Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:5 (2025) 2905–2932
DOI: 10.2140/agt.2025.25.2905
Published: 17 September 2025

Warped product metrics on hyperbolic and complex hyperbolic manifolds

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We study warped-product metrics on manifolds of the form $X \setminus Y$, where X denotes either $\mathbb{H}^n$ or $\mathbb{C}\mathbb{H}^n$, and Y is a totally geodesic submanifold with arbitrary codimension. The main results that we prove are curvature formulas for these metrics on $X \setminus Y$ expressed in spherical coordinates about Y. We also discuss past and potential future applications of these formulas.

53C20, 53C35; 53C56, 57R25

1 Introduction

1.1 Main results

Let $\mathbb{H}^n$ denote real hyperbolic space with real dimension n, and let $\mathbb{C}\mathbb{H}^n$ denote complex hyperbolic space with complex dimension n. In this paper, X will denote either $\mathbb{H}^n$ or $\mathbb{C}\mathbb{H}^n$, and Y will denote a totally geodesic submanifold of X. So if $X = \mathbb{H}^n$ then $Y = \mathbb{H}^k$, and if $X = \mathbb{C}\mathbb{H}^n$ then Y is either $\mathbb{H}^k$ or $\mathbb{C}\mathbb{H}^k$ for some $0 \leq k \leq n-1$. Let M be a Riemannian manifold, and N a totally geodesic submanifold of M. We say that the pair (M, N) is *modeled on* (X, Y) if X is the universal cover of M and, within this cover, Y corresponds to the universal cover of N.

The purpose of this paper is to develop curvature formulas for warped-product metrics on $X \setminus Y$ when the pair (X, Y) is one of $(\mathbb{H}^n, \mathbb{H}^k)$, $(\mathbb{C}\mathbb{H}^n, \mathbb{H}^n)$, or $(\mathbb{C}\mathbb{H}^n, \mathbb{C}\mathbb{H}^k)$. These cases are detailed in Sections 2, 3, and 4, respectively. In each case we write the metric on X in spherical coordinates about Y (Theorems 2.1, 3.1, and 4.1), we consider the corresponding warped product metric where we allow for variable coefficients in the metric tensor ((2-2), (3-3), and (4-2)), and we compute formulas for the components of the $(4,0)$ curvature tensor with respect to these coefficient functions (Theorems 2.2, 3.4, and 4.3). These last three theorems should be considered the main results of this paper.

1.2 Applications for these curvature formulas

Specific cases for these formulas are already known and have been used in various applications in the literature. Some examples are as follows. The case when $X = \mathbb{H}^n$ and $Y = \mathbb{H}^{n-2}$ was used by Gromov and Thurston in [7] (discussed further below) and by Belegradek in [1]. When $X = \mathbb{H}^n$ and $Y = \mathbb{H}^0$ is a point, this leads to the basis for the *Farrell and Jones warping deformation* used in [5]. This process is described by Ontaneda in [13] and used by the same author in [14]. The case when $X = \mathbb{CH}^n$ and $Y = \mathbb{CH}^0$ is a point was used by Farrell and Jones in [6], and the same X but with $Y = \mathbb{CH}^{n-1}$ was considered by Belegradek in [2]. Finally, the cases when (X, Y) are either $(\mathbb{CH}^2, \mathbb{H}^2)$ or $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$ were used by the author in [9] and [10].

While the author believes that the curvature formulas in Theorems 2.2, 3.4, and 4.3 will have many future uses, the primary motivation for the development of these curvature formulas was for the following application.

In [7] Gromov and Thurston famously construct pinched negatively curved manifolds which do not admit hyperbolic metrics. In this construction they consider pairs (M, N) modeled on $(\mathbb{H}^n, \mathbb{H}^{n-2})$ which satisfy a few special topological and geometric conditions. The pinched negatively curved manifold X which does not admit a hyperbolic metric is then the d-fold cyclic branched cover of M about N (where $d \in \mathbb{N}$ can take all but possibly finitely many values). The difficulty in all of this is showing that X exists, constructing a pinched negatively curved metric on X, and proving that X does not admit a hyperbolic metric.

It is an open question as to whether or not this construction can be extended to the locally symmetric pairs $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$ and $(\mathbb{CH}^2, \mathbb{H}^2)$. In a forthcoming paper [11] the author shows that the d-fold cyclic ramified cover of M about N for the case $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$ *does* admit an almost negatively $\frac{1}{4}$-pinched Riemannian metric. The fact that such a pair (M, N) can be realized so that the ramified cover is a smooth manifold for some integer $d > 2$ is a result of Stover and Toledo in [15]. The constructions of this Riemannian metric uses the curvature formulas proved in Theorems 2.2 and 4.3 below.

One last remark about these curvature formulas. In [1], [2], and [9] it is proved that the manifold $M \setminus N$, where (M, N) is modeled on one of $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-2})$, or $(\mathbb{CH}^2, \mathbb{H}^2)$, admits a complete, finite volume, negatively curved Riemannian metric. The curvature formulas developed in this paper are extensions of the curvature formulas computed and used in these three articles.

1.3 Obstructions to $M \setminus N$ admitting a complete, finite volume Riemannian metric of negative sectional curvature

Consider the finite volume manifold $M \setminus N$. The three cases where N has real codimension two in M are modeled on one of $(\mathbb{H}^n, \mathbb{H}^{n-2})$, $(\mathbb{CH}^n, \mathbb{CH}^{n-1})$, or $(\mathbb{CH}^2, \mathbb{H}^2)$. In all of these cases, the manifold $M \setminus N$ admits a complete, finite volume Riemannian metric whose sectional curvature is bounded above by a negative constant [1; 2; 9].

When the real codimension of N is greater than two, the manifold $M \setminus N$ should not admit a complete, finite volume, negatively curved metric because it generally will not be aspherical. This fact should be realized in the curvature equations in Theorems 2.2, 3.4, and 4.3. More specifically, there should be one or more equations which obstructs such a metric, but these curvature equations should vanish when N has codimension two.

In all cases except one "exceptional case" the obstruction is a sectional curvature equation of the form

$$\frac{1}{v^2} - \left(\frac{v'}{v}\right)^2 \tag{1-1}$$

where $v\colon \mathbb{R} \to \mathbb{R}$ is a positive, increasing real-valued function. In order to alter the metric on $M \setminus N$ to be complete, one needs to define a warping function for v that will make each component of N into the boundary of a cusp of the manifold. One easily checks that (1-1) is nonpositive if and only if $1 \le (v')^2$. But for the Riemannian metric to have any chance of having finite volume one needs $\lim_{r\to-\infty} v'(r) = 0$.

The one exceptional case is when (M, N) is modeled on $(\mathbb{C}\mathbb{H}^n, \mathbb{C}\mathbb{H}^{n-2})$. Here, all curvature equations of the form (1-1) vanish, and so this obstruction is more subtle. It should be noted that the vanishing of (1-1) is what leads to the metric developed in [10], which shows that a finite volume manifold of the form $M \setminus N$ where (M, N) is modeled on $(\mathbb{C}\mathbb{H}^n, \mathbb{C}\mathbb{H}^{n-2})$ admits a complete, finite volume metric which is negatively curved when restricted to a (nonintegrable) real codimension one distribution. The calculation in [10] is very complicated, whereas the work required to show that one cannot vary the curvature formulas in this setting to obtain global negative curvature (and finite volume) is pretty straightforward. A quick argument for this is given in Section 4.6.

1.4 Layout of this paper

In Section 2 we study manifolds of the form $\mathbb{H}^n \setminus \mathbb{H}^k$, in Section 3 we consider $\mathbb{C}\mathbb{H}^n \setminus \mathbb{H}^n$, and in Section 4 we analyze $\mathbb{C}\mathbb{H}^n \setminus \mathbb{C}\mathbb{H}^k$. The calculations in Sections 3 and 4 become very complicated. So in Section 3 we restrict our attention to $\mathbb{C}\mathbb{H}^3 \setminus \mathbb{H}^3$ and in Section 4 we restrict to $\mathbb{C}\mathbb{H}^5 \setminus \mathbb{C}\mathbb{H}^2$ to make these calculations simpler to follow. In each case, these are the smallest choices for n and k which capture all of the different formulas for the curvature tensor, up to the symmetries of the curvature tensor (and with respect to the frames chosen in each section). That is, from these cases one knows all of curvature formulas for general $\mathbb{C}\mathbb{H}^n \setminus \mathbb{H}^n$ and $\mathbb{C}\mathbb{H}^n \setminus \mathbb{C}\mathbb{H}^k$. Also, notice that we only consider $\mathbb{C}\mathbb{H}^n \setminus \mathbb{H}^n$ in Section 3 instead of the more general $\mathbb{C}\mathbb{H}^n \setminus \mathbb{H}^k$. The reason for this is due to simplicity: in general there are several ways that $\mathbb{H}^k$ can sit inside of $\mathbb{C}\mathbb{H}^n$ which requires a case-by-case analysis. But in all situations this copy of $\mathbb{H}^k$ is contained in a copy of $\mathbb{H}^n$, and then one can apply our formulas here to $\mathbb{C}\mathbb{H}^n \setminus \mathbb{H}^n$. Section 5 is a short section on some known formulas that are referenced throughout the paper, and Section 6 is devoted to computing values for Lie brackets from Section 3.

We end this section with the following three remarks which deal with notational differences between this paper and [1; 2; 9].

Remark 1.1 In this paper we scale the complex hyperbolic metric to have sectional curvatures in the interval $[-4, -1]$, whereas in the previous three references the curvatures were scaled to $\left[-1, -\frac{1}{4}\right]$. To adjust the formulas in [1; 2; 9], one simply multiplies the warping functions h, v, and h_r by $\frac{1}{2}$. With this adjustment (and the following remark), one sees that the formulas in these references agree with the codimension two versions of the formulas in Theorems 2.2, 3.4, and 4.3.

Remark 1.2 Another major notational difference between this paper and [1; 2] is the formula used for the curvature tensor. Let g be a Riemannian metric with Levi-Civita connection ∇, and let W, X, Y, and Z be vector fields. In this paper we follow [4] and use the notation

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{1-2}$$

for the curvature tensor R of g. The negative of this formula is used in [1; 2]. So, in particular, the $(4,0)$-curvature tensor $\langle R(X,Y)Z, W\rangle_g$ in this paper is equivalent to $\langle R(X,Y)W, Z\rangle_g$ in [1; 2].

Remark 1.3 We will be considering many different metrics throughout this paper. For a sectional curvature tensor R we will frequently use the notation R^g to emphasize that this tensor is with respect to the metric g. We use a superscript as subscripts will frequently be used for components of the curvature tensor. At times, we do just use R if the metric is to be understood from context.

2 Curvature formulas for warped product metrics on $\mathbb{H}^n \setminus \mathbb{H}^k$

2.1 Expressing the metric in $\mathbb{H}^n$ in spherical coordinates about $\mathbb{H}^k$

Let us first note that in Sections 2.1, 3.1, and 4.1 we closely follow the notation and terminology used in [2].

Let $\boldsymbol{h}_n$ denote the hyperbolic metric on $\mathbb{H}^n$. Since $\mathbb{H}^k$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{H}^n$, there exists an orthogonal projection map $\pi\colon \mathbb{H}^n \to \mathbb{H}^k$. This map π is a fiber bundle whose fibers are totally geodesic $(n-k)$-planes.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{H}^k$. Then $E(r)$ is a real hypersurface in $\mathbb{H}^n$, and consequently we can decompose $\boldsymbol{h}_n$ as

$$\boldsymbol{h}_n = (\boldsymbol{h}_n)_r + dr^2$$

where $(\boldsymbol{h}_n)_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r\colon E(r) \to \mathbb{H}^k$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^{n-k-1}$-bundle whose fiber over any point $q \in \mathbb{H}^k$ is the $(n-k-1)$-sphere of radius r in the totally geodesic $(n-k)$-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

It is well known (see [1] or [7] when $k = n-2$ and [13] for general k) that for an appropriate identification of $E(r) \cong \mathbb{H}^k \times \mathbb{S}^{n-k-1}$ the metric $(\boldsymbol{h}_n)_r$ can be written as

$$(\boldsymbol{h}_n)_r = \cosh^2(r)\boldsymbol{h}_k + \sinh^2(r)\boldsymbol{\sigma}_{n-k-1},$$

where $\boldsymbol{h}_k$ denotes the hyperbolic metric on $\mathbb{H}^k$ and $\boldsymbol{\sigma}_{n-k-1}$ denotes the round metric on the unit sphere $\mathbb{S}^{n-k-1}$. Note that $(\boldsymbol{h}_n)_r$ restricted to $\mathcal{H}(r)$ is $\cosh^2(r)\boldsymbol{h}_k$ and $(\boldsymbol{h}_n)_r$ restricted to $\mathcal{V}(r)$ is $\sinh^2(r)\boldsymbol{\sigma}_{n-k-1}$. We summarize this in the following theorem.

