Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:5 (2025) 2981–3033
DOI: 10.2140/agt.2025.25.2981
Published: 17 September 2025

On the homotopy groups of the suspended quaternionic projective plane and applications

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We determine the $(2,3)$-primary components of the homotopy groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for all $7 \le r \le 15$ and $k \ge 0$. Essentially, we give the determinations of the integral homotopy groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for all $0 \le r \le 15$ and $k \ge 0$, which in particular include the unstable ones. As applications, we first construct a suspended generalized Hopf fibration by a Toda bracket localized at 2. Then we provide two classification theorems for CW complexes of type $S^{4+k} \cup e^{8+k} \cup e^{12+k}$ (for $k \ge 1$ but $k \ne 4$) localized at 3, one of which states that the homotopy type of the space $(\Sigma^k \mathbb{H}P^3)_{(3)}$ as a CW complex of the above type depends only on its Steenrod module. Our results also yield some wedge decompositions of 3-local suspended self smash products of quaternionic projective spaces.

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1 Introduction

Homotopy groups occupy a central and foundational position in homotopy theory, encapsulating the essence of building spaces. In addition to the homotopy groups of spheres, the homotopy groups of finite CW complexes have been extensively studied. Experts have given many fundamental results, such as J H C Whitehead [29], A L Blakers and W S Massey [2], I M James [7], H Toda [26] and M E Mahowald [10]. Nowadays, the Blakers–Massey theorem, the James theorem [7, Theorem 2.1], the relative EHP sequence (see Whitehead [28]) and Toda bracket methods are still critical methods to determine the unstable homotopy groups of finite CW complexes.

Projective spaces are core objects in algebraic topology. The homotopy theory of quaternionic projective spaces has been studied by J F Adams [1], D Sullivan [23] and James [8]. The homotopy groups of the suspended quaternionic projective spaces are studied by A Liulevicius [9] and the second author [16]. It is worthy mentioning that at the time of [9], the homotopy groups $\pi_i(S^n)$ determined by Toda in [27] were not well known. Liulevicius did not use Toda's results but the Adams spectral sequence, and he determined many stable homotopy groups of projective spaces.

Let $\mathbb{H}P^n$ denote the quaternionic projective space of dimension n. It is the orbit space

$$(\mathbb{H}^{n+1}-\{0\})/(\mathbb{H}-\{0\}) = S^{4n+3}/S^3,$$

and admits a CW decomposition $S^4 \cup e^8 \cup \cdots \cup e^{4n}$. Here we determine the homotopy groups $\pi_{r+k}((\Sigma^k \mathbb{H}P^2)_{(p)})$ for all $7 \le r \le 15$ and all $k \ge 0$ (where $p \in \{2,3\}$); equivalently speaking, we determine the $(2,3)$-primary components of these groups.

Moreover, we study some unstable homotopy groups $\pi_{r+k}((\Sigma^k \mathbb{H}P^2)_{(2)})$ for a much greater r, that is, $r = 36$. These homotopy groups are significantly enigmatic, and their determinations are exceedingly arduous; see Theorems 6.8.2 and 6.8.6.

It is well known that $\pi_8(S^5) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/3$, which tells us for any prime $p \notin \{2,3\}$, $\Sigma\mathbb{H}P^2 \simeq S^5 \vee S^9$ localized at p. Further, we know for a prime $p \notin \{2,3\}$ that the homotopy groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ (for $r \le 15$ and $k \ge 0$) localized at p can be computed straightforwardly by Hilton and Milnor's formula and Toda's results, and so these groups localized at p are essentially known in history. For $r \le 6$, these groups are just homotopy groups of spheres which are also known. However, for the groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ localized at 2 or 3, especially for the unstable homotopy groups, the situations are more and more mysterious as r grows. In [5], Brayton Gray gives a method to determine the homotopy type of the homotopy fiber of the pinch map by his relative James construction. In some sense, Gray's method (our Proposition 3.1.1) gives another view to understand the relative homotopy group method, namely, the James theorem. Gray's method is one of our fundamental methods to determine the homotopy groups of the 2-cell complexes.

The first main theorem is stated as follows:

Theorem 1 *The $(2,3)$-primary components of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 15$ and $k \ge 0$ are summarized in Table 1*

It is noteworthy that, among the groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 15$ localized at 2 or 3, the determination of $\pi_{19}((\Sigma^4 \mathbb{H}P^2)_{(2)})$ requires the most effort, closely trailed by $\pi_{11}((\Sigma^2 \mathbb{H}P^2)_{(2)})$, $\pi_{13}((\Sigma^2 \mathbb{H}P^2)_{(2)})$ and $\pi^S_{15}((\mathbb{H}P^2)_{(3)})$. In particular, to determine $\pi^S_{15}((\mathbb{H}P^2)_{(3)})$, we use the (*strong*) *Jacobi identity* of Toda brackets, which is one of the trickiest tools in Toda bracket theory; see the proof of Theorem 5.3.1.

Since there are well-known isomorphisms $\pi_n(\mathbb{H}P^2) \approx \pi_n(S^{11}) \oplus \pi_{n-1}(S^3)$, we will omit all the proofs for the determinations of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ localized at 2 or 3 in the case $k = 0$.

$r\backslash k$	0	1	2	3	4	5	6	7	8	9	10
7	$4{+}3$	0	–								
8	2	∞	∞	–							
9	2	0	∞	2	–						
10	3	0	2	2	2	–					
11	$\infty{+}3$	$8{+}9$	$8{+}2$ $+9$	$8{+}4$ $+9$	$\infty{+}8$ $+4{+}9$	$16{+}4$ $+9$	–				
12	2^2	$\infty{+}2$	2^2	2^3	2^4	2^3	2^2	–			
13	2^3	2^3	$\infty{+}2^2$	2^3	2^4	2^3	$\infty{+}2^2$	2^2	–		
14	$8{+}4$ $+2{+}3^2$	$2^2\infty{+}2$ $+3$	$2^2{+}3$	$+3$	$2^2{+}3$	$2^2{+}3$	$2^2{+}3$	$2^2{+}3$	$2{+}3$	–	
15	$4{+}2^2$ $+3$	$16{+}8$ $+9{+}3$	$16{+}2^2$ $+9$	$16{+}2^2$ $+27$	$\infty{+}16$ $+2{+}27$	$32{+}27$	$64{+}27$	$128{+}27$	$\infty{+}128$ $+27$	$128{+}27$	–

Table 1: The $(2,3)$-primary components of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$. In this table, n indicates $\mathbb{Z}/n$, n^m indicates $(\mathbb{Z}/n)^m$ (for positive n), ∞ indicates $\mathbb{Z}$, 0 indicates the trivial group and $+$ indicates the direct sum. For $k \geq r-6$, the groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. The $(2,3)$-components of the groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $k=(r-6)+1$ are just denoted by bar "–"; the remaining entries in the table lying in the stable range are therefore omitted.

As a first application, in the following main theorem for which we refer the reader to Theorem 7.2.1, we construct a suspended generalized Hopf fibration by a Toda bracket localized at 2. Empirically speaking, a homotopy class with a particularly strong geometric meaning is often a generator of the homotopy group, but $\Sigma^\infty h$ localized at 2 is not, which provides an interesting counterexample.

Theorem 2 *After localization at* 2, *let* $h\colon S^{11} \to \mathbb{H}P^2$ *be the homotopy class of the Hopf fibration whose homotopy cofiber is* $\mathbb{H}P^3$. *Then, for some odd* t, Σh *is contained in the Toda bracket*

$$\{j_1, \nu_5, 2t\nu_8\},$$

where $S^5 \xrightarrow{j_1} \Sigma\mathbb{H}P^2$ *is the inclusion, and* $\nu_n \in \pi_{3+n}(S^n)$ *for* $n \geq 4$ *are the Hopf classes. Moreover,* Σh *generates* $\pi_{12}(\Sigma\mathbb{H}P^2) \approx \mathbb{Z}/8$, *but* $\Sigma^\infty h$ *cannot generate any direct summand of* $\pi_{11}^S(\mathbb{H}P^2) \approx \mathbb{Z}/16 \oplus \mathbb{Z}/4$.

We say that a CW complex X is a *homology* n*-dimensional quaternionic projective space* if $\widetilde{H}_*(X;\mathbb{Z}) \approx \widetilde{H}_*(\mathbb{H}P^n;\mathbb{Z})$ as graded groups. Up to homotopy, a simply connected 1-dimensional quaternionic projective space is just the sphere of dimension 4. (Note: the simply connected condition is necessary, as taking the wedge product with a noncontractible acyclic space preserves the reduced homology.) It is clear that the homotopy type of a p-local (for $p \in \{2,3\}$) suspended simply connected 2-dimensional quaternionic projective space, ie a CW complex $S^{4+n}_{(p)} \cup_f e^{8+n}_{(p)}$ for $n \geq 1$ and f some map, is only dependent on the order of $[f] \in \pi_{7+n}(S^{4+n}_{(p)}) \approx \mathbb{Z}/24 \otimes \mathbb{Z}_{(p)}$; see our Corollary 2.1.1. For suspended simply connected 3-dimensional quaternionic projective spaces, the classification becomes complicated.

As applications of Theorem 1, we provide two classification theorems for k-fold (where $k \geq 1$ but $k \neq 4$) suspended simply connected 3-dimensional quaternionic projective spaces localized at 3. (Note: For Theorem 3, in the case $k = 4$, we are unable to give the classification by current techniques. For Theorem 4, the assumption $k \neq 4$ is necessary; see Remark 7.2.4.)

Our third main theorem for, which we refer the reader to Theorem 7.2.1, is stated as follows:

Theorem 3 *After localization at* 3, *suppose $k \geq 1$ but $k \neq 4$. Then, up to homotopy, the k-fold suspension of simply connected homology* 3*-dimensional quaternionic projective spaces, that is, CW complexes of type $S^{4+k} \cup e^{8+k} \cup e^{12+k}$, can be classified as the following*:

$$\Sigma^k \mathbb{H}P^3, \quad \Sigma^k \mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k}, \quad \Sigma^k \mathbb{H}P^2 \vee S^{12+k}, \quad \Sigma^{k-1} A \vee S^{8+k},$$
$$S^{4+k} \vee \Sigma^{4+k} \mathbb{H}P^2, \quad (S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}, \quad S^{4+k} \vee S^{8+k} \vee S^{12+k}.$$

By a *Steenrod module* we mean a cohomology module over the Steenrod algebra, or a homology module over the dual algebra of the Steenrod algebra. We adhere to conventional terminology, that is, localized at a prime p, we refer to a simply connected CW complex X as being of finite type if $H_i(X; \mathbb{Z}_{(p)})$ is a finitely generated $\mathbb{Z}_{(p)}$-module for each i. As we see in the following main theorem, for which we refer the reader to Theorem 7.2.3, the homotopy type of $(\Sigma^k \mathbb{H}P^3)_{(3)}$ for $k \geq 1$ and $k \neq 4$ is completely determined by its simply connectedness, finite type property and Steenrod module structure.

Theorem 4 *After localization at* 3, *suppose X is a simply connected CW complex of finite type, and*

$$\widetilde{H}_*(X; \mathbb{Z}/3) \approx \widetilde{H}_*(\Sigma^k \mathbb{H}P^3; \mathbb{Z}/3)$$

as Steenrod modules where $k \geq 1$ but $k \neq 4$. Then $X \simeq \Sigma^k \mathbb{H}P^3$.

There is a connection between the wedge decompositions of self smashes of finite CW complexes and the modular representation theory. The homology of the functorial indecomposable factors of self smashes of a 2-cell suspension has been determined in [22] by P Selick and the third author. And the wedge decompositions of self smashes of real, complex, quaternionic and Cayley projective spaces localized at 2 are studied in [31, Proposition 3.4] by the third author. Here, as applications of Theorem 4, utilizing the methods in [31], we provide two wedge decompositions of suspended self smashes of quaternionic projective spaces localized at 3; see Theorems 7.3.2 and 7.3.3.

Theorem 5 *After localization at* 3, *there exist homotopy equivalences*

$$\Sigma \mathbb{H}P^2 \wedge \mathbb{H}P^2 \simeq S^{13} \vee \Sigma^5 \mathbb{H}P^3, \quad \Sigma \mathbb{H}P^3 \wedge \mathbb{H}P^3 \simeq \Sigma^9 \mathbb{H}P^3 \vee Y,$$

where Y is a 6*-cell CW complex and $\mathrm{sk}_{13}(Y) = \Sigma^5 \mathbb{H}P^2$.*

Acknowledgements This work is supported in part by the Natural Science Foundation of China (NSFC grant 11971144), High-level Scientific Research Foundation of Hebei Province. And this work is also supported in part by the start-up research fund from Beijing Institute of Mathematical Sciences and Applications.

2 Preliminaries

2.1 Notation and some fundamental facts

In this paper, all spaces, maps and homotopy classes are pointed, basepoints and constant maps are denoted by $*$, and homotopy classes of constant maps are denoted by 0. If we take the p-localization, we always use the original symbols of the spaces, maps and homotopy classes to denote them after localization at p, and $\widetilde{H}_*(-)$ denotes the mod p reduced homology. The homotopy fiber is called the fiber for short. For a map or a homotopy class f, we use C_f to denote the homotopy cofiber of f, and the homotopy cofiber is called the cofiber for short.

For a nonnegative integer m, let $\mathbb{Z}/m$ denote $\mathbb{Z}/m\mathbb{Z}$. For a prime p, let

$$\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z} \text{ are coprime and } p \nmid b\},$$

that is, the group or the ring of p-local integers; for a $\mathbb{Z}_{(p)}$-module A of form $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^k$, we use $G = A\{\mathbb{x}\}$ to denote a $\mathbb{Z}_{(p)}$-module G which is isomorphic to A and generated by $\mathbb{x}$. For example, $G = \mathbb{Z}/4\{\mathbb{x}\}$ stands for $G \approx \mathbb{Z}/4$ and G is generated by $\mathbb{x}$; $(\mathbb{Z}/m)^k$ denotes the direct sum of k-copies of $\mathbb{Z}/m$. We use $\oplus$ to denote both the internal direct sum and the external direct sum. And $\mathrm{ord}(\mathbb{x})$ denotes the order of the element $\mathbb{x}$ of a group.

Let $a_1, a_2, \dots, a_n \in A$ where A is a $\mathbb{Z}$- or $\mathbb{Z}_{(p)}$-module. Then $\langle a_1, a_2, \dots, a_n\rangle$ denotes the submodule generated by $a_1, a_2, \dots, a_n$. We know $\langle \alpha, \beta, \gamma\rangle$ also denotes a stable Toda bracket, but these two meanings of $\langle -, -, -\rangle$ are always easy to distinguish. We use $\mathbb{Z}_+$ to denote the set of positive integers. Suppose p is a prime. For a finitely generated abelian group G, the p-primary component of G is

$$G/\langle g \in G \mid \mathrm{ord}(g) = q^n \text{ for some prime } q \neq p \text{ and some } n \in \mathbb{Z}_+\rangle,$$

and the so-called $(2,3)$-primary component of G is

$$G/\langle g \in G \mid \mathrm{ord}(g) = q^n \text{ for some prime } q \notin \{2,3\} \text{ and some } n \in \mathbb{Z}_+\rangle.$$

Our result gives the homotopy groups after localization at $p \in \{2,3\}$, which naturally correspond to the $(2,3)$-primary components of these groups.

For a commutative unitary ring R, let

$$R\{a_1, a_2, \dots, a_n, \dots,\} \quad \text{where } |a_k| = j_k$$

denote the graded free R-module with basis $a_1, a_2, \dots, a_n, \dots$ and where a_k is of degree j_k.

We often use the obvious notation $X \xrightarrow{\subseteq} Y$ to denote the inclusion map from X to Y, where X and Y are spaces or modules. For a space X, let $X^{\wedge n}$ denote the n-fold self smash product of X, and $X^{\wedge 0} = S^0$.

We follow Toda's notation and directly use the symbols for the generators of $\pi_{n+k}(S^n)$ in [11; 12; 13; 14; 18; 27]. The only two differences are: we denote Toda's E by Σ, the suspension functor, and we denote

Toda's Δ by P, the boundary homomorphism of the EHP sequence. By abuse of notation, sometimes we use the same symbol to denote a map and its homotopy class. For a CW complex $Z = Y \cup_g e^{m+1}$ where $\dim(Y) \le m$, we denote the mapping cylinder M_g by M_Y to indicate $M_Y \simeq Y$, although M_Y/Y does not only depend on Y. Readers will see the benefit of this notation throughout this paper.

Let $\alpha \in \pi_n(X)$, $\beta \in \pi_m(S^n)$ where $n \ge 2$ and let $k \in \mathbb{Z}$. Commonly and reasonably, $\alpha\beta$ is the abbreviation of $\alpha \circ \beta$ and $k\alpha \circ \beta$ is the abbreviation of $(k\alpha) \circ \beta$; here we only use the symbol $k\alpha\beta$ to denote $k(\alpha\beta)$. So it is necessary to point out that

$$k\alpha\beta \ne k\alpha \circ \beta \quad \text{in general.}$$

Of course, $k\alpha\beta = k\alpha \circ \beta$ always holds if β is a suspension, or the codomain of α is S^7 or a group-like H-space [28, page 118]. However, to write the equation [27, Proposition 1.4, page 11] $\{\alpha, \beta, \gamma\} \circ \Sigma\delta = -(\alpha \circ \{\beta, \gamma, \delta\})$ briefly, we denote $-(\alpha \circ \{\beta, \gamma, \delta\})$ by $-\alpha \circ \{\beta, \gamma, \delta\}$, and denote $\{(-\alpha) \circ \varkappa \mid \varkappa \in \{\beta, \gamma, \delta\}\}$ only by $(-\alpha) \circ \{\beta, \gamma, \delta\}$. In this way, we can write the above equation conveniently as

$$\{\alpha, \beta, \gamma\} \circ \Sigma\delta = -\alpha \circ \{\beta, \gamma, \delta\},$$

and no confusion will arise.

By the *p-local Whitehead theorem* [30, Lemma 1.3], we have the following two corollaries:

Corollary 2.1.1 *Let p be a prime and $\alpha\colon \Sigma X \to Y$ be a homotopy class, where ΣX and Y are simply connected CW complexes. Then, after localization at p, for any invertible element t in the ring $\mathbb{Z}_{(p)}$, the cofibers of $t\alpha$ and α have the same homotopy type, that is, $C_{t\alpha} \simeq C_\alpha$.*

Proof Denote id_{S^1} by ι_1. We notice that, after localization at p, for any invertible element t in the ring $\mathbb{Z}_{(p)}$, $t(\mathrm{id}_{\Sigma X}) = \mathrm{id}_X \wedge t\iota_1 \in [\Sigma X, \Sigma X]$ has the inverse $t^{-1}(\mathrm{id}_{\Sigma X}) = \mathrm{id}_X \wedge t^{-1}\iota_1 \in [\Sigma X, \Sigma X]$. Then this corollary follows from the p-local Whitehead theorem and the naturality of cofiber sequences, by checking the $\mathbb{Z}_{(p)}$-homology. Consider the following commutative diagram with rows cofiber sequences; we leave it to the reader to verify that the homomorphism $H_k(C_{t\alpha}; \mathbb{Z}_{(p)}) \xrightarrow{\psi_*} H_k(C_\alpha; \mathbb{Z}_{(p)})$ is an isomorphism for each $k \ge 0$:

(2-1)
$$\begin{tikzcd}
\Sigma X \arrow[r, "t\alpha"] \arrow[d, "t(\mathrm{id}_{\Sigma X})"'] & Y \arrow[r] \arrow[d, "\mathrm{id}"'] & C_{t\alpha} \arrow[d, dashed, "\psi"] \\
\Sigma X \arrow[r, "\alpha"] & Y \arrow[r] & C_\alpha
\end{tikzcd}$$
□

Corollary 2.1.2 *After localization at a prime p, suppose $m, n_1, n_2 \ge 2$ are integers, $j_k\colon S^{n_k} \to S^{n_1} \vee S^{n_2}$ for $k = 1, 2$ are the inclusions and $\gamma_k \in \Sigma\pi_{m-1}(S^{n_k-1})$ for $k = 1, 2$. Then, for any two invertible numbers t_1 and t_2 in the ring $\mathbb{Z}_{(p)}$, the cofibers of $t_1 j_1\gamma_1 + t_2 j_2\gamma_2$ and $j_1\gamma_1 + j_2\gamma_2$ have the same homotopy type, that is,*

$$C_{t_1 j_1\gamma_1 + t_2 j_2\gamma_2} \simeq C_{j_1\gamma_1 + j_2\gamma_2}.$$

Proof Following Toda, we use ι_k to denote the homotopy class of the identity map of S^k. After localization at p, consider the following diagram:

$$\begin{tikzcd} S^m \arrow[r, "t_1 j_1\gamma_1 + t_2 j_2\gamma_2"] \arrow[d, "\iota_m"'] & S^{n_1} \vee S^{n_2} \arrow[d, "(1/t_1)\iota_{n_1} \vee (1/t_2)\iota_{n_2}"] \\ S^m \arrow[r, "j_1\gamma_1 + j_2\gamma_2"] & S^{n_1} \vee S^{n_2} \end{tikzcd}$$

By assumption, γ_1 and γ_2 are suspensions. Then

$$\begin{aligned}\Big(\frac{1}{t_1}\iota_{n_1} \vee \frac{1}{t_2}\iota_{n_2}\Big)(t_1 j_1\gamma_1 + t_2 j_2\gamma_2) &= \Big(\frac{1}{t_1}\iota_{n_1} \vee \frac{1}{t_2}\iota_{n_2}\Big)\circ t_1 j_1\gamma_1 + \Big(\frac{1}{t_1}\iota_{n_1} \vee \frac{1}{t_2}\iota_{n_2}\Big)\circ t_2 j_2\gamma_2 \\ &= \frac{1}{t_1}\iota_{n_1}\circ t_1\gamma_1 + \frac{1}{t_2}\iota_{n_2}\circ t_2\gamma_2 = j_1\gamma_1 + j_2\gamma_2,\end{aligned}$$

so the above diagram is commutative. Then this corollary follows immediately from the p-local Whitehead theorem and the naturality of cofiber sequences, by checking the $\mathbb{Z}/p$-homology. □

Some fundamental facts on homotopy groups are in the following.

Remark 2.1.3 Let X be an $(n-1)$-connected CW complex and $\pi_n(X) \neq 0$. Then $\pi_{d+k}(\Sigma^k X)$ is in the stable range if and only if $d+k < 2(n+k)-1$ if and only if $k \geq d-2n+2$. Hence $\pi_{d+k}(\Sigma^k \mathbb{H}P^m)$ for $1 \leq m \leq \infty$ is in the stable range if and only if $k \geq d-6$. Moreover, this relation still holds for $X = S^0$. In this case, taking $n=0$, we have that $\pi_{d+k}(S^k)$ is in the stable range if and only if $k \geq d-2n+2 = d+2$.

Roughly speaking, the following [3, Corollary 5.1] proposition tells us that $\pi_1(-)$ commutes with colimits for pointed and connected spaces:

Proposition 2.1.4 *Let I be a category with initial object and let $X : I \to S_*$ be a diagram of pointed and connected spaces. Then the fundamental group of the homotopy colimit is given as a colimit of groups*:

$$\pi_1(\mathrm{hocolim}_I\, X) \approx \mathrm{colim}_I\, \pi_1(X). \qquad \square$$

2.2 On the cell structures of the fibers and cofibers

Lemma 2.2.1 [4, page 237] *Let $f : X \to Y$ be a fibration where X and Y both have the homotopy type of CW complexes and Y is path connected. Then the fiber of f also has the homotopy type of a CW complex.* □

The following lemma is an immediate consequence of [4, Theorem 2.3.1, page 62].

Lemma 2.2.2 *Let $f : X \to Y$ be a cellular map between CW complexes. Then the mapping cylinder M_Y and the mapping cone C_f of f are both CW complexes. Moreover, X and Y are subcomplexes of M_Y, and Y is a subcomplex of C_f.* □

2.3 Extension problems of $\mathbb{Z}_{(p)}$-modules

We give the following lemma to prevent losing solutions when we meet extension problems. We use $\gcd(a,b)$ to denote the greatest common factor of integers a and b.

Lemma 2.3.1 *Suppose p is a prime.*

(1) *Let $0 \to A \to B \to \mathbb{Z}/p^r \to 0$ be an exact sequence of $\mathbb{Z}_{(p)}$-modules. Then such $\mathbb{Z}_{(p)}$-modules B are given by*

$$B \approx (A \oplus \mathbb{Z}_{(p)})/\langle(\zeta(x), -p^r x) \mid x \in \mathbb{Z}_{(p)}\rangle,$$

where $\zeta \in \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}, A)$; if $[\zeta] \in \mathrm{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^r, A)$ runs over $\mathrm{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^r, A)$, then, up to isomorphism, the formula gives all such $\mathbb{Z}_{(p)}$-modules B.

(2) *Let $m, n \in \mathbb{Z}_+$, $t = \min\{m,n\}$ and*

$$0 \to \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m \to B \to \mathbb{Z}/p^n \to 0$$

be an exact sequence of $\mathbb{Z}_{(p)}$-modules. Then, up to isomorphism, all such $\mathbb{Z}_{(p)}$-modules B are given by

$$B \approx \mathbb{Z}/p^m \oplus \mathbb{Z}/p^i \oplus \mathbb{Z}_{(p)} \quad \text{for } 0 \le i \le t-1,$$

or

$$B \approx \mathbb{Z}/p^{m+i-j} \oplus \mathbb{Z}/p^j \oplus \mathbb{Z}_{(p)} \quad \text{for } 0 \le j \le i \le n \text{ and } j \le t.$$

Proof (1) We only need to notice that

$$\cdots \to 0 \to \mathbb{Z}_{(p)} \xrightarrow{\times p^r} \mathbb{Z}_{(p)} \xrightarrow{\text{proj}} \mathbb{Z}/p^r \to 0$$

is a projective resolution of $\mathbb{Z}/p^r$. Then (1) follows immediately from [20, Theorem 7.30, page 425].

