AGT

Algebraic & Geometric Topology

Volume 25 (2025)

On the structure of the $RO(G)$-graded homotopy of $H\underline{M}$ for cyclic p-groups

Igor Sikora
Guoqi Yan

msp

msp

Algebraic & Geometric Topology **25**:5 (2025) 2933–2980
DOI: 10.2140/agt.2025.25.2933
Published: 17 September 2025

On the structure of the $RO(G)$-graded homotopy of $H\underline{M}$ for cyclic p-groups

IGOR SIKORA
GUOQI YAN

We study the structure of the $RO(G)$-graded homotopy Mackey functors of any Eilenberg–MacLane spectrum $H\underline{M}$ for G a cyclic p-group. When $\underline{R}$ is a Green functor, we define orientation classes u_V for $H\underline{R}$ and deduce a generalized gold relation. We deduce the a_V, u_V-isomorphism regions of the $RO(G)$-graded homotopy Mackey functors and prove two induction theorems. As applications, we compute the positive cone of $H\underline{\mathbb{A}}$, as well as the positive and negative cones of $H\underline{\mathbb{Z}}$. The latter two cones are essential to the slice spectral sequences of $MU^{((C_{2^n}))}$ and its variants.

55Q91

1. Introduction 2933
2. Basics 2939
3. Vanishing, a, u-isomorphism regions and the gold relation 2953
4. Induction theorems 2963
5. The positive cone of $H\underline{M}$ for arbitrary $\underline{M}$ 2966
6. The negative cone of $H\underline{M}$ for arbitrary $\underline{M}$ 2973
References 2979

1 Introduction

Let G be a finite group. In G-equivariant topology the role of ordinary cohomology is played by *Bredon cohomology*. These theories are easy to define, but making computations in them is more complicated than in their nonequivariant analogues. This comes mainly from two reasons. Firstly, Bredon theories take coefficients in diagrams of abelian groups, indexed over subgroups of G. Secondly, rather than graded over integers, Bredon theories are naturally graded over the ring of representations of G — they are $RO(G)$-graded.

These two reasons are connected. As shown in [16], a $\mathbb{Z}$-graded Bredon theory extends to $RO(G)$-graded theory if and only if the coefficients are in the form of a *Mackey functor*. As in nonequivariant topology,

© 2025 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

a Bredon theory with coefficients in a Mackey functor $\underline{M}$ is represented by the *Eilenberg–MacLane spectrum* $H\underline{M}$. Spectra of this form are ubiquitous in equivariant homotopy theory.

For instance, in the groundbreaking work of Hill, Hopkins and Ravenel [11] on the Kervaire invariant one problem, the authors studied the C_8-equivariant spectrum $MU^{((C_8))} = N_{C_2}^{C_8} MU_{\mathbb{R}}$, where $N_{C_2}^{C_8}$ is usually referred to as the HHR norm functor. With a suitably chosen class $D \in \pi_{19\rho_8}^{C_8} MU_{\mathbb{R}}$, the spectrum $\Omega = (D^{-1} MU^{((C_8))})^{C_8}$ satisfies the periodicity, gap and detection theorems, which makes it a powerful tool in detecting the Kervaire invariant one elements in the stable stems. The gap theorem is deduced from a Bredon cohomology computation,

$$H_G^*(S^{m\rho_{C_{2^k}}}; \underline{\mathbb{Z}}).$$

Hahn and Shi [10], showed that at the prime 2, all of the Morava E-theories receive real-orientations

$$MU_{\mathbb{R}} \to E_n$$

which lifts to G-equivariant maps $N_{C_2}^G MU_{\mathbb{R}} \to E_n$, where G is any finite subgroup of the Morava stabilizer group that contains C_2. With a suitably chosen class $D \in \pi_{*\rho_G}^G N_{C_2}^G MU_{\mathbb{R}}$, this orientation admits a factorization

$$\begin{array}{ccc} N_{C_2}^G MU_{\mathbb{R}} & \longrightarrow & E_n \\ \downarrow & \nearrow & \\ D^{-1} N_{C_2}^G MU_{\mathbb{R}} & & \end{array}$$

which turns the computations in chromatic homotopy theory to computations of the two equivariant spectra on the left. The main tool for the equivariant computations is the slice spectral sequence first invented by Dugger [3] and utilized to a great extent by Hill, Hopkins and Ravenel [11].

By the slice theorem of [11], the slices of $MU^{((C_{2^k}))}$ are of the forms

$$H\underline{\mathbb{Z}} \wedge C_{2^k +} \wedge_H S^{m\rho_H}, \quad e \neq H \subset C_{2^k},\ m \geq 0.$$

The trick of Hill, Hopkins and Ravenel [13, Corollary 10.4] can be used to prove that the slices of $D^{-1} N_{C_2}^G MU_{\mathbb{R}}$ are again of the above form, except that in this case, we allow m to be negative integers. Thus the inputs of the $RO(G)$-graded slice spectral sequences of the periodic $D^{-1} MU^{((C_{2^k}))}$ as well as the periodic versions of quotients of $MU^{((C_{2^k}))}$ are given by the positive and negative cones of the $RO(G)$-graded homotopy groups of $H\underline{\mathbb{Z}}$, ie the areas corresponding to homology and cohomology of actual representation spheres. Since the multiplicative structure of the slice spectral sequence is indispensable in determining the differentials, and the two kinds of most important classes, the Euler and orientation classes, are not in regular-representation-graded homotopy, knowing the entire positive cone and negative cone structure is essential even for computing the $\mathbb{Z}$-graded homotopy groups of $D^{-1} MU^{((C_{2^k}))}$.

The main goal of this paper is determining the structure of $H\underline{M}_\star^G$ when G is a cyclic group of a prime power order. As applications, we utilize our structural theorems to determine the positive and negative

cone of $H\underline{\mathbb{Z}}$, as well as the positive cone of $H\underline{\mathbb{A}}$. By working p-locally, it suffices to concentrate on the prime 2 as we will explain in Section 2. So for the rest of the introduction, $G = C_{2^k}$.

Cellular chains for representation spheres

One of the tools we use for the calculations of $RO(G)$-graded abelian group structure are the cellular chain and cochain complexes of representation spheres. We give explicit descriptions of differentials in these complexes for any representation V and coefficients $\underline{M}$ in Theorem 2.6, which can be summarized as follows:

Theorem *Let $G = C_{2^k}$, the cellular chain $C^G_*(S^V; \underline{M})$ and cochain $C^*_G(S^V; \underline{M})$ for the representation sphere S^V are completely determined for any G-Mackey functor $\underline{M}$ and follow from its structure.*

This theorem is a wide extension of the result in Section 3 of [13], which computed these differentials for the constant Mackey functor $\underline{\mathbb{Z}}$. Having this description we can compute two important entries: $H\underline{M}_{|V|-V}$ for V orientable and $H\underline{M}_{-V}$. The first one is equal to the top homology group of S^V and is computed in Proposition 2.23:

Proposition *The Mackey functor structure of $H\underline{M}_{|V|-V}$ is given as*

$$H\underline{M}_{|V|-V}(H) = \begin{cases} \underline{M}(G_V)^H & \text{if } G_V \subset H, \\ \underline{M}(H) & \text{otherwise.} \end{cases}$$

The structure maps are induced from $\underline{M}$ and from the fixed point Mackey functor of the G-module $\underline{M}(G_V)$.

An analogous computation for nonorientable V is given in Propositions 5.1 and 5.2. As noted above, we also compute $H\underline{M}_{-V}$, which is equal to the bottom homology group of the S^V. This is done in Proposition 2.19.

a_λ-periodicity

Let $S^0 \to S^V$ be the inclusion and denote the corresponding class in $\pi^G_{-V}(S^0)$ by a_V, as well as its Hurewicz image in $H\underline{R}^G_{-V}$ for any Green functor $\underline{R}$. For $G = C_{2^k}$, denote by λ its faithful irreducible representation. Then multiplication by a_λ in $H\underline{M}^G_\star$ together with the induction method allows us to reduce computations to groups graded over representations with the total degree -1, 0 and 1. We first note that we can easily describe multiplication by this element using Propositions 3.7 and 3.9, which can be summarized as follows:

Proposition *The multiplication map $a_\lambda \colon H\underline{M}^G_V \to H\underline{M}^G_{V-\lambda}$ is an isomorphism unless the total degree of V is either 0, 1 or 2. Moreover, if the total degree of V is equal to zero then a_λ is an epimorphism, and if the total degree of V is 2 then a_λ is a monomorphism.*

The implication of this proposition is strong: most classes in $H\underline{M}^G_\star$ are infinitely divisible by a_λ, or support infinite a_λ-towers. The Mackey functor structure of $H\underline{M}_V$ with $|V| = 0$ determines the Mackey functors $H\underline{M}_{V+\lambda}$ and $H\underline{M}_{V-\lambda}$ (for a precise statement, see Proposition 3.9). This allows us to concentrate on the degrees where $|V| = -1, 0, 1$.

Assume we want to compute $H\underline{M}^G_V$. Using the proposition above, we see that if V is not of total degree -1, 0 or 1, we can add/subtract λ from it (this corresponds to the division/multiplication by a_λ) until either we hit one of these degrees or the multiplicity of λ reaches zero. In the latter case, the obtained representation W is such that $G_W := \ker(W) \neq e$, and the computation can be reduced to a cyclic 2-group of smaller order. In the first case, although in general we can only fit the homotopy Mackey functors in total degrees -1 and 1 into long exact sequences (Proposition 4.1), we have complete control of them (as well as the homotopy Mackey functors in total degree 0) in the positive and negative cone, which enables us to determined the positive cones of $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{A}}$ and the negative cone of $H\underline{\mathbb{Z}}$. In general, these ideas lead to the two main induction theorems in this paper; see Theorems 4.3 and 4.4.

Induction method

A Mackey functor $\underline{M}$ over an abelian group G consists of an abelian group $\underline{M}(H)$ with an action of G/H for every subgroup H of G. These abelian groups are connected by *restrictions* and *transfers*. If $G = C_{2^k}$, a G-Mackey functor has simple structure, that can be presented by a Lewis diagram:

$$\begin{array}{c} \underline{M}(G) \\ \big(\ \ \big) \\ \underline{M}(G') \\ \big(\ \ \big) \\ \cdots \\ \big(\ \ \big) \\ \underline{M}(e) \end{array}$$

Downward looking arrows are the restrictions and upward looking arrows are transfers. Here G' is the unique subgroup of index two.

We observe that if V is a G-representation, the G-equivariant computations of $H\underline{M}^G_V$ can be reduced to the G/G_V-equivariant computations of the V^{th} homotopy group of an Eilenberg–MacLane spectrum with another coefficient $\underline{M}^{\sharp G_V}$. Here $G_V := \ker(V)$, and the new Mackey functor is called the *G_V-truncation of* $\underline{M}$ and is obtained from $\underline{M}$ by forgetting $\underline{M}(K)$, where K are proper subgroups of G_V. Therefore if $G_V \neq e$ we reduced the computation to a smaller group. Since in this paper we are concerned with the homotopy of *all* Eilenberg–MacLane spectra, $\underline{M}^{\sharp G_V}$ can belong to a different class of Mackey functors from that of $\underline{M}$. For example, the Burnside Mackey functor $\underline{\mathbb{A}}_{C_{2^k}}$ is not constant, and we have

$$(\underline{\mathbb{A}}_{C_{2^k}})^{\sharp C_2} \cong \underline{\mathbb{A}}_{C_{2^{k-1}}} \oplus \underline{\mathbb{Z}}^*_{C_{2^{k-1}}}.$$

Multiplicative structure

If the Mackey functor $\underline{M}$ is a *Green functor*, ie a commutative ring in the category of Mackey functors, then the coefficients $H\underline{M}_\star^G$ is a $RO(G)$-graded ring. The methods presented in the paper allow for the analysis of this ring. The key ingredients are Euler classes a_V, orientation classes u_V and relations between these elements. In Proposition 2.28 we prove that for any G-representations V, W the classes a_V and a_W commute, as well as classes u_V and u_W by Proposition 2.29. The relation between the classes a_V and u_W is called the *gold relation* and was first observed by Hill, Hopkins and Ravenel [13] in the case of the constant Mackey functor $\underline{\mathbb{Z}}$. We generalize this relation to the following in Proposition 3.16:

Proposition *Let V, W be chosen from λ_i for $1 \le i \le k$ such that $G_V \subset G_W$. Let R be a Green functor. Then there is the following relation in $H\underline{R}_\star^G$:*

$$a_V \cdot \mathrm{tr}_{G_V}^{G_W}(1) u_W = a_W \cdot u_V.$$

Positive and negative cone

We conclude the paper with computations of the positive cone and negative cone for any G-Mackey functor $\underline{M}$. This serves as a presentation how the induction method can be used for calculations. In particular, we provide descriptions of both cones in the case of $H\underline{\mathbb{Z}}$ in Theorems 5.4 and 6.6, as well as the positive cone of $H\underline{\mathbb{A}}$ in Theorem 5.5.

Related work

Computations of coefficients of Eilenberg–MacLane spectra over cyclic 2-groups have long history. For the group C_2, basing on an unpublished work of Stong, Lewis, Jr [17, Section 2] computed $H\underline{\mathbb{A}}_\star^{C_2}$, where $\underline{\mathbb{A}}$ is the Burnside Mackey functor. Calculations for the constant Mackey functor $\underline{\mathbb{F}}_2$ by Caruso can be found in [2] and by Hu and Kriz in [14, Proposition 6.2]. The computations for the constant Mackey functor $\underline{\mathbb{Z}}$ may be found in Dugger's work [3, Appendix B]. The full $RO(C_2)$-graded Mackey functor valued coefficients of $H\underline{M}$ for any Mackey functor $\underline{M}$ was given by Ferland [5] and may also be found in Ferland and Lewis, Jr [6, Chapter 8]. Using the method based on the Tate square as developed by Greenlees and May [9], Greenlees computed the coefficients of $H\underline{\mathbb{Z}}$ in [8]. Basing on this work, the first author computed coefficients of $H\underline{M}$ for any C_2-Mackey functor $\underline{M}$ with an insight into multiplicative structure in [21].

The computations for cyclic 2-groups of higher order were focused so far on the cases of $\underline{\mathbb{Z}}$ and $\underline{\mathbb{F}}_2$. For the first case, partial description is provided by Hill, Hopkins and Ravenel [13]. The full structure of $RO(G)$-graded ring $H\underline{\mathbb{Z}}_\star^{C_4}$ was computed by Nick Georgakopoulos [7]. The second author computed the same ring using the Tate square method in [24]. In [23], he also computed the ring structure of coefficients of $H\underline{\mathbb{F}}_2$ over any cyclic 2-group. Using Tate square methods, the coefficients of $H\underline{\mathbb{Z}}$ over groups C_{p^2} with p odd are computed by Zeng [25]. The work of Ayala, Mazel-Gee and Rozenblyum [1] provides an ∞-categorical calculation in the odd primary case.

1A Notation and conventions

Unless indicated otherwise, $G = C_{2^k}$ for $k \geq 1$, a cyclic group of order a power of 2. By G' we will denote the subgroup of G of index 2. A general Mackey functor over G will be denoted by $\underline{M}$, and a general Green functor will be denoted by $\underline{R}$.

For a Mackey functor $\underline{M}$, we denote by $H\underline{M}_\star := \pi_\star(H\underline{M})$, the Mackey functor valued $RO(G)$-graded homotopy groups of the G-spectrum $H\underline{M}$. By $H\underline{M}_V^H$ we mean the value of this Mackey functor at G/H-level. Structural maps of the Mackey functor $H\underline{M}_V$ for a given representation V will be denoted by $\mathrm{res}(V)_K^H$ and $\mathrm{tr}(V)_K^H$. When calculating G-homotopy groups of $H\underline{M}$, the structural maps of $\underline{M}$ will be denoted by res_K^H and tr_K^H for $K \subset H$.

Let $\underline{R}$ be a general Green functor. Then its associated Eilenberg–MacLane spectrum is a ring spectrum. In particular, there is a unit map $\eta\colon S^0 \to H\underline{R}$ and multiplication map $\mu\colon H\underline{R} \wedge H\underline{R} \to H\underline{R}$.

If $x \in \mathbb{Z}[G]$ and M is a $\mathbb{Z}[G]$-module, by ${}_xM$ we denote the submodule of x-torsion elements, ie

$$ {}_xM := \{m \in M \mid xm = 0\}.$$

In the more concrete computations of the positive and negative cones, we adopt the following notational conventions as in [24; 23; 25]: $[c]$ means a polynomial generator while $\langle a, b\rangle$ means additive generators. For example, $\mathbb{Z}/2[c]\langle a, b\rangle$ means we have elements of the form $c^i a, c^i b, i \geq 0$ all of whom are 2-torsion. By $\mathbb{Z}/2\langle a^i\rangle_{i\geq 1}\langle b^j\rangle_{j\geq 1}$ we mean that there are elements of the form $a^i b^j, i, j \geq 1$ all of whom are 2-torsion. The notation x^{-1} means divisibility, ie $x^{-1}y$ (or equally $\frac{y}{x}$) means an element that satisfies $x \cdot x^{-1}y = y$. The notation $\left[\frac{1}{x}\right]\langle z\rangle$ means we have the elements $z, \frac{z}{x}, \frac{z}{x^2}, \frac{z}{x^3}, \ldots$. Some elements have the form $2^i a$. In some cases they are just formal symbols, not 2^i times of an actual element a. The number 2^i in $2^i a$ means that its image under res_e^G (equally its image under the Borel completion map $H\underline{M} \to F(EG_+, H\underline{M})$) is 2^i times an actual class a.

1B General assumption

Our methods work for general Mackey functors $\underline{M}$ and Green functors $\underline{R}$ over the group C_{p^k}. For a simpler $RO(G)$-grading, in the whole paper we will work p-locally. See Section 2A for more details on the grading.

We note here that if $\underline{M}$ (or $\underline{R}$) has the property that $\underline{M}(G/H)$ is a finitely generated abelian group for each subgroup $H \subset G$, it can be argued using equivariant Künneth and universal coefficient spectral sequences, see Lewis, Jr and Mandell [18], that $H\underline{M}_V^H$ is also finitely generated for each $V \in RO(G)$. Therefore integral information can be completely recovered from local information. This includes the most interesting cases: the Burnside Green functor $\underline{\mathbb{A}}$, the constant Mackey functors $\underline{A}$ for A a finitely generated abelian group. $\underline{RO(G)}$ and $\underline{R(G)}$, are the real and complex representation Green functors, respectively. Note that in the $\underline{A}$ case, p-localization is not needed for a simpler grading by a theorem of Hu and Kriz; see [25, Proposition 4.25].

Acknowledgements

Sikora would like to thank Krakow University of Economics, who supported this work under the POTENCJAŁ program (project number 089/EIM/2024/POT), financed by a subsidy granted to the University. Both authors want to extend their thanks to the referee, whose deep insights and thorough review highly increased the readability and quality of the paper.

2 Basics

In this section we first introduce the basic notions which will be used in our computations. Then we completely determine the cellular chain and cochains of a representation sphere S^V. Using this result, we define Euler and orientations classes and discuss commutativity properties among them.

2A The $RO(G)$-grading

Irreducible real representations of the group C_{2^k} are as follows:

(1) The 1-dimensional trivial representation. We will denote this representation by 1.

(2) The 1-dimensional sign representation, denoted by α.

(3) 2-dimensional representations $\lambda(m)$ for $1 \leq m \leq 2^k - 1$ and $m \neq 2^{k-1}$. The action is given by rotation by an angle $\frac{2m\pi}{2^k}$.

Irreducible real representations of the group C_{p^k} for odd p are as follows:

(1) The 1-dimensional trivial representation. We will denote this representation by 1.

(2) 2-dimensional representations $\lambda(m)$ for $1 \leq m \leq p^k - 1$. The action is given by rotation by an angle $\frac{2m\pi}{p^k}$.

For all primes, put $\lambda_j := \lambda(p^{k-j})$ for $1 \leq j \leq k$. Note that $\lambda_1 = 2\alpha$ when $p = 2$. Our convention in this paper on enumerating the generating representations is chosen to support the induction on the group order. Therefore it is different from the one used in Hill, Hopkins and Ravenel's work [11; 12; 13], Zeng's work [25] as well as the second author's work in [24; 23] in that the ordering of the representations is reversed.

In this paper we will focus at $p = 2$, because it essentially contains the corresponding odd primary results. The reason is as follows. Since we work p-locally, we have the stable equivalence of representation spheres $S^{\lambda(rp^i)} \simeq S^{\lambda(p^i)}$ for $(r, p) = 1$; see [25, page 26]. This enables us to focus on $1, \lambda_j$, $1 \leq j \leq k$ (and α for $p = 2$) without losing any information. For odd p, the spheres involved in the computations will be S^1 and S^{λ_j}, $1 \leq j \leq k$. For $p = 2$, the spheres involved will be S^1, S^α and S^{λ_j}, $2 \leq j \leq k$, with $S^{2\alpha} = S^{\lambda_1}$ by convention. As a consequence, any result for $p = 2$ will contain the corresponding result for an odd p as a special case, where we only focus on those representations with even numbers of α in it.

The homotopy groups $\pi_\star H\underline{M}$ will be graded over the following free abelian group on representations:

$$RO(C_{2^k}) := \mathbb{Z}\{1, \alpha, \lambda_2, \dots, \lambda_k\};$$

this will be referred to as the $RO(G)$-*grading*. More precisely, we will study the structure of the direct sum of the following abelian groups for V running over elements in $RO(G)$:

$$H\underline{M}_V = [S^V, H\underline{M}]^G, \quad V \in \{a_0 + a_1\alpha + a_2\lambda_2 + \cdots + a_k\lambda_k \mid x, y, a_i \in \mathbb{Z} \text{ for all } 2 \le i \le k\}.$$

Here $[-,-]^G$ denotes the homotopy class of maps in genuine G-spectra, $\underline{M}$ is a Mackey functor over G and $H\underline{M}$ is the associated Eilenberg–MacLane spectrum.

Definition 2.1 (1) An element of $RO(G)$ is called a *virtual representation*. If

$$V = a_0 + a_1\alpha + \sum_{i=2}^{k} a_i\lambda_i$$

such that $a_i \ge 0$ for all $0 \le i \le k$, we will refer to it as an (*actual*) G-*representation*.

(2) The *total dimension/degree* (*underlying dimension/degree*) of $V = a_0 + a_1\alpha + \sum_{i=2}^{k} a_i\lambda_i$ is defined to be the sum $|V| = a_0 + a_1 + 2\sum_{i=2}^{k} a_i$.

(3) The *fixed dimension* of V as above is the number a_0.

(4) For $V \in RO(G)$ with $V^G = 0$, let $\lambda_{\min}(V)$ and $\lambda_{\max}(V)$ be the first and last irreducible G-representations that appear in V with nonzero coefficients in the list $\alpha, \lambda_2, \lambda_3, \dots, \lambda_k$. When V is clear, we simply use $\lambda_{\min}$ and $\lambda_{\max}$. If V is an actual G-representation, let G_V be its kernel $\{g \in G \mid gv = v \text{ for all } v \in V\}$. Note in fact we have $G_V = G_{\lambda_{\max}}$. For a general $V \in RO(G)$, define $G_V = G_{\lambda_{\max}}$.

(5) An actual G-representation V is *orientable* if the action of G on V factors through $SO(|V|)$. A virtual G-representation V is orientable if it can be written as $V = V' - V''$ such that both V' and V'' are orientable representations.

Definition 2.2 For $1 \le j \le k$ we define the elements of $\mathbb{Z}[G]$

$$\zeta_j := \sum_{0 \le i < 2^j} \gamma^i \quad \text{and} \quad \bar{\zeta}_j := \sum_{0 \le i < 2^j} (-\gamma)^i,$$

where γ is the generator of G. We also put $\zeta_0 = 1$.

