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Equivariant cohomology of projective spaces

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We compute the equivariant homology and cohomology of projective spaces with integer coefficients. More precisely, in the case of cyclic groups, we show that the cellular filtration of the projective space $P(k\rho)$ of lines inside copies of the regular representation yields a splitting of $H\underline{\mathbb{Z}} \wedge P(k\rho)_+$ as a wedge of suspensions of $H\underline{\mathbb{Z}}$. This is carried out both in the complex case, and also in the quaternionic case, and further, for the C_2-action on $\mathbb{C}P^n$ by complex conjugation. We also observe that these decompositions imply a degeneration of the slice tower in these cases. Finally, we describe the cohomology of the projective spaces when G is of prime power order, with explicit formulas for $\underline{\mathbb{Z}/p}$-coefficients. Letting $k = \infty$, this also describes the equivariant homology and cohomology of the classifying spaces of S^1 and S^3.

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1 Introduction

The purpose of this paper is to discuss new calculations for the equivariant cohomology of complex projective spaces. Given a complex representation V of a group G, one obtains a "linear" G-action on $P(V)$, the space of lines in V. The underlying space here is $\mathbb{C}P^{\dim(V)-1}$ whose homology computation is well known. The Borel-equivariant cohomology, which is the cohomology of the Borel construction, is easy to calculate as the space $P(V)$ has nonempty fixed points.

The equivariant cohomology used in this paper refers to an "ordinary" cohomology theory represented in the equivariant stable homotopy category [29, Chapter XIII, Section 4]. In this context, the word "ordinary" means that the equivariant homotopy groups of the representing spectrum *HM* are concentrated in degree 0. At this point, one notes that the equivariant stable homotopy category possesses additional structure which makes the equivariant homotopy groups part of a Mackey functor [29, Chapter IX, Section 4]. Conversely, given a Mackey functor $\underline{M}$, one has an associated "ordinary" equivariant cohomology theory represented by an Eilenberg–MacLane spectrum $H\underline{M}$ [14, Theorem 5.3], which is unique up to homotopy.

A Mackey functor $\underline{M}$ [9] comprises a pair of functors $(\underline{M}_*, \underline{M}^*)$ from the category $\mathcal{O}_G$ (of finite G-sets) to abelian groups taking disjoint unions to direct sums, such that $\underline{M}_*$ is covariant and $\underline{M}^*$ is contravariant, taking the same value on a given G-set S, denoted by $\underline{M}(S)$. These are required to be compatible in the sense of a double coset formula [32]. The covariant structure gives restriction maps

$$\mathrm{res}^H_K : \underline{M}(H) \to \underline{M}(K) \quad \text{for } K \subset H,$$

and the contravariant structure gives transfer maps

$$\mathrm{tr}^H_K : \underline{M}(K) \to \underline{M}(H) \quad \text{for } K \subset H.$$

Two important examples in the context of equivariant cohomology are the constant Mackey functor $\underline{\mathbb{Z}}$, and the Burnside ring Mackey functor $\underline{A}$. The Mackey functor $\underline{\mathbb{Z}}$ sends each G/H to $\mathbb{Z}$, with $\mathrm{res}^H_K = \mathrm{Id}$ and $\mathrm{tr}^H_K =$ multiplication by the index $[H:K]$. The Burnside ring Mackey functor $\underline{A}$ sends G/H to $A(H)$, the ring generated by isomorphism classes of finite H-sets. The restriction maps res^H_K for $\underline{A}$ are described as the restriction of the action of H to K, and the transfer maps tr^H_K are described by induction $S \mapsto H \times_K S$.

The category of Mackey functors is an abelian category. It also has a symmetric monoidal structure given by $\square$, whose unit object is the Burnside ring Mackey functor $\underline{A}$. The constant Mackey functor $\underline{\mathbb{Z}}$ is a commutative monoid, which implies that the cohomology groups with $\underline{\mathbb{Z}}$-coefficients possess a graded commutative ring structure. The spectrum $H\underline{A}$ is a homotopy commutative ring spectrum, and the commutative monoids give homotopy commutative $H\underline{A}$-algebras. However, if one tries to rigidify the construction of Eilenberg–MacLane spectra into a functor taking values in equivariant orthogonal spectra, there are obstructions coming from norm maps [18; 34].

Our focus in this paper is on computations of equivariant cohomology for G-spaces. A naive approach would be to break up the spaces into equivariant cells, and compute via cellular homology. An equivariant G-CW complex has cells of the form $G/H \times D^n$, which are attached along maps from $G/H \times S^{n-1}$ onto lower skeleta. Via this argument, one shows $H^n_G(X;\underline{\mathbb{Z}}) \cong H^n(X/G;\mathbb{Z})$. While working through concrete examples like the projective spaces $P(V)$, or for G-manifolds, we see that there is no systematic way of breaking these up into cells of the form $G/H \times D^n$ or identifying the space X/G. In these cases, the spaces may be more naturally built out of cells of the form $G \times_H D(V)$ [36], where V is a unitary H-representation, and $D(V)$ stands for the unit disk

$$D(V) = \{v \in V \mid \langle v, v\rangle \leq 1\}.$$

In the equivariant stable homotopy category, the representation spheres S^V (defined as the one-point compactification of V) are invertible in the sense that there is an S^{-V} such that $S^{-V} \wedge S^V \simeq S^0$. Consequently, the equivariant cohomology becomes $\mathrm{RO}(G)$-graded [29, Chapter XIII], and so, computations for the G-CW complexes above require the knowledge of $H^\alpha_G(G/H;\underline{M})$ for $\alpha \in \mathrm{RO}(G)$ and $H \subset G$. Such computations exist in the literature only for a handful of finite groups, namely, for $G = C_p$ for p prime [25], $G = C_{p_1 \cdots p_k}$ for distinct primes p_i [4; 6], and $G = C_{p^2}$ [39]. For restricted α belonging to certain sectors of $\mathrm{RO}(G)$ more computations are known [2; 18; 19; 24].

Equivariant cohomology of G-spaces has been studied for many of the groups above [3; 16; 23]. Equivariant cohomology of G-spaces has a rich algebraic structure in the sense that it is Mackey functor valued, and the complete understanding of the Mackey functor valued cohomology with $\mathrm{RO}(G)$-grading is often quite involved. Most explicit computations in the literature are done in the case where complete calculations for the $\mathrm{RO}(G)$-graded cohomology is known. Many of these occur in cases where the cohomology is a free module over $\pi_\star H\underline{\mathbb{Z}}$. There are various structural results which imply the conclusion that the cohomology is a free module [5; 28]. For the equivariant projective spaces, this is particularly relevant, and although a complete understanding of $\pi_\star^{C_n} H\underline{\mathbb{Z}}$ is still unknown, we are able to prove that the cohomology $P(V)$ is free when V is a direct sum of copies of the regular representation (see Theorems 4.7 and 4.10).

Theorem A *Let $G = C_n$. We have the following decompositions.*

(a) *Write $\phi_0 = 0$ and $\phi_i = \lambda^{-i}(1_{\mathbb{C}} + \lambda + \lambda^2 + \cdots + \lambda^{i-1})$ for $i > 0$. Then,*

$$H\underline{\mathbb{Z}} \wedge P(m\rho_{\mathbb{C}})_+ \simeq \bigvee_{i=0}^{nm-1} H\underline{\mathbb{Z}} \wedge S^{\phi_i}, \quad H\underline{\mathbb{Z}} \wedge B_G S^1_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_i}.$$

(b) *In the quaternionic case, we write $W_k = \lambda^{-k}\big(\sum_{i=0}^{k-1}(\lambda^i + \lambda^{-i})\big)$. Then*

$$H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(m\rho_{\mathbb{H}})_+ \simeq \bigvee_{i=0}^{mn-1} H\underline{\mathbb{Z}} \wedge S^{W_i}, \quad H\underline{\mathbb{Z}} \wedge B_G S^3_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{W_i}.$$

In the above expression, λ refers to the one-dimensional complex C_n-representation which sends a fixed generator g to $e^{\frac{2\pi i}{n}}$, and it's powers are taken with respect to the complex tensor product. The second implications above come from the identifications $B_G S^1 \simeq P(\mathcal{U}) \simeq \varinjlim_m P(m\rho)$, and $B_G S^3 \simeq P_{\mathbb{H}}(\mathbb{H} \otimes_{\mathbb{C}} \mathcal{U}) \simeq \varinjlim_m P_{\mathbb{H}}(m\rho_{\mathbb{H}})$, where $\mathcal{U}$ is a complete G-universe. We also carry out the computation for general G-representations V when the group G equals C_p (Theorem 4.3). One may view this as a simplification of the results in [25] in the case of the Mackey functor $\underline{\mathbb{Z}}$. For the group C_2, we consider $\mathbb{C}P^n_\tau$ where C_2-acts on $\mathbb{C}P^n$ by complex conjugation, and compute its equivariant homology (Theorem 4.8). In this context, one should note that results such as the above theorem are not expected for $\underline{A}$-coefficients once the group contains either C_{p^2} or $C_p \times C_p$ [11; 26, Remark 2.2].

As an application for the homology decomposition in Theorem A, we reprove a theorem of Caruso [7] stating that the cohomology operations expressible as a product of the even-degree Steenrod squares over $\mathbb{Z}/2$ do not occur as restriction of integer-degree C_2-equivariant cohomology operations. If we allow the more general $\mathrm{RO}(C_2)$-graded operations, there are those that restrict to Sq^i for every i [35]. For the group C_p, the same result holds for the products of the Steenrod powers P^i.

A careful analysis of the cellular filtration of the projective spaces $P(V)$ for $V = m\rho$ shows that the induced filtration on $\Sigma^2 H\underline{\mathbb{Z}} \wedge P(V)$ matches the slice filtration. The slice tower was defined as the equivariant analogue of the Postnikov tower using the localizing subcategory generated by $\{G/H_+ \wedge S^{k\rho_H} \mid k|H| \geq n\}$ instead of the spheres of the form $\{G/H_+ \wedge S^n\}$. The slice filtration played a critical role in the proof of the Kervaire invariant-one problem [18] and has been widely studied since. Usually, the slice tower is an involved computation even for the spectra $\Sigma^n H\underline{\mathbb{Z}}$. However, for the complex projective spaces and the quaternionic projective spaces, we discover that the slice tower for the $\underline{\mathbb{Z}}$-homology becomes amazingly simple. More precisely, we prove the following theorem in this regard (see Theorems 6.4 and 6.5).

Theorem B *Let $G = C_n$.*

(a) *The slice towers of $\Sigma^2 P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^2 P(m\rho)_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.*

(b) *The slice towers of $\Sigma^4 P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^4 P(m\rho_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.*

We proceed to describe the cohomology ring structure for the projective spaces. The basic approach towards the computation comes from [25], which deals with the case $G = C_p$. In this case, we observe that the generators are easy to describe, but the relations become complicated once the order of the group increases. For $G = C_n$, we show that $H^\star_G(P(V);\underline{\mathbb{Z}})$ is multiplicatively generated by classes α_{ϕ_d} for $d \mid n$, in degree $\sum_{i=-d}^{-1} \lambda^i$ (Proposition 7.3). However, the relations turn out to be difficult to write down in general, so we restrict our attention to prime powers n. In the process of figuring out the generators, we realize that there are exactly m relations ρ_j for $1 \le j \le m$ ($n = p^m$), of the form

$$u_{\lambda^{p^{j-1}} - \lambda^{p^j}} \alpha_{\phi_{p^j}} = \alpha^p_{\phi_{p^{j-1}}} + \text{lower-order terms.}$$

The explicit form turns out to be quite involved with $\underline{\mathbb{Z}}$-coefficients, so we determine them modulo p, and prove the following results (see Theorems 7.30, 7.32, and 7.34).

Theorem C (a) $H^\star_{C_2}(\mathbb{C}P^\infty_\tau;\underline{\mathbb{Z}}) \cong H^\star_{C_2}(\mathrm{pt})[\epsilon_{1+\sigma}]$.

(b) *As cohomology rings,*

$$H^\star_G(B_G S^1;\underline{\mathbb{Z}/p}) \cong H^\star_G(\mathrm{pt};\underline{\mathbb{Z}/p})[\alpha_{\phi_0},\dots,\alpha_{\phi_m}]/(\rho_1,\dots,\rho_m).$$

The relations ρ_r are described by

$$\rho_r = u_{\lambda^{p^{r-1}} - \lambda^{p^r}} \alpha_{\phi_{p^r}} - \mathcal{T}^p_{r-1} + a^{p-1}_{\lambda^{p^{r-1}}} \mathcal{T}_{r-1} \Bigl(\prod_{i=0}^{r-2} \mathcal{A}_i\Bigr)^{p-1},$$

where $\mathcal{T}_j$ and $\mathcal{A}_j$ are defined in (7.27).

(c) *As cohomology rings,*

$$H^\star_G(B_G S^3;\underline{\mathbb{Z}/p}) \cong H^\star_G(\mathrm{pt};\underline{\mathbb{Z}/p})[\beta_{2\phi_0},\dots,\beta_{2\phi_m}]/(\mu_1,\dots,\mu_m).$$

The relations μ_r are described by

$$\mu_r = (u_{\lambda^{p^{r-1}} - \lambda^{p^r}})^2 \beta_{2\phi_{p^r}} - \mathcal{L}^p_{r-1} + a^{2(p-1)}_{\lambda^{p^{r-1}}} \mathcal{L}_{r-1} \Bigl(\prod_{i=0}^{r-2} \mathcal{C}_i\Bigr)^{p-1},$$

where $\mathcal{L}_j$ and $\mathcal{C}_i$ are defined in Theorem 7.32.

