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The right angled Artin group functor as a categorical embedding

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It has long been known that the combinatorial properties of a graph Γ are closely related to the group theoretic properties of its *right angled Artin group* (raag). It's natural to ask if the graph *homomorphisms* are similarly related to the group homomorphisms between two raags. Our main result shows that there is a purely algebraic way to characterize the raags amongst groups, and the graph homomorphisms amongst the group homomorphisms. As a corollary we present a new algorithm for recovering Γ from its raag.

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1 Introduction

For us, a *graph* Γ with underlying vertex set V is a symmetric, reflexive relation on V. A *graph homomorphism* from a graph (V, Γ) to (W, Δ) is a function $\varphi\colon V \to W$ such that $(v_1, v_2) \in \Gamma \implies (\varphi v_1, \varphi v_2) \in \Delta$. These assemble into a category, which we call Gph.

Given a graph Γ with vertex set V, we can form a group $A\Gamma$, the *right angled Artin group* (raag) associated to Γ, defined as

$$A\Gamma \triangleq \langle v \in V \mid [v_1, v_2] = 1 \text{ whenever } (v_1, v_2) \in \Gamma \rangle.$$

For example, if K_n is a complete graph on n vertices then $AK_n \cong \mathbb{Z}^n$. If Δ_n is a discrete graph on n vertices then $A\Delta_n \cong \mathbb{F}_n$ is a free group on n generators. If $\square$ is the graph with four vertices a, b, c, and d and four edges (a, b), (b, c), (c, d), and (d, a), then $A\square \cong \langle a, c\rangle \times \langle b, d\rangle \cong \mathbb{F}_2 \times \mathbb{F}_2$. In this sense, raags allow us to *interpolate* between free and free abelian groups.

Raags are of particular interest to geometric group theorists because of their connections to the fundamental groups of closed hyperbolic 3-manifolds [39] and to the mapping class groups of hyperbolic surfaces [26]. Moreover, raags were instrumental in the resolution of the virtual Haken conjecture [3] due to their close connection with the CAT(0) geometry of cube complexes. See [8] for an overview.

Importantly, the combinatorial structure of Γ is closely related to the algebraic structure of $A\Gamma$, with useful information flow in both directions. For instance, the cohomology of $A\Gamma$ is the *exterior face algebra* of Γ [37], $A\Gamma$ factors as a direct product if and only if Γ factors as a join of two graphs [38], and we can compute the Bieri–Neumann–Strebel invariant $\Sigma^1(A\Gamma)$ from just information in Γ [30]. This correspondence can be pushed remarkably far, and recently it was shown that *expander graphs* (which are really sequences of graphs) can be recognized from the cohomology of their raags [18]! For more information about the close connection between the combinatorics of Γ and the algebra of $A\Gamma$, see [18; 28].

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With this context, it is natural to ask whether the combinatorics of graph homomorphisms are *also* closely connected to the algebra of group homomorphisms between raags. For a particular example, one might ask if there is a purely algebraic way to recognize when a group homomorphism between raags is $A\varphi$ for some homomorphism φ of their underlying graphs.

Our main result shows that the answer is *yes* in a very strong sense. We prove that the raag functor A is an equivalence between the category of graphs Gph and the category of groups equipped with a coalgebra structure (a kind of *descent data*) that we will describe shortly. As corollaries, we obtain a new way of recognizing the raags amongst the groups, and the graph homomorphisms amongst the group homomorphisms. This moreover gives a new algorithm for recovering the underlying graph of a raag from nothing but its isomorphism type.

Crucial for the proof of this theorem is the fact that $A\colon \mathsf{Gph} \to \mathsf{Grp}$ has a right adjoint, the *commutation graph* functor $C\colon \mathsf{Grp} \to \mathsf{Gph}$, which sends a group G to the graph whose vertices are elements of G and where $(g_1, g_2) \in CG \iff [g_1, g_2] = 1$. This is surely well known to experts[1] but is not often mentioned in the literature. This is likely because of the common convention that graphs have no self loops, whereas the adjunction requires us to work with graphs with a self loop at each vertex. Of course, this does not appreciably change the combinatorics, and we feel it is a small price to pay for the categorical clarity this adjunction provides.

Unsurprisingly, the commutation graph and related constructions have already been of interest to combinatorialists for many years [4; 6; 12; 16; 21], and the complement of the commutation graph was even the subject of a (now proven) conjecture of Erdős [34].

With the commutation graph functor C defined, we can state our main result:

Theorem *The right angled Artin group functor $A\colon \mathsf{Gph} \to \mathsf{Grp}$ is comonadic.*

That is, A is an equivalence of categories between Gph and the category Grp_{AC} of groups equipped with an AC-coalgebra structure, and group homomorphisms that are moreover AC-cohomomorphisms.

The group ACG is the group freely generated by symbols $[g]$ for each $g \in G$, with relations saying $[g][h] = [h][g]$ in ACG if and only if $gh = hg$ in G. Write $\epsilon_G\colon ACG \to G$ for the homomorphism sending each $[g]$ to g. Additionally, write $\delta\colon ACG \to AC(ACG)$ for the homomorphism sending each $[g]$ to $[[g]]$.