Remark 2.3 (1) Note that the element ζ_j gives the norm element in $\mathbb{Z}[G/C_{2^{k-j}}]$ via the quotient map $\mathbb{Z}[G] \to \mathbb{Z}[G/C_{2^{k-j}}]$. In particular, ζ_k is the norm element in $\mathbb{Z}[G]$ and will also be denoted by N. These elements satisfy the equations

$$\zeta_j(1-\gamma) = 1 - \gamma^{2^j}, \quad \bar{\zeta}_j(1+\gamma) = 1 - \gamma^{2^j}.$$

(2) For an odd prime p, we define the elements ζ_j in the same way, with 2 replaced by p. In this case, all representations are orientable, thus we do not need an analogue of $\bar{\zeta}_j$ for the differentials in Theorem 2.6.

2B Cellular chains and cochains of representation spheres

In this subsection we will describe the cellular structure of the representation spheres. We begin with recalling the *shearing isomorphism*, which will be used extensively throughout the paper.

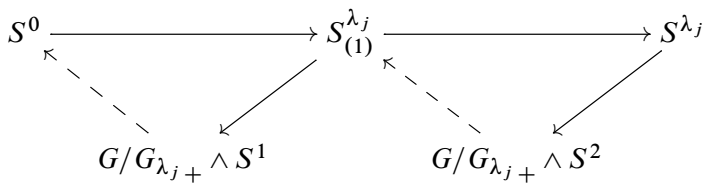
Proposition 2.4 (shearing isomorphism [19; 20, Section 4]) *Let G be a group and H its subgroup. Then for an H-space B and G-space A there is a G-homeomorphism,*

$$(G_+ \wedge_H B) \wedge A \cong G_+ \wedge_H (B \wedge \mathrm{res}^G_H A), \quad (g, b, a) \mapsto (g, b, g^{-1}a).$$

In particular, if $B = S^0$ and $A = S^V$ for some G-representation V, we have that

$$G/H_+ \wedge S^V \cong G_+ \wedge_H S^{\mathrm{res}^G_H V}.$$

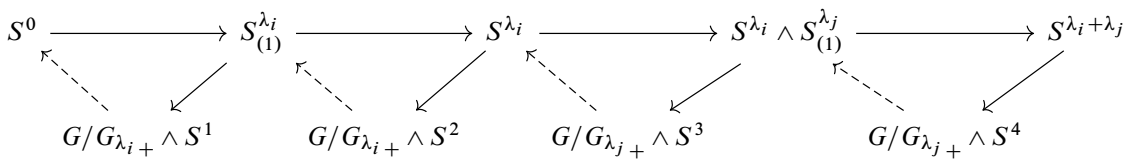
We now move to the description of the cellular structure of representation spheres. As observed in [12], the representation sphere S^V for an actual representation $V = y\alpha + \sum_{i=2}^k a_i\lambda_i$ has a simple cellular structure, since the stabilizers has a linear order. Consider $S^{\lambda_i+\lambda_j}$ with $i < j$. Recall that $\lambda_1 = 2\alpha$. The sphere S^{λ_j} has a cellular structure

(1)


where $X_{(m)}$ denotes the m-skeleton of a G-CW complex. Since $i < j$, we have that $G_{\lambda_j} \subset G_{\lambda_i}$, and

$$G/G_{\lambda_j+} \wedge S^{\lambda_i} \cong G/G_{\lambda_j+} \wedge S^2$$

by the shearing isomorphism (Proposition 2.4). Smashing S^{λ_i} with (1), and concatenating with the cellular structure on S^{λ_i}, we get the cellular filtration of $S^{\lambda_i+\lambda_j}$ as

(2)


This easily generalizes to a general G-representation $V = a_1\alpha + \sum_{i=2}^k a_i\lambda_i$. We record it as a lemma:

Lemma 2.5 *For an actual G-representation $V = a_1\alpha + \sum_{i=2}^k a_i\lambda_i$, the sphere S^V has a cellular structure that starts with $S^0, G/G_{\lambda_{\min}+} \wedge S^1$ and ends with $G/G_{\lambda_{\max}+} \wedge S^{|V|}$. It has one equivariant cell in each dimension and the stabilizers are in descending order.*

We will now state the theorem which will be fundamental for many computations in this paper. Let V be an actual G-representation with $V^G = 0$. Let S^V be equipped with the cellular structure as in the above lemma. For a better presentation of the result, write $V = V_0 + V_1 + \cdots + V_n$ with $V_0 \in \{0, \alpha\}$, $V_i \in \{\lambda_1, \ldots, \lambda_k\}$ for $1 \leq i \leq n$ and the kernels of the representations V_i are in descending order. Note that $V_0 = 0$ if V is orientable and $V_0 = \alpha$ if V is nonorientable. Since S^V has a canonical base point, the point at infinity, all cellular chains and cochains are reduced in this paper.

Theorem 2.6 *Let $G = C_{2^k}$ and let V be as above. Let $s_i = |V_0 + \cdots + V_i|$, $i \geq 0$. Then the (reduced) cellular chain complex $C^G_*(S^V; \underline{M})$ is as follows*:

(1) *If V is orientable, then $C^G_0 = \underline{M}(G)$, $C^G_{s_i} = C^G_{s_i-1} = \underline{M}(G_{V_i})$ for $i \geq 1$. For the differentials, we have for $i \geq 1$,*

$$\partial_{s_i} = 1 - \gamma, \quad \partial_{s_i - 1} = \zeta_{\log_2 |G/G_{V_{i-1}}|} \operatorname{tr}^{G_{V_{i-1}}}_{G_{V_i}}.$$

(2) *If V is nonorientable, then $C^G_0 = \underline{M}(G)$, $C^G_1 = \underline{M}(G')$ and $C^G_{s_i} = C^G_{s_i-1} = \underline{M}(G_{V_i})$ for $i \geq 1$. For the differentials, we have for $i \geq 1$,*

$$\partial_{s_i} = 1 + \gamma, \quad \partial_{s_i - 1} = \overline{\zeta}_{\log_2 |G/G_{V_{i-1}}|} \operatorname{tr}^{G_{V_{i-1}}}_{G_{V_i}}.$$

The (reduced) cellular cochain complex $C^*_G(S^V; \underline{M})$ is as follows:

(1) *If V is orientable, then $C^0_G = \underline{M}(G)$, $C^{s_i}_G = C^{s_i - 1}_G = \underline{M}(G_{V_i})$ for $i \geq 1$. For the differentials, we have for $i \geq 0$,*

$$\partial^{s_i + 1} = 1 - \gamma, \quad \partial^{s_i} = \zeta_{\log_2 |G/G_{V_i}|} \operatorname{res}^{G_{V_i}}_{G_{V_{i+1}}}.$$

(2) *If V is nonorientable, then $C^0_G = \underline{M}(G)$, $C^1_G = \underline{M}(G')$ and $C^{s_i}_G = C^{s_i - 1}_G = \underline{M}(G_{V_i})$ for $i \geq 1$. For the differentials, we have $\partial^0 = \operatorname{res}^G_{G'}$, and for $i \geq 0$,*

$$\partial^{s_i + 1} = 1 + \gamma, \quad \partial^{s_i} = \overline{\zeta}_{\log_2 |G/G_{V_i}|} \operatorname{res}^{G_{V_i}}_{G_{V_{i+1}}}.$$

If $G = C_{p^k}$ for p odd, then all representations are orientable. The results on the cellular chain and cochain still hold, with $\zeta_{\log_2 |G/G_{V_i}|}$ replaced by its odd primary analogue $\zeta_{\log_p |G/G_{V_i}|}$, see also Remark 2.3.

Proof We will only prove the part for chains. The proof for cochains is analogous. We are going to proceed by induction on the group order.

The theorem is true for $G = C_2$, where the cellular chain $C^{C_2}_*(S^{n\alpha}; \underline{M})$ is given by

$$\underline{M}(G) \xleftarrow{\operatorname{tr}^G_e} \underline{M}(e) \xleftarrow{1-\gamma} \underline{M}(e) \xleftarrow{1+\gamma} \cdots \leftarrow \underline{M}(e),$$

with the differentials interchanging between $1 - \gamma$ and $1 + \gamma$ except the beginning one.

Now fix $k \geq 2$ and assume that the theorem is true for groups C_{2^j} for $j < k$. Let $G = C_{2^k}$ and recall that G' is the subgroup of G such that $[G : G'] = 2$. Let $W = V_0 + \cdots + V_t$ be the biggest subrepresentation of V that does not contain λ_k. Note that in degrees $\leq |W|$ the complex $C^G_*(S^V, \underline{M})$ is the same as $C^G_*(S^W; \underline{M})$, and the latter is equal to $C^{G'}_*(S^W, \underline{M}^{\sharp C_2})$ (for the definition of $\underline{M}^{\sharp C_2}$, see Definition 2.12). Thus by induction we need to prove the claim for the tail of $C^G_*(S^V, \underline{M})$, consisting of the following terms:

$$\underline{M}(G_W) \leftarrow \underline{M}(e) \leftarrow \underline{M}(e) \leftarrow \cdots \leftarrow \underline{M}(e).$$

The idea is to make use of the Mackey functor structure on cellular chains. More precisely, let $\underline{S}^V$ denote the cellular structure given in Lemma 2.5, with one cell in each dimension. Then $i^*_{G'}\underline{S}^V$, the restricted cellular structure, has one 0-cell $G'/{G'}_+$, and two cells in each higher dimension. That is, if $G/H_+ \wedge S^i$ is an i-cell of $\underline{S}^V$ for $i \geq 1$, then the corresponding restricted cell to $i^*_{G'}\underline{S}^V$ is

$$i^*_{G'}(G/H_+ \wedge S^i) = G'/H_+ \wedge S^i \vee \gamma G'/H_+ \wedge S^i.$$

For $H \subset G'$ there is the folding map

$$G_+ \wedge_{G'} i^*_{G'} G/H_+ \cong G/{G'}_+ \wedge G/H_+ \to G/H_+.$$

Applying the contravariant part of $\underline{M}$ to this map and using additivity of a Mackey functor, we get an embedding which corresponds to $\mathrm{res}^G_{G'}$ in the cellular chain

$$\underline{M}(H) \xrightarrow{\Delta} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G']} i^*_{G'}\underline{M}(H) \cong \mathbb{Z}[G/G'] \otimes \underline{M}(H).$$

It sends $x \in \underline{M}(H)$ to $1 \otimes x + \gamma \otimes x$ in the middle term and then to $1 \otimes x + \gamma \otimes \gamma x$ on the right. These embeddings collectively form a chain map because they come from the natural transformation $G/{G'}_+ \wedge - \Rightarrow G/G_+ \wedge -$ applied to the cellular filtration of S^V. Because of this embedding, we can regard $C^G_*(\underline{S}^V; \underline{M})$ as a subchain complex of $C^{G'}_*(i^*_{G'}\underline{S}^V; \underline{M})$ and are left to show that the later has the stated differentials.

Now $i^*_{G'}S^V = S^{i^*_{G'}V}$ as a representation sphere for G' also has the cellular structure described in Lemma 2.5. We use $\underline{i^*_{G'}S^V}$ to refer to this cellular structure. Both complexes $C^{G'}_*(\underline{i^*_{G'}S^V}; \underline{M})$ and $C^{G'}_*(i^*_{G'}\underline{S}^V; \underline{M})$ compute the homology of $S^{i^*_{G'}V}$ with coefficient in $\underline{M}{\downarrow}^G_{G'}$ (the restricted Mackey functor to G', see Definition 3.1). By the inductive assumption we know how to compute the homology of the former, so this forces the differentials in the latter chain complex. Furthermore, we will construct a quasiisomorphism

$$\psi : C^{G'}_*(\underline{i^*_{G'}S^V}; \underline{M}) \xrightarrow{\simeq} C^{G'}_*(i^*_{G'}\underline{S}^V; \underline{M})$$

in Lemma 2.7. The differentials in $C^{G'}_*(i^*_{G'}\underline{S}^V; \underline{M})$ are also determined there. Then from the embedding

$$C^G_n(\underline{S}^V; \underline{M}) \xhookrightarrow{\Delta} C^{G'}_n(i^*_{G'}\underline{S}^V; \underline{M}) \cong \mathbb{Z}[G/G'] \otimes C^G_n(\underline{S}^V; \underline{M}), \quad n \geq 1, \tag{3}$$

we deduce the differentials in $C^G_*(\underline{S}^V; \underline{M})$. □

Lemma 2.7 *There exists a quasiisomorphism*

$$\psi : C_*^{G'}(i_{G'}^* S^V; \underline{M}) \xrightarrow{\simeq} C_*^{G'}(i_{G'}^* \underline{S}^V; \underline{M}).$$

Proof We proceed by the induction on the group order. For the group C_2, the subgroup G' is the trivial group. Let $V = m\alpha$, $m \geq 1$. For $i_e^* S^V = S^{|m|}$, recall that we use the reduced cellular structure such that there is only one cell in dimension m. We have the diagram

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \cdots \longleftarrow & 0 & \longleftarrow & \underline{M}(e) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi_m} \\
\underline{M}(e) & \xleftarrow{\nabla} & \mathrm{Ind}_e^{C_2} \underline{M}(e) & \xleftarrow{1-\gamma} & \mathrm{Ind}_e^{C_2} \underline{M}(e) & \xleftarrow{1+\gamma} & \mathrm{Ind}_e^{C_2} \underline{M}(e) & \longleftarrow \cdots \longleftarrow & \mathrm{Ind}_e^{C_2} \underline{M}(e) & \longleftarrow & \mathrm{Ind}_e^{C_2} \underline{M}(e)
\end{array}$$

where ψ_m is the m^{th} component of ψ. It is defined by

$$\psi_m(z) = \begin{cases} (1,\gamma)(z) = 1 \otimes z + \gamma \otimes \gamma z & \text{for } m \text{ even}, \\ (1,-\gamma)(z) = 1 \otimes z - \gamma \otimes \gamma z & \text{for } m \text{ odd}. \end{cases}$$

The differentials in the second row are given by

$$\begin{aligned}
\nabla(1 \otimes z) &= z, & \nabla(\gamma \otimes z) &= z, \\
(1-\gamma)(1 \otimes z) &= 1 \otimes z - \gamma \otimes \gamma z, & (1-\gamma)(\gamma \otimes z) &= \gamma \otimes z - 1 \otimes \gamma z, \\
(1+\gamma)(1 \otimes z) &= 1 \otimes z + \gamma \otimes \gamma z, & (1+\gamma)(\gamma \otimes z) &= \gamma \otimes z + 1 \otimes \gamma z.
\end{aligned}$$

It is easily seen that ψ is a quasiisomorphism.

Now let $G = C_{2^k}$ and assume that ψ has been constructed for groups C_{2^j} with $j < k$. Let W be as in Theorem 2.6 so that $G_W \neq e$. Note that W can be regarded as a $G' \cong G/C_2$-representation. By induction, we can assume that the maps

$$\psi_n : C_n^{G'}(i_{G'}^* S^W; \underline{M}) \to C_n^{G'}(i_{G'}^* \underline{S}^W; \underline{M})$$

are constructed for $n \leq |W|$.

To construct ψ_i for $i > |W|$, we are going to consider three cases. We firstly consider the case when $V_0 = 0$. If $W \neq 0$, suppose that $G_W = C_{2^{k-i}}$. Recall that $\zeta_j = \sum_{0 \leq i < 2^j} \gamma^i$ (see Definition 2.2) and $\mathrm{Ind}_H^G M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$ for M a H-module. We are to determine the differentials in the second row and construct vertical chain maps that are a quasiisomorphism in the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \xleftarrow{1-\gamma^2} & \underline{M}(C_{2^{k-i}}) & \xleftarrow{\zeta_{i-1}(\gamma^2)\,\mathrm{tr}_e^{C_{2^{k-i}}}} & \underline{M}(e) & \xleftarrow{1-\gamma^2} & \underline{M}(e) & \xleftarrow{\zeta_{k-1}(\gamma^2)} & \underline{M}(e) & \xleftarrow{1-\gamma^2} & \underline{M}(e) \longleftarrow \cdots \\
 & & \downarrow{\scriptstyle (1,\gamma)} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xleftarrow{1-\gamma} & \mathrm{Ind}_{G'}^G \underline{M}(C_{2^{k-i}}) & \longleftarrow & \mathrm{Ind}_{G'}^G \underline{M}(e) & \longleftarrow & \mathrm{Ind}_{G'}^G \underline{M}(e) & \longleftarrow & \mathrm{Ind}_{G'}^G \underline{M}(e) & \longleftarrow & \mathrm{Ind}_{G'}^G \underline{M}(e) \longleftarrow \cdots
\end{array}$$

Here $\zeta_j(\gamma^2) = \sum_{0 \leq i < 2^j} (\gamma^2)^i$.

Note that the induction Ind_H^G is with respect to the Weyl G/H-action on $\underline{M}(H)$, which comes from the $1 \times \gamma$ on action on $G/G' \times G/H$. On the other hand, there is a Weyl G/G'-action on $C_*^{G'}(i_{G'}^* \underline{S}^W; \underline{M})$, which comes from the $\gamma \times 1$-action on $G/G' \times G/H$. We have, for $H \subset G'$,

$$G/G' \times G/H = G/H_{(e,e)} \sqcup G/H_{(\gamma,e)},$$

where $1\times\gamma$ send (e,e) to $(e,\gamma)=\gamma(\gamma,e)$. Thus it acts as permuting the two factors with an internal twist. And $\gamma\times 1$ acts by permuting the two factors without any internal twist. We note the differentials in the second row are $\gamma\times 1$-equivariant.

Denote by $(1,\gamma)$ the map sending $x\in\underline{M}(C_{2^{k-i}})$ to $1\otimes x+\gamma\otimes\gamma x$, with

$$\underline{M}(C_{2^{k-i}})=C^{G'}_{d_{k-1}}(i^*_{\underline{G'}}S^V;\underline{M}).$$

To make sure that the second row has the same homology as the first row at degree $|W|$, the first unknown differential in the second row must be $\zeta_i\operatorname{tr}_e^{C_{2^{k-i}}}$. By further comparing higher homology, we know the following differentials must be interchanging between $1-\gamma$ and ζ_k. Since the dashed arrows should be chain maps, the only possibility is that they interchange between $(1,0)$ and $(1,\gamma)$. The completed diagram is

$$\begin{array}{ccccccccccccc}
\cdots & \xleftarrow{1-\gamma^2} & \underline{M}(C_{2^{k-i}}) & \xleftarrow{\zeta_{i-1}(\gamma^2)\operatorname{tr}_e^{C_{2^{k-i}}}} & \underline{M}(e) & \xleftarrow{1-\gamma^2} & \underline{M}(e) & \xleftarrow{\zeta_{k-1}(\gamma^2)} & \underline{M}(e) & \xleftarrow{1-\gamma^2} & \underline{M}(e) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle (1,\gamma)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,\gamma)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,\gamma)} & & \\
\cdots & \xleftarrow{1-\gamma} & \operatorname{Ind}_{G'}^{G}\underline{M}(C_{2^{k-i}}) & \xleftarrow{\zeta_i\operatorname{tr}_e^{C_{2^{k-i}}}} & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{1-\gamma} & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{\zeta_k} & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{1-\gamma} & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \longleftarrow & \cdots
\end{array}$$

Here by γ we mean the $1\times\gamma$-action as mentioned above. The differentials in the second row are given by

$$\begin{aligned}
(\zeta_i\operatorname{tr}_e^{C_{2^{k-i}}})(1\otimes x)&=1\otimes\zeta_{i-1}(\gamma^2)\operatorname{tr}_e^{C_{2^{k-i}}}(x)+\gamma\otimes\gamma\zeta_{i-1}(\gamma^2)\operatorname{tr}_e^{C_{2^{k-i}}}(x),\\
(\zeta_i\operatorname{tr}_e^{C_{2^{k-i}}})(\gamma\otimes y)&=1\otimes\gamma\zeta_{i-1}(\gamma^2)\operatorname{tr}_e^{C_{2^{k-i}}}(y)+\gamma\otimes\zeta_{i-1}(\gamma^2)\operatorname{tr}_e^{C_{2^{k-i}}}(y),\\
(1-\gamma)(1\otimes x)&=1\otimes x-\gamma\otimes\gamma x,\\
(1-\gamma)(\gamma\otimes y)&=\gamma\otimes y-1\otimes\gamma y,\\
\zeta_k(1\otimes x)&=1\otimes\zeta_{k-1}(\gamma^2)x+\gamma\otimes\gamma\zeta_{k-1}(\gamma^2)x,\\
\zeta_k(\gamma\otimes y)&=1\otimes\gamma\zeta_{k-1}(\gamma^2)x+\gamma\otimes\zeta_{k-1}(\gamma^2)x.
\end{aligned}$$

The vertical maps are given by

$$(1,\gamma)(z)=1\otimes z+\gamma\otimes\gamma z,\quad (1,0)(z)=1\otimes z.$$

The embedding (3) is induced by $*\times G/H\xleftarrow{*\times 1}G/G'\times G/H$ for $H\subset G'$. In particular, it sends $x\in\underline{M}(e)$ to $1\otimes x+\gamma\otimes x\in\operatorname{Ind}_{G'}^{G}\underline{M}(e)$, from where we deduce all the differentials of $C^G_*(\underline{S}^V;\underline{M})$.

For $V_0=0$ and $W=0$, that is $S^V=S^{a_k\lambda_k}$, we start from

$$\begin{array}{ccccccccc}
\underline{M}(G') & \xleftarrow{\operatorname{tr}_e^{G'}} & \underline{M}(e) & \xleftarrow{1-\gamma^2} & \underline{M}(e) & \xleftarrow{\zeta_{k-1}(\gamma^2)} & \underline{M}(e) & \longleftarrow & \cdots \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,\gamma)} & & \downarrow{\scriptstyle (1,0)} & & \\
\underline{M}(G') & \xleftarrow{\operatorname{tr}_e^{G'}(1\oplus\gamma)} & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \longleftarrow & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \longleftarrow & \operatorname{Ind}_{G'}^{G}\underline{M}(e) & \longleftarrow & \cdots
\end{array}$$

and deduce the differentials and chain maps as in the previous case. Here the map $\operatorname{tr}_e^{G'}(1\oplus\gamma)$ sends $1\otimes x+\gamma\otimes y$ to $\operatorname{tr}_e^{G'}(x+\gamma y)$. It is deduced from the map $G/G'\times *\xleftarrow{1\times *}G/G'\times G/e$.

Now for the $V_0 = \alpha$ case. Assume $G_W = C_{2^{k-i}}$, then it is contained in G'. Using the same method, we deduce the completed diagram

$$\begin{array}{ccccccccccccc}
\cdots & \xleftarrow{1+\gamma^2} & \underline{M}(C_{2^{k-i}}) & \xleftarrow{\bar{\xi}_{i-1}(\gamma^2)\,\mathrm{tr}_e^{C_{2^{k-i}}}} & \underline{M}(e) & \xleftarrow{1+\gamma^2} & \underline{M}(e) & \xleftarrow{\bar{\xi}_{k-1}(\gamma^2)} & \underline{M}(e) & \xleftarrow{1+\gamma^2} & \underline{M}(e) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle (1,-\gamma)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,-\gamma)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,-\gamma)} & & \\
\cdots & \xleftarrow{1+\gamma} & \mathrm{Ind}_{G'}^{G}\underline{M}(C_{2^{k-i}}) & \xleftarrow{\bar{\xi}_i\,\mathrm{tr}_e^{C_{2^{k-i}}}} & \mathrm{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{1-\gamma} & \mathrm{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{\xi_k} & \mathrm{Ind}_{G'}^{G}\underline{M}(e) & \xleftarrow{1-\gamma} & \mathrm{Ind}_{G'}^{G}\underline{M}(e) & \longleftarrow & \cdots
\end{array}$$

where $(1,-\gamma)(z) = 1\otimes z - \gamma\otimes\gamma z$. □

Remark 2.8 We have also showed how to deduce the homology Mackey functors using the cellular chains $C_*^H(i_H^* S^V;\underline{M})$ for different $H\subset G$. Note that these chains have one cell in each dimension, therefore are easier to compute. Since restrictions and transfers are transitive, we can take $H = G'$. The maps Δ and ∇ in the following diagram are usually used to deduce $\mathrm{res}_{G'}^G$ and $\mathrm{tr}_{G'}^G$:

$$\begin{array}{ccc}
C_*^G(\underline{S}^V;\underline{M}) & & \\
\downarrow & \overset{\Delta}{\searrow} & \\
C_*^{G'}(i_{G'}^*\underline{S}^V;\underline{M}) & \underset{\psi}{\longrightarrow} & C_*^{G'}(i_{G'}^*\underline{S}^V;\underline{M})
\end{array}$$

It is not obvious that there exist restrictions and transfers between the two chains of abelian groups on the left. In fact, there is no chain map in the direction of the dashed arrow that makes the triangle commute. However, since $H_*(\psi)$ is an isomorphism, we can deduce the restrictions and transfers between the homology of the two chains on the left by first computing $H_*(\Delta)$ and $H_*(\nabla)$.