1.1 Organization In Section 2, we recall results in equivariant homotopy theory that are useful from the viewpoint of ordinary cohomology. In Section 3, we recall previously known computations for $\underline{\mathbb{Z}}$-coefficients, and extend them as necessary for the following sections. We prove the homology decompositions for projective spaces in Section 4. These are applied to cohomology operations in Section 5, and the slice tower in Section 6. The ring structures are described in Section 7.

1.2 Notation We use the following notation throughout the paper.

- Throughout this paper, G denotes the cyclic group of order n, and g denotes a fixed generator of G. The unit sphere of an orthogonal G-representation V is denoted by $S(V)$, the unit disk by $D(V)$, and S^V the one-point compactification, which is homeomorphic to $D(V)/S(V)$.

- We write $1_{\mathbb{C}}$ for the trivial complex representation and 1 for the real trivial representation, and the regular representation ρ to mean $\rho_{\mathbb{C}}$ (the complex regular representation) if not specified. The irreducible complex representations of G are one dimensional, and up to isomorphism are listed as $1_{\mathbb{C}}, \lambda, \lambda^2, \dots, \lambda^{n-1}$ where λ sends g to $e^{2\pi i/n}$, the n^{th} root of unity.

- In the case n is even, we denote the sign representation by σ. This is a one-dimensional real representation.

- The nontrivial real irreducible representations other than σ are the underlying real representations of the complex irreducible representations. The realization of λ^i is also denoted by the same notation. Note that λ^i and λ^{n-i} are conjugate and hence their realizations are isomorphic by the natural $\mathbb{R}$-linear map $z \mapsto \bar{z}$ which reverses orientation.

- Unless specified, the cohomology groups are taken with $\underline{\mathbb{Z}}$-coefficients and suppressed from the notation.

- The notation $\mathcal{U}$ is used for the complete G-universe.

- The notation p is used for a prime not necessarily odd. The notation $\mathcal{A}_p$ denotes the nonequivariant (mod p) Steenrod algebra.

- The linear combinations $\ell - \left(\sum_{d_i \mid n} b_i \lambda^{d_i}\right)$ with $\ell \in \mathbb{Z}$ and $b_i \in \mathbb{Z}_{\geq}0$ are denoted by $\star^e \subset \mathrm{RO}(G)$. In the case n is even, we denote by $\star_{\text{div}}$ the gradings of the form $\ell - \left(\sum_{d_i \mid n} b_i \lambda^{d_i}\right) - \epsilon\sigma$ where $\ell \in \mathbb{Z}$, $b_i \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \{0, 1\}$.

- The notation $\mathbb{C}P^{\infty}_{\tau}$ denotes the complex projective space with the C_2-action given by conjugation.

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2 Equivariant cohomology with integer coefficients

We recall some features of equivariant cohomology with coefficients in a Mackey functor, with a particular emphasis on $\underline{\mathbb{Z}}$-coefficients, restricting our attention to cyclic groups G. For such G, a G-Mackey functor $\underline{M}$ consists of abelian groups $\underline{M}(G/H)$ with G/H-action, for every subgroup $H \le G$, and they are related via the following maps:

(1) the *transfer* map $\mathrm{tr}_K^H\colon \underline{M}(G/K) \to \underline{M}(G/H)$,

(2) the *restriction* map $\mathrm{res}_K^H\colon \underline{M}(G/H) \to \underline{M}(G/K)$

for $K \le H \le G$. The composite $\mathrm{res}_L^H \mathrm{tr}_K^H$ satisfies a double coset formula (see [18, Section 3]).

2.1 Example The Burnside ring Mackey functor $\underline{A}$ is defined by $\underline{A}(G/H) = A(H)$. Here $A(H)$ is the Burnside ring of H, ie the group completion of the monoid of finite H-sets up to isomorphism. The transfer maps are defined by inducing up the action $S \mapsto H \times_K S$ for $K \le H$, and the restriction maps are given by restricting the action. For the K-set K/K, the double coset formula takes the form $\mathrm{res}_L^H \mathrm{tr}_K^H(K/K) = \mathrm{res}_L^H(H/K) =$ union of double cosets $L\backslash H/K$.

2.2 Example The constant G-Mackey functors are defined as follows. For an abelian group C, the constant G-Mackey functor $\underline{C}$ is defined as

$$\underline{C}(G/H) = C, \quad \mathrm{res}_K^H = \mathrm{Id}, \quad \mathrm{tr}_K^H = [H:K]$$

for $K \le H \le G$. The double coset formula is given by $\mathrm{res}_L^H \mathrm{tr}_K^H(x) = [H:K]x$, for an element $x \in C$. We may also define its dual Mackey functor $\underline{C}^*$ by

$$\underline{C}^*(G/H) = C, \quad \mathrm{res}_K^H = [H:K], \quad \mathrm{tr}_K^H = \mathrm{Id}.$$

2.3 Example For the group C_p, the Mackey functor $\langle \mathbb{Z}/p \rangle$ is defined by

$$\langle \mathbb{Z}/p \rangle(C_p/C_p) = \mathbb{Z}/p, \quad \langle \mathbb{Z}/p \rangle(C_p/e) = 0, \quad \mathrm{res}_e^{C_p} = 0, \quad \mathrm{tr}_e^{C_p} = 0.$$

The following Mackey functor will appear in Section 5.

2.4 Example For the group C_2, we have the Mackey functor $\langle \Lambda \rangle$ described by

$$\langle \Lambda \rangle(C_2/e) = \mathbb{Z}/2, \quad \langle \Lambda \rangle(C_2/C_2) = 0, \quad \mathrm{res}_e^{C_2} = 0, \quad \mathrm{tr}_e^{C_2} = 0.$$

The equivariant stable homotopy category is the homotopy category of equivariant orthogonal spectra [27]. The Eilenberg–MacLane spectra are those whose integer graded homotopy groups vanish except in degree 0. The following result shows that Eilenberg–MacLane spectra, and thus equivariant ordinary cohomology theories, are uniquely determined from Mackey functors.

2.5 Theorem [14, Theorem 5.3] *For every Mackey functor $\underline{M}$, there is an Eilenberg–MacLane G-spectrum $H\underline{M}$ which is unique up to isomorphism in the equivariant stable homotopy category.*

For a G-spectrum X, the equivariant homotopy groups have the structure of a Mackey functor $\underline{\pi}_n^G(X)$, which on objects assigns the value

$$\underline{\pi}_n^G(X)(G/H) := \pi_n(X^H).$$

The grading may be extended to $\alpha \in \mathrm{RO}(G)$, the real representation ring of G, as

$$\underline{\pi}_\alpha^G(X)(G/K) \cong [S^\alpha \wedge G/K_+, X]_G,$$

which is isomorphic to $\pi_\alpha^K(X)$. Analogously, equivariant homology and cohomology theories are $\mathrm{RO}(G)$-graded and Mackey functor valued, which on objects is defined by

$$\underline{H}_G^\alpha(X;\underline{M})(G/K) \cong [X \wedge G/K_+, \Sigma^\alpha H\underline{M}]_G,$$
$$\underline{H}^G_\alpha(X;\underline{M})(G/K) \cong [S^\alpha \wedge G/K_+, X \wedge H\underline{M}]_G.$$

For a constant Mackey functor $\underline{C}$, the integer graded groups at orbit G/G compute the cohomology of the orbit space of X under the G-action, ie $H_G^n(X;\underline{C}) = H^n(X/G;C)$.

As $H\underline{\mathbb{Z}}$ is a ring spectrum, the Mackey functor $\underline{\mathbb{Z}}$ has a multiplicative structure, ie it is a commutative Green functor [29, Chapter XIII.5]. As a consequence, $\underline{H}_G^\alpha(X;\underline{\mathbb{Z}})$ are $\underline{\mathbb{Z}}$-modules. These modules satisfy the following property.

2.6 Proposition [38, Theorem 4.3] *For any $\underline{\mathbb{Z}}$-module G-Mackey functor $\underline{M}$, $\mathrm{tr}_K^H \mathrm{res}_K^H$ is multiplication by the index $[H:K]$ for $K \le H \le G$.*

For an element $\alpha \in \mathrm{RO}(G)$ such that $|\alpha^H| = 0$ for all $H \le G$, the representation sphere S^α belongs to the Picard group of G-spectra. We note from [1, Theorem B] that, for such α,

$$\underline{\pi}_0^G(H\underline{\mathbb{Z}} \wedge S^\alpha) \cong \underline{\mathbb{Z}}. \tag{2.7}$$

We recall a few important classes which generate a portion of the ring $\pi_\star^G(H\underline{\mathbb{Z}})$.

2.8 Definition [18, Section 3] For a G-representation V, consider the inclusion $S^0 \hookrightarrow S^V$. Composing with the unit map $S^0 \to H\underline{\mathbb{Z}}$, we obtain $S^0 \hookrightarrow S^V \to S^V \wedge H\underline{\mathbb{Z}}$ which represents an element in $\pi_{-V}^G(H\underline{\mathbb{Z}}) \cong \tilde{H}_G^V(S^0;\underline{\mathbb{Z}})$. This class is denoted by a_V.

2.9 Definition [18, Section 3] For an oriented G-representation of dimension n, $\underline{\pi}_{n-V}^G(H\underline{\mathbb{Z}}) = \underline{\mathbb{Z}}$ [18, Example 3.10]. Define $u_V \in \pi_{n-V}^G(H\underline{\mathbb{Z}}) \cong \tilde{H}_n^G(S^V;\underline{\mathbb{Z}}) \cong \mathbb{Z}$ to be the generator that restricts to the choice of orientation in $\underline{H}_n^G(S^V;\underline{\mathbb{Z}})(G/e) \cong \tilde{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$.

The following result simplifies calculations for $\pi_\star^G(H\underline{\mathbb{Z}})$.

2.10 Proposition [3, Theorem 3.6] *If* $\gcd(k,n) = 1$, *then*

$$H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}}.$$

Proposition 2.10 implies $\Sigma^{\lambda^i-\lambda^{ki}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}$ if $\gcd(k,n)=1$. That is, in the graded commutative ring $\pi_\star^G H\underline{\mathbb{Z}}$, there are invertible elements $u_{\lambda^{ki}-\lambda^i}$ in degrees $\lambda^i-\lambda^{ki}$ whenever $\gcd(k,n)=1$. Therefore to determine the ring $\pi_\star^G H\underline{\mathbb{Z}}$ it is enough to consider the gradings which are linear combinations of λ^d for $d \mid n$.

The linear combinations $\ell - \left(\sum_{d_i\mid n} b_i\lambda^{d_i}\right) + \epsilon\sigma \in \mathrm{RO}(G)$ with $\ell, b_i \in \mathbb{Z}$ and $\epsilon \in \{0,1\}$ are denoted by $\star_{\mathrm{div}}$. The last term $\epsilon\sigma$ occurs only when $|G|$ is even. In the case of $H\underline{\mathbb{Z}}$, the ring $\pi^G_{\star_{\mathrm{div}}}(H\underline{\mathbb{Z}})$ is also obtained from $\pi_\star^G(H\underline{\mathbb{Z}})$ by identifying all the $u_{\lambda^{ki}-\lambda^i}$ with 1. More precisely,

$$\pi^G_{\star_{\mathrm{div}}}(H\underline{\mathbb{Z}}) \cong \pi^G_\star(H\underline{\mathbb{Z}})/(u_{\lambda^{ki}-\lambda^i}-1) \quad\text{and}\quad \pi^G_\star(H\underline{\mathbb{Z}}) \cong \pi^G_{\star_{\mathrm{div}}}(H\underline{\mathbb{Z}})[u_{\lambda^{ki}-\lambda^i} \mid \gcd(k,n)=1].$$

In fact, we choose $u_{\lambda^{ki}-\lambda^i}$ such that $\mathrm{res}^G_e(u_{\lambda^{ki}-\lambda^i}) = 1 \in H^0(\mathrm{pt})$. This implies that $u_{\lambda^{ki}-\lambda^i}\cdot u_{\lambda^i} = u_{\lambda^{ki}}$. The following proposition describes the relation between $a_{\lambda^{ki}}$ and a_{λ^i} in $\pi^G_{\star_{\mathrm{div}}}(H\underline{\mathbb{Z}})$ in the case G is of prime power order (see Definition 2.8 for a description of the class $a_{\lambda^{ki}}$).