(2) We will use some basic techniques in homological algebra; see [20, page 370]. For the projective resolution of $\mathbb{Z}/p^n$,

$$\cdots \to 0 \xrightarrow{d_2} \mathbb{Z}_{(p)} \xrightarrow[\times p^n]{d_1} \mathbb{Z}_{(p)} \to \mathbb{Z}/p^n \to 0,$$

the deleted projective resolution is

$$\cdots \to 0 \xrightarrow{d_2} \mathbb{Z}_{(p)} \xrightarrow[\times p^n]{d_1} \mathbb{Z}_{(p)} \to 0.$$

Applying $\mathrm{Hom}(-, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m)$, we have the cochain complex,

$$0 \to \mathrm{Hom}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \xrightarrow{d_1^*} \mathrm{Hom}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \xrightarrow{d_2^*} 0 \to \cdots.$$

Then

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^n, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) = \mathrm{Ker}(d_2^*)/\mathrm{Im}(d_1^*) &= \mathrm{Hom}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m)/\mathrm{Im}(d_1^*) \\ &\approx \mathbb{Z}/p^n \oplus \mathbb{Z}/p^t \quad \text{for } t = \min\{m,n\}.\end{aligned}$$

For $\varepsilon_1', \varepsilon_2' \in \mathrm{Hom}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m)$ where $\varepsilon_1'(1) = (1,0)$ and $\varepsilon_2'(1) = (0,1)$, we put $\varepsilon_1 = \varepsilon_1' + \mathrm{Im}(d_1^*)$ and $\varepsilon_2 = \varepsilon_2' + \mathrm{Im}(d_1^*)$. Then

$$\mathrm{Ext}^1_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^n, \mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) = \mathbb{Z}/p^n\{\varepsilon_1\} \oplus \mathbb{Z}/p^t\{\varepsilon_2\}.$$

Notice that if $\gcd(\lambda, p) = \gcd(\mu, p) = 1$ for $\lambda, \mu \in \mathbb{Z}$, then

$$\begin{aligned}((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}_{(p)})/\langle((\lambda p^i \varepsilon_1'(x), \mu p^j \varepsilon_2'(x)), -p^n x) \mid x \in \mathbb{Z}_{(p)}\rangle \\ = ((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}_{(p)})/\langle((p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \mid x \in \mathbb{Z}_{(p)}\rangle.\end{aligned}$$

So, by (1) of this lemma, all such $\mathbb{Z}_{(p)}$-modules B are given by

$$B \approx ((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}_{(p)})/\langle((p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \mid x \in \mathbb{Z}_{(p)}\rangle,$$

where $0 \le i \le n$ and $0 \le j \le t = \min\{m,n\}$. For such integers i and j, we distinguish two cases.

Case 1 ($i < j$, equivalently $0 \le i < j \le t = \min\{m,n\}$, also equivalently $0 \le i \le t-1$) Then

$$\begin{aligned}&((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}_{(p)})/\langle((p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \mid x \in \mathbb{Z}_{(p)}\rangle \\ &\quad\approx \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, p^i a + p^j b - p^n c\rangle} = \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, p^i(a + p^{j-i} b - p^{n-i} c)\rangle} = \frac{\mathbb{Z}_{(p)}\{a + p^{j-i} b - p^{n-i} c, b, c\}}{\langle p^m b, p^i(a + p^{j-i} b - p^{n-i} c)\rangle} \\ &\quad\approx \mathbb{Z}/p^m \oplus \mathbb{Z}/p^i \oplus \mathbb{Z}_{(p)}.\end{aligned}$$

Case 2 ($i \ge j$, equivalently $0 \le j \le i \le n$ and $j \le t$) Let

$$b' = p^{i-j} a + b - p^{n-j} c.$$

Then $((\mathbb{Z}_{(p)} \oplus \mathbb{Z}/p^m) \oplus \mathbb{Z}_{(p)})/\langle((p^i \varepsilon_1'(x), p^j \varepsilon_2'(x)), -p^n x) \mid x \in \mathbb{Z}_{(p)}\rangle$ is isomorphic to

$$\begin{aligned}\frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, p^i a + p^j b - p^n c\rangle} &= \frac{\mathbb{Z}_{(p)}\{a,b,c\}}{\langle p^m b, p^j(p^{i-j} a + b - p^{n-j} c)\rangle} = \frac{\mathbb{Z}_{(p)}\{a, p^{i-j} a + b - p^{n-j} c, c\}}{\langle p^m b, p^j(p^{i-j} a + b - p^{n-j} c)\rangle} \\ &\approx \frac{\mathbb{Z}_{(p)}\{a, b', c\}}{\langle p^{m+n-j} c - p^{m+i-j} a, p^j b'\rangle} = \frac{\mathbb{Z}_{(p)}\{a, b', c\}}{\langle p^{m+i-j}(p^{n-i} c - a), p^j b'\rangle} \\ &= \frac{\mathbb{Z}_{(p)}\{p^{n-i} c - a, b', c\}}{\langle p^{m+i-j}(p^{n-i} c - a), p^j b'\rangle} \approx \mathbb{Z}/p^{m+i-j} \oplus \mathbb{Z}/p^j \oplus \mathbb{Z}_{(p)}.\end{aligned}$$

Combining the above two cases, we infer the desired result. □

2.4 The Toda bracket

Toda brackets are the art of constructing homotopy liftings and homotopy extensions of maps. They play a fundamental role in dealing with composition relations of homotopy classes, and are deeply studied in [27]. We will freely use the well-known properties of Toda brackets shown in [27, pages 9–12], especially Propositions 1.2, 1.4 and 1.6 therein.

To give a detailed proof of our Corollary 2.4.2, we need to state some definitions about Toda brackets.

For the sequence of spaces and maps

$$L \xrightarrow{f} M \xrightarrow{g} N \quad \text{where } g \circ f \simeq *,$$

if a map $\bar{g}\colon C_f \to N$ makes the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\subseteq"'] & N \\ C_f \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

strictly commutative, we say $\bar{g}$ is an extension of g with respect to f. A coextension of f with respect to g, that is, $\tilde{f}\colon \Sigma L = C^+L \cup C^-L \to C_g$, is defined as the following: Using the cone functor $C(-) = - \wedge I$ where I has basepoint 1, define $\tilde{f}^+$ to be the composition

$$C^+L \xrightarrow{\mathrm{id}} CL \xrightarrow{Cf} CM \xrightarrow[j_{CM}]{\subseteq} N \vee CM \xrightarrow{q} C_g, \tag{2.4a}$$

where q is the quotient map and the map $C^f = f \wedge \mathrm{id}_I$ is extended over f. Define $\tilde{f}^-$ as a composition of the form

$$C^-L \xrightarrow{\mathrm{id}} CL \xrightarrow{\overline{gf}} N \xrightarrow[j_N]{\subseteq} C_g,$$

where $\overline{gf}$ is an extension of gf with respect to id_L. Then $\tilde{f}^+|_L = \tilde{f}^-|_L$, and finally define $\tilde{f}\colon \Sigma L = C^+L \cup C^-L \to C_g$ to be $\tilde{f} = \tilde{f}^+ \cup \tilde{f}^-$.

In general, up to homotopy, neither the extension nor the coextension of a map is unique. To indicate another map, the above map $\bar{g}$ is also denoted by $\mathrm{ext}_f(g)$, and the above map $\tilde{f}$ is also denoted by $\mathrm{coext}_g(f)$. Suppose $\alpha \in [L, M]$, $\beta \in [M, N]$ and $\beta\alpha = 0$. An extension of α with respect to β, which is denoted by $\mathrm{ext}_\alpha(\beta)$, is defined to be $[\mathrm{ext}_a(b)]$ for some choice of $\mathrm{ext}_a(b)$, where a can be any representative map of α and b can be any representative map of β; a coextension of β with respect to α, which is denoted by $\mathrm{coext}_\beta(\alpha)$, is defined to be $[\mathrm{coext}_{b'}(a')]$ for some choice of $\mathrm{coext}_{b'}(a')$, where a' can be any representative map of α and b' can be any representative map of β.

The following is a paraphrase of [27, Proposition 1.7, page 13], which is essentially the definition of Toda brackets.

Definition 2.4.1 [27] Given the sequence of spaces and homotopy classes

$$W \xleftarrow{\alpha} \Sigma^n X \xleftarrow{\Sigma^n\beta} \Sigma^n Y \xleftarrow{\Sigma^n\gamma} \Sigma^n Z,$$

where $\alpha \circ \Sigma^n\beta = \beta\gamma = 0$, the Toda bracket indexed by n, denoted by $\{\alpha, \Sigma^n\beta, \Sigma^n\gamma\}_n$, is defined to be the collection of all compositions of the form

$$(-1)^n \,\mathrm{ext}_{\Sigma^n\beta}(\alpha) \circ \Sigma^n \,\mathrm{coext}_\beta(\gamma),$$

that is, the collection of homotopy classes of the form

$$(-1)^n [\mathrm{ext}_{\Sigma^n b}(a) \circ \Sigma^n \,\mathrm{coext}_b(c)] \quad \text{where } a \in \alpha,\ b \in \beta \text{ and } c \in \gamma.$$

The Toda bracket $\{\alpha, \beta, \gamma\}_0$ is denoted by $\{\alpha, \beta, \gamma\}$ for short. We shall give a detailed proof of the following useful corollary, which is well known to experts, and will use the same notation to denote a map and its homotopy class if there is no ambiguity.

Corollary 2.4.2 *For any cofiber sequence*

$$\cdots \leftarrow \Sigma X \xleftarrow{p} C_f \xleftarrow{i} Y \xleftarrow{f} X,$$

the relation $\mathrm{id}_{\Sigma X} \in \{p, i, f\}$ *always holds.*

Proof We regard p as the composition

$$C_f \xrightarrow{\text{pinch}} C_f/Y \xrightarrow{\text{id}} CX/X \underset{\cong}{\xrightarrow{\psi}} C^+X \cup C^-X,$$

where ψ is the inverse of $\phi\colon C^+X \cup C^-X \xrightarrow{\cong} CX/X$ given by $\phi(a) = *$ for any $a \in C^+X$ and $\phi(x \wedge t) = [x \wedge t]$ for any $x \wedge t \in C^-X$. We know C_i retracts to ΣX by pinching CY to $*$, taking

$$\bar{p}\colon C_i = (Y \cup_f CX) \cup_i CY \twoheadrightarrow CX/X \xrightarrow{\psi} C^+X \cup C^-X$$

as this retraction. So $\bar{p}|_{C_f} = p$, that is, $\bar{p}$ is an extension of p with respect to i. In the following, we explain that we can construct a map $\tilde{f}\colon \Sigma X \to C_i$ which is a coextension of f with respect to i, and $\tilde{f}$ is exactly a right inverse of $\bar{p}$.

An extension of $i \circ f$ with respect to id_X, that is, $\overline{i \circ f}$, is taken as the composition $C^-X \hookrightarrow Y \vee CX \twoheadrightarrow Y \cup_f CX$. Then we take $\tilde{f} = \tilde{f}^+ \cup \tilde{f}^-$, where $\tilde{f}^- = j_{C_f} \circ \overline{i \circ f}$ and $\tilde{f}^+$ follows the definition given by (2.4a). Thus $\bar{p} \circ \tilde{f}^+$ is the constant map. Successively, $\bar{p} \circ \tilde{f}$ is the composition

$$C^+X \cup C^-X \xrightarrow{\phi} CX/X \xrightarrow{\psi} C^+X \cup C^-X,$$

which is $\mathrm{id}_{\Sigma X}$. □

We introduce some common terms on dealing with Toda brackets:

Remark 2.4.3 (1) Suppose $n \geq 0$, $\alpha \in [\Sigma^n Y, Z]$, $\beta \in [X, Y]$ and $\gamma \in [W, X]$ satisfy $\alpha \circ \Sigma^n \beta = \beta \circ \gamma = 0$, and $[\Sigma^{n+1} W, Z]$ is abelian. Then

$$\varkappa \in \{\alpha, \Sigma^n \beta, \Sigma^n \gamma\}_n \quad \bmod A$$

means that $\{\alpha, \Sigma^n \beta, \Sigma^n \gamma\}_n$ contains $\varkappa$ and it is a coset of the subgroup A. If A is generated by $\{a_\lambda\}_{\lambda \in \Lambda}$, then mod A is also denoted by $\bmod a_\lambda$ for $\lambda \in \Lambda$.

For $\mathbb{a} \in G/H$ where G is an abelian group and H is a subgroup, we say H the indeterminacy of the coset $\mathbb{a}$, denoted by $\mathrm{Ind}(\mathbb{a}) = H$. So

$$\mathrm{Ind}(f \circ \{\alpha, \Sigma^n \beta, \Sigma^n \gamma\}_n \circ \Sigma g) = f \circ \mathrm{Ind}(\{\alpha, \Sigma^n \beta, \Sigma^n \gamma\}_n) \circ \Sigma g.$$

(2) Suppose G is an abelian group, $\{K_i\}$ is a family of subgroups of G, $\mathbb{a}_i \in G/K_i$ and

$$g \in \mathbb{a}_{j_1} \subseteq \mathbb{a}_{j_2} \subseteq \mathbb{a}_{j_3} \subseteq \cdots \subseteq \mathbb{a}_{j_m} \supseteq \mathbb{a}_{k_1} \supseteq \mathbb{a}_{k_2} \supseteq \cdots \supseteq \mathbb{a}_{k_n} \ni g'.$$

We write

$$g \in \mathbb{a}_{j_1} \subseteq \mathbb{a}_{j_2} \subseteq \mathbb{a}_{j_3} \subseteq \cdots \subseteq \mathbb{a}_{j_m} \supseteq \mathbb{a}_{k_1} \supseteq \mathbb{a}_{k_2} \supseteq \cdots \supseteq \mathbb{a}_{k_n} \ni g' \quad \bmod A,$$

if $\mathrm{Ind}(\mathbb{a}_{j_m}) = A$. That is, mod A corresponds to the largest coset.

(3) Suppose G is an abelian group, B is a subgroup of G and $g, g' \in G$. Then

$$g \equiv g' \bmod B \overset{\text{def}}{\Longleftrightarrow} g - g' \in B.$$

Suppose $U \subseteq G$ and $\mathbb{x} \in G$. Then

$$\mathbb{x} \equiv U \bmod B \overset{\text{def}}{\Longleftrightarrow} \text{there exists } \mathbb{y} \in U \text{ such that } \mathbb{x} \equiv \mathbb{y} \bmod B; \quad 0 \equiv U \overset{\text{def}}{\Longleftrightarrow} 0 \in U.$$

(4) A Toda bracket consisting of only one element is usually identified with its element.

3 On the fiber of the pinch map

3.1 Gray's relative James construction

Suppose A is a closed subspace of X. Following nowadays' popular notation $J(X)$ for the ordinary James construction, we use $J(X, A)$ to denote the relative James construction instead of Gray's $(X, A)_\infty$ [5].

Proposition 3.1.1 *Let X be a path-connected CW complex and A be its path-connected subcomplex. Let $i : A \hookrightarrow X$ be the inclusion. Successively, we have the inclusion $i : (A, A) \hookrightarrow (X, A)$. Then the following statements hold.*

(1) *There exists a fiber sequence*

$$J(A) \xrightarrow[J(i)]{\subseteq} J(X, A) \to X \cup_i CA \xrightarrow{p} \Sigma A,$$

where p is the pinch map, $J(A)$ is the ordinary James construction and $J(i)$ is the inclusion extended over $i : (A, A) \hookrightarrow (X, A)$.

Moreover, if $K \xrightarrow{f} L \xrightarrow{j_L} C_f \xrightarrow{q} \Sigma K$ is a cofiber sequence where K and L are path-connected CW complexes, then there exists a fiber sequence

$$J(K) \xrightarrow[J(i_f)]{\subseteq} J(M_L, K) \to C_f \xrightarrow{q} \Sigma K,$$

where M_L is the mapping cylinder of f and $J(i_f)$ is the inclusion extended by the inclusion $i_f : (K, K) \hookrightarrow (M_L, K)$.

(2) *If X/A is path-connected, and $\Omega\Sigma A$ and X/A are CW complexes of finite types, then*

$$\widetilde{H}_*(J(X, A); \Bbbk) \approx \widetilde{H}_*(X; \Bbbk) \otimes H_*(\Omega\Sigma A; \Bbbk) \quad \textit{for } \Bbbk \textit{ a field.}$$

(3) *There exists a filtration*

$$J_0(X, A) \subseteq J_1(X, A) \subseteq J_2(X, A) \subseteq \cdots, \quad J(X, A) = \bigcup_{n \ge 0} J_n(X, A),$$

where $J_0(X, A) = *$ and $J_1(X, A) = X$.

(4) *If $X = \Sigma X'$ and $A = \Sigma A'$, then $J_2(X, A) = X \cup_{[1_X, i]} C(X \wedge A')$, where $[1_X, i]$ is the Whitehead product.*

(5) *$J_n(X, A)/J_{n-1}(X, A) = X \wedge A^{\wedge(n-1)}$ and $\Sigma J(X, A) \simeq \Sigma \bigvee_{n \ge 0} X \wedge A^{\wedge n}$.*

Proof By Lemmas 2.2.1 and 2.2.2, and since any point of a CW complex is nondegenerate, $U \hookrightarrow V$ is a closed cofibration if and only if (V, U) is an NDR pair. Hence this proposition is essentially given by Gray:

(1) from [5, Theorems 2.11, and 4.2, and Lemma 4.1], regarding C_f as $M_L \cup CK$,

(2) from [5, Theorem 5.1],

(3) from [5, page 498],

(4) from [5, Corollary 5.8],

(5) from [5, page 499 and Theorem 5.1]. □

Corollary 3.1.2 *Suppose $2 \leq n \leq m$ and*

$$S^m \xrightarrow{f} S^n \to C_f \xrightarrow{q} S^{m+1}$$

is a cofiber sequence. Then there exists a fiber sequence

$$J(M_{S^n}, S^m) \to C_f \xrightarrow{q} S^{m+1},$$

and $J(M_{S^n}, S^m)$ admits a CW decomposition as

$$J(M_{S^n}, S^m) \simeq S^n \cup e^{n+m} \cup e^{n+2m} \cup e^{n+3m} \cup \cdots \qquad \textit{(infinitely many cells)}.$$

Proof By Proposition 3.1.1(1), for the cofiber sequence

$$S^m \xrightarrow{f} S^n \to C_f \xrightarrow{q} S^{m+1},$$

there exists a fiber sequence $J(M_{S^n}, S^m) \to C_f \xrightarrow{q} S^{m+1}$. By Proposition 3.1.1(5),

$$\Sigma J(M_{S^n}, S^m) \simeq \Sigma \bigvee_{r \geq 0} S^n \wedge S^{mr} = \Sigma \bigvee_{r \geq 0} S^{n+mr}.$$

Thus $\widetilde{H}_*(J(M_{S^n}, S^m)) \approx \bigoplus_{r \geq 0} \widetilde{H}_*(S^{n+mr})$, which is isomorphic to

$$\mathbb{Z}\{y_n, y_{n+m}, y_{n+2m}, y_{n+3m}, \dots\} \quad \text{where } |y_{n+mr}| = n + mr.$$

Notice $2 \leq n \leq m$. As the fiber of the pinch map $S^n \cup e^{n+m+1} \xrightarrow{q} S^{m+1}$, $J(M_{S^n}, S^m)$ is simply connected. Then, $J(M_{S^n}, S^m)$ admits a CW decomposition as

$$J(M_{S^n}, S^m) \simeq S^n \cup e^{n+m} \cup e^{n+2m} \cup e^{n+3m} \cup \cdots \quad \text{(infinitely many cells)}. \ \square$$

Similarly, we have the following corollary:

Corollary 3.1.3 *Given a cofiber sequence $X \xrightarrow{f} Y \to C_f \xrightarrow{q} \Sigma X$, where X and Y are path-connected CW complexes, there exists an isomorphism of graded groups*

$$\widetilde{H}_*(J(M_Y, X); \mathbb{Z}) \approx \widetilde{H}_*\Big(\bigvee_{n \geq 0} Y \wedge X^{\wedge n}; \mathbb{Z}\Big).$$ □

Recall in Proposition 3.1.1(1), for a map $K \xrightarrow{f} L$, $J(i_f)\colon J(K) \hookrightarrow J(M_L, K)$ is the inclusion extended over the inclusion $i_f\colon (K, K) \hookrightarrow (M_L, K)$.

Lemma 3.1.4 *Suppose $2 \leq n \leq m$ and $S^m \xrightarrow{f} S^n \to C_f \xrightarrow{q} S^{m+1}$ is a cofiber sequence. Then there exists a commutative diagram for each $k \geq 0$, where $j_{S^{n+1}}: S^{n+1} \to \Sigma J(M_{S^n}, S^m)$ is the inclusion*:

$$\begin{tikzcd}
\pi_k(J(S^m)) \arrow[r, "(J(i_f))_*"] \arrow[d, "\approx"'] & \pi_k(J(M_{S^n}, S^m)) \arrow[d, "\Sigma"] \\
\pi_{k+1}(S^{m+1}) \arrow[r] \arrow[d, "(\Sigma f)_*"'] & \pi_{k+1}(\Sigma J(M_{S^n}, S^m)) \arrow[d, "\mathrm{id}"] \\
\pi_{k+1}(S^{n+1}) \arrow[r, "(j_{S^{n+1}})_*"] & \pi_{k+1}(\Sigma J(M_{S^n}, S^m))
\end{tikzcd}$$

Proof There exists a commutative diagram

$$\begin{tikzcd}
J(S^m) \arrow[r, "J(i_f)", "\subseteq"'] \arrow[d, "\mathrm{id}"'] & J(M_{S^n}, S^m) \arrow[d, "\subseteq"] \\
J(S^m) \arrow[r, "\subseteq"] \arrow[d, "\subseteq"'] & J(J(M_{S^n}, S^m)) \arrow[d, "\mathrm{id}"] \\
J(M_{S^n}) \arrow[r, "\subseteq"] & J(J(M_{S^n}, S^m))
\end{tikzcd}$$

Since, in the homotopy category of pointed path-connected CW complexes, there exists an isomorphism of functors $J(-) \cong \Omega\Sigma(-)$, $S^n \hookrightarrow M_{S^n}$ is a homotopy equivalence, and the inclusion $J(M_{S^n}) \hookrightarrow J(J(M_{S^n}, S^m))$ is extended by $j_{M_{S^n}}: M_{S^n} \hookrightarrow J(M_{S^n}, S^m)$. Let $j_{S^n}: S^n \to J(M_{S^n}, S^m)$ be the inclusion. Then we have the homotopy-commutative diagram

$$\begin{tikzcd}
\Omega S^{m+1} \arrow[r, "\Omega\Sigma i_f"] \arrow[d, "\mathrm{id}"'] & J(M_{S^n}, S^m) \arrow[d, "\Omega\Sigma(\mathrm{id})"] \\
\Omega S^{m+1} \arrow[r] \arrow[d, "\Omega\Sigma f"'] & \Omega\Sigma J(M_{S^n}, S^m) \arrow[d, "\mathrm{id}"] \\
\Omega S^{n+1} \arrow[r, "\Omega\Sigma j_{S^n}"] & \Omega\Sigma J(M_{S^n}, S^m)
\end{tikzcd}$$

Applying the functor $\pi_k(-)$ to this diagram, we obtain the result. □

Corollary 3.1.5 *Let $m \geq 2$ be an integer, and $Z = Y \cup_f e^{m+1}$ for some map f where Y is a CW complex satisfying*:

(i) *Y is simply connected and not contractible.*

(ii) *$H_k(Y; \mathbb{Z})$ is finitely generated for each $k \geq 0$, and the cell structure of Y is minimal, that is, the cell structure of Y is consistent with its homology.*

(iii) $\dim(Y) \leq m$.

Let $r = \min\{q \in \mathbb{Z}_+ \mid H_q(Y; \mathbb{Z}) \neq 0\}$, namely, the dimension of the bottom cell(s) of Y. Then, for the fiber $J(M_Y, S^m)$ of $Z \xrightarrow{\text{pinch}} S^{m+1}$, we have

$$\mathrm{sk}_j(J(M_Y, S^m)) = J_t(M_Y, S^m) \quad \text{for } j = m(t-1) + r - 1.$$

Proof By the first equation of Proposition 3.1.1(5), after checking homology, we notice the steps of attaching higher cells (by induction on the dimensions of the skeletons) give an isomorphism of functors, which are the direct diagrams used to define the colimits $\mathrm{colim}_n J_n(M_Y, S^m)$ and $\mathrm{colim}_n \mathrm{sk}_n(J(M_Y, S^m))$. Notice that isomorphic functors have isomorphic colimits. Hence the result holds. $\square$

The following lemma is well known. For completeness, we give a detailed proof.

Lemma 3.1.6 *After localization at* 2, $\Sigma\mathbb{H}P^2 \simeq S^5 \cup_{\nu_5} e^9$. *After localization at* 3, $\Sigma\mathbb{H}P^2 \simeq S^5 \cup_{\alpha_1(5)} e^9$.

Proof We know that $\mathbb{H}P^2$ is the cofiber of the map $f_{\mathrm{Hopf}}\colon S^7 \to S^4$. By checking the cohomology ring, we see that f_{Hopf} has the Hopf invariant 1.

After localization at 2, f_{Hopf} has the Hopf invariant 1. By [27, Proposition 8.1, page 82], we have $[f_{\mathrm{Hopf}}] \equiv \nu_4 \bmod 2\nu_4, \Sigma\nu'$. Thus $\Sigma[f_{\mathrm{Hopf}}] \equiv \nu_5 \bmod 2\nu_5$. By Corollary 2.1.1, $C_{\nu_5} \simeq C_{t\nu_5}$ for any odd integer t. Then we obtain the result of the first part.

After localization at 3, we know $\widetilde{H}_*(\Sigma\mathbb{H}P^2) = \mathbb{Z}/3\{a, b\}$ for $|a| = 5$ and $|b| = 9$ with Steenrod operation $P^1_*(b) = a$. Since $\pi_8(S^5) = \mathbb{Z}/3\{\alpha_1(5)\}$ [27] and $P^1_*(b) = a$ suggests $\Sigma\mathbb{H}P^2 \not\simeq S^5 \vee S^9$, we have $\Sigma[f_{\mathrm{Hopf}}] = \pm\alpha_1(5)$. By Corollary 2.1.1, $\Sigma\mathbb{H}P^2 \simeq S^5 \cup_{\alpha_1(5)} e^9$. $\square$

Remark 3.1.7 In the proof of Lemma 3.1.6, we point out a fundamental fact: on spheres of dimension 1, 3 or 7, for the generalized Hopf invariants defined by Toda, the generalized Hopf invariants defined by G W Whitehead, and the original Hopf invariants defined by Hopf, the first two correspond to each other up to sign, and the first two are truly generalized by the third. On this fact, see [24, page 60; 27, Proposition 8.1, page 82; 28, Theorem 8.17, page 540] for more details.

Remark 3.1.8 By Proposition 3.1.1(1), taking $f = \Sigma^k f_{\mathrm{Hopf}}$, we derive that the fiber of the pinch map $p_k\colon \Sigma^k\mathbb{H}P^2 \to S^{8+k}$ is $J(M_{S^{4+k}}, S^{7+k})$.

From now on, we fix the symbol F_k to denote the fiber of the pinch map $\Sigma^k\mathbb{H}P^2 \xrightarrow{p_k} S^{8+k}$, that is, $F_k = J(M_{S^{4+k}}, S^{7+k})$, and let $p = \Sigma^\infty p_1$.