Theorem 2.1 *The hyperbolic manifold $\mathbb{H}^n \setminus \mathbb{H}^k$ can be written as $E \times (0,\infty)$ where $E \cong \mathbb{H}^k \times \mathbb{S}^{n-k-1}$ equipped with the metric*

$$\boldsymbol{h}_n = \cosh^2(r)\boldsymbol{h}_k + \sinh^2(r)\sigma_{n-k-1} + dr^2. \tag{2-1}$$

2.2 The warped product metric and curvature formulas

For some positive, increasing real-valued functions $h, v\colon (0,\infty) \to \mathbb{R}$, we define

$$\boldsymbol{\lambda}_r := h^2(r)\boldsymbol{h}_k + v^2(r)\boldsymbol{\sigma}_{n-k-1} \quad \text{and} \quad \boldsymbol{\lambda} := \boldsymbol{\lambda}_r + dr^2. \tag{2-2}$$

Of course, $\boldsymbol{\lambda} = \boldsymbol{h}_n$ when $h = \cosh(r)$ and $v = \sinh(r)$.

Fix $p \in E(r)$ for some r and let $q = \pi(p) \in \mathbb{H}^k$. Let $\{\check{X}_i\}_{i=1}^k$ be an orthonormal frame of $\mathbb{H}^k$ with respect to $\boldsymbol{h}_k$ near q which satisfies $[\check{X}_i, \check{X}_j]_q = 0$ for all $1 \le i, j \le k$. These vector fields can be extended to a collection of orthogonal vector fields $\{X_i\}_{i=1}^k$ in a neighborhood of p via the inclusion $\mathbb{H}^k \to E \times (0,\infty)$. Analogously, define an orthonormal frame $\{\check{X}_j\}_{j=k+1}^{n-1}$ of $\mathbb{S}^{n-k-1}$ near (the projection of) p which satisfies $[\check{X}_i, \check{X}_j]_p = 0$ for all $k+1 \le i, j \le n-1$, and extend this frame to vector fields $\{X_j\}_{j=k+1}^{n-1}$ in a neighborhood of p via the inclusion $\mathbb{S}^{n-k-1} \to E \times (0,\infty)$. Lastly, let $X_n = \frac{\partial}{\partial r}$.

The orthogonal collection of vector fields $\{X_i\}_{i=1}^n$ satisfies the following:

(1) $\langle X_i, X_i \rangle_\lambda = h^2$ for $1 \le i \le k$.

(2) $\langle X_i, X_i \rangle_\lambda = v^2$ for $k+1 \le i \le n-1$.

(3) $\langle X_n, X_n \rangle_\lambda = 1$.

(4) $[X_i, X_j]_p = 0$ for all i, j.

It should be noted that property (4) is special to the real hyperbolic case and will not be true in Sections 3 and 4 below.

Now define the corresponding orthonormal frame near p by $Y_i = \frac{1}{h}X_i$ for $1 \le i \le k$, $Y_j = \frac{1}{v}X_j$ for $k+1 \le j \le n-1$, and $Y_n = X_n$. This frame satisfies the property that $[Y_i, Y_j]_p = 0$ for $1 \le i, j \le n-1$. We can then apply formulas (5-4) through (5-7) to write the $(4,0)$ curvature tensor R^λ in terms of R^{λ_r} as follows, where $1 \le a, b \le k$ and $k+1 \le c, d \le n-1$:

$$K^\lambda(Y_a, Y_b) = K^{\lambda_r}(Y_a, Y_b) - \left(\frac{h'}{h}\right)^2, \quad K^\lambda(Y_c, Y_d) = K^{\lambda_r}(Y_c, Y_d) - \left(\frac{v'}{v}\right)^2,$$
$$K^\lambda(Y_a, Y_c) = K^{\lambda_r}(Y_a, Y_c) - \frac{h'v'}{hv}, \quad K^\lambda(Y_a, Y_n) = -\frac{h''}{h}, \quad K^\lambda(Y_c, Y_n) = -\frac{v''}{v}.$$

In the above equations, we use the notation

$$K(X,Y) = \langle R(X,Y)X, Y \rangle$$

to denote the sectional curvature of the 2-plane spanned by X and Y. The above equations are the only terms that appear (up to the symmetries of the curvature tensor). So, in particular, all mixed terms of R^λ are identically zero.

Now, the $(4,0)$ curvature tensor R^{λ_r} is simple to calculate. Since both $h(r)\mathbb{H}^k$ and $v(r)\mathbb{S}^{n-k-1}$ have constant curvature, and $h(r)\mathbb{H}^k \times v(r)\mathbb{S}^{n-k-1}$ is metrically a product, we have that for $1 \le a, b, \le k$ and $k+1 \le c, d \le n-1$,

$$K^{\lambda_r}(Y_a, Y_b) = -\frac{1}{h^2}, \quad K^{\lambda_r}(Y_c, Y_d) = \frac{1}{v^2}, \quad K^{\lambda_r}(Y_a, Y_c) = 0.$$

Putting this all together yields the following.

Theorem 2.2 *Up to the symmetries of the curvature tensor, the only nonzero terms of the* $(4,0)$ *curvature tensor* R^λ *are*

$$K^\lambda(Y_a, Y_b) = -\frac{1}{h^2} - \left(\frac{h'}{h}\right)^2, \quad K^\lambda(Y_c, Y_d) = \frac{1}{v^2} - \left(\frac{v'}{v}\right)^2,$$
$$K^\lambda(Y_a, Y_c) = -\frac{h'v'}{hv}, \quad K^\lambda(Y_a, Y_n) = -\frac{h''}{h}, \quad K^\lambda(Y_c, Y_n) = -\frac{v''}{v},$$

where $1 \le a, b \le k$ *and* $k+1 \le c, d \le n-1$.

One easily checks that plugging in the values $v(r) = \sinh(r)$ and $h(r) = \cosh(r)$ gives all sectional curvatures of -1.

3 Curvature formulas for warped product metrics on $\mathbb{CH}^n \setminus \mathbb{H}^n$

As mentioned in the introduction, for simplicity we are going to restrict ourselves to the case when $n = 3$. This is exactly the smallest dimension which captures every nonzero component of the curvature tensor, and so nothing is lost with this restriction (see Remark 3.5).

3.1 Expressing the metric in $\mathbb{CH}^3$ in spherical coordinates about $\mathbb{H}^3$

Let $\boldsymbol{c}_3$ denote the complex hyperbolic metric on $\mathbb{CH}^3$ normalized to have constant holomorphic sectional curvature -4. Since $\mathbb{H}^3$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{CH}^3$, there exists an orthogonal projection map $\pi : \mathbb{CH}^3 \to \mathbb{H}^3$. This map π is a fiber bundle whose fibers are totally real totally geodesic 3-planes, and therefore have constant sectional curvature -1.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{H}^3$. Then $E(r)$ is a real hypersurface in $\mathbb{CH}^3$, and consequently we can decompose $\boldsymbol{c}_3$ as

$$\boldsymbol{c}_3 = (\boldsymbol{c}_3)_r + dr^2,$$

where $(\boldsymbol{c}_3)_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r : E(r) \to \mathbb{H}^3$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^2$-bundle whose fiber over any point $q \in \mathbb{H}^3$ is the 2-sphere of radius r in the totally real totally geodesic 3-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the 2-sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$.

The orthogonal projection π induces a unique geodesic flow on $\mathbb{CH}^3$ as follows. Let $p \in \mathbb{CH}^3$ and let $q = \pi(p)$. Then there exists a unique unit-speed geodesic from p to q and, moreover, this geodesic is contained in $\pi^{-1}(q)$ since this is a totally geodesic copy of $\mathbb{H}^3$. We define the geodesic flow

$$\phi\colon [0,\infty) \times \mathbb{CH}^3 \to \mathbb{CH}^3$$

by just moving along this geodesic for the given amount of time (and $\phi(t,p) = q$ for all $t \geq d(p,q)$).

For $r, s > 0$ there exists a diffeomorphism $\phi_{sr}\colon E(s) \to E(r)$ induced by this geodesic flow. Fix $p \in E(r)$ arbitrary, let $q = \pi(p) \in \mathbb{H}^3$, and let γ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_p E(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $E(r)$ and the totally real totally geodesic 3-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$. Since $\exp_p^{-1}(\pi^{-1}(q))$ is a totally real 3-plane, there exists a suitable identification $\pi^{-1}(q) \cong \mathbb{S}^2 \times (0,\infty)$ where the metric $\boldsymbol{c}_3$ restricted to $\pi^{-1}(q)$ can be written as

$$\sinh^2(r)\sigma^2 + dr^2.$$

Here, σ^2 is the round metric on the unit 2-sphere.

Let

$$\check{X}_4 = \frac{\partial}{\partial\theta}, \quad \check{X}_5 = \frac{1}{\sin\theta}\frac{\partial}{\partial\psi} \tag{3-1}$$

be an orthonormal frame on a neighborhood of (the projection of) p in $\mathbb{S}^2$, where θ and ψ denote the standard spherical coordinates on $\mathbb{S}^2$. Extend these to orthogonal vector fields $\{X_4, X_5\}$ on $\pi^{-1}(q)$ via the inclusion $\mathbb{S}^2 \to \pi^{-1}(q)$. Note that both X_4 and X_5 are invariant under $d\phi_{sr}$. Let $X_6 = \frac{\partial}{\partial r}$.

Let J denote the complex structure on $\mathbb{CH}^3$. It is well known that J_p preserves complex subspaces in $T_p\mathbb{CH}^3$ and maps real subspaces into their orthogonal complement. Since (X_4, X_5, X_6) spans a real 3-plane in $T_p\mathbb{CH}^3$, its orthogonal complement $\mathcal{H}_p(r)$ is spanned by (JX_4, JX_5, JX_6). In what follows we define vector fields X_1, X_2, and X_3 which are just scaled copies of JX_4, JX_5, and JX_6, respectively.

3.1.1 The vector fields X_1 and X_2 First note that (JX_4, X_6) spans a real 2-plane in $T_p\mathbb{CH}^3$ (since its J-image is contained in its orthogonal complement). So $P = \exp_p(\operatorname{span}(JX_4, X_6))$ is a totally real totally geodesic 2-plane in $\mathbb{CH}^3$ which intersects $\mathbb{H}^3$ orthogonally. Since this intersection is orthogonal, P is preserved by the geodesic flow ϕ. Therefore, $\operatorname{span}(JX_4)$ is preserved by $d\phi$.

The set $P \cap \mathbb{H}^3$ is a (real) geodesic. Let $\alpha(s)$ denote this geodesic parameterized with respect to arc length so that $\alpha(0) = q$. Then define $(X_1)_p = (d\pi)_p^{-1}\alpha'(0)$. There exists a positive real-valued function

$a(r,s)$ such that the metric $\boldsymbol{c}_3$ restricted to P is of the form $dr^2 + a^2(r,s)ds^2$. But since $\mathbb{R}$ acts by isometries on P via translation along α, the function $a(r,s)$ is independent of s. Then since the curvature of a real 2-plane is -1, we have that $a(r) = \cosh(r)$.

We analogously define X_2 by replacing X_4 with X_5 in the above description. All conclusions follow in an identical manner. Thus, we can write the metric $\boldsymbol{c}_3$ restricted to $\exp_p(X_1, X_2, X_6)$ as

$$\cosh^2(r)(dX_1^2 + dX_2^2) + dr^2.$$

3.1.2 The vector field X_3 This is also mostly analogous to the definition of X_1. But this time note that (JX_6, X_6) spans a complex line in $T_p\mathbb{CH}^3$ (since it is preserved by its J-image). So $Q = \exp_p(\operatorname{span}(JX_6, X_6))$ is a complex geodesic in $\mathbb{CH}^3$ which intersects $\mathbb{H}^3$ orthogonally. Since this intersection is orthogonal, Q is preserved by the geodesic flow ϕ. Therefore, $\operatorname{span}(JX_6)$ is preserved by $d\phi$.

The set $Q \cap \mathbb{H}^3$ is a (real) geodesic. Let $\beta(t)$ denote this geodesic parameterized with respect to arc length so that $\beta(0) = q$. Then define $(X_3)_p = (d\pi)^{-1}\beta'(0)$. There exists a positive real-valued function $b(r,t)$ such that the metric $\boldsymbol{c}_3$ restricted to Q is of the form $dr^2 + b^2(r,t)dt^2$. But since $\mathbb{R}$ acts by isometries on Q via translation along β, the function $b(r,t)$ is independent of t. Then since the curvature of a complex geodesic is -4, we have that $b(r) = \cosh(2r)$.

3.1.3 Conclusion

Theorem 3.1 *The complex hyperbolic manifold $\mathbb{CH}^3 \setminus \mathbb{H}^3$ can be written as $E \times (0,\infty)$ where $E \cong \mathbb{H}^3 \times \mathbb{S}^2$ equipped with the metric*

$$\boldsymbol{c}_3 = \cosh^2(r)(dX_1^2 + dX_2^2) + \cosh^2(2r)dX_3^2 + \sinh^2(r)(dX_4^2 + dX_5^2) + dr^2. \tag{3-2}$$

In (3-2), dX_1 through dX_5 denote the covector fields dual to the vector fields X_1 through X_5, respectively. Lastly, notice that $dX_1^2 + dX_2^2$ is the hyperbolic metric with constant sectional curvature -1, and $dX_4^2 + dX_5^2$ is the spherical metric with constant sectional curvature 1.

