θ_1 and θ_2 such that the diagram

$$\begin{CD} p_n^{1,W_1} @>{f_1}>> p_n^{1,W_1'} \\ @V{\theta_1}VV @VV{\theta_2}V \\ p_n^{2,W_2} @>>{f_2}> p_n^{2,W_2'} \end{CD}$$

commutes for a pair of automorphisms (f_1, f_2), then the DGA diagrams $X = (p_n^{1,W_1}, p_n^{2,W_2}, \theta_1)$ and $Y = (p_n^{1,W_1'}, p_n^{2,W_2'}, \theta_2)$ are weakly equivalent. In particular, if we take their injective envelopes $\mathcal{I}_X$ and $\mathcal{I}_Y$, by Proposition 5.1 there is a map $\mathcal{I}_X \to \mathcal{I}_Y$ which is a weak equivalence.

Moreover, p_n^{1,W_1}, p_n^{2,W_2}, $p_n^{1,W_1'}$, and $p_n^{2,W_2'}$ are minimal algebras in the nonequivariant sense; any quasi-isomorphism between them is an isomorphism.

We now define an equivalence relation on the set $\mathcal{O}_{l_1,l_2}$ as follows. Two elements (W_1, W_2) and (W_1', W_2') in $\mathcal{O}_{l_1,l_2}$ are equivalent if there is a pair of DGA isomorphisms (f_1, f_2) such that either one of the following diagrams commutes:

$$\begin{CD} p_n^{1,W_1} @>{f_1}>> p_n^{1,W_1'} \\ @V{\theta_1}VV @VV{\theta_2}V \\ p_n^{2,W_2} @>>{f_2}> p_n^{2,W_2'} \end{CD} \qquad \begin{CD} p_n^{1,W_1'} @>{f_2}>> p_n^{1,W_1} \\ @V{\theta_2}VV @VV{\theta_1}V \\ p_n^{2,W_2'} @>>{f_2}> p_n^{2,W_2} \end{CD}$$

We denote this relation by κ.

Using Propositions 4.5 and 5.1 and Construction 5.2, we have the following theorem.

Theorem 5.4 *Let the diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$, where p is a prime number, be such that the structure map $\mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is a retract. Let $\mathcal{M}$ be the minimal model for $\mathcal{A}^*$. The set of isomorphism classes of minimal algebras $\mathcal{N}_k$ with the properties*

(1) *$\mathcal{N}_k$ contains the minimal algebra $\mathcal{M}_{k-1}$ satisfying $(1)_k$ and $(2)_k$,*

(2) *$\mathcal{N}_k(G/H)$ is minimal in the nonequivariant sense for every subgroup H of C_p,*

corresponds bijectively to the disjoint union of orbit spaces

$$X_k = \bigsqcup \mathcal{O}_{l_1,l_2}/\kappa,$$

where κ is the equivalent relation mentioned in Construction 5.2.

Remark 5.5 Let the diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$, where p is a prime number, be such that the structure map is a retract. If $\mathcal{A}^*$ is equivariantly $(n-1)$-intrinsically formal and $H^i(N_{C_p}) = H^i(\mathcal{A}) = 0$ for $i \geq n$, then $H^i(p_n^{2,W_2}) = 0$ for $i \geq n$ and one has Theorem 5.6.

Theorem 5.6 *Let the diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$, where p is a prime number, be such that the structure map $\mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is a retract. Let $\mathcal{M}$ be the minimal model for $\mathcal{A}^*$. If $\mathcal{A}^*$ is equivariantly $(n-1)$-intrinsically formal and $N^i_{C_p} = 0$ for $i \geq n$, where N_H is the minimal model for $\mathcal{A}^*(C_p/H)$ for every subgroup H of C_p, then the set of isomorphism classes of the minimal system $\mathcal{K}_n$ containing the minimal system $\mathcal{M}_{n-1}$ is determined by the set $\mathcal{M}_{\mathcal{A}^*(C_p/e)}$.*

Moreover, if $\mathcal{A}^*$ is equivariantly $(n-1)$-IF and $\mathcal{A}^*(C_p/e) = 0$ for $j > n+1$, then the cardinality of $\mathcal{M}^{C_p}_{\mathcal{A}^*}$ is the same as the cardinality of $\mathcal{M}_{\mathcal{A}^*(C_p/e)}$.