Now merely unwinding the category theoretic definitions gives the following corollary:

Corollary (main corollary) *An abstract group G is isomorphic to a raag if and only if it admits a group homomorphism $\mathfrak{g}\colon G \to ACG$ such that the following two diagrams commute:*

$$\begin{tikzcd} G \arrow[r, "\mathfrak{g}"] \arrow[dr, "1_G"'] & ACG \arrow[d, "\epsilon_G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathfrak{g}"] \arrow[d, "\mathfrak{g}"'] & ACG \arrow[d, "AC\mathfrak{g}"] \\ ACG \arrow[r, "\delta"'] & AC(ACG) \end{tikzcd}$$

[1] It's implicit in the "universal property of raags" given in [27], for instance, and is stated as such in [38].

Moreover, a group homomorphism $f : G \to H$ between raags is $A\varphi$ for some graph homomorphism φ if and only if it respects these structure maps in the sense that

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\mathfrak{g}"] & H \arrow[d, "\mathfrak{h}"] \\ ACG \arrow[r, "ACf"'] & ACH \end{tikzcd}$$

commutes.

Remark In particular, there is a purely algebraic way to recognize the raags amongst the groups and the image of the graph homomorphisms amongst the group homomorphisms between raags.

This additionally gives us a new way to recover Γ from the abstract isomorphism class of $A\Gamma$, and shows it is decidable (even efficient!) to check whether any particular group homomorphism between raags came from a graph homomorphism.

Our proof uses some category theory that might not be familiar to all readers, so in Section 3 we will briefly review the machinery of *comonadic descent*, which is the main technical tool for the proof (which is the subject of Section 4). First, though, in Section 2 we give an example to show that category theory is not needed in order to apply our results. This section might also be of interest to those learning category theory looking for toy examples of comonadic descent, since it is usually applied in more complicated situations than this.[2] Lastly, in Section 5 we discuss the algorithmic consequences of the main result.

Throughout this paper, we make the notational convention that graph theoretic concepts are written with Greek letters and group theoretic concepts with roman letters. The coalgebraic structure maps are written in Fraktur font.

2 An simple example

It's important to note that applying this result requires no knowledge of the deep category theory used in its proof. Let's begin with a simple example of how the result can be used to detect whether a group homomorphism came from a graph homomorphism or not. This is meant only to illustrate the technique, rather than to solve a novel problem. In Section 5 we give a more interesting example.

Let $\Gamma = \{v\}$ and $\Delta = \{w\}$ be two one-vertex graphs. Then $A\Gamma = \langle v \rangle$ and $A\Delta = \langle w \rangle$, and we want to detect when a homomorphism between these groups came from a homomorphism of their underlying graphs.

Recall that CG, the commutation graph of G, has a vertex $[g]$ for each $g \in G$, with an edge relating $[g]$ and $[h]$ exactly when g and h commute in G. So $C\langle v \rangle$ is a complete graph on $\mathbb{Z}$-many vertices labeled by $[v^n]$.

[2] Indeed, this is how the author came upon this result.

Then the group ACG is freely generated by the symbols $[g]$, for $g \in G$, subject to relations saying $[g][h] = [h][g]$ in ACG if and only if $gh = hg$ in G. So $AC\langle v\rangle$ is the free abelian group with generators $[v^n]$.

It's not hard to see that the map $\mathfrak{v}\colon \langle v\rangle \to AC\langle v\rangle$ given by $v \mapsto [v^1]$ satisfies the axioms from the main corollary. The existence of such a $\mathfrak{v}$ tells us that $\langle v\rangle$ must be a raag, which it is.[3]

Let's first look at a map that *does* come from a graph homomorphism, for instance $f\colon \langle v\rangle \to \langle w\rangle$ given by $fv = w$.

The corollary says to consult the square

$$\begin{tikzcd}
\langle v\rangle \arrow[r, "v\mapsto w"] \arrow[d, "v\mapsto[v^1]"'] & \langle w\rangle \arrow[d, "w\mapsto[w^1]"'] \\
\langle [v^n] \mid [v^n][v^m] = [v^m][v^n]\rangle \arrow[r, "{[v^n]\mapsto[w^n]}"] & \langle [w^n] \mid [w^n][w^m] = [w^m][w^n]\rangle
\end{tikzcd}$$

and since this is quickly seen to commute, we learn that f is of the form $A\varphi$ for some graph homomorphism (as indeed it is).

Next, let's look at a map which *doesn't* come from a graph homomorphism, like $f\colon \langle v\rangle \to \langle w\rangle$ given by $fv = w^2$.

Now our square is

$$\begin{tikzcd}
\langle v\rangle \arrow[r, "v\mapsto w^2"] \arrow[d, "v\mapsto[v^1]"'] & \langle w\rangle \arrow[d, "w\mapsto[w^1]"'] \\
\langle [v^n] \mid [v^n][v^m] = [v^m][v^n]\rangle \arrow[r, "{[v^n]\mapsto[w^{2n}]}"] & \langle [w^n] \mid [w^n][w^m] = [w^m][w^n]\rangle
\end{tikzcd}$$

which does *not* commute (even though it seems to at first glance). Indeed, if we chase the image of v around the top right of the square, then we see

$$v \mapsto w^2 \mapsto [w^1]^2.$$

If instead we chase around the lower left of the square, we get

$$v \mapsto [v^1] \mapsto [w^2].$$

Since $[w^1]^2 \neq [w^2]$ in this group (recall $AC\langle w\rangle$ is freely generated by the symbols $[w^n]$), we have successfully detected that f did *not* come from a graph homomorphism!