By Proposition 3.1.1(4), Corollary 3.1.5 and Lemma 3.1.6, we have the following lemma:

Lemma 3.1.9 *Suppose* $k \in \mathbb{Z}_+$. *Then, up to homotopy, after localization at* 2,

$$\mathrm{sk}_{18+3k}(F_k) = S^{4+k} \cup_{f_k} e^{11+2k},$$

where $f_k = [\iota_{4+k}, \nu_{4+k}]$ *is the Whitehead product; after localization at* 3,

$$\mathrm{sk}_{18+3k}(F_k) = S^{4+k} \cup_{g_k} e^{11+2k},$$

where $g_k = [\iota_{4+k}, \alpha_1(4+k)]$ *is the Whitehead product, and* $\alpha_1(4+k)$ *generates* $\pi_{7+k}(S^{4+k}) \approx \mathbb{Z}/3$. $\square$

Lemma 3.1.10 [28, Theorem 8.18, page 484] *Suppose $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $\gamma \in \pi_m(S^p)$, $\delta \in \pi_n(S^q)$ and write $\iota_1 = [\mathrm{id}_{S^1}]$. Then*

$$[\alpha \circ \Sigma\gamma, \beta \circ \Sigma\delta] = [\alpha, \beta] \circ (\gamma \wedge \delta \wedge \iota_1) = [\alpha, \beta] \circ \Sigma(\gamma \wedge \delta). \qquad \square$$

Corollary 3.1.11 *Suppose $k \in \mathbb{Z}_+$. Then, after localization at 2,*

$$f_k = \pm[\iota_{4+k}, \iota_{4+k}] \circ \nu_{2k+7} = \pm P(\iota_{2k+9}) \circ \nu_{2k+7};$$

after localization at 3,

$$g_k = \pm[\iota_{4+k}, \iota_{4+k}] \circ \alpha_1(2k+7).$$

Proof By Lemma 3.1.10, after localization at 2,

$$\begin{aligned} f_k = [\iota_{4+k}, \nu_{4+k}] &= [\iota_{4+k} \circ \Sigma\iota_{3+k}, \iota_{4+k} \circ \Sigma\nu_{3+k}] = [\iota_{4+k}, \iota_{4+k}] \circ \Sigma(\iota_{3+k} \wedge \nu_{3+k}) \\ &= \pm[\iota_{4+k}, \iota_{4+k}] \circ \nu_{2k+7} \end{aligned}$$

on $[\iota_{4+k}, \iota_{4+k}] = \pm P(\iota_{2k+9})$ and $P(\Sigma^2\alpha_{2m-1}) = P(\iota_{2m+1}) \circ \alpha_{2m-1}$, where α_{2m-1} is contained in $\pi_*(S^{2m-1})$; see [27, Proposition 2.5]. After localization at 3,

$$\begin{aligned} g_k = [\iota_{4+k}, \alpha_1(4+k)] &= [\iota_{4+k} \circ \Sigma\iota_{3+k}, \iota_{4+k} \circ \Sigma\alpha_1(3+k)] = [\iota_{4+k}, \iota_{4+k}] \circ \Sigma(\iota_{3+k} \wedge \alpha_1(3+k)) \\ &= \pm[\iota_{4+k}, \iota_{4+k}] \circ \alpha_1(2k+7). \end{aligned} \qquad \square$$

Lemma 3.1.12 (1) *After localization at 2, up to homotopy,*

$$\begin{array}{lll} \mathrm{sk}_{20}(F_1) = S^5 \vee S^{13}, & \mathrm{sk}_{23}(F_2) = S^6 \cup_{2\bar{\nu}_6} e^{15}, & \mathrm{sk}_{26}(F_3) = S^7 \vee S^{17}, \\ \mathrm{sk}_{29}(F_4) = S^8 \cup_{f_4} e^{19}, & \mathrm{sk}_{32}(F_5) = S^9 \cup_{\bar{\nu}_9\nu_{17}} e^{21}, & \mathrm{sk}_{35}(F_6) = S^{10} \cup_{P(\nu_{21})} e^{23}, \\ \mathrm{sk}_{38}(F_7) = S^{11} \cup_{\sigma_{11}\nu_{18}^2} e^{25}, & \mathrm{sk}_{41}(F_8) = S^{12} \cup_{P(\nu_{23})} e^{27}, & \end{array}$$

where $f_4 = \nu_8\sigma_{11} - 2t'\sigma_8\nu_{15}$ for some odd t'.

(2) *After localization at 3, up to homotopy,*

$$\begin{array}{lll} \mathrm{sk}_{20}(F_1) = S^5 \vee S^{13}, & \mathrm{sk}_{23}(F_2) = S^6 \cup_{g_2} e^{15}, & \mathrm{sk}_{26}(F_3) = S^7 \vee S^{17}, \\ \mathrm{sk}_{29}(F_4) = S^8 \cup_{g_4} e^{19}, & \mathrm{sk}_{32}(F_5) = S^9 \vee S^{21}, & \mathrm{sk}_{35}(F_6) = S^{10} \cup_{g_6} e^{23}. \end{array}$$

Here $g_k = \pm[\iota_{4+k}, \iota_{4+k}] \circ \alpha_1(2k+7) \in \pi_{11+2k}(S^{4+k})$ for $k = 2, 4$ or 6 is of order 3.

Proof (1) We will freely use some well-known results on $[\iota_m, \iota_m]$ and some relations of the generators of $\pi_*(S^n)$ due to Toda in [27]. However, more conveniently, the reader can also find the results on $[\iota_m, \iota_m]$ in [19, (1.3), page 3], and find the relations of the generators of $\pi_*(S^n)$ in [19, table, page 104]. After localization at 2:

$$f_1 = [\iota_5, \iota_5] \circ \nu_9 = \nu_5\eta_8\nu_9 = 0, \quad f_2 = [\iota_6, \nu_6] = -2\bar{\nu}_6, \quad f_3 = [\iota_7, \iota_7] \circ \nu_{13} = 0 \circ \nu_{13} = 0,$$
$$\pm f_4 = [\iota_8, \iota_8] \circ \nu_{15} = (\Sigma\sigma' - 2\sigma_8) \circ \nu_{15} = (\Sigma\sigma') \circ \nu_{15} - 2\sigma_8\nu_{15} = x\nu_8\sigma_{11} - 2\sigma_8\nu_{15},$$

for x odd. For t odd, f_4 and tf_4 have the same cofiber, so we can replace the original f_4 by $f_4 = \nu_8\sigma_{11} - 2t'\sigma_8\nu_{15}$ for t' odd. Then

$$f_5 = [\iota_9, \iota_9] \circ \nu_{17} = (\sigma_9\eta_{16} + \varepsilon_9 + \bar{\nu}_9)\nu_{17} = \bar{\nu}_9\nu_{17}, \quad f_6 = \pm P(\nu_{21}),$$
$$f_7 = [\iota_{11}, \iota_{11}] \circ \nu_{21} = (\sigma_{11}\nu_{18})\nu_{21} = \sigma_{11}\nu_{18}^2.$$

(2) After localization at 3, we have

$$\pm g_k = [\iota_{4+k}, \alpha_1(4+k)] = [\iota_{4+k}, \iota_{4+k}] \circ \alpha_1(2k+7) \in \pi_{11+2k}(S^{4+k}) \quad \text{for } k \geq 1.$$

If $4+k \equiv 1 \bmod 2$, by [27, (13.1)], we know

$$\Sigma \colon \pi_{11+2k}(S^{4+k}) \to \pi_{12+2k}(S^{5+k})$$

is monomorphic for each above k. Notice the Whitehead product is in $\mathrm{Ker}(\Sigma)$. Thus $g_1 = g_3 = g_5 = 0$. If $4+k \equiv 0 \bmod 2$, by [27, (13.1)] there is an isomorphism sending the order-3 element $\alpha_1(2k+7)$ to g_k, and so $\mathrm{ord}(g_k) = 3$. □

4 Main tools to compute the cokernels and kernels

4.1 The diagram Long(m,k,t), sequence Long(m,k) and so on

Recall from Remark 3.1.8 that p_k and F_k denote the pinches and fibers, respectively.

Definition 4.1.1 We fixed the symbols i_k, ∂_k, φ_{k+t}, w_{k+t} and θ_{k+t} to denote the maps of the following homotopy commutative diagram with rows fiber sequences, where ∂_k is the composition $\Omega S^{8+k} \xrightarrow{\simeq} JS^{7+k} \hookrightarrow J(M_{S^{4+k}}, S^{7+k}) = F_k$ and we define $i = \Sigma^\infty i_1$:

$$\begin{tikzcd}
\Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d] & F_k \arrow[r, "i_k"] \arrow[d, "\varphi_{k+t}"] & \Sigma^k \mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "w_{k+t}=\Omega^t\Sigma^t"'] & S^{8+k} \arrow[d, "\theta_{k+t}=\Omega^t\Sigma^t"'] \\
\Omega^{t+1} S^{8+k+t} \arrow[r, "\Omega^t\partial_{k+t}"] & \Omega^t F_{k+t} \arrow[r, "\Omega^t i_{k+t}"] & \Omega^t \Sigma^{k+t}\mathbb{H}P^2 \arrow[r, "\Omega^t p_{k+t}"] & \Omega^t S^{8+k+t}
\end{tikzcd}$$

For a map or a homotopy class $K \xrightarrow{f} L$, the inclusion $(K,K) \hookrightarrow (M_L, K)$ is denoted by i_f as in Proposition 3.1.1(1).

Definition 4.1.2 (1) There exists the following a commutative diagrams with exact rows induced by the above. If $k \geq 1$, then $\partial_{k*} := (J(\mathbb{i}))_* \circ \mathbb{t}$ where $\mathbb{t}$ is the isomorphism $\pi_{m+1}(S^{8+k}) \to \pi_m(J(S^{8+k}))$, $\mathbb{i} = i_{\nu_{4+k}}$ in the 2-local case, and $\mathbb{i} = i_{\alpha_1(4+k)}$ in the 3-local case. This diagram is called Long(m,k,t):

$$\begin{tikzcd}
\pi_{m+1}(S^{8+k}) \arrow[r] \arrow[d, "\Sigma^t"] & \pi_m(F_k) \arrow[r, "i_{k*}"] \arrow[d, "\varphi_{k+t*}"] & \pi_m(\Sigma^k\mathbb{H}P^2) \arrow[r, "p_{k*}"] \arrow[d, "w_{k+t*}=\Sigma^t"] & \pi_m(S^{8+k}) \arrow[r, "\partial_{k*m}"] \arrow[d, "\Sigma^t"] & \pi_{m-1}(F_k) \arrow[d, "\varphi_{k+t*}"] \\
\pi_{m+1+t}(S^{8+k+t}) \arrow[r] & \pi_{m+t}(F_{k+t}) \arrow[r, "i_{k+t*}"] & \pi_{m+t}(\Sigma^{k+t}\mathbb{H}P^2) \arrow[r, "p_{k+t*}"] & \pi_{m+t}(S^{8+k+t}) \arrow[r] & \pi_{m-1+t}(F_{k+t})
\end{tikzcd} \tag{4-1}$$

(2) Long(m,k,t) induces the following commutative diagram with short exact rows, called Short(m,k,t). Here i_{k_*} induces $\mathrm{Coker}(\partial_{k*_{m+1}}) \approx i_{k_*}(\pi_m(F_k))$. By abuse of notation, $i_{k_*}(\pi_m(F_k))$ is also denoted by $\mathrm{Coker}(\partial_{k*_{m+1}})$.

$$\begin{tikzcd} 0 \arrow[r] & i_{k_*}(\pi_m(F_k)) \arrow[r, "\subseteq"] \arrow[d, "\mathcal{A}(m,k,t)=\Sigma^t"] & \pi_m(\Sigma^k \mathbb{H}P^2) \arrow[r, "p_{k*}"] \arrow[d, "\mathcal{B}(m,k,t)=\Sigma^t"] & \mathrm{Ker}(\partial_{k*_m}) \arrow[r] \arrow[d, "\mathcal{C}(m,k,t)=\Sigma^t"] & 0 \\ 0 \arrow[r] & i_{k+t_*}(\pi_{m+t}(F_{k+t})) \arrow[r, "\subseteq"] & \pi_{m+t}(\Sigma^{k+t}\mathbb{H}P^2) \arrow[r, "p_{k+t*}"] & \mathrm{Ker}(\partial_{k*_{m+t}}) \arrow[r] & 0 \end{tikzcd} \tag{4-2}$$

(3) The following short exact sequence is called Short(m,k):

$$0 \to \mathrm{Coker}(\partial_{k_{*m+1}}) \to \pi_m(\Sigma^k \mathbb{H}P^2) \to \mathrm{Ker}(\partial_{k*_m}) \to 0. \tag{4-3}$$

The following short exact sequence is still called Short(m,k):

$$0 \to i_{k_*}(\pi_m(F_k)) \xrightarrow{\subseteq} \pi_m(\Sigma^k \mathbb{H}P^2) \to \mathrm{Ker}(\partial_{k*_m}) \to 0.$$

(4) The following exact sequence is called Long(m,k):

$$\pi_{m+1}(S^{8+k}) \xrightarrow{\partial_{k*_{m+1}}} \pi_m(F_k) \xrightarrow{i_{k*}} \pi_m(\Sigma^k \mathbb{H}P^2) \xrightarrow{p_{k*}} \pi_m(S^{8+k}) \xrightarrow{\partial_{k*_m}} \pi_{m-1}(F_k). \tag{4-4}$$

4.2 Some diagrams on skeletons of the fibers

Lemma 4.2.1 *Let $m \in \mathbb{Z}_+$, suppose $K = Y \cup_f e^{m+1}$ is a CW complex, where $Y = \mathrm{sk}_m(K)$ is a path-connected CW complex, and suppose ∂ is the composition*

$$\Omega S^{m+1} \xrightarrow{\simeq} J(S^m) \xrightarrow{\subseteq} J(M_Y, S^m).$$

Then, for the fiber sequence

$$\Omega S^{m+1} \xrightarrow{\partial} J(M_Y, S^m) \to K \to S^{m+1},$$

there exists a homotopy-commutative diagram

$$\begin{tikzcd} S^m \arrow[r, "f"] \arrow[d, "\subseteq"'] & Y \arrow[d, "\subseteq"] \\ \Omega S^{m+1} \arrow[r, "\partial"] & J(M_Y, S^m) \end{tikzcd}$$

Proof Using Proposition 3.1.1(1), the lemma follows by the homotopy-commutativity of the following diagram:

$$\begin{tikzcd} S^m \arrow[drrr, "E"] & & & \\ S^m \arrow[u, "\mathrm{id}"] \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] \arrow[dl, "f"'] & J(S^m) \arrow[rr, "\simeq"'] \arrow[d, "\subseteq"'] & & \Omega S^{m+1} \arrow[dll, "\partial"] \\ Y \arrow[r, "\subseteq", "\simeq"'] & M_Y \arrow[r, "\subseteq"] & J(M_Y, S^m) & \end{tikzcd}$$

$\square$

Lemma 4.2.2 *Suppose $k \geq 1$. Then there exists a homotopy-commutative diagram*

$$\begin{tikzcd} S^{4+k} \arrow[r, "j_k"] \arrow[d, "\Omega^t \Sigma^t"'] & F_k \arrow[d, "\phi_{k+t}"] \\ \Omega^t S^{4+k+t} \arrow[r, "\Omega^t j_{k+t}"] & \Omega^t F_{k+t} \end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd} \pi_m(S^{4+k}) \arrow[r, "j_{k*}"] \arrow[d, "\Sigma^t"'] & \pi_m(F_k) \arrow[d, "\phi_{k+t*}"] \\ \pi_{m+t}(S^{4+k+t}) \arrow[r, "j_{k+t*}"] & \pi_{m+t}(F_{k+t}) \end{tikzcd} \tag{4-5}$$

This diagram of homotopy groups is called $\mathrm{COM}(m,k,t)$.

Proof For any triad (m,k,t), we consider $\mathrm{Long}(4+k,k,t)$ given by (4-1). In this case, ϕ_{k+t} induces $\pi_{4+k}(F_k) \approx \pi_{4+t}(\Omega^t F_{k+t})$.

By the naturality of Hurewicz isomorphisms and the adjoin, we have the commutative diagram

$$\begin{tikzcd} H_{4+k+t}(S^{4+k+t}) \arrow[r, "j_{k+t*}"] \arrow[d, "\approx"'] & H_{4+k+t}(F_{4+k+t}) \arrow[d, "\approx"] \\ \pi_{4+k+t}(S^{4+k+t}) \arrow[r, "j_{k+t*}"] \arrow[d, "\approx"'] & \pi_{4+k+t}(F_{4+k+t}) \arrow[d, "\approx"] \\ \pi_{4+k}(\Omega^t S^{4+k+t}) \arrow[r, "(\Omega^t j_{k+t})_*"] & \pi_{4+k+t}(\Omega^t F_{4+k+t}) \end{tikzcd}$$

By Corollary 3.1.2, $F_{4+k+t} = S^{4+k+t} \cup e^{11+2k+2t} \cup \cdots$. Thus

$$H_{4+k+t}(S^{4+k+t}) \xrightarrow{j_{k+t*}} H_{4+k+t}(F_{4+k+t})$$

is an isomorphism. Successively,

$$\pi_{4+k}(\Omega^t S^{4+k+t}) \xrightarrow{j_{k+t*}} \pi_{4+k+t}(\Omega^t F_{4+k+t})$$

is an isomorphism. Therefore all the 4-maps in the following diagram induce isomorphisms between the $(4+k)^{\text{th}}$-homotopy groups:

$$\begin{tikzcd} S^{4+k} \arrow[r, "j_k"] \arrow[d, "\Omega^t \Sigma^t"'] & F_k \arrow[d, "\phi_{k+t}"] \\ \Omega^t S^{4+k+t} \arrow[r, "\Omega^t j_{k+t}"] & \Omega^t F_{k+t} \end{tikzcd}$$

By the cellular approximation theorem, any map $f : S^{4+k} \to \Omega^t F_{k+t}$ decomposes as the following, up to homotopy:

$$S^{4+k} \xrightarrow{(x(f))\mathbf{1}_{S^{4+k}}} S^{4+k} \xrightarrow[\subseteq]{\text{incl}} \Omega^t F_{k+t} \quad \text{for } x(f) \in \mathbb{Z}.$$

(Regard $x(-)$ as a $\mathbb{Z}$-value function.) Thus $\pi_{4+k}(\Omega^t F_{k+t}) = \mathbb{Z}\{[\mathrm{incl}]\}$ and f induces

$$f_*\colon \pi_{4+k}(S^{4+k}) \to \pi_{4+k}(\Omega^t F_{k+t}), \quad \iota_{4+k} \mapsto x(f)\cdot[\mathrm{incl}].$$

Therefore

$$x(\phi_{k+t}\circ j_k) = \pm 1, \quad x(\Omega^t j_{k+t}\circ(\Omega^t\Sigma^t)) = \pm 1.$$

If $x(\phi_{k+t}\circ j_k) = x(\Omega^t j_{k+t}\circ(\Omega^t\Sigma^t))$, then the result holds; if $x(\phi_{k+t}\circ j_k) = -x(\Omega^t j_{k+t}\circ(\Omega^t\Sigma^t))$, replace j_k by $-j_k$. □

Remark 4.2.3 We fix the notation j_k and j'_k to denote the inclusions of the following homotopy-commutative diagram:

$$\begin{tikzcd} S^{4+k} \arrow[d, "j'_k"'] \arrow[dr, "j_k"] & \\ F_k \arrow[r, "i_k"'] & \Sigma^k \mathbb{H}P^2 \end{tikzcd}$$

We use the notation $\hat{j}_k$ and $\tilde{j}_k$ to denote the inclusions of the following homotopy-commutative diagram

$$\begin{tikzcd} S^{4+k} \arrow[r, "\tilde{j}_k"] \arrow[dr, "j'_k"'] & sk_{3k+17}(F_k) \arrow[d, "\hat{j}_k"] \\ & F_k \end{tikzcd}$$

By abuse of notation, without giving rise to ambiguity,

$$j_k,\ j'_k,\ \hat{j}_k,\ \tilde{j}_k \quad \text{and} \quad \Sigma^\infty j_1,\ \Sigma^\infty j'_1,\ \Sigma^\infty \hat{j}_1,\ \Sigma^\infty \tilde{j}_1$$

are all denoted by j for simplicity.

4.3 The diagram BUND$(m+1,k)$

By Lemmas 3.1.6 and 4.2.1, we have the following:

Corollary 4.3.1 *Suppose $\Sigma y \in \pi_8(S^5)$ and $k \in \mathbb{Z}_+$. After localization at $p = 2$ or 3, there exists a homotopy-commutative diagram*

$$\begin{tikzcd} S^{7+k} \arrow[r, "\Sigma^k y"] \arrow[d, "\Omega\Sigma"'] & S^{4+k} \arrow[d, "j_k"] \\ \Omega S^{8+k} \arrow[r, "\partial_k"] & F_k \end{tikzcd}$$

taking $\Sigma^k y = \nu_{4+k}$ in the 2-local case, and taking $\Sigma^k y = \alpha_1(4+k)$ in the 3-local case. It induces a commutative diagram after localization at $p = 2$ or 3:

(4-6)
$$\begin{tikzcd} \pi_m(S^{7+k}) \arrow[r, "(\Sigma^k y)_*"] \arrow[d, "\Sigma"'] & \pi_m(S^{4+k}) \arrow[d, "j_{k*}"] \\ \pi_{m+1}(S^{8+k}) \arrow[r, "\partial_{k*m+1}"] & \pi_m(F_k) \end{tikzcd}$$

We call this diagram BUND$(m+1,k)$. □

4.4 The sequence PISK(m,k) and diagram $D(m+1,k)$

Similar to Corollary 4.3.1, we have the following:

Corollary 4.4.1 *Let $k \in \mathbb{Z}_+$. For each k, there exists a fiber sequence*

$$\Omega S^{11+2k} \xrightarrow{d} J(M_{S^{4+k}}, S^{10+2k}) \to \mathrm{sk}_{3k+17}(F_k) \xrightarrow{q} S^{11+2k},$$

inducing the following exact sequence, denoted by PISK(m,k):

$$\pi_{m+1}(S^{11+2k}) \xrightarrow{d_*} \pi_m(J(M_{S^{4+k}}, S^{10+2k})) \to \pi_m(\mathrm{sk}_{3k+17}(F_k)) \xrightarrow{q_*} \pi_m(S^{11+2k}) \xrightarrow{d_*} \pi_{m-1}(J(M_{S^{4+k}}, S^{10+2k})). \tag{4-7}$$

Here $d_* = d_{k*m+1}, d_{k*m}$, respectively. Moreover, there exists the following commutative diagram, denoted by $D(m+1,k)$, taking $\mathbb{b}_k = f_k$ in the 2-local case, and taking $\mathbb{b}_k = g_k$ in the 3-local case, (the properties of f_k and g_k are in the proof of Lemma 3.1.12):

$$\begin{tikzcd} \pi_m(S^{10+2k})_\Sigma \arrow[r, "\mathbb{b}_k"] \arrow[d] & \pi_m(S^{4+k}) \arrow[d, "j_*"] \\ \pi_{m+1}(S^{11+2k}) \arrow[r, "d_* = d_{k*m+1}"] & \pi_m(J(M_{S^{4+k}}, S^{10+2k})) \end{tikzcd} \tag{4-8}$$

□

5 $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ localized at 3

In this section, *all spaces, maps and homotopy classes are localized at* 3 whether we say "after localization at 3" or not. And $\widetilde{H}_*(-)$ denotes reduced homology with $\mathbb{Z}/3$ coefficients. For an element $\mathbb{x} \in \pi_*(X)$, a lifting of $\mathbb{x}$ up to sign is denoted by $\overline{\mathbb{x}}$ if there is no confusion.

5.1 Determination of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 13$ but $r \ne 11$

As the reader will see in the following theorem, the homotopy groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 13$ but $r \ne 11$ are easy to determine.

Theorem 5.1.1 *After localization at* 3,

$$\pi_{7+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0 & \text{if } k \ge 1, \\ \mathbb{Z}/3 & \text{if } k = 0, \end{cases} \qquad \pi_{8+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(3)} & \text{if } k \ge 1, \\ 0 & \text{if } k = 0, \end{cases}$$

$$\pi_{9+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(3)} & \text{if } k = 2, \\ 0 & \text{otherwise}, \end{cases} \qquad \pi_{10+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0 & \text{if } k \ge 1, \\ \mathbb{Z}/3 & \text{if } k = 0, \end{cases}$$

$$\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0 & \text{if } k \ge 0 \text{ and } k \ne 1, \\ \mathbb{Z}_{(3)} & \text{if } k = 1, \end{cases} \qquad \pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0 & \text{if } k \ge 0 \text{ and } k \notin \{2,6\}, \\ \mathbb{Z}_{(3)} & \text{if } k = 2 \text{ or } 6. \end{cases}$$

Proof This theorem follows from observing the corresponding Long(m,k) given by (4-4) and utilizing Lemmas 3.1.12 and 5.2.2 together with the group structures of $\pi_*(S^n)$ in [27]. In these cases, the $\mathbb{Z}_{(3)}$-module extension problems are all trivial. □

5.2 Determination of $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$

The determinations of $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ are involved, in some sense because these homotopy groups are related to the generalized Hopf fibration $S^{11} \to \mathbb{H}P^2$ whose mapping cone is $\mathbb{H}P^3$.

Recall from [27, (13.4) and Theorem 13.4, page 176] that, after localization at 3, $\pi_3^S(S^0) = \mathbb{Z}/3\{\alpha_1\}$, $\pi_7^S(S^0) = \mathbb{Z}/3\{\alpha_2\}$ and $\alpha_2 = \langle \alpha_1, 3\iota, \alpha_1\rangle$.

Theorem 5.2.1 *After localization at* 3,

$$\pi_{11}^S(\mathbb{H}P^2) = \mathbb{Z}/9\{\bar{\alpha}_1\},$$

where $\bar{\alpha}_1 \in \langle j, \alpha_1, \alpha_1\rangle$ *for* $j \circ \alpha_2 = \pm 3\bar{\alpha}_1$.

Proof For the cofiber sequence $S^8 \xrightarrow{\alpha_1(5)} S^5 \xrightarrow{j_1} \Sigma\mathbb{H}P^2 \xrightarrow{p_1} S^9 \to \cdots$, applying $\pi_{12}^S(-)$, we have an exact sequence

$$0 \to \pi_7^S(S^0) \xrightarrow{j_{1*}} \pi_{11}^S(\mathbb{H}P^2) \xrightarrow{p_{1*}} \pi_3^S(S^0) \to 0,$$

where

$$\pi_7^S(S^0) = \mathbb{Z}/3\{\alpha_2\}, \quad \pi_3^S(S^0) = \mathbb{Z}/3\{\alpha_1\} \quad \text{and} \quad \alpha_2 = \langle \alpha_1, 3\iota, \alpha_1\rangle.$$

Since $\alpha_1 \circ \alpha_1 \in \pi_6^S(S^0) = 0$, we have that $\langle j, \alpha_1, \alpha_1\rangle$ is well defined. By [27, (3.9), (i) and (ii), page 33], we have $\langle \alpha_1, \alpha_1, 3\iota\rangle = -\alpha_2 = -\langle \alpha_1, 3\iota, \alpha_1\rangle$. Thus

$$j_{1*}(\alpha_2) \in j_{1*}\langle \alpha_1, 3\iota, \alpha_1\rangle = j \circ \langle \alpha_1, 3\iota, \alpha_1\rangle = -j \circ \langle \alpha_1, \alpha_1, 3\iota\rangle = \pm\langle j, \alpha_1, \alpha_1\rangle \circ 3\iota = \pm 3\langle j, \alpha_1, \alpha_1\rangle.$$

Therefore $j_{1*}(\alpha_2)$ is divisible by 3. Taking $\bar{\alpha}_1 \in \langle j, \alpha_1, \alpha_1\rangle$ such that $j_{1*}(\alpha_2) = j \circ \alpha_2 = \pm 3\bar{\alpha}_1$, we infer the result. □

In the proof of the above theorem, without constructing $\bar{\alpha}_1$, to determine the group extension $\pi_{11}^S(\mathbb{H}P^2) \approx \mathbb{Z}/9$ or $\mathbb{Z}/3 \oplus \mathbb{Z}/3$, we only need to notice the result of Liulevicius, namely, $\pi_{11}^S(\mathbb{H}P^3) = \pi_{11}^S(\mathbb{H}P^\infty) = 0$ [9, Theorem II.17] together with the cofibration $S^{11} \to \mathbb{H}P^2 \to \mathbb{H}P^3$, by which we see that the induced homomorphism $\pi_{11}^S(S^{11}) \to \pi_{11}^S(\mathbb{H}P^2)$ is onto.