2.11 Proposition *Let $G = C_{p^m}$. If $\gcd(k,p)=1$, then*

$$u_{\lambda^i-\lambda^{ki}} a_{\lambda^{ki}} = k a_{\lambda^i}.$$

Thus, in the ring $\pi^G_{\star_{\mathrm{div}}}(H\underline{\mathbb{Z}})$,

$$a_{\lambda^{ki}} = k a_{\lambda^i}.$$

Proof By Proposition 2.10 we have

$$H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \simeq H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}}. \tag{2.12}$$

Hence there exists $u_{\lambda^{ki}-\lambda^i} \in \pi_0^G(H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}-\lambda^i})$ such that the composition

$$\psi\colon H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \xrightarrow{id\wedge u_{\lambda^{ki}-\lambda^i}} H\underline{\mathbb{Z}} \wedge S^{\lambda^i} \wedge H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}-\lambda^i} \to H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}}$$

induces the equivalence (2.12). Since all the fixed-point dimensions of $\lambda^{ki}-\lambda^i$ are zero,

$$\underline{\pi}_0^G(H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}-\lambda^i}) \cong \underline{\mathbb{Z}}$$

by (2.7). Choose $u_{\lambda^{ki}-\lambda^i} \in \pi_0^G(H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}-\lambda^i})$ to be the element such that $\mathrm{res}^G_e(u_{\lambda^{ki}-\lambda^i}) = 1$. Multiplication by $u_{\lambda^{ki}-\lambda^i}$,

$$\psi_*\colon \pi^G_\alpha(H\underline{\mathbb{Z}} \wedge S^{\lambda^i}) \to \pi^G_\alpha(H\underline{\mathbb{Z}} \wedge S^{\lambda^{ki}}),$$

sends a_{λ^i} to $u_{\lambda^{ki}-\lambda^i}\cdot a_{\lambda^i}$. Consider the map

$$\phi_{\lambda^i-\lambda^{ki}}\colon S^{\lambda^i} \to S^{\lambda^{ki}},$$

under which $z \mapsto z^k$, that is, the underlying degree of $\phi_{\lambda^i-\lambda^{ki}}$ is k. Since $\gcd(k,p)=1$, we may choose k' such that $k\cdot k' = 1+tp^m$. Then

$$u_{\lambda^{ki}-\lambda^i} = k'\cdot \phi^{H\underline{\mathbb{Z}}}_{\lambda^i-\lambda^{ki}} - t\cdot \mathrm{tr}^G_e(1)$$

as $\mathrm{res}^G_e(u_{\lambda^{ki}-\lambda^i}) = k\cdot k' - tp^m = 1$, where $\phi^{H\underline{\mathbb{Z}}}_{\lambda^i-\lambda^{ki}}$ is the Hurewicz image of $\phi_{\lambda^i-\lambda^{ki}}$. This implies

$$u_{\lambda^{ki}-\lambda^i}\cdot a_{\lambda^i} = k'\cdot\phi_{\lambda^i-\lambda^{ki}}\cdot a_{\lambda^i} - t\cdot\mathrm{tr}^G_e(1)\cdot a_{\lambda^i} = k'\cdot a_{\lambda^{ki}}.$$

Consequently $a_{\lambda^{ki}} = k\cdot a_{\lambda^i}$. □

In particular, we obtain the following again assuming $|G| = p^m$.

2.13 Proposition *Let $d = p^k$ be a divisor of p^m and $1 \le i < d$. Note that $i - d$ and i have the same p-adic valuation. Then*

$$a_{\lambda^{i-d}} = \Theta_{i,d} \cdot a_{\lambda^i},$$

where $\Theta_{i,d} = \frac{i-d}{i}$ which is well defined in $\mathbb{Z}/p^m$.

The following computations of $\underline{\pi}_\star^{C_p}(H\underline{\mathbb{Z}/p})$ [10, Appendix B] will help us in the following sections. For an odd prime p we have

$$\textbf{(2.14)} \qquad \underline{\pi}_\alpha^{C_p}(H\underline{\mathbb{Z}/p}) = \begin{cases} \underline{\mathbb{Z}/p} & \text{if } |\alpha| = 0, |\alpha^{C_p}| \ge 0, \\ \underline{\mathbb{Z}/p}^* & \text{if } |\alpha| = 0, |\alpha^{C_p}| < 0, \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0, |\alpha^{C_p}| \ge 0, \text{and } |\alpha| \text{ even}, \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0, |\alpha^{C_p}| < -1, \text{and } |\alpha| \text{ odd}, \\ 0 & \text{otherwise}, \end{cases}$$

and for the group C_2,

$$\textbf{(2.15)} \qquad \underline{\pi}_\alpha^{C_2}(H\underline{\mathbb{Z}/2}) = \begin{cases} \underline{\mathbb{Z}/2} & \text{if } |\alpha| = 0, |\alpha^{C_2}| \ge 0, \\ \underline{\mathbb{Z}/2}^* & \text{if } |\alpha| = 0, |\alpha^{C_2}| < 0, \\ \langle \Lambda \rangle & \text{if } |\alpha| = 0, |\alpha^{C_2}| = -1, \\ \langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| < 0, |\alpha^{C_2}| \ge 0, \\ \langle \mathbb{Z}/2 \rangle & \text{if } |\alpha| > 0, |\alpha^{C_2}| < -1, \\ 0 & \text{otherwise}. \end{cases}$$

2.16 Anderson duality Let $I_{\mathbb{Q}}$ and $I_{\mathbb{Q}/\mathbb{Z}}$ be the spectra representing the cohomology theories given by $X \mapsto \mathrm{Hom}(\pi_{-*}^G(X), \mathbb{Q})$ and $X \mapsto \mathrm{Hom}(\pi_{-*}^G(X), \mathbb{Q}/\mathbb{Z})$, respectively. The natural map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces the spectrum map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$, and the homotopy fiber is denoted by $I_{\mathbb{Z}}$. For a G-spectrum X, the *Anderson dual* $I_{\mathbb{Z}}X$ of X is the function spectrum $F(X, I_{\mathbb{Z}})$. For $X = H\underline{\mathbb{Z}}$, one easily computes $I_{\mathbb{Z}} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^* \simeq \Sigma^{2-\lambda} H\underline{\mathbb{Z}}$ [6; 39] in the case G is a cyclic group.

In general, for G-spectra E, X, and $\alpha \in \mathrm{RO}(G)$, there is short exact sequence

$$\textbf{(2.17)} \qquad 0 \to \mathrm{Ext}_L(\underline{E}_{\alpha-1}(X), \mathbb{Z}) \to \underline{I_{\mathbb{Z}}(E)}^\alpha(X) \to \mathrm{Hom}_L(\underline{E}_\alpha(X), \mathbb{Z}) \to 0.$$

In (2.17), Ext_L and Hom_L refer to levelwise Ext and Hom, which turn out to be Mackey functors. In particular, for $E = H\underline{\mathbb{Z}}$ and $X = S^0$, we have the equivalence $\underline{E}_\alpha(X) \cong \underline{\pi}_\alpha^G(S^0; \underline{\mathbb{Z}})$. Therefore, one may rewrite (2.17) as

$$\textbf{(2.18)} \qquad 0 \to \mathrm{Ext}_L(\underline{\pi}_{\alpha+\lambda-3}^G(H\underline{\mathbb{Z}}), \mathbb{Z}) \to \underline{\pi}_{-\alpha}^G(H\underline{\mathbb{Z}}) \to \mathrm{Hom}_L(\underline{\pi}_{\alpha+\lambda-2}^G(H\underline{\mathbb{Z}}), \mathbb{Z}) \to 0$$

for each $\alpha \in \mathrm{RO}(G)$.

3 $\pi_\star^G(H\underline{\mathbb{Z}})$ for cyclic groups

This section describes various structural results of $\pi_\star^G(H\underline{\mathbb{Z}})$ which helps us to construct the homology decompositions in the later sections. With Burnside ring coefficients, Lewis [25] first described $\pi_\star^{C_p}(H\underline{A})$. The portion of the $\mathrm{RO}(C_{p^n})$-graded homotopy of $H\underline{\mathbb{Z}}$ in dimensions of the form $k-V$ was described in [19; 20]. Using the Tate square, $\pi_\star^{C_2}(H\underline{\mathbb{Z}})$ was determined in [13] and $\pi_\star^{C_p}(H\underline{\mathbb{Z}/p})$ in [5]. For groups of square free order $\pi_\star^G(H\underline{\mathbb{Z}})$ was explored in [6], and for the group C_{p^2}, $\pi_\star^G(H\underline{\mathbb{Z}})$ appeared in [39].

We use the notation $\pi_{\star^e}^G(H\underline{\mathbb{Z}})$ to denote the part of $\pi_{\star_{\mathrm{div}}}^G(H\underline{\mathbb{Z}})$ in gradings of the form $k-V$ where V does not contain the sign representation σ. That is, $\star^e \subset \mathrm{RO}(G)$ consists of $l-\sum_{d|n} b_d\lambda^d$ with $b_d \geq 0$.

3.1 Theorem [2] *The subalgebra $\pi_{\star^e}^G(H\underline{\mathbb{Z}})$ of $\pi_\star^G(H\underline{\mathbb{Z}})$ is generated over $\mathbb{Z}$ by the classes $a_{\lambda^d}, u_{\lambda^d}$ where d is a divisor of n, $d \neq n$ with relations*

(3.2)
$$\frac{n}{d}a_{\lambda^d} = 0,$$

(3.3)
$$\frac{d}{\gcd(d,s)}a_{\lambda^s}u_{\lambda^d} = \frac{s}{\gcd(d,s)}a_{\lambda^d}u_{\lambda^s}.$$

For a general cyclic group G and $\alpha \in \star^e$, we observe that the expression above implies that $\pi_\alpha^G(H\underline{\mathbb{Z}})$ is cyclic. This is generated by a product of the corresponding u-classes and a-classes, and (3.3) implies that they assemble together into a cyclic group. The order of this is the least common multiple of the orders of a product of a-classes and u-classes occurring in $\pi_\alpha^G(H\underline{\mathbb{Z}})$.

In $\underline{\mathbb{Z}/p}$-coefficients, (3.3) simplifies to the following.

3.4 Proposition *In $\underline{\mathbb{Z}/p}$-coefficients,*

(3.5)
$$a_{\lambda^{pd}}u_{\lambda^d} = pa_{\lambda^d}u_{\lambda^{pd}} = 0.$$

The space $S(\lambda^{p^r})$ fits into a cofiber sequence

$$G/C_{p^r+} \xrightarrow{1-g} G/C_{p^r+} \to S(\lambda^{p^r})_+.$$

It follows that

(3.6)
$$\underline{\pi}_\alpha^{C_{p^r}}(H\underline{\mathbb{Z}}) = 0 \text{ and } \underline{\pi}_{\alpha-1}^{C_{p^r}}(H\underline{\mathbb{Z}}) = 0 \implies \underline{\pi}_\alpha^G(S(\lambda^{p^r})) = 0.$$

In the case $r=0$, we may make a complete computation to obtain [6]

(3.7)
$$\underline{\pi}_\alpha(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^* & \text{if } |\alpha| = 0, \\ \underline{\mathbb{Z}} & \text{if } |\alpha| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose α satisfies $|\alpha^H| > 0$ for all subgroups H. This means that S^α has a cell structure with cells of the type $G/H \times D^n$ for $n \geq |\alpha^H|$. Therefore,

(3.8)
$$|\alpha^H| > 0 \text{ for all subgroups } H \implies \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0.$$

Now if α satisfies $|\alpha^H| < 0$ for all subgroups H, $\beta = -\alpha$ satisfies the above condition. As S^α is the Spanier–Whitehead dual of S^β, we may construct it using cells of the type $G/H \times D^n$ for $n < 0$. Again, $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0$. Thus,

$$\text{(3.9)} \qquad |\alpha^H| < 0 \text{ for all subgroups } H \implies \underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0.$$

Using Anderson duality, we extend these relations slightly in the following proposition. We say that a representation α is even if all the fixed points of α are even dimensional.

3.10 Proposition *Let $\alpha \in \mathrm{RO}(G)$ be such that α is even, $|\alpha| > 0$, and $|\alpha^K| \geq 0$ for all subgroups $K \neq e$. Then $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0$.*

Proof By (2.18),

$$0 \to \mathrm{Ext}_L(\underline{\pi}^G_{-\alpha+\lambda-3}(H\underline{\mathbb{Z}}), \mathbb{Z}) \to \underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) \to \mathrm{Hom}_L(\underline{\pi}^G_{-\alpha+\lambda-2}(H\underline{\mathbb{Z}}), \mathbb{Z}) \to 0.$$

Since $|\alpha| > 0$, we have $|-\alpha+\lambda-2| < 0$. Hence the right-hand side term is zero as the Mackey functor $\underline{\pi}^G_{-\alpha+\lambda-2}(H\underline{\mathbb{Z}})$ only features torsion elements. Also, $|(-\alpha+\lambda-3)^H| < 0$ for all $H \leq G$. Hence $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0$. □

The following theorem may be viewed as another extension of (3.8). It also provides the necessary input in proving homology decompositions.

3.11 Theorem *Let $\alpha \in \mathrm{RO}(G)$ be such that $|\alpha^H|$ is odd for all subgroups H, and $|\alpha^H| > -1$ implies $|\alpha^K| \geq -1$ for all subgroups $K \supseteq H$. Then $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0$.*

The second condition stated here may be equivalently expressed as $|\alpha^H| < -1$ implies $|\alpha^K| \leq -1$ for all subgroups K of H. The first condition implies that α does not contain any multiples of the sign representation in the case $|G|$ is even.

Proof It suffices to prove this for G of prime power order, via Proposition 2.6. For groups of odd prime power order, this is proved in [3].

Let $\mathcal{F}_G = \{\alpha \in \mathrm{RO}(G) \mid \forall H \subseteq G, |\alpha^H| > -1 \implies |\alpha^K| \geq -1 \text{ for all } H \subset K\}$. We would like to show that $\alpha \in \mathcal{F}_G$ implies $\underline{\pi}_\alpha(H\underline{\mathbb{Z}}) = 0$. If $\alpha \in \mathcal{F}_G$ with $|\alpha^G| < -1$, the hypothesis implies $|\alpha^H| \leq -1$ for all subgroups H of G. For these α, $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) = 0$ by (3.9) as all the fixed points are negative.