Remark 2.9 The motivations for the differentials in Theorem 2.6 are simple: they are the universal formulas for all Mackey functors, thus they should satisfy the least amount of restrictions that make the cellular chain/cochain complexes.

Example 2.10 The chain complex $C_*(S^{\alpha+2\lambda_k};\underline{M})$ is given by

$$0 \leftarrow \underline{M}(G) \xleftarrow{\mathrm{tr}_{G'}^G} \underline{M}(G') \xleftarrow{(1-\gamma)\,\mathrm{tr}_e^{G'}} \underline{M}(e) \xleftarrow{1+\gamma} \underline{M}(e) \xleftarrow{\bar{\xi}_k} \underline{M}(e) \xleftarrow{1+\gamma} \underline{M}(e).$$

Example 2.11 The cochain complex $C^*(S^{2\alpha+2\lambda_2+2\lambda_3};\underline{M})$ with $k\geq 3$ in C_{2^k} is given by

$$\begin{aligned}
0 \to \underline{M}(G) \xrightarrow{\mathrm{res}_{G'}^G} \underline{M}(G') \xrightarrow{1-\gamma} \underline{M}(G') \xrightarrow{(1+\gamma)\,\mathrm{res}_{C_{2^{k-2}}}^{G'}} \underline{M}(C_{2^{k-2}}) \xrightarrow{1-\gamma} \underline{M}(C_{2^{k-2}}) \xrightarrow{\xi_2} \underline{M}(C_{2^{k-2}}) \\
\xrightarrow{1-\gamma} \underline{M}(C_{2^{k-2}}) \xrightarrow{\xi_2\,\mathrm{res}_{C_{2^{k-3}}}^{C_{2^{k-2}}}} \underline{M}(C_{2^{k-3}}) \xrightarrow{1-\gamma} \underline{M}(C_{2^{k-3}}) \xrightarrow{\xi_3} \underline{M}(C_{2^{k-3}}) \xrightarrow{1-\gamma} \underline{M}(C_{2^{k-3}}).
\end{aligned}$$

2C Induction method

In this subsection we will describe the induction method, which enables us to reduce computations to smaller groups when $G_V \neq e$.

Definition 2.12 Let $\underline{M}$ be a G-Mackey functor and let $H \leq G$. Then an *H-truncation of $\underline{M}$* is the G/H-Mackey functor $\underline{M}^{\sharp H}$ defined by

$$\underline{M}^{\sharp H}(K/H) := \underline{M}(K)$$

for $H \leq K \leq G$. The transfers, restrictions, and Weyl group actions are the respective structural maps of $\underline{M}$.

Remark 2.13 Intuitively, the H-truncation of $\underline{M}$ amounts to forgetting all of the information below the H-level of the Mackey functor $\underline{M}$.

Example 2.14 Let $\underline{\mathbb{A}}(C_4)$ be the Burnside Mackey functor over C_4. Then $\underline{\mathbb{A}}(C_4)^{\sharp C_2}$ is the C_4/C_2-Mackey functor

$$\begin{array}{c} \mathbb{A}(C_4) \\ {\scriptstyle \mathrm{res}_{\underline{\mathbb{A}}}}\big(\quad \big){\scriptstyle \mathrm{tr}_{\underline{\mathbb{A}}}} \\ \mathbb{A}(C_2) \end{array}$$

The basis for retrieving G-information on $H\underline{M}^G_\star$ from the H and G/H computations is the following result:

Proposition 2.15 *Let $H \leq G$ and let $(H\underline{M})^H$ be the H-categorical fixed points spectrum of $H\underline{M}$. Then there is the following equivalence of G/H-spectra*:

$$(H\underline{M})^H \simeq H(\underline{M}^{\sharp H}).$$

Proof This comes from the following chain of isomorphisms, for $H \leq K \leq G$:

$$\pi_n^{K/H}(H\underline{M}^H) = [S^n, H\underline{M}^H]^{K/H} \cong [S^n, H\underline{M}]^K = \pi_n^K(H\underline{M}).$$

For $n = 0$, we then obtain that

$$\pi_0^{K/H}(H\underline{M}^H) \cong \pi_0^K(H\underline{M}) = \underline{M}(K) = \underline{M}^{\sharp H}(K/H) = \pi_0^{K/H}(H\underline{M}^{\sharp H}). \qquad \square$$

Now consider a G-representation V. Since G_V acts trivially on it, we can regard it as a G/G_V-representation. If $G_V \neq e$, then $a_k = 0$ and G/G_V is a cyclic 2-group of smaller order. Then we can reduce to G/G_V-computations by the following corollary:

Corollary 2.16 *Let V be such that $G_V \neq e$. Then*

$$H\underline{M}^G_V \cong (H\underline{M}^{\sharp G_V})^{G/G_V}_V.$$

The above isomorphism also preserves the Mackey functor structure. For $G_V \subset K \subset H$, we have

$$\mathrm{res}^H_K \leftrightarrow \mathrm{res}^{H/G_V}_{K/G_V} \quad \text{and} \quad \mathrm{tr}^H_K \leftrightarrow \mathrm{tr}^{H/G_V}_{K/G_V}.$$

2D Classes a_V and u_V

In this section we will define the classes a_V and u_V. We start with the definition of Euler classes a_V. Recall that $\underline{M}$ denotes an arbitrary Mackey functor and $\underline{R}$ is an arbitrary Green functor.

Definition 2.17 Let V be an actual irreducible G-representation and consider the inclusion of fixed points $i : S^0 \to S^V$. We define the element $a_V \in H\underline{R}^G_{-V}$ to be the class of the map

$$S^0 \xrightarrow{i} S^V \cong S^V \wedge S^0 \xrightarrow{1\wedge\eta} S^V \wedge H\underline{R}.$$

It factors through the ring maps $S^0 \to H\underline{\mathbb{A}} \to H\underline{R}$. This class is well-known as the Euler class of V. Note that $a_V = 0$ if $V^G \neq 0$, since $S^0 \to S^V$ will be equivariantly null-homotopic.

Remark 2.18 Let $W \in RO(G)$, V be an irreducible representation and let $x \in H\underline{M}^G_W$. Then multiplication by the element a_V is given by

$$S^W \cong S^0 \wedge S^W \xrightarrow{a_V\wedge x} S^V \wedge H\underline{M},$$

or

$$S^W \cong S^0 \wedge S^W \xrightarrow{a_V\wedge x} S^V \wedge H\underline{\mathbb{A}} \wedge H\underline{M} \xrightarrow{1\wedge\mu} S^V \wedge H\underline{M}.$$

where the second map uses the $H\underline{\mathbb{A}}$-module structure on $H\underline{M}$.

We can compute the homotopy Mackey functors at the degrees of the Euler classes:

Proposition 2.19 *Let $V = a_1\alpha + \sum_{i=2}^k a_i\lambda_i$ be an actual G-representation. Recall that $\lambda_{\min}$ is the first irreducible representation with nonzero coefficient in the list $\alpha, \lambda_2, \dots, \lambda_k$ (see Definition 2.1). Then $H\underline{M}^G_{-V} \cong \underline{M}(G)/\mathrm{tr}^G_{G_{\lambda_{\min}}}$. For the Mackey functor structure, we have*

$$H\underline{M}_{-V}(H) = \begin{cases} \mathrm{coker}(\mathrm{tr}^H_{G_{\lambda_{\min}}}) & \text{if } G_{\lambda_{\min}} \subsetneq H, \\ 0 & \text{otherwise.} \end{cases}$$

For $K \subset H$, if $K \subset G_{\lambda_{\min}}$, then $\mathrm{res}^H_K, \mathrm{tr}^H_K$ are both 0. For $G_{\lambda_{\min}} \subset K \subset H$, the maps res^H_K and tr^H_K are induced from that of $\underline{M}$.

Proof The first claim directly comes from the cellular chain of S^V from Lemma 2.5.

For the Mackey functor structure, if $H \subset G_{\lambda_{\min}}$, then $H\underline{M}^{G_{\lambda_{\min}}}_{-V} \cong H\underline{M}^{G_{\lambda_{\min}}}_{-|V|} = 0$ by the shearing isomorphism (see Proposition 2.4) and the definition of $H\underline{M}$.

Now take $G_{\lambda_{\min}} \subset K \subset H$. The first two terms of the G/H and G/K-level cellular chains are

$$\begin{array}{ccccc} 0 & \longleftarrow & \underline{M}(G/H \times G/G) & \xleftarrow{1\times\nabla} & \underline{M}(G/H \times G/G_{\lambda_{\min}}) \\ & & \Big\downarrow{\scriptstyle \mathrm{res}^H_K} \quad \Big\uparrow{\scriptstyle \mathrm{tr}^H_K} & & \\ 0 & \longleftarrow & \underline{M}(G/K \times G/G) & \xleftarrow{1\times\nabla} & \underline{M}(G/K \times G/G_{\lambda_{\min}}) \end{array}$$

where $\nabla\colon G/G_{\lambda_{\min}} \to G/G$ is the canonical projection of G-sets. We have the identifications

$$\underline{M}(G/H \times G/G_{\lambda_{\min}}) \cong \underline{M}(G \times_H i_H^* G/G_{\lambda_{\min}}) = \underline{M}\Big(G \times_H \coprod_{G/H} H/G_{\lambda_{\min}}\Big) \cong \bigoplus_{G/H} \underline{M}(G/G_{\lambda_{\min}})$$

and the map $1 \times \nabla$ is the sum of the maps $\mathrm{tr}^H_{G_{\lambda_{\min}}}$. Similarly for the G/K-level chain complex. Taking the 0^{th} homology proves the claim. □

Remark 2.20 It will be proved later that the map $a_V\colon H\underline{M}^G_0 \to H\underline{M}^G_{-V}$ is the projection on the cokernel of $\mathrm{tr}^G_{G_V}$ (see Proposition 3.10). For a Green functor $\underline{R}$, we have $a_V = \bar{1} \in \underline{R}(G)/\mathrm{tr}^G_{G_V}$, the class represented by $1 \in \underline{R}(G)$.

Example 2.21 If $G = C_4$, $\underline{R} = \underline{\mathbb{A}}$, we have

$$H\underline{\mathbb{A}}_{-\alpha} \xleftarrow{=} \underline{\mathbb{A}}(C_4)/\mathrm{tr}^{C_4}_{C_2} \qquad H\underline{\mathbb{A}}_{-\lambda} \xleftarrow{=} \underline{\mathbb{A}}(C_4)/\mathrm{tr}^{C_4}_e$$

$$\Big\downarrow{\scriptstyle 0} \quad \Big\uparrow{\scriptstyle 0} \qquad\qquad \Big\downarrow{\scriptstyle \overline{\mathrm{res}^{C_4}_{C_2}}} \quad \Big\uparrow{\scriptstyle \overline{\mathrm{tr}^{C_4}_{C_2}}}$$

$$0 \qquad\qquad \underline{\mathbb{A}}(C_2)/\mathrm{tr}^{C_2}_e$$

$$\Big\downarrow{\scriptstyle 0} \quad \Big\uparrow{\scriptstyle 0} \qquad\qquad \Big\downarrow{\scriptstyle 0} \quad \Big\uparrow{\scriptstyle 0}$$

$$0 \qquad\qquad 0$$

Let $\underline{\mathbb{A}}(C_4) = \mathbb{Z}\langle 1, c, d\rangle$, where $c = [C_4/C_2]$, $d = [C_4/e]$, and $\mathbb{A}(C_2) = \mathbb{Z}\langle 1, \omega\rangle$, where $\omega = [C_2/e]$. Then $H\underline{\mathbb{A}}_{-\lambda}(G/G) = \mathbb{Z}\langle a_\lambda\rangle \oplus \mathbb{Z}\langle ca_\lambda\rangle$, and $H\underline{\mathbb{A}}_{-\lambda}(G/C_2) = \mathbb{Z}\langle \mathrm{res}^{C_4}_{C_2} a_\lambda\rangle$. We have $\overline{\mathrm{tr}^{C_4}_{C_2}}(\mathrm{res}^{C_4}_{C_2} a_\lambda) = ca_\lambda$.

Now we turn our attention to the second family of generators, the orientation classes u_V which will be defined in Definition 2.25. Recall the definition of orientable representations in Definition 2.1.

Remark 2.22 Note that the only irreducible nonorientable representation of G is the sign representation α. In particular, $V = a_0 + a_1\alpha + \sum_{2\le i\le k} a_i\lambda_i \in RO(G)$ is nonorientable if and only if a_1 is odd and every nonorientable representation V can be written as $V = V' \pm \alpha$ with orientable V'.

We first study the Mackey functor structure on the $(|V|-V)$-degree.

Proposition 2.23 *Let $G = C_{2^k}$. Let V be an orientable (not necessarily irreducible) G-representation. Then $H\underline{M}^G_{|V|-V} \cong \underline{M}(G_V)^G$. For the Mackey functor structure, we have*

$$H\underline{M}_{|V|-V}(H) = \begin{cases} \underline{M}(G_V)^H & \text{if } G_V \subsetneq H, \\ \underline{M}(H) & \text{otherwise.} \end{cases}$$

For $K \subset H \subset G_V$, the maps $\mathrm{res}^H_K, \mathrm{tr}^H_K$ are induced from that of $\underline{M}$. If $G_V \subset K \subset H$, then res^H_K is induced by inclusion of fixed points, and tr^H_K is given by $\mathrm{tr}^{C_{2^i}}_{C_{2^{i-1}}}(x) = (1+\gamma^{2^{k-i}})\cdot x$ and transitivity.

Proof S^V has a cellular filtration with cells in top two dimensions given by $G/G_{V+} \wedge S^{|V|-1}$ and $G/G_{V+} \wedge S^{|V|}$. So its chain complex with coefficient in $\underline{M}$ has the following form in top dimensions:

$$\cdots \leftarrow \underline{M}(G/G_V) \xleftarrow{1-\gamma} \underline{M}(G/G_V).$$

The kernel of the top differential is $\underline{M}(G_V)^G$. Thus the first claim follows.

For the claim about the Mackey functor structure, notice that the lattice of subgroups of G has a linear order. By transitivity of restrictions and transfers, we may assume that $[H : K] = 2$.

When $H \subset G_V$, the shearing isomorphism $G/H_+ \wedge S^{V-|V|} \cong G/H_+ \wedge S^0$ (see Proposition 2.4) implies an isomorphism of Mackey functors $H\underline{M}_0{\downarrow}^G_{G_V} \cong H\underline{M}_{|V|-V}{\downarrow}^G_{G_V}$. Now let $G_V \subset K \subset H = C_{2^i} = \langle \gamma^{2^{k-i}} \rangle$. Thus $K = C_{2^{i-1}} = \langle \gamma^{2^{k-i+1}} \rangle$. Let $\nabla\colon G/K \to G/H$ be the natural projection. In the commutative diagram

$$\begin{array}{ccc} \underline{M}(G/H \times G/G_V) & \xleftarrow{1\times(1-\gamma)} & \underline{M}(G/H \times G/G_V) \\ {\scriptstyle \nabla\times 1}\downarrow & & \downarrow{\scriptstyle \nabla\times 1} \\ \underline{M}(G/K \times G/G_V) & \xleftarrow{1\times(1-\gamma)} & \underline{M}(G/K \times G/G_V) \end{array}$$

the first and second rows are the differential $C_{|V|-1}(S^V; \underline{M}) \xleftarrow{1-\gamma} C_{|V|}(S^V; \underline{M})$ on the G/H and G/K-levels, respectively. We have the identifications

$$\begin{aligned} \underline{M}(G/H \times G/G_V) \cong [G/H_+ \wedge G/G_{V+}, H\underline{M}]^G &\cong [G_+ \wedge_H i_H^*(G/G_{V+}), H\underline{M}]^G \\ &\cong [i_H^*(G/G_{V+}), H\underline{M}]^H \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* \underline{M}(G_V) \end{aligned}$$

and similarly for $\underline{M}(G/K \times G/G_V)$. Here the notation $i_H^*\underline{M}(G_V)$ means we are restricting the Weyl G/G_V-action to an H/G_V-action on $\underline{M}(G_V)$. To understand the differential $1 \times (1-\gamma)$, we have to understand the action of γ on $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* \underline{M}(G_V)$ coming from $\gamma\colon G/G_V \to G/G_V$. Tracing through the identifications we conclude that it acts by left multiplication on $\mathbb{Z}[G]$.

The commutative diagram above becomes

$$\begin{array}{ccc} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* \underline{M}(G_V) & \xleftarrow{(1-\gamma)} & \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} i_H^* \underline{M}(G_V) \\ \downarrow & & \downarrow \\ \mathbb{Z}[G] \otimes_{\mathbb{Z}[K]} i_K^* \underline{M}(G_V) & \xleftarrow{(1-\gamma)} & \mathbb{Z}[G] \otimes_{\mathbb{Z}[K]} i_K^* \underline{M}(G_V) \end{array}$$

where both $(1-\gamma)$ are given by left multiplication on $\mathbb{Z}[G]$. The kernels of these two $(1-\gamma)$ are

$$(\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} i_H^*\underline{M}(G_V))^G \cong \mathbb{Z}[G/H]^G \otimes \underline{M}(G_V)^H \cong \underline{M}(G_V)^H,$$

and

$$(\mathbb{Z}[G]\otimes_{\mathbb{Z}[K]} i_H^*\underline{M}(G_V))^G \cong \mathbb{Z}[G/K]^G \otimes \underline{M}(G_V)^K \cong \underline{M}(G_V)^K.$$

The induced maps on kernels is the map induced from the G-equivariant map $\mathbb{Z}[G/H]\to\mathbb{Z}[G/K]$ by sending 1 to $1+\gamma^{2^{k-i}}$, which can be easily seen to be $\mathbb{Z}\xrightarrow{1}\mathbb{Z}$ upon taking $(-)^G$. Thus restrictions are induced by inclusion of fixed points.

For the transfers, we have the double-coset formula

$$\mathrm{res}^H_K \circ \mathrm{tr}^H_K(x) = (1+\gamma^{2^{k-i}})\cdot x.$$

Since restrictions are inclusions of subgroups, we deduce $\mathrm{tr}^H_K(x) = (1+\gamma^{2^{k-i}})\cdot x$. □

Construction 2.24 For an orientable G-representation V, consider the commutative diagram

$$\begin{array}{ccccc} & & \underline{M}(G) & & \\ & \swarrow & \downarrow{\scriptstyle \mathrm{res}^G_{G_V}} & & \\ H\underline{M}^G_{|V|-V} \cong \underline{M}(G_V)^G & \longrightarrow & \underline{M}(G_V) & \xrightarrow{1-\gamma} & \underline{M}(G_V) \end{array}$$

The map $(1-\gamma)$ is the top differential of $C_*(S^V;\underline{M})$. The dashed arrow is induced from $\mathrm{res}^G_{G_V}$, since for any $H\subset G$, we have $\mathrm{res}^G_H(\underline{M}(G))\subset\underline{M}(H)^G$. We make the following definition of orientation classes for Green functors.

Definition 2.25 For V an orientable G-representation, $\underline{R}$ a Green functor, we define the *orientation class* u_V to be the element $\mathrm{res}^G_{G_V}(1)\in H\underline{R}^G_{|V|-V}$.

Remark 2.26 (1) Note that by Construction 2.24 we obtain that the multiplication by

$$u_V\colon H\underline{R}^G_0 \to H\underline{R}^G_{|V|-V}$$

is the restriction map $\mathrm{res}^G_{G_V}$, by the following commutative diagram

$$\begin{array}{ccc} H\underline{R}^G_0 & \xrightarrow{u_V} & H\underline{R}^G_{|V|-V} \\ {\scriptstyle \mathrm{res}^G_{G_V}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}(|V|-V)^G_{G_V}} \\ H\underline{R}^{G_V}_0 & \xrightarrow{u_V} & H\underline{R}^{G_V}_{|V|-V} \end{array}$$

where $\mathrm{res}(|V|-V)^G_{G_V}$ is inclusion of G-fixed points, and the horizontal map in the second row is an isomorphism.

(2) Definition 2.25 is restricted to Green functors. However, since every Mackey functor $\underline{M}$ is a module over Burnside Mackey functor $\underline{\mathbb{A}}$, we can consider the induced action of $H\underline{\mathbb{A}}^G_\star$ on $H\underline{M}^G_\star$. We can still deduce $u_V\colon H\underline{M}^G_0 \to H\underline{M}^G_{|V|-V}$ is $\mathrm{res}^G_{G_V}$ as above. If $\underline{R}$ is a Green functor, its orientation class coincides with the image of the orientation class of $H\underline{\mathbb{A}}$ through the canonical ring map $H\underline{\mathbb{A}} \to H\underline{R}$.

Example 2.27 (1) When $\underline{R} = \underline{\mathbb{Z}}$, $H\underline{\mathbb{Z}}_{|V|-V} = \underline{\mathbb{Z}}$, and the definition of orientation classes here agrees with the orientation classes used widely in previous computations of [11; 13; 25; 7; 24].

(2) When $\underline{R} = \underline{\mathbb{A}}$ and $G = C_4$, the Weyl actions are always trivial, and we have

$$\begin{array}{ccc} H\underline{\mathbb{A}}_{2-2\alpha} & \xleftarrow{=} & \underline{\mathbb{A}}(C_2) \\ & & {\scriptstyle 1}\big\downarrow \big\uparrow{\scriptstyle 2} \\ & & \underline{\mathbb{A}}(C_2) \\ & & {\scriptstyle \mathrm{res}^{C_2}_e}\big\downarrow \big\uparrow{\scriptstyle \mathrm{tr}^{C_2}_e} \\ & & \underline{\mathbb{A}}(e) \end{array} \qquad \begin{array}{ccc} H\underline{\mathbb{A}}_{2-\lambda} & \xleftarrow{=} & \underline{\mathbb{A}}(e) \\ & & {\scriptstyle 1}\big\downarrow \big\uparrow{\scriptstyle 2} \\ & & \underline{\mathbb{A}}(e) \\ & & {\scriptstyle 1}\big\downarrow \big\uparrow{\scriptstyle 2} \\ & & \underline{\mathbb{A}}(e) \end{array}$$

If $\mathbb{A}(C_2) = \mathbb{Z}\langle 1, \omega\rangle$, where $\omega = [C_2/e]$, then $u_{2\alpha} \in H\underline{\mathbb{A}}_{2-2\alpha}(G/G)$ correspond the generator 1 and $H\underline{\mathbb{A}}_{2-2\alpha}(G/G) = \mathbb{Z}\langle u_{2\alpha}\rangle \oplus \mathbb{Z}\langle \omega u_{2\alpha}\rangle$. This agrees with a general definition of orientation classes for all finite groups of Kriz [15].

The following two propositions allow us to restrict our attention only to classes coming from irreducible representations.