The following lemma provides important information about the suspended generalized Hopf fibration $S^{11+k} \to \Sigma^k \mathbb{H}P^2$:

Lemma 5.2.2 *Suppose* $k \geq 1$ *and* $\mathbb{H}P^3 = \mathbb{H}P^2 \cup_h e^{12}$, *where* $h\colon S^{11} \to \mathbb{H}P^2$ *is the homotopy class of the attaching map. Then*

$$\Sigma^k h \neq 3\varkappa \quad \text{for all } \varkappa \in \pi_{11+k}(\Sigma^k \mathbb{H}P^2).$$

Proof To obtain a contradiction, assume that there exists $\varkappa \in \pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ such that $\Sigma^k h = 3\varkappa$. Let $q\colon \Sigma^k \mathbb{H}P^2 \to \Sigma^k \mathbb{H}P^2/\Sigma^k \mathbb{H}P^1$ be the pinch map and consider the space $\Sigma^k \mathbb{H}P^3/\Sigma^k \mathbb{H}P^1 = S^{8+k} \cup_{q \circ \Sigma^k h} e^{12+k}$. By assumption and $\pi_{11+k}(S^{8+k}) \approx \mathbb{Z}/3$ when $k \geq 1$ [27], we have

$$q \circ \Sigma^k h = q \circ 3\varkappa = 3(q \circ \varkappa) = 0.$$

Then $\Sigma^k \mathbb{H}P^3/\Sigma^k \mathbb{H}P^1 \simeq S^{8+k} \vee S^{12+k}$.

On the one hand, $\widetilde{H}_*(\Sigma^k \mathbb{H}P^3/\Sigma^k \mathbb{H}P^1)$ =$\mathbb{Z}/3\{a,b\}$ for $|a| = 12+k$ and $|b| = 8+k$, with Steenrod operation $P^1_*(a) = b$; see [6, Example 4.L.4, page 492].

On the other hand, $\widetilde{H}_*(S^{8+k} \vee S^{12+k})$ is splitting as a Steenrod module.

Then $\Sigma^k \mathbb{H}P^3/\Sigma^k \mathbb{H}P^1$ is not homotopy equivalent to $S^{8+k} \vee S^{12+k}$, which is a contradiction. This forces $\Sigma^k h \neq 3\varkappa$ for all $\varkappa \in \pi_{11+k}(\Sigma^k \mathbb{H}P^2)$. □

We can now determine $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ for all $k \geq 0$.

Theorem 5.2.3 *After localization at* 3,

$$\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/9 & \text{if } k \geq 1 \text{ but } k \neq 4, \\ \mathbb{Z}/9 \oplus \mathbb{Z}_{(3)} & \text{if } k = 4, \\ \mathbb{Z}/3 \oplus \mathbb{Z}_{(3)} & \text{if } k = 0. \end{cases}$$

Proof For $k \geq 5$, the groups $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. By Theorem 5.2.1, we have $\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \mathbb{Z}/9$ for $k \geq 5$.

For $k \in \{3,2,1\}$, consider $\mathrm{Long}(11+k,k)$ given by (4-4). By Lemmas 3.1.12 and 5.2.2, and the groups $\pi_*(S^n)$ in [27], we infer that $\pi_{14}(\Sigma^3 \mathbb{H}P^2)$, $\pi_{13}(\Sigma^2 \mathbb{H}P^2)$ and $\pi_{12}(\Sigma \mathbb{H}P^2)$ are all groups of order 9, and the elements $\Sigma^k h$ cannot be divisible by 3. Thus the elements $\Sigma^k h$ are all of order 9. Therefore these three groups are all isomorphic to $\mathbb{Z}/9$.

For $k = 4$, consider $\mathrm{Long}(15,4)$ given by (4-4). By the above paragraph, $\mathrm{ord}(\Sigma^3 h) = \mathrm{ord}(\Sigma^5 h) = 9$, so $\pi_{15}(\Sigma^4 \mathbb{H}P^2)$ contains the element $\Sigma^4 h$ of order 9. By Lemma 3.1.12 and the groups $\pi_*(S^n)$ in [27],

$$\pi_{14}(F_4) = \pi_{16}(S^{12}) = 0.$$

Since

$$\pi_{15}(F_4) = j \circ \pi_{15}(S^8) = j \circ (\mathbb{Z}/3\{\alpha_2(8)\} \oplus \mathbb{Z}_{(3)}\{[\iota_8,\iota_8]\}) = \mathbb{Z}/3\{j \circ \alpha_2(8)\} \oplus \mathbb{Z}_{(3)}\{j \circ [\iota_8,\iota_8]\}$$

and $\pi_{15}(S^{12}) = \mathbb{Z}/3\{\alpha_1(12)\}$, we have

$$\pi_{15}(\Sigma^4 \mathbb{H}P^2) = \mathbb{Z}_{(3)}\{\varkappa\} \oplus \mathbb{Z}/9\{\Sigma^4 h\} \quad \text{for some } \varkappa.$$

We know

$$p_{4*}(\Sigma^4 h) = \pm\Sigma(p_{3*}(\Sigma^3 h)) = \pm\Sigma\alpha_1(11) = \pm\alpha_1(12),$$

and

$$i_{4*}(j \circ \alpha_2(8)) = \pm\Sigma(i_{3*}(j \circ \alpha_2(7))) = \pm\Sigma(3\Sigma^3 h) = \pm 3\Sigma^4 h.$$

This gives the following relation for some integer $r \geq 0$ and some invertible ℓ in the ring $\mathbb{Z}_{(3)}$:

$$i_{4*}(j \circ [\iota_8,\iota_8]) \equiv 3^r \ell \varkappa \mod \Sigma^4 h.$$

Notice that for any $b \in \mathbb{Z}/9$, there exists an isomorphism,

$$(\mathbb{Z}_{(3)} \oplus \mathbb{Z}/9)/\langle (3^r,b),(0,3)\rangle \approx \mathbb{Z}/3^r \oplus \mathbb{Z}/3.$$

By exactness $r \geq 1$ is impossible, so $r = 0$. Therefore $\pi_{15}(\Sigma^4 \mathbb{H}P^2) = \mathbb{Z}_{(3)}\{j_4 \circ [\iota_8,\iota_8]\} \oplus \mathbb{Z}/9\{\Sigma^4 h\}$. □

5.3 Determination of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$

As before, let us determine the stable homotopy group first.

Theorem 5.3.1 *After localization at* 3,

$$\pi_{15}^S(\mathbb{H}P^2) = \mathbb{Z}/27\{\overline{\alpha}_2\} \quad \text{for } \overline{\alpha}_2 \in \langle j, \alpha_1, \alpha_2 \rangle.$$

Proof For the cofiber sequence $S^8 \xrightarrow{\alpha_1(5)} S^5 \xrightarrow{j_1} \Sigma \mathbb{H}P^2 \xrightarrow{p_1} S^9 \to \cdots$, applying $\pi_{16}^S(-)$, we have an exact sequence

$$0 \to \pi_{11}^S(S^0) \xrightarrow{j_{1*}} \pi_{15}^S(\mathbb{H}P^2) \xrightarrow{p_{1*}} \pi_7^S(S^0) \xrightarrow{\alpha_1(5)_*} \pi_{10}^S(S^0),$$

where

$$\pi_7^S(S^0) = \mathbb{Z}/3\{\alpha_2\}, \quad \mathrm{Im}(\alpha_1(5)_*) = \alpha_1 \circ \pi_7^S(S^0) = \langle \alpha_1 \circ \alpha_2 \rangle = 0,$$

by [27, Theorem 13.9, page 180]. It follows that there is a short exact sequence

$$0 \to \pi_{11}^S(S^0) \xrightarrow{j_{1*}} \pi_{15}^S(\mathbb{H}P^2) \xrightarrow{p_{1*}} \pi_7^S(S^0) \to 0.$$

By [25, Theorem 14.14 (ii)], we know

$$\pi_{11}^S(S^0) = \mathbb{Z}/9\{\alpha_3'\}, \alpha_3' \in -\langle \alpha_2, \alpha_1, 3\iota \rangle.$$

Notice

$$\alpha_1 \circ \langle \alpha_1, 3\iota, \alpha_1 \rangle = \alpha_1 \circ \alpha_2 = 0, \quad 3\langle \alpha_1, 3\iota, \alpha_1 \rangle = 3\alpha_2 = 0.$$

Then by the strong Jacobi identity [27, (3.7), page 33] for $(\alpha_1, \alpha_1, 3\iota, \alpha_1, 3\iota)$,

$$0 \equiv \langle \langle \alpha_1, \alpha_1, 3\iota \rangle, \alpha_1, 3\iota \rangle - \langle \alpha_1, \langle \alpha_1, 3\iota, \alpha_1 \rangle, 3\iota \rangle + \langle \alpha_1, \alpha_1, \langle 3\iota, \alpha_1, 3\iota \rangle \rangle.$$

In the proof of Theorem 5.2.1, we have already shown $\langle \alpha_1, \alpha_1, 3\iota \rangle = -\alpha_2$. Hence

$$0 \equiv \alpha_3' - \langle \alpha_1, \alpha_2, 3\iota \rangle \mod 3\alpha_3',$$

which tells that all elements in $\langle \alpha_1, \alpha_2, 3\iota \rangle$ are of order 9, and so any element in $\langle \alpha_1, \alpha_2, 3\iota \rangle$ generates $\pi_{11}(S^0) \approx \mathbb{Z}/9$. Choosing $\overline{\alpha}_2 \in \langle j, \alpha_1, \alpha_2 \rangle$, we have,

$$3\overline{\alpha}_2 \in \langle j, \alpha_1, \alpha_2 \rangle \circ 3\iota = \pm j \circ \langle \alpha_1, \alpha_2 3\iota \rangle.$$

Thus $\mathrm{ord}(3\overline{\alpha}_2) = 9$, successively $\mathrm{ord}(\overline{\alpha}_2) = 27$. So $\pi_{15}^S(\mathbb{H}P^2) = \mathbb{Z}/27\{\overline{\alpha}_2\}$. □

We give the following lemma to construct an element to compute $\pi_{18}(\Sigma^3 \mathbb{H}P^2)$:

Lemma 5.3.2 *There exists* $\overline{\alpha}_2(7) \in \{j_3, \alpha_1(7), \alpha_2(10)\} \subseteq \pi_{18}(\Sigma^3 \mathbb{H}P^2)$ *such that*

$$\Sigma^\infty \overline{\alpha}_2(7) \equiv \pm \overline{\alpha}_2 \mod 3\overline{\alpha}_2 \qquad \text{and} \qquad \mathrm{ord}(\Sigma^n \overline{\alpha}_2(7)) \geq 27 \quad \text{for all } n \geq 0.$$

Proof By $\pi_{17}(S^7) \approx \mathbb{Z}/3$ [27, Theorems 13.8 and 13.9, page 180], we have $\alpha_1(7) \circ \alpha_2(10) = 0$. So $\{j_3, \alpha_1(7), \alpha_2(10)\}$ is well defined, taking $\overline{\alpha}_2(7)$ from this Toda bracket. By Theorem 5.3.1, $\pi_{11}^S(S^0) \approx \mathbb{Z}/9 = \mathbb{Z}/3^2$ and $\mathrm{ord}(\alpha_2) = 3$, we have

$$\Sigma^\infty \overline{\alpha}_2(7) \in \pm \langle j, \alpha_1, \alpha_2 \rangle \ni \pm \overline{\alpha}_2$$

modulo $j\circ\pi_{11}^S(S^0)+\pi_8^S(\mathbb{H}P^2)\circ\alpha_2\subseteq\langle 3^{3-2}\overline{\alpha}_2\rangle=\langle 3\overline{\alpha}_2\rangle$. Thus

$$\Sigma^\infty\overline{\alpha}_2(7)\equiv\pm\overline{\alpha}_2\mod 3\overline{\alpha}_2.$$

So $\Sigma^\infty\overline{\alpha}_2(7)$ is of order 27, successively, $\mathrm{ord}(\Sigma^n\overline{\alpha}_2(7))\geq 27$ for all $n\geq 0$. □

The following two propositions are used to determine the cokernels and kernels.

Proposition 5.3.3 *After localization at* 3,

$$\pi_{18}(F_4)=\mathbb{Z}/3\{j\beta_1(8)\},\quad \pi_{19}(F_4)=\mathbb{Z}/9\{j\alpha_1'(8)\}\oplus\mathbb{Z}_{(3)}\{j\circ\overline{3\iota_{19}}\}\quad\textit{and}\quad q_*(\overline{3\iota_{19}})=3\iota_{19}.$$

Proof Recall that $\mathrm{PISK}(m,k)$ and $D(m,k)$ denote (4-7) and (4-8), respectively.

For $\pi_{18}(F_4)$, by $\mathrm{PISK}(18,4)$, we have $\pi_{18}(F_4)=\mathbb{Z}/3\{j\beta_1(8)\}$.

For $\pi_{19}(F_4)$, we consider $\mathrm{PISK}(19,4)$. By Corollary 3.1.2,

$$\mathrm{sk}_{15}(J(M_{S^8},S^{18}))=S^8.$$

By $D(19,4)$ and since g_4 is of order 3,

$$\mathrm{Ker}(d_*\colon\pi_{19}(S^{19})\to\pi_{18}(F_4))=\mathbb{Z}_{(3)}\{3\iota_{19}\}.$$

It follows that we have a splitting exact sequence

$$0\to\pi_{19}(S^8)\xrightarrow{j_*}\pi_{19}(\mathrm{sk}_{29}(F_4))\xrightarrow{q_*}\mathbb{Z}_{(3)}\{3\iota_{19}\}\to 0.$$

Therefore $\pi_{19}(F_4)=\mathbb{Z}/9\{j\alpha_1'(8)\}\oplus\mathbb{Z}_{(3)}\{j\circ\overline{3\iota_{19}}\}$, where $q_*(\overline{3\iota_{19}})=3\iota_{19}$. □

Proposition 5.3.4 *After localization at* 3,

(1) $\pi_{17}(F_2)=\mathbb{Z}/9\{j\alpha_1'(6)\}$ *and* $\pi_{16}(F_2)=\mathbb{Z}/9\{j\beta_1(6)\}$.

(2) $\mathrm{Ker}(\partial_{1*_{16}}\colon\pi_{16}(S^9)\to\pi_{15}(F_1))=0$ *and* $\mathrm{Ker}(\partial_{2*_{17}}\colon\pi_{17}(S^{10})\to\pi_{16}(F_2))=0$.

Proof (1) Consider $\mathrm{PISK}(18,2)$ given by (4-7). From Corollary 3.1.2,

$$\mathrm{sk}_{19}(J(M_{S^6},S^{14}))=S^6.$$

Through $D(18,2)$ given by (4-8), the following diagram is commutative:

$$\begin{tikzcd}
\pi_{17}(S^{14}) \arrow[r, "g_{2*}"] \arrow[d, "\Sigma"', "\approx"] & \pi_{17}(S^6) \arrow[d, "j_*"', "\approx"] \\
\pi_{18}(S^{15}) \arrow[r, "d_*"] & \pi_{17}(J(M_{S^6},S^{14}))
\end{tikzcd}$$

Notice that

$$g_{2*}(\pi_{17}(S^{14}))=g_2\circ\pi_{17}(S^{14})=[\iota_6,\iota_6]\circ\alpha_1(11)\circ\langle\alpha_1(14)\rangle=[\iota_6,\iota_6]\circ\langle\alpha_1(11)\circ\alpha_1(14)\rangle=0.$$

Thus $d_*(\pi_{18}(S^{15}))=0$. It follows that there is a short exact sequence induced by $\mathrm{PISK}(18,2)$

$$0\to\pi_{17}(S^6)\xrightarrow{=}j_*\pi_{17}(\mathrm{sk}_{23}(F_2))\to 0.$$

Hence $\pi_{17}(\mathrm{sk}_{23}(F_2)) = \mathbb{Z}/9\{j\alpha_1'(6)\}$. The homotopy group $\pi_{16}(F_2) = \mathbb{Z}/9\{j\beta_1(6)\}$ follows immediately from observing PISK(16, 2) given by (4-7).

(2) Recall that ∂_{k*_r} is the homomorphism $\partial_{k*_r} : \pi_r(S^{8+k}) \to \pi_{r-1}(F_k)$.

- For determining $\mathrm{Ker}(\partial_{1*_{16}})$, consider BUND(16, 1) given by (4-6). By [27, (13.1), page 172 and Theorem 13.9, page 180], the restriction $\Sigma|_{\langle\alpha_2(8)\rangle}$ is an isomorphism and j_{1*} is a monomorphism (Lemma 3.1.12). Moreover, by [27, Lemma 13.8, page 180], we know the restriction $\alpha_1(5)_*|_{\langle\alpha_2(8)\rangle}$ is a monomorphism. Thus $\mathrm{Ker}(\partial_{1*_{16}}) = 0$.
- For determining $\mathrm{Ker}(\partial_{2*_{17}})$, consider BUND(17, 2) in (4-6). Since the homomorphism Σ is an isomorphism j_{2*} is a monomorphism, and [27, (13.1), page 172 and Lemma 13.8, page 180] gives $\alpha_1(6)_*(\alpha_2(9)) = -3\beta_1(6)$. Thus $\alpha_1(6)_*$ is monomorphic, and so $\mathrm{Ker}(\partial_{2*_{17}}) = 0$. □

We can now give the determinations of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ for all $k \geq 0$.

Theorem 5.3.5 *After localization at* 3,

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/27 & \text{if } k \geq 9 \text{ or } k \in \{7,6,5,3\}, \\ \mathbb{Z}/27 \oplus \mathbb{Z}_{(3)} & \text{if } k = 8 \text{ or } 4, \\ \mathbb{Z}/9 & \text{if } k = 2, \\ \mathbb{Z}/9 \oplus \mathbb{Z}/3 & \text{if } k = 1, \\ \mathbb{Z}/3 & \text{if } k = 0. \end{cases}$$

Proof For $k \geq 9$, the groups $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range, so by Theorem 5.3.1 we have $\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \mathbb{Z}/27$ for $k \geq 9$.

For $k \in \{7,6,5,3\}$, by Lemma 5.3.2 $\Sigma^{k-3}\bar{\alpha}_2(7) \in \pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ with $\mathrm{ord}(\Sigma^{k-3}\bar{\alpha}_2(7)) \geq 27$. Then, by Long$(15+k, k)$ together with Lemma 3.1.12, $\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \mathbb{Z}/27$ with $\mathrm{ord}(\Sigma^{k-3}\bar{\alpha}_2(7)) = 27$.

For $k = 8$, from the above, $\mathrm{ord}(\bar{\alpha}_2(7)) = \mathrm{ord}(\Sigma^6\bar{\alpha}_2(7)) = 27$, so $\Sigma^5\bar{\alpha}_2(7)$ is of order 27, that is, $\pi_{23}(\Sigma^8 \mathbb{H}P^2)$ contains $\Sigma^5\bar{\alpha}_2(7)$ of order 27. Through Long(23, 8) given by (4-4), $\pi_{23}(\Sigma^8 \mathbb{H}P^2) \approx \mathbb{Z}/27 \oplus \mathbb{Z}_{(3)}$.

For $k = 4$, by the proof for $k \in \{7,6,5,3\}$, we have $\mathrm{ord}(\Sigma^4\bar{\alpha}_2(7)) = 27$ and so $\pi_{19}(\Sigma^4 \mathbb{H}P^2)$ contains an order-27 element $\Sigma^4\bar{\alpha}_2(7)$. By Long(19, 4) together with Proposition 5.3.3, $\pi_{19}(\Sigma^4 \mathbb{H}P^2) \approx \mathbb{Z}/27 \oplus \mathbb{Z}_{(3)}$.

For $k = 2$, by Proposition 5.3.4 and Short(17, 2) given by (4-3), $\pi_{17}(\Sigma^2 \mathbb{H}P^2) \approx \mathbb{Z}/9$.

For $k = 1$, by Proposition 5.3.4 and Short(16, 1), $\pi_{16}(\Sigma \mathbb{H}P^2) \approx \pi_{16}(S^5 \vee S^{13}) \approx \mathbb{Z}/9 \oplus \mathbb{Z}/3$. □

We show a corollary used to determine $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$:

Corollary 5.3.6 *If* $k \geq 5$ *or* $k = 3$, *then*

$$\partial_{k*_{15+k}} : \pi_{15+k}(S^{8+k}) \to \pi_{14+k}(F_k)$$

is trivial.

Proof The assertion follows from Long$(15+k, k)$ given by (4-4), Theorem 5.3.5 and the exactness. □

5.4 Determination of $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$

We need the following lemmas to compute the homotopy groups.

Lemma 5.4.1 $\pi_{18}(F_4) = \mathbb{Z}/3\{j\beta_1(8)\}$.

Proof Using a similar method to the proof of Proposition 5.3.4(1), we obtain this lemma. □

Lemma 5.4.2 $$\mathrm{Coker}(\partial_{4*_{19}}) \approx \mathrm{Coker}(\partial_{2*_{17}}) \approx \mathrm{Coker}(\partial_{1*_{16}}) \approx \mathbb{Z}/3.$$

Proof We notice that
$$\alpha_1(5) \circ \alpha_2(8) = -3\beta_1(5)$$
by [27, Lemma 13.8, page 180], and
$$3\pi_{17+n}(S^{7+n}) = 0 \quad \text{for any } n \geq 0.$$
Then this lemma follows from observing the corresponding $\mathrm{BUND}(m+1,k)$ given by (4-6) and making use of Proposition 5.3.4(1) together with Lemma 5.4.1. □

After the above preparation, we can determine $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$ for $k \geq 0$:

Theorem 5.4.3 *After localization at* 3,
$$\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/3 & \text{if } k \geq 4 \text{ or } k \in \{2,1\}, \\ \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3 & \text{if } k = 3, \\ \mathbb{Z}/3 \oplus \mathbb{Z}/3 & \text{if } k = 0. \end{cases}$$

Proof Recall $\mathrm{Short}(m,k)$ stands for (4-3).

For $k \geq 5$ or $k = 3$, we use $\mathrm{Short}(14+k,k)$. By Corollary 5.3.6 and Lemma 3.1.12, we obtain the result. Here the groups $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range when $k \geq 8$.

For $k = 4, 2$ or 1, by $\mathrm{Short}(14+k,k)$ and Lemma 5.4.2, the result holds. □

6 $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ localized at 2

In this section, unless otherwise specified, *all spaces, maps and homotopy classes are localized at* 2 whether we emphasize "after localization at 2" or not. And $\widetilde{H}_*(-)$ denotes the reduced homology with $\mathbb{Z}/2$ coefficients. For an element $\mathbb{x} \in \pi_*(X)$, a lifting of $\mathbb{x}$ up to sign is denoted by $\overline{\mathbb{x}}$ if there is no confusion. And $\pi_{n+k}(S^k)$ localized at 2 is also denoted by π^k_{n+k}, which naturally corresponds to Toda's π^k_{n+k} in [27]. We denote Toda's E by Σ, the suspension functor, and we denote Toda's Δ by P, where Δ denotes boundary homomorphisms of the EHP sequence. In this section, we often freely use the well-known group structures of π^k_{n+k} for $n \leq 11$ in [27]. For the convenience of readers, in the next paragraph we shall give a brief introduction of Toda's naming convention for the generators of $\pi_*(S^n;2)$;

this naming convention is also adopted by [11; 12; 13; 14; 18]. Since Toda's 2-primary component method in [27] naturally corresponds to the 2-localization method, we use the language of the 2-localization to state the facts on the 2-primary components.

Roughly speaking, for $\mathbb{x}$ representing a Greek letter, $\mathbb{x}_n$ denotes one of the generators of $\pi_{n+r}(S^n)$ for some r; the subscript n indicates the codomain of $\mathbb{x}_n$. Moreover, $\mathbb{x}_{n+k} := \Sigma^k \mathbb{x}_n$, $\mathbb{x} := \Sigma^\infty \mathbb{x}_n$ and $\mathbb{x}_n^l$ is the abbreviation of $\mathbb{x}_n \circ \mathbb{x}_{n+r} \circ \cdots \circ \mathbb{x}_{n+(l-1)r}$ (l factors). The usages of $\overline{\mathbb{x}}_n$ and $\mathbb{x}_n^*$ are similar to above. In $\pi_{j+r}(S^j)$ (not a stable homotopy group), if a generator is written without a subscript, then this generator does not survive in the stable homotopy group $\pi_r^S(S^0)$ or its Σ^∞-image is divisible by 2. For example, for $\theta \in \pi_{24}(S^{12})$ and $\sigma''' \in \pi_{12}(S^5)$, their Σ^∞-images satisfy $\Sigma^\infty\theta = 0$ and $\Sigma^\infty\sigma''' = 8\sigma$ of order 2. There is an advantage of this naming convention, that is, we can examine the commutativity of unstable compositions conveniently,

$$\mathbb{x}_n \circ \mathbb{y}_i = \pm \mathbb{y}_n \circ \mathbb{x}_j \quad \text{for some } i \text{ and } j \text{ if } n \geq a+b,$$

where $\{\mathbb{x}_k\}$ was born in $\pi_*(S^a)$ and $\{\mathbb{y}_l\}$ was born in $\pi_*(S^b)$; see [27, Proposition 3.1]. For example, for the elements in [27]

$$\sigma_n \in \pi_{7+n}(S^n) \quad \text{for } n \geq 8 \qquad \text{and} \qquad \mu_n \in \pi_{9+n}(S^n) \quad \text{for } n \geq 3,$$

$\sigma_{8+3}\mu_i = \pm\mu_{8+3}\sigma_j$, successively, $\sigma_{11}\mu_{18} = \mu_{11}\sigma_{20}$ ($\pm$ is not necessary, since μ_3 is of order 2). But $\sigma_{10}\mu_{17} \neq \mu_{10}\sigma_{19}$; see [27, page 156]. Some common generators are summarized in [27, page 189; 19, (1.1), page 66].

6.1 Some relations of generators of π_*^n and Toda brackets

The following fundamental relations of generators of π_*^n and Toda brackets are critical to compute the cokernels and kernels:

Proposition 6.1.1 (1) $P(\iota_{13}) \in \{\nu_6, \eta_9, 2\iota_{10}\}$.

(2) $\Sigma\sigma''' = 2\sigma''$, $\Sigma\sigma'' = \pm 2\sigma'$ *and* $\Sigma^2\sigma' = 2\sigma_9$.

(3) $\sigma''' = \{\nu_5, 2\nu_8, 4\iota_{10}\}$.

(4) $\nu_9\sigma_{12} = 2x\sigma_9\nu_{16}$ *for x odd, and* $\nu_{11}\sigma_{14} = 0$.

(5) $\nu_5\Sigma\sigma' = 2\nu_5\sigma_8$ *and* $\nu_5(\Sigma\sigma')\eta_{15} = 0$.