3.2 The warped product metric and curvature formulas in $\mathbb{CH}^3 \setminus \mathbb{H}^3$

For some positive, increasing real-valued functions $h, h_r, v \colon (0,\infty) \to \mathbb{R}$ define

$$\begin{aligned} \boldsymbol{\mu}_r &:= h^2(r)(dX_1^2 + dX_2^2) + h_r^2(r)dX_3^2 + v^2(r)(dX_4^2 + dX_5^2), \\ \boldsymbol{\mu} &:= \boldsymbol{\mu}_r + dr^2. \end{aligned} \tag{3-3}$$

Of course, $\boldsymbol{\mu} = \boldsymbol{c}_3$ when $h = \cosh(r)$, $h_r = \cosh(2r)$, and $v = \sinh(r)$.

Define an orthonormal basis $\{Y_i\}_{i=1}^6$ with respect to μ by

$$Y_1 = \frac{1}{h}X_1, \quad Y_2 = \frac{1}{h}X_2, \quad Y_3 = \frac{1}{h_r}X_3, \quad Y_4 = \frac{1}{v}X_4, \quad Y_5 = \frac{1}{v}X_5, \quad Y_6 = X_6. \tag{3-4}$$

Our goal is to compute formulas for the components of the $(4,0)$ curvature tensor R^{μ} in terms of the warping functions h, h_r, and v (this is the content of Theorem 3.4). As a first step, we need to compute the components of the $(4,0)$ curvature tensor $R^{\boldsymbol{c}_3}$ of the complex hyperbolic metric with respect to the orthonormal basis given above. We can do this with the help of formula (5-1). To use this formula note that, by construction, we have that $JY_4 = Y_1$, $JY_5 = Y_2$, and $JY_6 = Y_3$ (again, when the metric is $\boldsymbol{c}_3$, that is, when $h = \cosh(r)$, $h_r = \cosh(2r)$, and $v = \sinh(r)$). Lastly, we use the notation

$$R^{\boldsymbol{c}_3}_{ijkl} := \langle R^{\boldsymbol{c}_3}(Y_i, Y_j)Y_k, Y_l\rangle_{\boldsymbol{c}_3}.$$

Then, up to the symmetries of the curvature tensor, the nonzero components of the $(4,0)$ curvature tensor $R^{\boldsymbol{c}_3}$ are

$$-4 = R^{\boldsymbol{c}_3}_{1414} = R^{\boldsymbol{c}_3}_{2525} = R^{\boldsymbol{c}_3}_{3636}, \tag{3-5}$$

$$\begin{aligned} -1 = R^{\boldsymbol{c}_3}_{1212} = R^{\boldsymbol{c}_3}_{1313} = R^{\boldsymbol{c}_3}_{1515} = R^{\boldsymbol{c}_3}_{1616} = R^{\boldsymbol{c}_3}_{2323} = R^{\boldsymbol{c}_3}_{2424} \\ = R^{\boldsymbol{c}_3}_{2626} = R^{\boldsymbol{c}_3}_{3434} = R^{\boldsymbol{c}_3}_{3535} = R^{\boldsymbol{c}_3}_{4545} = R^{\boldsymbol{c}_3}_{4646} = R^{\boldsymbol{c}_3}_{5656}, \end{aligned} \tag{3-6}$$

$$-2 = R^{\boldsymbol{c}_3}_{1425} = R^{\boldsymbol{c}_3}_{1436} = R^{\boldsymbol{c}_3}_{2536}, \tag{3-7}$$

$$-1 = R^{\boldsymbol{c}_3}_{1245} = R^{\boldsymbol{c}_3}_{1346} = R^{\boldsymbol{c}_3}_{2356} = R^{\boldsymbol{c}_3}_{1524} = R^{\boldsymbol{c}_3}_{1634} = R^{\boldsymbol{c}_3}_{2635}. \tag{3-8}$$

3.3 Lie brackets

We now need to compute the values of the Lie brackets of the orthogonal basis $\{X_i\}_{i=1}^{6}$. A first observation is that, by construction, each of these vector fields is invariant under the flow of $\frac{\partial}{\partial r}$. This implies that $[X_i, X_6] = 0$ for all $1 \le i \le 6$. From this we can deduce that

$$[Y_1, Y_6] = \frac{h'}{h}Y_1, \quad [Y_2, Y_6] = \frac{h'}{h}Y_2, \quad [Y_3, Y_6] = \frac{h_r'}{h_r}Y_3, \quad [Y_4, Y_6] = \frac{v'}{v}Y_4, \quad [Y_5, Y_6] = \frac{v'}{v}Y_5.$$

Next, we know that each Lie bracket is tangent to the level surfaces of r. Thus, for all $1 \le i, j \le 6$, the Lie bracket $[X_i, X_j]$ has no X_6 term. For all $1 \le i, j, k \le 5$ define structure constants c_{ij}^k by

$$[X_i, X_j] = \sum_{k=1}^{5} c_{ij}^k X_k. \tag{3-9}$$

Two quick observations about the structure constants. The first is that $c_{ij}^k = -c_{ji}^k$ due to the antisymmetry of the Lie bracket. The second observation is about the values of c_{45}^4 and c_{45}^5. Recall the definitions for $\check{X}_4$ and $\check{X}_5$ from (3-1). Then

$$[\check{X}_4, \check{X}_5] = \left[\frac{\partial}{\partial\theta}, \frac{1}{\sin(\theta)}\frac{\partial}{\partial\psi}\right] = \frac{-\cos(\theta)}{\sin^2(\theta)}\frac{\partial}{\partial\psi} = -\cot(\theta)\check{X}_5. \tag{3-10}$$

We therefore conclude that $c_{45}^4 = 0$ and $c_{45}^5 = -\cot(\theta)$.

The following theorem gives almost a full description of the values of the Lie brackets. Some quantities are only defined up to sign, but this is sufficient to compute the curvature formulas in Theorem 3.4. The interested reader can find the proof of Theorem 3.2 in Section 6.

Theorem 3.2 *The values for the Lie brackets in* (3-9) *are*

$$\begin{aligned}
&[X_1,X_2]=\pm X_1, && [X_1,X_3]=\mp\cot(\theta)X_2+X_4,\\
&[X_1,X_4]=X_3\mp X_5, && [X_1,X_5]=-\cot(\theta)X_2\pm X_4,\\
&[X_2,X_3]=\pm\cot(\theta)X_1+X_5, && [X_2,X_4]=0,\\
&[X_2,X_5]=\cot(\theta)X_1+X_3, && [X_3,X_4]=-X_1\pm\cot(\theta)X_5,\\
&[X_3,X_5]=-X_2\mp\cot(\theta)X_4, && [X_4,X_5]=-\cot(\theta)X_5.
\end{aligned}$$

In the above equations, all of the $\pm$ *and* $\mp$ *signs are related. For example, if it is the case that* $[X_1,X_2]=X_1$, *then* $[X_1,X_4]=X_3-X_5$ *and so on.*

3.4 The Levi-Civita connection and formulas for the $(4,0)$ curvature tensor R^μ

In this subsection we first compute the Levi-Civita connection ∇ associated to the metric μ with respect to the frame $(Y_i)_{i=1}^6$. The difficult part in all of this is computing the Lie brackets in Theorem 3.2. From there it is now a simple calculation using formula (5-3) to prove the following theorem.

Theorem 3.3 *The Levi-Civita connection* ∇ *compatible with* μ *is determined by the following* 36 *equations*:

$$\begin{aligned}
&\nabla_{Y_1}Y_1=\mp\frac{1}{h}Y_2-\frac{h'}{h}Y_6, && \nabla_{Y_1}Y_2=\pm\frac{1}{h}Y_1,\\
&\nabla_{Y_1}Y_3=\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_4, && \nabla_{Y_1}Y_4=-\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_3\mp\frac{1}{h}Y_5,\\
&\nabla_{Y_1}Y_5=\pm\frac{1}{h}Y_4, && \nabla_{Y_1}Y_6=\frac{h'}{h}Y_1,\\
&\nabla_{Y_2}Y_1=0, && \nabla_{Y_2}Y_2=-\frac{h'}{h}Y_6,\\
&\nabla_{Y_2}Y_3=\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_5, && \nabla_{Y_2}Y_4=0,\\
&\nabla_{Y_2}Y_5=-\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_3, && \nabla_{Y_2}Y_6=\frac{h'}{h}Y_2,\\
&\nabla_{Y_3}Y_1=\pm\frac{1}{h_r}\cot(\theta)Y_2+\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{vh}-\frac{v}{hh_r}\right)Y_4, && \nabla_{Y_3}Y_2=\mp\frac{1}{h_r}\cot(\theta)Y_1+\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}-\frac{v}{hh_r}\right)Y_5,\\
&\nabla_{Y_3}Y_3=-\frac{h_r'}{h_r}Y_6, && \nabla_{Y_3}Y_4=-\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}-\frac{v}{hh_r}\right)Y_1\pm\frac{1}{h_r}\cot(\theta)Y_5,\\
&\nabla_{Y_3}Y_5=-\frac{1}{2}\left(\frac{h}{h_rv}-\frac{h_r}{hv}-\frac{v}{hh_r}\right)Y_2\mp\frac{1}{h_r}\cot(\theta)Y_4, && \nabla_{Y_3}Y_6=\frac{h_r'}{h_r}Y_3,\\
&\nabla_{Y_4}Y_1=-\frac{1}{2}\left(\frac{h}{h_rv}+\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_3, && \nabla_{Y_4}Y_2=0,\\
&\nabla_{Y_4}Y_3=\frac{1}{2}\left(\frac{h}{h_rv}+\frac{h_r}{hv}+\frac{v}{hh_r}\right)Y_1, && \nabla_{Y_4}Y_4=-\frac{v'}{v}Y_6,\\
&\nabla_{Y_4}Y_5=0, && \nabla_{Y_4}Y_6=\frac{v'}{v}Y_4,
\end{aligned}$$

$$\nabla_{Y_5}Y_1 = \frac{1}{v}\cot(\theta)Y_2, \qquad \nabla_{Y_5}Y_2 = -\frac{1}{v}\cot(\theta)Y_1 - \frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right)Y_3,$$
$$\nabla_{Y_5}Y_3 = \frac{1}{2}\left(\frac{h}{h_r v} + \frac{h_r}{hv} + \frac{v}{hh_r}\right)Y_2, \qquad \nabla_{Y_5}Y_4 = \frac{1}{v}\cot(\theta)Y_5,$$
$$\nabla_{Y_5}Y_5 = -\frac{1}{v}\cot(\theta)Y_4 - \frac{v'}{v}Y_6, \qquad \nabla_{Y_5}Y_6 = \frac{v'}{v}Y_5,$$
$$\nabla_{Y_6}Y_1 = \nabla_{Y_6}Y_2 = \nabla_{Y_6}Y_3 = \nabla_{Y_6}Y_4 = \nabla_{Y_6}Y_5 = \nabla_{Y_6}Y_6 = 0.$$

By combining Theorem 3.3 with (1-2), and remembering that $Y_6 = X_6 = \frac{\partial}{\partial r}$ and $X_4 = \frac{\partial}{\partial \theta}$, we compute the following formulas for the $(4,0)$ curvature tensor R^μ. As in equations (3-5) through (3-8) we use the notation

$$R^\mu_{ijkl} := \langle R^\mu(Y_i, Y_j)Y_k, Y_l\rangle_\mu.$$

Theorem 3.4 *In terms of the basis given in* (3-4), *the only independent nonzero components of the* $(4,0)$ *curvature tensor* R^μ *are the following*:

$$R^\mu_{1212} = -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2},$$
$$R^\mu_{4545} = -\left(\frac{v'}{v}\right)^2 + \frac{1}{v^2},$$
$$R^\mu_{1515} = R^\mu_{2424} = -\frac{h'v'}{hv},$$
$$R^\mu_{1414} = R^\mu_{2525} = -\frac{v'h'}{vh} - \left(\frac{-v^2}{4h^2h_r^2} - \frac{h^2}{4v^2h_r^2} + \frac{3h_r^2}{4v^2h^2} - \frac{1}{2v^2} + \frac{1}{2h^2} - \frac{1}{2h_r^2}\right),$$
$$R^\mu_{3434} = R^\mu_{3535} = -\frac{v'h_r'}{vh_r} - \left(\frac{-v^2}{4h^2h_r^2} + \frac{3h^2}{4v^2h_r^2} - \frac{h_r^2}{4v^2h^2} - \frac{1}{2v^2} - \frac{1}{2h^2} + \frac{1}{2h_r^2}\right),$$
$$R^\mu_{1313} = R^\mu_{2323} = -\frac{h'h_r'}{hh_r} - \left(\frac{3v^2}{4h^2h_r^2} - \frac{h^2}{4v^2h_r^2} - \frac{h_r^2}{4v^2h^2} + \frac{1}{2v^2} + \frac{1}{2h^2} + \frac{1}{2h_r^2}\right),$$
$$R^\mu_{1616} = R^\mu_{2626} = -\frac{h''}{h},$$
$$R^\mu_{3636} = -\frac{h_r''}{h_r},$$
$$R^\mu_{4646} = R^\mu_{5656} = -\frac{v''}{v},$$
$$R^\mu_{1436} = R^\mu_{2536} = \frac{1}{2h_r}\left[\left(\frac{h}{v}\right)' - \left(\frac{v}{h}\right)' - \left(\frac{h_r^2}{vh}\right)'\right],$$
$$R^\mu_{1634} = R^\mu_{2635} = \frac{1}{2h}\left[-\left(\frac{h_r}{v}\right)' + \left(\frac{v}{h_r}\right)' + \left(\frac{h^2}{vh_r}\right)'\right],$$
$$R^\mu_{1346} = R^\mu_{2356} = \frac{-1}{2v}\left[\left(\frac{h}{h_r}\right)' + \left(\frac{h_r}{h}\right)' + \left(\frac{v^2}{hh_r}\right)'\right],$$

$$R^{\mu}_{1425} = \frac{1}{2v^2} - \frac{1}{2h^2} - \frac{h_r^2}{2h^2v^2},$$
$$R^{\mu}_{1245} = -\frac{1}{4}\left(\frac{h^2}{h_r^2v^2} + \frac{h_r^2}{h^2v^2} + \frac{v^2}{h^2h_r^2} + \frac{2}{h^2} + \frac{2}{h_r^2} - \frac{2}{v^2}\right),$$
$$R^{\mu}_{1524} = -\frac{1}{4}\left(\frac{-h^2}{h_r^2v^2} + \frac{h_r^2}{h^2v^2} - \frac{v^2}{h^2h_r^2} - \frac{2}{h_r^2}\right).$$