Proposition 2.28 *Let $G = C_{2^k}$ and V, W be two G-representations. Then the following equalities hold in $\pi^G_\star S^0$, thus hold in $H\underline{R}^G_\star$:*

$$a_V a_W = a_W a_V = a_{V+W} = a_{W+V}.$$

Proof $RO(G)$-graded commutativity is studied by Dugger in [4]. Using his theory, we will show that these equalities actually hold in the G-equivariant stable stems. Since for any two G-representation V_1, V_2 we have $a_{V_1}a_{V_2} = a_{V_1+V_2}$, we can assume that V, W are irreducible. Using the methods in [24, Appendix B], we can compute for each λ_i, $\tau(S^{\lambda_i}) = 1$ and $\tau(S^\alpha) = 1 - \mathrm{tr}^G_{G'} \circ \mathrm{res}^G_{G'}$. Here $\tau(X) \in \pi^G_0 S^0$ is the trace of identity of a finite G-complex X studied in [4, Section 4.19].

Now we will use [4, Proposition 1.2]. Take any $x \in \pi^G_{-V} S^0$, $y \in \pi^G_{-W} S^0$. If $V = \lambda_i$ and $W = \lambda_j$ for some $1 \le i, j \le k$, then $xy = \tau(S^{\lambda_i})^{1\cdot 1} yx = yx$ if $i = j$, and $xy = \tau(S^{\lambda_i})^0 \tau(S^{\lambda_j})^0 yx = yx$ if $i \ne j$.

If $V = \lambda_i$ for $1 \le i \le k$, and $W = \alpha$, then $xy = \tau(S^{\lambda_i})^0 \tau(S^\alpha)^0 yx = yx$.

If $V = W = \alpha$, then $xy = \tau(S^\alpha)^{1\cdot 1} yx = (1 - \mathrm{tr}^G_{G'} \circ \mathrm{res}^G_{G'}) yx$. The class $\mathrm{tr}^G_{G'} \circ \mathrm{res}^G_{G'}\, yx$ may not be zero in general. But if $x = y = a_\alpha$, it becomes zero because $\mathrm{res}^G_{G'} a_\alpha = 0$. □

Proposition 2.29 *Let V, W are two orientable G-representations. Assume $G_V \subset G_W$. Regard $H\underline{M}$ as an $H\underline{\mathbb{A}}$-module. Then*:

(1) *The map $u_V\colon H\underline{M}^G_{|W|-W} \to H\underline{M}^G_{|V|+|W|-V-W}$ is the restriction map $\mathrm{res}^{G_W}_{G_V}$.*

(2) *The map $u_W\colon H\underline{M}^G_{|V|-V} \to H\underline{M}^G_{|V|+|W|-V-W}$ is an isomorphism.*

(3) *We have that $u_V u_W = u_W u_V$ in $H\underline{\mathbb{A}}_\star$, and thus in $H\underline{R}_\star$ for all Green functors $\underline{R}$.*

Proof Consider the following commutative diagram where horizontal maps are restriction maps in $H\underline{M}_{|W|-W}$ and $H\underline{M}_{|V|+|W|-V-W}$, and vertical maps are multiplication by u_V and its restrictions:

$$\begin{array}{ccccc}
H\underline{M}^G_{|W|-W} & \xrightarrow{\mathrm{res}^G_{G_W}} & H\underline{M}^{G_W}_{|W|-W} & \xrightarrow{\mathrm{res}^{G_W}_{G_V}} & H\underline{M}^{G_V}_{|W|-W} \\
\downarrow{\scriptstyle u_V} & & \downarrow{\scriptstyle u_V} & & \downarrow{\scriptstyle u_V}\cong \\
H\underline{M}^G_{|V|+|W|-V-W} & \xrightarrow{\mathrm{res}^G_{G_W}} & H\underline{M}^{G_W}_{|V|+|W|-V-W} & \xrightarrow{\mathrm{res}^{G_W}_{G_V}} & H\underline{M}^{G_V}_{|V|+|W|-V-W}
\end{array}$$

By Theorem 2.6, the above diagram is

$$\begin{array}{ccccc}
\underline{M}(G_W)^G & \xrightarrow{i} & \underline{M}(G_W) & \xrightarrow{\mathrm{res}^{G_W}_{G_V}} & \underline{M}(G_V) \\
\downarrow & & \downarrow & & \downarrow\cong \\
\underline{M}(G_V)^G & \xrightarrow{i} & \underline{M}(G_V)^{G_W} & \xrightarrow{i} & \underline{M}(G_V)
\end{array}$$

All maps labeled i are inclusion of fixed points, and we deduce the first vertical map is induced from $\mathrm{res}^{G_W}_{G_V}$. This proves (1). An analogous argument will prove (2).

Then we see that the following isomorphic diagrams commute:

$$\begin{array}{ccc}
H\underline{M}^G_0 & \xrightarrow{u_W} & H\underline{M}^G_{|W|-W} \\
\downarrow{\scriptstyle u_V} & & \downarrow{\scriptstyle u_V} \\
H\underline{M}^G_{|V|-V} & \xrightarrow{u_W} & H\underline{M}^G_{|V|+|W|-V-W}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{M}(G) & \xrightarrow{\mathrm{res}^G_{G_W}} & \underline{M}(G_W)^G \\
\downarrow{\scriptstyle \mathrm{res}^G_{G_V}} & & \downarrow{\scriptstyle \mathrm{res}^{G_W}_{G_V}} \\
\underline{M}(G_V)^G & \xrightarrow{1} & \underline{M}(G_V)^G
\end{array}$$

When $\underline{M} = \underline{\mathbb{A}}$, the two images of $1 \in H\underline{\mathbb{A}}^G_0$ in $H\underline{\mathbb{A}}^G_{|V|+|W|-V-W}$ agree, which proves (3). □

3 Vanishing, a, u-isomorphism regions and the gold relation

In this section we determine the parts of the $RO(G)$-graded homotopy Mackey functors $H\underline{M}_\star$ where multiplication by different Euler and orientation classes are isomorphism. We also study the parts when they vanish and deduce a gold relation for Green functors $\underline{R}$, generalizing the original gold relation of [13].

3A $a_{\lambda_s}, u_{\lambda_s}$-isomorphism regions

In this subsection we study the $RO(G)$-graded regions where multiplications by a_{λ_s}, a_α and u_{λ_s} are isomorphisms, respectively. For simplicity, we write $\lambda = \lambda_k$, as it will play a very important role. Recall that we are working over $G = C_{2^k}$, thus λ is the faithful representation.

We start with definitions of Mackey functors which will be used for the description of the homology of a unit representation sphere.

Definition 3.1 [22] For $H \subset G = C_{2^k}$, let $\underline{M}\downarrow_H^G$ be the H-Mackey functor defined by

$$\underline{M}\downarrow_H^G(H/K) = M(G/K), \quad K \subset H,$$

with restrictions and transfers induced from that of $\underline{M}$. This is the *restricted* Mackey functor. The restriction functor $\downarrow_H^G : \mathrm{Mack}_G \to \mathrm{Mack}_H$ is the right adjoint of the induction functor $\uparrow_H^G$.

Definition 3.2 [22] Let $G = C_{2^k}$ and let M be a $\mathbb{Z}[G]$-module.

(1) The *fixed-point Mackey functor* $FP(M)$ is defined by

$$FP(M)(G/H) = M^H.$$

Since restrictions and transfers are transitive, we specify them for $H = C_{2^i} = \langle \gamma^{2^{k-i}} \rangle$ and $K = C_{2^{i-1}}$ for $1 \le i \le k$. We have $\mathrm{res}_K^H = 1$ and $\mathrm{tr}_K^H = 1 + \gamma^{2^{k-i}}$.

(2) The *fixed-quotient Mackey functor* $FQ(M)$ is defined by

$$FQ(M)(G/H) = M/H.$$

Since restrictions and transfers are transitive, we specify them for $H = C_{2^i} = \langle \gamma^{2^{k-i}} \rangle$ and $K = C_{2^{i-1}}$ for $1 \le i \le k$. We have $\mathrm{res}_K^H = 1 + \gamma^{2^{k-i}}$ and $\mathrm{tr}_K^H = 1$.

Recall that we have the forgetful functor $U : \mathrm{Mack}_G \to \mathbb{Z}[G]\text{-mod}$, $\underline{M} \mapsto \underline{M}(e)$. Then FP is the right adjoint of U and FQ is the left adjoint.

Definition 3.3 Let $G = C_{2^k}$ and let M be a $\mathbb{Z}[G]$-module. Define a new $\mathbb{Z}[G]$-module structure on M, denoted by $\widetilde{M}$, via the rule $\gamma * m := -\gamma \cdot m$, where the right-hand-side action is the original G-action on M. Then we define

(1) the *twisted fixed-point Mackey functor* $TFP(M)$ by

$$TFP(M) = FP(\widetilde{M}),$$

(2) the *twisted fixed-quotient Mackey functor* $TFQ(M)$ by

$$TFQ(M) = FQ(\widetilde{M}).$$

Note that $\gamma^2 * m = \gamma^2 \cdot m$. This implies $TFP(M)\downarrow^G_{G'} = FP(M)\downarrow^G_{G'}$ and $TFQ(M)\downarrow^G_{G'} = FQ(M)\downarrow^G_{G'}$. The top two levels of $TFP(M)$ and $TFQ(M)$ look like

$$\begin{array}{c} {}^{(1+\gamma)}M \\ {}_{1}\big\downarrow \big\uparrow_{1-\gamma} \\ {}^{(1-\gamma^2)}M \end{array} \qquad \text{and} \qquad \begin{array}{c} M/(1+\gamma) \\ {}_{1-\gamma}\big\downarrow \big\uparrow_{1} \\ M/(1-\gamma^2) \end{array}$$

Remark 3.4 (1) The action on $\widetilde{M}$ is the same as the diagonal action on $\widetilde{\mathbb{Z}} \otimes_{\mathbb{Z}} M$, where $\widetilde{\mathbb{Z}}$ is the sign representation of G. To avoid confusion, if $\underline{M}$ is a Mackey functor, we will always use $\gamma \cdot m$ to denote the canonical Weyl action for $m \in \underline{M}(H)$, $H \subset G$.

(2) The adjective "twisted" comes from the fact that they arise from $H\underline{M}^G_U$ when U is nonorientable. The top differential in this case is $1+\gamma$ on $\underline{M}(e)$, which corresponds to $1-\gamma$ on $\widetilde{\underline{M}(e)}$.

Consider now the cofiber sequence

$$S(V)_+ \to S^0 \to S^V.$$

We can use this cofiber sequence to compute the groups $H\underline{M}^G_{nV}$ for an irreducible actual G-representation and $n \in \mathbb{Z}$ and the action of the element a_λ on $H\underline{M}^G_\star$. We will first need the following lemma.

Lemma 3.5 *The $RO(G)$-graded homotopy Mackey functor $H\underline{M}_V(S(\lambda)_+)$ is given as*

$$H\underline{M}_V(S(\lambda)_+) = \begin{cases} FQ(\underline{M}(e)) & \text{if } |V| = 0 \text{ and } V \text{ is orientable,} \\ TFQ(\underline{M}(e)) & \text{if } |V| = 0 \text{ and } V \text{ is nonorientable,} \\ FP(\underline{M}(e)) & \text{if } |V| = 1 \text{ and } V \text{ is orientable,} \\ TFP(\underline{M}(e)) & \text{if } |V| = 1 \text{ and } V \text{ is nonorientable,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof We first compute the G/G-level abelian group and claim it is as

$$H\underline{M}^G_V(S(\lambda)_+) = \begin{cases} \underline{M}(e)/(1-\gamma) & \text{if } |V| = 0 \text{ and } V \text{ is orientable,} \\ \underline{M}(e)/(1+\gamma) & \text{if } |V| = 0 \text{ and } V \text{ is nonorientable,} \\ \underline{M}(e)^G & \text{if } |V| = 1 \text{ and } V \text{ is orientable,} \\ {}^{(1+\gamma)}\underline{M}(e) & \text{if } |V| = 1 \text{ and } V \text{ is nonorientable,} \\ 0 & \text{otherwise.} \end{cases}$$

The unit sphere $S(\lambda)$ is the homotopy cofiber in the sequence

$$G/e \xrightarrow{1-\gamma} G/e \longrightarrow S(\lambda).$$

By the long exact sequence in homotopy associated to this cofiber sequence we get

$$H\underline{M}^G_{V+1}(G/e_+) \to H\underline{M}^G_{V+1}(S(\lambda)_+) \to H\underline{M}^G_V(G/e_+) \xrightarrow{1-\gamma} H\underline{M}^G_V(G/e_+) \to H\underline{M}^G_V(S(\lambda)_+) \to H\underline{M}^G_{V-1}(G/e_+).$$

For any G-representation W we have that $H\underline{M}^G_W(G/e_+) \cong H\underline{M}^e_{|W|}$. In particular, it is zero if $|W| \neq 0$ and equal to $M(e)$ if $|W| = 0$ as abelian groups. As $\mathbb{Z}[G]$-modules, the action depends on the orientability of W. If W is such that $|W| = 0$ and orientable, then $H\underline{M}^G_W(G/e_+) = \underline{M}(e)$; if W is nonorientable, then $H\underline{M}^G_W(G/e_+) = \widetilde{\underline{M}(e)}$. We can see this fact by looking at the shearing isomorphisms (see Proposition 2.4)

$$\begin{array}{ccc} G/e_+ \wedge S^W & \xrightarrow{\gamma\wedge 1} & G/e_+ \wedge S^W \\ \downarrow\cong & & \downarrow\cong \\ G_+ \wedge_e S^{|W|} & \longrightarrow & G_+ \wedge_e S^{|W|} \end{array} \qquad \begin{array}{ccc} (g,a) & \longmapsto & (\gamma g, a) \\ \downarrow & & \downarrow \\ (g, g^{-1}a) & \longmapsto & (\gamma g, g^{-1}\gamma^{-1}a) \end{array}$$

Therefore in the sequence above the third and fourth arguments are zero unless $|V| = 0$. In the latter case we have that the outer groups are equal to zero and the long exact sequence above takes the form

$$0 \longrightarrow H\underline{M}^G_{V+1}(S(\lambda)_+) \longrightarrow \underline{M}(e) \xrightarrow{1-\gamma} \underline{M}(e) \longrightarrow H\underline{M}^G_V(S(\lambda)_+) \longrightarrow 0$$

or

$$0 \longrightarrow H\underline{M}^G_{V+1}(S(\lambda)_+) \longrightarrow \widetilde{\underline{M}(e)} \xrightarrow{1-\gamma} \widetilde{\underline{M}(e)} \longrightarrow H\underline{M}^G_V(S(\lambda)_+) \longrightarrow 0,$$

depending on the orientability of V. The conclusion on the abelian group structure follows. The Mackey functor structure can be computed as in Remark 2.8. □

Remark 3.6 Using the same cofiber sequence we can compute the $H\underline{M}$-cohomology Mackey functor of $S(\lambda)_+$ and obtain that $H\underline{M}^{-V+1}(S(\lambda)_+) \cong H\underline{M}_V(S(\lambda)_+)$ as Mackey functors. This comes from the fact that $S(\lambda)_+ \simeq S^1 \wedge D(S(\lambda)_+)$.

Proposition 3.7 (a_λ-isomorphism region) *Let V be a G-representation such that $|V| \neq 0, 1, 2$. Then the multiplication map $a_\lambda\colon H\underline{M}_V \to H\underline{M}_{V-\lambda}$ is an isomorphism of Mackey functors.*

Proof Consider the cofiber sequence $S(\lambda)_+ \to S^0 \to S^\lambda$. Taking the V^{th} homology we obtain

$$H\underline{M}^G_V(S(\lambda)_+) \longrightarrow H\underline{M}^G_V \longrightarrow H\underline{M}^G_{V-\lambda} \longrightarrow H\underline{M}^G_{V-1}(S(\lambda)_+).$$

If $|V| \neq 0, 1, 2$, the outer groups are zero by Lemma 3.5. This is also true when we replace G by any subgroup of G. Thus the claim follows. □

Remark 3.8 Note that traditionally the term "a_λ-periodic classes" is used for classes that support an infinite a_λ-tower. From Proposition 3.7 we know nontrivial classes of total dimensions ≤ -1 are a_λ-periodic.

Proposition 3.9 *Let V be a G-representation such that $|V| = 0$ and recall that* res(V) *and* tr(V) *are structure maps of $H\underline{M}_V$. Then*:

(1) $H\underline{M}^G_{V-\lambda} \cong \operatorname{coker}(\operatorname{tr}(V)^G_e)$ *and the map $a_\lambda\colon H\underline{M}^G_V \to H\underline{M}^G_{V-\lambda}$ is the projection onto the cokernel.*

(2) $H\underline{M}^G_{V+\lambda} \cong \ker(\operatorname{res}(V)^G_e)$ *and the map $a_\lambda\colon H\underline{M}^G_{V+\lambda} \to H\underline{M}^G_V$ is the inclusion of the kernel.*

The Mackey functor structures on $H\underline{M}^G_{V-\lambda}$ and $H\underline{M}^G_{V+\lambda}$ are induced from that of $H\underline{M}^G_V$.

Proof (1) Consider the long exact sequence obtained by smashing cofiber sequence $S(\lambda)_+ \to S^0 \to S^\lambda$ with $S^V \wedge H\underline{M}$:

$$H\underline{M}^G_V(S(\lambda)_+) \longrightarrow H\underline{M}^G_V \longrightarrow H\underline{M}^G_{V-\lambda} \longrightarrow H\underline{M}^G_{V-1}(S(\lambda)_+).$$

Since $|V| = 0$, by Lemma 3.5 the rightmost group is 0. To understand the first map in the sequence above, consider the commutative diagram of cofiber sequences

$$\begin{array}{ccccc} G/e_+ & \longrightarrow & S^0 & \longrightarrow & S^\lambda_{(1)} \\ \downarrow & & \downarrow & & \downarrow \\ S(\lambda)_+ & \longrightarrow & S^0 & \longrightarrow & S^\lambda \end{array}$$

where $S^\lambda_{(1)}$ denotes the 1-skeleton of S^λ. The left vertical map is the inclusion of the bottom cell. Taking the V^{th} homology we obtain the commutative diagram

$$\begin{array}{ccccccc} M(e) & \longrightarrow & H\underline{M}^G_V & \longrightarrow & H\underline{M}^G_V(S^\lambda_{(1)}) & \longrightarrow & 0 \\ \downarrow & & \downarrow{=} & & \downarrow & & \\ M(e)/(1-\gamma) & \longrightarrow & H\underline{M}^G_V & \longrightarrow & H\underline{M}^G_{V-\lambda} & \longrightarrow & 0 \end{array}$$

The first top horizontal map exhibits the transfer in the Mackey functor $H\underline{M}^\bullet_V$. Therefore the bottom left map is the map on $M(e)/1-\gamma$ induced by $\mathrm{tr}(V)$. Thus the claim follows.

(2) This comes from a similar analysis by taking the V^{th} cohomology.

For the Mackey functor structure in (1), by induction, we only need to consider the restriction and transfer between adjacent subgroups. In the following diagram

$$\begin{array}{ccc} H\underline{M}^G_V & \longrightarrow & H\underline{M}^G_V/\mathrm{tr}(V)^G_e \\ {\scriptstyle \mathrm{tr}^G_{G'}}\uparrow \downarrow{\scriptstyle \mathrm{res}^G_{G'}} & & {\scriptstyle \overline{\mathrm{tr}^G_{G'}}}\uparrow \downarrow{\scriptstyle \overline{\mathrm{res}^G_{G'}}} \\ H\underline{M}^{G'}_V & \longrightarrow & H\underline{M}^{G'}_V/\mathrm{tr}(V)^{G'}_e \end{array}$$

both horizontal arrows are the natural surjections, and the result follows. The Mackey functor structure in (2) is completely analogous. □

Combining Propositions 3.7 and 3.9 we can deduce groups $H\underline{M}^G_{nV}$ for V orientable and irreducible.

Proposition 3.10 *Let V be an irreducible actual representation. Then we have that*

$$H\underline{M}^G_{nV} = \begin{cases} \underline{M}(G) & \text{if } n = 0, \\ \mathrm{coker}(\mathrm{tr}^G_{G_V}) & \text{if } n < 0, \\ \ker(\mathrm{res}^G_{G_V}) & \text{if } n > 0. \end{cases}$$

Moreover, we have that

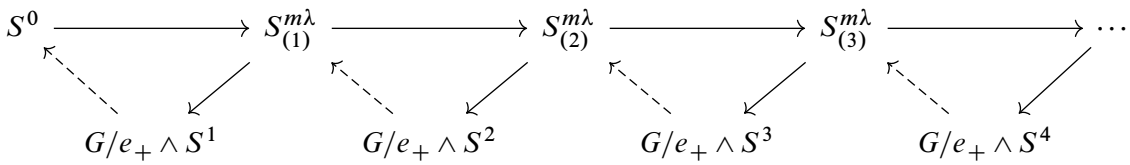
- *the map $a_V \colon H\underline{M}^G_0 \to H\underline{M}^G_{-V}$ is the projection onto the cokernel;*
- *the map $a_V \colon H\underline{M}^G_V \to H\underline{M}^G_0$ is the inclusion of the kernel;*
- *the multiplication $a_V \colon H\underline{M}^G_{nV} \to H\underline{M}^G_{(n-1)V}$ is an isomorphism for $n \in \mathbb{Z}$ such that $n \neq 0, 1$.*

Proof We first consider the case $V = \lambda$, the faithful representation. We then have that $H\underline{M}_0^G = \underline{M}(G)$ by the defining property of Eilenberg–MacLane spectra. For $n < 0$, the formula follows from Proposition 3.9 in case of $n = -1$ and from Proposition 3.7 for $n < -1$. The case $n > 0$ follows analogously.

Now assume that V is an irreducible representation. Then we have that $H\underline{M}_{nV}^G \cong H(\underline{M}^{\sharp G_V})_{nV}^{G/G_V}$. Assume that $V \neq \alpha$. The representation V is G/G_V-faithful, so by the previous paragraph we get that

$$H(\underline{M}^{\sharp G_V})_{nV}^{G/G_V} = \begin{cases} \underline{M}(G) & \text{if } n = 0, \\ \operatorname{coker}(\operatorname{tr}(\underline{M}^{\sharp G_V})_e^{G/G_V}) & \text{if } n < 0, \\ \ker(\operatorname{res}(\underline{M}^{\sharp G_V})_e^{G/G_V}) & \text{if } n > 0. \end{cases}$$

However, since the transfers and restrictions in the Mackey functor $\underline{M}^{\sharp G_V}$ are defined as transfers and restrictions in $\underline{M}$, so we obtain the abelian group structures as stated. For the multiplications by a_V, as above, we can assume $V = \lambda$, the faithful representation. Consider the following cellular filtration of $S^{m\lambda}$, $m \geq 1$:



Smash the diagram with $H\underline{M}$ and take π_0. Since $\pi_0^G G/e_+ \wedge S^i \wedge H\underline{M} = 0$ for $i \neq 0$, we see that the first horizontal map induce the projection onto the cokernel of tr_e^G, and the other horizontal maps induce isomorphisms. This proves the claim about $a_V \colon H\underline{M}_{nV}^G \to H\underline{M}_{(n-1)V}^G$ when $n \leq 0$. For $n > 0$, we can dualize the cellular structure.

The case $V = \alpha$ follows from the similar considerations. □

Using the multiplication by the element a_λ, we can prove the following vanishing result.

Lemma 3.11 (vanishing) *Let $V = a_0 + a_1\alpha + \sum_{i=2}^k a_i\lambda_i$ be such that one of the following conditions is satisfied*:

(1) $a_0 > 0$, $a_0 + a_1 > 0$ *and* $a_0 + a_1 + \sum_{i=2}^j a_i > 0$ *for all* $2 \leq j \leq k$, *or*

(2) $a_0 < 0$, $a_0 + a_1 < 0$ *and* $a_0 + a_1 + \sum_{i=2}^j a_i < 0$ *for all* $2 \leq j \leq k$.