(6) $\nu_6\bar{\nu}_9 = \nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$ *and* $\nu_7\bar{\nu}_{10} = \nu_7\varepsilon_{10} = 0$.

(7) $\zeta_5 \in \{\nu_5, 8\iota_8, \Sigma\sigma'\}_1$.

(8) $\eta^2\mu = 4\zeta$.

(9) $\nu_6\eta_9 = 0$.

(10) $\nu_6\sigma_9\eta_{16} = \nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$.

(11) $P(\iota_{31}) = 2\sigma_{15}^2$.

Proof (1) See [27, Lemma 5.10, page 45].

(2) See [27, Lemma 5.14, page 48].

(3) By [18, Proposition 2.2 (9), page 84], we have

$$\sigma''' \in \{\nu_5, 2\nu_8, 4\iota_{10}\}_3 \subseteq \{\nu_5, 2\nu_8, 4\iota_{10}\}, \quad \mathrm{Ind}\{\nu_5, 2\nu_8, 4\iota_{10}\} = \nu_5 \circ 0 + 4\pi_{12}^5 = 0.$$

Hence the result holds.

(4) The first is from [27, (7.19), page 71] and the second from [27, (7.20), page 72].

(5) See [27, (7.16), page 69].

(6) By [27, (7.17) and (7.18), page 70], $\bar{\nu}_7\nu_{15} \in \pi_{18}^7$ is of order 2.

(7) See [27, (v), page 59].

(8) See [27, (7.14), page 69].

(9) This follows from $\pi_{10}^6 = 0$.

(10) $\nu_6\sigma_9\eta_{14} = \nu_6\varepsilon_9$ is from [27, page 152], and $\nu_6\varepsilon_9 = 2\bar{\nu}_6\nu_{14}$ is just a part of (6).

(11) See [27, (10.10), page 102]. □

Some known stable homotopy groups of quaternionic projective spaces are stated in the following.

Proposition 6.1.2 *After localization at* 2:

(1) $\pi_7^S(\mathbb{H}P^\infty) = 0$, $\pi_8^S(\mathbb{H}P^\infty) \approx \mathbb{Z}_{(2)}$ *and* $\pi_9^S(\mathbb{H}P^\infty) \approx \pi_{10}^S(\mathbb{H}P^\infty) \approx \mathbb{Z}/2$.

(2) $\pi_{11}^S(\mathbb{H}P^2) \approx \mathbb{Z}/4 \oplus \mathbb{Z}/16$, $\pi_{13}^S(\mathbb{H}P^2) \approx (\mathbb{Z}/2)^2$, $\pi_{14}^S(\mathbb{H}P^2) \approx \mathbb{Z}/2$ *and* $\pi_{15}^S(\mathbb{H}P^2) = \mathbb{Z}/128\{\hat{a}\}$ *for* $\hat{a} \in \langle j, \nu, \sigma \rangle$.

Proof (1) See [9, Theorem II.9].

(2) $\pi_{11}^S(\mathbb{H}P^2)$ is from [16, page 198]; notice that $\pi_{14}^S(Q_2^3) = \pi_{11}^S(\mathbb{H}P^2)$. $\pi_{13}^S(\mathbb{H}P^2)$ and $\pi_{14}^S(\mathbb{H}P^2)$ are from [16, Lemma 2.7]. $\pi_{15}^S(\mathbb{H}P^2)$ is from [16, Lemma 3.3]. □

6.2 Determination of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 10$

As usual, we need to compute cokernels and kernels to transform long exact sequences into short exact sequences.

Lemma 6.2.1 (1) $\mathrm{Coker}(\partial_{1*_{10}}) = 0$ *and* $\mathrm{Ker}(\partial_{1*_9}) = \mathbb{Z}_{(2)}\{8\iota_8\}$.

(2) $\mathrm{Coker}(\partial_{1*_{11}}) = \mathrm{Ker}(\partial_{1*_{10}}) = 0$.

(3) $\mathrm{Coker}(\partial_{1*_{12}}) = \mathrm{Coker}(\partial_{2*_{13}}) = \mathrm{Coker}(\partial_{3*_{14}}) = \mathrm{Ker}(\partial_{1*_{11}}) = 0$ *and* $\mathrm{Ker}(\partial_{2*_{12}}) \approx \mathrm{Ker}(\partial_{3*_{13}}) \approx \mathbb{Z}/2$.

Proof The result follows immediately from the corresponding diagrams $\mathrm{BUND}(m+1,k)$ given by (4-6) and the groups π_*^n in [27]. □

To compute $\pi_{11}(\Sigma^2\mathbb{H}P^2)$, the homotopy groups of spheres we need are only those like $\pi_{11}(S^6)$ and $\pi_{11}(S^{10})$, which are of quite low dimension. But the computation is exceedingly arduous, necessitating a highly subtle technique.

Proposition 6.2.2 *After localization at* 2, $\pi_{11}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}$.

Proof By Proposition 3.1.1(1)–(2), we have the fiber sequence

$$\Omega\Sigma^2\mathbb{H}P^2\xrightarrow{d}J(M_{S^9},\Sigma\mathbb{H}P^2)\to S^6\xrightarrow{j_2}\Sigma^2\mathbb{H}P^2,$$

where $\mathrm{sk}_{13}(J(M_{S^9},\Sigma\mathbb{H}P^2))=S^9$. It induces the following exact sequence (here the homomorphism can be replaced by ν_{6*}, because for the cofiber sequence $S^9\xrightarrow{\nu_6}S^6\xrightarrow{j_2}\Sigma^2\mathbb{H}P^2$, ν_6 can extend to the fiber of j_2; alternatively we can use the James theorem [7, Theorem 2.1]):

$$\pi_{11}^9\xrightarrow{\nu_{6*}}\pi_{11}^6\xrightarrow{j_{2*}}\pi_{11}(\Sigma^2\mathbb{H}P^2)\xrightarrow{d_*}\pi_{10}^9\to 0.$$

By $\nu_{6*}(\pi_{11}^9)=\langle\nu_6\eta_9^2\rangle=0$ and Lemma 4.2.1, we have an exact sequence,

$$0\to\pi_{11}^6\xrightarrow{j_{2*}}\pi_{11}(\Sigma^2\mathbb{H}P^2)\xrightarrow{p_*}\pi_{11}^{10}\to 0.$$

We know [27]

$$\pi_{11}^6=\mathbb{Z}_{(2)}\{P(\iota_{13})\},\ \pi_{11}^{10}=\mathbb{Z}/2\{\eta_{10}\}.$$

By Proposition 6.1.1(1), $P(\iota_{13})\in\{\nu_6,\eta_9,2\iota_{10}\}$, so

$$j_{2*}(P(\iota_{13}))\in j_2\circ\{\nu_6,\eta_9,2\iota_{10}\}=-\{j_2,\nu_6,\eta_9\}\circ 2\iota_{11}=-2\{j_2,\nu_6,\eta_9\}.$$

Thus $j_{2*}(P(\iota_{13}))$ is divisible by 2. Hence, $\pi_{11}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}$. □

We can now determine $\pi_{r+k}(\Sigma^k\mathbb{H}P^2)$ for $7\le r\le 10$ and all $k\ge 0$.

Theorem 6.2.3 *After localization at* 2,

$$\pi_{7+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}0 & \text{if } k\ge 1,\\ \mathbb{Z}/4 & \text{if } k=0,\end{cases}\qquad \pi_{8+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}_{(2)} & \text{if } k\ge 1,\\ \mathbb{Z}/2 & \text{if } k=0,\end{cases}$$

$$\pi_{9+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}/2 & \text{if } k\ge 3 \text{ or } k=0,\\ \mathbb{Z}_{(2)} & \text{if } k=2,\\ 0 & \text{if } k=1,\end{cases}\qquad \pi_{10+k}(\Sigma^k\mathbb{H}P^2)\approx\begin{cases}\mathbb{Z}/2 & \text{if } k\ge 2,\\ 0 & \text{if } k=1 \text{ or } 0.\end{cases}$$

Proof We recall that $\mathrm{Short}(m,k)$ denotes (4-3), and $\mathrm{Long}(m,k)$ denotes (4-4).

For $\pi_{7+k}(\Sigma^k\mathbb{H}P^2)$, in the case $k\ge 1$, the groups $\pi_{7+k}(\Sigma^k\mathbb{H}P^2)$ are in the stable range. So

$$\pi_{7+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_7^S(\mathbb{H}P^2)\approx\pi_7^S(\mathbb{H}P^\infty)\quad\text{for } k\ge 1.$$

By Proposition 6.1.2(1), the result holds.

For $\pi_{8+k}(\Sigma^k\mathbb{H}P^2)$, in the case $k\ge 2$, the groups $\pi_{8+k}(\Sigma^k\mathbb{H}P^2)$ are in the stable range. So

$$\pi_{8+k}(\Sigma^k\mathbb{H}P^2)\approx\pi_8^S(\mathbb{H}P^2)\approx\pi_8^S(\mathbb{H}P^\infty)\quad\text{for } k\ge 2.$$

By Proposition 6.1.2(1), we obtain the result in the case $k \geq 2$. In the case $k = 1$, we consider Short(9, 1). By Lemma 6.2.1(1), we obtain the result.

For $\pi_{9+k}(\Sigma^k \mathbb{H}P^2)$, in the case $k \geq 3$, the groups $\pi_{9+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. So

$$\pi_{9+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_9^S(\mathbb{H}P^2) \approx \pi_9^S(\mathbb{H}P^\infty) \quad \text{for } k \geq 3.$$

By Proposition 6.1.2(1), we obtain the result in the case $k \geq 3$. In the case $k = 2$, it is just Proposition 6.2.2. In the case $k = 1$, we consider Short(10, 1). By Lemma 6.2.1(2), the result follows.

For $\pi_{10+k}(\Sigma^k \mathbb{H}P^2)$, in the case $k \geq 4$, the groups are in the stable range. By Proposition 6.1.2(1), we obtain the result. In the case $k \in \{3, 2, 1\}$, we consider Short$(10+k, k)$. By Lemma 6.2.1(3), we obtain the result. □

6.3 Determination of $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$

As usual, to determine the homotopy groups, we compute the cokernels and kernels first.

Lemma 6.3.1 (1) $\mathrm{Coker}(\partial_{3*_{15}}) = \mathbb{Z}/8\{j\sigma'\}$ *and* $\mathrm{Ker}(\partial_{3*_{14}}) = \mathbb{Z}/4\{2\nu_{11}\}$.

(2) $\mathrm{Coker}(\partial_{4*_{16}}) = \mathbb{Z}_{(2)}\{j\sigma_8\} \oplus \mathbb{Z}_8\{j\Sigma\sigma'\}$ *and* $\mathrm{Ker}(\partial_{4*_{15}}) = \mathbb{Z}/4\{2\nu_{12}\}$.

(3) $\mathrm{Coker}(\partial_{2*_{14}}) = \mathbb{Z}/4\{j\sigma''\}$ *and* $\mathrm{Ker}(\partial_{2*_{13}}) = \mathbb{Z}/4\{2\nu_{10}\}$.

(4) $\mathrm{Coker}(\partial_{1*_{13}}) = \mathbb{Z}/2\{j\sigma'''\}$ *and* $\mathrm{Ker}(\partial_{1*_{12}}) = \mathbb{Z}/4\{2\nu_9\}$.

Proof This lemma follows straightforwardly from observing the corresponding diagram BUND$(m+1, k)$ in (4-6) and making use of Proposition 6.1.1(2) together with the corresponding group structures of π_*^n in [27]. □

Next we determine the groups $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ for $k \geq 0$ which are related to the generalized Hopf fibration whose cofiber is $\mathbb{H}P^3$. Like the 3-local case, their computation is arduous.

Theorem 6.3.2 *After localization at* 2,

$$\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/16 \oplus \mathbb{Z}/4 & \text{if } k \geq 5, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}_{(2)} & \text{if } k = 4, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 & \text{if } k = 3, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/2 & \text{if } k = 2, \\ \mathbb{Z}/8 & \text{if } k = 1, \\ \mathbb{Z}_{(2)} & \text{if } k = 0. \end{cases}$$

Proof For $k \geq 5$, the groups $\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. Hence $\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{11}^S(\mathbb{H}P^2)$ for $k \geq 5$. By Proposition 6.1.2(2), we obtain the result.

For $k = 3$, we consider Short(14, 3). By Lemma 6.3.1(1), we have an exact sequence

$$0 \to \mathbb{Z}/8\{j\sigma'\} \to \pi_{14}(\Sigma^3 \mathbb{H}P^2) \xrightarrow{p_{3*}} \mathbb{Z}/4\{2\nu_{11}\} \to 0.$$

Since

$$p_3\circ\{j,\nu_7,2\nu_{10}\}=-\{p_3,j,\nu_7\}\circ 2\nu_{11}\ni -2\nu_{11},$$

there exists

$$\mathbb{x}\in\{j,\nu_7,2\nu_{10}\}\quad\text{such that}\quad p_{3*}(\mathbb{x})=-2\nu_{11},$$

and so

$$4\mathbb{x}\in\{j,\nu_7,2\nu_{10}\}\circ 4\iota_{14}=-j\circ\{\nu_7,2\nu_{10},4\iota_{13}\}.$$

By Proposition 6.1.1(2)(3),

$$4\sigma'\in\{\nu_7,2\nu_{10},4\iota_{13}\}\bmod 4\sigma',$$

that is,

$$\{\nu_7,2\nu_{10},4\iota_{13}\}=\mathbb{Z}/2\{4\sigma'\}.$$

Thus

$$-j\circ\{\nu_7,2\nu_{10},4\iota_{13}\}=\mathbb{Z}/2\{4j\sigma'\}.$$

Hence there exists $t\in\mathbb{Z}$ such that $4\mathbb{x}=t(4j\sigma')$. Let $\mathbb{x}'=\mathbb{x}-tj\sigma'$. So $4\mathbb{x}'=0$ and $p_{3*}(\mathbb{x}')=-2\nu_{11}$. Thus $\mathrm{ord}(\mathbb{x}')=4$. Then

$$\pi_{14}(\Sigma^3\mathbb{H}P^2)=\mathbb{Z}/8\{j\sigma'\}\oplus\mathbb{Z}/4\{\mathbb{x}'\}.$$

For $k=4$, we consider Short$(15,4)$. By Lemma 6.3.1(2), we have an exact sequence

$$0\to\mathbb{Z}/8\{j\Sigma\sigma'\}\oplus\mathbb{Z}_{(2)}\{j\sigma_8\}\to\pi_{15}(\Sigma^4\mathbb{H}P^2)\xrightarrow{p_{4*}}\mathbb{Z}/4\{2\nu_{12}\}\to 0.$$

Thus $\pi_{15}(\Sigma^4\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}\oplus A$, where

$$A=\mathbb{Z}/8,\mathbb{Z}/16,\mathbb{Z}/8\oplus\mathbb{Z}/2,\mathbb{Z}/8\oplus\mathbb{Z}/4,\mathbb{Z}/16\oplus\mathbb{Z}/2\quad\text{or}\quad\mathbb{Z}/32.$$

We consider Short$(14,3,1)$ given by (4-2). Through COM$(14,3,1)$ given by (4-5), we know the homomorphism $\mathcal{A}(14,3,1)$ in Short$(14,3,1)$ is monomorphic, and obviously $\mathcal{C}(14,3,1)$ is monomorphic. By the snake lemma, we know the homomorphism $\Sigma\colon\pi_{14}(\Sigma^3\mathbb{H}P^2)\to\pi_{15}(\Sigma^4\mathbb{H}P^2)$ is monomorphic. Finally, by the result of $\pi_{14}(\Sigma^3\mathbb{H}P^2)$, we have $\pi_{15}(\Sigma^4\mathbb{H}P^2)\approx\mathbb{Z}_{(2)}\oplus\mathbb{Z}/8\oplus\mathbb{Z}/4$.

For $k=2$, we consider Short$(13,2)$. By Lemma 6.3.1(3), we have an exact sequence

$$0\to\mathbb{Z}/4\{j\sigma''\}\to\pi_{13}(\Sigma^2\mathbb{H}P^2)\xrightarrow{p_{2*}}\mathbb{Z}/4\{2\nu_{10}\}\to 0.$$

Thus $\pi_{13}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}/16$, $\mathbb{Z}/8\oplus\mathbb{Z}/2$ or $\mathbb{Z}/4\oplus\mathbb{Z}/4$. Since

$$p_2\circ\{j,\nu_6,2\nu_9\}=-\{p_2,j,\nu_6\}\circ 2\nu_{10}\ni -2\nu_{10},$$

there exists $\mathbb{y}\in\{j,\nu_7,2\nu_{10}\}$ such that $p_{2*}(\mathbb{y})=-2\nu_{10}$. By Proposition 6.1.1(2)–(3),

$$4\mathbb{y}\in\{j,\nu_6,2\nu_9\}\circ 4\iota_{13}=-j\circ\{\nu_6,2\nu_9,4\iota_{12}\}\ni 2j\sigma''\bmod j\circ(\nu_6\circ 0+4\pi_{13}^6)=0,$$

so $4\mathbb{y}=2j\sigma''$ is of order 2. Then $\mathbb{y}$ is of order 8. Thus $\pi_{13}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}/8\oplus\mathbb{Z}/2$. Notice

$$p_{2*}(2\mathbb{y}+j\sigma'')=p_{2*}(2\mathbb{y})=4\nu_{10}\neq 0,2(2\mathbb{y}+j\sigma'')=4\mathbb{y}+2j\sigma''=2j\sigma''+2j\sigma''=0.$$

Hence $\operatorname{ord}(2\mathbb{y} + j\sigma'') = 2$. We show $\langle \mathbb{y} \rangle + \langle 2\mathbb{y} + j\sigma'' \rangle$ is a direct sum. Suppose

$$a\mathbb{y} + b(2\mathbb{y} + j\sigma'') = 0 \quad \text{for some } a, b \in \mathbb{Z}. \tag{6-1}$$

Applying p_*, we have

$$a + 2b \equiv 0 \mod 4. \tag{6-2}$$

Multiplying (6-1) by 2, we have $2a \equiv 0 \bmod 8$, equivalently saying $a \equiv 0 \bmod 4$. Substituting it into (6-2), we have $2b \equiv 0 \bmod 4$, that is, $b \equiv 0 \bmod 2$. Substituting this into (6-1), we obtain $a \equiv 0 \bmod 8$. So the sum is a direct sum. Therefore $\pi_{13}(\Sigma^2 \mathbb{H}P^2) = \mathbb{Z}/8\{\mathbb{y}\} \oplus \mathbb{Z}/2\{2\mathbb{y} + j\sigma''\}$.

For $k = 1$, we consider Short$(12, 1)$. By Lemma 6.3.1(4), we have an exact sequence

$$0 \to \mathbb{Z}/2\{j\sigma'''\} \to \pi_{12}(\Sigma \mathbb{H}P^2) \xrightarrow{p_{1*}} \mathbb{Z}/4\{2\nu_9\} \to 0.$$

So $\pi_{12}(\Sigma \mathbb{H}P^2) \approx \mathbb{Z}/8$ or $\mathbb{Z}/4 \oplus \mathbb{Z}/2$. Since

$$p_1 \circ \{j, \nu_5, 2\nu_8\} = -\{p_1, j, \nu_5\} \circ 2\nu_9 \ni -2\nu_9,$$

there exists $\mathbb{a} \in \{j, \nu_5, 2\nu_8\}$ such that $p_{1*}(\mathbb{a}) = -2\nu_9$. By Proposition 6.1.1(3),

$$4\mathbb{a} \in \{j, \nu_5, 2\nu_8\} \circ 4\iota_{12} = -j \circ \{\nu_5, 2\nu_8, 4\iota_{11}\} = j\sigma'''.$$

So $4\mathbb{a} = j\sigma'''$ is of order 2. Then $\operatorname{ord}(\mathbb{a}) = 8$, and so $\pi_{12}(\Sigma \mathbb{H}P^2) \approx \mathbb{Z}/8$. □

6.4 Determination of $\pi_{12+k}(\Sigma^k \mathbb{H}P^2)$

The following lemma is a preparation for using Long$(14, 2)$.

Lemma 6.4.1

$$\pi_{14}(F_2) \approx (\mathbb{Z}/2)^2.$$

Proof By Lemma 3.1.12, $\mathrm{sk}_{23}(F_2) = S^6 \cup_{2\bar{\nu}_6} e^{15}$; by [27],

$$\pi^6_{14} = \mathbb{Z}/8\{\bar{\nu}_6\} \oplus \mathbb{Z}/2\{\varepsilon_6\}.$$

So $\pi_{14}(F_2) \approx \pi_{14}(S^6 \cup_{2\bar{\nu}_6} e^{15}) \approx (\mathbb{Z}/8\{\bar{\nu}_6\} \oplus \mathbb{Z}/2\{\varepsilon_6\})/(\mathbb{Z}/4\{2\bar{\nu}_6\}) \approx (\mathbb{Z}/2)^2$. □

Theorem 6.4.2 *After localization at* 2,

$$\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} (\mathbb{Z}/2)^2 & \text{if } k \ge 6 \text{ or } k \in \{2, 0\}, \\ (\mathbb{Z}/2)^3 & \text{if } k = 5 \text{ or } 3, \\ (\mathbb{Z}/2)^4 & \text{if } k = 4, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 & \text{if } k = 1. \end{cases}$$

Proof We recall that Long(m, k) stands for (4-4).

For $k \ge 6$, the $\pi_{12+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. Hence

$$\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \pi^S_{12}(\mathbb{H}P^2) \quad \text{for } k \ge 6.$$

Consider Long$(12 + k, k)$. Since $\pi^S_4(S^0) = \pi^S_5(S^0) = 0$,

$$\pi_{18}(\Sigma^6 \mathbb{H}P^2) \approx \pi_{18}(F_6) \approx \pi^{10}_{18} \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

For $1 \le k \le 5$, we consider Long$(12+k,k)$. Since $\pi_4^S(S^0) = \pi_5^S(S^0) = 0$, we have $\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{12+k}(F_k)$ for $1 \le k \le 5$. By Lemmas 6.4.1 and 3.1.12, we obtain the result. $\square$

6.5 Determination of $\pi_{13+k}(\Sigma^k \mathbb{H}P^2)$

The following lemma is a preparation for using BUND$(r+14,r)$.

Lemma 6.5.1 $\pi_{15}(F_2) = \mathbb{Z}/2\{j\nu_6^3\} \oplus \mathbb{Z}/2\{j\eta_6\varepsilon_7\} \oplus \mathbb{Z}/2\{j\mu_6\} \oplus \mathbb{Z}_{(2)}\{j \circ \overline{4\iota}_{15}\}$.

Proof We consider PISK$(15,2)$ given by (4-7),

$$\pi_{16}^{15} \xrightarrow{d_*} \pi_{15}(J(M_{S^6}, S^{14})) \to \pi_{15}(\mathrm{sk}_{23}(F_2)) \to \pi_{15}^{15} \xrightarrow{d_*} \pi_{14}(J(M_{S^6}, S^{14})).$$

Through $D(16,2)$ given by (4-8), we have

$$\mathrm{Coker}(d_* \colon \pi_{16}^{15} \to \pi_{15}(J(M_{S^6}, S^{14}))) = \mathbb{Z}/2\{j\nu_6^3\} \oplus \mathbb{Z}/2\{j\eta_6\varepsilon_7\} \oplus \mathbb{Z}/2\{j\mu_6\}.$$

By $D(15,2)$,

$$\mathrm{Ker}(d_* \colon \pi_{15}^{15} \to \pi_{14}(J(M_{S^6}, S^{14}))) = \mathbb{Z}_{(2)}\{4\iota_{15}\}.$$

Then $\pi_{15}(\mathrm{sk}_{23}(F_2)) = \mathbb{Z}/2\{j\nu_6^3\} \oplus \mathbb{Z}/2\{j\eta_6\varepsilon_7\} \oplus \mathbb{Z}/2\{j\mu_6\} \oplus \mathbb{Z}_{(2)}\{\overline{4\iota}_{15}\}$. $\square$

We recall that ∂_{k*_r} is the homomorphism $\partial_{k*_r} \colon \pi_r(S^{8+k}) \to \pi_{r-1}(F_k)$.

Lemma 6.5.2 *We have*

$$\begin{array}{lll} \mathrm{Coker}(\partial_{6*_{20}}) \approx (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)}, & \mathrm{Coker}(\partial_{5*_{19}}) \approx (\mathbb{Z}/2)^3, & \mathrm{Coker}(\partial_{4*_{18}}) \approx (\mathbb{Z}/2)^4, \\ \mathrm{Coker}(\partial_{3*_{17}}) \approx (\mathbb{Z}/2)^3, & \mathrm{Coker}(\partial_{2*_{16}}) \approx (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)}, & \mathrm{Coker}(\partial_{1*_{15}}) \approx (\mathbb{Z}/2)^3. \end{array}$$

Proof For Coker$(\partial_{r*_{r+14}})$ with $r \ne 2$ listed in this lemma, the result follows immediately from observing the corresponding BUND$(r+14,r)$. For Coker$(\partial_{2*_{16}})$, it follows from observing BUND$(16,2)$ and making use of Lemma 6.5.1. $\square$

After the above preparations, we can now determine $\pi_{13+k}(\Sigma^k \mathbb{H}P^2)$:

Theorem 6.5.3 *After localization at 2,*

$$\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} (\mathbb{Z}/2)^2 & \text{if } k \ge 7, \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)} & \text{if } k = 6 \text{ or } 2, \\ (\mathbb{Z}/2)^3 & \text{if } k = 5, 3, 1 \text{ or } 0, \\ (\mathbb{Z}/2)^4 & \text{if } k = 4. \end{cases}$$

Proof For $k \ge 7$, the groups $\pi_{13+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. So $\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{13}^S(\mathbb{H}P^2)$ for $k \ge 7$. By Proposition 6.1.2(2), we obtain the result.

For $1 \le k \le 6$, consider Long$(13+k,k)$. Since $\pi_5^S(S^0)$ is trivial, $\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \mathrm{Coker}(\partial_{k*_{14+k}})$ for $1 \le k \le 6$. By Lemma 6.5.2, the result follows. $\square$

6.6 Determination of $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$

We show the following lemmas to compute $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$:

Lemma 6.6.1 (1) $\pi_{16}(F_2) = \mathbb{Z}/8\{j\nu_6\sigma_9\} \oplus \mathbb{Z}/2\{j\eta_{10}\mu_{11}\} \oplus \mathbb{Z}/2\{j \circ \bar{\eta}_{15}\}$.