Now let $\alpha \in \mathcal{F}_G$, and $H = C_{p^r}$ is a subgroup such that $|\alpha^H| < -1$. This implies that for all $K \subset H$, $|\alpha^K| \leq -1$. For such an α, $\alpha - \lambda^{p^s} \in \mathcal{F}_G$ for $s \leq r$. Also the cofiber sequence

$$S(\lambda^{p^s})_+ \to S^0 \to S^{\lambda^{p^s}}$$

gives the long exact sequence of Mackey functors

$$\text{(3.12)} \qquad \underline{\pi}^G_\alpha(S(\lambda^{p^s})_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}^G_\alpha(H\underline{\mathbb{Z}}) \to \underline{\pi}^G_{\alpha-\lambda^{p^s}}(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-1}(S(\lambda^{p^s})_+ \wedge H\underline{\mathbb{Z}}).$$

The given condition on α implies that $\alpha - t$ for $t \in \{0,1,2\}$ has negative-dimensional fixed points for subgroups of C_{p^s}. It follows from (3.6) and (3.9) that $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_{\alpha-\lambda^{p^s}}(H\underline{\mathbb{Z}})$. In this way by adding and subtracting copies of λ^{p^s} for $s \le r$ while adhering to the condition $|\alpha^K| \le -1$ for all subgroups K of H, we may find a new $\beta \in \mathcal{F}_G$, satisfying $|\beta^K| = -1$ for all $K = C_{p^s}$ for $s \le r$, and $\underline{\pi}_\beta^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$. A consequence of this maneuver is that it suffices to prove the result for those $\alpha \in \mathcal{F}_G$ such that $|\alpha^H| \ge -1$ for all subgroups H. Call this collection $\mathcal{F}_G^{\ge -1} \subset \mathcal{F}_G$.

A small observation will now allow us to assume $|\alpha| \ge 1$ in $\mathcal{F}_G^{\ge -1}$. For, we have the long exact sequence

$$\underline{\pi}_\alpha^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha-1}(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}),$$

by putting $s = 0$ in (3.12). Applying the computation of (3.7), we deduce $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong \underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}})$ if $|\alpha| \ge 3$, and if $|\alpha| = 1$, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$ implies $\underline{\pi}_{\alpha-\lambda}^G(H\underline{\mathbb{Z}}) = 0$. The last conclusion is true because for $\nu = \alpha - 1$, $|\nu| = 0$, and the map

$$\underline{\mathbb{Z}}^* \cong \underline{\pi}_\nu^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}})) \to \underline{\pi}_\nu^G(H\underline{\mathbb{Z}})$$

is an isomorphism at G/e, and hence injective at all levels.

Suppose that $\alpha \in \mathcal{F}_G^{\ge -1}$ and $|\alpha| \ge 1$. By Proposition 2.6, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ only features torsion elements as $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})(G/e) = 0$. Applying Anderson duality (2.18), we obtain

$$\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \cong \mathrm{Ext}_L(\underline{\pi}_{\lambda-\alpha-3}^G(H\underline{\mathbb{Z}}), \mathbb{Z}).$$

Now note that all the fixed points of $\lambda - \alpha - 3$ are negative if $\alpha \in \mathcal{F}_G^{\ge -1}$ and $|\alpha| \ge 1$. Therefore, $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) = 0$ and the proof is complete. □

We also have a calculation of $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ if all the fixed points are ≥ 0.

3.13 Proposition *Let $\alpha \in \mathrm{RO}(G)$ be such that $|\alpha| = 0$, and $|\alpha^K| \ge 0$ even for all subgroups $K \ne e$. Then $\underline{\pi}_\alpha^G(H\underline{\mathbb{Z}})$ is isomorphic to the Mackey functor $\underline{\mathbb{Z}}$.*

Proof The cofiber sequence

$$S(\lambda)_+ \to S^0 \to S^\lambda$$

implies the long exact sequence (by taking Mackey functor valued homotopy groups after smashing with $H\underline{\mathbb{Z}}$)

$$\cdots \to \underline{\pi}_{\alpha+\lambda}^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda}^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_\alpha^G(H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda-1}^G(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \to \underline{\pi}_{\alpha+\lambda-1}^G(H\underline{\mathbb{Z}}) \to \cdots$$

putting $s = 0$ in (3.12). Note that $\alpha + \lambda - 1$ satisfies the hypothesis of Theorem 3.11. The element $\alpha + \lambda$ has dimension 2, so by Proposition 2.6, the Mackey functor $\underline{\pi}_{\alpha+\lambda}^G(H\underline{\mathbb{Z}})$ features only torsion elements. By Anderson duality (2.18), we have

$$\underline{\pi}_{\alpha+\lambda}^G(H\underline{\mathbb{Z}}) \cong \mathrm{Ext}_L(\underline{\pi}_{-3-\alpha}^G(H\underline{\mathbb{Z}}), \mathbb{Z}).$$

Clearly from the given hypothesis, all the fixed points of $-3-\alpha$ are negative, therefore by (3.9), $\underline{\pi}^G_{\alpha+\lambda}(H\underline{\mathbb{Z}}) = 0$. We obtain

$$\underline{\pi}^G_{\alpha}(H\underline{\mathbb{Z}}) \cong \underline{\pi}^G_{\alpha+\lambda-1}(S(\lambda)_+ \wedge H\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}},$$

by (3.7). □

This helps us define the following classes.

3.14 Definition Let j be a multiple of i. Then by Proposition 3.13, the Mackey functor $\underline{\pi}^G_{\lambda^j-\lambda^i}(H\underline{\mathbb{Z}})$ is isomorphic to $\underline{\mathbb{Z}}$. Define the class $u_{\lambda^i-\lambda^j} \in \pi^G_{\lambda^j-\lambda^i}(H\underline{\mathbb{Z}})$ to be the element which under restriction to the orbit G/e corresponds to $1 \in \mathbb{Z}$.

The multiplication of the class $u_{\lambda^k-\lambda^{dk}}$ with $a_{\lambda^{dk}}$ is a multiple of a_{λ^k}. A similar description also appeared in [20, page 395].

3.15 Proposition *We have*

$$u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = d a_{\lambda^k}$$

and

$$u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}} = u_{\lambda^k}.$$

Proof Let $a_{\lambda^{dk}/\lambda^k}$ denote the map

$$a_{\lambda^{dk}/\lambda^k} : S^{\lambda^k} \to S^{\lambda^{dk}},$$

under which $z \mapsto z^d$. Then, the underlying degree of this map is d. Moreover,

$$a_{\lambda^{dk}/\lambda^k}\, a_{\lambda^k} = a_{\lambda^{dk}}.$$

Hence $u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}/\lambda^k}\, a_{\lambda^k}$, where

$$u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}/\lambda^k} \in \underline{H}^0_G(S^0;\underline{\mathbb{Z}})(G/G) \cong \mathbb{Z}.$$

Since $\mathrm{res}^G_e(u_{\lambda^k-\lambda^{dk}}) = 1$ and $\mathrm{res}^G_e(a_{\lambda^{dk}/\lambda^k}) = d$, we obtain

$$u_{\lambda^k-\lambda^{dk}}\, a_{\lambda^{dk}} = d a_{\lambda^k}.$$

Similarly, since $\mathrm{res}^G_e(u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}}) = \mathrm{res}^G_e(u_{\lambda^k}) = 1$, we have

$$u_{\lambda^k-\lambda^{dk}}\, u_{\lambda^{dk}} = u_{\lambda^k}.$$

□

The following will be used in subsequent sections.

3.16 Proposition *Let* $\alpha = \lambda^{i_1} + \cdots + \lambda^{i_k}$. *Then*

$$H^{-\alpha}_G(S^0) = 0.$$

Proof For a representation λ^{i_s}, we have the cofiber sequence $S(\lambda^{i_s})_+ \xrightarrow{r} S^0 \to S^{\lambda^{i_s}}$. If H_s is the kernel of the representation λ^{i_s}, then we also have the cofiber sequence

$$G/H_{s+} \xrightarrow{1-g} G/H_{s+} \to S(\lambda^{i_s})_+.$$

To see $H_G^{-\lambda^{i_1}}(S^0) = 0$, consider the long exact sequence

$$0 \to H_G^{-1}(S(\lambda^{i_1})_+) \to H_G^{-\lambda^{i_1}}(S^0) \to H_G^0(S^0) \xrightarrow{r^*} H_G^0(S(\lambda^{i_1})_+) \to \cdots.$$

The first term is zero as $H_G^j(G/H_{1+}) = 0$ for $j \leq -1$. Also, $H_G^0(S(\lambda^{i_1})_+) \cong H_G^0(G/H_{1+}) \cong \mathbb{Z}$. Moreover, under this identification, the map r^* is the restriction map $\mathrm{res}^G_{H_1}(\underline{\mathbb{Z}})$, and hence r^* is an isomorphism. Thus $H_G^{-\lambda^{i_1}}(S^0) = 0$. Using similar arguments, the result follows by induction. □

3.17 Example The Mackey functor $\underline{H}^0_{C_2}(S^\sigma; \underline{\mathbb{Z}})$ is zero. To see this consider the cofiber sequence

$$C_2/e_+ \to S^0 \to S^\sigma$$

and the associated long exact sequence in cohomology

$$0 \to \underline{H}^0_{C_2}(S^\sigma; \underline{\mathbb{Z}}) \to \underline{H}^0_{C_2}(S^0; \underline{\mathbb{Z}}) \to \underline{H}^0_{C_2}(C_2/e_+; \underline{\mathbb{Z}}) \to \underline{H}^1_{C_2}(S^\sigma; \underline{\mathbb{Z}}) \to \cdots.$$

The restriction map $\underline{H}^0_{C_2}(S^0; \underline{\mathbb{Z}}) \to \underline{H}^0_{C_2}(C_2/e_+; \underline{\mathbb{Z}})$ is injective. Thus we have $\underline{H}^0_{C_2}(S^\sigma; \underline{\mathbb{Z}}) = 0$.

Next we describe a homology decomposition theorem for a cyclic group by generalizing Lewis's approach.

3.18 Theorem *Let X be a finite type (that is, with finitely many cells of each dimension) generalized G-cell complex with only even-dimensional cells of the form $D(W)$. Suppose further that for cells $D(W)$, $D(V)$ we have the condition*

$$\dim W^H < \dim V^H \implies |W^K| \leq |V^K| \text{ for every subgroup } K \text{ containing } H.$$

Let $\mathcal{C}$ denote the collection of cells of X under the above description. Then,

$$H\underline{\mathbb{Z}} \wedge X_+ \simeq H\underline{\mathbb{Z}} \vee \bigvee_{D(W) \in \mathcal{C}} H\underline{\mathbb{Z}} \wedge S^W.$$

Proof The main step of the proof involves a pushout diagram of the form

$$\begin{array}{ccc} S(V) & \longrightarrow & D(V) \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \tag{3.19}$$

where we know that $H\underline{\mathbb{Z}} \wedge X \simeq \bigvee_{i=1}^k H\underline{\mathbb{Z}} \wedge S^{W_i}$. By looking at the cofiber sequence

$$H\underline{\mathbb{Z}} \wedge X \to H\underline{\mathbb{Z}} \wedge Y \to H\underline{\mathbb{Z}} \wedge S^V,$$

we see that the connecting map goes from $H\underline{\mathbb{Z}} \wedge S^V$ to $\bigvee_{i=1}^k H\underline{\mathbb{Z}} \wedge S^{W_i+1}$. Let $\alpha = V - W_i - 1$. Then for each i, $H\underline{\mathbb{Z}} \wedge S^V \to H\underline{\mathbb{Z}} \wedge S^{W_i+1}$ is in $\underline{\pi}^G_\alpha(H\underline{\mathbb{Z}})$. Using Theorem 3.11 for this α we get the connecting map to be 0. Hence the result follows. □

4 Additive homology decompositions for projective spaces

In this section, we show that $H\underline{\mathbb{Z}} \wedge P(V)$ is a wedge of suspensions of $H\underline{\mathbb{Z}}$ in many examples. Along the way, we also construct suitable bases for the homology which are used in later sections.

4.1 Cellular filtration of complex projective spaces For a complex representation V of G, the equivariant complex projective space $P(V)$ is the set of complex lines in V. It is constructed by attaching even-dimensional cells of the type $D(W)$ for representations W. We note that $P(V)$ and $P(V \otimes \phi)$ are homeomorphic as G-spaces for a one-dimensional G-representation ϕ. As G is abelian, the complex representation V is a direct sum of ϕ_i where $\dim(\phi_i) = 1$. If we write $V = V' \oplus \phi$ for a one-dimensional representation ϕ, we have a cofiber sequence

$$P(V') \to P(V) \to S^{\phi^{-1} \otimes V'}.$$

As a consequence, we obtain a cellular filtration of $P(V)$, which we proceed to describe now. Write $V = \phi_1 + \phi_2 + \cdots + \phi_n$ and $V_i = \phi_1 + \phi_2 + \cdots + \phi_i$. The cellular filtration is given by

$$P(V_1) \subseteq P(V_2) \subseteq \cdots \subseteq P(V_n) = P(V)$$

with cofiber sequences

$$P(V_i) \to P(V_{i+1}) \to S^{\phi_{i+1}^{-1} \otimes V_i}.$$

This shows that $P(V_{i+1})$ is obtained from $P(V_i)$ by attaching a cell of the type $D(W_i)$ for $W_i = \phi_{i+1}^{-1} \otimes V_i$. This filtration depends on the choice of the ordering of the ϕ_i. Via Theorem 3.18, we are looking for decompositions of $H\underline{\mathbb{Z}} \wedge P(V)$ as a wedge of suspensions of $H\underline{\mathbb{Z}}$.