Then $H\underline{M}_V = 0$.

Proof We will proceed by induction on the order of G. The claim is true for the trivial group. Now assume that the statement is correct for $G' = C_{2^{k-1}}$. Thus the vanishing is true for all subgroup level of the Mackey functor. Consider the first case with all sums simultaneously greater than 0. We then have that

$$H\underline{M}_{a_0+a_1\alpha+a_2\lambda_2+\cdots+a_{k-1}\lambda_{k-1}}^G \cong (H\underline{M}^{\sharp C_2})_{a_0+a_1\alpha+a_2\lambda_2+\cdots+a_{k-1}\lambda_{k-1}}^{G/C_2},$$

and the latter is 0 by the inductive assumption (note that the last index in the sum missing). By Proposition 3.7 the map

$$a_{\lambda_k}^{a_k} : H\underline{M}_V^G \to H\underline{M}_{a_0+a_1\alpha+a_2\lambda_2+\cdots+a_{k-1}\lambda_{k-1}}^G$$

is an isomorphism, because the target has total dimension greater than 0. Thus the claim follows. The proof of the second case is completely analogous. □

Using vanishing (Lemma 3.11) we can describe a_{λ_i}-isomorphism regions for $2 \le i \le k$.

Corollary 3.12 (a_{λ_s}-isomorphism region) *For C_{p^k} with p odd, let $V = a_0 + \sum_{i=1}^k a_i\lambda_i$. For each s such that $1 \le s \le k$, multiplication by $a_{\lambda_s} : H\underline{M}_V \to H\underline{M}_{V-\lambda_s}$ is an isomorphism of Mackey functors when one of the following conditions is satisfied*:

(1) $a_0 + 2\sum_{i=1}^j a_i > 2$ *for all* $s \le j \le k$, *or*

(2) $a_0 + 2\sum_{i=1}^j a_i < 0$ *for all* $s \le j \le k$.

For C_{2^k}, let

$$V = a_0 + a_1\alpha + \sum_{i=2}^k a_i\lambda_i.$$

For each s such that $2 \le s \le k$, multiplication by $a_{\lambda_s} : H\underline{M}_V \to H\underline{M}_{V-\lambda_s}$ is an isomorphism of Mackey functors when one of the following conditions is satisfied:

(1) $a_0 + a_1 + 2\sum_{i=2}^j a_i > 2$ *for all* $s \le j \le k$, *or*

(2) $a_0 + a_1 + 2\sum_{i=2}^j a_i < 0$ *for all* $s \le j \le k$.

Multiplication by $a_\alpha : H\underline{M}_V \to H\underline{M}_{V-\alpha}$ is an isomorphism if one of the following conditions is satisfied:

(1) $a_0 + a_1 > 0, a_0 + a_1 + 2\sum_{i=2}^j a_i > 1$ *for all* $2 \le j \le k$, *or*

(2) $a_0 + a_1 < 0, a_0 + a_1 + 2\sum_{i=2}^j a_i < 0$ *for all* $2 \le j \le k$.

Proof The proof for p odd and $p = 2$ are analogous. We will assume $p = 2$, and use the notation $\lambda_1 = 2\alpha$. For each $2 \le s \le k$, the sphere $S(\lambda_s)$ is a two-cell complex, and we have a cofiber sequence

$$C_{2^k}/C_{2^{k-s}+} \to S(\lambda_s)_+ \to C_{2^k}/C_{2^{k-s}+} \wedge S^1.$$

We get the following long exact sequence of Mackey functors

$$\cdots \to H\underline{M}_V(C_{2^k}/C_{2^{k-s}}) \to H\underline{M}_V(S(\lambda_s)) \to H\underline{M}_{V-1}(C_{2^k}/C_{2^{k-s}}) \to \cdots.$$

The two Mackey functors on the outside only cares about the $C_{2^{k-s}}$-representation

$$i^*_{C_{2^{k-s}}}(V) = a_0 + a_1 + 2a_2 + \cdots + 2a_s + a_{s+1}\lambda_1 + \cdots + a_k\lambda_{k-s}.$$

By the previous lemma, we deduce that $\underline{H_V(S(\lambda_s))} = 0$ when one of the following is satisfied:

(1) $a_0 + a_1 + 2\sum_{i=2}^j a_i > 1$ for all $s \le j \le k$, or

(2) $a_0 + a_1 + 2\sum_{i=2}^j a_i < 0$ for all $s \le j \le k$.

Then the cofiber sequence

$$S(\lambda_s)_+ \to S^0 \to S^{\lambda_s}$$

provides us with the long exact sequence

$$\cdots \to H\underline{M}_V(S(\lambda_s)) \to H\underline{M}_V \xrightarrow{a_{\lambda_s}} H\underline{M}_{V-\lambda_s} \to H\underline{M}_{V-1}(S(\lambda_s)) \to \cdots .$$

The map a_{λ_s} is an isomorphism when the two outside Mackey functors vanish. And we have finished the proof for a_{λ_s}. For a_α, the only difference is that the sphere $S(\alpha) = C_{2^k}/C_{2^{k-1}}$ is a one-cell complex. All the proofs are along the same line. □

Using the vanishing property, we can also deduce when multiplication by u_{λ_s} is an isomorphism. From Theorem 2.6 we deduce

$$\pi_*^G H\underline{M} \wedge S^{\lambda_i - 2} = H^G_{2+*}(S^{\lambda_i}; \underline{M}) = \begin{cases} \underline{M}(C_{2^{k-i}})^G & \text{if } * = 0, \\ \ker(\mathrm{tr}^G_{C_{2^{k-i}}})/(1-\gamma) & \text{if } * = -1, \\ \mathrm{coker}(\mathrm{tr}^G_{C_{2^{k-i}}}) & \text{if } * = -2, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then we look at the cofiber sequence

$$H\underline{M} \xrightarrow{u_{\lambda_i}} H\underline{M} \wedge S^{\lambda_i - 2} \to C_{u_{\lambda_i}}. \tag{5}$$

In general, the cofiber $C_{u_{\lambda_i}}$ is not an Eilenberg–MacLane spectrum (it is one in the special case of $\underline{M} = \underline{\mathbb{Z}}$), but it has a finite Postnikov tower, which enables us to extend the vanishing results to this case. Before that, first let us notice that the map u_{λ_i} is an $C_{2^{k-i}}$-equivariant equivalence, thus $i^*_{C_{2^{k-i}}} C_{u_{\lambda_i}} \simeq *$. Since S^{λ_s} is built from S^0 out of cells of orbit type $C_{2^k}/C_{2^{k-s}}$, we deduce:

Lemma 3.13 *The spectrum $C_{u_{\lambda_i}}$ is $a_{\lambda_i}, a_{\lambda_{i+1}}, \ldots, a_{\lambda_k}$-local.*

Corollary 3.14 (u_{λ_s}-isomorphism region) *For p odd, let $V = a_0 + \sum_{i=1}^k a_i\lambda_i$. For each s such that $1 \le s \le k$, multiplication by $u_{\lambda_s}\colon H\underline{M}_V \to H\underline{M}_{V+2-\lambda_s}$ is an isomorphism of Mackey functors when one of the following conditions is satisfied*:

(1) *$a_0 > 0$ and $a_0 + 2\sum_{i=1}^j a_i > 0$ for all $1 \le j \le s-1$, or*

(2) *$a_0 < -3$ and $a_0 + 2\sum_{i=1}^j a_i < -3$ for all $1 \le j \le s-1$.*

If $p = 2$, let

$$V = a_0 + a_1\alpha + \sum_{i=2}^k a_i\lambda_i.$$

For each s such that $2 \le s \le k$, multiplication by $u_{\lambda_s}\colon H\underline{M}_V \to H\underline{M}_{V+2-\lambda_s}$ is an isomorphism of Mackey functors when one of the following conditions is satisfied:

(1) *$a_0 > 0$ and $a_0 + a_1 > 0, a_0 + a_1 + 2\sum_{i=2}^j a_i > 0$ for all $2 \le j \le s-1$, or*

(2) *$a_0 < -3$ and $a_0 + a_1 < -3, a_0 + a_1 + 2\sum_{i=2}^j a_i < -3$ for all $2 \le j \le s-1$.*

Multiplication by $u_{2\alpha}\colon H\underline{M}_V \to H\underline{M}_{V+2-2\alpha}$ is an isomorphism when $a_0 > 0$ or $a_0 < -3$.

Proof Again we assume $p = 2$ and the odd primary case is analogous. We also assume $2 \le s \le k$ since the $u_{2\alpha}$ case is proved in the same way. Consider the long exact sequence

$$\underline{\pi}_{V+1} C_{u_{\lambda_s}} \to H\underline{M}_V \xrightarrow{u_{\lambda_s}} H\underline{M}_{V+2-\lambda_s} \to \underline{\pi}_V C_{u_{\lambda_s}}.$$

We want to find the range where the two outside Mackey functors vanish. By $a_{\lambda_s}, \dots, a_{\lambda_k}$-periodicity of $\pi_\star C_{u_{\lambda_s}}$, we can assume $V = x + y\alpha + \sum_{i=2}^{s-1} a_i \lambda_i$. From the computation (4), we know $\pi_* C_{u_{\lambda_s}}$ is concentrated at $* = -2, -1, 0$, and we get its Postnikov tower

$$\begin{array}{ccc} C_{u_{\lambda_i}} = P^0 & \longleftarrow & H\underline{M}_3 \\ \downarrow & & \\ P^{-1} & \longleftarrow & \Sigma^{-1} H\underline{M}_2 \\ \downarrow & & \\ P^{-2} & \xleftarrow[\simeq]{} & \Sigma^{-2} H\underline{M}_1 \end{array}$$

where $\underline{M}_i$, $i = 1, 2, 3$ are some Mackey functors. Then we get a strongly convergent spectral sequence

$$\underline{E}^1_{V,s} = \underline{\pi}_V \bar{P}^s \Rightarrow \underline{\pi}_V C_{u_{\lambda_s}},$$

where $\bar{P}^s$ is the fiber of $P^s \to P^{s-1}$. The conditions stated are the conditions for $\underline{E}^1_{V,s} = 0$ and $\underline{E}^1_{V+1,s} = 0$. □

Remark 3.15 For $\underline{M} = \underline{\mathbb{Z}}$, from the formula (4) and the fact that $u_{\lambda_s} : \underline{\mathbb{Z}} = H\underline{\mathbb{Z}}_0 \xrightarrow{\cong} H\underline{\mathbb{Z}}_{2-\lambda_s} = \underline{\mathbb{Z}}$ we know $C_{u_{\lambda_s}} \simeq \Sigma^{-2} H\underline{M}_1$. We can get a finer condition than the one stated in the theorem, which is left to the reader.

3B *au*-relation (aka gold relation)

In this subsection, we generalize the gold relation [13, Lemma 3.6] for $\underline{R} = \underline{\mathbb{Z}}$ to the general case.

Proposition 3.16 (au-relation) *Let V, W be chosen from λ_i for $1 \le i \le k$ such that $G_V \subset G_W$. Let $\underline{R}$ be a Green functor. Then there is the following relation in $H\underline{R}^G_\star$:*

$$a_V \cdot \mathrm{tr}^{G_W}_{G_V}(1) u_W = a_W \cdot u_V.$$

Since $H\underline{R}^G_{2-W} = \underline{R}(G_W)^G$ by Proposition 2.23, and $\mathrm{tr}^{G_W}_{G_V}(1) \in \underline{R}(G_W)^G$, the class $\mathrm{tr}^{G_W}_{G_V}(1) u_W \in H\underline{R}^G_{2-W}$ makes sense.

Lemma 3.17 *Let W be one of λ_i for $1 \le i \le k$. Then* $H\underline{M}^G_{2-W-\lambda} \cong \dfrac{\underline{M}(G_W)^G}{\zeta_{\log_2 |G/G_W|} \mathrm{tr}^{G_W}_e(\underline{M}(e))}$.

Proof This follows from Theorem 2.6. □

Lemma 3.18 *Let W be one of λ_i, $1 \le i \le k$, then we have the following relation in $H\underline{R}^G_\star$:*

$$a_\lambda \cdot \mathrm{tr}^{G_W}_e(1) u_W = a_W \cdot u_\lambda.$$

Proof We only need to prove for the universal case $\underline{R} = \underline{\mathbb{A}}$. We claim that the following commutative diagrams are isomorphic:

$$\begin{array}{ccc} H\underline{\mathbb{A}}_0^G & \xrightarrow{a_\lambda} & H\underline{\mathbb{A}}_{-\lambda}^G \\ \downarrow{\scriptstyle u_\lambda} & & \downarrow{\scriptstyle u_W} \\ H\underline{\mathbb{A}}_{2-\lambda}^G & \xrightarrow{a_W} & H\underline{\mathbb{A}}_{2-\lambda-W}^G \end{array} \qquad \begin{array}{ccc} \underline{\mathbb{A}}(G) & \xrightarrow{\text{proj}} & \underline{\mathbb{A}}(G)/\operatorname{tr}_e^G \\ \downarrow{\scriptstyle \operatorname{res}_e^G} & & \downarrow{\scriptstyle \operatorname{res}_{G_W}^G} \\ \underline{\mathbb{A}}(e) & \xrightarrow{\operatorname{tr}_e^{G_W}} & \underline{\mathbb{A}}(G_W)/|G/G_W|\operatorname{tr}_e^{G_W} \end{array}$$

The corresponding abelian groups are isomorphic by previous computations, and that $\underline{\mathbb{A}}(H)$ has trivial Weyl action for each $H \subset G$. That a_λ is projection and u_λ is res_e^G are proved in Proposition 3.9 and Remark 2.26, respectively.

To show that u_W is induced from $\operatorname{res}_{G_W}^G$, we look at the following isomorphic commutative diagrams:

$$\begin{array}{ccc} H\underline{\mathbb{A}}_{-\lambda}^G & \xrightarrow{\operatorname{res}_{G_W}^G} & H\underline{\mathbb{A}}_{-\lambda}^{G_W} \\ \downarrow{\scriptstyle u_W} & & \downarrow{\scriptstyle u_W} \\ H\underline{\mathbb{A}}_{2-\lambda-W}^G & \xrightarrow{\operatorname{res}_{G_W}^G} & H\underline{\mathbb{A}}_{2-\lambda-W}^{G_W} \end{array} \qquad \begin{array}{ccc} \underline{\mathbb{A}}(G)/\operatorname{tr}_e^G & \xrightarrow{\operatorname{res}_{G_W}^G} & \underline{\mathbb{A}}(G_W)/\operatorname{tr}_e^{G_W} \\ \downarrow{\scriptstyle \operatorname{res}_{G_W}^G} & & \downarrow{\scriptstyle 1} \\ \underline{\mathbb{A}}(G_W)/|G/G_W|\operatorname{tr}_e^{G_W} & \xrightarrow{f} & \underline{\mathbb{A}}(G_W)/\operatorname{tr}_e^{G_W} \end{array}$$

The map f is induced from the identity of $\underline{\mathbb{A}}(G_W)$ thus send $\bar{x}$ to $\bar{x}$, and we deduce that u_W is induced from $\operatorname{res}_{G_W}^G$.

We are left to show that a_W is induced from $\operatorname{tr}_e^{G_W}$. We consider the following filtration of $S^\lambda \wedge S^W$

$$\begin{array}{ccccccccc} S^0 & \longrightarrow & S^\lambda_{(1)} & \longrightarrow & S^\lambda & \longrightarrow & S^\lambda \wedge S^W_{(1)} & \longrightarrow & S^\lambda \wedge S^W \\ & \nwarrow\ \swarrow & & \nwarrow\ \swarrow & & \nwarrow\ \swarrow & & \nwarrow\ \swarrow & \\ & G/e_+ \wedge S^1 & & G/e_+ \wedge S^2 & & G/G_{W+} \wedge S^\lambda \wedge S^1 & & G/G_{W+} \wedge S^\lambda \wedge S^2 & \end{array}$$

The map $a_W \colon S^\lambda \to S^\lambda \wedge S^W$ sits in the first row. By running a spectral sequence, we can see that the classes that contribute to $\pi_2^G(S^\lambda \wedge S^W \wedge H\underline{\mathbb{A}})$ are subquotients of $\pi_2^G(G/e_+ \wedge S^2 \wedge H\underline{\mathbb{A}})$, $\pi_2^G(G/G_{W+} \wedge S^\lambda \wedge S^1 \wedge H\underline{\mathbb{A}})$ and $\pi_2^G(G/G_{W+} \wedge S^\lambda \wedge S^2 \wedge H\underline{\mathbb{A}})$. We have

$$\begin{gathered} \pi_2^G(G/e_+ \wedge S^2 \wedge H\underline{\mathbb{A}}) = \underline{\mathbb{A}}(e), \\ \pi_2^G(G/G_{W+} \wedge S^\lambda \wedge S^1 \wedge H\underline{\mathbb{A}}) = H\underline{\mathbb{A}}_1^{G_W}(S^\lambda) = \ker(\operatorname{tr}_e^{G_W})/(1-\gamma) = 0, \\ \pi_2^G(G/G_{W+} \wedge S^\lambda \wedge S^2 \wedge H\underline{\mathbb{A}}) = H\underline{\mathbb{A}}_0^{G_W}(S^\lambda) = \underline{\mathbb{A}}(G_W)/\operatorname{tr}_e^{G_W}. \end{gathered}$$

We already know that

$$\pi_2^G(S^\lambda \wedge S^W \wedge H\underline{\mathbb{A}}) = H\underline{\mathbb{A}}_{2-W-\lambda}^G \cong \frac{\underline{\mathbb{A}}(G_W)}{|G/G_W|\operatorname{tr}_e^{G_W}} \cong \underline{\mathbb{A}}(G_W)/x \oplus \mathbb{Z}/|G/G_W|\langle x \rangle$$

by the previous lemma, where x is represented by the free G_W-set $G_W/e = \operatorname{tr}_e^{G_W}(1)$. The differential

$$d_1 \colon \underline{\mathbb{A}}(e) = \pi_2^G(G/G_{W+} \wedge S^\lambda \wedge H\underline{\mathbb{A}}) \to \pi_2(G/e_+ \wedge S^2 \wedge H\underline{\mathbb{A}}) = \underline{\mathbb{A}}(e)$$

must be $|G/G_W|$, and we have the following extension:

$$0 \to \underline{\mathbb{A}}(e)/|G/G_W| \xrightarrow{h} H\underline{\mathbb{A}}^G_{2-\lambda-W} \to \underline{\mathbb{A}}(G_W)/\operatorname{tr}_e^{G_W} \to 0,$$

where h is induced from $\operatorname{tr}_e^{G_W}$. Thus a_W is induced from $\operatorname{tr}_e^{G_W}$.

In $\underline{\mathbb{A}}(G_W)/x \oplus \mathbb{Z}/|G/G_W|\langle x\rangle$ we have that $a_\lambda u_W = (1,0)$ and $a_W u_\lambda = (0,x)$, where 1 is G_W/G_W. We deduce

$$a_W u_\lambda = a_\lambda \cdot x u_W = a_\lambda \cdot \operatorname{tr}_e^{G_W}(1) u_W. \qquad \square$$

Proof of Proposition 3.16 Let V, W be chosen from λ_i, $1 \le i \le k$ with $e \neq G_V \subset G_W$. Note $\underline{\mathbb{A}}^{\sharp G_V}$ is a Green functor for G/G_V. Regard V, W as G/G_V-representations and by abuse of notation, we will still write V, W. By induction on k, we get

$$a_V \cdot \operatorname{tr}_e^{G_W/G_V}(1) u_W = a_W u_V$$

in $(H\underline{\mathbb{A}}^{\sharp G_V})_\star$ since V is a faithful G/G_V-representation. We have

$$\operatorname{tr}_e^{G_W/G_V}(1) = \operatorname{tr}_{G_V}^{G_W}(1),$$

where the transfer on the right is in $\underline{\mathbb{A}}$. Thus the claim follows. $\square$

4 Induction theorems

In this section, we prove our induction theorems. The results fit $H\underline{M}_\star$ in exact sequences where the rest of the terms are computable in terms of $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$. Thus we can carry out the computations by induction on the order of $G = C_{2^k}$. The initial input is the structure of $H\underline{M}^{C_2}_\star$, which is completely determined by the first author in [21].

Recall that Proposition 3.7 reduces our consideration of $H\underline{M}_V$ to those $V \in RO(G)$ such that $|V| = 2, 1, 0, -1, -2$. Proposition 3.9 then tells us how to further reduce to those V with $|V| = 1, 0, -1$. We first relate the homotopy groups in degrees V with $|V| = \pm 1$.

Proposition 4.1 *Let $V \in RO(G)$ with $|V| = 1$. If V is orientable, then we have the exact sequence of abelian groups*

$$0 \to \operatorname{Im}(\operatorname{res}(V+1-\lambda)^G_e) \to \underline{M}(e)^G \to H\underline{M}^G_V \xrightarrow{a_\lambda} H\underline{M}^G_{V-\lambda} \to \underline{M}(e)/(1-\gamma) \to \operatorname{Im}(\operatorname{tr}(V-1)^G_e) \to 0.$$

If V is nonorientable, then we have the exact sequence of abelian groups

$$0 \to \operatorname{Im}(\operatorname{res}(V+1-\lambda)^G_e) \to_{\zeta_1} \underline{M}(e) \to H\underline{M}^G_V \xrightarrow{a_\lambda} H\underline{M}^G_{V-\lambda} \to \underline{M}(e)/(1+\gamma) \to \operatorname{Im}(\operatorname{tr}(V-1)^G_e) \to 0.$$

Each exact sequence can be upgraded to an exact sequence of Mackey functors.

Proof By Lemma 3.5, when V is orientable, the long exact sequence on homology of the cofiber sequence

$$S(\lambda)_+ \to S^0 \xrightarrow{a_\lambda} S^\lambda$$

gives us

$$\underline{HM}^G_{V+1} \xrightarrow{a_\lambda} \underline{HM}^G_{V+1-\lambda} \xrightarrow{\mathrm{res}(V+1-\lambda)^G_e} \underline{M}(e)^G \to \underline{HM}^G_V \xrightarrow{a_\lambda} \underline{HM}^G_{V-\lambda} \to \underline{M}(e)/(1-\gamma) \xrightarrow{\mathrm{tr}(V-1)^G_e} \underline{HM}^G_{V-1} \xrightarrow{a_\lambda} \underline{HM}^G_{V-1-\lambda}.$$

By Proposition 3.9, the first a_λ can be identified with the inclusion of $\ker(\mathrm{res}(V+1-\lambda)^G_e)$, whose cokernel is $\mathrm{Im}(\mathrm{res}(V+1-\lambda)^G_e)$. The third a_λ can be identified with projection to $\mathrm{coker}(\mathrm{tr}(V-1)^G_e)$, whose kernel is $\mathrm{Im}(\mathrm{tr}(V-1)^G_e)$. Replacing G by $H \subset G$, we get and exact sequence of Mackey functors.

The case for nonorientable V is completely analogous. □

Remark 4.2 Proposition 4.1 tells us that the homotopy Mackey functors with $|V|=1$ and $|V|=-1$ determine each other up to extension if we can figure out the maps in the long exact sequences and understand the $|V|=0$ grading, since both $(V+1-\lambda)$ and $(V-1)$ have underlying degree 0 there.

The following two theorems are the main theorems in this section. For simplicity, if $U \in RO(G)$, we use the same notation for its restriction to subgroups. If $G_U \neq e$, we also use the same notation for the corresponding G/G_U-representation. For definitions of Mackey functors FP, FQ and TFP, TFQ, see Definitions 3.2 and 3.3, respectively.