Proof By Proposition 4.5 it follows that the minimal model of $\mathcal{A}^*$ is levelwise minimal. Throughout the proof, we use G to denote C_p.

We first show that if $\mathcal{B}$ is any system of DGAs such that $H^*(\mathcal{B}) = \mathcal{A}^*$ and $\mathcal{B}_{n-1} = \mathcal{M}_{n-1}$ then $\mathcal{B}$ is weakly equivalent to a system of DGAs which is minimal at each G/H in the nonequivariant sense.

To show this, let N_H denote the nonequivariant minimal model for $\mathcal{M}(G/H)$. Let L denote the minimal model for $\mathcal{B}(G/e)$. We define

$$\mathcal{V} := \begin{cases} L & \text{at } G/e, \\ N_G & \text{at } G/G. \end{cases}$$

We claim that $\mathcal{V}$ is a system of DGAs, and it is weakly equivalent to $\mathcal{B}$.

First, note that $\mathcal{B}(G/G)$ is quasi-isomorphic to $N_G = \mathcal{V}(G/G)$. This is true since $\mathcal{B}_{n-1}(G/G)$ contains $\mathcal{M}_{n-1}(G/G)$ as a sub-DGA in the nonequivariant sense. Also, $H^k(\mathcal{B}(G/G)) = H^k(N_G) = \mathcal{A}(G/G)$ and $H^i(\mathcal{B}(G/G)) = 0$ for $i \geq n$. Thus, $\mathcal{B}(G/G)$ is obtained by adding an acyclic vector space to N_G, and hence the inclusion map $N_G \to \mathcal{B}(G/G)$ is a quasi-isomorphism. Now observe that, $(\mathcal{V}(G/e))^{n-1} = L^{n-1} = N_e^{n-1}$ and $N_G^i = 0$ for $i \geq n$, so we can define the map $\theta \colon \mathcal{V}(G/e) \to \mathcal{V}(G/G)$ by $\theta|\mathcal{V}(G/e)^{n-1}$ as the map $\mathcal{M}_{n-1}(G/e) = N_e^{n-1} \to \mathcal{M}_{n-1}(G/G) = N_G^{n-1}$ and zero at $(\mathcal{V}(G/e)^i)$ for $i \geq n$, so that $\mathcal{V}$ is a diagram DGAs.

Next, we show that $\mathcal{V}$ is a system of DGAs. Note that $\mathcal{V}_{n-1} = \mathcal{M}_{n-1}$ is already a system of DGAs. Since $\mathcal{V}^i(G/G) = N_G^i = 0$ for $i \geq n$, so the maps $\mathcal{V}^i(G/e) \to \mathcal{V}^i(G/G)$ are zero maps for $i \geq n$ and hence surjective. $\mathcal{V}_{n-1} = \mathcal{M}_{n-1}$ being a system, the map $\mathcal{V}^j(G/e) \to \mathcal{V}^j(G/G)$ is surjective for $j \leq n-1$; thus, combining we get that $L \to N_G$ is surjective and hence, by Proposition 4.1, $\mathcal{V}$ is a system.