Importantly, this same approach works even if we merely know the coalgebra structures on G and H. Thus we don't need to know their underlying graphs to detect the graph homomorphisms![4]

[3] More generally, if we know Γ, then the map $A\Gamma \to AC(A\Gamma)$ sending each generator γ to $[\gamma]$ will always satisfy the axioms.

[4] Though we will see later that the coalgebra structure actually lets us recover the underlying graphs as well.

As a last aside, let's mention what the structure map $\mathfrak{g}\colon G \to ACG$ does. Elements of ACG are formal words in the elements of G. Then, intuitively, $\mathfrak{g}(g) = [\gamma_1][\gamma_2]\cdots[\gamma_k]$ decomposes g as a formal product of the vertices making up g. This means that we can recover the vertices of Γ as those g such that $\mathfrak{g}(g) = [g]$ is a word of length 1, as we prove in Section 5

3 A brief review of comonadic descent

Recall that an *adjunction* $(L\colon \mathcal{C} \to \mathcal{D}) \dashv (R\colon \mathcal{D} \to \mathcal{C})$ is a pair of (covariant) functors equipped with a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(LC, D) \cong \mathrm{Hom}_{\mathcal{C}}(C, RD).$$

Of particular interest for us is the adjunction $A \dashv C$ specifying the universal property of raags.

Recall moreover that a *comonad* $W\colon \mathcal{D} \to \mathcal{D}$ is a functor equipped with natural transformations $\epsilon\colon W \Rightarrow 1_{\mathcal{D}}$ and $\delta\colon W \Rightarrow WW$ such that the following diagrams of natural transformations commute:

$$\begin{tikzcd}
 & W \arrow[dl, Rightarrow, "1_W"'] \arrow[d, Rightarrow, "\delta"] \arrow[dr, Rightarrow, "1_W"] & \\
W & WW \arrow[l, Rightarrow, "1_W\cdot\epsilon"] \arrow[r, Rightarrow, "\epsilon\cdot 1_W"'] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & WW \arrow[d, Rightarrow, "1_W\cdot\delta"] \\
WW \arrow[r, Rightarrow, "\delta\cdot 1_W"'] & WWW
\end{tikzcd}$$

Dually, a *monad* is a functor $M\colon \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\eta\colon 1_{\mathcal{C}} \Rightarrow M$ and $\mu\colon MM \Rightarrow M$ satisfying diagrams opposite those above. A precise definition can be found in Chapter 4 of [10].

Every adjunction $L \dashv R$ gives rise to a monad RL and a comonad LR. In particular, the raag adjunction gives us a comonad $AC\colon \mathsf{Grp} \to \mathsf{Grp}$, which is our primary object of study.

Monads and comonads find application in settings as varied as algebraic geometry and number theory [23; 10; 11], universal algebra [10; 2; 9; 5; 24], probability theory [20; 15; 29], and computer science [33; 35; 19; 36]. Relevant for us is the theory of *(co)monadic descent*, which comes from gluing conditions in algebraic geometry, and is reviewed in this context in Section 4.7 of [10].

Given a comonad W, a *W-coalgebra* is an object $G \in \mathcal{D}$ equipped with an arrow $\mathfrak{g}\colon G \to WG$ such that the diagrams in Figure 1 commute. A *W-cohomomorphism* between coalgebras $(G, \mathfrak{g})$ and $(H, \mathfrak{h})$ is an arrow $f\colon G \to H$ in $\mathcal{D}$ compatible with $\mathfrak{g}$ and $\mathfrak{h}$, in the sense that Figure 2 commutes. When W is clear

$$\begin{tikzcd}
G \arrow[r, "\mathfrak{g}"] \arrow[dr, "1_G"'] & WG \arrow[d, "\epsilon_G"] \\
 & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\mathfrak{g}"] \arrow[d, "\mathfrak{g}"'] & WG \arrow[d, "W\mathfrak{g}"] \\
WG \arrow[r, "\delta"'] & WWG
\end{tikzcd}$$

Figure 1: The defining diagrams for a coalgebra.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\mathfrak{g}"] & H \arrow[d, "\mathfrak{h}"] \\ WG \arrow[r, "Wf"'] & WH \end{tikzcd}$$

Figure 2: The defining diagram for a cohomomorphism.

from context, we simply call these coalgebras and cohomomorphisms, and they assemble into a category $\mathcal{D}_W$ which admits a faithful functor $U : \mathcal{D}_W \to \mathcal{D}$ that simply forgets the structure map $\mathfrak{g}$.

Abstract nonsense shows that for any adjunction $L \dashv R$, the essential image of L lands inside the category of coalgebras $\mathcal{D}_{LR}$. That is, every object $LX \in \mathcal{D}$ is an LR-coalgebra, where the structure map is given by $L\eta_X : LX \to LRLX$, and every $L\varphi$ is an LR-cohomomorphism.