It is a tedious exercise in hyperbolic trigonometric identities to check that, when $h=\cosh(r)$, $h_r=\cosh(2r)$, and $v=\sinh(r)$, the above formulas reduce to the constants in (3-5) through (3-8). Also, note that the first nine equations above give the sectional curvatures of the coordinate planes, while the last six equations are formulas for the nonzero mixed terms.

Remark 3.5 Let us explain how the above curvature formulas contain all of the formulas that arise in the case of $\mathbb{CH}^n \setminus \mathbb{H}^n$ for generic n. In general, one can write the complex hyperbolic metric $\boldsymbol{c}_n$ as

$$\boldsymbol{c}_n = \cosh^2(r)\boldsymbol{h}_{n-1} + \cosh^2(2r)dX_n^2 + \sinh^2(r)\boldsymbol{\sigma}_{n-1} + dr^2$$

and the corresponding warped-product metric as

$$\boldsymbol{\mu}_n = h^2(r)\boldsymbol{h}_{n-1} + h_r^2(r)dX_n^2 + v^2(r)\boldsymbol{\sigma}_{n-1} + dr^2$$

where $\boldsymbol{\sigma}_{n-1}$ is the round metric on $\mathbb{S}^{n-1}$ and the vector field X_n is defined in the same manner as X_3. As above, we choose an orthonormal basis $\{X_i\}_{i=n+1}^{2n-1}$ for the $\mathbb{S}^{n-1}$ factor. We use these vectors to define X_i for all $1 \le i \le n-1$ in the same manner as above which defines an orthogonal basis for the orthogonal complement to X_n in the $\mathbb{H}^n$ factor. We then define an orthonormal basis $\{Y_i\}_{i=1}^{2n}$ exactly as in (3-4).

Curvature formulas for the nonzero components of R^{μ_n} corresponding to the base $\mathbb{H}^n$ are of the form $R^{\mu_n}_{ijij}$ for $1 \le i, j \le n$ and are encoded in the formulas for R^{μ}_{1212} and R^{μ}_{1313}. Note that there are no mixed terms here since $\mathbb{H}^n$ does not contain any *holomorphic pairs*, that is, a pair of unit vectors (A, B) which satisfy that $JA = \pm B$. All curvature formulas for the $\mathbb{S}^{n-1}$ factor, which are of the form $R^{\mu_n}_{k\ell k\ell}$ for $n+1 \le k, \ell \le 2n-1$, are contained in the term R^{μ}_{4545}. Again there are no mixed terms here since there are no holomorphic pairs of vectors. The curvature formulas R^{μ}_{1414} and R^{μ}_{3434} above give formulas for $\mathbb{R}^{\mu_n}_{ikik}$ where $1 \le i \le n$ and $n+1 \le k \le 2n-1$. The terms R^{μ}_{1425}, R^{μ}_{1245}, and R^{μ}_{1524} give formulas for all mixed terms of the metric $h^2(r)\boldsymbol{h}_{n-1} + h_r^2(r)dX_n^2 + v^2(r)\boldsymbol{\sigma}_{n-1}$. Finally, all of the formulas above containing a “6” give the rest of the curvature formulas for μ_n.

4 Curvature formulas for warped product metrics on $\mathbb{CH}^n \setminus \mathbb{CH}^k$

As mentioned in the introduction, for simplicity we are going to restrict ourselves to the case when $n=5$ and $k=2$. These are the smallest choices for n and k which capture every formula for the curvature tensor in the general case, so nothing is lost with this restriction (see Remark 4.4).

4.1 Expressing the metric in $\mathbb{CH}^5$ in spherical coordinates about $\mathbb{CH}^2$

Let $\boldsymbol{c}_5$ denote the complex hyperbolic metric on $\mathbb{CH}^5$ normalized to have constant holomorphic sectional curvature -4. Since $\mathbb{CH}^2$ is a complete totally geodesic submanifold of the negatively curved manifold $\mathbb{CH}^5$, there exists an orthogonal projection map $\pi\colon \mathbb{CH}^5 \to \mathbb{CH}^2$. This map π is a fiber bundle whose fibers are totally geodesic 6-planes isometric to $\mathbb{CH}^3$.

For $r > 0$ let $E(r)$ denote the r-neighborhood of $\mathbb{CH}^2$. Then $E(r)$ is a real hypersurface in $\mathbb{CH}^5$, and consequently we can decompose $\boldsymbol{c}_5$ as

$$\boldsymbol{c}_5 = (\boldsymbol{c}_5)_r + dr^2,$$

where $(\boldsymbol{c}_5)_r$ is the induced Riemannian metric on $E(r)$. Let $\pi_r\colon E(r) \to \mathbb{CH}^2$ denote the restriction of π to $E(r)$. Note that π_r is an $\mathbb{S}^5$-bundle whose fiber over any point $q \in \mathbb{CH}^2$ is (topologically) the 5-sphere of radius r in the totally geodesic 6-plane $\pi^{-1}(q)$. The tangent bundle splits as an orthogonal sum $\mathcal{V}(r) \oplus \mathcal{H}(r)$ where $\mathcal{V}(r)$ is tangent to the 5-sphere $\pi_r^{-1}(q)$ and $\mathcal{H}(r)$ is the orthogonal complement to $\mathcal{V}(r)$. Note that this copy of $\mathbb{S}^5$ does *not* have constant sectional curvature equal to 1, but rather it is an example of a *Berger sphere*. This will be discussed further below.

One can use the orthogonal projection π to define a geodesic flow on $\mathbb{CH}^5$ towards the copy of $\mathbb{CH}^2$ in a completely analogous manner as to what was done in Section 3.1. For $r, s > 0$ there exists a diffeomorphism $\phi_{sr}\colon E(s) \to E(r)$ induced by this geodesic flow along the totally geodesic 6-planes orthogonal to $\mathbb{CH}^2$. Fix $p \in E(r)$ arbitrary, let $q = \pi(p) \in \mathbb{CH}^2$, and let γ be the unit speed geodesic such that $\gamma(0) = q$ and $\gamma(r) = p$. In what follows, all computations are considered in the tangent space $T_pE(r)$.

Note that $\mathcal{V}(r)$ is tangent to both $E(r)$ and the totally geodesic 6-plane $\pi^{-1}(q)$. Then since $\pi^{-1}(q)$ is preserved by the geodesic flow, we have that $d\phi_{sr}$ takes $\mathcal{V}(s)$ to $\mathcal{V}(r)$. Consider the complex geodesic $P = \exp_p\bigl(\operatorname{span}\bigl(\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\bigr)\bigr)$. P intersects $E(r)$ orthogonally, and $P \cap E(r)$ is isometric to a circle of radius r. Thus, since a complex geodesic has curvature -4, there exists a suitable identification $P \cong \mathbb{S}^1 \times (0, \infty)$ where the metric $\boldsymbol{c}_5$ restricted to P can be written as

$$\tfrac{1}{4}\sinh^2(2r)d\theta^2 + dr^2,$$

where $d\theta^2$ denotes the round metric on the unit circle $\mathbb{S}^1$. Note that the presence of the "$1/4$" is to make the metric complete when extended to the core $\mathbb{CH}^2$.

Notice that $\frac{\partial}{\partial\theta}$ is a vector field on the five sphere $\mathbb{S}^5$ mentioned above. More generally, thinking of $\mathbb{S}^5$ as the unit sphere in $\mathbb{C}^3$ with respect to the usual Hermitian metric, there is an obvious free action of the circle $\mathbb{S}^1$ on $\mathbb{S}^5$. The unit tangent vector field with respect to this action corresponds to the vector field $\frac{\partial}{\partial\theta}$ above. This action fibers $\mathbb{S}^5$ over the complex projective plane $\mathbb{CP}^2$, and the Riemannian submersion metric on this fiber bundle is an example of a Berger sphere (see [6, page 59] for more details). Let $\alpha(t)$ be a unit-speed geodesic in $\mathbb{S}^5$ orthogonal to $J\frac{\partial}{\partial r}$ such that $\alpha(0) = p$. Then $\exp_p(\alpha'(0), \partial r)$ forms a totally real totally geodesic 2-plane in $\mathbb{CH}^5$. Thus the curvature of this 2-plane is -1. Since the direction of α orthogonal to $J\frac{\partial}{\partial r}$ was arbitrary, we can write the Riemannian metric $(\boldsymbol{c}_5)_r$ restricted to $\mathcal{V}(r)$ as

$$\sinh^2(r)\boldsymbol{p}_2 + \tfrac{1}{4}\sinh^2(2r)d\theta^2$$

where $\boldsymbol{p}_2$ denotes the complex projective metric on $\mathbb{CP}^2$.

Now let $\beta(t)$ be any unit speed geodesic in $\mathbb{CH}^2$ such that $\beta(0)=q$. Then $Q=\exp_q(\operatorname{span}(\beta'(0),\gamma'(0)))$ is a totally real totally geodesic submanifold of $\mathbb{CH}^5$, and thus $K(\beta',\gamma')=-1$. Therefore, the metric $\boldsymbol{c}_5$ restricted to Q can be written as $\cosh^2(r)dt^2+dr^2$. But since γ was arbitrary, we can write the metric on the 5-dimensional submanifold determined by $\mathbb{CH}^2$ and $\frac{\partial}{\partial r}$ as $\cosh^2 \boldsymbol{c}_2 + dr^2$. This leads to the following.

Theorem 4.1 *The complex hyperbolic manifold $\mathbb{CH}^5 \setminus \mathbb{CH}^2$ can be written as $E\times(0,\infty)$ where $E\cong \mathbb{CH}^2\times\mathbb{S}^5$ equipped with the metric*

$$\boldsymbol{c}_5 = \cosh^2(r)\boldsymbol{c}_2 + \sinh^2(r)\boldsymbol{p}_2 + \tfrac{1}{4}\sinh^2(2r)d\theta^2 + dr^2. \tag{4-1}$$

4.2 The warped product metric, orthonormal basis, and curvature formulas in $\mathbb{CH}^5\setminus\mathbb{CH}^2$

For some positive, increasing real-valued functions $h, v, v_r\colon (0,\infty)\to\mathbb{R}$ define the Riemannian metrics

$$\begin{aligned}\boldsymbol{\gamma}_{r,\theta} &= h^2(r)\boldsymbol{c}_2 + v^2(r)\boldsymbol{p}_2,\\ \boldsymbol{\gamma}_r &:= \boldsymbol{\gamma}_{r,\theta} + \frac{1}{4}v_r^2(r)d\theta^2,\\ \boldsymbol{\gamma} &:= \boldsymbol{\gamma}_r + dr^2.\end{aligned}\tag{4-2}$$

Of course, $\boldsymbol{\gamma}=\boldsymbol{c}_5$ when $h=\cosh(r)$, $v=\sinh(r)$, and $v_r=\sinh(2r)$.

For the remainder of this section, fix $p=(q_1,\bar q, r)\in\mathbb{CH}^2\times\mathbb{S}^5\times(0,\infty)\cong\mathbb{CH}^5\setminus\mathbb{CH}^2$, and write $\bar q\in\mathbb{S}^5$ as (q_2,θ) where $q_2\in\mathbb{CP}^2$ and $\theta\in\mathbb{S}^1$. Let $(\check X_1,\check X_2,\check X_3,\check X_4)$ be an orthonormal collection of vector fields near $q_1\in\mathbb{CH}^2$ which satisfies:

(1) $[\check X_i,\check X_j]_{q_1}=0$ for all $1\le i,j\le 4$.
(2) $J\check X_2|_{q_1}=\check X_1|_{q_1}$ and $J\check X_4|_{q_1}=\check X_3|_{q_1}$.