Theorem 4.3 (induction theorem A) *Let $U = a_0 + a_1\alpha + \Sigma_{i=2}^k a_i\lambda_i \in RO(G)$ with $|U|=0$. Then*:

(1) *When $a_k = 0$, $\underline{HM}_U$ can be computed from $H(\underline{M}\downarrow^G_{C_2})_U$ and $H(\underline{M}^{\sharp C_2})_U$.*

(2) *When $a_k \leq -1$, if U is orientable, we have the exact sequence of Mackey functors*

$$0 \to \underline{HM}_{U+\lambda_k} \xrightarrow{a_\lambda} \underline{HM}_U \xrightarrow{\mathrm{res}^{(-)}_e} FP(\underline{M}(e)) \to \underline{HM}_{U-1+\lambda_k}.$$

If U is nonorientable, we have the exact sequence of Mackey functors

$$0 \to \underline{HM}_{U+\lambda_k} \xrightarrow{a_\lambda} \underline{HM}_U \xrightarrow{\mathrm{res}^{(-)}_e} TFP(\underline{M}(e)) \to \underline{HM}_{U-1+\lambda_k}.$$

Here $\mathrm{res}^{(-)}_e$ is the map induced from res^H_e on the G/H-level. In each sequence, the first and last Mackey functors can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

(3) *When $a_k \geq 1$, if U is orientable, we have the exact sequence of Mackey functors*

$$\underline{HM}_{U-\lambda_k+1} \to FQ(\underline{M}(e)) \xrightarrow{\mathrm{tr}^{(-)}_e} \underline{HM}_U \xrightarrow{a_\lambda} \underline{HM}_{U-\lambda_k} \to 0.$$

If U is nonorientable, we have the exact sequence of Mackey functors

$$\underline{HM}_{U-\lambda_k+1} \to TFQ(\underline{M}(e)) \xrightarrow{\mathrm{tr}^{(-)}_e} \underline{HM}_U \xrightarrow{a_\lambda} \underline{HM}_{U-\lambda_k} \to 0.$$

Here $\mathrm{tr}^{(-)}_e$ is the map that is induced from tr^H_e on the G/H-level. In each sequence, the first and last Mackey functors can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

Proof The case $a_k = 0$ follows from Corollary 2.16. It describes the restrictions and transfers between subgroups that are not equal to e. The rest of the structure can be deduced from $H(\underline{M}\downarrow^G_{C_2})_U$.

Consider the cofiber sequence

$$S(\lambda_k)_+ \to S^0 \to S^{\lambda_k}.$$

Since $|U| = 0$, we have that $|U + \lambda_k| = 2$ and $|U - \lambda_k - 1| = -3$. Thus

$$\underline{\pi}_{U+\lambda_k} S(\lambda_k)_+ \wedge H\underline{M} = \underline{\pi}_{U-\lambda_k-1} S(\lambda_k)_+ \wedge H\underline{M} = 0.$$

The homology long exact sequence together with Lemma 3.5 give us the four exact sequences.

Now we show that groups other than $H\underline{M}^G_U$ in the exact sequences can be reduced to C_2- and G/C_2-equivariant computations. Since the proofs for orientable and nonorientable U are completely analogous, we assume U is orientable.

In part (2), if $a_k = -1$, then $U + \lambda_k$ is a G/C_2-representation and we know the first and last Mackey functors via G/C_2-computations. If $a_k \leq -2$, let $W = U - a_k\lambda_k$. By Proposition 3.7, $H\underline{M}_W \cong H\underline{M}_{U+\lambda_k}$ and the former Mackey functor can be computed G/C_2-equivariantly. Similarly for $H\underline{M}_{W-1} \cong H\underline{M}_{U+\lambda_k-1}$. Then we get the structures of the relevant homotopy Mackey functors above the group C_2. The rest of the structure can be deduced from $H(\underline{M}\downarrow^G_{C_2})_\star$.

In part (3), if $a_k = 1$, the computations of the first and last Mackey functors can be reduced to computations over G/C_2. If $a_k \geq 2$, let $W = U - a_k\lambda_k$. We have $H\underline{M}_{U-\lambda_k} \cong H\underline{M}_W$ and $H\underline{M}_{U-\lambda_k+1} \cong H\underline{M}_{W+1}$, thus the computations above the group C_2 can be reduced to G/C_2. $H(\underline{M}\downarrow^G_{C_2})_\star$ tells us the rest of the structure. □

For the cases $|U| = \pm 1$, we have the following result:

Theorem 4.4 (induction theorem B) *Let $U = a_0 + a_1\alpha + \Sigma_{i=2}^k a_i\lambda_i \in RO(G)$.*

When $|U| = 1$:

(1) *If $a_k \leq 0$, then $H\underline{M}^G_U$ can be computed from $H(\underline{M}\downarrow^G_{C_2})_\star$ and $H(\underline{M}^{\sharp C_2})_\star$.*

(2) *If $a_k \geq 1$ and U is orientable, then we have the exact sequence of Mackey functors*

$$FP(\underline{M}(e)) \to H\underline{M}_U \xrightarrow{a_\lambda} H\underline{M}_{U-\lambda_k} \to FQ(\underline{M}(e)),$$

where the third Mackey functor can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

(3) *If $a_k \geq 1$ and U is nonorientable, then we have the exact sequence of Mackey functors*

$$TFP(\underline{M}(e)) \to H\underline{M}_U \xrightarrow{a_\lambda} H\underline{M}_{U-\lambda_k} \to TFQ(\underline{M}(e)),$$

where the third Mackey functor can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

When $|U| = -1$:

(1) *If $a_k \geq 0$, then $\underline{\pi}_U H\underline{M}$ can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.*

(2) *If $a_k \leq -1$ and U is orientable, then we have the exact sequence of Mackey functors*
$$FP(\underline{M}(e)) \to H\underline{M}_{U+\lambda_k} \xrightarrow{a_\lambda} H\underline{M}_U \to FQ(\underline{M}(e)),$$
where the second Mackey functor can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

(3) *If $a_k \leq -1$ and U is nonorientable, then we have the exact sequence of Mackey functors*
$$TFP(\underline{M}(e)) \to H\underline{M}_{U+\lambda_k} \xrightarrow{a_\lambda} H\underline{M}_U \to TFQ(\underline{M}(e)),$$
where the second Mackey functor can be computed from $H(\underline{M}^{\sharp C_2})_\star$ and $H(\underline{M}\downarrow^G_{C_2})_\star$.

Proof The proof for the cases $|U| = \pm 1$ are analogous, and we only prove it for $|U| = 1$. The case $a_k = 0$ follows from Corollary 2.16. The exact sequences comes from the cofiber sequence $S(\lambda)_+ \to S^0 \to S^\lambda$.

If $a_k \leq -1$, let $W = U - a_k\lambda_k$. By Proposition 3.7, we have the isomorphism $H\underline{M}_U \cong H\underline{M}_W$, and the latter Mackey functor can be reduced to the homotopy of $H\underline{M}^{\sharp C_2}$. Then we understand the Mackey functor structure above C_2. $H(\underline{M}\downarrow^G_{C_2})_\star$ tells us the rest.

If $a_k \geq 1$, let $W = U - a_k\lambda_k$. By Proposition 3.7, $H\underline{M}_{U-\lambda_k} \cong H\underline{M}_W$ and the latter Mackey functor can be reduced to the homotopy of $H\underline{M}^{\sharp C_2}$. The rest of the structure can be deduced from $H(\underline{M}\downarrow^G_{C_2})_\star$. □

5 The positive cone of $H\underline{M}$ for arbitrary $\underline{M}$

In this section we will show how to use the methods described in previous sections to describe the positive cone of $H\underline{M}^G_\star$ for an arbitrary Mackey functor $\underline{M}$. Here the positive cone $H\underline{M}^G_{\text{pos}}$ is the subgroup of $H\underline{M}^G_\star$ indexed by $V \in RO(G)$ such that $a_i \leq 0$ for $1 \leq i \leq k$. As examples, we completely compute the positive cone of $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{A}}$.

5A Structure of the positive cone for arbitrary $\underline{M}$

Note that in the positive cone, all indexing representations will have $|V| \leq 0$, since a n-dimensional G-CW complex cannot have a nonzero $(n+1)^{\text{th}}$ homology. By Proposition 3.7 and 3.9, multiplication by a_{λ_k} is an isomorphism when $|V| \leq -1$, and it is projection to $\operatorname{coker}(\operatorname{tr}^G_e)$ when $|V| = 0$. So the V^{th} homotopy groups with $|V| = 0, -1$ are the only groups we need to describe. Note that when V is orientable with $|V| = 0$, this is discussed in Proposition 2.23.

5A1 $H\underline{M}^G_V$ for nonorientable V with $|V| = 0$ We first describe $H\underline{M}^G_V$ for nonorientable V with $|V| = 0$.

Proposition 5.1 *We have that*
$$H\underline{M}^G_{1-\alpha} = \ker(\operatorname{tr}^G_{G'}) \quad \textit{and}$$
$$H\underline{M}^G_{n-n\alpha} = {}_{\zeta_1}\underline{M}(G') \quad \textit{for odd } n > 1, \textit{ where } \zeta_1 = 1 + \gamma \textit{ (see Definition 2.2).}$$

Proof This directly follows from Theorem 2.6. □

Notation Let $V=\sum_{1\le i\le k} a_i\lambda_i$ be an actual orientable representation (recall that $\lambda_1=2\alpha$). Recall also that elements $u_W, u_{W'}$ commute for irreducible (thus general) orientable W and W' (see Proposition 2.29). We will write

$$u_V := u_{\lambda_1}^{c_1} u_{\lambda_2}^{c_2} \dots u_{\lambda_k}^{c_k} \in H\underline{\mathbb{A}}^G_{|V|-V}.$$

Note that $u_V\colon H\underline{M}^G_0 \to H\underline{M}^G_{|V|-V} \cong \underline{M}(G_V)^G$ is the restriction $\mathrm{res}^G_{G_V}$.

Proposition 5.2 *Let $V=1-\alpha+|V'|-V'$ be a nonorientable representation with $V'=\sum_{1\le i\le k} a_i\lambda_i \neq 0$. Then*

(1) $H\underline{M}^G_V = {}_{\zeta_1}\underline{M}(G_{V'})$,

(2) *multiplication* $u_{V'}\colon H\underline{M}^G_{1-\alpha} \cong \ker(\mathrm{tr}^G_{G'}) \to H\underline{M}^G_V \cong {}_{\zeta_1}\underline{M}(G_{V'})$ *is the restriction map* $\mathrm{res}^{G'}_{G_V}$.

Note that in point (2) the image of $\mathrm{res}^{G'}_{G_V}$ always lies in the codomain when restricted to the domain, by the double-coset formula.

Proof (1) This follows from Theorem 2.6.

(2) Let $v\colon S^1 \to S^\alpha \wedge H\underline{M}$ be a representative of an element in $H\underline{M}^G_{1-\alpha}$. Then consider the commutative diagram

$$\begin{array}{ccccc} S^{|V'|} & \xrightarrow{v\wedge\mathrm{id}} & S^{\alpha-1}\wedge S^{|V'|}\wedge H\underline{M} & \xrightarrow{\mathrm{id}\wedge u_{V'}} & S^{\alpha-1}\wedge S^{V'}\wedge H\underline{M} \\ & & & \searrow & \downarrow \\ & & & & S^{\alpha-1}\wedge G/G'_{V+}\wedge S^{|V'|}\wedge H\underline{M} \end{array}$$

The diagonal map displays the restriction of the class in $H\underline{M}^G_{1-\alpha}$ to $H\underline{M}^{G_{V'}}_{1-\alpha} \cong H\underline{M}^{G_{V'}}_0 = \underline{M}(G_{V'})$, and the vertical map is the inclusion of ${}_{\zeta_1}\underline{M}(G_{V'})$ in $\underline{M}(G_{V'})$ when taking $\pi^G_{|V'|}(-)$.

The restriction map $\mathrm{res}(1-\alpha)^G_{G'_V}$ is induced by the restriction in the Mackey functor $\underline{M}$. This can be seen as follows. Let $v\colon S^1 \to S^\alpha \wedge H\underline{M}$ be a representative of a class in $H\underline{M}^G_{1-\alpha}$. Then its restriction to $H\underline{M}^{G_{V'}}_{1-\alpha}$ is given by the composite

$$G/G_{V'+}\wedge S^{1-\alpha} \to S^{1-\alpha} \to H\underline{M}.$$

Note that $G/G_{V'+}\wedge S^{1-\alpha} \cong G/G_{V'+}$ by the shearing isomorphism. Thus restriction of v gives a class in $\underline{M}(G_{V'})$, as expected.

Therefore since the diagonal map is induced by $\mathrm{res}^G_{G_{V'}}$ and the vertical map is an inclusion, the multiplication by u_V has to be the restriction. □

5A2 $H\underline{M}^G_V$ for V with $|V|=-1$ Now we will compute the entries in the positive cone graded over V with $|V|=-1$. We first observe that if V has $|V|=-1$, then there exists a unique orientable G-representation U with $U^G=0$ such that $V=|U|-U-1$ or $|U|-U-\alpha$ depending on the orientability of V.

Proposition 5.3 *Let $U=\sum_{i=1}^{j} a_i\lambda_i, a_i\geq 0$ and $a_j\neq 0$. Let $W=U-\lambda_j$ and s be the biggest integer such that λ_s appears in W with nonzero coefficient. Then*

$$H\underline{M}^G_{|U|-1-U}=\begin{cases}\dfrac{\zeta_j\,\underline{M}(G_U)}{(1-\gamma)\underline{M}(G_U)} & \text{if } a_j\geq 2,\\[2ex] \dfrac{\ker(\zeta_s\,\mathrm{tr}^{G_W}_{G_U})}{(1-\gamma)\underline{M}(G_U)} & \text{if } a_j=1 \text{ and } W\neq 0,\\[2ex] \dfrac{\ker(\mathrm{tr}^{G}_{G_U})}{(1-\gamma)\underline{M}(G_U)} & \text{if } a_j=1 \text{ and } W=0.\end{cases}$$

and

$$H\underline{M}^G_{|U|-\alpha-U}=\begin{cases}\dfrac{\bar{\zeta}_j\,\underline{M}(G_U)}{(1+\gamma)\underline{M}(G_U)} & \text{if } a_j\geq 2,\\[2ex] \dfrac{\ker(\bar{\zeta}_s\,\mathrm{tr}^{G_W}_{G_U})}{(1+\gamma)\underline{M}(G_U)} & \text{if } a_j=1 \text{ and } W\neq 0,\\[2ex] \dfrac{\ker(\bar{\zeta}_1\,\mathrm{tr}^{G'}_{G_U})}{(1+\gamma)\underline{M}(G_U)} & \text{if } a_j=1 \text{ and } W=0.\end{cases}$$

Proof This follows directly from Theorem 2.6. □

In the next two subsections, we will use the theory described above to derive the positive cones of the most important Eilenberg–MacLane spectra $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{A}}$ over C_{2^k}. Note that the whole $RO(G)$-graded homotopy of $H\underline{\mathbb{F}}_2$, in particular the positive cone, is determined by the second author [23].

5B The constant Mackey functor $\underline{\mathbb{Z}}$

In this subsection we will describe the structure of the positive cone of $H\underline{\mathbb{Z}}$. Recall the Mackey functor $\underline{\mathbb{Z}}$ is given by $\underline{\mathbb{Z}}(H)=\mathbb{Z}$ for $H\subset G$. Using transitivity, the restrictions and transfers are determined by $\mathrm{res}^H_K=1$ and $\mathrm{tr}^H_K=2$ when $[H:K]=2$. Since $\underline{\mathbb{Z}}$ is actually a Green functor, the positive cone of $H\underline{\mathbb{Z}}$ has a ring structure. The reader can compare the following result with [12].

Theorem 5.4 *Let $G=C_{2^k}$, $k\geq 1$. Then we have*

$$H\underline{\mathbb{Z}}^G_{\mathrm{pos}}=\frac{\mathbb{Z}\big[a_\alpha,\ \{a_{\lambda_i}\mid 2\leq i\leq k\},\ \{u_{\lambda_i}\mid 1\leq i\leq k\}\big]}{2a_\alpha,\ \{2^i a_{\lambda_i}\mid 2\leq i\leq k\},\ \{2^{i-j}a_{\lambda_i}u_{\lambda_j}-a_{\lambda_j}u_{\lambda_i}\mid 1\leq j<i\leq k\}}.$$

Proof The result is true for $k=1$ by [3; 8; 25]. It is true for $k=2$ by [24]. Assume it is true for $k-1$. For $G=C_{2^k}$, the relations can be readily deduced from Propositions 2.19 and 3.16. We are left to show that we have the additive splitting

$$\begin{aligned}(6)\quad H\underline{\mathbb{Z}}^G_{\mathrm{pos}}=\mathbb{Z}[a_{\lambda_k}][a_\alpha,\dots,a_{\lambda_{k-1}},u_{2\alpha},\dots,u_{\lambda_{k-1}}]/\{2^k a_{\lambda_k},\ \text{relations for } C_{2^{k-1}}\}\\ \oplus\mathbb{Z}\langle u^i_{\lambda_k}\rangle_{i\geq 1}[a_{\lambda_k}][u_{2\alpha},\dots,u_{\lambda_{k-1}}]\langle 1,a_\alpha\rangle/\{2a_\alpha,2^k a_{\lambda_k}\}.\end{aligned}$$

For the notation, see the introduction (Section 1A). Let V be in the positive cone. Then we get nontrivial classes only when $|V| \leq 0$, since a n-dimensional G-CW complex cannot have $(n+1)$-dimensional homology. By Corollary 2.16 and the induction assumption, we get the classes

$$\mathbb{Z}[a_\alpha, \dots, a_{\lambda_{k-1}}, u_{2\alpha}, \dots, u_{\lambda_{k-1}}]/\{\text{relations for } C_{2^{k-1}}\} \tag{7}$$

when V does not contain any copies of λ_k. Write $V = V^\sharp + a_k\lambda_k$, where $V^\sharp$ does not contain λ_k. Note $a_k \geq 0$ since V is in the positive cone. We use $H\underline{\mathbb{Z}}^G_{\text{pos}^\sharp}$ to denote the subring of the first summand in (6) with a_{λ_k} having 0^{th} power. That is, $\text{pos}^\sharp$ is the direct sum of all representations in the positive cone without any copy of λ_k.

If $|V^\sharp| \leq 0$, then $H\underline{\mathbb{Z}}^G_{\text{pos}^\sharp}$ is known as in (7), and Propositions 3.9 and 3.7 gives us the first direct summand of (6), since multiplication by a_{λ_k} will either be an isomorphism or a surjection. If $|V^\sharp| > 0$, let c be the unique integer such that $|V - c\lambda_k| = 0$ or -1 depending on whether $|V|$ is even or odd. Let $d = a_k - c$, then we have $V = V^\sharp - c\lambda_k - d\lambda_k$. Write $W = V^\sharp - c\lambda_k$. We will show that the groups $H\underline{\mathbb{Z}}^G_V$ with $|V^\sharp| > 0$ consist of precisely the second summand in (6).

By Proposition 2.23 and an easy generalization of Proposition 2.29, we know the elements u_{λ_i} generate a subring $\mathbb{Z}[\{u_{\lambda_i} \mid 1 \leq i \leq k\}]$ which gives us the groups $H\underline{\mathbb{Z}}^G_W = 0$ with $|W| = 0$, since $H\underline{\mathbb{Z}}_{n-n\alpha} = 0$ for odd n by Proposition 5.1. Multiplying by powers of a_{λ_k} gives us all the classes in the gradings with $|V^\sharp| > 0$ and $|W| = 0$ by Propositions 3.9 and 3.7.

By Proposition 5.3, for U an actual G-representation, we have that

$$H\underline{\mathbb{Z}}^G_{|U|-U-1} = 0 \quad \text{and} \quad H\underline{\mathbb{Z}}^G_{|U|-U-\alpha} = \mathbb{Z}/2.$$

We look at the following exact sequence:

$$H\underline{\mathbb{Z}}^{G'}_{|U|-U} \xrightarrow{\text{tr}^G_{G'}} H\underline{\mathbb{Z}}^G_{|U|-U} \xrightarrow{a_\alpha} H\underline{\mathbb{Z}}^G_{|U|-U-\alpha} \longrightarrow H\underline{\mathbb{Z}}^{G'}_{|U|-U-1}.$$

Since $\text{tr}^G_{G'} = 2$, we see a_α is the projection $\mathbb{Z} \to \mathbb{Z}/2$, and we obtain the classes $\mathbb{Z}/2[\{u_{\lambda_i} \mid 1 \leq i \leq k\}]\langle a_\alpha\rangle$. Multiplying by powers of a_{λ_k} gives us all the classes in gradings with $|V^\sharp| > 0$ and $|W| = -1$ by Propositions 3.9 and 3.7. □

5C Burnside Mackey functor $\underline{\mathbb{A}}$

In this subsection we will describe the structure of the positive cone for $H\underline{\mathbb{A}}$, where $\underline{\mathbb{A}}$ is the Burnside Mackey functor (actually a Green functor). Recall that for a general finite group G, the Burnside Green functor for $H \subset G$ is given by

$$\underline{\mathbb{A}}(H) = \text{the Grothendieck ring on isomorphism classes of finite } H\text{-sets},$$

where the addition is disjoint union and product is the Cartesian product. In particular, it has a free $\mathbb{Z}$-basis given by the isomorphism classes of H/L, $L \subset H$. We will use $[H/L]$ to denote the isomorphism class of H/L. For $K \subset H \subset G$, res^H_K is given by restricting the H-action to a K-action, and tr^H_K is given by $H \times_K (-)$.

Fix $G = C_{2^k}$ and consider $\underline{\mathbb{A}}(C_{2^k})$. The $\mathbb{Z}$-basis will be denoted by $\omega_j^{(k)}$, where $\omega_j^{(k)}$ is the class corresponding to the isomorphism class $[C_{2^k}/C_{2^{k-j}}]$. In particular, $\omega_k^{(k)}$ corresponds to the free G-set and $\omega_0^{(k)}$ is the unity of the ring. Similarly we define $\omega_j^{(k)}$ for subgroups of G.

The restrictions are explicitly given by

$$\operatorname{res}^{C_{2^i}}_{C_{2^{i-1}}}(\omega_j^{(i)}) = \begin{cases} 2\omega_{j-1}^{(i-1)} & \text{if } 1 \le j \le i, \\ \omega_0^{(i-1)} & \text{if } j = 0. \end{cases}$$

The transfers are given by

$$\operatorname{tr}^{C_{2^i}}_{C_{2^{i-1}}}(\omega_j^{(i-1)}) = \omega_{j+1}^{(i)}$$

for $0 \le j \le i-1$.

By Proposition 2.23 and the fact that Weyl actions in $\underline{\mathbb{A}}$ are trivial, we have

$$H\underline{\mathbb{A}}_{2-\lambda_j}(H) = \begin{cases} \underline{\mathbb{A}}(C_{2^{k-j}}) & \text{if } C_{2^{k-j}} \subset H, \\ \underline{\mathbb{A}}(H) & \text{otherwise.} \end{cases}$$

It is the constant Mackey functor with value $\underline{\mathbb{A}}(C_{2^{k-j}})$ from the G/G-level to the $G/C_{2^{k-j}}$-level. At the $G/C_{2^{k-j}}$-level and below, it is the Burnside Mackey functor for the group $C_{2^{k-j}}$.

At the $C_{2^{k-j}}$-level, we have the actual products $\omega_i^{(k-j)} \cdot u_{\lambda_j}$ for $0 \le i \le k-j$ and they form a $\mathbb{Z}$-basis of $H\underline{\mathbb{A}}_{2-\lambda_j}(G/C_{2^{k-j}})$. Since $\operatorname{res}^G_{C_{2^{k-j}}} = 1$ in this degree, we denote the lifts of these classes to the G/G-level by $[\omega_i^{(k-j)} u_{\lambda_j}]$, since they are not actual products anymore.

For a general orientable $V = \Sigma_{i=1}^j a_i\lambda_i$ with $a_j \neq 0$, the group $H\underline{\mathbb{A}}^G_{|V|-V}$ has a $\mathbb{Z}$-basis $[\omega_i^{(k-j)} u_V]$, $0 \le i \le k-j$.