4.2 Example For $G = C_p$ where p is odd, consider $P(V)$ for $V = \lambda^0 + 2\lambda + \lambda^2$. We write $V = \lambda^0 + \lambda + \lambda + \lambda^2$ and obtain the corresponding cellular filtration on $P(V)$. The corresponding cells are $D(W_m)$ where $W_m = \phi_m^{-1} \otimes V_{m-1}$ for $m \leq 4$. Observe that $|W_3| < |W_4|$ but $|W_3^{C_p}| = 2 > 0 = |W_4^{C_p}|$ which means that the hypothesis of Theorem 3.18 is not satisfied. However a simple rearrangement allows us to write down a homology decomposition. Writing $V = \lambda^0 + \lambda + \lambda^2 + \lambda$, we see that the resulting W_i satisfy the hypothesis of Theorem 3.18. This implies

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \vee (H\underline{\mathbb{Z}} \wedge S^{\lambda}) \vee (H\underline{\mathbb{Z}} \wedge S^{2\lambda}) \vee (H\underline{\mathbb{Z}} \wedge S^{2\lambda+2}).$$

In the following theorem, let $V = n_0\lambda^0 + n_1\lambda^1 + \cdots + n_{p-1}\lambda^{p-1}$ be any C_p-representation. Except for the fact that the n_i's are nonnegative, no other condition is imposed on n_i. We may assume $n_0 \geq n_i$ by replacing V with $V \otimes \lambda^{-j}$ if necessary.

4.3 Theorem *Let $V = n_0\lambda^0 + n_1\lambda^1 + \cdots + n_{p-1}\lambda^{p-1}$ be a complex C_p-representation and $n_0 \geq n_i \geq 0$ for all i. Then*

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \vee \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \vee \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \vee \cdots \vee \bigvee_{i=(\sum_{j=1}^{n_0-1} a_j)-(n_0-1)}^{(\sum_{j=1}^{n_0} a_j)-n_0} \Sigma^{i\lambda+2(n_0-1)} H\underline{\mathbb{Z}},$$

where a_i is the cardinality of the set $\{n_j : n_j \geq i\}$.

Proof We arrange the irreducible representations in V in such a way that

$$V = A_1 + A_2 + \cdots + A_{n_0},$$

where $A_1 = \sum_{n_i \geq 1} \lambda^i, A_2 = \sum_{n_i \geq 2} \lambda^i, \ldots, A_{n_0} = \sum_{n_i \geq n_0} \lambda^i$. Then, a_i is the number of summands appearing in A_i.

We consider the cell complex structure on $P(V)$ associated to $V = \sum_{i=1}^{\dim_{\mathbb{C}} V} \phi_i$ where ϕ_i's are defined by

$$\phi_{(\sum_{l=1}^{j-1} a_l)+1} + \phi_{(\sum_{l=1}^{j-1} a_l)+2} + \cdots + \phi_{(\sum_{l=1}^{j-1} a_l)+a_j} = A_j = \sum_{n_i \geq j} \lambda^i$$

for $j \geq 1$, assuming $a_0 = 0$, and the powers of λ above are arranged in increasing order from 0 to $p-1$. To prove the statement, we use induction on the sum $n_0 + n_1 + \cdots + n_{p-1} = \dim_{\mathbb{C}} V$.

When $n_0 + \cdots + n_{p-1} = 1$, that is, $n_0 = 1$ and $n_i = 0$ for all $1 \leq i \leq p-1$, then $V = \lambda^0$. Thus $P(V)_+ = S^0$ and

$$H\underline{\mathbb{Z}} \wedge P(V)_+ \simeq H\underline{\mathbb{Z}} \wedge S^0 \simeq H\underline{\mathbb{Z}}.$$

Now suppose that the statement is true for integers less than $n_0 + n_1 + \cdots + n_{p-1}$. Using the notation $V_k = \sum_{i=1}^{k} \phi_i$ as above, the inductive hypothesis implies the result for $X = P(V_k)$ whenever $k < \dim_{\mathbb{C}} V$. In particular, letting $m = \sum n_i - 1$,

$$V = V_m + \phi_{\dim_{\mathbb{C}} V} = V_m + \lambda^s$$

for some integer s. Let a_i', n_i' and A_i' denote the values for V_m that correspond to a_i, n_i and A_i for V. Observe that $a_i' = a_i$ if $i < n_0$, $a_{n_0}' = a_{n_0} - 1$, and our choice of the ϕ_i implies that $n_s = n_0$. The induction hypothesis implies

$$\textbf{(4.4)}\quad H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \vee \bigvee_{i=1}^{a_1'-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \vee \bigvee_{i=a_1'-1}^{a_1'+a_2'-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \vee \cdots \vee \bigvee_{i=(\sum_{j=1}^{n_0'-1} a_j')-(n_0'-1)}^{(\sum_{j=1}^{n_0'} a_j')-n_0'} \Sigma^{i\lambda+2(n_0'-1)} H\underline{\mathbb{Z}}.$$

We see that either $s = 0$ or if $s \neq 0$, $n_s = n_0$. We first consider the latter case. Then $n_0' = n_0$, $a_i' = a_i$ whenever $i < n_0$ and $a_{n_0}' = a_{n_0} - 1$. Thus (4.4) reduces to

$$H\underline{\mathbb{Z}} \wedge P(V_m)_+ \simeq H\underline{\mathbb{Z}} \vee \bigvee_{i=1}^{a_1-1} \Sigma^{i\lambda} H\underline{\mathbb{Z}} \vee \bigvee_{i=a_1-1}^{a_1+a_2-2} \Sigma^{i\lambda+2} H\underline{\mathbb{Z}} \vee \cdots \vee \bigvee_{i=(\sum_{j=1}^{n_0-1} a_j)-(n_0-1)}^{(\sum_{j=1}^{n_0} a_j)-(n_0+1)} \Sigma^{i\lambda+2(n_0-1)} H\underline{\mathbb{Z}}.$$

As $n_0 = n_s$, the coefficient of λ^s in $V_m = V - \lambda^s$ is $n_s - 1 = n_0 - 1$, which in turn implies the coefficient of λ^0 in $\lambda^{-s} \otimes V_m$ is $n_0 - 1$. Thus $|(\lambda^{-s} \otimes V_m)^{C_p}| = 2(n_0 - 1)$. We look at representations $i\lambda + j\lambda^0$ where $i \in \{0, 1, \ldots, a_0 + a_1 + \cdots + a_{n_0-1} - 1 - n_0\}$ and $j \in \{0, 1, \ldots, n_0 - 1\}$. Then,

$$|(i\lambda + j\lambda^0)^{C_p}| = 2j \leq 2(n_0 - 1) = |(\lambda^{-s} \otimes V_m)^{C_p}|.$$

Therefore the hypothesis of Theorem 3.18 is satisfied in this case.

When $s=0$ we have $n_0'=n_0-1$, $a_i'=a_i$ for all $i<n_0'$. In this case, (4.4) reduces to

$$H\underline{\mathbb{Z}}\wedge P(V_m)_+\simeq H\underline{\mathbb{Z}}\vee\bigvee_{i=1}^{a_1-1}\Sigma^{i\lambda}H\underline{\mathbb{Z}}\vee\bigvee_{i=a_1-1}^{a_1+a_2-2}\Sigma^{i\lambda+2}H\underline{\mathbb{Z}}\vee\cdots\vee\bigvee_{i=(\sum_{j=1}^{n_0-2}a_j)-(n_0-2)}^{(\sum_{j=1}^{n_0-1}a_j)-(n_0-1)}\Sigma^{i\lambda+2(n_0-2)}H\underline{\mathbb{Z}}.$$

Note that $a_{n_0}=1$, that is, $a_{n_0}-1=0$. So we can rewrite the equation as

$$H\underline{\mathbb{Z}}\wedge P(V_m)_+\simeq H\underline{\mathbb{Z}}\vee\bigvee_{i=1}^{a_1-1}\Sigma^{i\lambda}H\underline{\mathbb{Z}}\vee\bigvee_{i=a_1-1}^{a_1+a_2-2}\Sigma^{i\lambda+2}H\underline{\mathbb{Z}}\vee\cdots\vee\bigvee_{i=(\sum_{j=1}^{n_0-2}a_j)-(n_0-2)}^{(\sum_{j=1}^{n_0}a_j)-n_0}\Sigma^{i\lambda+2(n_0-2)}H\underline{\mathbb{Z}}.$$

Since $s=0$, we have $\lambda^{-s}\otimes V_m=V_m$. Now for all representations of the form $i\lambda+j\lambda^0$ where $i\in\{0,1,\dots,(a_0+a_1+\cdots+a_{n_0-1}-n_0)\}$ and $j\in\{0,1,\dots,(n_0-2)\}$ we have

$$|(i\lambda+j\lambda^0)^{C_p}|=2j\le 2(n_0-2)<2(n_0-1)=|V_m^{C_p}|=|(\lambda^{-s}\otimes V_m)^{C_p}|.$$

These calculations imply that both cases satisfy the hypothesis of Theorem 3.18. Thus, we obtain

$$H\underline{\mathbb{Z}}\wedge P(V)_+\simeq(H\underline{\mathbb{Z}}\wedge P(V_m)_+)\vee(H\underline{\mathbb{Z}}\wedge S^{\lambda^{-s}\otimes V_m}).$$

Using the facts $H\underline{\mathbb{Z}}\wedge S^{\lambda^i}\simeq H\underline{\mathbb{Z}}\wedge S^{\lambda}$ and $H\underline{\mathbb{Z}}\wedge S^{\lambda^0}\simeq H\underline{\mathbb{Z}}\wedge S^2$,

$$H\underline{\mathbb{Z}}\wedge P(V)_+\simeq(H\underline{\mathbb{Z}}\wedge P(V_m)_+)\vee(H\underline{\mathbb{Z}}\wedge S^{(\sum_{j=1}^{n_0}a_j-n_0)\lambda+2(n_0-1)})$$

for both $s=0$ and $s\neq 0$. □

We elaborate this theorem with an example below.

4.5 Example Let $V=3\lambda^0+2\lambda+4\lambda^2$ be a complex C_p-representation. Since among the three coefficients appearing in the expression for V here, 4 is the largest, we consider $V\otimes\lambda^{-2}=4\lambda^0+3\lambda^{p-2}+2\lambda^{p-1}$. For simplicity, we call this also as V. We now write V as a sum of A_i, that is, $V=A_1+A_2+A_3+A_4$, where $A_1=\lambda^0+\lambda^{p-2}+\lambda^{p-1}$, $A_2=\lambda^0+\lambda^{p-2}+\lambda^{p-1}$, $A_3=\lambda^0+\lambda^{p-2}$, $A_4=\lambda^0$.

Note that $a_1=3$, $a_2=3$, $a_3=2$ and $a_4=1$. From Theorem 4.3, we conclude that

$$H\underline{\mathbb{Z}}\wedge P(V)_+\simeq(H\underline{\mathbb{Z}}\wedge P(V_m)_+)\vee(H\underline{\mathbb{Z}}\wedge S^{V_m}),$$

which is the same as

$$H\underline{\mathbb{Z}}\vee\bigvee_{i=1}^{2}\Sigma^{i\lambda}H\underline{\mathbb{Z}}\vee\bigvee_{i=2}^{4}\Sigma^{i\lambda+2}H\underline{\mathbb{Z}}\vee\bigvee_{i=4}^{5}\Sigma^{i\lambda+4}H\underline{\mathbb{Z}}\vee\bigvee\Sigma^{5\lambda+6}H\underline{\mathbb{Z}}.$$

4.6 Decompositions over general cyclic groups We now proceed towards the equivariant homology decomposition of complex projective spaces where the group G is any cyclic group C_n. Note that a complete C_n-universe $\mathcal{U}$ may be constructed as

$$\mathcal{U}=\varinjlim_m m\rho,$$

where ρ is the regular representation. In the remaining part of the section, we stick to these representations to avoid the involved expressions as in Theorem 4.3 for the general cyclic groups.