(2) $\pi_{18}(F_4) = \mathbb{Z}/8\{j\sigma_8\nu_{15}\} \oplus \mathbb{Z}/2\{j\eta_8\mu_9\}$ *and* $j\nu_8\sigma_{11} = 2t'\sigma_8\nu_{15}$ *for* t' *odd.*

Proof (1) By Lemma 3.1.12, $\mathrm{sk}_{23}(F_2) = S^6 \cup_{2\bar{\nu}_6} e^{15}$. We consider PISK(17, 2) given by (4-7), and we observe $D(18,2)$ and $D(17,2)$ given by (4-8). Then $d_*(\pi^{15}_{17}) = d_*(\pi^{15}_{16}) = 0$. Since

$$\Sigma(S^6 \cup_{2\bar{\nu}_6} e^{15}) = \Sigma(S^6 \cup_{\pm[\iota_6,\nu_6]} e^{15}) = S^7 \vee S^{16},$$

we have the homotopy-commutative diagram with rows fiber sequences

$$\begin{tikzcd}
J(M_{S^6}, S^{14}) \arrow[r] \arrow[d] & \mathrm{sk}_{23}(F_2) \arrow[r] \arrow[d, "\Omega\Sigma"'] & S^{15} \arrow[d, "\Omega\Sigma"] \\
\Omega J(M_{S^7}, S^{15}) \arrow[r] & \Omega(S^7 \vee S^{16}) \arrow[r] & \Omega S^{16}
\end{tikzcd}$$

So we have the commutative diagram with exact rows (by the Hilton–Milnor formula, the second row splits)

$$\begin{tikzcd}
0 \arrow[r] & \pi^6_{16} \arrow[r] \arrow[d, "\Sigma"'] & \pi_{16}(\mathrm{sk}_{23}(F_2)) \arrow[r] \arrow[d, "\Sigma"'] & \pi^{15}_{16} \arrow[r] \arrow[d, "\Sigma"', "\approx"] & 0 \\
0 \arrow[r] & \pi^7_{17} \arrow[r] & \pi^7_{17} \oplus \pi^{16}_{17} \arrow[r] & \pi^{16}_{17} \arrow[r] & 0
\end{tikzcd}$$

By [27, Theorem 7.3, page 66], $\pi^6_{16} \xrightarrow{\Sigma} \pi^7_{17}$ is an isomorphism. By the five lemma, $\pi_{16}(\mathrm{sk}_{23}(F_2)) \xrightarrow{\Sigma} \pi^7_{17} \oplus \pi^{16}_{17}$ is an isomorphism, and the first row also splits.

(2) In a similar manner to the proof of Lemma 6.4.1, we obtain the result. □

Lemma 6.6.2 (1) $\mathrm{Coker}(\partial_{7*_{22}}) \approx (\mathbb{Z}/2)^2$.

(2) $\mathrm{Coker}(\partial_{6*_{21}}) \approx (\mathbb{Z}/2)^2$ *and* $\mathrm{Coker}(\partial_{5*_{20}}) \approx (\mathbb{Z}/2)^2$.

(3) $\mathrm{Coker}(\partial_{4*_{19}}) \approx (\mathbb{Z}/2)^2$, $\mathrm{Coker}(\partial_{3*_{18}}) \approx \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}$ *and* $\mathrm{Coker}(\partial_{2*_{17}}) \approx (\mathbb{Z}/2)^2$.

(4) $\mathrm{Coker}(\partial_{1*_{16}}) \approx (\mathbb{Z}/2)^2$.

(5) $\mathrm{Ker}(\partial_{r*_{r+14}}) = 0$ *for* $1 \le r \le 7$.

Proof We recall that BUND$(m+1,k)$ denotes (4-6).

(1) Consider BUND(22, 7). By $\nu_{11}\sigma_{14} = 0$ in Proposition 6.1.1(4), we obtain the result.

(2) Consider the corresponding BUND$(r+15,r)$. By $\nu_9\sigma_{12} = 2x\sigma_9\nu_{16}$ for x odd in Proposition 6.1.1(4), we obtain the result.

(3) The result follows from observing the corresponding BUND$(r+15,r)$ and using Lemma 6.6.1.

(4) The result follows from examining BUND(16, 1) and making use of Proposition 6.1.1(5).

(5) The result follows from observing the corresponding BUND$(r+14,r)$, and Lemma 6.5.1. □

We can now determine $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$.

Theorem 6.6.3 *After localization at* 2,

$$\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2 & \text{if } k \geq 8, \\ (\mathbb{Z}/2)^2 & \text{if } k = 7, 6, 5, 4, 2 \text{ or } 1, \\ \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} & \text{if } k = 3, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 & \text{if } k = 0. \end{cases}$$

Proof For $k \geq 8$, the groups $\pi_{14+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. So $\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{14}^S(\mathbb{H}P^2)$ for $k \geq 8$. By Proposition 6.1.2(2), we obtain the result.

For $1 \leq k \leq 7$, consider $\mathrm{Short}(14+k, k)$ given by (4-3). By Lemma 6.6.2, we infer the result (in these cases, the $\mathbb{Z}_{(2)}$-module extension problems are all trivial). □

6.7 Determination of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$

We have now reached the most intricate part of determining the $(2,3)$-primary components of the homotopy groups $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \leq r \leq 15$ but $k \geq 0$, that is, $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ localized at 2.

Let us first construct some elements which connect the stable homotopy group $\pi_{15}^S(\mathbb{H}P^2)$ and the unstable homotopy groups.

Lemma 6.7.1 (1) *There exists $\bar{\sigma}_{15} \in \{j_7, \nu_{11}, \sigma_{14}\} \subseteq \pi_{22}(\Sigma^7 \mathbb{H}P^2)$, and $\mathrm{ord}(\Sigma^n \bar{\sigma}_{15}) \geq 128$ for all $n \geq 0$.*

(2) *There exists $\overline{2\sigma}_{14} \in \{j_6, \nu_{10}, 2\sigma_{13}\} \subseteq \pi_{21}(\Sigma^6 \mathbb{H}P^2)$, and $\mathrm{ord}(\Sigma^n \overline{2\sigma}_{14}) \geq 64$ for all $n \geq 0$.*

(3) *There exists $\overline{4\sigma}_{13} \in \{j_5, \nu_9, 4\sigma_{12}\} \subseteq \pi_{20}(\Sigma^5 \mathbb{H}P^2)$, and $\mathrm{ord}(\overline{4\sigma}_{13}) \geq 32$.*

(4) *There exists $\overline{8\sigma}_9 \in \{j_1, \nu_5, 8\sigma_8\} \subseteq \pi_{16}(\Sigma \mathbb{H}P^2)$, and $\mathrm{ord}(\Sigma^n \overline{8\sigma}_9) \geq 16$ for all $n \geq 0$.*

Proof (1) By Proposition 6.1.1(4), the Toda bracket $\{j_7, \nu_{11}, \sigma_{14}\}$ is well defined. Choose $\bar{\sigma}_{15}$ from it. By Proposition 6.1.2(2), we have

$$\Sigma^\infty \bar{\sigma}_{15} \in \pm\langle j, \nu, \sigma\rangle \ni \pm\hat{a}$$

modulo $j \circ \pi_{11}^S(S^0) + \pi_8^S(\mathbb{H}P^2) \circ \sigma \subseteq 8\pi_{15}^S(\mathbb{H}P^2) = \langle 8\hat{a}\rangle$. Thus

$$\Sigma^\infty \bar{\sigma}_{15} \equiv \pm\hat{a} \bmod 8\hat{a}.$$

Since $\mathrm{ord}(\hat{a}) = 128$, $\Sigma^\infty \bar{\sigma}_{15}$ is also of order 128.

(2) By Proposition 6.1.1(4) and since $\sigma_{10}\nu_{17} \in \pi_{20}^{10}$ is of order 4, the Toda bracket $\{j_6, \nu_{10}, 2\sigma_{13}\}$ is well defined. Choose $\overline{2\sigma}_{14}$ from it. By Proposition 6.1.2(2), we have

$$\Sigma^\infty \overline{2\sigma}_{14} \in \pm\langle j, \nu, 2\sigma\rangle \ni \pm 2\hat{a}$$

modulo $j\circ\pi_{11}^S(S^0)+\pi_8^S(\mathbb{H}P^2)\circ 2\sigma\subseteq 16\pi_{15}^S(\mathbb{H}P^2)=\langle 16\hat{a}\rangle$. Thus

$$\Sigma^\infty\overline{2\sigma}_{14}\equiv\pm 2\hat{a}\mod 16\hat{a}.$$

Since $\mathrm{ord}(\hat{a})=128$, $\Sigma^\infty\overline{2\sigma}_{14}$ is of order 64, which completes the proof of (2).

(3) By Proposition 6.1.1(4) and since $\sigma_9\nu_{16}\in\pi_{19}^9$ is of order 8, the Toda bracket $\{j_5,\nu_9,4\sigma_{12}\}$ is well defined. Take $\overline{4\sigma}_{13}$ from it. By Proposition 6.1.2(2), we have

$$\Sigma^\infty\overline{4\sigma}_{13}\in\pm\langle j,\nu,4\sigma\rangle\ni\pm 4\hat{a}$$

modulo $j\circ\pi_{11}^S(S^0)+\pi_8^S(\mathbb{H}P^2)\circ 4\sigma\subseteq 16\pi_{15}^S(\mathbb{H}P^2)=\langle 16\hat{a}\rangle$. Thus

$$\Sigma^\infty\overline{4\sigma}_{13}\equiv\pm 4\hat{a}\mod 16\hat{a}.$$

Since $\mathrm{ord}(\hat{a})=128$, $\Sigma^\infty\overline{4\sigma}_{13}$ is of order 32, which is the desired conclusion.

(4) Since $\nu_5\sigma_8\in\pi_{15}^5$ is of order 8, the Toda bracket $\{j_5,\nu_5,8\sigma_8\}$ is well defined. We take $\overline{8\sigma}_9$ from it. By Proposition 6.1.2(2), we have

$$\Sigma^\infty\overline{8\sigma}_9\in\pm\langle j,\nu,8\sigma\rangle\ni\pm 8\hat{a}$$

modulo $j\circ\pi_{11}^S(S^0)+\pi_8^S(\mathbb{H}P^2)\circ 8\sigma\subseteq\langle 16\hat{a}\rangle$. Thus $\Sigma^\infty\overline{8\sigma}_9\equiv\pm 8\hat{a}\mod 16\hat{a}$. Since $\mathrm{ord}(\hat{a})=128$, $\Sigma^\infty\overline{8\sigma}_9$ is of order 16. □

As before, the following two lemmas are used to prepare for examining $\mathrm{BUND}(m,k)$, and compute the cokernels and kernels, respectively.

Lemma 6.7.2 (1) $\pi_{20}(F_5)=\mathbb{Z}/8\{j\zeta_9\}$.

(2) $\pi_{19}(F_4)=\mathbb{Z}/8\{j\zeta_8\}\oplus\mathbb{Z}/2\{j\bar{\nu}_8\nu_{16}\}\oplus\mathbb{Z}_{(2)}\{j\circ\overline{8\iota}_{19}\}$.

(3) $\pi_{17}(F_2)=\mathbb{Z}/8\{j\zeta_6\}\oplus\mathbb{Z}/2\{j\bar{\nu}_6\nu_{14}\}\oplus\mathbb{Z}/2\{j\circ\overline{\eta^2_{15}}\}$.

Proof (1) Since

$$\pi_{20}(\mathrm{sk}_{32}(F_5))=\pi_{20}(S^9\cup_{\bar{\nu}_9\nu_{17}}e^{21})=j\circ\pi_{21}^9\approx\frac{\pi_{21}^9}{\langle\bar{\nu}_9\nu_{17}\rangle}=\frac{\mathbb{Z}/8\{\zeta_9\}\oplus\mathbb{Z}/2\{\bar{\nu}_9\nu_{17}\}}{\langle\bar{\nu}_9\nu_{17}\rangle}\approx\mathbb{Z}/8,$$

the result follows.

(2) We examine $\mathrm{PISK}(19,4)$ given by (4-7). By Corollary 3.1.2, $\mathrm{sk}_{25}(J(M_{S^8},S^{18}))=S^8$. Considering $D(20,4)$ given by (4-8), we have

$$\mathrm{Coker}\big(d_*\colon\pi_{20}^{19}\to\pi_{19}(J(M_{S^8},S^{18}))\big)=\mathbb{Z}/8\{j\zeta_8\}\oplus\mathbb{Z}/2\{j\bar{\nu}_8\nu_{16}\}.$$

Here we notice $f_{4*}(\eta_{18})=\nu_8\sigma_{11}\eta_{18}-2t'\sigma_8\nu_{15}\eta_{18}=(\nu_8\eta_{11})\sigma_{12}-0=0$. Through $D(19,4)$ given by (4-8),

$$\mathrm{Ker}\big(d_*\colon\pi_{19}^{19}\to\pi_{18}(J(M_{S^8},S^{18}))\big)=\mathbb{Z}_{(2)}\{8\iota_{19}\}.$$

Hence PISK(19, 4) induces a short exact sequence,

$$0 \to \mathbb{Z}/8\{j\zeta_8\} \oplus \mathbb{Z}/2\{j\bar{\nu}_8\nu_{16}\} \xrightarrow{\subseteq} \pi_{29}(\mathrm{sk}_{19}(F_4)) \to \mathbb{Z}_{(2)}\{8\iota_{19}\} \to 0,$$

which obviously splits and gives the desired result.

(3) Consider PISK(17, 2). By Corollary 3.1.2, we have $\mathrm{sk}_{19}(J(M_{S^6}, S^{14})) = S^6$. By $D(18, 2)$, we have

$$\mathrm{Coker}\big(d_*\colon \pi_{18}^{15} \to \pi_{17}(J(M_{S^6}, S^{14}))\big) = \mathbb{Z}/8\{j\zeta_6\} \oplus \mathbb{Z}/2\{j\bar{\nu}_6\nu_{14}\};$$

by $D(17, 2)$, we have

$$\mathrm{Ker}(d_*\colon \pi_{17}^{15} \to \pi_{16}(J(M_{S^6}, S^{14})) = \mathbb{Z}/2\{\eta_{15}^2\}.$$

Hence PISK(17, 2) induces a short exact sequence,

$$0 \to \mathbb{Z}/8\{j\zeta_6\} \oplus \mathbb{Z}/2\{j\bar{\nu}_6\nu_{14}\} \xrightarrow{\subseteq} \pi_{17}(\mathrm{sk}_{23}(F_2)) \to \mathbb{Z}/2\{\eta_{15}^2\} \to 0.$$

Similar to the proof of Lemma 6.6.1(1), this sequence splits and then we deduce the result. □

Lemma 6.7.3 (1) *We have*

$\mathrm{Coker}(\partial_{8*_{24}}) \approx \mathbb{Z}/8 \oplus \mathbb{Z}_{(2)}$,	$\mathrm{Ker}(\partial_{8*_{23}}) \approx \mathbb{Z}/16$,	$\mathrm{Coker}(\partial_{6*_{22}}) \approx \mathbb{Z}/8$,	$\mathrm{Ker}(\partial_{6*_{21}}) \approx \mathbb{Z}/8$,
$\mathrm{Coker}(\partial_{5*_{21}}) \approx \mathbb{Z}/8$,	$\mathrm{Ker}(\partial_{5*_{20}}) \approx \mathbb{Z}/4$,	$\mathrm{Coker}(\partial_{3*_{19}}) \approx \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$,	$\mathrm{Ker}(\partial_{3*_{18}}) \approx \mathbb{Z}/2$.

(2) $\mathrm{Coker}(\partial_{4*_{20}}) = \pi_{19}(F_4) = \mathbb{Z}/8\{j\zeta_8\} \oplus \mathbb{Z}/2\{j\bar{\nu}_8\sigma_{16}\} \oplus \mathbb{Z}_{(2)}\{j \circ \overline{8\iota_{19}}\}$ *and* $\mathrm{Ker}(\partial_{4*_{19}}) \approx \mathbb{Z}/4$.

(3) $\mathrm{Coker}(\partial_{2*_{18}}) \approx \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$ *and* $\mathrm{Ker}(\partial_{2*_{17}}) \approx \mathbb{Z}/2$.

(4) $\mathrm{Coker}(\partial_{1*_{17}}) \approx (\mathbb{Z}/8)^2$ *and* $\mathrm{Ker}(\partial_{1*_{16}}) \approx \mathbb{Z}/2$.

Proof (1) The result follows from the corresponding diagrams BUND$(m+1, k)$ and Proposition 6.1.1(4) and (6).

(2) Consider BUND(20, 4) and BUND(19, 4). By Lemmas 6.7.2(2) and 6.6.1(2), and Proposition 6.1.1(4), we derive the result.

(3) Consider BUND(18, 2) and BUND(17, 2). By Lemmas 6.7.2(3) and 6.6.1(1), and Proposition 6.1.1(6) and (10), we infer the result.

(4) By Proposition 6.1.1(6) and (10) and the group structures of π_{16}^5 and π_{17}^6, we obtain

$$\nu_5\sigma_8\eta_{15} \equiv 0 \mod \nu_5\bar{\nu}_8, \nu_5\varepsilon_8$$

(check its Σ-image).

Next consider BUND(17, 1) and BUND(16, 1). By Proposition 6.1.1(5), we deduce the result. □

We can now give the determination of $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ localized at 2, the most arduous part of determining $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ for $7 \le r \le 15$ localized at 2 or 3.

Theorem 6.7.4 *After localization at* 2,

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/128 & \text{if } k \geq 7 \text{ and } k \neq 8, \\ \mathbb{Z}/128 \oplus \mathbb{Z}_{(2)} & \text{if } k = 8, \\ \mathbb{Z}/64 & \text{if } k = 6, \\ \mathbb{Z}/32 & \text{if } k = 5, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2 & \text{if } k = 4, \\ \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2 & \text{if } k = 3 \text{ or } 2, \\ \mathbb{Z}/16 \oplus \mathbb{Z}/8 & \text{if } k = 1, \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4 & \text{if } k = 0. \end{cases}$$

Proof For $k \geq 9$, the groups $\pi_{15+k}(\Sigma^k \mathbb{H}P^2)$ are in the stable range. Hence

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \pi_{15}^S(\mathbb{H}P^2) \quad \text{for } k \geq 9.$$

By Proposition 6.1.2(2), we obtain the result.

For $k = 7$, consider $\mathrm{Long}(22, 7)$. By Lemma 6.7.1(1), we know $\pi_{22}(\Sigma^7 \mathbb{H}P^2)$ contains the element $\bar{\sigma}_{15}$ of order at least 128. Since $\pi_{22}(F_7) \approx \pi_{22}^{11} \approx \mathbb{Z}/8$ we have $\pi_{22}^{15} \approx \mathbb{Z}/16$, so, in $\mathrm{Long}(22, 7)$, i_{7*} is monomorphic and p_{7*} is epimorphic. Therefore $\pi_{22}(\Sigma^7 \mathbb{H}P^2) = \mathbb{Z}/128\{\bar{\sigma}_{15}\}$.

For $k = 8$, consider $\mathrm{Short}(23, 8)$. By $\pi_{22}(\Sigma^7 \mathbb{H}P^2) = \mathbb{Z}/128\{\bar{\sigma}_{15}\}$ and Lemma 6.7.1(1), we know $\pi_{23}(\Sigma^8 \mathbb{H}P^2)$ contains the element $\Sigma\bar{\sigma}_{15}$ of order 128. By Lemma 6.7.3(1),

$$\mathrm{Coker}(\partial_{8*24}) \approx \mathbb{Z}/8 \oplus \mathbb{Z}_{(2)}, \quad \mathrm{Ker}(\partial_{8*23}) \approx \mathbb{Z}/16.$$

Since $\pi_{22}(\Sigma^7 \mathbb{H}P^2)$ contains an element of order 128, $\pi_{23}(\Sigma^8 \mathbb{H}P^2) \approx \mathbb{Z}/128 \oplus \mathbb{Z}_{(2)}$.

For $k = 6$, consider $\mathrm{Short}(21, 6)$. By Lemma 6.7.1(2), $\pi_{21}(\Sigma^6 \mathbb{H}P^2)$ contains the element $\overline{2\sigma}_{14}$ of order at least 64. By Lemma 6.7.3(1), $\pi_{21}(\Sigma^6 \mathbb{H}P^2) \approx \mathbb{Z}/64$.

For $k = 5$, consider $\mathrm{Short}(20, 5)$. By Lemma 6.7.1(3), $\pi_{20}(\Sigma^5 \mathbb{H}P^2)$ contains the element $\overline{4\sigma}_{13}$ of order at least 32. Then, by Lemma 6.7.3(1), $\pi_{20}(\Sigma^5 \mathbb{H}P^2) \approx \mathbb{Z}/32$.

For $k = 1$, consider $\mathrm{Short}(16, 1)$. By Lemma 6.7.1(4), $\pi_{16}(\Sigma \mathbb{H}P^2)$ contains the element $\overline{8\sigma}_9$ of order at least 16; by Lemma 6.7.3(4), $\pi_{16}(\Sigma \mathbb{H}P^2) \approx \mathbb{Z}/16 \oplus \mathbb{Z}/8$. Successively, by Lemma 6.7.1(4), we know the elements $\Sigma^n \overline{8\sigma}_9 (n \geq 0)$ are all order 16.

For $k = 3$ or 2, we consider $\mathrm{Short}(15 + k, k)$. By the above proof for $k = 1$, $\pi_{18}(\Sigma^3 \mathbb{H}P^2)$ contains the element $\Sigma^2 \overline{8\sigma}_9$ of order 16 and $\pi_{17}(\Sigma^2 \mathbb{H}P^2)$ contains the element $\Sigma \overline{8\sigma}_9$ of order 16. Then, by Lemma 6.7.3(1) and (3), respectively, we deduce the result.

For $k = 4$, consider $\mathrm{Short}(19, 4)$. By Lemmas 6.7.3(2) and 2.3.1(2), $\pi_{19}(\Sigma^4 \mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus K$, where $K \in \{\mathbb{Z}/8 \oplus \mathbb{Z}/2, \mathbb{Z}/8 \oplus \mathbb{Z}/4, \mathbb{Z}/16 \oplus \mathbb{Z}/2, \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2, \mathbb{Z}/32 \oplus \mathbb{Z}/2, (\mathbb{Z}/8)^2, \mathbb{Z}/8 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4\}$.

By the proof for $k = 1$, we know $\Sigma^3 \overline{8\sigma}_9 \in \pi_{19}(\Sigma^4 \mathbb{H}P^2)$ is of order 16. Then

$$\pi_{19}(\Sigma^4 \mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/2 \text{ or } \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2.$$

To exclude the extra solution, consider the following fiber sequence, where $\mathcal{U}_4 = J(M_{S^{11}}, \Sigma^3\mathbb{H}P^2)$ denotes the fiber of the inclusion $j_4\colon S^8 \hookrightarrow \Sigma^4\mathbb{H}P^2$:

$$\mathcal{U}_4 \to S^8 \xrightarrow{j_4} \Sigma^4\mathbb{H}P^2.$$

By checking homology, $\Sigma\mathcal{U}_4$ is splitting and $\Sigma\colon \pi_{17}(S^{11}) \to \pi_{18}(S^{12})$ is an isomorphism, so we obtain $\mathrm{sk}_{21}(\mathcal{U}_4) = S^{11} \vee S^{18}$. So there is a fiber sequence up to dimension 20 (here $\boldsymbol{a} = \nu_8 \vee f$ for some homotopy class f, since ν_8 can extend to $\mathcal{U}_4$),

$$S^{11} \vee S^{18} \xrightarrow{\boldsymbol{a}=\nu_8\vee f} S^8 \xrightarrow{j_4} \Sigma^4\mathbb{H}P^2,$$

which induces an exact sequence,

$$\pi_{19}(S^8) \xrightarrow{j_{4*}} \pi_{19}(\Sigma^4\mathbb{H}P^2) \to \pi_{18}(S^{11}) \oplus \pi_{18}(S^{18}) \xrightarrow[(\nu_{8*}, f_*)]{\boldsymbol{a}_*} \pi_{18}(S^8).$$

Denote $(x\sigma_{11}, y\iota_{18}) \in \pi_{18}(S^{11}) \oplus \pi_{18}(S^{18})$ by (x, y) for simplicity, for $x, y \in \mathbb{Z}$.

By the result for $\mathrm{Coker}(\partial_{4*20})$ in Lemma 6.7.3(2), we know the homomorphism j_{4*} is monomorphic. So $j_{4*}(\pi_{19}(S^8)) \approx \pi_{19}(S^8) \approx \mathbb{Z}/8 \oplus \mathbb{Z}/2$. Through observing $\nu_{8*}\colon \mathbb{Z}/16\{\sigma_{11}\} \to \mathbb{Z}/8\{\nu_8\sigma_{11}\} \oplus (\text{else})$, we derive $\boldsymbol{a}_*(1,0) = \nu_8\sigma_{11}$. Suppose $\boldsymbol{a}_*(0,1) \equiv m\nu_8\sigma_{11} \mod \sigma_8\nu_{15}, \eta_8\mu_9$ for some $m \in \mathbb{Z}$. Then

$$\boldsymbol{a}_*(-m, 1) = l\sigma_8\nu_{15} + l'\eta_8\mu_9 \quad \text{for some } l, l' \in \mathbb{Z}.$$

Since

$$\pi_{18}(S^{11}) \oplus \pi_{18}(S^{18}) = \langle (1,0)\rangle \oplus \langle (-m,1)\rangle \approx \mathbb{Z}/16 \oplus \mathbb{Z}_{(2)},$$
$$\pi_{18}(S^8) = \mathbb{Z}/8\{\nu_8\sigma_{11}\} \oplus \mathbb{Z}/8\{\sigma_8\nu_{15}\} \oplus \mathbb{Z}/2\{\eta_8\mu_9\},$$

by [27],

$$\mathrm{Ker}(\boldsymbol{a}_*) = \langle (8,0)\rangle \oplus \langle b(-m,1)\rangle \approx \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)},$$

where $b = \mathrm{ord}(l\sigma_8\nu_{15} + l'\eta_8\mu_9) \in \mathbb{Z}_+$. Hence we have an exact sequence induced by the above,

$$0 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2 \to \pi_{19}(\Sigma^4\mathbb{H}P^2) \to \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)} \to 0.$$

This tells us that any element of finite order contained in $\pi_{19}(\Sigma^4\mathbb{H}P^2)$ is of order at most 16. Recall that we have already shown

$$\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/2 \text{ or } \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2.$$

Thus $\pi_{19}(\Sigma^4\mathbb{H}P^2) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2$. □

6.8 Determination of some unstable $\pi_{36+k}(\Sigma^k\mathbb{H}P^2)$

In this subsection, we will study some homotopy groups $\pi_{r+k}(\Sigma^k\mathbb{H}P^2)$ for a much greater r, namely, $r = 36$. These homotopy groups are much more mysterious than those for $7 \le r \le 15$ and their determinations pose exceedingly formidable challenges.

$$\pi_{36+k}(\Sigma^k\mathbb{H}P^2) \text{ is in the stable range} \iff k \ge 36 - 8 + 2 = 30.$$

We will determine $\pi_{36+29}(\Sigma^{29}\mathbb{H}P^2)$, $\pi_{36+28}(\Sigma^{28}\mathbb{H}P^2)$ and $\pi_{36+27}(\Sigma^{27}\mathbb{H}P^2)$, and will examine $\pi_{36+11}(\Sigma^{11}\mathbb{H}P^2)$.

In this subsection, the results

$$\pi^{32}_{64} \approx (\mathbb{Z}/2)^6, \quad \pi^{31}_{63} \approx \pi^{33}_{65} \approx (\mathbb{Z}/2)^5, \quad \pi^{34}_{66} \approx (\mathbb{Z}/2)^4,$$

are used freely; see [14, Theorem 1.1]. Also $\pi^S_{28}(S^0) = \mathbb{Z}/2\{\varepsilon\bar{\kappa}\}$ and $\pi^S_{29}(S^0) = 0$; see [18, Theorem 2(b), page 81 and Theorem 3(a), page 105].