Theorem 5.5 *Let $G = C_{2^k}$. We have*

$$H\underline{\mathbb{A}}^G_{\mathrm{pos}} = \underline{\mathbb{A}}(G)\big[a_\alpha, \{a_{\lambda_i} \mid 2 \le i \le k\}, \{[\omega_i^{(k-j)} u_{\lambda_j}] \mid 1 \le j \le k,\ 0 \le i \le k-j\}\big]/S,$$

where S is the set of relations consisting of the following:

(1) *For all $1 \le j \le k$ we have that $\omega_j^{(k)} a_\alpha = 0$.*

(2) *For all $2 \le i \le j \le k$ we have that $\omega_j^{(k)} a_{\lambda_i} = 0$.*

(3) *For all $1 \le j_1 \le j_2 \le k$ and $0 \le i_1 \le k-j_1$, $0 \le i_2 \le k-j_2$, we have that*

$$[\omega_{i_1}^{(k-j_1)} u_{\lambda_{j_1}}] \cdot [\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_2}}] = \begin{cases} 2^{i_1}[\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_2}}] u_{\lambda_{j_1}} & \text{if } i_1 + j_1 \le j_2, \\ 2^{j_2-j_1+i_1}[\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_2}}] u_{\lambda_{j_1}} & \text{if } j_2 \le i_1 + j_1 \le i_2 + j_2, \\ 2^{j_2-j_1+i_2}[\omega_{i_1+j_1-j_2}^{(k-j_2)} u_{\lambda_{j_2}}] u_{\lambda_{j_1}} & \text{if } i_1 + j_1 > i_2 + j_2. \end{cases}$$

In particular, we have that for all $1 \le l \le j \le k$,

$$u_{\lambda_l} \cdot [\omega_i^{(k-j)} u_{\lambda_j}] = [\omega_i^{(k-j)} u_{\lambda_l} u_{\lambda_j}].$$

(4) *For all $1 \le i, j \le k$, we have that*

$$\omega_{i_1}^{(k)} \cdot [\omega_{i_2}^{(k-j)} u_{\lambda_i}] = \begin{cases} 2^{i_1+i_2}[\omega_{k-j-i_1}^{(k-j)} u_{\lambda_i}] & \text{if } i_1 \le j, \text{ and } i_1 \le k-j-i_2, \\ 2^{k-j}[\omega_{i_2}^{(k-j)} u_{\lambda_i}] & \text{if } i_1 \le j, \text{ and } i_1 > k-j-i_2, \\ 2^{j+i_2}[\omega_{k-j-i_1}^{(k-j)} u_{\lambda_i}] & \text{if } i_1 > j, \text{ and } i_1 \le k-j-i_2, \\ 2^{k-i_1}[\omega_{i_2}^{(k-j)} u_{\lambda_i}] & \text{if } i_1 > j, \text{ and } i_1 > k-j-i_2. \end{cases}$$

(5) *For all $1 \le j < i \le k$ we have that $a_{\lambda_i}[\omega_{i-j}^{(k-j)} u_{\lambda_j}] = a_{\lambda_j} u_{\lambda_i}$.*

Proof The case when $k = 1$ is shown in [21]. Assume the result is true for $k-1$. We use the same notation $V = V^\sharp - a_k\lambda_k$ and $W = V^\sharp - c\lambda_k$ when $|V^\sharp| > 0$ as in Theorem 5.4.

We first deduce the relations. Since $\underline{\mathbb{A}}$ is a Green functor, we have the Frobenius relation: for any $a \in H\underline{\mathbb{A}}_V(K)$ and $b \in H\underline{\mathbb{A}}_W(H)$,

$$\mathrm{tr}_K^H(a) \cdot b = \mathrm{tr}_K^H(a \cdot \mathrm{res}_K^H(b)).$$

The relations (1) and (2) are direct consequences of the Frobenius relation. Relation (5) follows from the gold relation in Proposition 3.16. More precisely, we have $G_{\lambda_i} = C_{2^{k-i}}$, $G_{\lambda_j} = C_{2^{k-j}}$ and $\mathrm{tr}_{C_{2^{k-i}}}^{C_{2^{k-j}}}(1) = \omega_{i-j}^{(k-j)}$.

For (3), we look at the following diagram coming from the pairing $H\underline{\mathbb{A}}_{2-\lambda_{j_1}} \square H\underline{\mathbb{A}}_{2-\lambda_{j_2}} \xrightarrow{\mu} H\underline{\mathbb{A}}_{4-\lambda_{j_1}-\lambda_{j_2}}$:

$$\begin{array}{ccc} \underline{\mathbb{A}}(C_{2^{k-j_1}}) \otimes \underline{\mathbb{A}}(C_{2^{k-j_2}}) & \xrightarrow{\mu} & \underline{\mathbb{A}}(C_{2^{k-j_2}}) \\ {\scriptstyle 1\otimes 1}\downarrow & & \downarrow{\scriptstyle 1} \\ \underline{\mathbb{A}}(C_{2^{k-j_1}}) \otimes \underline{\mathbb{A}}(C_{2^{k-j_2}}) & \xrightarrow{\mu} & \underline{\mathbb{A}}(C_{2^{k-j_2}}) \\ {\scriptstyle \mathrm{res}_{C_{2^{k-j_2}}}^{C_{2^{k-j_1}}} \otimes 1}\downarrow & & \downarrow{\scriptstyle 1} \\ \underline{\mathbb{A}}(C_{2^{k-j_2}}) \otimes \underline{\mathbb{A}}(C_{2^{k-j_2}}) & \xrightarrow{\mu} & \underline{\mathbb{A}}(C_{2^{k-j_2}}) \end{array}$$

Since the target has restrictions given by 1 above $G/C_{2^{k-j_2}}$-level by Proposition 2.23, to understand the product on the G/G-level, it suffices to restrict it to the $G/C_{2^{k-j_2}}$-level. We notice

$$\mathrm{res}_{C_{2^{k-j_2}}}^{C_{2^{k-j_1}}}(\omega_{i_1}^{(k-j_1)}) = \begin{cases} 2^{i_1} & \text{if } i_1 + j_1 \le j_2, \\ 2^{j_2-j_1}\omega_{i_1+j_1-j_2}^{(k-j_2)} & \text{if } i_1 + j_1 > j_2. \end{cases}$$

Together with

$$\omega_{i_1+j_1-j_2}^{(k-j_2)} \cdot \omega_{i_2}^{(k-j_2)} = \begin{cases} 2^{i_2}\omega_{i_1+j_1-j_2}^{(k-j_2)} & \text{if } i_1 + j_1 > i_2 + j_2, \\ 2^{i_1}\omega_{i_2}^{(k-j_2)} & \text{if } i_1 + j_1 \le i_2 + j_2, \end{cases}$$

we deduce that

$$\mathrm{res}_{C_{2^{k-i_2}}}^{G}([\omega_{i_1}^{(k-j_1)} u_{\lambda_{j_1}}] \cdot [\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_2}}]) = \begin{cases} 2^{i_1}\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_1}} u_{\lambda_{j_2}} & \text{if } i_1 + j_1 \le j_2, \\ 2^{j_2-j_1+i_1}\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_1}} u_{\lambda_{j_2}} & \text{if } j_2 \le i_1 + j_1 \le i_2 + j_2, \\ 2^{j_2-j_1+i_2}\omega_{i_1+j_1-j_2}^{(k-j_2)} u_{\lambda_{j_1}} u_{\lambda_{j_2}} & \text{if } i_1 + j_1 > i_2 + j_2. \end{cases}$$

Since $\mathrm{res}^G_{C_{2^k-i_2}} = 1$ in $H\underline{\mathbb{A}}_{4-\lambda_1-\lambda_2}$, relation (3) follows. Note that since $G_{\lambda_{j_2}} \subset G_{\lambda_{j_1}}$,

$$[\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_1}} u_{\lambda_{j_2}}] = [\omega_{i_2}^{(k-j_2)} u_{\lambda_{j_2}}] u_{\lambda_{j_1}}$$

by Proposition 2.29. For relation (4), we use the pairing $H\underline{\mathbb{A}}_0 \square H\underline{\mathbb{A}}_{2-\lambda_i} \xrightarrow{\mu} H\underline{\mathbb{A}}_{2-\lambda_i}$ and the argument is similar to the that of (3).

Using the gold relation (see Proposition 3.16), to prove the structure of $H\underline{\mathbb{A}}^G_{\mathrm{pos}}$, we are left to show that we have the additive splitting

$$(8)\quad H\underline{\mathbb{A}}^G_{\mathrm{pos}} = \frac{\underline{\mathbb{A}}(G)[a_{\lambda_k}]\big[a_\alpha,\ \{a_{\lambda_i} | 2 \le i \le k-1\},\ \{[\omega_i^{(k-j)} u_{\lambda_j}] \mid 1 \le j \le k-1,\ 0 \le i \le k-j\}\big]}{\widetilde{S}} \oplus \frac{\underline{\mathbb{A}}(G)\langle u^i_{\lambda_k}\rangle_{i\ge 1}[a_{\lambda_k}]\big[\{[\omega_i^{(k-j)} u_{\lambda_j}] \mid 1 \le j \le k-1,\ 0 \le i \le k-j\}\big]\langle 1, a_\alpha\rangle}{2a_\alpha}.$$

The relations in $\widetilde{S}$ consists of the relations (1), (2), as well as the relations (3), (4), (5) where only the elements u_i with $i \neq k$ are involved. The proof of the second summand in (8), which corresponds to the gradings with $|V^\sharp| > 0$, is completely analogous to the case of $H\underline{\mathbb{Z}}$. We only need to consider the first summand in (8), which correspond to $|V^\sharp| \le 0$.

Note that

$$(9)\quad (\underline{\mathbb{A}}_G)^{\sharp C_2} \cong \underline{\mathbb{A}}_{G'} \oplus \underline{\mathbb{Z}}^*_{G'},$$

where $\underline{M}_H$ is to indicate that $\underline{M}$ is the relevant H-equivariant Mackey functor. $\underline{\mathbb{Z}}^*$ is the levelwise dual of $\underline{\mathbb{Z}}$ generated by the free sets, ie $\underline{\mathbb{Z}}^*(C_{2^i}) = \mathbb{Z}\langle\omega_0^{(i)}\rangle$ and $\mathrm{res}^H_K = 2$ and $\mathrm{tr}^H_K = 1$ when $[H : K] = 2$. Under the splitting, $\omega_i^{(k-j)} \in \underline{\mathbb{A}}(G)$ with $0 \le i < k-j$ on the left correspond to $\omega_i^{(k-j-1)} \in \underline{\mathbb{A}}(G')$ on the right.

Thus we have the splitting of G'-equivariant spectra

$$(H\underline{\mathbb{A}}_G)^{C_2} \simeq H\underline{\mathbb{A}}_{G'} \vee H\underline{\mathbb{Z}}^*_{G'}.$$

We consider the subring $H\underline{\mathbb{A}}^G_{\mathrm{pos}^\sharp}$ of the first summand in (8), where $\mathrm{pos}^\sharp$ are all representations of the positive cone without any copy of a_{λ_k}. It corresponds to the classes with a_{λ_k} having 0^{th} power and additively decomposes as

$$(10)\quad H\underline{\mathbb{A}}^G_{\mathrm{pos}^\sharp} \cong (H\underline{\mathbb{A}}_{G'})^{G'}_{\mathrm{pos}} \oplus (H\underline{\mathbb{Z}}^*_{G'})^{G'}_{\mathrm{pos}}$$

via Corollary 2.16, with

$$(H\underline{\mathbb{A}}_{G'})^{G'}_{\mathrm{pos}} = \frac{\underline{\mathbb{A}}(G')\big[a_\alpha, \{a_{\lambda_i} \mid 2 \le i \le k-1\},\ \{[\omega_i^{(k-1-j)} u_{\lambda_j}] \mid 1 \le j \le k-1,\ 0 \le i \le k-j\}\big]}{S'}$$

by induction. Here S' is the set of relations for the group G'. We notice that, via the identification (10), the difference between this ring and $H\underline{\mathbb{A}}^G_{\mathrm{pos}^\sharp}$ consists of the groups

$$\mathbb{Z}\big\langle\{[\omega_{k-j}^{(k-j)} u_U] \mid U = \Sigma_{i=1}^j a_i\lambda_j \text{ with } a_j \neq 0,\ 1 \le j \le k-1\}\big\rangle$$

since $\omega_k^{(k)}$ kills any Euler class by the Frobenius relation. These groups comes from $(H\underline{\mathbb{Z}}^*_{G'})^{G'}_{\text{pos}}$ via Proposition 2.23. Taking into account the particular case of relation (3), we get a surjective ring map

$$\underline{\mathbb{A}}(G)\big[a_\alpha, \{a_{\lambda_i} \mid 2 \le i \le k\},\ \{[\omega_i^{(k-j)} u_{\lambda_j}] \mid 1 \le j \le k,\ 0 \le i \le k-j\}\big] \twoheadrightarrow H\underline{\mathbb{A}}^G_{\text{pos}},$$

and the relation set S is proved above. □

Remark 5.6 As mentioned in Section 2A, this result essentially contains the odd primary result since we are working locally. Let p be an odd prime, and $G = C_{p^n}$. Recall the convention that $\lambda_1 = 2\alpha$, p-locally, the positive cone in this case would be

$$H\underline{\mathbb{A}}^G_{\text{pos}} = \underline{\mathbb{A}}(G)\big[\{a_{\lambda_i} \mid 1 \le i \le k\},\ \{[\omega_i^{(k-j)} u_{\lambda_j}] \mid 1 \le j \le k,\ 0 \le i \le k-j\}\big] \,/\, S,$$

In the definition of $\omega_j^{(k)}$ and in the relations (1)–(5), 2 should be changed to p.

6 The negative cone of $H\underline{M}$ for arbitrary $\underline{M}$

In this section we determine the structure of the negative cone $H\underline{M}^G_{\text{neg}}$ of $H\underline{M}^G_\star$, which is the subgroup indexed by $V \in RO(G)$ such that $a_i \ge 0$ for $1 \le i \le k$. As an example, we completely compute the negative cone of $H\underline{\mathbb{Z}}$.

6A Structure of the negative cone for arbitrary $\underline{M}$

The homotopy groups with $|V| \le -1$ in the negative cone are 0 since an n-dimensional CW-complex does not have $n+1$-dimensional cohomology. As a consequence, we only need to care about the V^{th} homotopy groups where $|V| = 0, 1$.

6A1 $H\underline{M}^G_V$ for orientable V with $|V| = 0$

Proposition 6.1 *Let V be an orientable (not necessarily irreducible) G-representation. Then $H\underline{M}^G_{V-|V|} \cong \underline{M}(G_V)/G$. For the Mackey functor structure, we have*

$$H\underline{M}_{V-|V|}(H) = \begin{cases} \underline{M}(G_V)/H & \text{if } G_V \subset H, \\ \underline{M}(H) & \text{otherwise.} \end{cases}$$

For $K \subset H \subset G_V$, the maps $\operatorname{res}^H_K, \operatorname{tr}^H_K$ are induced from that of $\underline{M}$. If $G_V \subset K \subset H$, then tr^H_K is induced by the canonical projection, and res^H_K is given by $\operatorname{res}^{C_{2^i}}_{C_{2^{i-1}}}(x) = (1+\gamma^{2^{k-i}})\cdot x$ and transitivity. Moreover, the map $u_V \colon \underline{M}(G_V)/G \cong H\underline{M}^G_{V-|V|} \to H\underline{M}^G_0 = \underline{M}(G)$ is induced by $\operatorname{tr}^G_{G_V}$.

Proof The proof is dual to that of Proposition 2.23. □

6A2 $H\underline{M}_V^G$ for nonorientable V with $|V| = 0$

Proposition 6.2 (1) *The Mackey functor structure of $H\underline{M}_{\alpha-1}^G$ is given by*

$$H\underline{M}_{\alpha-1}(H) = \begin{cases} \underline{M}(G')/\mathrm{Im}(\mathrm{res}_{G'}^G) & \text{if } H = G, \\ \widetilde{\underline{M}(H)} & \text{otherwise.} \end{cases}$$

The map $\mathrm{tr}_{G'}^G$ is the canonical projection, and $\mathrm{res}_{G'}^G(x) = (1-\gamma)\cdot x$. Transfers and restrictions on other levels are induced from that of $\underline{M}$.

(2) *The Mackey functor structure of $H\underline{M}_{n\alpha-n}^G$ for odd $n > 1$ is given by*

$$H\underline{M}_{n\alpha-n}(H) = \begin{cases} \underline{M}(G')/\zeta_1 \cdot \underline{M}(G') & \text{if } H = G, \\ \widetilde{\underline{M}(H)} & \text{otherwise.} \end{cases}$$

The map $\mathrm{tr}_{G'}^G$ is the canonical projection, and $\mathrm{res}_{G'}^G(x) = (1-\gamma)\cdot x$. Transfers and restrictions on other levels are induced from that of $\underline{M}$.

Proof We first note that since $\mathrm{res}_{G'}^G \alpha = 1$, we have $H\underline{M}_{n\alpha-n}(H) \cong \underline{M}(H)$ for $H \subset G'$. Now for each $H \subset G'$, the Weyl action on $H\underline{M}_n^H(S^{n\alpha})$ comes from the action on $S^{n\alpha}$ (sign action for n odd) as well as the action on $\underline{M}(H)$. We have that there is an isomorphism

$$H\underline{M}_{n\alpha-n}{\downarrow_{G'}^G} \cong \widetilde{\mathbb{Z}} \otimes (H\underline{M}_0{\downarrow_{G'}^G}) \cong \widetilde{\mathbb{Z}} \otimes (\underline{M}{\downarrow_{G'}^G}) \quad \text{(levelwise tensoring).}$$

For the Mackey functor structure between the levels G and G', our proof works for both $n = 1$ and odd $n > 1$, so we will assume $n = 1$. Replacing the top differential by ζ_1 in the cellular cochain complex from Theorem 2.6 provides the proof for odd $n > 1$. The sphere S^α fits into the cofiber sequence

$$G/G'_+ \to G/G_+ \to S^\alpha.$$

Its cellular cochain complex with coefficient in $\underline{M}$ is

$$0 \to \underline{M}(G) \xrightarrow{\mathrm{res}_{G'}^G} \underline{M}(G')$$

with differential induced from the folding map $\nabla : G/G' \to G/G$. To deduce the Mackey functor structure, we have to apply $G/G' \times -$ to this map and consider the following commutative diagram, where the vertical arrows induce the restrictions and transfers in the Mackey functor structures:

$$\begin{array}{ccc} G/G \times G/G & \xleftarrow{1\times\nabla} & G/G \times G/G' \\ \uparrow{\scriptstyle \nabla\times 1} & & \uparrow{\scriptstyle \nabla\times 1} \\ G/G' \times G/G & \xleftarrow{1\times\nabla} & G/G' \times G/G' \end{array} \qquad \begin{array}{ccc} \underline{M}(G) & \xrightarrow{\mathrm{res}_{G'}^G} & \underline{M}(G') \\ \downarrow{\scriptstyle \mathrm{res}_{G'}^G} & & \downarrow{\scriptstyle \Delta} \\ \underline{M}(G') & \xrightarrow{(1,\gamma)} & \underline{M}(G') \oplus M(G') \end{array}$$

Here we made the identification

$$G/G' \times G/G' \cong G/G'_{(e,e)} \amalg G/G'_{(\gamma,1)},$$

and the vertical maps correspond to the restriction $C^*_G(S^\alpha; \underline{M}) \to C^*_{G'}(i^*_{G'}\underline{S}^\alpha; \underline{M})$. In the right square, the map Δ is the diagonal, and $(1,\gamma)(x) = (x, \gamma x)$. For any $x \in \underline{M}(G')$ from the top right corner,

$$\Delta(x) = (x,x) = (x,0) + (0,x) = (0,-\gamma x) + (0,x) = (0,(1-\gamma)x)$$

modulo the image of $(1,\gamma)$. From the construction of ψ_1 in the proof of Theorem 2.6, we deduce that in $H\underline{M}_{\alpha-1}$, we have $\mathrm{res}^G_{G'}(x) = (1-\gamma)\cdot x$. Note that the G-module structure on $H\underline{M}^{G'}_{\alpha-1}$ is actually $\widetilde{\mathbb{Z}} \otimes \underline{M}(G')$. By the double-coset formula, we deduce $\mathrm{tr}^G_{G'} = 1$. □

Proposition 6.3 *We have that*

$$H\underline{M}^G_{\alpha-1} = \mathrm{coker}(\mathrm{res}^G_{G'}) \quad \text{and} \quad H\underline{M}^G_{n\alpha-n} = \underline{M}(G')/\zeta_1 \cdot \underline{M}(G')$$

for odd $n > 1$.

Proof This follows directly from Theorem 2.6. □

Proposition 6.4 *Let $V = \alpha - 1 + V' - |V'|$ be a nonorientable representation with $V' = \sum_{1\le i\le k} c_i\lambda_i \neq 0$. Then*

(1) $H\underline{M}^G_V = \dfrac{\underline{M}(G_V)}{\zeta_1 \underline{M}(G_V)}$,

(2) *multiplication $u_{V'}\colon H\underline{M}^G_V \cong \dfrac{\underline{M}(G_V)}{\zeta_1\underline{M}(G_V)} \to H\underline{M}^G_{\alpha-1} \cong \mathrm{coker}(\mathrm{res}^G_{G'})$ is the transfer map $\mathrm{tr}^{G'}_{G_V}$.*

Proof The proof is dual to that of Proposition 5.2. □

6A3 $H\underline{M}^G_V$ for V with $|V| = 1$

Proposition 6.5 *Let $U = \sum_{i=1}^j a_i\lambda_i$, $a_i \ge 0$ and $a_j \neq 0$. Let $W = U - \lambda_j$ and s be the biggest integer such that λ_s appears in W with nonzero coefficient. Then*

$$H\underline{M}^G_{U-|U|+1} = \begin{cases} \dfrac{\underline{M}(G_U)^G}{\zeta_j\,\underline{M}(G_U)} & \text{if } a_j \ge 2, \\ \dfrac{\underline{M}(G_U)^G}{\mathrm{Im}(\zeta_s\,\mathrm{res}^{G_W}_{G_U})} & \text{if } a_j = 1 \text{ and } W \neq 0, \\ \dfrac{\underline{M}(G_U)^G}{\mathrm{Im}(\mathrm{res}^G_{G_U})} & \text{if } a_j = 1 \text{ and } W = 0, \end{cases}$$

and

$$H\underline{M}^G_{U-|U|+\alpha} = \begin{cases} \dfrac{\zeta_1\underline{M}(G_U)}{\bar{\zeta}_j\,\underline{M}(G_U)} & \text{if } a_j \ge 2, \\ \dfrac{\zeta_1\underline{M}(G_U)}{\mathrm{Im}(\bar{\zeta}_s\,\mathrm{res}^{G_W}_{G_U})} & \text{if } a_j = 1 \text{ and } W \neq 0, \\ \dfrac{\zeta_1\underline{M}(G_U)}{\mathrm{Im}(\bar{\zeta}_1\,\mathrm{res}^{G'}_{G_U})} & \text{if } a_j = 1 \text{ and } W = 0. \end{cases}$$

Proof This follows directly from Theorem 2.6. □

6B The constant Mackey functor $\underline{\mathbb{Z}}$

In the following theorem we compute the negative cone of $H\underline{\mathbb{Z}}$. The notation comes from the previous computations via Tate squares in [24; 23; 25]. For the methods to deduce the entire Mackey functor structure, we refer the reader to [24, Section 6]. Briefly speaking, by induction, we are only concerned with $\mathrm{tr}^G_{G'}$ and $\mathrm{res}^G_{G'}$, which can be deduced from a_α multiplications by a trick of Hill, Hopkins and Ravenel [13, Lemma 4.2].