Also, we have a map from $\mathcal{V} \to \mathcal{B}$ which is the identity up to the $(n-1)^{\text{st}}$ stage and since $\mathcal{V}^i(G/G) = 0$ for $i \geq n$ so $\mathcal{V}^i(G/e) \to \mathcal{B}^i(G/e)$ the quasi-isomorphisms will fit in to the diagram $\mathcal{V} \to \mathcal{B}$:

$$\begin{array}{ccc} \mathcal{V}^i(G/e) & \longrightarrow & \mathcal{B}^i(G/e) \\ {\scriptstyle \mathcal{V}(\hat{e}_{e,G})}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\hat{e}_{e,G})} \\ \mathcal{V}^i(G/G) & \longrightarrow & \mathcal{B}^i(G/G) \end{array}$$

Thus, using the converse of Proposition 5.1 we get that $\mathcal{V}$ and $\mathcal{B}$ are weakly equivalent.

Thus, any such DGA diagram $\mathcal{H}$ of the form

$$\mathcal{H} := \begin{cases} m & \text{at } G/e, \\ N_G & \text{at } G/G, \end{cases}$$

with the property that m is a minimal algebra, and $\mathcal{H}_{n-1} = \mathcal{M}_{n-1}$ is a system of DGAs. Thus, the plural homotopy types are determined by m. The cardinality of the set of such m is determined by the set $\mathcal{M}_{\mathcal{A}^*(G/e)}$. □

Corollary 5.7 *Let the diagram of graded algebras $\mathcal{A}^*$ over $\mathcal{O}_{C_p}$, where p is a prime number, be such that its structure map is a retract. Let $\mathcal{M}$ be the minimal model for $\mathcal{A}^*$. If $\mathcal{A}^*$ is equivariantly $(n-1)$-intrinsically formal, $N^i_{C_p} = 0$ for $i \geq n$ and N_e is formal, then $\mathcal{A}^*$ is equivariantly intrinsically formal. In particular, $\mathcal{M}$ is equivariantly formal.*

Proof By Theorem 5.6, $\mathcal{M}_{n-1}$ is determined by the set $\mathcal{M}_{\mathcal{A}^*(C_p/e)}$ and since $H^*(N_e)$ is formal, the cardinality of this set in one. This holds for each n and hence $\mathcal{A}^*$ is equivariantly IF. □

6 Examples

Example 6.1 Let $n \geq 2$ and p be a prime. Let $X = S^n \times \cdots \times S^n$ be the product of p copies of S^n with the $G = C_p$ action $t(a_1, a_2, \ldots, a_p) = (a_2, a_3, \ldots, a_{p-1}, a_1)$ where t is a generator of C_p. Then we have the following two cases.

(1) Let n be odd. From Example 3.15 we conclude that the cohomology diagram is equivariantly n-IF. Using Corollary 5.7 we conclude that the product of odd spheres under the above action is equivariantly formal.

(2) Let n be an even integer. From Example 3.16 the cohomology diagram is equivariantly $(2n-1)$-IF. So by using Corollary 5.7 we conclude that the product of even spheres under the above action is equivariantly formal.

Example 6.2 Consider $X = (S^3 \vee S^3) \times S^5$ with diagonal action of $G = C_2$, where C_2 acts on $(S^3 \vee S^3)$ by switching copies and acts trivially on S^5.

We denote the cohomology diagram by $\mathcal{A}$, the generators of the cohomology by x, y, and z. Note that $\deg(x) = 3 = \deg(y)$ and $\deg(z) = 5$, and

$$\mathcal{A} := \begin{cases} \wedge(x, y)/\langle xy \rangle \otimes \wedge(z) & \text{at } G/e, \\ \wedge(z) & \text{at } G/G. \end{cases}$$

First, note that $\mathcal{A}$ is injective by Proposition 4.1. We claim that the cohomology diagram is equivariantly 6-IF.

To see this, first, note that the cohomology diagram is equivariantly 3-IF since $\mathcal{A}$ is 2-connected and the minimal system of DGAs,

$$\mathcal{M}_3 = \begin{cases} \wedge(a_3, b_3) & \text{at } G/e, \\ \mathbb{Q} & \text{at } G/G, \end{cases}$$

which is equivariantly IF.

Next, if we consider any minimal system $\mathcal{M}'$ with $H^*(\mathcal{M}') = \mathcal{A}$ then $\mathcal{M}'_3 = \mathcal{M}_3$. Now $\mathcal{M}'_4 = \mathcal{M}_3(\underline{V})$.