Lemma 6.8.1 $\{\nu_{31+n}, \varepsilon_{34+n}\bar{\kappa}_{42+n}, 2\iota_{62+n}\} = 0$ *for any* $n \geq 0$.

Proof Since $\pi^S_{12}(S^0) = 0$, for any $n \geq 0$ we have

$$\begin{aligned}\{\nu_{31+n}, \varepsilon_{34+n}\bar{\kappa}_{42+n}, 2\iota_{62+n}\} \supseteq \{\nu_{31+n}, \varepsilon_{34+n}, 2\bar{\kappa}_{42+n}\} &\supseteq \{\nu_{31+n}, \varepsilon_{34+n}, 2\iota_{42+n}\} \circ \bar{\kappa}_{43+n} \\ &\subseteq \pi^{31+n}_{43+n} \circ \bar{\kappa}_{43+n} = 0 \circ \bar{\kappa}_{43+n} = 0,\end{aligned}$$

so $\mathrm{Ind}\{\nu_{31+n}, \varepsilon_{34+n}\bar{\kappa}_{42+n}, 2\iota_{62+n}\} = \nu_{31+n} \circ \pi^{34+n}_{63+n} + 2\pi^{31+n}_{63+n} = \nu_{31+n} \circ 0 + 0 = 0.$ □

Theorem 6.8.2 *After localization at* 2,

$$\pi_{36+k}(\Sigma^k\mathbb{H}P^2) \approx \begin{cases} (\mathbb{Z}/2)^6 & \text{if } k = 29, \\ (\mathbb{Z}/2)^7 & \text{if } k = 28, \\ (\mathbb{Z}/2)^6 & \text{if } k = 27. \end{cases}$$

Proof We only show the case $k = 27$; similar proofs work for the other two cases.

For $\pi_{63}(\Sigma^{27}\mathbb{H}P^2)$, we use $\mathrm{Long}(63, 27)$ given by (4-4). By Lemma 3.1.9(1), we have $\mathrm{sk}_{64}(F_{11}) = S^{31}$. Since $\pi^{35}_{64} = \pi^S_{29}(S^0) = 0$, in $\mathrm{Long}(63, 27)$, i_{27*} is monomorphic. Recall that $\pi^S_{28}(S^0) = \mathbb{Z}/2\{\varepsilon\bar{\kappa}\}$ where $\nu\varepsilon = 0$. By observing $\mathrm{BUND}(63, 27)$ given by (4-6), we have $\partial_{27*63} = 0$. So, in $\mathrm{Long}(63, 27)$, p_{27*} is epimorphic. Then $\mathrm{Long}(63, 27)$ gives a short exact sequence

$$0 \to (\mathbb{Z}/2)^5 \to \pi_{63}(\Sigma^{27}\mathbb{H}P^2) \xrightarrow{p_{27*}} \mathbb{Z}/2\{\varepsilon_{35}\bar{\kappa}_{43}\} \to 0.$$

Since

$$p_{27*}\{j_{27}, \nu_{31}, \varepsilon_{34}\bar{\kappa}_{42}\} = p_{27} \circ \{j_{27}, \nu_{31}, \varepsilon_{34}\bar{\kappa}_{42}\} = \{p_{27}, j_{27}, \nu_{31}\} \circ \varepsilon_{35}\bar{\kappa}_{43} \ni \varepsilon_{35}\bar{\kappa}_{43},$$

there exists $\overline{\varepsilon_{35}\bar{\kappa}_{43}} \in \{j_{27}, \nu_{31}, \varepsilon_{34}\bar{\kappa}_{42}\}$ such that $p_{27*}(\overline{\varepsilon_{35}\bar{\kappa}_{43}}) = \varepsilon_{35}\bar{\kappa}_{43}$. Lemma 6.8.1 tells us that

$$2\overline{\varepsilon_{35}\bar{\kappa}_{43}} \in \{j_{27}, \nu_{31}, \varepsilon_{34}\bar{\kappa}_{42}\} \circ 2\iota_{63} = -j_{27} \circ \{\nu_{31}, \varepsilon_{34}\bar{\kappa}_{42}, 2\iota_{62}\} = 0.$$

Thus $2\overline{\varepsilon_{35}\bar{\kappa}_{43}} = 0$, and so $\overline{\varepsilon_{35}\bar{\kappa}_{43}}$ is of order 2. Therefore $\pi_{63}(\Sigma^{27}\mathbb{H}P^2)$ is isomorphic to $(\mathbb{Z}/2)^6$. □

Similar to the above, $\pi^S_{36}(\mathbb{H}P^2)$ can also be obtained.

For the following lemmas, we point out that, N Oda denotes the generator $\bar{\beta} \in \pi^{19}_{20+19}$ defined in [13] by C_2, that is, $C_2 = \bar{\beta}$; see [18, (2.1), page 52].

We recall from [18, Theorem 4.3 (a), page 104] that

$$\pi_{48}^{19} = \mathbb{Z}/2\{C_2\mu_{39}\} \oplus \mathbb{Z}/2\{\sigma_{19}^*\sigma_{41}\} \oplus \mathbb{Z}/2\{\Sigma^3 P_1\}.$$

To obtain $\nu_{16} \circ \pi_{48}^{19}$ and $\operatorname{Coker}(\partial_{11*48})$, we give the following lemmas:

Lemma 6.8.3 (1) $\nu_{16}C_2 \in \mathbb{Z}/2\{P(\varepsilon_{33})\} \oplus \mathbb{Z}/2\{P(\bar{\nu}_{33})\}$ *and*

$$\nu_{16}C_2\mu_{39} = \begin{cases} 0 & \text{if } \nu_{16}C_2 \equiv 0 \bmod P(\bar{\nu}_{33}), \\ \Sigma\bar{\zeta}^* & \text{if } \nu_{16}C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33}). \end{cases}$$

(2) $\nu_{16}\sigma_{19}^*\sigma_{41} = \xi_{16}\sigma_{24}^2 \in \pi_{48}^{16}$.

(3) $\nu_{13} \circ P_1 \equiv \lambda\kappa_{31} \bmod \Sigma\pi_{44}^{12}$ *and* $\nu_{16} \circ \Sigma^3 P_1 \equiv (\Sigma^3\lambda)\kappa_{34} \bmod \Sigma^4\pi_{44}^{12}$.

(4) $[\iota_{15}, \nu_{15}] = 0$ *and* $\mathrm{sk}_{50}(F_{11}) = S^{15} \vee S^{33}$.

(5) $\pi_{47}(F_{11}) = j(\hat{\imath}_{15} \circ \pi_{47}^{15} \oplus \hat{\imath}_{33} \circ \pi_{47}^{33} \oplus [\hat{\imath}_{15}, \hat{\imath}_{33}] \circ \pi_{47}^{47}) \approx (\mathbb{Z}/2)^8 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)}$, *generated by*

$$(j \circ \hat{\imath}_{15})_*\{\overline{\nu\bar{\kappa}}_{15}, \bar{\bar{\sigma}}_{15}, \eta^{*\prime}\eta_{31}^*, \mu^{*\prime\prime}, \Sigma\bar{\zeta}^*, (\Sigma^2\lambda)\kappa_{33}, \xi_{15}\sigma_{33}^2, \sigma_{15}\mu_{3,22}\},$$
$$(j \circ \hat{\imath}_{33})_*\{\kappa_{33}, \sigma_{33}^2\} \quad \text{and} \quad (j \circ [\hat{\imath}_{15}, \hat{\imath}_{33}])_*\iota_{47}.$$

Also

$$\pi_{48}(\Sigma F_{11}) = (\Sigma(j \circ \hat{\imath}_{15})) \circ \pi_{48}^{16} \oplus (\Sigma(j \circ \hat{\imath}_{33})) \circ \pi_{48}^{34} \approx \pi_{48}^{16} \oplus \pi_{48}^{34}.$$

Here $\hat{\imath}_m \colon S^m \hookrightarrow S^{15} \vee S^{33}$ *for* $m = 15, 33$ *are the inclusions.*

(6) *If* $\nu_{16}C_2 \equiv 0 \bmod P(\bar{\nu}_{33})$, *then*

$$\operatorname{Coker}(\partial_{11*} \colon \pi_{48}^{19} \to \pi_{47}(F_{11})) \approx (\mathbb{Z}/2)^6 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)},$$

generated by

$$(j \circ \hat{\imath}_{15})_*\{\overline{\nu\bar{\kappa}}_{15}, \bar{\bar{\sigma}}_{15}, \eta^{*\prime}\eta_{31}^*, \mu^{*\prime\prime}, \Sigma\bar{\zeta}^*, \sigma_{15}\mu_{3,22}\}, \quad (j \circ \hat{\imath}_{33})_*\{\kappa_{33}, \sigma_{33}^2\} \quad \text{and} \quad (j \circ [\hat{\imath}_{15}, \hat{\imath}_{33}])_*\iota_{47}.$$

If $\nu_{16}C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33})$, *then*

$$\operatorname{Coker}(\partial_{11*} \colon \pi_{48}^{19} \to \pi_{47}(F_{11})) \approx (\mathbb{Z}/2)^5 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)},$$

generated by

$$(j \circ \hat{\imath}_{15})_*\{\overline{\nu\bar{\kappa}}_{15}, \bar{\bar{\sigma}}_{15}, \eta^{*\prime}\eta_{31}^*, \mu^{*\prime\prime}, \sigma_{15}\mu_{3,22}\}, \quad (j \circ \hat{\imath}_{33})_*\{\kappa_{33}, \sigma_{33}^2\} \quad \text{and} \quad (j \circ [\hat{\imath}_{15}, \hat{\imath}_{33}])_*\iota_{47}.$$

Proof (1) By [13, Lemma 16.4, page 53], we know $\Sigma C_2 = P(\eta_{41})$. Then

$$\nu_{17}\Sigma C_2 = \nu_{17} \circ P(\eta_{41}) = \nu_{17} \circ [\iota_{20}, \iota_{20}]\eta_{39} = [\iota_{17}, \iota_{17}]\nu_{33}^2\eta_{39} = 0.$$

So

$$\nu_{16}C_2 \in \operatorname{Ker}(\Sigma \colon \pi_{39}^{16} \to \pi_{40}^{17}) = P(\pi_{41}^{33}).$$

By [12, (3.4), page 12], $P(\pi_{41}^{33}) = \mathbb{Z}/2\{P(\varepsilon_{33})\} \oplus \mathbb{Z}/2\{P(\bar{\nu}_{33})\}$. By [27, Theorem 14.1], $\varepsilon\mu = \eta^2\rho$ and $\bar{\nu}\mu = 0$. By [14, (5.8)], $P(\eta_{33}^2\rho_{35}) = \Sigma\bar{\zeta}^*$. Hence $P(\varepsilon_{33}) \circ \mu_{39} = \Sigma\bar{\zeta}^*$ and $P(\bar{\nu}_{33}) \circ \mu_{39} = 0$.

(2) By [18, Proposition 3.5 (3), page 60] we know the two relations $\nu_{13}\sigma_{16}^* \equiv t\xi_{13}\sigma_{31} \bmod \lambda\sigma_{31}$ for t odd, and $(\Sigma^3\lambda)\sigma_{34} = 0$.

Then $\nu_{16}\sigma^*_{19} \equiv t\xi_{16}\sigma_{34}$ mod 0 for t odd. Hence

$$\nu_{16}\sigma^*_{19}\sigma_{41} = t\xi_{16}\sigma^2_{34} = \xi_{16}\sigma^2_{34} \quad \text{for } \operatorname{ord}(\sigma^2_{34}) = 2.$$

(3) Notice two facts: $H(P_1) = \kappa_{31}$ [18, Proposition 3.3 (4), page 90], and $H(\lambda\kappa_{31}) = \nu^2_{25}\kappa_{31}$ [14, proof of π^{13}_{45}]. Then $H(\nu_{13}P_1) = \Sigma(\nu_{12} \wedge \nu_{12}) \circ \kappa_{31} = \nu^2_{25}\kappa_{31}$. Hence $\nu_{13}P_1 \equiv \lambda\kappa_{31}$ mod $\Sigma\pi^{12}_{44}$. Therefore, $\nu_{16}\Sigma^3 P_1 \equiv (\Sigma^3\lambda)\kappa_{34}$ mod $\Sigma^4\pi^{12}_{44}$.

(4) By Proposition 6.1.1(11) and (4), $[\iota_{15}, \nu_{15}] = \pm P(\iota_{31}) \circ \nu_{29} = 2\sigma^2_{15}\nu_{29} = 0$. By Lemma 3.1.9, $\mathrm{sk}_{50}(F_{11}) = S^{15} \vee S^{33}$.

(5) By the Hilton–Milnor formula,

$$\Omega(S^{15} \vee S^{33}) \simeq \Omega S^{15} \times \Omega S^{33} \times \Omega S^{47} \times \Omega S^{61} \times \cdots.$$

Then, up to isomorphism,

$$\pi_{47}(\mathrm{sk}_{50}(F_{11})) = \pi_{46}(\Omega S^{15} \times \Omega S^{33} \times \Omega S^{47}) = \pi^{15}_{47} \oplus \pi^{33}_{47} \oplus \pi^{47}_{47},$$

where the isomorphism $\pi^{15}_{47} \oplus \pi^{33}_{47} \oplus \pi^{47}_{47} \to \pi_{47}(\mathrm{sk}_{50}(F_{11}))$ is induced by the inclusions and iterated Whitehead products; see [28, Theorem 8.1, page 533]. By [14, Theorem 1.1], $\pi^{15}_{47} \approx (\mathbb{Z}/2)^8$, generated by $\overline{\nu\bar{\kappa}}_{15}$, $\bar{\bar{\sigma}}_{15}$, $\eta^{*\prime}\eta^*_{31}$, $\mu^{*\prime\prime}$, $\Sigma\bar{\zeta}^*$, $(\Sigma^2\lambda)\kappa_{33}$, $\xi_{15}\sigma^2_{33}$ and $\sigma_{15}\mu_{3,22}$. By [27], $\pi^{33}_{47} = \mathbb{Z}/2\{\kappa_{33}\} \oplus \mathbb{Z}/2\{\sigma^2_{33}\}$. Hence

$$\pi_{47}(\mathrm{sk}_{50}(F_{11})) \approx (\mathbb{Z}/2)^8 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)},$$

generated by

$$(j \circ \hat{i}_{15})_*\{\overline{\nu\bar{\kappa}}_{15}, \bar{\bar{\sigma}}_{15}, \eta^{*\prime}\eta^*_{31}, \mu^{*\prime\prime}, \Sigma\bar{\zeta}^*, (\Sigma^2\lambda)\kappa_{33}, \xi_{15}\sigma^2_{33}, \sigma_{15}\mu_{3,22}\},$$
$$(j \circ \hat{i}_{33})_*\{\kappa_{33}, \sigma^2_{33}\} \quad \text{and} \quad (j \circ [\hat{i}_{15}, \hat{i}_{33}])_*\iota_{47}.$$

Since $\mathrm{sk}_{51}(\Sigma F_{11}) = S^{16} \vee S^{34}$,

$$\pi_{48}(\Sigma F_{11}) = (\Sigma(j \circ \hat{i}_{15})) \circ \pi^{16}_{48} \oplus (\Sigma(j \circ \hat{i}_{33})) \circ \pi^{34}_{48} \approx \pi^{16}_{48} \oplus \pi^{34}_{48}.$$

(6) By Lemma 3.1.4, there exists a commutative diagram

$$\begin{tikzcd}
\pi^{19}_{48} \arrow[r, "\partial_{11*}"] \arrow[d, "\nu_{16*}"'] & \pi_{47}(F_{11}) \arrow[d, "\Sigma"] \\
\pi^{16}_{48} \arrow[r, "(\Sigma j)_*"] & \pi_{48}(\Sigma F_{11})
\end{tikzcd}$$

that is,

$$\begin{tikzcd}
\pi^{19}_{48} \arrow[r, "\partial_{11*}"] \arrow[d, "\nu_{16*}"'] & j \circ \hat{i}_{15} \circ \pi^{15}_{47} \oplus j \circ \hat{i}_{33} \circ \pi^{33}_{47} \oplus j \circ [\hat{i}_{15}, \hat{i}_{33}] \circ \pi^{47}_{47} \arrow[d, "\Sigma"] \\
\pi^{16}_{48} \arrow[r, "(\Sigma j)_*"] & \Sigma(j \circ \hat{i}_{15}) \circ \pi^{16}_{48} \oplus \Sigma(j \circ \hat{i}_{33}) \circ \pi^{34}_{48}
\end{tikzcd}$$

By [14, Theorem 1.1], $\pi^{15}_{32+15} \xrightarrow{\Sigma} \pi^{16}_{32+16}$ is monomorphic. Notice

$$(\operatorname{Im} \partial_{11*}) \cap (j \circ [\hat{i}_{15}, \hat{i}_{33}] \circ \pi^{47}_{47}) = 0.$$

Then, by (1)–(3) and (5) of this lemma, the result follows. □

Recall from [18, Theorem 2 (b), page 81] that

$$\pi_{47}^{19} = \mathbb{Z}/2\{C_2^{(1)}\} \oplus \mathbb{Z}/2\{\Sigma \boldsymbol{F}_2\} \oplus \mathbb{Z}/2\{\varepsilon_{19}\bar{\kappa}_{27}\},$$

and from [12] that

$$\pi_{38}^{15} \approx \mathbb{Z}/16 \oplus \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^4$$

generated by $\bar{\rho}_{15}$, $\nu_{15}\bar{\kappa}_{18}$, ϕ_{15}, ψ_{15}, $\bar{\varepsilon}^{*\prime}$ and $\bar{\nu}^{*\prime}$. ($\boldsymbol{F}_2 \in \pi_{46}^{18}$ is the generator, not the fiber F_2.)

To study $\nu_{16} \circ \pi_{47}^{19}$ and $\mathrm{Ker}(\partial_{11_*47})$, we give the following two lemmas:

Lemma 6.8.4 (1) $C_2^{(1)} \in \{C_2, 2\iota_{39}, 8\sigma_{39}\}_1$.

(2) $D_2^{(1)}\sigma_{39} \equiv \{\bar{\varepsilon}^{*\prime}, 2\iota_{38}, 8\sigma_{38}\}_1 \bmod \Sigma\pi_{46}^{15}$.

Proof (1) This is just the definition of $C_2^{(1)}$ given by Oda; see [17, Definition 3.6].

(2) We notice the following facts:

$$D_1^{(1)} \equiv \mu^{*\prime} \bmod \Sigma\pi_{38}^{14} \qquad \text{(by [18, (2.3), page 52])},$$

$$H(\mu^{*\prime}) = \eta_{29}\mu_{30} \qquad \text{(by [12, (3.11)])}.$$

It is easy to obtain $\eta\mu\sigma = \varepsilon\mu$ by [27, Theorem 14.1, page 190]. Then

$$H(D_2^{(1)}\sigma_{39}) = \eta_{29}\mu_{30}\sigma_{39} = \varepsilon_{29}\mu_{37}.$$

Since $H(\bar{\varepsilon}^{*\prime}) = \eta_{29}\varepsilon_{30} = \varepsilon_{29}\eta_{37}$ [12, (3.4)], $\mu \in \langle \eta, 2\iota, 8\sigma \rangle$ [27, page 189].

Then

$$H\{\bar{\varepsilon}^{*\prime}, 2\iota_{38}, 8\sigma_{38}\}_1 \subseteq \{\varepsilon_{29}\eta_{37}, 2\iota_{38}, 8\sigma_{38}\}_1 \supseteq \varepsilon_{29} \circ \{\eta_{37}, 2\iota_{38}, 8\sigma_{38}\}_1 \ni \varepsilon_{29}\mu_{37}.$$

Notice $\varepsilon^2 = \varepsilon\bar{\nu} = 0$ [27, Theorem 14.1], and $8\pi_{29+10}^{29} = 0$. Thus

$$\mathrm{Ind}\{\varepsilon_{29}\eta_{37}, 2\iota_{38}, 8\sigma_{38}\}_1 = \varepsilon_{29}\eta_{37} \circ \pi_{46}^{38} = \varepsilon_{29}\eta_{37} \circ \langle \varepsilon_{38}, \bar{\nu}_{38} \rangle = 0.$$

Hence $H\{\bar{\varepsilon}^{*\prime}, 2\iota_{38}, 8\sigma_{38}\}_1 = \varepsilon_{29}\mu_{37}$. Then the result follows from the exactness of the EHP sequence. □

Lemma 6.8.5 (1) $\nu_{15}\boldsymbol{F}_2 = \nu_{16}\varepsilon_{19}\bar{\kappa}_{27} = 0$.

(2) *We have*

$$\nu_{16}C_2^{(1)} \begin{cases} = 0 & \textit{if } \nu_{16}C_2 \equiv 0 \bmod P(\bar{\nu}_{33}), \\ \equiv (\Sigma D_1^{(1)}) \circ \sigma_{40} \bmod \omega_{16}^*, \kappa_{16}^*, \rho_{3,16} & \textit{if } \nu_{16}C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33}). \end{cases}$$

(3) $\mathrm{Ker}(\partial_{11_*} : \pi_{47}^{19} \to \pi_{48}(F_{11})) = \mathbb{Z}/2\{\Sigma\boldsymbol{F}_2\} \oplus \mathbb{Z}/2\{\varepsilon_{19}\bar{\kappa}_{27}\} \oplus G$, *where*

$$G = \begin{cases} \mathbb{Z}/2\{C_2^{(1)}\} & \textit{if } \nu_{16}C_2 \equiv 0 \bmod P(\bar{\nu}_{33}), \\ 0 & \textit{if } \nu_{16}C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33}). \end{cases}$$

Proof (1) $\nu_{16}\varepsilon_{19}\bar{\kappa}_{27}=0$ is obvious; see Proposition 6.1.1(6). Next, we consider $\nu_{15}\boldsymbol{F}_2$. We know $H(\boldsymbol{F}_2)=a\zeta_{35}$ for some odd a by [18, Proposition 3.2(2), page 89]. By [27, Proposition 2.2, page 18 and Proposition 3.1, page 25], we have

$$H(\nu_{15}\boldsymbol{F}_2)=\Sigma(\nu_{14}\wedge\nu_{14})\circ a\zeta_{35}=a\nu_{29}^2\zeta_{35}=a\nu_{29}(\nu_{32}\zeta_{35}).$$

By [27, Theorem 14.1(ii)], $\nu\zeta=0$, so $H(\nu_{15}\boldsymbol{F}_2)=0$. Then $\nu_{15}\boldsymbol{F}_2\in\Sigma\pi_{45}^{14}$.

Finally, by [18, Theorem 3 (c), page 106], $\mathrm{ord}(\omega_{15}^*)=2$ and

$$\mathrm{Ker}(\Sigma^\infty\colon\pi_{45}^{14}\to\pi_{31}^S(S^0))=\mathbb{Z}/4\{2\omega_{14}^*\}.$$

Notice

$$\Sigma^\infty(\nu_{15}\boldsymbol{F}_2)=\nu\circ\Sigma^\infty\boldsymbol{F}_2\in\nu\circ\pi_{28}^S(S^0)=\langle\nu\varepsilon\bar{\kappa}\rangle=0.$$

Then $\nu_{15}\boldsymbol{F}_2\in\langle 2\omega_{15}^*\rangle=0$.

(2) By Lemma 6.8.4(1), $\nu_{16}C_2^{(1)}\in\nu_{16}\circ\{C_2,2\iota_{39},8\sigma_{39}\}_1$. The proof falls into two cases.

Case 1 ($\nu_{16}C_2\equiv 0 \bmod P\bar{\nu}_{33}$) We have $\nu_{16}C_2=tP\bar{\nu}_{33}$ for some $t\in\mathbb{Z}$. By [18, Proposition 3.2 (1), page 57], we have $\langle\bar{\nu},2\iota,8\sigma\rangle=0$. Then

$$\nu_{16}C_2^{(1)}\in\nu_{16}\circ\{C_2,2\iota_{39},8\sigma_{39}\}_1\subseteq tP(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1\supseteq P\{t(\bar{\nu}_{33}),2\iota_{41},8\sigma_{41}\}_3=0.$$

Since $8\pi_{40}^{16}=0$,

$$\mathrm{Ind}\{tP(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1=tP(\bar{\nu}_{33})\circ\langle\varepsilon_{39},\bar{\nu}_{39}\rangle+\pi_{40}^{16}\circ 8\sigma_{40}=0,$$

so $\{tP(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1=0$. And so

$$\nu_{16}C_2^{(1)}=0\quad\text{if }\nu_{16}C_2\equiv 0 \bmod P\bar{\nu}_{33}.$$

Essentially, we have also proved $\{P(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1=0$ (the method of the proof for the triviality of the above Toda bracket carries over to any $t\in\mathbb{Z}$).

Case 2 ($\nu_{16}C_2\equiv P\varepsilon_{33} \bmod P\bar{\nu}_{33}$) By Lemma 6.8.4(2) and $P(\varepsilon_{33})=\Sigma\bar{\varepsilon}^{*\prime}$ [12, (3.4)], we have, for some $z\in\mathbb{Z}$,

$$\begin{aligned}\nu_{16}C_2^{(1)}&\in\nu_{16}\circ\{C_2,2\iota_{39},8\sigma_{39}\}_1\subseteq\{P(\varepsilon_{33})+zP(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1\\&\subseteq\{P(\varepsilon_{33}),2\iota_{39},8\sigma_{39}\}_1+\{zP(\bar{\nu}_{33}),2\iota_{39},8\sigma_{39}\}_1=\{P(\varepsilon_{33}),2\iota_{39},8\sigma_{39}\}_1+0\\&=\{P(\varepsilon_{33}),2\iota_{39},8\sigma_{39}\}_1=\{\Sigma\bar{\varepsilon}^{*\prime},2\iota_{39},8\sigma_{39}\}_1\supseteq\Sigma\{\bar{\varepsilon}^{*\prime},2\iota_{38},8\sigma_{38}\},\\&\ni\Sigma(D_1^{(1)}\sigma_{39}+\Sigma\varkappa)=(\Sigma D_1^{(1)})\sigma_{40}+\Sigma^2\varkappa\quad\text{for some }\varkappa\in\pi_{45}^{15},\end{aligned}$$

and so $\mathrm{Ind}\{P(\varepsilon_{33}),2\iota_{39},8\sigma_{39}\}_1=P(\varepsilon_{33})\circ\langle\varepsilon_{39},\bar{\nu}_{39}\rangle+\pi_{40}^{16}\circ 8\sigma_{40}=0$. Then

$$\nu_{16}C_2^{(1)}\equiv(\Sigma D_1^{(1)})\sigma_{40}\mod\Sigma^2\pi_{45}^{14}$$

(if $\nu_{16}C_2\equiv P\varepsilon_{33} \bmod P\bar{\nu}_{33}$).