Define an analogous collection of vector fields $(\check X_5,\check X_6,\check X_7,\check X_8)$ about $q_2\in\mathbb{CP}^2$ so that $J\check X_6|_{q_2}=\check X_5|_{q_2}$, $J\check X_8|_{q_2}=\check X_7|_{q_2}$, and $[\check X_i,\check X_j]_{q_2}=0$ for all $5\le i,j\le 8$. Extend both collections to vector fields $(X_1,\dots,X_8)$ near p. Lastly, let $X_9=\frac{\partial}{\partial\theta}$ and $X_{10}=\frac{\partial}{\partial r}$.

Define an orthonormal basis $\{Y_i\}_{i=1}^{8}$ with respect to γ by

$$\begin{aligned}&Y_1=\frac{1}{h}X_1,\quad Y_2=\frac{1}{h}X_2,\quad Y_3=\frac{1}{h}X_3,\quad Y_4=\frac{1}{h}X_4,\qquad Y_5=\frac{1}{v}X_5,\\ &Y_6=\frac{1}{v}X_6,\quad Y_7=\frac{1}{v}X_7,\quad Y_8=\frac{1}{v}X_8,\quad Y_9=\frac{1}{\frac{1}{2}v_r}X_9,\quad Y_{10}=X_{10}.\end{aligned}\tag{4-3}$$

Our goal is to compute formulas for the components of the $(4,0)$ curvature tensor R^γ in terms of the warping functions h, v, and v_r. As a first step, we need to compute the components of the $(4,0)$ curvature tensor $R^{\boldsymbol{c}_5}$ of the complex hyperbolic metric with respect to the orthonormal basis given above. Just as in Section 3 we can do this with the help of formula (5-1). To use this formula note that, by construction, we have that $JY_2 = Y_1$, $JY_4 = Y_3$, $JY_6 = Y_5$, $JY_8 = Y_7$, and $JY_{10} = Y_9$ at the point p (and again, when the metric is $\boldsymbol{c}_5$, so when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$). Lastly, we use the notation

$$R^{\boldsymbol{c}_5}_{ijkl} := \langle R^{\boldsymbol{c}_5}(Y_i, Y_j)Y_k, Y_l\rangle_{\boldsymbol{c}_5}.$$

Then, up to the symmetries of the curvature tensor, the nonzero components of the $(4,0)$ curvature tensor $R^{\boldsymbol{c}_5}$ are

$$-4 = R^{\boldsymbol{c}_5}_{1212} = R^{\boldsymbol{c}_5}_{3434} = R^{\boldsymbol{c}_5}_{5656} = R^{\boldsymbol{c}_5}_{7878} = R^{\boldsymbol{c}_5}_{9,10,9,10}, \tag{4-4}$$

$$-1 = R^{\boldsymbol{c}_5}_{ijij} \quad \text{where } \{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\}\}, \tag{4-5}$$

$$-2 = R^{\boldsymbol{c}_5}_{ijkl} \quad \text{where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\}, \tag{4-6}$$

$$-1 = R^{\boldsymbol{c}_5}_{ikjl} \quad \text{where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\}, \tag{4-7}$$

$$1 = R^{\boldsymbol{c}_5}_{iljk} \quad \text{where } (i,j) \neq (k,l) \in \{(1,2),(3,4),(5,6),(7,8),(9,10)\}, \tag{4-8}$$

Let us quickly note that, since $\mathbb{CP}^2$ is dual to $\mathbb{CH}^2$, we have the following curvature formulas for $R^{\boldsymbol{p}_2}$:

$$\begin{aligned} 4 &= R^{\boldsymbol{p}_2}_{5656} = R^{\boldsymbol{p}_2}_{7878}, \\ 1 &= R^{\boldsymbol{p}_2}_{5757} = R^{\boldsymbol{p}_2}_{5858} = R^{\boldsymbol{p}_2}_{6767} = R^{\boldsymbol{p}_2}_{6868}, \\ 2 &= R^{\boldsymbol{p}_2}_{5678} = 2R^{\boldsymbol{p}_2}_{5768} = -2R^{\boldsymbol{p}_2}_{5867}. \end{aligned}$$

In the above formulas, $R^{\boldsymbol{p}_2}_{ijkl} := \langle R^{\boldsymbol{p}_2}(Y_i, Y_j)Y_k, Y_l\rangle_{\boldsymbol{p}_2}$ and with the abuse of notation of Y_i denoting the restriction of Y_i to $\mathbb{CP}^2$.

4.3 Lie brackets and curvature formulas for $\gamma_{r,\theta}$

The vector fields $\{X_i\}_{i=1}^{10}$ form an orthogonal frame near p which satisfies the following properties (at p):

1. $[X_i, X_j]$ is tangent to the level surfaces of r for $1 \le i, j \le 9$.
2. $[X_i, X_j]$ is tangent to $\mathbb{CH}^2 \times \exp_p(J\frac{\partial}{\partial r})$ for $1 \le i, j \le 4$.
3. $[X_i, X_j]$ is tangent to $\mathbb{S}^5$ for $5 \le i, j \le 8$.
4. $[X_i, X_{10}] = 0$ since X_i is invariant under the flow of $\frac{\partial}{\partial r}$ for $1 \le i \le 9$.
5. $[X_i, X_9] = 0$ since X_i is invariant under the flow of $\frac{\partial}{\partial \theta}$ for $1 \le i \le 8$.
6. $[X_i, X_j] = 0$ for $i \in \{1,2,3,4\}$ and $j \in \{5,6,7,8\}$ since these vector fields were defined via inclusion.

By the above points, and since $[\check{X}_i, \check{X}_j]_p = 0$ for all $1 \le i, j \le 8$, there exist *structure constants* c_{ij} such that $[X_i, X_j]_p = c_{ij}X_9$. Note that $c_{ij} = -c_{ji}$. The following lemma provides the values for the structure constants.

Lemma 4.2 *The values for the structure constants are* $c_{12} = c_{34} = c_{56} = c_{78} = 2$, *and all other* (*independent*) *structure constants are equal to zero.*

A quick note is that by "independent" structure constants we just mean that, obviously,

$$c_{21} = c_{43} = c_{65} = c_{87} = -2 \neq 0.$$

Proof All of the structure constants can be found by combining formula (5-7) with the curvature formulas (4-6) through (4-8). To see that $c_{12} = 2$, we combine (4-6) with (5-7) to obtain

$$4 = 2R^{\boldsymbol{c}_5}_{10,9,1,2} = 0 + 0 + \langle [Y_1, Y_2], Y_9\rangle_{\boldsymbol{c}_5}\left(\ln\left[\frac{\frac{1}{4}\sinh^2(2r)}{\cosh^2(r)}\right]\right)' = \frac{c_{12}\sinh(2r)}{\cosh^2(r)}\left(\frac{\cosh(r)}{\sinh(r)}\right) = 2c_{12}.$$

An analogous argument shows that $c_{34} = c_{56} = c_{78} = 2$. To see that

$$c_{13} = c_{14} = c_{23} = c_{24} = c_{57} = c_{58} = c_{67} = c_{68} = 0$$

we use the same equations as above, but note that the left hand side is now 0 instead of 4.

Lastly, to see that $c_{15} = 0$, note that

$$\begin{aligned} 0 = R^{\boldsymbol{c}_5}_{10,1,5,9} &= 0 + \langle [Y_5, Y_1], Y_9\rangle_{\boldsymbol{c}_5}\left(\ln\left[\frac{\frac{1}{2}\sinh(2r)}{\sinh(r)}\right]\right)' + 0 \\ &= \frac{-\frac{1}{2}c_{15}\sinh(2r)}{\sinh(r)\cosh(r)}(\ln 2\cosh(r))' = -c_{15}\tanh(r). \end{aligned}$$

The argument that the remaining structure constants are 0 is identical to the argument above. □

We now, for some fixed r and θ, compute the components of the $(4,0)$ curvature tensor $R^{\boldsymbol{\lambda}_{r,\theta}}$ with respect to the orthonormal frame $\{Y_i\}_{i=1}^8$. Since $[X_i, X_j] = 0$ for $i \in \{1,2,3,4\}$ and $j \in \{5,6,7,8\}$, the metric $\boldsymbol{\gamma}_{r,\theta}$ is a product metric. Then since the $(4,0)$ curvature tensor scales like the metric, up to the symmetries of the curvature tensor the only nonzero components of $R^{\boldsymbol{\gamma}_{r,\theta}}$ are

$$\begin{aligned} &R^{\boldsymbol{\gamma}_{r,\theta}}_{1212} = R^{\boldsymbol{\gamma}_{r,\theta}}_{3434} = -\frac{4}{h^2}, && R^{\boldsymbol{\gamma}_{r,\theta}}_{1313} = R^{\boldsymbol{\gamma}_{r,\theta}}_{1414} = R^{\boldsymbol{\gamma}_{r,\theta}}_{2323} = R^{\boldsymbol{\gamma}_{r,\theta}}_{2424} = -\frac{1}{h^2}, \\ &R^{\boldsymbol{\gamma}_{r,\theta}}_{5656} = R^{\boldsymbol{\gamma}_{r,\theta}}_{7878} = \frac{4}{v^2}, && R^{\boldsymbol{\gamma}_{r,\theta}}_{5757} = R^{\boldsymbol{\gamma}_{r,\theta}}_{5858} = R^{\boldsymbol{\gamma}_{r,\theta}}_{6767} = R^{\boldsymbol{\gamma}_{r,\theta}}_{6868} = \frac{1}{v^2}, \\ &R^{\boldsymbol{\gamma}_{r,\theta}}_{1234} = 2R^{\boldsymbol{\gamma}_{r,\theta}}_{1324} = -2R^{\boldsymbol{\gamma}_{r,\theta}}_{1423} = -\frac{2}{h^2}, && R^{\boldsymbol{\gamma}_{r,\theta}}_{5678} = 2R^{\boldsymbol{\gamma}_{r,\theta}}_{5768} = -2R^{\boldsymbol{\gamma}_{r,\theta}}_{5867} = \frac{2}{v^2}. \end{aligned}$$

In particular, note that mixed terms of the form $R^{\boldsymbol{\gamma}_{r,\theta}}_{1256}$ are 0.

4.4 Curvature formulas for $\boldsymbol{\gamma_r}$

Formulas (5-4) through (5-7) allow us to compute the $(4,0)$ curvature tensor R^{γ} in terms of R^{γ_r}. We use a very different approach from Section 3 to compute the nonzero components of R^{γ_r}. The background for our current computations can be found in Section 5.5, all of which comes from [3, pages 235–242]. The metric γ_r is a Riemannian submersion metric with (horizontal) base $\gamma_{r,\theta}$ and (vertical) fiber $\frac{1}{4}v_r^2 d\theta^2$.

So our approach is to compute the A and T tensors of γ_r, and to then use Theorem 5.5 to compute the components of R^{γ_r}. The computations in this subsection are very similar to [2, Section 6].

First, the T-tensor is identically zero by Remark 5.2 and since the vertical $\mathbb{S}^1$-fibers are totally geodesic. The argument for why this fiber is totally geodesic is identical to that in [2, Section 6].

We now compute the A tensor associated with γ_r. By Theorem 5.4 we have that

$$A_{X_1}X_2 = \tfrac{1}{2}\mathcal{V}[X_1, X_2] = X_9.$$

Analogously, $A_{X_3}X_4 = A_{X_5}X_6 = A_{X_7}X_8 = X_9$ and $A_{X_i}X_j = 0$ if $\{i, j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$. Also, by (5-9) we have $A_{X_9}X_i = 0$ for $1 \le i \le 8$.