4.7 Theorem *We have the decomposition*

$$H\underline{\mathbb{Z}} \wedge P(m\rho)_+ \simeq \bigvee_{i=0}^{nm-1} H\underline{\mathbb{Z}} \wedge S^{\phi_i},$$

where $\phi_0 = 0$ *and* $\phi_i = \lambda^{-i}(1_{\mathbb{C}} + \lambda + \lambda^2 + \cdots + \lambda^{i-1})$ *for* $i > 0$. *Passing to the homotopy colimit, for a* C_n*-universe* $\mathcal{U}$, *we obtain*

$$H\underline{\mathbb{Z}} \wedge P(\mathcal{U})_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_i}.$$

Proof We use induction on k to show that

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$$

for $V_k = \sum_{i=0}^{k} \lambda^i$. The statement holds for $k = 0$ since $V_0 = \lambda^0 = 1_{\mathbb{C}}$ and $H\underline{\mathbb{Z}} \wedge P(V_0)_+ \simeq H\underline{\mathbb{Z}} \wedge S^0$. Now let the statement be true for V_k, ie $H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i}$. We have the cofiber sequence

$$P(V_k)_+ \to P(V_{k+1})_+ \to S^{\phi_{k+1}}.$$

Smashing with $H\underline{\mathbb{Z}}$ we have

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \to H\underline{\mathbb{Z}} \wedge P(V_{k+1})_+ \to H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}}.$$

Thus the connecting map goes from $H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}}$ to $\bigvee_{i=0}^{k} H\underline{\mathbb{Z}} \wedge S^{\phi_i+1}$. For each $i \le k$ the map $H\underline{\mathbb{Z}} \wedge S^{\phi_{k+1}} \to H\underline{\mathbb{Z}} \wedge S^{\phi_i+1}$ belongs to $\pi_0^{C_n}(H\underline{\mathbb{Z}} \wedge S^{\phi_i+1-\phi_{k+1}}) \simeq \pi^{C_n}_{\phi_{k+1}-\phi_i-1}(H\underline{\mathbb{Z}})$. Taking $\alpha = \phi_{k+1} - \phi_i - 1$, we have $|\alpha|$ is odd. Note that

$$|\alpha^H| = |\phi_{k+1}^H| - |\phi_i^H| - 1 = \left| \left(\sum_{j=i+1}^{k+1} \lambda^j \right)^H \right| - 1 \ge -1.$$

By Theorem 3.11 the connecting map is 0. Thus

$$H\underline{\mathbb{Z}} \wedge P(V_{k+1})_+ \simeq \bigvee_{i=0}^{k+1} H\underline{\mathbb{Z}} \wedge S^{\phi_i}.$$ □

We may also consider a variant in the case $G = C_2$ which acts on $\mathbb{C}P^n$ by complex conjugation. The resulting C_2-space is denoted by $\mathbb{C}P^n_\tau$ for $1 \le n \le \infty$. Then, $(\mathbb{C}P^n_\tau)^{C_2} \simeq \mathbb{R}P^n$ which shows that this example is homotopically different from the example above.

4.8 Theorem *There is a decomposition*

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^n_{\tau\,+} \simeq \bigvee_{i=0}^{n} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma} \simeq \bigvee_{i=0}^{n} H\underline{\mathbb{Z}} \wedge S^{i\rho}.$$

Passing to the homotopy colimit,

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{\infty}_{\tau}{}_{+} \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma} \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{i\rho}.$$

Proof The method of the proof is exactly same as Theorem 4.7. The main step involves the cofiber sequence

$$\mathbb{C}P^{n-1}_{\tau}{}_{+} \hookrightarrow \mathbb{C}P^{n}_{\tau}{}_{+} \to S^{n+n\sigma}.$$

Smashing with $H\underline{\mathbb{Z}}$ gives

$$H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{n-1}_{\tau}{}_{+} \to H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{n}_{\tau}{}_{+} \to H\underline{\mathbb{Z}} \wedge S^{n+n\sigma}.$$

The connecting homomorphism goes from

$$H\underline{\mathbb{Z}} \wedge S^{n+n\sigma} \to \bigvee_{i=0}^{n-1} H\underline{\mathbb{Z}} \wedge S^{i+i\sigma+1},$$

as $H\underline{\mathbb{Z}} \wedge \mathbb{C}P^{n-1}_{\tau}{}_{+} \simeq \bigvee_{i=0}^{n-1} H\underline{\mathbb{Z}} \wedge S^{n+n\sigma}$, by the inductive hypothesis. For each $0 \leq i \leq n-1$, up to homotopy, the map $H\underline{\mathbb{Z}} \wedge S^{n+n\sigma} \to H\underline{\mathbb{Z}} \wedge S^{i+i\sigma+1}$ belongs to

$$\pi_0(S^{i+1-n+(i-n)\sigma} \wedge H\underline{\mathbb{Z}}) \cong H_G^{i+1-n+(i-n)\sigma}(S^0),$$

which is 0 when $i < n-1$ as all the fixed-point dimensions are negative. At $i = n-1$, we get the Mackey functor $\underline{\pi}_0(S^{-\sigma} \wedge H\underline{\mathbb{Z}})$ equals $\underline{H}_{C_2}^{-\sigma}(S^0; \underline{\mathbb{Z}})$, which is zero by Example 3.17. ◻

4.9 Quaternionic projective spaces As in the complex case, the quaternionic projective spaces may be equipped with a cell structure that turns out to be useful from the perspective of homology decompositions [8]. The quaternionic C_n-representation ψ^r is given as multiplication by $e^{\frac{2\pi i r}{n}}$ on $\mathbb{H}$. As a complex C_n-representation, $\psi^r \cong \lambda^r + \lambda^{-r}$. The equivariant projective space $P_{\mathbb{H}}(V)$ for a quaternionic representation V is the set of lines in V, that is, $P_{\mathbb{H}}(V) = V \setminus \{0\}/\sim$ where $v \sim hv$ for all $v \in V \setminus \{0\}$ and for all $h \in \mathbb{H} \setminus \{0\}$. Define $\rho_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} \rho_G$, and note that as $\mathbb{H}$-representations,

$$\rho_{\mathbb{H}} = \sum_{i=0}^{n-1} \psi^i.$$

We write $V_k = \sum_{i=0}^{k-1} \psi^i$ and $W_k = \lambda^{-k} \otimes_{\mathbb{C}} V_k$. We recall from [8] that $P_{\mathbb{H}}(m\rho_{\mathbb{H}})$ is a G-cell complex with cells of the form $D(W_k)$ for $k \leq mn-1$.

4.10 Theorem *Let $G = C_n$. We have the splitting*

$$H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(m\rho_{\mathbb{H}})_{+} \simeq \bigvee_{i=0}^{mn-1} H\underline{\mathbb{Z}} \wedge S^{W_i}.$$

Passing to the homotopy colimit, we obtain

$$H\underline{\mathbb{Z}} \wedge B_G S^3_+ \simeq H\underline{\mathbb{Z}} \wedge P_{\mathbb{H}}(\mathcal{U}_{\mathbb{H}})_+ \simeq \bigvee_{i=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{W_i}.$$

Proof We proceed as in Theorem 4.7 by showing via induction on k that

$$H\underline{\mathbb{Z}} \wedge P(V_k)_+ \simeq \bigvee_{i=0}^{k-1} H\underline{\mathbb{Z}} \wedge S^{W_i}.$$

Along the way we are required to check $|W_k^H - W_i^H - 1| \geq -1$ for $i < k$ which implies Theorem 3.11 applies to prove the result. Indeed, it is true as

$$|W_i^{C_d}| = \begin{cases} \lfloor \frac{2i-1}{d} \rfloor + 1 & \text{if } d \mid i, \\ \lfloor \frac{2i-1}{d} \rfloor & \text{if } d \nmid i, \end{cases}$$

is a nondecreasing function of i. □

4.11 Construction of a homology basis We now define the classes α_{ϕ_ℓ} which serve as additive generators of $H^\star_G(P(\mathcal{U}))$. Let W_ℓ and ϕ_ℓ be the representations

(4.12) $$W_\ell := 1_{\mathbb{C}} + \lambda + \cdots + \lambda^\ell \quad \text{and} \quad \phi_\ell := \lambda^{-\ell}(1_{\mathbb{C}} + \lambda + \cdots + \lambda^{\ell-1}).$$

Consider the cofiber sequence

$$P(W_{\ell-1})_+ \hookrightarrow P(W_\ell)_+ \xrightarrow{\chi} S^{\phi_\ell}.$$

At $\deg \phi_\ell$, the associated long exact sequence is

$$\cdots \to \widetilde{H}_G^{\phi_\ell - 1}(P(W_{\ell-1})_+) \to \widetilde{H}_G^{\phi_\ell}(S^{\phi_\ell}) \xrightarrow{\chi^*} \widetilde{H}_G^{\phi_\ell}(P(W_\ell)_+) \to \widetilde{H}_G^{\phi_\ell}(P(W_{\ell-1})_+) \to \cdots.$$

Note that $\underline{H}_G^{\phi_\ell-1}(P(W_{\ell-1})_+)(G/e) \cong H^{2\ell-1}(P(\mathbb{C}^\ell)) \cong 0$. So, the restriction of the map χ^* at the orbit G/e is an isomorphism. Hence, the Mackey functor diagram says that the image of $1 \in \widetilde{H}_G^{\phi_\ell}(S^{\phi_\ell}) \cong H^0_G(\mathrm{pt}) \cong \mathbb{Z}$ under the map χ^* is nonzero. Define $\alpha_{\phi_\ell}^{W_\ell}$ to be the element $\chi^*(1)$. We often omit the superscript and simply write α_{ϕ_ℓ}.

Now we lift α_{ϕ_ℓ} by induction to get the generator $\alpha_{\phi_\ell}^{\mathcal{U}}$ (or simply α_{ϕ_ℓ}) which belongs to $H_G^{\phi_\ell}(P(\mathcal{U}))$. For this we successively add representations (one at a time) to W_ℓ in a proper order to reach $\mathcal{U}$. Let $U' \subseteq \mathcal{U}$ be a representation containing W_ℓ. Assume that for U' the class α_{ϕ_ℓ} has been defined for $H_G^{\phi_\ell}(P(U'))$. Suppose $U = U' + \lambda^j$. Consider the cofiber sequence

$$P(U')_+ \xrightarrow{\theta} P(U)_+ \to S^{\lambda^{-j} U'}$$

and thus the long exact sequence

(4.13) $$\cdots \to \widetilde{H}_G^{\phi_\ell}(S^{\lambda^{-j}U'}) \to \widetilde{H}_G^{\phi_\ell}(P(U)_+) \xrightarrow{\theta^*} \widetilde{H}_G^{\phi_\ell}(P(U')_+) \to \widetilde{H}_G^{\phi_\ell+1}(S^{\lambda^{-j}U'}) \to \cdots.$$

By Proposition 3.16 and Theorem 3.11, the first and the fourth term in (4.13) are zero. So the map θ^* is an isomorphism. Hence a unique lift of α_{ϕ_ℓ} exists in $H_G^{\phi_\ell}(P(U))$ along the map

$$\theta^*\colon \widetilde{H}_G^{\phi_\ell}(P(U)_+) \to \widetilde{H}_G^{\phi_\ell}(P(U')_+).$$

Since the restriction of the map χ^* to the orbit G/e is an isomorphism, we get

$$\mathrm{res}_e^G(\alpha_{\phi_\ell}) = x^\ell. \tag{4.14}$$

4.15 Additive generators of $H_G^\star(P(\mathcal{U}))$ By Theorem 4.7, we may express the additive structure of the cohomology of $P(\mathcal{U})$ as

$$H_G^\star(P(\mathcal{U})) = \bigoplus_{k=0}^{\infty} \bigoplus_{i=0}^{n-1} H_G^{\star - k\phi_n - \phi_i}(\mathrm{pt}), \tag{4.16}$$

where $\phi_i = \lambda^{-i}(1_{\mathbb{C}} + \lambda + \cdots + \lambda^{i-1})$ and $\phi_0 = 0$. The above construction defines the generators $\alpha_{k\phi_n+\phi_i} \in H_G^{k\phi_n+\phi_i}(P(\mathcal{U}))$ which corresponds to the factor $H_G^{\star-k\phi_n-\phi_i}(\mathrm{pt})$ in (4.16). Summarizing the above, we get the following.

4.17 Proposition *Additively, the classes $\alpha_{k\phi_n+\phi_i}$ generate $H_G^\star(P(\mathcal{U}))$ as a module over $H_G^\star(\mathrm{pt})$.*

5 Application: cohomology operations

Let $G = C_p$ (p not necessarily odd). Recall that $\mathcal{A}_p$ is the mod p Steenrod algebra. We consider

$$\mathcal{A}_G^n = [H\underline{\mathbb{Z}/p}, \Sigma^n H\underline{\mathbb{Z}/p}]_G,$$

for $n \in \mathbb{Z}$, and the map

$$\Omega\colon \mathcal{A}_G^* \to \mathcal{A}_p,$$

as its restriction to the identity subgroup. We demonstrate that the additive decomposition of Section 4 recovers the following result of Caruso [7].

5.1 Theorem *Let θ be a degree-r (r is even) cohomology operation not involving the Bockstein β. For such θ, there does not exist an equivariant cohomology operation*

$$\tilde{\theta}\colon H\underline{\mathbb{Z}/p} \to \Sigma^r H\underline{\mathbb{Z}/p}$$

such that $\Omega(\tilde{\theta}) = \theta$.

Before going into the proof, let us look at some examples.