We claim that the inclusion $\mathcal{M}_3 \to \mathcal{M}'_4$ is a 4-isomorphism, so $\mathcal{A}$ becomes equivariantly 4-IF.

Since $\mathcal{M}'_4 \to \mathcal{M}'$ is 4-isomorphism, the map $H^i(\mathcal{M}'_4) \to H^i(\mathcal{M}') \cong H^i(\mathcal{M}) = \mathcal{A}^i$ is isomorphism for $i \le 4$ and monomorphism for $i = 5$.

Therefore, $H^i(\mathcal{M}_3) \to H^i(\mathcal{M}'_4)$ is isomorphism for $i \le 4$ as $H^4(\mathcal{M}') = 0$. Hence, we need to show $H^5(\mathcal{M}_3) \to H^5(\mathcal{M}'_4)$ is monomorphism, i.e. we must show $H^5(\mathcal{M}'_4) = 0$.

Assume to the contrary that there exists $[\beta] \in H^5(\mathcal{M}'_4)$ nonzero. Then $\beta \in (\mathcal{M}_3(\underline{V}))^5$. Now we consider the corresponding resolution of $\underline{V}$:

$$\begin{array}{ccccccccc} \underline{V} & \longrightarrow & \underline{V}_0 & \xrightarrow{w_0} & \underline{V}_1 & \xrightarrow{w_1} & \underline{V}_2 & \longrightarrow & \cdots \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & \\ \underline{Z}^4(\mathcal{M}_3) & \longrightarrow & \mathcal{M}_3^4 & \xrightarrow[d]{} & \mathcal{M}_3^5 & \xrightarrow[d]{} & \mathcal{M}_3^6 & \longrightarrow & \cdots \end{array}$$

Note that $\underline{Z}^4(\mathcal{M}_3) = \mathcal{M}_3^4 = \mathcal{M}_3^5 = 0$. Thus, $\alpha_0 = \alpha_1 = 0$ and the differential is dependent only on w_0 and w_1 at the $\underline{V}_0$ and $\underline{V}_1$ level, respectively. The element $\beta \in \underline{V}_1$ is nonzero in the cohomology so it is not in the image of w_0. In particular, $\beta \in \underline{V}_1/\mathrm{Im}(w_0)$ and $\underline{V}_1/\mathrm{Im}(w_0) \to \underline{V}_2$ is an injection, which implies that $d(\beta) \neq 0$ and β is not a cohomology class. This implies $H^5(\mathcal{M}'_4) = 0$. Hence, $H^5(\mathcal{M}_3) \to H^5(\mathcal{M}'_4)$ is a monomorphism. So the map $\mathcal{M}_3 \to \mathcal{M}'_4$ is a 4-isomorphism and hence $\mathcal{A}$ is equivariantly 4-IF.

Next, we claim that $\mathcal{A}$ is equivariantly 5-IF.

We note that

$$\mathcal{M}_5 = \begin{cases} \wedge(a_3, b_3, c_5, d_5) & \text{at } G/e, \\ \wedge(c_5) & \text{at } G/G. \end{cases}$$

So if $\mathcal{M}'$ is any minimal system with $H^*(\mathcal{M}') = \mathcal{A}$, then by previous argument we get $\mathcal{M}'_3 = \mathcal{M}_3$ and $\mathcal{M}_4 \to \mathcal{M}'_4$ is a 4-isomorphism.

We claim that $\mathcal{M}_5 \to \mathcal{M}'_5$ is a 5-isomorphism.

Since $\mathcal{M}'_5 \to \mathcal{M}'$ is a 5-isomorphism, there are elements $p, q \in \mathcal{M}'_5$ of degree 5 such that $[p]$ maps to z and the differential of q kills the product xy and $H^6(\mathcal{M}'_5) = 0$.