In our special case, $\eta : \Gamma \to CA\Gamma$ is the map sending each $v \in \Gamma$ to $v^1 \in CA\Gamma$. Then the above says that the functor $A : \mathsf{Gph} \to \mathsf{Grp}$ factors through the category of coalgebras Grp_{AC} as follows:

$$\mathsf{Gph} \xrightarrow{A} \mathsf{Grp}_{AC} \xrightarrow{U} \mathsf{Grp}, \quad \Gamma \mapsto (A\Gamma, A\eta) \mapsto A\Gamma$$

We will show that A is actually an equivalence of categories $\mathsf{Gph} \simeq \mathsf{Grp}_{AC}$. This tells us that a group is of the form $A\Gamma$ if and only if it's a coalgebra, and a group homomorphism is of the form $A\varphi$ if and only if it's a cohomomorphism!

The main tool for proving this equivalence is Beck's famed *(co)monadicity theorem*,[5] which says

Theorem (Beck, 1968) *To show that a left adjoint $(L : \mathcal{C} \to \mathcal{D}) \dashv (R : \mathcal{D} \to \mathcal{C})$ witnesses L as an equivalence of categories*[6] *$\mathcal{C} \simeq \mathcal{D}_{LR}$, it suffices to show*

(1) *L reflects isomorphisms (that is, whenever $L\varphi : L\Gamma \cong L\Delta$ is an isomorphism in $\mathcal{D}$, then φ must have already been an isomorphism in $\mathcal{C}$),*

(2) *$\mathcal{C}$ has, and L preserves, equalizers of coreflexive pairs.*[7]

This gives us our outline for proving the main theorem:

Theorem (main theorem) *The right angled Artin group functor $A : \mathsf{Gph} \to \mathsf{Grp}$ restricts to an equivalence of categories $A : \mathsf{Gph} \simeq \mathsf{Grp}_{AC}$ between the category of graphs and the full subcategory of groups equipped with an AC-coalgebra structure.*

[5] The original manuscript due to Beck was unpublished, but widely distributed. A scan is available at [7], but this is also proven as Theorem 4.4.4 in [10]. Both of these prove the statement for *monads*, which is then dualized to give the *co*monadicity theorem we use.

[6] Such an adjunction $L \dashv R$ is called *comonadic*.

[7] We will recall the definition of both equalizers and coreflexive pairs in Section 4.

Proof By Beck's comonadicity theorem, it suffices to check the two conditions above.

Condition (1) is a classical result due to Droms [17], so it remains to check (2). It's well known that Gph is complete,[8] and thus has all equalizers.

In the next section we'll recall the definition of a coreflexive pair, and show that A really does preserve their equalizers. This will complete the proof. □

4 The raag functor preserves equalizers of coreflective pairs

A *coreflexive pair* is a pair of arrows with a common retract. That is, a diagram

$$\begin{tikzcd} \Gamma \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \Delta \arrow[l, "\rho"] \end{tikzcd}$$

where

$$\rho\alpha = 1_\Gamma = \rho\beta.$$

An *equalizer* of a pair of arrows $\alpha, \beta\colon X \rightrightarrows Y$ is an arrow $\iota\colon E \to X$ such that every arrow $\varphi\colon Z \to X$ with $\alpha\varphi = \beta\varphi$ factors uniquely through ι. In Gph, this is (the inclusion of the) induced subgraph of X on the vertices x with $\alpha x = \beta x$. In Grp, this is the (inclusion of the) subgroup of X of all x's with $\alpha x = \beta x$.

Now, we want to show that if Θ is the equalizer of α and β, as computed in Gph, then $A\Theta$ should still be the equalizer of $A\alpha$ and $A\beta$, as computed in Grp. For ease of notation, we will confuse α and β with $A\alpha$ and $A\beta$, since $(A\alpha)(v_1^{n_1} v_2^{n_2} \cdots v_k^{n_k}) = (\alpha v_1)^{n_1} (\alpha v_2)^{n_2} \cdots (\alpha v_k)^{n_k}$.

Note that Θ is the full subgraph of Γ on the vertices where $\alpha v = \beta v$. So then $A\Theta = \langle v \mid \alpha v = \beta v \rangle \leq A\Gamma$. If instead we compute the equalizer of $A\alpha$ and $A\beta$ in Grp, we get $G = \{g \mid \alpha g = \beta g\} \leq A\Gamma$.

So showing that $A\Theta = G$ amounts to showing that, provided α and β admit a common retract ρ, each g with $\alpha g = \beta g$ is a word in those vertices v with $\alpha v = \beta v$.

Theorem 1 *The right angled Artin group functor A preserves equalizers of coreflexive pairs.*

Proof Since ρ is a graph homomorphism, we see that v and w are Γ-related if and only if αv and αw (equivalently βv and βw, equivalently αv and βw) are Δ-related. Thus v and w commute in $A\Gamma$ if and only if their images under α and β commute in $A\Delta$.