Now, by (5-12) we see that

$$\langle A_{X_1}X_9, X_2\rangle_{\gamma_r} = -\langle A_{X_1}X_2, X_9\rangle_{\gamma_r} = -\tfrac{1}{4}v_r^2.$$

By this same equation we know that there are no other nonzero components of $A_{X_1}X_9$. Therefore,

$$A_{X_1}X_9 = -\frac{1}{4}\frac{v_r^2}{h^2}X_2.$$

Analogously, we have that

$$A_{X_2}X_9 = \frac{1}{4}\frac{v_r^2}{h^2}X_1, \quad A_{X_3}X_9 = -\frac{1}{4}\frac{v_r^2}{h^2}X_4, \quad A_{X_4}X_9 = \frac{1}{4}\frac{v_r^2}{h^2}X_3,$$
$$A_{X_5}X_9 = -\frac{1}{4}\frac{v_r^2}{v^2}X_6, \quad A_{X_6}X_9 = \frac{1}{4}\frac{v_r^2}{v^2}X_5, \quad A_{X_7}X_9 = -\frac{1}{4}\frac{v_r^2}{v^2}X_8, \quad A_{X_8}X_9 = \frac{1}{4}\frac{v_r^2}{v^2}X_7.$$

We are now ready to use Theorem 5.5 to compute the nonzero components of R^{γ_r}. By (5-15) we have that

$$\langle R^{\gamma_r}(X_1, X_9)X_1, X_9\rangle_{\gamma_r} = \langle A_{X_1}X_9, A_{X_1}X_9\rangle_{\gamma_r} = \frac{1}{16}\frac{v_r^4}{h^2},$$

and thus

$$R^{\gamma_r}_{1919} = \frac{4}{h^2 v_r^2}\langle R^{\gamma_r}(X_1, X_9)X_1, X_9\rangle_{\gamma_r} = \frac{v_r^2}{4h^4}. \tag{4-9}$$

Identically, $R^{\gamma_r}_{2929} = R^{\gamma_r}_{3939} = R^{\gamma_r}_{4949} = \frac{v_r^2}{4h^4}$. Also, a completely analogous computation shows that

$$R^{\gamma_r}_{5959} = R^{\gamma_r}_{6969} = R^{\gamma_r}_{7979} = R^{\gamma_r}_{8989} = \frac{v_r^2}{4v^4}. \tag{4-10}$$

By (5-18) we have that

$$\begin{aligned}\langle R^{\gamma_r}(X_1, X_2)X_1, X_2\rangle_{\gamma_r} &= \langle R^{\gamma_{r,\theta}}(X_1, X_2)X_1, X_2\rangle_{\gamma_{r,\theta}} - 3\langle A_{X_1}X_2, A_{X_1}X_2\rangle_{\gamma_r}\\ &= \langle R^{\gamma_{r,\theta}}(X_1, X_2)X_1, X_2\rangle_{\gamma_{r,\theta}} - \tfrac{3}{4}v_r^2,\end{aligned}$$

and thus

$$R^{\gamma_r}_{1212} = R^{\gamma_{r,\theta}}_{1212} - \frac{3v_r^2}{4h^4} = -\frac{4}{h^2} - \frac{3v_r^2}{4h^4} = R^{\gamma_r}_{3434}. \tag{4-11}$$

An identical argument shows that

$$R^{\gamma_r}_{5656} = R^{\gamma_r}_{7878} = \frac{4}{v^2} - \frac{3v_r^2}{4v^4}. \tag{4-12}$$

Since $A_{X_i}X_j = 0$ if $\{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$, the above argument also provides

$$R^{\gamma_r}_{ijij} = R^{\gamma_{r,\theta}}_{ijij} \quad \text{if } \{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}. \tag{4-13}$$

We now compute the mixed terms of R^{γ_r}:

$$\begin{aligned}\langle R^{\gamma_r}(X_1,X_2)X_3,X_4\rangle_{\gamma_r} &= \langle R^{\gamma_{r,\theta}}(X_1,X_2)X_3,X_4\rangle_{\gamma_{r,\theta}} - 2\langle A_{X_1}X_2, A_{X_3}X_4\rangle_{\gamma_r}\\ &= \langle R^{\gamma_{r,\theta}}(X_1,X_2)X_3,X_4\rangle_{\gamma_{r,\theta}} - \tfrac{1}{2}v_r^2,\end{aligned}$$

and therefore

$$R^{\gamma_r}_{1234} = R^{\gamma_{r,\theta}}_{1234} - \frac{v_r^2}{2h^4} = -\frac{2}{h^2} - \frac{v_r^2}{2h^4} = 2R^{\gamma_r}_{1324} = -2R^{\gamma_r}_{1423}. \tag{4-14}$$

Identically

$$R^{\gamma_r}_{5678} = R^{\gamma_{r,\theta}}_{5678} - \frac{v_r^2}{2v^4} = \frac{2}{v^2} - \frac{v_r^2}{2v^4} = 2R^{\gamma_r}_{5768} = -2R^{\gamma_r}_{5867}. \tag{4-15}$$

Now

$$\langle R^{\gamma_r}(X_1,X_2)X_5,X_6\rangle_{\gamma_r} = \langle R^{\gamma_{r,\theta}}(X_1,X_2)X_5,X_6\rangle_{\gamma_{r,\theta}} - 2\langle A_{X_1}X_2, A_{X_5}X_6\rangle_{\gamma_r} = -\tfrac{1}{2}v_r^2,$$

and hence

$$R^{\gamma_r}_{1256} = -\frac{v_r^2}{2h^2v^2} = R^{\gamma_r}_{1278} = R^{\gamma_r}_{3456} = R^{\gamma_r}_{3478}. \tag{4-16}$$

Finally, the same argument yields

$$R^{\gamma_r}_{1526} = R^{\gamma_r}_{1728} = R^{\gamma_r}_{3546} = R^{\gamma_r}_{3748} = -\frac{v_r^2}{4h^4v^4}, \tag{4-17}$$

$$R^{\gamma_r}_{1625} = R^{\gamma_r}_{1827} = R^{\gamma_r}_{3645} = R^{\gamma_r}_{3847} = \frac{v_r^2}{4h^4v^4}. \tag{4-18}$$

The final thing that needs to be done is to show that all other mixed terms are 0. Any mixed term contains 0, 1, or 2 vertical vectors (since this is just Y_9). A mixed term with two vertical vectors has the form $R^{\gamma_r}_{i9j9}$ with $i \neq j$. By (5-15) and since the T-tensor is identically 0 we have that

$$\langle R(X_i,X_9)X_j,X_9\rangle = \langle A_{X_i}X_9, A_{X_j}X_9\rangle = \langle f(r)X_i, g(r)X_j\rangle = 0$$

for some functions f and g of r. If there is one vertical vector, then the mixed term is 0 by (5-17) (again using the fact that the T-tensor is identically 0).

This just leaves the case of no vertical vectors. Recall that a pair of unit vectors (A,B) is a *holomorphic pair* if $JA = \pm B$. If the collection $\{Y_i,Y_j,Y_k,Y_\ell\}$ contains two holomorphic pairs, then we have seen above that this component of the curvature tensor is (potentially) nonzero. We need to show that it is 0 in all other cases.

The applicable formula for 0 vertical vectors is (5-18), which reduces to

$$\langle R^{\gamma_r}(X,Y)Z,Z'\rangle = -2\langle A_X Y, A_Z Z'\rangle + \langle A_Y Z, A_X Z'\rangle - \langle A_X Z, A_Y Z'\rangle. \tag{4-19}$$

But recall from above that $A_{X_i}X_j = 0$ if $\{i,j\} \notin \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$, that is, if (Y_i, Y_j) is not a holomorphic pair. Therefore, if the collection $\{Y_i, Y_j, Y_k, Y_\ell\}$ does not contain two holomorphic pairs, then every term on the right-hand side of (4-19) is 0. Hence, all remaining mixed terms are identically 0.

4.5 Curvature formulas for γ

Combining (4-9) through (4-18) with formulas (5-4) through (5-7) proves the following theorem.

Theorem 4.3 *In terms of the basis given in* (4-3), *the only independent nonzero components of the* (4, 0) *curvature tensor* R^γ *are given by the following formulas*, *where* $i \in \{1,2,3,4\}$, $k \in \{5,6,7,8\}$, $(i,j) \in \{(1,2),(3,4)\}$, *and* $(k,l) \in \{(5,6),(7,8)\}$:

$$\begin{aligned}
R^\gamma_{1212} &= R^\gamma_{3434} = -\left(\frac{h'}{h}\right)^2 - \frac{4}{h^2} - \frac{3v_r^2}{4h^4},\\
R^\gamma_{5656} &= R^\gamma_{7878} = -\left(\frac{v'}{v}\right)^2 + \frac{4}{v^2} - \frac{3v_r^2}{4v^4},\\
R^\gamma_{i9i9} &= -\frac{h'v_r'}{hv_r} + \frac{v_r^2}{4h^4},\\
R^\gamma_{k9k9} &= -\frac{v'v_r'}{vv_r} + \frac{v_r^2}{4v^4},\\
R^\gamma_{ikik} &= -\frac{h'v'}{hv},\\
R^\gamma_{1313} &= R^\gamma_{1414} = R^\gamma_{2323} = R^\gamma_{2424} = -\left(\frac{h'}{h}\right)^2 - \frac{1}{h^2},\\
R^\gamma_{5757} &= R^\gamma_{5858} = R^\gamma_{6767} = R^\gamma_{6868} = -\left(\frac{v'}{v}\right)^2 + \frac{1}{v^2},\\
R^\gamma_{i,10,i,10} &= -\frac{h''}{h},\\
R^\gamma_{k,10,k,10} &= -\frac{v''}{v},\\
R^\gamma_{9,10,9,10} &= -\frac{v_r''}{v_r},\\
R^\gamma_{1234} &= 2R^\gamma_{1324} = -2R^\gamma_{1423} = -\frac{2}{h^2} - \frac{v_r^2}{2h^4},\\
R^\gamma_{5678} &= 2R^\gamma_{5768} = -2R^\gamma_{5867} = \frac{2}{v^2} - \frac{v_r^2}{2v^4},\\
R^\gamma_{ijkl} &= 2R^\gamma_{ikjl} = -2R^\gamma_{iljk} = -\frac{v_r^2}{2h^2v^2},
\end{aligned}$$

$$R^{\gamma}_{i,j,9,10} = 2R^{\gamma}_{i,9,j,10} = -2R^{\gamma}_{i,10,j,9} = -\frac{v_r}{h^2}\left(\ln\frac{v_r}{h}\right)',$$
$$R^{\gamma}_{k,l,9,10} = 2R^{\gamma}_{k,9,l,10} = -2R^{\gamma}_{k,10,l,9} = -\frac{v_r}{v^2}\left(\ln\frac{v_r}{v}\right)'.$$

Unlike Section 3, this time is a much simpler exercise in hyperbolic trigonometric identities to check that, when $h = \cosh(r)$, $v = \sinh(r)$, and $v_r = \sinh(2r)$, the above formulas reduce to the constants in equations (4-4) through (4-8).

Remark 4.4 Here we explain how the above curvature formulas contain all of the formulas that arise in the case for $\mathbb{CH}^n \setminus \mathbb{CH}^m$ for generic n and m with $n > m$.[1] In general, one can write the complex hyperbolic metric $\boldsymbol{c}_n$ as

$$\boldsymbol{c}_n = \cosh^2(r)\boldsymbol{c}_m + \sinh^2(r)\boldsymbol{p}_{n-m-1} + \tfrac{1}{4}\sinh^2(2r)d\theta^2 + dr^2$$

and the corresponding warped-product metric as

$$\boldsymbol{\gamma}_n = h^2(r)\boldsymbol{c}_m + v^2(r)\boldsymbol{p}_{n-m-1} + \tfrac{1}{4}v_r^2(r)d\theta^2 + dr^2$$

where $\boldsymbol{p}_{n-m-1}$ is the complex projective metric on $\mathbb{CP}^{n-m-1}$ and $\frac{\partial}{\partial\theta}$ is defined in the same manner as above. Choose an orthonormal basis $\{X_i\}_{i=1}^{2m}$ for the $\mathbb{CH}^m$ factor in such a way that $X_i = JX_{i+1}$ for all odd i with $1 \le i \le 2m$. Analogously choose an orthonormal basis $\{X_j\}_{j=2m+1}^{2n-2}$ of the $\mathbb{CP}^{n-m-1}$ factor. We then define an orthonormal basis $\{Y_i\}_{i=1}^{2n}$ exactly as in (4-3).

The nonzero components of the curvature tensor R^{γ_n} for the base $\mathbb{CH}^m$ are of the form $R^{\gamma_n}_{ijij}$ for $1 \le i, j \le 2m$ or mixed terms of the form $R^{\gamma_n}_{i,i+1,j,j+1}$ for i and j odd (or permutations of these indices). The curvature formulas from these components are encoded in the formulas for R^{γ}_{1212}, R^{γ}_{1313}, and R^{γ}_{1234} above. The analogous curvature formulas for the $\mathbb{CP}^{n-m-1}$ component are contained in the above formulas for R^{γ}_{5656}, R^{γ}_{5757}, and R^{γ}_{5678}. If $1 \le i \le 2m$ and $2m+1 \le k \le 2n-2$, formulas for all components of the form $R^{\gamma_n}_{ikik}$ are given by the same indices above. The mixed terms between the $\mathbb{CH}^m$ and $\mathbb{CP}^{n-m-1}$ components are given by the R^{γ}_{ijkl} formula. Finally, all of the formulas above containing either a "9" or a "10" give the formulas for all components of R^{γ_n} that contain either Y_{2n-1} or Y_{2n}.

4.6 The exceptional case $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2}$

Notice that, when $k = n-2$, there are no sectional curvatures of γ of the form

$$-\left(\frac{v'}{v}\right)^2 + \left(\frac{1}{v}\right)^2.$$

That is because we can write $\mathbb{CH}^n \setminus \mathbb{CH}^{n-2} \cong \mathbb{CH}^{n-2} \times \mathbb{S}^3 \times (0,\infty)$, and $\mathbb{CP}^1$ (the base of the Hopf fibration) has constant holomorphic curvature 4. So the purpose of this subsection is to prove the following:

[1] We use $\mathbb{CH}^m$ instead of $\mathbb{CH}^k$ since k is also used as an index in Theorem 4.3.