We define the map $\mathcal{M}_5 \to \mathcal{M}'_5$ which when restricted to $\mathcal{M}_3$ is the previous map and sends a, b to p, q respectively. The induced map indeed is an isomorphism $H^i(\mathcal{M}_5) \to H^i(\mathcal{M}'_5)$ for $i \leq 5$ and monomorphism for $i = 6$.

Hence, $\mathcal{A}$ is equivariantly 5-IF. Since $\mathcal{A}^6 = \mathcal{A}^7 = 0$ from Proposition 3.6 we conclude that $\mathcal{A}$ is equivariantly 6-IF.

The minimal system up to 6^{th} stage is given by

$$\mathcal{M}_6 = \begin{cases} \wedge(a_3, b_3, c_5, d_5) & \text{at } G/e, \\ \wedge(c_5) & \text{at } G/G. \end{cases}$$

The differential is $d(a_3) = 0 = d(b_3)$, $d(d_5) = ab$, $d(c_5) = 0$, and $\rho\colon \mathcal{M}_6 \to \mathcal{A}$ maps $c_5 \to z$ and $d_5 \to 0$.

Thus, the cohomology diagram satisfies the hypothesis of Theorem 5.6. Hence, the set of isomorphism classes of the minimal system containing $\mathcal{M}_6$ is in bijection with $\mathcal{M}_{\mathcal{A}^*(G/e)}$. From [13] and Theorem 5.6, it follows that the $\mathcal{M}^{C_2}_{\mathcal{A}^*}$ consists of three points. The description is given in [13, Example 5].

Example 6.3 Consider the space $X = (S^2 \vee S^2) \times S^3$ with the action of $G = C_2$, by switching copies of $(S^3 \vee S^3)$ and acts on S^5 trivially. We denote the cohomology diagram by $\mathcal{A}$, and the generators of the cohomology by x, y, and z. Note that $\deg(x) = 2 = \deg(y)$ and $\deg(z) = 3$. The cohomology diagram is given by

$$\mathcal{A} = \begin{cases} \wedge(x, y)/\langle xy \rangle \otimes \wedge(z) & \text{at } G/e, \\ \wedge(z) & \text{at } G/G. \end{cases}$$

By Proposition 4.1, we conclude that $\mathcal{A}$ is injective. The cohomology diagram is equivariantly 3-IF. The minimal system up to 3^{rd} stage is given by

$$\mathcal{M}_3 = \begin{cases} \wedge(a_2, b_2, c_3, d_3, e_a, e_b) & \text{at } G/e, \\ \wedge(c_3) & \text{at } G/G. \end{cases}$$

The differential is given by $d(a_2) = 0 = d(b_2)$, $d(d_3) = ab$, $d(c_3) = 0$, $d(e_a) = a^2$, $d(e_b) = b^2$ and $\rho\colon \mathcal{M}_3 \to \mathcal{A}$ maps $(c_3, a_2, b_2) \to (z, x, y)$ and $(d_3, e_a, e_b) \to (0, 0, 0)$.

Thus, the cohomology diagram satisfies the hypothesis of Theorem 5.6. Hence, the set of isomorphism classes of the minimal system containing $\mathcal{M}_4$ is determined by the set $\mathcal{M}_{\mathcal{A}^*(G/e)}$. From Theorem 5.6, we conclude that $\mathcal{M}^{C_2}_{\mathcal{A}^*}$ is the same as $\mathcal{M}_{\mathcal{A}^*(G/e)}$. By [13, Example 4] then we conclude that $\mathcal{M}^{C_2}_{\mathcal{A}^*}$ contains exactly two points.

Example 6.4 There is a diagonal G action on $X = S^3 \vee S^3 \vee S^5$ where $G = C_2$ acts on $(S^3 \vee S^3)$ by switching copies and acts on S^5 trivially.