(3) We have shown that $\mathrm{sk}_{50}(F_{11}) = S^{15} \vee S^{33}$ in Lemma 6.8.3(4). By use of Lemma 3.1.4, there exists the following commutative diagram (here, by abuse of notation, j denotes the inclusions, although they are different inclusions; in this case, these two j and these two Σj all induce monomorphisms):

$$\begin{tikzcd}
\pi_{47}^{19} \arrow[r, "\partial_{11*}"] \arrow[d, "\nu_{16*}"'] & j \circ \pi_{46}^{15} \oplus j \circ \pi_{46}^{33} \arrow[d, "\Sigma"] \\
\pi_{47}^{16} \arrow[r, "(\Sigma j)_*"] & (\Sigma j) \circ \pi_{47}^{16} \oplus (\Sigma j) \circ \pi_{47}^{34}
\end{tikzcd}$$

By [18, Theorem 3 (c), page 106], we know $\pi_{46}^{15} \xrightarrow{\Sigma} \pi_{47}^{16}$ is a monomorphism. Thus $j \circ \pi_{46}^{15} \oplus j \circ \pi_{46}^{33} \xrightarrow{\Sigma} (\Sigma j) \circ \pi_{47}^{16} \oplus (\Sigma j) \circ \pi_{47}^{34}$ is a monomorphism. So $\mathrm{Ker}(\nu_{16*}) = \mathrm{Ker}(\partial_{11*})$. Then, by (1) and (2) of this lemma, we obtain the result. □

Recall that $\nu_{16} C_2 \in \mathbb{Z}/2\{P(\varepsilon_{33})\} \oplus \mathbb{Z}/2\{P(\bar{\nu}_{33})\}$. Equivalently speaking, either $\nu_{16} C_2 \equiv 0 \bmod P(\bar{\nu}_{33})$ or $\nu_{16} C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33})$.

By Lemmas 6.8.3(6) and 6.8.5(3), we have the following theorem:

Theorem 6.8.6 *After localization at 2, if $\nu_{16} C_2 \equiv 0 \bmod P(\bar{\nu}_{33})$, then $\pi_{47}(\Sigma^{11}\mathbb{H}P^2)$ satisfies the short exact sequence*

$$0 \to (\mathbb{Z}/2)^8 \oplus \mathbb{Z}_{(2)} \xrightarrow{\subseteq} \pi_{47}(\Sigma^{11}\mathbb{H}P^2) \xrightarrow{p_{11*}} \mathbb{Z}/2\{\Sigma \boldsymbol{F}_2\} \oplus \mathbb{Z}/2\{\varepsilon_{19}\bar{\kappa}_{27}\} \oplus \mathbb{Z}/2\{C_2^{(1)}\} \to 0.$$

If $\nu_{16} C_2 \equiv P(\varepsilon_{33}) \bmod P(\bar{\nu}_{33})$, then $\pi_{47}(\Sigma^{11}\mathbb{H}P^2)$ satisfies the short exact sequence

$$0 \to (\mathbb{Z}/2)^7 \oplus \mathbb{Z}_{(2)} \xrightarrow{\subseteq} \pi_{47}(\Sigma^{11}\mathbb{H}P^2) \xrightarrow{p_{11*}} \mathbb{Z}/2\{\Sigma \boldsymbol{F}_2\} \oplus \mathbb{Z}/2\{\varepsilon_{19}\bar{\kappa}_{27}\} \to 0.$$

Here $(\mathbb{Z}/2)^8 \oplus \mathbb{Z}_{(2)}$ and $(\mathbb{Z}/2)^7 \oplus \mathbb{Z}_{(2)}$ in these two sequences are the abbreviations of $\mathrm{Coker}(\partial_{11*_{48}})$ in Lemma 6.8.3(6). □

7 Applications

7.1 The Hopf fibration $S^{11} \to \mathbb{H}P^2$ localized at 2

We notice on [8, page 38], $P_{n,k} := \mathbb{H}P^{n+1}/\mathbb{H}P^{n+1-k}$; see [8, page 21]. Then we have the following proposition:

Proposition 7.1.1 [8, page 38] *Let $h_n\colon S^{4n+3} \to \mathbb{H}P^n$ be the homotopy class of the Hopf fibration and $p_n^{\mathbb{H}}\colon \mathbb{H}P^n \to S^{4n}$ be the pinch map. Then*

$$p_n^{\mathbb{H}} \circ h_n = n(\Sigma^{4n-4}\varkappa),$$

where $\varkappa \in \pi_7(S^4)$ is the homotopy class of the Hopf fibration. Of course, after localization at 2, we have $\varkappa \equiv \pm\nu_4 \bmod \Sigma\nu'$.

Theorem 7.1.2 *After localization at* 2, *for the homotopy class of the Hopf fibration* $h\colon S^{11}\to \mathbb{H}P^2$, *the relation*

$$\Sigma h = x\mathbb{a}$$

holds for some odd x, *where* $\mathbb{a}\in\{j_1,\nu_5,2\nu_8\}$ *generates* $\pi_{12}(\Sigma\mathbb{H}P^2)\approx\mathbb{Z}/8$. *But* $\Sigma^\infty h$ *cannot generate any direct summand of* $\pi_{11}^S(\mathbb{H}P^2)\approx\mathbb{Z}/16\oplus\mathbb{Z}/4$.

Proof By [15, Remark 3 (3)], we obtain, $\operatorname{ord}(\Sigma^\infty h)=\frac{6!}{2}=8\cdot 9\cdot 5$ (without taking the localization). Then, after localization at 2, $\operatorname{ord}(\Sigma^\infty h)=8$. By Theorem 6.3.2, we have $\pi_{12}(\Sigma\mathbb{H}P^2)\approx\mathbb{Z}/8\{\mathbb{a}\}$ for $\mathbb{a}\in\{j_1,\nu_5,2\nu_8\}$. So Σh is of order 8 and $\Sigma h=x\mathbb{a}$ for some odd x. By using Theorem 6.3.2, we have $\pi_{11}^S(\mathbb{H}P^2)\approx\mathbb{Z}/16\oplus\mathbb{Z}/4$. □

Corollary 7.1.3 *After localization at* 2,

$$\Sigma\mathbb{H}P^3\simeq\Sigma\mathbb{H}P^2\cup_{\mathbb{a}} e^{13},$$

where $\mathbb{a}\in\{j_1,\nu_5,2\nu_8\}\subseteq\pi_{12}(\Sigma\mathbb{H}P^2)=\mathbb{Z}/8\{\mathbb{a}\}$. □

7.2 Two classification theorems

In this subsection, for convenience, after localization at 3, we use $P^n_*[i]$ to denote $P^n_*\colon \widetilde{H}_i(-)\to\widetilde{H}_{i-4n}(-)$, that is, the dual of the Steenrod operation.

As already stated in the introduction, given a CW complex X, we say that X is a *homology* n*-dimensional quaternionic projective space* if $\widetilde{H}_*(X;\mathbb{Z})\approx\widetilde{H}_*(\mathbb{H}P^n;\mathbb{Z})$ as graded groups.

We recall that, after localization at 3, $\pi_{12}(S^5)=\mathbb{Z}/3\{\alpha_2(5)\}$, $\pi_{12}(S^9)=\mathbb{Z}/3\{\alpha_1(9)\}$, $h\colon S^{11}\to\mathbb{H}P^2$ is the attaching class of $\mathbb{H}P^3=\mathbb{H}P^2\cup_h e^{12}$. Recall from Theorem 5.2.3 that

$$\pi_{15}(\Sigma^4\mathbb{H}P^2)=\mathbb{Z}_{(3)}\{j_4\circ[\iota_8,\iota_8]\}\oplus\mathbb{Z}/9\{\Sigma^4 h\}$$

localized at 3. After localization at 3, let $A=S^5\cup_{\alpha_2(5)}e^{13}$, $i_5'\colon S^5\to S^5\vee S^9$ and $i_9'\colon S^5\to S^5\vee S^9$ be the inclusions, $c_k=\Sigma^{k-1}(i_5'\circ\alpha_2(5)+i_9'\circ\alpha_1(9))$. By use of this notation, we give the following classification theorems:

Theorem 7.2.1 *After localization at* 3, *up to homotopy, the* k*-fold* (*for* $k\geq 1$ *but* $k\neq 4$) *suspension of the simply connected homology* 3*-dimensional quaternionic projective spaces can be classified as the following*:

$$\Sigma^k\mathbb{H}P^3,\quad \Sigma^k\mathbb{H}P^2\cup_{3\Sigma^k h}e^{12+k},\quad \Sigma^k\mathbb{H}P^2\vee S^{12+k},\quad \Sigma^{k-1}A\vee S^{8+k},$$
$$S^{4+k}\vee\Sigma^{4+k}\mathbb{H}P^2,\quad (S^{4+k}\vee S^{8+k})\cup_{c_k}e^{12+k},\quad S^{4+k}\vee S^{8+k}\vee S^{12+k}.$$

Proof Suppose $k\geq 1$ but $k\neq 4$ and spaces are localized at 3. The proof will be divided into two steps.

Step 1 Notice that $\pi_{11+k}(\Sigma^k \mathbb{H}P^2) = \mathbb{Z}/9\{\Sigma^k h\}$; see Theorem 5.2.3. By Corollaries 2.1.1 and 2.1.2, any k-fold suspension of a simply connected homology 3-dimensional quaternionic projective space, that is, any CW complex of type $S^{4+k} \cup e^{8+k} \cup e^{12+k}$, is homotopy equivalent to some CW complex listed in our theorem.

Step 2 We first examine the Steenrod module structures of the CW complexes listed in our theorem.

•For $\Sigma^k \mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k}$, $P^1_*[12+k] = 0$ and $P^1_*[8+k] \neq 0$. On $P^1_*[12+k] = 0$, because

$$(\Sigma^k \mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k})/S^{4+k} \simeq S^{8+k} \vee S^{12+k}.$$

•For $(S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$, $P^1_*[12+k] \neq 0$ and $P^1_*[8+k] = 0$. Notice

$$((S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k})/S^{4+k} \simeq \Sigma^{k+4} \mathbb{H}P^2.$$

•For $\Sigma^{k-1} A \vee S^{8+k}$, $P^1_*[12+k] = P^1_*[8+k] = 0$.

Then we consider the homotopy groups,

$$\begin{aligned}
\pi_{11+k}(\Sigma^{k-1} A \vee S^{8+k}) &= \pi_{11+k}((S^{4+k} \cup_{\alpha_2(4+k)} e^{12+k}) \vee S^{8+k}) \approx \mathbb{Z}/3, \\
\pi_{11+k}(S^{4+k} \vee S^{8+k} \vee S^{12+k}) &\approx (\mathbb{Z}/3)^2, \\
\pi_{11+k}(\Sigma^k \mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k}) &\approx \frac{\mathbb{Z}/9\{\Sigma^k h\}}{\langle 3\Sigma^k h\rangle} \approx \mathbb{Z}/3, \\
\pi_{11+k}(\Sigma^k \mathbb{H}P^2 \vee S^{12+k}) &\approx \mathbb{Z}/9.
\end{aligned}$$

Hence we only need to show that

$$S^{4+k} \vee \Sigma^{4+k} \mathbb{H}P^2 \not\simeq (S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}.$$

Since

$$\begin{gathered}
(S^{4+k} \vee \Sigma^{4+k} \mathbb{H}P^2)/S^{8+k} \simeq S^{4+k} \vee S^{12+k}, \\
((S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k})/S^{8+k} \simeq S^{4+k} \cup_{\alpha_2(4+k)} e^{12+k}, \\
\pi_{11+k}(S^{4+k} \vee S^{12+k}) \approx \mathbb{Z}/3, \\
\pi_{11+k}(S^{4+k} \cup_{\alpha_2(4+k)} e^{12+k}) = 0,
\end{gathered}$$

$S^{4+k} \vee \Sigma^{4+k} \mathbb{H}P^2$ is not homotopy equivalent to $(S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$.

The above facts imply the CW complexes listed in our theorem are not homotopy equivalent to each other. □

The following lemma is easy; we leave its proof to the reader. Notice that there is a diagram (similar to (2-1)) which contains a map $K \xrightarrow{\Psi} C_f$ and Sq^{m-n} commutes with Ψ^*.

Lemma 7.2.2 *Let $f \in \pi_{m-1}(S^n)$ where $m \geq n+2$ and let $K = C_{2f}$. Then $\mathrm{Sq}^{m-n} \colon H^n(K; \mathbb{Z}/2) \to H^m(K; \mathbb{Z}/2)$ is trivial.* □

The Steenrod square operations control $(\Sigma^k\mathbb{H}P^2)_{(2)}$ very strongly. Up to homotopy, $(\Sigma^k\mathbb{H}P^2)_{(2)}$ for $k\geq 0$ is determined by its simply connectedness, finite type property and Steenrod module structure. This interesting result is essentially shown by the third author in [31, Lemma 3.9], utilizing the theory of the Cohen group. Notice Lemma 7.2.2, Corollary 2.1.1 and $\pi_{7+n}(S^{4+n}_{(2)})\approx\mathbb{Z}/8$ for $n\geq 1$.

But the Steenrod square operations control $(\Sigma^k\mathbb{H}P^3)_{(2)}$ poorly. In fact, setting

$$\widetilde{H}_*((\mathbb{H}P^3)_{(2)})=\mathbb{Z}/2\{u_4,u_8,u_{12}\}\quad\text{for } |u_i|=i,$$

we know $\mathrm{Sq}^8_*(u_{12})=\mathrm{Sq}^4_*(u_{12})=0$ and $\mathrm{Sq}^4_*(u_8)=u_4$; see [6, Example 4L.4, page 492].

The situation in the 3-local case is quite different. The Steenrod power operations command a deep influence over the space $(\Sigma^k\mathbb{H}P^3)_{(3)}$. That is, as the reader will see in the following theorem, $(\Sigma^k\mathbb{H}P^3)_{(3)}$ for $k\geq 1$ but $k\neq 4$ is entirely dictated by its simply connectedness, finite type property and Steenrod module structure, up to homotopy.

Theorem 7.2.3 *After localization at 3, suppose Z is a k-fold (for $k\geq 1$ but $k\neq 4$) suspension of a simply connected homology 3-dimensional quaternionic projective space.*

1. *If $\widetilde{H}_*(Z)$ has nontrivial Steenrod operations P^1_* of dimension $8+k$ and $12+k$, then $Z\simeq\Sigma^k\mathbb{H}P^3$.*
2. *If $\widetilde{H}_*(Z)\approx\widetilde{H}_*(\Sigma^k\mathbb{H}P^3)$ as Steenrod modules, then $Z\simeq\Sigma^k\mathbb{H}P^3$. Moreover, if X is a 3-local simply connected CW complex such that $\widetilde{H}_i(X;\mathbb{Z}_{(3)})$ is a finitely generated $\mathbb{Z}_{(3)}$-module for each i, and $\widetilde{H}_*(X)\approx\widetilde{H}_*(\Sigma^k\mathbb{H}P^3)$ as Steenrod modules, then $X\simeq\Sigma^k\mathbb{H}P^3$.*

Proof (1) After localization at 3, suppose $k\geq 1$ but $k\neq 4$. Since $P^1_*[8+k]\neq 0$ and $\pi_{7+k}(S^{4+k})\approx\mathbb{Z}/3$, we take

$$\mathrm{sk}_{11+k}(Z)=\Sigma^k\mathbb{H}P^2\quad\text{and}\quad Z=\Sigma^k\mathbb{H}P^2\cup_f e^{12+k},$$

where $f\in\pi_{11+k}(\Sigma^k\mathbb{H}P^2)=\mathbb{Z}/9\{\Sigma^k h\}$. Now, to obtain a contraction, suppose f is divisible by 3, that is, there exists $\varkappa\in\pi_{11+k}(\Sigma^k\mathbb{H}P^2)$ such that $f=3\varkappa$. Notice that $\pi_{11+k}(S^{8+k})\approx\mathbb{Z}/3$. Then

$$Z/S^{4+k}\simeq S^{8+k}\cup_{q\circ 3\varkappa}e^{12+k}\simeq S^{8+k}\cup_0 e^{12+k}\simeq S^{8+k}\vee S^{12+k}.$$

Here $q\colon\Sigma^k\mathbb{H}P^2\to(\Sigma^k\mathbb{H}P^2)/S^{4+k}$ is the pinch. By assumption, for $\widetilde{H}_*(Z)$, $P^1_*[12+k]\neq 0$. Then $\widetilde{H}_*(Z/S^{4+k})\approx\widetilde{H}_*(S^{8+k}\vee S^{12+k})$ satisfies $P^1_*[12+k]\neq 0$, which is impossible. This forces f to be of order 9. By Corollary 2.1.1, we deduce the result.

(2) The first part of (2) is an immediate consequence of (1); in fact, the hypotheses for Z of (1) and (2) are equivalent as $P^2_*=-P^1_*P^1_*$. The second part of (2) follows directly from the first part. □

Remark 7.2.4 For Theorem 7.2.3, the condition $k\neq 4$ is indispensable. For example, $C_{j_4\circ[\iota_8,\iota_8]+\Sigma^4 h}\not\simeq\Sigma^4\mathbb{H}P^3$ after localization at 3. Although they both have nontrivial Steenrod operations P^1_* of dimension 12 and 16, we see that they are not homotopy equivalent by checking the 15th homotopy groups.

7.3 Wedge decompositions of suspended self smashes

Suppose $\{A_n\}_{n=1}^{\infty}$ is a family of spaces and $\{\mathbb{f}_n\colon A_n \to A_{n+1}\}_{n=1}^{\infty}$ is a family of maps. We denote the homotopy colimit of the sequence

$$A_1 \xrightarrow{\mathbb{f}_1} A_2 \xrightarrow{\mathbb{f}_2} A_3 \xrightarrow{\mathbb{f}_3} \cdots$$

by $\operatorname{hocolim}_{\mathbb{f}_n} A_n$. If $A_n = A_1$ and $\mathbb{f}_n = \mathbb{f}_1$ for all $n \geq 1$, then $\operatorname{hocolim}_{\mathbb{f}_n} A_n$ is also denoted by $\operatorname{hocolim}_{\mathbb{f}_1} A_1$. For example, given a map $g\colon B \to B$, the notation $\operatorname{hocolim}_g B$ stands for the homotopy colimit of the sequence $B \xrightarrow{g} B \xrightarrow{g} B \to \cdots$.

Let X be a p-local path-connected CW complex for p prime, and let the symmetric group $\mathcal{S}_n$ act on $X^{\wedge n}$ by permuting positions. Thus, for each $\tau \in \mathcal{S}_n$, we have a map $\tau\colon X^{\wedge n} \to X^{\wedge n}$. Using the construction of the classical group structure of $[\Sigma X^{\wedge n}, \Sigma X^{\wedge n}]$, we obtain a map $\|k\|\colon \Sigma X^{\wedge n} \to \Sigma X^{\wedge n}$ for any k in the group ring $\mathbb{Z}_{(p)}[\mathcal{S}_n]$. Then the $\mathbb{Z}/p$ coefficient reduced homology $\widetilde{H}_*(\Sigma X^{\wedge n})$ becomes a module over $\mathbb{Z}_{(p)}[\mathcal{S}_n]$; the module structure is decided by permuting factors in the graded sense for each $\tau \in \mathcal{S}_n$. Let $1 = \Sigma_\alpha e_\alpha$ be an orthogonal decomposition of the identity in $\mathbb{Z}_{(p)}[\mathcal{S}_n]$ in terms of primitive idempotents and μ' be the classical comultiplication of the suspension as a co-H space. Then the composition

$$\Sigma X^{\wedge n} \xrightarrow{\mu'} \bigvee_\alpha \Sigma X^{\wedge n} \to \bigvee_\alpha \operatorname{hocolim}_{\|e_\alpha\|} \Sigma X^{\wedge n}$$

is a homotopy equivalence because its induced map on the singular chains over $\mathbb{Z}_{(p)}$ is a natural homotopy equivalence with respect to X; see [31, Section 3.1, pages 32–34; 21, pages 11–12] for more details.

By abuse of notation, for $k \in \mathbb{Z}_{(p)}[\mathcal{S}_n]$, the map $\|k\|$ and the element k are both denoted by k. The following is just a special and the most simple case of the above by taking $n = 2$, p an odd prime and using $1 = \frac{1}{2}(1 + (12)) + \frac{1}{2}(1 - (12))$.

Lemma 7.3.1 *After localization at an odd prime p, suppose X is a path-connected CW complex.*

(1) [31] *There exists a natural decomposition with respect to X,*

$$\Sigma X^{\wedge 2} \simeq \operatorname{hocolim}_{(1+(12))/2} \Sigma X^{\wedge 2} \bigvee \operatorname{hocolim}_{(1-(12))/2} \Sigma X^{\wedge 2}.$$

(2) [31] *$\widetilde{H}_*(\Sigma X^{\wedge 2})$ becomes a module over the group ring $\mathbb{Z}_{(p)}[\mathcal{S}_2]$. The module structures are given by: for $(12) \in \mathcal{S}_2 \subseteq \mathbb{Z}_{(p)}[\mathcal{S}_2]$ which decides a map $\Sigma X^{\wedge 2} \xrightarrow{(12)} \Sigma X^{\wedge 2}$,*

$$(12)_*\colon \widetilde{H}_*(\Sigma X^{\wedge 2}) \to \widetilde{H}_*(\Sigma X^{\wedge 2}), \quad \sigma(a \otimes b) \mapsto (-1)^{|a|\cdot|b|}\sigma(b \otimes a);$$

for $k \in \mathbb{Z}_{(p)} \subseteq \mathbb{Z}_{(p)}[\mathcal{S}_2]$ which decides a map $\Sigma X^{\wedge 2} \xrightarrow{k} \Sigma X^{\wedge 2}$,

$$k_*\colon \widetilde{H}_*(\Sigma X^{\wedge 2}) \to \widetilde{H}_*(\Sigma X^{\wedge 2}), \quad \sigma(a \otimes b) \mapsto k(\sigma(a \otimes b)).$$

Here $\sigma\colon \widetilde{H}_*(-) \to \widetilde{H}_{*+1}(\Sigma -)$ denotes the suspension isomorphism.

(3) [21] $\widetilde{H}_*(\operatorname{hocolim}_{(1\pm(12))/2} \Sigma X^{\wedge 2}) = \operatorname{Im}\big(\big(\tfrac{1}{2}(1 \pm (12))\big)_*\colon \widetilde{H}_*(\Sigma X^{\wedge 2}) \to \widetilde{H}_*(\Sigma X^{\wedge 2})\big)$. □

The following decomposition is surprising:

Theorem 7.3.2 *After localization at* 3, $\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2\simeq S^{13}\vee\Sigma^5\mathbb{H}P^3$.

Proof We denote $a\otimes b$ by ab for short. After localization at 3, $\widetilde{H}_*(\mathbb{H}P^2)$=$\mathbb{Z}/3\{x,y\}:=V$, where $|x|=4$, $|y|=8$ and $P^1_*(y)=x$. Then

$$\widetilde{H}_*(\mathbb{H}P^2\wedge\mathbb{H}P^2)=V\otimes V=\mathbb{Z}/3\{xx,xy,yx,yy\}.$$

By Lemma 7.3.1, taking $X=\mathbb{H}P^2$, we have

$$\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2\simeq\mathrm{hocolim}_{(1+(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2)\vee\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2).$$

Then $\widetilde{H}_*(\mathrm{hocolim}_{(1+(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2))=\mathrm{Im}\big(\tfrac{1}{2}(1+(12))\big)_*=\mathbb{Z}/3\{\sigma(xx),\sigma(xy+yx),\sigma(yy)\}$, where

$$|\sigma(xx)|=9,\quad |\sigma(xy+yx)|=13,\quad |\sigma(yy)|=17,$$
$$P^1_*(\sigma(yy))=\sigma(xy+yx),\quad P^1_*(\sigma(xy+yx))=-\sigma(xx),$$
$$\widetilde{H}_*(\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2))=\mathrm{Im}\big(\tfrac{1}{2}(1-(12))\big)_*=\mathbb{Z}/3\{\sigma(xy-yx)\}.$$

Here $|\sigma(xy-yx)|=13$. By applying $\widetilde{H}_0(-;\mathbb{Z})$ and $\pi_1(-)$ (see Proposition 2.1.4), we know the spaces $\mathrm{hocolim}_{(1\pm(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2)$ are simply connected. So $\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2)\simeq S^{13}$. By Theorem 7.2.3(1), we obtain

$$\mathrm{hocolim}_{(1+(12))/2}(\Sigma\mathbb{H}P^2\wedge\mathbb{H}P^2)\simeq\Sigma^5\mathbb{H}P^3.\qquad\square$$

We show a similar decomposition with respect to $\mathbb{H}P^3$:

Theorem 7.3.3 *After localization at* 3,

$$\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3\simeq\Sigma^9\mathbb{H}P^3\vee Y,$$

where Y is a 6*-cell CW complex and* $\mathrm{sk}_{13}(Y)=\Sigma^5\mathbb{H}P^2$.

Proof We denote $a\otimes b$ by ab. After localization at 3,

$$\widetilde{H}_*(\mathbb{H}P^3)=\mathbb{Z}/3\{x,y,z\}:=V,\quad P^1_*(z)=-y,\quad P^2_*(z)=x,\quad P^1_*(y)=x,$$

where $|x|$=4, $|y|$=8 and $|z|$=12. And

$$\widetilde{H}_*(\mathbb{H}P^3\wedge\mathbb{H}P^3)=V\otimes V=\mathbb{Z}/3\{xx,xy,xz,yx,yy,yz,zx,zy,zz\}.$$

By Lemma 7.3.1, taking $X=\mathbb{H}P^3$, we have

$$\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3\simeq\mathrm{hocolim}_{(1+(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3)\vee\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3).$$

By applying $\widetilde{H}_0(-;\mathbb{Z})$ together with $\pi_1(-)$ (see Proposition 2.1.4), $\mathrm{hocolim}_{(1\pm(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3)$ are simply connected. Consider

$$\widetilde{H}_*(\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3))=\mathrm{Im}\big(\tfrac{1}{2}(1-(12))\big)_*=\mathbb{Z}/3\{\sigma(xy-yx),\sigma(xz-zx),\sigma(zy-yz)\},$$

where $|\sigma(xy-yx)|=13$, $|\sigma(xz-zx)|=17$ and $|\sigma(zy-yz)|=21$. There are nontrivial Steenrod operations

$$P^1_*(\sigma(zy-yz))=-\sigma(xz-zx),\quad P^1_*(\sigma(xz-zx))=\sigma(yx-xy).$$

By Theorem 7.2.3(1), we obtain

$$\mathrm{hocolim}_{(1-(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3)\simeq\Sigma^9\mathbb{H}P^3.$$

Let $Y=\mathrm{hocolim}_{(1+(12))/2}(\Sigma\mathbb{H}P^3\wedge\mathbb{H}P^3)$. Then $\widetilde{H}_*(Y)$ is isomorphic to

$$\mathrm{Im}\big(\tfrac{1}{2}(1+(12))\big)_*=\mathbb{Z}/3\{\sigma(xy+yx),\sigma(xz+zx),\sigma(zy+yz),\sigma(xx),\sigma(yy),\sigma(zz)\}.$$

The degrees of the basis elements $\sigma(xy+yx)$, $\sigma(xz+zx)$, $\sigma(zy+yz)$, $\sigma(xx)$, $\sigma(yy)$ and $\sigma(zz)$ are 13, 17, 21, 9, 17 and 25, respectively. Notice $P^1_*(\sigma(xy+yx))=-\sigma(xx)$. Then $\mathrm{sk}_{13}(Y)\simeq\Sigma^5\mathbb{H}P^2$. Replacing Y by a CW complex homotopy equivalent to Y and having $\Sigma^5\mathbb{H}P^2$ as its skeleton of dimension 13, we infer the result. □

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Received: 13 November 2023 Revised: 17 May 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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Algebraic & Geometric Topology

Volume 25 Issue 5 (pages 2527–3144) 2025