Lemma 4.5 *There do not exist functions h, v, and v_r that, when inserted into* (4-2), *yield a complete finite volume Riemannian metric on $\mathbb{C}\mathbb{H}^n \setminus \mathbb{C}\mathbb{H}^{n-2}$ with nonpositive sectional curvature, and which interpolate to the values*

$$h(r) = \cosh(r), \quad v(r) = \sinh(r), \quad v_r(r) = \sinh(2r)$$

near $r = 0$.

Proof Consider a finite volume manifold $M \setminus N$ where (M, N) is modeled on $(\mathbb{C}\mathbb{H}^n, \mathbb{C}\mathbb{H}^{n-2})$. The ambient complex hyperbolic metric on $M \setminus N$ will not be complete since we have removed a copy of N. To make the metric complete at N we need to turn the normal $\mathbb{S}^1$-bundle over N into a cusp of $M \setminus N$. Geometrically, thinking of the cusp occurring as r approaches $-\infty$, this is equivalent to defining the functions h, v, and v_r over $\mathbb{R}$ instead of $[0, \infty)$. We then need these functions to agree with the values in Lemma 4.5 for all values of r larger than the normal injectivity radius of N.

The functions h, v, and v_r need to be positive for the metric to be Riemannian, and they need to be nondecreasing for there to be any chance of nonpositive curvature. Since these functions are positive, nondecreasing, and must eventually agree with the values in Lemma 4.5, we must have that all three of the following limits are zero:

$$\lim_{r \to -\infty} h', v', v_r' = 0.$$

Lastly, even though we will not need this below, note that in order to have finite volume, at least one of h, v, and v_r must also approach 0 as $r \to -\infty$.

Now, from the formula for the R_{5656} term in Theorem 4.3 we must have that

$$\frac{4 - (v')^2}{v^2} - \frac{3v_r^2}{4v^4} \leq 0 \iff \frac{16 - 4(v')^2}{3} \leq \left(\frac{v_r}{v}\right)^2. \tag{4-20}$$

In particular, since $(v') \to 0$ as $r \to -\infty$, we see that a necessary requirement for nonpositive curvature is that

$$\lim_{r \to -\infty} \frac{v_r}{v} > 1. \tag{4-21}$$

From the formula for the R_{k9k9} term in Theorem 4.3 we must have that

$$-\frac{v'v_r'}{vv_r} + \frac{v_r^2}{4v^4} \leq 0 \implies \frac{3v_r^2}{4v^4} \leq \frac{3v'v_r'}{vv_r}. \tag{4-22}$$

Comparing (4-20) and (4-22), we see that

$$\frac{4 - (v')^2}{v^2} \leq \frac{3v'v_r'}{vv_r} \implies 4 - (v')^2 \leq (3v'v_r') \cdot \frac{v}{v_r}$$

is also a necessary requirement for nonpositive curvature. But as $r \to -\infty$, we know that $4 - (v')^2 \to 4$ and $3v'v_r' \to 0$. Thus, we must have that

$$\lim_{r \to -\infty} \frac{v}{v_r} = \infty \implies \lim_{r \to -\infty} \frac{v_r}{v} = 0. \tag{4-23}$$

Equations (4-21) and (4-23) provide a contradiction, proving the lemma. □

5 Preliminaries

5.1 Formula for the curvature tensor of $\mathbb{CH}^n$ in terms of the complex structure J

The components of the (4,0) curvature tensor of the complex hyperbolic metric g can be expressed in terms of g and the complex structure J. The following formula can be found in [8] or in [2, Section 5] (recall Remark 1.2 from the introduction). In this formula X, Y, Z, and W are arbitrary vector fields:

$$\begin{aligned}\langle R^g(X,Y)Z,W\rangle_g &= \langle X,W\rangle_g\langle Y,Z\rangle_g - \langle X,Z\rangle_g\langle Y,W\rangle_g \\ &\quad + \langle X,JW\rangle_g\langle Y,JZ\rangle_g - \langle X,JZ\rangle_g\langle Y,JW\rangle_g + 2\langle X,JY\rangle_g\langle W,JZ\rangle_g.\end{aligned} \tag{5-1}$$

5.2 Koszul's formula for the Levi-Civita connection

Let X, Y, and Z denote vector fields on a Riemannian manifold (M,g). The following is the well-known "Koszul formula" for the values of the Levi-Civita connection ∇ (which can be found in [4, page 55]):

$$\langle \nabla_Y X, Z\rangle_g = \tfrac{1}{2}\big(X\langle Y,Z\rangle_g + Y\langle Z,X\rangle_g - Z\langle X,Y\rangle_g - \langle [X,Z],Y\rangle_g - \langle [Y,Z],X\rangle_g - \langle [X,Y],Z\rangle_g\big). \tag{5-2}$$

In this paper we will usually be considering an orthonormal frame (Y_i). In this setting we know that $\langle Y_i, Y_j\rangle_g = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta. Therefore the first three terms on the right hand side of formula (5-2) are all zero. Thus, in an orthonormal frame, formula (5-2) reduces to

$$\langle \nabla_Y X, Z\rangle_g = -\tfrac{1}{2}\big(\langle [X,Z],Y\rangle_g + \langle [Y,Z],X\rangle_g + \langle [X,Y],Z\rangle_g\big). \tag{5-3}$$

5.3 General curvature formulas for warped product metrics

The curvature formulas below, which were worked out by Belegradek in [1] and stated in [2, Appendix B], apply to metrics of the form $g = g_r + dr^2$ on manifolds of the form $E \times I$ where I is an open interval and E is a manifold. The formulas are true provided that for each point $q \in E$ there exists a neighborhood U_q in E of q and a local frame $\{X_i\}$ defined on U_q such that, for any $r \in (0,\infty)$, the collection $\{X_i\}$ is g_r-orthogonal. So, as r varies, the g-lengths of vectors may change, but g-orthogonality does not. Such a family of metrics (E, g_r) is called *simultaneously diagonalizable*. Let

$$h_i(r) := \sqrt{g_r(X_i, X_i)}.$$

Then the local frame $\{Y_i\}$ defined by

$$Y_i = \frac{1}{h_i} X_i$$

is a g_r-orthonormal frame on U_q for any value of r. We then have the following formulas for the (4,0) curvature tensor R^g in terms of the (4,0) curvature tensor R^{g_r}, the collection $\{h_i\}$, and the Lie brackets $[Y_i, Y_j]$. Note that $\langle\,\cdot\,,\cdot\,\rangle$ is used to denote the metric g and $\partial r = \frac{\partial}{\partial r}$:

(5-4) $\langle R^g(Y_i,Y_j)Y_i,Y_j\rangle = \langle R^{g_r}(Y_i,Y_j)Y_i,Y_j\rangle - \frac{h_i'h_j'}{h_ih_j}$,

(5-5) $\langle R^g(Y_i,Y_j)Y_k,Y_l\rangle = \langle R^{g_r}(Y_i,Y_j)Y_k,Y_l\rangle$ if $\{i,j\}\neq\{k,l\}$,

(5-6) $\langle R^g(Y_i,\partial r)Y_i,\partial r\rangle = -\frac{h_i''}{h_i}$ and $\langle R^g(Y_i,\partial r)Y_j,\partial r\rangle = 0$ if $i\neq j$,

(5-7) $2\langle R^g(\partial r,Y_i)Y_j,Y_k\rangle = \langle [Y_i,Y_k],Y_j\rangle\left(\ln\frac{h_j}{h_k}\right)' + \langle [Y_j,Y_i],Y_k\rangle\left(\ln\frac{h_k}{h_j}\right)' + \langle [Y_j,Y_k],Y_i\rangle\left(\ln\frac{h_i^2}{h_jh_k}\right)'$.

5.4 The Nijenhuis tensor

In Sections 3 and 4 we explicitly dealt with $\mathbb{CH}^n$. Since the almost complex structure on $\mathbb{CH}^n$ is integrable, we have that the *Nijenhuis tensor* is identically equal to zero. Explicitly, for any vector fields X and Y on $\mathbb{CH}^n$, we have that

$$0 = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \tag{5-8}$$

where J denotes the complex structure on $\mathbb{CH}^n$.

5.5 The A and T tensors of a Riemannian submersion

All of this subsection comes from [3, pages 236–241], but where we make small changes in the notation to fit the notation in this paper. The original source of most of this material is [12].

Let (M,g) and $(B,\check{g})$ be two Riemannian manifolds and $\phi\colon M\to B$ a smooth submersion. Let $p\in M$ and $q=\phi(p)$. By F_p we denote the fiber $\phi^{-1}(q)$. The *vertical distribution* at p, denoted by $\mathcal{V}_p$, is the tangent space to F_p within T_pM. The *horizontal distribution* $\mathcal{H}_p$ is the orthogonal complement to $\mathcal{V}_p$ in T_pM. Note that, by construction, the vertical distribution is always integrable whereas the horizontal distribution may or may not be integrable.

The vertical distribution is equal to the kernel of the map $\phi_*\colon T_pM\to T_qB$ and thus ϕ_* induces an isomorphism from $\mathcal{H}_p$ to T_qB. If this map is an isometry then we call ϕ a *Riemannian submersion*. If B is a submanifold of M and the Riemannian submersion ϕ is the identity on B (which implies that $\check{g}$ is just g restricted to B), we call g a *Riemannian submersion metric*.

Let $v\in T_pM$. The vector $\mathcal{V}v\in T_pM$ denotes the projection of v onto $\mathcal{V}_p$ and similar for $\mathcal{H}v$. Let D denote the Levi-Civita connection of the Riemannian metric g on M.

For the remainder of this subsection, U, V, and W will always denote vertical vector fields in T_pM while X, Y, and Z will denote horizontal vector fields. With this notation, we are now prepared to define the T and A tensors.

Definition 5.1 [3, Definition 9.17] The $(2,1)$ tensor field T on M is defined by

$$T_{E_1}E_2 = \mathcal{H}D_{\mathcal{V}E_1}(\mathcal{V}E_2) + \mathcal{V}D_{\mathcal{V}E_1}(\mathcal{H}E_2)$$

where E_1 and E_2 are vector fields on M.

Remark 5.2 Notice that $T_U V$ gives the second fundamental form for the fiber. From this observation one sees that, if the fiber is totally geodesic, then the T-tensor is identically zero.

Definition 5.3 [3, Definition 9.20] The $(2,1)$ tensor field A on M is defined by

$$A_{E_1} E_2 = \mathcal{H} D_{\mathcal{H} E_1}(\mathcal{V} E_2) + \mathcal{V} D_{\mathcal{H} E_1}(\mathcal{H} E_2)$$

where E_1 and E_2 are vector fields on M.

The following are properties of the A-tensor that follow from the definition and Theorem 5.4 below:

$$A_U X = A_U V = 0, \tag{5-9}$$

$$A_X U = \mathcal{H} D_X U \quad \text{and} \quad A_X Y = \mathcal{V} D_X Y, \tag{5-10}$$

$$A_X Y = -A_Y X, \tag{5-11}$$

$$A_X \text{ is alternating, so } g(A_X Y, U) = -g(A_X U, Y). \tag{5-12}$$

We will need the following theorem in Section 4.4, whose proof is in [3].

Theorem 5.4 [3, Proposition 9.24] *For all horizontal vector fields X and Y,*

$$A_X Y = \tfrac{1}{2}\mathcal{V}[X, Y].$$

Thus, the A-tensor measures the obstruction to integrability of the horizontal distribution.

The last theorem that we need from [3] contains formulas for the components of the sectional curvature tensor R of g with respect to the A and T-tensors.

Theorem 5.5 [3, Theorem 9.28] *Let $\phi\colon (M, g) \to (B, \check{g})$ be a Riemannian submersion. Let R be the curvature tensor with respect to g, $\check{R}$ the curvature tensor with respect to $\check{g}$, and $\hat{R}$ the curvature tensor of g restricted to each vertical fiber. We then have*

$$g(R(U,V)W,W') = g(\hat{R}(U,V)W,W') - g(T_U W, T_V W') + g(T_V W, T_U W'), \tag{5-13}$$

$$g(R(U,V)W,X) = g((D_V T)_U W, X) - g((D_U T)_V W, X), \tag{5-14}$$

$$g(R(X,U)Y,V) = g((D_X T)_U V, Y) - g(T_U X, T_V Y) + g((D_U A)_X Y, V) + g(A_X U, A_Y V), \tag{5-15}$$

$$\begin{aligned} g(R(U,V)X,Y) = {} & g((D_U A)_X Y, V) - g((D_V A)_X Y, U) + g(A_X U, A_Y V) \\ & - g(A_X V, A_Y U) - g(T_U X, T_V Y) + g(T_V X, T_U Y), \end{aligned} \tag{5-16}$$

$$g(R(X,Y)Z,U) = g((D_Z A)_X Y, U) + g(A_X Y, T_U Z) - g(A_Y Z, T_U X) - g(A_Z X, T_U Y). \tag{5-17}$$

$$g(R(X,Y)Z,Z') = g(\check{R}(X,Y)Z,Z') - 2g(A_X Y, A_Z Z') + g(A_Y Z, A_X Z') - g(A_X Z, A_Y Z'). \tag{5-18}$$

6 Computations for the Lie brackets for $\mathbb{CH}^3 \setminus \mathbb{H}^3$

The whole purpose of this section is to prove Theorem 3.2.