Note that the fixed point set X^G is homeomorphic to S^5. We denote the cohomology diagram by $\mathcal{A}$, and the generators of the cohomology by x, y, and z. Note that, $\deg(x) = 3 = \deg(y)$ and $\deg(z) = 5$, and

$$\mathcal{A} = \begin{cases} \mathbb{Q}(x,y,z)/\langle x^2, z^2, xy, xz, yz\rangle & \text{at } G/e, \\ \mathbb{Q}(z)/\langle z^2\rangle & \text{at } G/G. \end{cases}$$

This diagram of graded algebras is equivariantly 5-IF,

$$\mathcal{M}_5 = \begin{cases} \wedge(a_3, b_3, c_5, d_5) & \text{at } G/e, \\ \wedge(c_5) & \text{at } G/G. \end{cases}$$

The differential is $d(a_3) = 0 = d(b_3)$, $d(d_5) = ab$, and $d(c_5) = 0$.

Thus, the cohomology diagram satisfies all the hypotheses of Theorem 5.6 and hence up to isomorphism the number of minimal algebras containing $\mathcal{M}_5$ with the same cohomology diagram can be computed similarly.

Example 6.5 Let $n > 1$ be an integer and let $X = (S^n \vee S^n) \times S^{2n-1}$. Then there is a $G = C_2$ action on X switching the two copies of S^n and keeping the S^{2n-1} copy fixed. We denote the cohomology diagram by $\mathcal{A}$ and the generators of the cohomology by x, y, and z. Note that, $\deg(x) = n = \deg(y)$ and $\deg(z) = 2n-1$, and

$$\mathcal{A} := \begin{cases} \wedge(x,y)/\langle xy\rangle \otimes \wedge(z) & \text{at } G/e, \\ \wedge(z) & \text{at } G/G. \end{cases}$$

Since the map $\mathcal{A}(G/e) \to \mathcal{A}(G/G)$ is onto, we see that the given cohomology diagram is injective.

(1) If n is odd, the cohomology diagram is equivariantly $2n$-intrinsically formal. The minimal system up to $2n$ stage is given by

$$\mathcal{M}_{2n} = \begin{cases} \wedge(a_n, b_n, c_{2n-1}, d_{2n-1}) & \text{at } G/e, \\ \wedge(c_{2n-1}) & \text{at } G/G. \end{cases}$$

The differential is $d(a_n) = 0 = d(b_n)$, $d(d_{2n-1}) = ab$, $d(c_{2n-1}) = 0$, and $\rho\colon \mathcal{M}_{2n} \to \mathcal{A}$ maps $c_{2n-1} \to z$ and $d_{2n-1} \to 0$.

Thus, the cohomology diagram satisfies the hypothesis of Theorem 5.6. Hence, the set of isomorphism classes of the minimal system containing $\mathcal{M}_6$ is obtained similarly.

(2) If n is even, the cohomology diagram is equivariantly $(2n-1)$-IF. The minimal system up to the $(2n-1)^{\text{st}}$ stage is given by

$$\mathcal{M}_{2n-1} = \begin{cases} \wedge(a_n, b_n, c_{2n-1}, d_{2n-1}, e_a, e_b) & \text{at } G/e, \\ \wedge(c_{2n-1}) & \text{at } G/G. \end{cases}$$

The differential is $d(a_n) = 0 = d(b_n)$, $d(d_{2n-1}) = ab$, $d(c_{2n-1}) = 0$, $d(e_a) = a^2$, $d(e_b) = b^2$ and $\rho\colon \mathcal{M}_{2n-1} \to \mathcal{A}$ maps $(c_{2n-1}, a_n, b_n) \to (z, x, y)$ and $(d_{2n-1}, e_a, e_b) \to (0,0,0)$.

Thus, the cohomology diagram satisfies the hypothesis of Theorem 5.6. Hence, the set of isomorphism classes of the minimal system containing $\mathcal{M}_{2n}$ is in bijection with the set $\mathcal{M}_{\mathcal{A}^*(G/e)}$. The case $n = 2$ is computed in [13].

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