Theorem 6.6 *Let $G = C_{2^k}$, $k \geq 1$, then $H\underline{\mathbb{Z}}^G_{\mathrm{neg}}$ can be written as the direct sum of the following $3k - 1 + \frac{1}{2}k(k-1)$ summands. (For simplicity, all indices i, j have range ≥ 1.)*

(1) *The torsion-free part (k summands):*

$$\mathbb{Z}\langle 2u_{2\alpha}^{-i}\rangle \oplus \mathbb{Z}[u_{2\alpha}^{-1}]\langle 4u_{\lambda_2}^{-i}\rangle \oplus \mathbb{Z}[u_{\lambda_2}^{-1}, u_{2\alpha}^{-1}]\langle 8u_{\lambda_3}^{-i}\rangle \oplus \cdots \oplus \mathbb{Z}[u_{\lambda_{k-1}}^{-1}, \ldots, u_{2\alpha}^{-1}]\langle 2^k u_{\lambda_k}^{-i}\rangle.$$

(2) *The 2-torsion part (involving a_α, k summands):*

$$\begin{aligned}\mathbb{Z}/2[u_{2\alpha}^{-1}][a_\alpha^{-1}, a_{\lambda_2}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_\alpha^{-1}u_{2\alpha}^{-1}\rangle &\oplus \mathbb{Z}/2[u_{\lambda_2}^{-1}][a_{\lambda_3}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_2}^{-i}u_{2\alpha}^{-j}\rangle\langle a_\alpha\rangle \\ &\oplus \mathbb{Z}/2[u_{\lambda_2}^{-1}, u_{\lambda_3}^{-1}][a_{\lambda_4}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_3}^{-i}u_{2\alpha}^{-j}\rangle\langle a_\alpha\rangle \\ &\oplus \cdots \oplus \mathbb{Z}/2[u_{\lambda_2}^{-1}, \ldots, u_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_k}^{-i}u_{2\alpha}^{-j}\rangle\langle a_\alpha\rangle;\end{aligned}$$

and (not involving a_α, $k-1$ summands):

$$\mathbb{Z}/2[a_{\lambda_3}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_2}^{-i}u_{2\alpha}^{-j}\rangle \oplus \mathbb{Z}/2[a_{\lambda_4}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_3}^{-i}u_{2\alpha}^{-j}\rangle \oplus \cdots \oplus \mathbb{Z}/2\langle \Sigma^{-1}a_{\lambda_k}^{-i}u_{2\alpha}^{-j}\rangle.$$

(3) *For a fixed s where $2 \leq s \leq k$, the 2^s-torsion part ($k-s+1$ summands):*

$$\begin{aligned}\mathbb{Z}/2^s[u_{2\alpha}^{-1}, \ldots, u_{\lambda_{s-1}}^{-1}][a_{\lambda_{s+1}}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_s}^{-i}u_{\lambda_s}^{-j}\rangle &\\ \oplus \mathbb{Z}/2^s[u_{2\alpha}^{-1}, \ldots, u_{\lambda_{s-1}}^{-1}][a_{\lambda_{s+2}}^{-1}, \ldots, a_{\lambda_k}^{-1}]\langle \Sigma^{-1}a_{\lambda_{s+1}}^{-i}u_{\lambda_s}^{-j}\rangle &\\ \oplus \cdots \oplus \mathbb{Z}/2^s[u_{2\alpha}^{-1}, \ldots, u_{\lambda_{s-1}}^{-1}]\langle \Sigma^{-1}a_{\lambda_k}^{-i}u_{\lambda_s}^{-j}\rangle.&\end{aligned}$$

Proof The theorem is true for $k = 1$ by [3; 8; 25]. It is true for $k = 2$ by [24]. Assume it is true for $k-1$. Note that since $\underline{\mathbb{Z}}$ is a constant Mackey functor, for any $H \subset G$, we have that $(H\underline{\mathbb{Z}}_G)^H = H\underline{\mathbb{Z}}_{G/H}$, where we use $H\underline{\mathbb{Z}}_K$ to emphasize the K-equivariant $H\underline{\mathbb{Z}}$. When $V \in RO(G)$ does not contain any copy of λ_k, by Corollary 2.16, we get an isomorphism which we will denote by

$$\varepsilon^*\colon H\underline{\mathbb{Z}}_V^{C_{2^{k-1}}} \xrightarrow{\cong} H\underline{\mathbb{Z}}^G_V.$$

Here, for simplicity, we used V for both the $C_{2^{k-1}}$ and the C_{2^k}-representation. Under this map, we have $\varepsilon^*(a_{\lambda_i}) = a_{\lambda_i}$ and $\varepsilon^*(u_{\lambda_i}) = u_{\lambda_i}$.

Write $V = V^\sharp + a_k\lambda_k$ and note that $a_k \geq 0$ since we are in the negative cone. There are several different cases to consider.

When $|V^\sharp| \geq 1$, then by Proposition 3.7, classes in $H\underline{\mathbb{Z}}^G_{V^\sharp}$ are all infinitely a_{λ_k}-divisible. For torsion classes in degrees with $|V^\sharp| = 0$, by induction, we know they are all infinitely divisible by $a_{\lambda_{k-1}}$. Since $\mathrm{res}^G_e(a_{\lambda_{k-1}}) = 0$, these classes restrict to 0 under res^G_e. By Proposition 3.9, they are divisible by a_{λ_k}.

They are actually infinitely divisible by a_{λ_k}, since for any $x \in H\underline{\mathbb{Z}}_V^G$ with $|V^\sharp| = 0$, $a_{\lambda_k}^{-1}x$ is in total degree ≥ 2, thus infinitely a_{λ_k}-divisible. As a result, we deduce that the torsion classes in $H\underline{\mathbb{Z}}_V^G$ with $|V^\sharp| \geq 0$ are $\varepsilon^* H\underline{\mathbb{Z}}_V^{C_{2^{k-1}}}[a_{\lambda_k}^{-1}]$.

For torsion-free classes in $H\underline{\mathbb{Z}}_V^G$ with $|V^\sharp| \geq 0$, we know that $H\underline{\mathbb{Z}}_{V^\sharp}^G$ consist of all the "negative powers" of the u_{λ_i}'s as in part (1), with no u_{λ_k} involved. By induction, we know that the Mackey functors they generate have $\mathbb{Z}$ in each level, with res_K^H equal to 1 or 2 when $[H:K] = 2$. Thus restrictions are injective on these classes and they are not divisible by a_{λ_k}.

If $|V^\sharp| < 0$, then let c be the unique positive integer such that the total degree of $W = V^\sharp + c\lambda_k$ equals 0 or 1 depending on whether $|V^\sharp|$ is even or odd. Let $d_k = a_k - c$, so that we have $V = W + d_k\lambda_k$.

If $|W| = 0$ and orientable, we get the torsion-free classes

$$\mathbb{Z}[u_{\lambda_{k-1}}^{-1}, \dots, u_{2\alpha}^{-1}]\langle 2^k u_{\lambda_k}^{-i}\rangle$$

by Proposition 6.1. They are not a_{λ_k}-divisible as before. This completes the inductive proof of part (1). If $|W| = 0$ and nonorientable, we get the classes

$$\mathbb{Z}/2[u_{\lambda_2}^{-1}, \dots, u_{\lambda_k}^{-1}]\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i}\rangle\langle a_\alpha\rangle$$

by Proposition 6.4. We will justify their names later. Since $H\underline{\mathbb{Z}}_W^e = \underline{\mathbb{Z}}$, the restrictions of these classes to the trivial subgroup are 0, thus they are infinitely divisible by a_{λ_k}. This finishes the inductive proof of the classes in part (2) involving a_α.

Now we consider the gradings with $|W| = 1$. By Proposition 6.5, we have $H\underline{\mathbb{Z}}_{U-|U|+\alpha}^G = 0$ for any orientable U. Thus we only need to look at the groups $H\underline{\mathbb{Z}}_W^G$ with $W = U - |U| + 1$. From the cofiber sequence

$$G/G'_+ \to S^0 \xrightarrow{a_\alpha} S^\alpha,$$

we can derive long exact sequences for some orientable G-representation U

$$(*)\quad \cdots \to H\underline{\mathbb{Z}}_{U+2-|U|}^{G'} \to H\underline{\mathbb{Z}}_{U+\alpha+1-|U|}^G \xrightarrow{a_\alpha} H\underline{\mathbb{Z}}_{U-|U|+1}^G \xrightarrow{\mathrm{res}_{G'}^G} H\underline{\mathbb{Z}}_{U-|U|+1}^{G'} \xrightarrow{\mathrm{tr}_{G'}^G} H\underline{\mathbb{Z}}_{U-|U|+\alpha}^G \to \cdots,$$

and

$$(**)\quad \cdots \to H\underline{\mathbb{Z}}_{U+1-|U|}^{G'} \to H\underline{\mathbb{Z}}_{U+1-|U|}^G \xrightarrow{a_\alpha} H\underline{\mathbb{Z}}_{U-|U|+1-\alpha}^G \xrightarrow{\mathrm{res}_{G'}^G} H\underline{\mathbb{Z}}_{U-|U|}^{G'} \xrightarrow{\mathrm{tr}_{G'}^G} H\underline{\mathbb{Z}}_{U-|U|}^G \to \cdots.$$

We focus on the groups $H\underline{\mathbb{Z}}_{U-|U|+1}^G$ and $H\underline{\mathbb{Z}}_{U-|U|+1-\alpha}^G$ in the middle.

In the sequence $(*)$, the first group is 0 by Theorem 2.6, since in the cellular cochain of S^U, $d^{|U-2|} = 2^i$ for some i and $\mathbb{Z}$ is torsion-free. The last group is 0 by Proposition 6.5. The group $H\underline{\mathbb{Z}}_{U+\alpha+1-|U|}^G = \mathbb{Z}/2$ since in the cellular cochain of $S^{U+\alpha}$, $d^{|U|-1} = \bar{\zeta}_i$ for some i which is 0, and $d^{|U|-2} = 1 + \gamma = 2$. Since each group is cyclic, as can be seen easily from the cellular cochain, we know the sequence $(*)$ is always of the form

$$(11)\qquad 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{a_\alpha} \mathbb{Z}/2^i \xrightarrow{\mathrm{res}_{G'}^G} \mathbb{Z}/2^{i-1} \longrightarrow 0$$

for $1 \leq i \leq k$. More concretely, from the cellular cochain, we have

$$H\underline{M}_W^G = \begin{cases} \mathbb{Z}/2^k & \text{if } c \geq 2, \\ \mathbb{Z}/2^i & \text{if } c = 1 \text{ and } \lambda_{\max}(V^\sharp) = \lambda_i, \\ 0 & \text{if } c = 1 \text{ and } V^\sharp = |V^\sharp|. \end{cases}$$

Thus all the 2^{i-1}-torsion groups in $H\underline{\mathbb{Z}}_V^{G'}$ with $|W| \geq 1$ lift to $H\underline{\mathbb{Z}}_V^G$ and become 2^i-torsion, when $i \geq 2$. For example, we look at the 4-torsion classes in $H\underline{\mathbb{Z}}_V^G$ for V in the negative cone. There are two sources of these classes. One is from the quotient group $C_{2^{k-1}}$ through ε^* when we already have $|V^\sharp| \geq 0$. Another is from the subgroup G' lifted via $\mathrm{res}^G_{G'}$, when $|V^\sharp| < 0$ but $|W| = 1$. By induction, we have the 4-torsion classes in $H\underline{\mathbb{Z}}_{\mathrm{neg}}^{G'}$ ($k-2$ summands),

$$\mathbb{Z}/4[u_{2\alpha}^{-1}][a_{\lambda_3}^{-1}, \dots, a_{\lambda_{k-1}}^{-1}]\langle \Sigma^{-1} a_{\lambda_2}^{-i} u_{\lambda_2}^{-j} \rangle \oplus \mathbb{Z}/4[u_{2\alpha}^{-1}][a_{\lambda_4}^{-1}, \dots, a_{\lambda_{k-1}}^{-1}]\langle \Sigma^{-1} a_{\lambda_3}^{-i} u_{\lambda_2}^{-j} \rangle$$
$$\oplus \cdots \oplus \mathbb{Z}/4[u_{2\alpha}^{-1}]\langle \Sigma^{-1} a_{\lambda_{k-1}}^{-i} u_{\lambda_2}^{-j} \rangle.$$

They give rise to classes in $H\underline{\mathbb{Z}}_{\mathrm{neg}}^G$ via ε^*, and we should join $[a_{\lambda_k}^{-1}]$ since they are infinitely a_{λ_k}-divisible. These are the first $k-2$ summands of the 4-torsion classes in part (3) for C_{2^k}. The last summand of the 4-torsion classes are lifted via $\mathrm{res}^G_{G'}$ from the classes

$$\mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_{k-1}}^{-i} u_{2\alpha}^{-j} \rangle$$

in $H\underline{\mathbb{Z}}_{\mathrm{neg}}^{G'}$. These classes lifts to

$$\mathbb{Z}/4[u_{2\alpha}^{-1}]\langle \Sigma^{-1} a_{\lambda_k}^{-i} u_{\lambda_2}^{-j} \rangle.$$

Notice that we added $[u_{2\alpha}^{-1}]$ since $\mathrm{res}^G_{G'}(u_{2\alpha})$ is a unit. The 2^s-torsion classes for $2 \leq s \leq k$ are proved the same way, and we finished the inductive proof of part (3).

We are left with the proof of the 2-torsion classes that does not involve a_α. The first $k-2$ summands come from ε^* followed by a_{λ_k}-divisibility. The last summand

$$\mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_k}^{-i} u_{2\alpha}^{-j} \rangle$$

comes for the sequence (11) when $i = 1$. From the cellular cochain, we know in the case of $|V^\sharp| < 0$ and $|W| = 1$, $H\underline{\mathbb{Z}}_W^G = \mathbb{Z}/2$ only if $W = * + a_1\lambda_1 + \lambda_k, a_1 \neq 0$. In these case, these groups are generated by

$$\mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-j} \rangle.$$

Since they are a_{λ_k}-divisible, we will finish the proof once we justify the notation.

As in [24], the classes

$$\mathbb{Z}/2[u_{2\alpha}^{-1}]\langle \mathrm{tr}^{C_{2^k}}_{C_{2^{k-1}}}(e_{3\alpha}) a_{\lambda_k}^{-1} \rangle$$

can be also written as

$$\mathbb{Z}/2\langle \Sigma^{-1} a_\alpha^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle.$$

The exact sequence $(*)$ tells us

$$\mathbb{Z}/2\langle \Sigma^{-1} a_\alpha^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle \cdot a_\alpha = \mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle$$

since $H\underline{\mathbb{Z}}_{U-|U|+1}^{G'} = 0$. To justify the names of the classes

$$\mathbb{Z}/2[u_{\lambda_2}^{-1}, \dots, u_{\lambda_k}^{-1}]\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle \langle a_\alpha \rangle$$

we look at the exact sequence $(**)$. We get

$$\mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle \cdot a_\alpha = \mathbb{Z}/2\langle \Sigma^{-1} a_{\lambda_k}^{-1} u_{2\alpha}^{-i} \rangle \langle a_\alpha \rangle.$$

This is because in the degrees $U - |U|$, transfers are either 1 or 2, and $\ker(\mathrm{tr}(U-|U|)_{G'}^{G}) = 0$. □

Remark 6.7 The result has been verified additively by Anderson duality used in [25]. We have $I_{\mathbb{Z}}(H\underline{\mathbb{Z}}) \simeq \Sigma^{2-\lambda_k} H\underline{\mathbb{Z}}$. Anderson duality gives the short exact sequence of abelian groups

$$0 \to \mathrm{Ext}^1(H\underline{\mathbb{Z}}^G_{\star-1}, \mathbb{Z}) \to H\underline{\mathbb{Z}}^G_{\lambda_k-2-\star} \to \mathrm{Hom}_{\mathrm{Ab}}(H\underline{\mathbb{Z}}^G_{\star}, \mathbb{Z}) \to 0.$$

Torsion and torsion-free classes in $H\underline{\mathbb{Z}}^G_{\mathrm{neg}}$ will dualize to $H\underline{\mathbb{Z}}^G_{\mathrm{pos}}$ (sometimes outside the positive cone) via $\mathrm{Ext}^1(-,\mathbb{Z})$ and $\mathrm{Hom}_{\mathrm{Ab}}(-,\mathbb{Z})$, respectively. By an abuse of notation, we denote this duality by D. For example, we have the dualities

$$\begin{aligned}
D(2^k u_{\lambda_k}^{-1}) &= 1,\\
D(2^i u_{\lambda_i}^{-1}) &= 2^{k-i} u_{\lambda_i} u_{\lambda_k}^{-1},\\
D(\Sigma^{-1} a_{\lambda_i}^{-1} u_{2\alpha}^{-1}) &= a_\alpha^2 a_{\lambda_i} a_{\lambda_k}^{-1},\\
D(\Sigma^{-1} a_\alpha a_{\lambda_i}^{-1} u_{2\alpha}^{-1}) &= a_\alpha a_{\lambda_i} a_{\lambda_k}^{-1},\\
D(\Sigma^{-1} a_{\lambda_k}^{-1} u_{\lambda_k}^{-1}) &= a_{\lambda_k},\\
D(\Sigma^{-1} a_{\lambda_t}^{-1} u_{\lambda_s}^{-1}) &= a_{\lambda_t} a_{\lambda_s} a_{\lambda_k}^{-1} \qquad (t \geq s \geq 2).
\end{aligned}$$

The divisibility relations between the Euler and orientation classes on the right hand sides of the equations can be seen by cellular computations. See for example [12].

References

[1] **D Ayala**, **A Mazel-Gee**, **N Rozenblyum**, *Derived Mackey functors and C_{p^n}-equivariant cohomology*, preprint (2021) arXiv 2105.02456

[2] **J L Caruso**, *Operations in equivariant $\mathbb{Z}/p$-cohomology*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 521–541 MR

[3] **D Dugger**, *An Atiyah–Hirzebruch spectral sequence for KR-theory*, K-Theory 35 (2005) 213–256 MR

[4] **D Dugger**, *Coherence for invertible objects and multigraded homotopy rings*, Algebr. Geom. Topol. 14 (2014) 1055–1106 MR

[5] **K K Ferland**, *On the $RO(G)$-graded equivariant ordinary cohomology of generalized G-cell complexes for $G = \mathbb{Z}/p$*, PhD thesis, Syracuse University (1999) Available at `https://www.proquest.com/docview/304562761`

[6] **K K Ferland**, **L G Lewis, Jr**, *The $RO(G)$-graded equivariant ordinary homology of G-cell complexes with even-dimensional cells for $G = \mathbb{Z}/p$*, Mem. Amer. Math. Soc. 794, Amer. Math. Soc., Providence, RI (2004) MR

[7] **N Georgakopoulos**, *The $RO(C_4)$ integral homology of a point*, preprint (2019) arXiv 1912.06758

[8] **J P C Greenlees**, *Four approaches to cohomology theories with reality*, from "An alpine bouquet of algebraic topology", Contemp. Math. 708, Amer. Math. Soc., Providence, RI (2018) 139–156 MR

[9] **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR

[10] **J Hahn**, **X D Shi**, *Real orientations of Lubin–Tate spectra*, Invent. Math. 221 (2020) 731–776 MR

[11] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR

[12] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *The slice spectral sequence for certain $RO(C_{p^n})$-graded suspensions of $H\underline{\mathbb{Z}}$*, Bol. Soc. Mat. Mex. 23 (2017) 289–317 MR

[13] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *The slice spectral sequence for the C_4 analog of real K-theory*, Forum Math. 29 (2017) 383–447 MR

[14] **P Hu**, **I Kriz**, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence*, Topology 40 (2001) 317–399 MR

[15] **S Kriz**, *On completion and the evenness conjecture for homotopical equivariant cobordism*, preprint (2022) Available at `https://krizsophie.github.io/EvennessConjecture23071.pdf`

[16] **G Lewis**, **J P May**, **J McClure**, *Ordinary $RO(G)$-graded cohomology*, Bull. Amer. Math. Soc. 4 (1981) 208–212 MR

[17] **L G Lewis, Jr**, *The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions*, from "Algebraic topology and transformation groups", Lecture Notes in Math. 1361, Springer (1988) 53–122 MR

[18] **L G Lewis, Jr**, **M A Mandell**, *Equivariant universal coefficient and Künneth spectral sequences*, Proc. Lond. Math. Soc. 92 (2006) 505–544 MR

[19] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Region. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR

[20] **S Schwede**, *Lectures on equivariant stable homotopy theory*, lecture notes (2023) Available at `https://www.math.uni-bonn.de/people/schwede/equivariant.pdf`

[21] **I Sikora**, *On the $RO(Q)$-graded coefficients of Eilenberg–MacLane spectra*, J. Homotopy Relat. Struct. 17 (2022) 525–568 MR

[22] **J Thévenaz**, **P Webb**, *The structure of Mackey functors*, Trans. Amer. Math. Soc. 347 (1995) 1865–1961 MR

[23] **G Yan**, *The $RO(C_{2^n})$-graded homotopy of $H\underline{\mathbb{F}}_2$*, Proc. Amer. Math. Soc. 153 (2025) 1803–1816 MR

[24] **G Yan**, *The $RO(C_{2^n})$-graded homotopy of $H\underline{\mathbb{Z}}$ through generalized Tate squares*, Matematica 4 (2025) 364–418

[25] **M Zeng**, *Equivariant Eilenberg–Mac Lane spectra in cyclic p-groups*, preprint (2017) arXiv 1710.01769

IS: *Kraków University of Economics*
Krakow, Poland

GY: *Department of Mathematics, University of Notre Dame*
Notre Dame, IN, United States

Current address: *Shanghai Center for Mathematical Sciences, Fudan University*
Shanghai, China

`sikorai@uek.krakow.pl`, `yanguoqi1993@outlook.com`

Received: 18 October 2023 Revised: 21 April 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Thomas Koberda	University of Virginia thomas.koberda@virginia.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Markus Land	LMU München markus.land@math.lmu.de
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Octav Cornea	Université' de Montreal cornea@dms.umontreal.ca	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Kristen Hendricks	Rutgers University kristen.hendricks@rutgers.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

© 2025 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 5 (pages 2527–3144) 2025

The homotopy type of the PL cobordism category, I 2527
MAURICIO GÓMEZ LÓPEZ

The homotopy type of the PL cobordism category, II 2613
MAURICIO GÓMEZ LÓPEZ

Pullbacks of metric bundles and Cannon–Thurston maps 2667
SWATHI KRISHNA and PRANAB SARDAR

Product set growth in virtual subgroups of mapping class groups 2757
ALICE KERR

Surgery sequences and self-similarity of the Mandelbrot set 2807
DANNY CALEGARI

Diffeomorphisms of the 4-sphere, Cerf theory and Montesinos twins 2817
DAVID T GAY

Equivariant intrinsic formality 2851
REKHA SANTHANAM and SOUMYADIP THANDAR

Generating the liftable mapping class groups of cyclic covers of spheres 2883
PANKAJ KAPARI, KASHYAP RAJEEVSARATHY and APEKSHA SANGHI

Warped product metrics on hyperbolic and complex hyperbolic manifolds 2905
BARRY MINEMYER

On the structure of the $RO(G)$-graded homotopy of $H\underline{M}$ for cyclic p-groups 2933
IGOR SIKORA and GUOQI YAN

On the homotopy groups of the suspended quaternionic projective plane and applications 2981
JUXIN YANG, JUNO MUKAI and JIE WU

The right angled Artin group functor as a categorical embedding 3035
CHRIS GROSSACK

Equivariant cohomology of projective spaces 3049
SAMIK BASU, PINKA DEY and APARAJITA KARMAKAR

Nonuniform lattices of large systole containing a fixed 3-manifold group 3089
PAIGE HILLEN

Noncommutative divergence and the Turaev cobracket 3103
TOYO TANIGUCHI

Building weight-free Følner sets for Yu's property A in coarse geometry 3133
GRAHAM A NIBLO, NICK WRIGHT and JIAWEN ZHANG