5.2 Example Let p be odd. We claim that there does not exist an equivariant cohomology operation

$$\widetilde{P^1}\colon H\underline{\mathbb{Z}/p} \to \Sigma^{2p-2} H\underline{\mathbb{Z}/p}$$

such that $\Omega(\widetilde{P^1}) = P^1$, where P^1 is the power operation. The existence of such a $\widetilde{P^1}$ would lead to a map of Mackey functors

$$\underline{H}^{\alpha}_G(X; \underline{\mathbb{Z}/p}) \to \underline{H}^{\alpha+2p-2}_G(X; \underline{\mathbb{Z}/p}) \quad \text{for } \alpha \in \mathrm{RO}(G),$$

which is natural in X. In particular, let us take $X = P(\mathcal{U})$ and $\alpha = \lambda$. We observe that

$$\textbf{(5.3)} \qquad \begin{array}{ccc} \underline{H}^{\lambda}_G(P(\mathcal{U}); \underline{\mathbb{Z}/p}) & \xrightarrow{\widetilde{P^1}} & \underline{H}^{\lambda+2p-2}_G(P(\mathcal{U}); \underline{\mathbb{Z}/p}) \\ \downarrow\cong & & \downarrow\cong \\ \underline{\mathbb{Z}/p} & & \underline{\mathbb{Z}/p}^{*} \end{array}$$

An explanation for this is as follows. For the group C_p, the additive decomposition in Theorem 4.7 tells us

$$\textbf{(5.4)} \qquad P(\mathcal{U}) \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{k=0}^{\infty} \bigvee_{i=0}^{p-1} S^{k\phi_p + i\lambda} \wedge H\underline{\mathbb{Z}} \quad \text{such that } (k,i) \neq (0,0).$$

As a result,

$$\widetilde{H}^{\star}_G(P(\mathcal{U}); \underline{\mathbb{Z}/p}) \cong \bigoplus_{k=0}^{\infty} \bigoplus_{i=0}^{p-1} \widetilde{H}^{\star-k\phi_p-i\lambda}_G(S^0; \underline{\mathbb{Z}/p}), \quad (k,i) \neq (0,0).$$

Let $\alpha = \lambda + 2p - 2 - k\phi_p - i\lambda$. So $|\alpha| = 2p - 2kp - 2i$ and $|\alpha^{C_p}| = 2p - 2 - 2k$. Applying (2.14), we conclude that $\underline{H}^{\lambda+2p-2}_G(P(\mathcal{U}); \underline{\mathbb{Z}/p}) \cong \underline{\mathbb{Z}/p}^{*}$.

As $P^1(x) = x^p$ for a generator $x \in H^2(\mathbb{C}P^{\infty})$, the diagram (5.3) yields a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p & \xrightarrow{\widetilde{P^1}} & \mathbb{Z}/p \\ {\scriptstyle 1}\downarrow \;\uparrow{\scriptstyle 0} & & {\scriptstyle 0}\downarrow \;\uparrow{\scriptstyle 1} \\ \mathbb{Z}/p & \xrightarrow{\cong} & \mathbb{Z}/p \end{array}$$

which is a contradiction.

The technique used in Example 5.2 does not work when $p = 2$. This is because the Mackey functor $\underline{\mathbb{Z}/2}$ may now appear in the right-hand side of the diagram (5.3), and so the contradiction drawn out by comparing the Mackey functor diagram fails. Below we argue differently to show that Sq^2 is not in the image of Ω.

5.5 Example Let $p = 2$. There does not exist an equivariant cohomology operation

$$\widetilde{\mathrm{Sq}^2} \colon H\underline{\mathbb{Z}/2} \to \Sigma^2 H\underline{\mathbb{Z}/2}$$

such that $\Omega(\widetilde{\mathrm{Sq}^2}) = \mathrm{Sq}^2$. The existence of such an equivariant cohomology operation would lead to a map of Mackey functors

$$\underline{H}^{\alpha}_G(X; \underline{\mathbb{Z}/2}) \to \underline{H}^{\alpha+2}_G(X; \underline{\mathbb{Z}/2}) \quad \text{for } \alpha \in \mathrm{RO}(G).$$

Let $X = \mathbb{C}P^2_\tau$, the complex projective space with the conjugation action. Taking $\alpha = \rho = 1 + \sigma$,

$$\text{(5.6)} \qquad \begin{array}{ccc} \underline{H}^{\rho}_G(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) & \xrightarrow{\widetilde{\mathrm{Sq}^2}} & \underline{H}^{\rho+2}_G(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) \\ \downarrow \cong & & \downarrow \cong \\ \underline{\mathbb{Z}/2} & & \langle \Lambda \rangle \end{array}$$

To see this recall from Theorem 4.8 that

$$\mathbb{C}P^2_\tau \wedge H\mathbb{Z} \simeq \bigvee_{i=1}^{2} S^{i+i\sigma} \wedge H\underline{\mathbb{Z}}.$$

Hence

$$\widetilde{H}^{\star}_{C_2}(\mathbb{C}P^2_\tau; \underline{\mathbb{Z}/2}) \cong \bigoplus_{i=1}^{2} \widetilde{H}^{\star - i - i\sigma}_{C_2}(S^0; \underline{\mathbb{Z}/2}).$$

Applying (2.15), we obtain the required Mackey functors in (5.6). Since $\mathrm{Sq}^2(x) = x^2$ for a generator $x \in H^2(\mathbb{C}P^\infty)$, the diagram (5.6) yields the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{\widetilde{\mathrm{Sq}^2}} & 0 \\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 0}\downarrow\uparrow{\scriptstyle 0} \\ \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 \end{array}$$

which is a contradiction.

Now we demonstrate the result in general. Let $P(\mathcal{U})^{\wedge r}$ be the smash product of r copies of $P(\mathcal{U})$. Equation (5.4) gives us

$$\begin{aligned} F(P(\mathcal{U})^{\wedge r}, H\underline{\mathbb{Z}/p}) &\simeq F_{H\underline{\mathbb{Z}}\text{-mod}}(P(\mathcal{U})^{\wedge r} \wedge H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}/p}) \\ &\simeq \bigvee_{k_j=0}^{\infty} \bigvee_{i_j=0}^{p-1} S^{-(k_1\phi_p + i_1\lambda + \cdots + k_r\phi_p + i_r\lambda)} \wedge H\underline{\mathbb{Z}/p}, \end{aligned}$$

where $j \in \{1, \dots, r\}$ and $(k_j, i_j) \neq (0,0)$. The last equivalence comes from the fact that $P(\mathcal{U})^{\wedge r} \wedge H\underline{\mathbb{Z}}$ is a wedge of suspensions of $H\underline{\mathbb{Z}}$ with finitely many $\Sigma^V H\underline{\mathbb{Z}}$ of a given dimension V. Hence

$$\widetilde{H}^{\star}_{C_p}(P(\mathcal{U})^{\wedge r}; \underline{\mathbb{Z}/p}) \cong \bigoplus_{k_j=0}^{\infty} \bigoplus_{i_j=0}^{p-1} \widetilde{H}^{\star - (k_1\phi_p + i_1\lambda + \cdots + k_r\phi_p + i_r\lambda)}_G(S^0; \underline{\mathbb{Z}/p}),$$

where $j \in \{1, \dots, r\}$ and $(k_j, i_j) \neq (0,0)$. We are now ready to prove Theorem 5.1.

Proof of the Theorem 5.1 when p is odd Let θ be a cohomology operation of degree r for r even, such that θ does not involve the Bockstein. This condition implies $(p-1) \mid r$. Let $s = \frac{r}{p-1}$. Consider the element $x_1 \otimes \cdots \otimes x_s \in \widetilde{H}^{2s}(\mathbb{C}P^{\infty \wedge s}; \mathbb{Z}/p)$, where each x_i is a generator of $\widetilde{H}^2(\mathbb{C}P^\infty; \mathbb{Z}/p)$. By an argument analogous to [30, Chapter 3, Theorem 2], we obtain $\theta(x_1 \otimes \cdots \otimes x_s) \neq 0$. Now suppose we have a map

$$\tilde{\theta}\colon H\underline{\mathbb{Z}/p} \to \Sigma^r H\underline{\mathbb{Z}/p}$$

such that $\Omega(\tilde{\theta}) = \theta$. This will lead to a map of Mackey functors

$$\tilde{\theta}\colon \underline{H}^{\alpha}_G(X;\underline{\mathbb{Z}/p}) \to \underline{H}^{\alpha+r}_G(X;\underline{\mathbb{Z}/p}).$$

Let us take $X = P(\mathcal{U})^{\wedge s}$ and $\alpha = s\lambda$. We observe that

$$\begin{array}{ccccc} \underline{H}^{s\lambda}_G(P(\mathcal{U})^{\wedge s};\underline{\mathbb{Z}/p}) & \xrightarrow{\tilde{\theta}} & \underline{H}^{s\lambda+r}_G(P(\mathcal{U})^{\wedge s};\underline{\mathbb{Z}/p}) & \xrightarrow{\text{proj}} & \underline{\mathbb{Z}/p}^* \\ \downarrow\cong & & \downarrow\cong & & \\ \underline{\mathbb{Z}/p} & & \underline{\mathbb{Z}/p}^{*\oplus t} \oplus \langle\mathbb{Z}/p\rangle^{\oplus \ell} & & \end{array} \tag{5.7}$$

for some integer $t \geq 1$ and $\ell \geq 0$. To see this, let $\alpha = s\lambda - k_1\phi_p - i_1\lambda - \cdots - k_s\phi_p - i_s\lambda$ and $\widetilde{\alpha} = s\lambda + r - \tilde{k}_1\phi_p - \tilde{i}_1\lambda - \cdots - \tilde{k}_s\phi_p - \tilde{i}_s\lambda$. The condition $(k_j, i_j) \neq (0,0)$ implies

$$|\alpha| = 2s - 2p(k_1 + \cdots + k_s) - 2(i_1 + \cdots + i_s) \leq 0.$$

If $|\alpha| < 0$, then the Mackey functor is zero by (2.14). So the left-hand side of diagram (5.7) turns out to be $\underline{\mathbb{Z}/p}$. However the Mackey functor $\underline{\mathbb{Z}/p}$ cannot appear in the right-hand side as the condition $|\widetilde{\alpha}| = 0$ forces $|\widetilde{\alpha}^{C_p}|$ to be > 0. This is because

$$|\widetilde{\alpha}| = 2s + r - 2p(\tilde{k}_1 + \cdots + \tilde{k}_s) - 2(\tilde{i}_1 + \cdots + \tilde{i}_s) = 0$$

implies some $i_j \neq 0$. Now if $|\widetilde{\alpha}^{C_p}| = r - 2(\tilde{k}_1 + \cdots + \tilde{k}_r) \leq 0$ then

$$|\widetilde{\alpha}| \leq 2s + (2-2p)(\tilde{k}_1 + \cdots + \tilde{k}_r) - 2(\tilde{i}_1 + \cdots + \tilde{i}_r) < 0.$$

So $|\widetilde{\alpha}^{C_p}| > 0$. Since $\theta(x_1 \otimes \cdots \otimes x_s) \neq 0$, the map $\underline{\mathbb{Z}/p}(G/e) \to \underline{\mathbb{Z}/p}^*(G/e)$ is an isomorphism. As in the Example 5.2, this gives a contradiction. □

As before, the $p = 2$ case requires a different argument which we detail below.

Proof of Theorem 5.1 when $p = 2$ Let θ be a cohomology operation of degree $2r$, such that θ does not involve the Bockstein. By an argument analogous to [30, Chapter 3, Theorem 2], we obtain

$$\theta(x_1 \otimes \cdots \otimes x_r) \neq 0,$$

where each x_i is a generator of $\widetilde{H}^2(\mathbb{C}P^\infty;\mathbb{Z}/p)$ and $x_1 \otimes \cdots \otimes x_r \in \widetilde{H}^{2r}(\mathbb{C}P^{\infty\wedge r};\mathbb{Z}/p)$. Now suppose we have a map

$$\tilde{\theta}\colon H\underline{\mathbb{Z}/2} \to \Sigma^{2r} H\underline{\mathbb{Z}/2}$$

such that $\Omega(\tilde{\theta}) = \theta$. This will lead to a map of Mackey functors

$$\tilde{\theta}\colon \underline{H}^{\alpha}_G(X;\underline{\mathbb{Z}/2}) \to \underline{H}^{\alpha+2r}_G(X;\underline{\mathbb{Z}/2}).$$

Let us take $X = \mathbb{C}P_\tau^{\infty\wedge r}$ and $\alpha = r\rho = r + r\sigma$. With the help of Theorem 4.8 and (2.15) we derive that

$$\begin{array}{ccccc} \underline{H}_G^{r\rho}(\mathbb{C}P_\tau^{\infty\wedge r};\underline{\mathbb{Z}/2}) & \xrightarrow{\tilde{\theta}} & \underline{H}_G^{r\rho+2r}(\mathbb{C}P_\tau^{\infty\wedge r};\underline{\mathbb{Z}/2}) & \xrightarrow{\text{proj}} & \underline{\mathbb{Z}/2}^* \\ \downarrow\cong & & \downarrow\cong & & \\ \underline{\mathbb{Z}/2} & & \underline{\mathbb{Z}/2}^* \oplus \langle\mathbb{Z}/2\rangle^{\oplus\ell} & & \end{array}$$

for some integer $\ell \geq 0$. Since $\theta(x_1 \otimes \cdots \otimes x_r) \neq 0$, the map $\underline{\mathbb{Z}/2}(G/e) \to \underline{\mathbb{Z}/2}^*(G/e)$ is an isomorphism. As in the examples, this gives a contradiction. □

6 The slice tower of $P(V)_+ \wedge H\mathbb{Z}$

The slice filtration in the equivariant stable homotopy category was introduced by Hill, Hopkins and Ravenel in their proof of the Kervaire invariant-one problem [18] (see also [17]). The associated slice tower is an equivariant analogue of the Postnikov tower. We use the regular slice filtration on equivariant spectra (see [33]), which differs from the original formulation by a shift of one [33, Proposition 3.1].

6.1 Definition Let $\tau_{\geq n}$ denote the localizing subcategory of genuine G-spectra which is generated by G-spectra of the form $G_+ \wedge_H S^{k\rho_H}$, where ρ_H is the (real) regular representation of H and $k|H| \geq n$.

Let E be a G-spectrum. Then E is said to be *slice n-connective* (written as $E \geq n$) if $E \in \tau_{\geq n}$, and E is said to be *slice n-coconnective* (written as $E < n$) if

$$[S^{k\rho_H + r}, E]^H = 0$$

for all subgroups $H \leq G$ such that $k|H| \geq n$ and for all $r \geq 0$. We say E is an *n-slice* if $n \leq E \leq n$.