In Theorem 3.9 of her thesis [22], Green proves that elements of $A\Gamma$ have a normal form as words in the vertices of Γ.[9] Following the exposition of Koberda [27] and others, we call a word $w \in A\Gamma$ *central* if the letters in w pairwise commute. This happens if and only if the letters in w form a clique in Γ. We say

[8] One quick way to see this is to note that it's *topologically concrete* in the sense of [1].

[9] In fact, she proves something slightly more general.

that w is in *central form* if it is a product of central words $w = w_1 w_2 \cdots w_k$. If we stipulate that we are "left greedy" in the sense that no letter in w_{i+1} commutes with each letter of w_i (so that we first make w_1 as long as possible, then make w_2 as long as possible, and so on), then the central form is unique up to commuting the letters in each w_i. See also Section 3.3 of [14] for a summary.

Now suppose that $\alpha g = \beta g$. Fix such a central form $g = w_0 w_1 \dots w_k$, and look at

$$(\alpha w_0)(\alpha w_1)\dots(\alpha w_k) = (\beta w_0)(\beta w_1)\dots(\beta w_k).$$

These representations of $\alpha g = \beta g$ are both minimal length, as we could hit a shorter representation with ρ in order to get a shorter representation for g. Then uniqueness of the central form says that each αw_i and βw_i are equal up to permuting the letters in each.

We restrict attention to each $w_i = \gamma_1^{n_1}\gamma_2^{n_2}\dots\gamma_k^{n_k}$ separately, say

$$(\alpha\gamma_1^{n_1})(\alpha\gamma_2^{n_2})\dots(\alpha\gamma_k^{n_k}) = \delta_1^{n_1}\delta_2^{n_2}\dots\delta_k^{n_k} = (\beta\gamma_1^{n_1})(\beta\gamma_2^{n_2})\dots(\beta\gamma_k^{n_k}).$$

If we can show that actually $\alpha\gamma_i = \beta\gamma_i$ for each i, then we'll be done.

But α and β give injections from $\{\gamma_1, \dots, \gamma_k\}$ to $\{\delta_1, \dots, \delta_k\}$, which are in fact bijections since we're dealing with finite sets of the same cardinality.

Moreover, by assumption ρ provides an inverse for α *and* for β!

Then α and β must be the same map on this set, and in particular each γ_i satisfies $\alpha\gamma_i = \beta\gamma_i$, as desired. □

5 Can we really compute these?

It is well known that the problem "is a finitely presented group G isomorphic to a raag" is undecidable. Indeed, being isomorphic to a raag is a *Markov property* in the sense of Definition 3.1 in [31], so Theorem 3.3 in the same paper guarantees this problem is undecidable.

Let's work with the next best thing, then, and suppose we're given a finite presentation of a group G and a promise that G *is* a raag (though we are not given its underlying graph). How much can we learn about the combinatorics of its underlying graph from just G?

First, we must find an AC-coalgebra structure on G — that is, a group homomorphism $\mathfrak{g}\colon G \to ACG$ satisfying the conditions from Figure 1. Since ACG is a raag, it has solvable word problem, so we can enumerate all homomorphisms $G \to ACG$ and check if they satisfy the axioms. We will eventually find such a $\mathfrak{g}$ since we were promised that G is abstractly isomorphic to a raag, so this algorithm terminates.

Once we know the coalgebra structures on G and H, we can already efficiently check whether a group homomorphism $f: G \to H$ came from a graph homomorphism. Indeed, the existence of an AC-coalgebra

structure on G and H is enough to guarantee that G and H are raags, even without having direct access to the underlying graphs[10] of G and H.

Theorem 2 *Given a homomorphism $f: G \to H$ between finitely presented groups*[11] *where $(G, \mathfrak{g})$ and $(H, \mathfrak{h})$ are moreover AC-coalgebras, then there is an algorithm deciding whether f is $A\varphi$ for φ a graph homomorphism of the graphs presenting G and H.*

Proof By the equivalence $\mathsf{Gph} \simeq \mathsf{Grp}_{AC}$, this amounts to checking if f is a cohomomorphism — that is, whether the square

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\mathfrak{g}"'] & H \arrow[d, "\mathfrak{h}"] \\ ACG \arrow[r, "ACf"'] & ACH \end{tikzcd}$$

commutes. Of course, we can check this on the (finitely many) generators of G, and the claim now follows from the fact that ACH is a raag,[12] and thus has solvable word problem [14]. □

Corollary 1 *Given* **any** *finite presentation of G and a promise that G really is a raag, there is an algorithm to recover a graph Γ and an isomorphism $G \cong A\Gamma$.*

Proof As discussed earlier in this section, given a finite presentation of G and a promise that G is a raag, we can algorithmically find a coalgebra structure $\mathfrak{g}: G \to ACG$.

Now, we know that the vertices of Γ are in bijection with graph homomorphisms from the one-vertex graph 1 to Γ. By the equivalence $\mathsf{Gph} \simeq \mathsf{Grp}_{AC}$, this amounts to cohomomorphisms $\mathbb{Z} \to G$, which one can explicitly calculate to be those elements $g \in G$ such that $\mathfrak{g}(g) = [g]$.

Since we know that the number of vertices of Γ is equal to the rank of the abelianization G^{ab}, we can keep checking elements of G to see if $\mathfrak{g}(g) = [g]$. This algorithm terminates because once we've found $\mathrm{rk}(G^{\mathrm{ab}})$-many such elements, we must have found all of them.