Proof of Theorem 3.2 There are $5 \times 10 = 50$ structure constants to compute from (3-9). From (3-10) we know that $c_{45}^4 = 0$ and $c_{45}^5 = -\cot(\theta)$, leaving 48 unknown structure constants.

We can combine formula (5-7) with equations (3-5) through (3-8) to compute many of the constants. As a first example, note that

$$0 = 2R_{6131}^{\boldsymbol{c}_3} = 0 + 2\langle [Y_3, Y_1], Y_1\rangle_{\boldsymbol{c}_3}\left(\ln \frac{h}{h_r}\right)' = -\frac{2c_{13}^1}{\cosh(2r)}\left(\ln \frac{\cosh(r)}{\cosh(2r)}\right)',$$

and thus $c_{13}^1 = 0$. We can analogously show

$$0 = c_{14}^1 = c_{15}^1 = c_{23}^2 = c_{24}^2 = c_{25}^2 = c_{13}^3 = c_{23}^3 = c_{34}^3 = c_{35}^3 = c_{14}^4 = c_{24}^4 = c_{34}^4 = c_{15}^5 = c_{25}^5 = c_{35}^5.$$

This narrows us down to 32 unknown constants.

Continuing with the same formula and equations, we have that

$$0 = 2R_{6145}^{\boldsymbol{c}_3} = 0 + 0 + \langle [Y_4, Y_5], Y_1\rangle_{\boldsymbol{c}_3}\left(\ln \frac{\cosh^2(r)}{\sinh^2(r)}\right)' = \frac{c_{45}^1 h}{v^2}\left(\ln \frac{\cosh^2(r)}{\sinh^2(r)}\right)',$$

and therefore $c_{45}^1 = 0$. Analogously, $c_{45}^2 = c_{45}^3 = c_{12}^3 = c_{12}^4 = c_{12}^5 = 0$. This reduces us to 26 unknowns. But we can also use the same curvature formulas here, but with the indices permuted, to derive some simple equations relating some of the constants. For example,

$$\begin{aligned} 0 = 2R_{6415}^{\boldsymbol{c}_3} &= 0 + \langle [Y_1, Y_4], Y_5\rangle_{\boldsymbol{c}_3}\left(\ln \frac{\sinh(r)}{\cosh(r)}\right)' + \langle [Y_1, Y_5], Y_4\rangle_{\boldsymbol{c}_3}\left(\ln \frac{\sinh(r)}{\cosh(r)}\right)' \\ &= \frac{1}{h}(c_{14}^5 + c_{15}^4)\left(\ln \frac{\sinh(r)}{\cosh(r)}\right)', \end{aligned}$$

and thus $c_{14}^5 = -c_{15}^4$. Analogously, we have the identities

$$c_{24}^5 = -c_{25}^4, \quad c_{34}^5 = -c_{35}^4, \quad c_{13}^2 = -c_{23}^1, \quad c_{14}^2 = -c_{24}^1, \quad c_{15}^2 = -c_{25}^1.$$

Combining formula (5-7) with the fact that $2R_{6413}^{\boldsymbol{c}_3} = 2$ gives that

$$\begin{aligned} 2 &= \langle [Y_4, Y_3], Y_1\rangle_{\boldsymbol{c}_3}\left(\ln \frac{h}{h_r}\right)' + \langle [Y_1, Y_4], Y_3\rangle_{\boldsymbol{c}_3}\left(\ln \frac{h_r}{h}\right)' + \langle [Y_1, Y_3], Y_4\rangle_{\boldsymbol{c}_3}\left(\ln \frac{v^2}{hh_r}\right)' \\ &= -\frac{c_{34}^1 h}{h_r v}\left(\ln \frac{h}{h_r}\right)' + \frac{c_{14}^3 h_r}{hv}\left(\ln \frac{h_r}{h}\right)' + \frac{c_{13}^4 v}{hh_r}\left(\ln \frac{v^2}{hh_r}\right)' \\ &= \left(\frac{-c_{34}^1 \cosh(r)}{\cosh(2r)\sinh(r)} - \frac{c_{14}^3 \cosh(2r)}{\cosh(r)\sinh(r)}\right)(\tanh(r) - 2\tanh(2r)) \\ &\qquad + \frac{c_{13}^4 \sinh(r)}{\cosh(r)\cosh(2r)}(2\coth(r) - \tanh(r) - 2\tanh(2r)). \end{aligned} \tag{6-1}$$

It is an exercise in hyperbolic trigonometric identities (or one can consult [9, (5.9)]) to check that the values

$$c_{13}^4 = 1, \quad c_{14}^3 = 1, \quad c_{34}^1 = -1 \tag{6-2}$$

satisfy (6-1). In theory, there could be other solutions for these structure constants. But any potential solution must hold for all positive values of r. Plugging in the values $r = 2$, 3, and 4 yields the following three equations (constants rounded to four decimal places):

$$\begin{aligned} 2 &= 0.0393c_{34}^1 + 2.0707c_{14}^3 - 0.0313c_{13}^4, \\ 2 &= 0.005c_{34}^1 + 2.0099c_{14}^3 - 0.0049c_{13}^4, \\ 2 &= 0.0007c_{34}^1 + 2.0013c_{14}^3 - 0.0007c_{13}^4. \end{aligned}$$

One can check that these three equations are independent and so the solutions (6-2) are unique (note that the values in (6-2) do not perfectly satisfy the above three equations due to roundoff error). In exactly the same manner we can use $R^{\boldsymbol{c}_3}_{6253}$ with [9, (5.9)] to compute

$$c_{23}^5 = 1, \quad c_{25}^3 = 1, \quad c_{35}^2 = -1.$$

This leaves 20 unknowns together with the 6 identities listed above. Now, using $R^{\boldsymbol{c}_3}_{6135}$, we have that

$$\begin{aligned} 0 &= \langle [Y_1, Y_5], Y_3 \rangle_{\boldsymbol{c}_3} \left(\ln \frac{h_r}{v} \right)' + \langle [Y_3, Y_1], Y_5 \rangle_{\boldsymbol{c}_3} \left(\ln \frac{v}{h_r} \right)' + \langle [Y_3, Y_5], Y_1 \rangle_{\boldsymbol{c}_3} \left(\ln \frac{h^2}{h_r v} \right)' \\ &= \left(\frac{c_{15}^3 \cosh(2r)}{\cosh(r)\sinh(r)} + \frac{c_{13}^5 \sinh(r)}{\cosh(r)\cosh(2r)} \right) \Big(2\tanh(2r) - \coth(r) \Big) \\ &\qquad + \frac{c_{35}^1 \cosh(r)}{\cosh(2r)\sinh(r)} (2\tanh(r) - 2\tanh(2r) - \coth(r)). \end{aligned}$$

Just like for (6-1) above, one can obtain an independent system of equations by inserting different values for r into the above equation. One can check that the only solution to this equation is $c_{13}^5 = c_{15}^3 = c_{35}^1 = 0$. Analogously, we can use $R^{\boldsymbol{c}_3}_{6234}$ to show that $c_{23}^4 = c_{24}^3 = c_{34}^2 = 0$. These equations reduce us to 14 unknowns.

This is as much information as we can gain from formula (5-7). So we next turn to the Nijenhuis tensor (5-8). First applying this to (Y_1, Y_2), we have

$$\begin{aligned} 0 &= [Y_1, Y_2] - J[Y_4, Y_2] - J[Y_1, Y_5] - [Y_4, Y_5] \\ &= \frac{1}{h}(c_{12}^1 Y_1 + c_{12}^2 Y_2) + J\left(\frac{c_{24}^1}{v} Y_1 + \frac{c_{24}^5}{h} Y_5 \right) - J\left(\frac{c_{15}^2}{v} Y_2 + \frac{c_{15}^4}{h} Y_4 \right) + \frac{1}{v}\cot(\theta) Y_5 \\ &= \frac{1}{h}(c_{12}^1 - c_{15}^4) Y_1 + \frac{1}{h}(c_{12}^2 + c_{24}^5) Y_2 - \frac{c_{24}^1}{v} Y_4 + \frac{1}{v}(c_{15}^2 + \cot(\theta)) Y_5. \end{aligned}$$

Therefore,

$$c_{24}^1 = 0 = -c_{14}^2, \quad c_{15}^2 = -\cot(\theta) = -c_{25}^1, \quad c_{12}^1 = c_{15}^4 = -c_{14}^5, \quad c_{12}^2 = -c_{24}^5.$$

We can also apply the Nijenhuis tensor to the pairs (Y_1, Y_3) and (Y_2, Y_3), but these are much less productive. These applications only give us the pair of identities

$$c_{13}^2 = -c_{34}^5, \quad c_{23}^1 = -c_{35}^4,$$

the former of which comes from the pair (Y_1, Y_3), and the latter from the pair (Y_2, Y_3).

At this stage, we have reduced our 10 Lie brackets as follows:

$$\begin{aligned} [X_1,X_2] &= c_{12}^1 X_1 + c_{12}^2 X_2, & [X_1,X_3] &= c_{13}^2 X_2 + X_4,\\ [X_1,X_4] &= X_3 - c_{12}^1 X_5, & [X_1,X_5] &= -\cot(\theta) X_2 + c_{12}^1 X_4,\\ [X_2,X_3] &= -c_{13}^2 X_1 + X_5, & [X_2,X_4] &= -c_{12}^2 X_5,\\ [X_2,X_5] &= \cot(\theta) X_1 + X_3 + c_{12}^2 X_4, & [X_3,X_4] &= -X_1 - c_{13}^2 X_5,\\ [X_3,X_5] &= -X_2 + c_{13}^2 X_4, & [X_4,X_5] &= -\cot(\theta) X_5. \end{aligned}$$

Notice that, using the known identities, we can reduce the system to three unknowns: c_{12}^1, c_{12}^2, and c_{13}^2. All that is left is to show that $c_{12}^1 = \pm 1$, $c_{12}^2 = 0$, and $c_{13}^2 = \mp\cot(\theta)$.

At this point we have exhausted all of our "easy" options. The only way to obtain new relationships between the structure constants is to compute new components of $R^{\boldsymbol{c}_3}$. To do this, one needs to first use (5-2) with the values for the Lie brackets given above to compute the Levi-Civita connection ∇ compatible with $\boldsymbol{c}_3$. Of course, these formulas will contain the constants c_{12}^1, c_{12}^2, and c_{13}^2. But when the correct values for these constants are inserted, these formulas will reduce to those of Theorem 3.3. Then once one has computed ∇, they can use those values to compute the components of $R^{\boldsymbol{c}_3}$.

The first component that will be useful is $R^{\boldsymbol{c}_3}_{1212}$:

$$\begin{aligned} -1 = R^{\boldsymbol{c}_3}_{1212} &= \langle \nabla_{Y_2}\nabla_{Y_1} Y_1 - \nabla_{Y_1}\nabla_{Y_2} Y_1 + \nabla_{[Y_1,Y_2]} Y_1, Y_2\rangle_{\boldsymbol{c}_3}\\ &= \left\langle \nabla_{Y_2}\left(\frac{-c_{12}^1}{\cosh(r)} Y_2 - \tanh(r) Y_6\right) - \nabla_{Y_1}\left(\frac{-c_{12}^2}{\cosh(r)} Y_2\right) + \frac{c_{12}^1}{\cosh(r)} \nabla_{Y_1} Y_1 + \frac{c_{12}^2}{\cosh(r)} \nabla_{Y_2} Y_1, Y_2\right\rangle_{\boldsymbol{c}_3}\\ &= -\frac{\sinh^2(r)}{\cosh^2(r)} - \frac{((c_{12}^1)^2 + (c_{12}^2)^2)}{\cosh^2(r)} \implies (c_{12}^1)^2 + (c_{12}^2)^2 = 1. \end{aligned}$$

The next component that we use is $R^{\boldsymbol{c}_3}_{1512}$. We will skip the details and just note that

$$0 = R^{\boldsymbol{c}_3}_{1512} = \frac{c_{12}^2}{\sinh(r)\cosh(r)} \cdot \cot(\theta),$$

which implies that $c_{12}^2 = 0$. Combining this with the first equation shows that $c_{12}^1 = \pm 1$. Finally, to compute c_{13}^2 we use $R^{\boldsymbol{c}_3}_{1412}$:

$$0 = R^{\boldsymbol{c}_3}_{1412} = \frac{-c_{13}^2}{\sinh(r)\cosh(r)} - \frac{c_{12}^1}{\sinh(r)\cosh(r)} \cdot \cot(\theta).$$

Therefore,

$$c_{13}^2 = -(\pm 1)\cot(\theta) = \mp\cot(\theta). \qquad \square$$

Acknowledgements

The author would like to thank J F Lafont for both suggesting this problem and for many helpful discussions about the work contained in this paper. The author would also like to thank I Belegradek, J Meyer, and

B Tshishiku for various comments which aided in this research. Finally, the author is grateful for the excellent work done by the referee. Many explanations and notational conventions in this paper were much improved by the referee's report.

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Received: 14 September 2023 Revised: 11 July 2024

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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