Finally, we see that the following conditions are equivalent:

(1) Two elements g_1 and g_2 represent adjacent elements in Γ.

(2) g_1 and g_2 commute in G.

(3) $[g_1]$ and $[g_2]$ commute in ACG.

(4) There is a cohomomorphism from $A(\bullet\!-\!\bullet)$ to G sending the two vertices to g_1 and g_2. □

[10] Though, as we show in Corollary 1, we can recover the underlying graph from the coalgebra structure, and obviously we can find a coalgebra structure given the underlying graph (the unique homomorphism sending each generator $\gamma \in A\Gamma$ to $[\gamma] \in ACA\Gamma$ works). So the data of the coalgebra structure is equivalent to the data of the underlying graph!

[11] Recall that these presentations may have nothing to do with the underlying graphs.

[12] We have to be a bit careful, since CH is infinite, so that ACH is not finitely generated. However, the images of each generator of G will land in a finite subgraph of CH, so we can do our computation inside the raag associated to that finite subgraph.

Lastly, we give a more interesting example of this machinery in action:

Say we're given the following group presentations[13] and we're promised both G and H are raags:

$$G = \langle x_1, x_2, x_3, x_4 \mid x_1 x_4^{-1} x_1^{-1} x_4 x_3, x_3 x_2^2 x_3^{-1} x_2^{-3}, x_2^2 x_3^{-1} x_2^{-1} x_3^2 x_2 \rangle$$
$$H = \langle y_1, y_2, y_3, y_4, y_5 \mid y_4^{-1} y_2^{-1} y_4 y_2 y_4^{-1}, y_3 y_1 y_4^{-1} y_3^{-1} y_1^{-1}, y_5 y_3 y_2 y_5^{-1} y_3^{-1}, y_2^{-1} y_4^{-1} y_2 y_4 y_2^{-1} \rangle$$

Then the algorithm described at the start of this section will terminate and provide the AC-coalgebra structures, given on generators by

$$\mathfrak{g}(x_1) = [x_1], \quad \mathfrak{g}(x_2) = [\,], \quad \mathfrak{g}(x_3) = [\,], \quad \mathfrak{g}(x_4) = [x_4 x_1^{-1}][x_1],$$
$$\mathfrak{h}(y_1) = [y_1], \quad \mathfrak{h}(y_2) = [\,], \quad \mathfrak{h}(y_3) = [y_3], \quad \mathfrak{h}(y_4) = [\,], \quad \mathfrak{h}(y_5) = [y_5],$$

where we write $[\,]$ for the identity elements of ACG and ACH.

Now we consider two group homomorphisms, $\varphi, \psi \colon G \to H$, one of which comes from a graph homomorphism, and one doesn't. These are given on generators by

$$\varphi(x_1) = y_1, \quad \varphi(x_2) = y_2, \quad \varphi(x_3) = y_2 y_4, \quad \varphi(x_4) = y_3 y_1,$$
$$\psi(x_1) = y_1, \quad \psi(x_2) = y_2, \quad \psi(x_3) = y_2 y_4, \quad \psi(x_4) = y_3.$$

These homomorphisms are superficially very similar, so it's not obvious which comes from a graph homomorphism! Especially since we don't know what the underlying graphs should *be*!

But if we chase the generators around the squares

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\mathfrak{g}"'] & H \arrow[d, "\mathfrak{h}"] \\ ACG \arrow[r, "AC\varphi"'] & ACH \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\mathfrak{g}"'] & H \arrow[d, "\mathfrak{h}"] \\ ACG \arrow[r, "AC\psi"'] & ACH \end{tikzcd}$$

we find they both commute for x_1, x_2, and x_3. The case of x_4 is instructive. Indeed, for φ, we want to check if the following square commutes:

$$\begin{tikzcd} G \arrow[r, "x_4 \mapsto y_3 y_1"] \arrow[d, dashed, "x_4 \mapsto [x_4 x_1^{-1}][x_1]"'] & H \arrow[d, dashed, "y_3 y_1 \mapsto [y_3][y_1]"] \\ ACG \arrow[r, "{[x_4 x_1^{-1}][x_1] \mapsto [y_3 y_1 y_1^{-1}][y_1]}"'] & ACH \end{tikzcd}$$

Since $[y_3 y_1 y_1^{-1}] = [y_3]$, this square *does* commute, and we learn that φ is of the form Af for some graph homomorphism f on the underlying graphs of G and H (which we still don't know!)

[13] These presentations came from taking a "typical" presentation of the simplest nontrivial raags we could think of (G is the raag on $\bullet\!-\!\bullet$ and H is on $\bullet\!-\!\bullet\!-\!\bullet$), adding variables for the commutators, and then adding interesting relations forcing those commutator variables to be trivial using the theorems in [32]. Then, in G, we used a and ab as generators rather than a and b to show what happens when one of the vertices isn't a generator in the given presentation. Lastly we obfuscated and shuffled the variable names.

Now if we consider ψ, we want to check the following square:

$$\begin{tikzcd} G \arrow[r, "x_4 \mapsto y_3"] \arrow[d, "x_4 \mapsto [x_4 x_1^{-1}][x_1]"'] & H \arrow[d, "y_3 \mapsto [y_3]"] \\ ACG \arrow[r, "{[x_4 x_1^{-1}][x_1] \mapsto [y_3 y_1^{-1}][y_1]}"'] & ACH \end{tikzcd}$$

As in the example from Section 2, since $[y_3 y_1^{-1}][y_1] \neq [y_3]$, this square *doesn't* commute, which tells us that ψ was not induced from a graph homomorphism!

From here it's a natural question to try and find the underlying graphs, and Corollary 1 tells us how!

We know that the abelianization of G has rank 2. This tells us that we're looking for two elements $a, b \in G$ such that $\mathfrak{g}(a) = [a]$ and $\mathfrak{g}(b) = [b]$. An exhaustive search quickly finds $a = x_1$ and $b = x_4 x_1^{-1}$ do the job. Now since a and b commute in G, they must be joined by an edge, so that

$$G = A(\bullet\!-\!\bullet).$$

Similarly, the abelianization of H has rank 3, so we want three elements $p, q, r \in H$ with $\mathfrak{h}(p) = [p]$, $\mathfrak{h}(q) = [q]$, and $\mathfrak{h}(r) = [r]$. Again, we find $p = y_1$, $q = y_3$, and $r = y_5$ work, and we can check that p and q commute, q and r commute, but p and r don't commute, so that

$$H = A(\bullet\!-\!\bullet\!-\!\bullet).$$

From here, we can see that φ is just the map extending $\varphi(a) = p$ and $\varphi(b) = q$ (which, as predicted, *does* come from a graph homomorphism) while ψ is the map extending $\psi(a) = p$ and $\psi(b) = qp^{-1}$ (which, as predicted does *not* come from a graph homomorphism).

6 Conclusion

It has been well known for some time now that the combinatorics of a graph Γ are reflected in the algebra of its raag $A\Gamma$, but the question of how the combinatorics of graph homomorphisms relates to group homomorphisms between raags remains fertile ground. Here we've shown that the connection remains strong, by showing that the category of (reflexive) graphs embeds faithfully as an explicit subcategory of the category of groups.

More speculatively, while this paper focused on the comonad $AC : \mathsf{Grp} \to \mathsf{Grp}$, we suspect there is a future role to be played by the monad $CA\colon \mathsf{Gph} \to \mathsf{Gph}$. Indeed, Kim and Koberda conjecture in [25] that embeddings $A\Gamma \to A\Delta$ exist exactly when Γ embeds into a graph Δ^e, which they call the *extension graph*. This graph is closely related to the monad graph $CA\Delta$ (indeed, it's the full subgraph of $CA\Delta$ on the conjugates of generators), as we might expect since maps $A\Gamma \to A\Delta$ are in natural bijection with maps $\Gamma \to CA\Delta$.

While the extension graph conjecture is now known to be false in general [13], it is true for many classes of graphs. In some sense this is likely "because of" the close connection of the extension graph with the monad graph. It would be interesting to see if category theoretic techniques can be brought to bear on a new version of this conjecture, by finding a combinatorial condition which picks out those embeddings $\Gamma \to CA\Delta$ which transpose to an embedding of raags.

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References

[1] **J Adámek**, **H Herrlich**, **G E Strecker**, *Abstract and concrete categories: the joy of cats*, Wiley, New York (1990) MR

[2] **J Adámek**, **J Rosický**, **E M Vitale**, *Algebraic theories: a categorical introduction to general algebra*, Cambridge Tracts in Math. 184, Cambridge Univ. Press (2011) MR

[3] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR

[4] **V Arvind**, **P J Cameron**, *Recognizing the commuting graph of a finite group*, preprint (2022) arXiv 2206.01059

[5] **S Awodey**, **J Hughes**, *The coalgebraic dual of Birkhoff's variety theorem*, technical report 109, Carnegie Mellon University (2000) Available at `https://www.cmu.edu/dietrich/philosophy/docs/tech-reports/109_Awodey.pdf`

[6] **B Baumeister**, **A Stein**, *Commuting graphs of odd prime order elements in simple groups*, preprint (2009) arXiv 0908.2583

[7] **J Beck**, *Monadicity theorem*, unpublished notes (1968) Available at `https://ncatlab.org/nlab/files/Beck_MonadicityTheorem.pdf`

[8] **M Bestvina**, *Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision*, Bull. Amer. Math. Soc. 51 (2014) 53–70 MR

[9] **M Bojańczyk**, *Recognisable languages over monads*, from "Developments in language theory" (I Potapov, editor), Lecture Notes in Comput. Sci. 9168, Springer (2015) 1–13 MR

[10] **F Borceux**, *Handbook of categorical algebra, II: Categories and structures*, Encyclopedia of Mathematics and its Applications 51, Cambridge Univ. Press (1994) MR

[11] **F Borceux**, **S Caenepeel**, **G Janelidze**, *Monadic approach to Galois descent and cohomology*, Theory Appl. Categ. 23 (2010) 92–112 MR

[12] **P J Cameron**, *Graphs defined on groups*, Int. J. Group Theory 11 (2022) 53–107 MR

[13] **M Casals-Ruiz**, **A Duncan**, **I Kazachkov**, *Embedddings between partially commutative groups: two counterexamples*, J. Algebra 390 (2013) 87–99 MR

[14] **R Charney**, *An introduction to right-angled Artin groups*, Geom. Dedicata 125 (2007) 141–158 MR

[15] **J Culbertson**, **K Sturtz**, *A categorical foundation for Bayesian probability*, Appl. Categ. Structures 22 (2014) 647–662 MR

[16] **G Dolinar**, **B Kuzma**, **P Oblak**, *On maximal distances in a commuting graph*, Electron. J. Linear Algebra 23 (2012) 243–256 MR

[17] **C Droms**, *Isomorphisms of graph groups*, Proc. Amer. Math. Soc. 100 (1987) 407–408 MR

[18] **R Flores**, **D Kahrobaei**, **T Koberda**, *Expanders and right-angled Artin groups*, J. Topol. Anal. 16 (2024) 155–179 MR

[19] **N Ghani**, **C Lüth**, **F de Marchi**, **J Power**, *Algebras, coalgebras, monads and comonads*, Electron. Notes Theor. Comput. Sci. 44 (2001) 128–145

[20] **M Giry**, *A categorical approach to probability theory*, from "Categorical aspects of topology and analysis" (B Banaschewski, editor), Lecture Notes in Math. 915, Springer (1982) 68–85 MR

[21] **M Giudici**, **A Pope**, *The diameters of commuting graphs of linear groups and matrix rings over the integers modulo m*, Australas. J. Combin. 48 (2010) 221–230 MR

[22] **E R Green**, *Graph products of groups*, PhD thesis, University of Leeds (1990) Available at `https://etheses.whiterose.ac.uk/id/eprint/236`

[23] **A Grothendieck**, *Technique de descente et théorèmes d'existence en géométrie algébrique, I: Généralités. Descente par morphismes fidèlement plats*, from "Séminaire Bourbaki, 1958–1960", Benjamin, New York (1959) exposé 190 MR Reprinted in "Séminaire Bourbaki 5", Soc. Math. France, Paris (1995) 249–276

[24] **M Hyland**, **J Power**, *The category theoretic understanding of universal algebra: Lawvere theories and monads*, from "Computation, meaning, and logic: articles dedicated to Gordon Plotkin" (L Cardelli, M Fiore, G Winskel, editors), Electron. Notes Theor. Comput. Sci. 172, Elsevier, Amsterdam (2007) 437–458 MR

[25] **S-h Kim**, **T Koberda**, *Embedability between right-angled Artin groups*, Geom. Topol. 17 (2013) 493–530 MR

[26] **S-H Kim**, **T Koberda**, *The geometry of the curve graph of a right-angled Artin group*, Internat. J. Algebra Comput. 24 (2014) 121–169 MR

[27] **T Koberda**, *Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups*, Geom. Funct. Anal. 22 (2012) 1541–1590 MR

[28] **T Koberda**, *Geometry and combinatorics via right-angled Artin groups*, from "In the tradition of Thurston, II: Geometry and groups" (K Ohshika, A Papadopoulos, editors), Springer (2022) 475–518 MR

[29] **T Leinster**, *Codensity and the ultrafilter monad*, Theory Appl. Categ. 28 (2013) 332–370 MR

[30] **J Meier**, **L Van Wyk**, *The Bieri–Neumann–Strebel invariants for graph groups*, Proc. London Math. Soc. 71 (1995) 263–280 MR

[31] **C F Miller, III**, *Decision problems for groups—survey and reflections*, from "Algorithms and classification in combinatorial group theory" (G Baumslag, I Miller, C F, editors), Math. Sci. Res. Inst. Publ. 23, Springer (1992) 1–59 MR

[32] **C F Miller, III**, **P E Schupp**, *Some presentations of the trivial group*, from "Groups, languages and geometry" (R H Gilman, editor), Contemp. Math. 250, Amer. Math. Soc., Providence, RI (1999) 113–115 MR

[33] **E Moggi**, *Notions of computation and monads*, Inform. and Comput. 93 (1991) 55–92 MR

[34] **B H Neumann**, *A problem of Paul Erdős on groups*, J. Austral. Math. Soc. Ser. A 21 (1976) 467–472 MR

[35] **V de Paiva**, *Dialectica comonads*, from "9th Conference on Algebra and Coalgebra in Computer Science" (F Gadducci, A Silva, editors), Leibniz Int. Proc. Inform. 211, Schloss Dagstuhl, Wadern (2021) art. id. 3 MR

[36] **J J M M Rutten**, *Universal coalgebra: a theory of systems*, Theoret. Comput. Sci. 249 (2000) 3–80 MR

[37] **L Sabalka**, *On rigidity and the isomorphism problem for tree braid groups*, Groups Geom. Dyn. 3 (2009) 469–523 MR

[38] **H Servatius**, *Automorphisms of graph groups*, J. Algebra 126 (1989) 34–60 MR

[39] **H Servatius**, **C Droms**, **B Servatius**, *Surface subgroups of graph groups*, Proc. Amer. Math. Soc. 106 (1989) 573–578 MR

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