In [22], the authors provide an alternative criterion for a G-spectrum being slice connective using the geometric fixed-point functor.

6.2 Theorem [22, Theorem 3.2] *The representation sphere S^V is in $\tau_{\geq n}$ if and only if for all $H \subset G$,*

$$\dim V^H \geq \frac{n}{|H|}.$$

We note the following result from [17, Proposition 2.23].

6.3 Proposition *If X is in $\tau_{\geq 0}$ and Y is in $\tau_{\geq n}$, then $X \wedge Y$ is in $\tau_{\geq n}$.*

There has been a large number of computations of slices for equivariant spectra. They have been either carried out in the case of *MU* or its variants [18; 20; 21] or for spectra of the form $\Sigma^n H\underline{\mathbb{Z}}$ [12; 15; 19; 31; 37]. We show that our cellular filtration yields the slice filtration for $P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ up to a suspension. In addition, the additive decomposition proves that the slice tower is degenerate in the sense that the maps possess sections.

6.4 Theorem *The slice towers of $\Sigma^2 P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^2 P(n\rho)_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.*

Proof Theorem 4.7 allows us to write

$$P(\mathcal{U})_+ \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{\ell=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\phi_\ell},$$

where $\phi_\ell = \lambda^{-l}(\lambda + \cdots + \lambda^{\ell-1})$ and $\phi_0 = 0$. We claim that each of $\Sigma^2 S^{\phi_\ell} \wedge H\underline{\mathbb{Z}}$ is a $2\ell+2$-slice. Let $H = C_m$, a subgroup of G. We verify

$$[S^{k\rho_H + r}, S^{\mathrm{res}_H(\phi_\ell)+2} \wedge H\underline{\mathbb{Z}}]^H \cong H_H^{\alpha}(S^0; \underline{\mathbb{Z}}) = 0$$

for $k|H| > 2\ell+2$ and $\alpha = \mathrm{res}_H(\phi_\ell) + 2 - r - k\rho_H$. We may write $\ell = qm + s$, where $s < m$, and thus $\mathrm{res}_H(\phi_\ell) = 2q\rho_H + \lambda + \cdots + \lambda^s$. Since $km > 2\ell + 2 = 2qm + 2s + 2$, we get $k > 2q$. Let $k = 2q + j$, so that

$$\alpha = -r - (\lambda^{s+1} + \cdots + \lambda^{m-1}) - (j-1)\rho_H.$$

Therefore either by Proposition 3.16, or using the fact that all the fixed-point dimensions of α are negative, we have $H_H^{\alpha}(S^0; \underline{\mathbb{Z}}) = 0$.

To show $S^{\phi_\ell+2} \wedge H\underline{\mathbb{Z}} \geq 2\ell+2$, using Proposition 6.3, it is enough to prove $S^{\phi_\ell+2} \in \tau_{\geq 2\ell+2}$ as $H\underline{\mathbb{Z}}$ is a 0-slice [18, Proposition 4.50]. Appealing to Theorem 6.2, $S^{\phi_\ell+2} \in \tau_{\geq 2\ell+2}$ as

$$\frac{2\ell+2}{m} = 2q + \frac{2s+2}{m} \leq 2q + 2 = \dim(S^{\phi_\ell+2})^H \quad \text{for all } H \subset G. \qquad \square$$

6.5 Theorem *The slice towers of $\Sigma^4 P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ and $\Sigma^4 P(n\rho_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}}$ are degenerate and these spectra are a wedge of slices of the form $S^V \wedge H\underline{\mathbb{Z}}$.*

Proof Theorem 4.10 allows us to write

$$P(\mathcal{U}_{\mathbb{H}})_+ \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{\ell=0}^{\infty} H\underline{\mathbb{Z}} \wedge S^{\xi_\ell},$$

where $\xi_\ell = \psi^1 + \cdots + \psi^\ell$ and $\xi_0 = 0$. Let $H = C_m$, a subgroup of G. We may write $\ell = qm + s$, where $s < m$. Consequently, $\mathrm{res}_H(\xi_\ell) = 4q\rho_H + \lambda + \cdots + \lambda^s$. Proceeding as in the case of Theorem 6.4, we deduce that each of $\Sigma^4 S^{\xi_\ell} \wedge H\underline{\mathbb{Z}}$ is a $4\ell+4$-slice. $\square$

7 Ring structure of $B_G S^1$

The objective of this section is to compute the equivariant cohomology ring $H_G^{\star}(B_G S^1)$ for $G = C_{p^m}$, p prime and $m \geq 1$. We use the identification $B_G S^1 \simeq P(\mathcal{U})$ where $\mathcal{U}$ is a complete G-universe. The additive decomposition of Section 4 already provides a basis for the cohomology as a module over $H_G^{\star}(\mathrm{pt}) = \pi_{-\star}^G(H\underline{\mathbb{Z}})$.

7.1 Multiplicative generators of $H^\star_G(P(\mathcal{U}))$ The fixed-point space $P(\mathcal{U})^G$ is a disjoint union of n-copies of $\mathbb{C}P^\infty$ which are included in $P(\mathcal{U})$ as $P(\infty\lambda^i) = \mathrm{colim}_k P(k\lambda^i)$ for $0 \le i \le n-1$. Let $q_i \colon P(\infty\lambda^i) \to P(\mathcal{U})$ denote the inclusion. In particular, we have

$$q_0 \colon P(\mathbb{C}^\infty) \to P(\mathcal{U}). \tag{7.2}$$

For the trivial G-action on $\mathbb{C}P^\infty$, we get

$$H^\star_G(\mathbb{C}P^\infty) = H^*(\mathbb{C}P^\infty) \otimes H^\star_G(\mathrm{pt}) = H^\star_G(\mathrm{pt})[x],$$

where x is the multiplicative generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$.

Recall from Section 4.11 that the classes $\alpha_{k\phi_n+\phi_i}$ form an additive generating set for $H^\star_G(P(\mathcal{U}))$. The following notes the multiplicative generating set.

7.3 Proposition *The collection $\{\alpha_{\phi_d} \mid d \text{ divides } n\}$ generates $H^\star_G(P(\mathcal{U}))$ as an algebra over $H^\star_G(\mathrm{pt})$.*

Proof We show by induction that each $\alpha_{k\phi_n+\phi_i}$ may be expressed as a sum of monomials on the α_{ϕ_d}. Suppose we know that all generators with degree lower than $\alpha_{k\phi_n+\phi_i}$ can be written in terms of α_{ϕ_d}'s. Let $i = \sum_{j=0}^{\ell} r_j p^j$ where $0 \le r_j \le p-1$. The class $\alpha^k_{\phi_n}\alpha^{r_\ell}_{\phi_{p^\ell}} \cdots \alpha^{r_0}_{\phi_1}$ also belongs to $H^{k\phi_n+\phi_i}_G(P(\mathcal{U}))$. So we may express the class as

$$\alpha^k_{\phi_n}\alpha^{r_\ell}_{\phi_{p^\ell}} \cdots \alpha^{r_0}_{\phi_1} = c_{\phi_1}\alpha_{\phi_1} + \cdots + c_{k\phi_n+\phi_i}\alpha_{k\phi_n+\phi_i}, \quad \text{where } c_{\phi_j} \in H^\star_G(\mathrm{pt}). \tag{7.4}$$

Note that in (7.4), a generator $\alpha_{t\phi_n+\phi_s}$ with degree greater than the degree of $\alpha_{k\phi_n+\phi_i}$ cannot appear; this is because for $\zeta = k\phi_n + \phi_i - (t\phi_n + \phi_s)$, the group $H^\zeta_G(\mathrm{pt})$ equals 0 by (3.8) as all the fixed-point dimensions of ζ are negative. Now, except the class $c_{k\phi_n+\phi_i}$, the degree of the other c_ϕ must be greater than zero; hence their restriction to G/e is zero. Thus, $\mathrm{res}^G_e(\alpha^k_{\phi_n}\alpha^{r_\ell}_{\phi_{p^\ell}} \cdots \alpha^{r_0}_{\phi_1}) = x^{n+i} = \mathrm{res}^G_e(\alpha_{k\phi_n+\phi_i})$ by (4.14). Therefore, $c_{k\phi_n+\phi_i}$ must be 1. ∎

7.5 Example Let $G = C_p$. The two classes α_{ϕ_1} and α_{ϕ_p} generate $H^\star_{C_p}(P(\mathcal{U}))$. The computations of Lewis [25, Section 5] may be adapted to prove

$$H^\star_G(P(\mathcal{U})) \cong H^\star_G(\mathrm{pt})[\alpha_{\phi_1},\alpha_{\phi_p}]\Big/\Big(u_\lambda\alpha_{\phi_p} - \alpha_{\phi_1}\prod_{i=1}^{p-1}(ia_\lambda + \alpha_{\phi_1})\Big). \tag{7.6}$$

The relation is obtained by restriction to various fixed points.

The proof of Proposition 7.3 also demonstrates that one may change the basis of $H^\star_G(P(\mathcal{U}))$ over $H^\star_G(\mathrm{pt})$ from $\{\alpha_{k\phi_n+\phi_i}\}$ to $\{\alpha^k_{\phi_n}\alpha^{r_\ell}_{\phi_{p^\ell}} \cdots \alpha^{r_0}_{\phi_1}\}$ where $0 \le r_j \le p-1$. Therefore, there exists a relation of the form

$$\alpha^p_{\phi_{p^j}} = c\alpha_{\phi_{p^{j+1}}} + \text{lower-order terms}.$$

By restricting to G/e, we see that c must be $u_{\lambda^{p^j}-\lambda^{p^{j+1}}}$. The lower-order terms will be calculated by restriction to fixed points. The next result describes $q_0^*(\alpha_{\phi_1})$.

7.7 Proposition $q_0^*(\alpha_{\phi_1}) = u_\lambda x.$

Proof At $\deg\lambda$, $H_G^\star(\mathbb{C}P^\infty)$ has a basis given by a_λ and $u_\lambda x$. So $q_0^*(\alpha_{\phi_1}) = c_1 a_\lambda + c_2 u_\lambda x$. In the notation of (4.12), $W_1 = 1_{\mathbb{C}} + \lambda$. Consider the map $i\colon P(W_1) = S^{\lambda^{-1}} \hookrightarrow P(\mathcal{U})$ and $f\colon \mathrm{pt} \hookrightarrow \mathbb{C}P^\infty$. Consider the commutative diagram

$$\begin{array}{ccccc} H^2(\mathbb{C}P^\infty) & \xleftarrow{\mathrm{res}_e^G} & H_G^\lambda(P(\mathcal{U})) & \xrightarrow{i^*} & H_G^\lambda(S^{\lambda^{-1}}) \\ \big\downarrow{\cong} & & \big\downarrow{q_0^*} & & \big\downarrow{q_0^*} \\ H^2(\mathbb{C}P^\infty) & \xleftarrow{\mathrm{res}_e^G} & H_G^\lambda(\mathbb{C}P^\infty) & \xrightarrow{f^*} & H_G^\lambda(\mathrm{pt}) \end{array}$$

By (4.14), $\mathrm{res}_e^G(\alpha_{\phi_1}) = x$. So the left commutative square implies c_2 must be 1. Next, observe that the map i^* sends α_{ϕ_1} to the generator corresponding to $1 \in H_G^0(S^0) \simeq \widetilde{H}_G^\lambda(S^\lambda) \subseteq H_G^\lambda(S^\lambda)$. The cofiber sequence $P(1_{\mathbb{C}})_+ \simeq S^0 \xrightarrow{q_0} P(W_1)_+ \to S^\lambda$ implies $q_0^* i^*(\alpha_{\phi_1}) = 0$. So $c_1 = 0$, and thus $q_0^*(\alpha_{\phi_1}) = u_\lambda x$. □

7.8 Restrictions to fixed points We adapt the approach of Lewis [25] to our case for calculating $q_0^*(\alpha_{\phi_d})$. For a subset I of $\underline{d-1} := \{1, 2, \dots, d-1\}$, define

$$\omega_I = \lambda^{-d}\Big(1_{\mathbb{C}} + \sum_{i\in I} \lambda^i\Big) \quad \text{and} \quad V_{I,k} = 1_{\mathbb{C}} + \sum_{i\in I} \lambda^i + \lambda^d + k \cdot 1_{\mathbb{C}},$$

for $k \geq 0$. Consider the cofiber sequence

$$P(V_{I,0} - \lambda^d)_+ \to P(V_{I,0})_+ \xrightarrow{\chi} S^{\omega_I},$$

which implies the long exact sequence

$$\cdots \to \widetilde{H}_G^{\omega_I - 1}(P(V_{I,0} - \lambda^d)_+) \to \widetilde{H}_G^{\omega_I}(S^{\omega_I}) \xrightarrow{\chi^*} \widetilde{H}_G^{\omega_I}(P(V_{I,0})_+) \to \widetilde{H}_G^{\omega_I}(P(V_{I,0} - \lambda^d)_+) \to \cdots.$$

Define $\Delta_{\omega_I}^{V_{I,0}} \in H_G^{\omega_I}(P(V_{I,0}))$ to be $\chi^*(1)$. Next we lift the class $\Delta_{\omega_I}^{V_{I,0}}$ uniquely to the class $\Delta_{\omega_I}^{V_{I,k}} \in H_G^{\omega_I}(P(V_{I,k}))$ with the help of the cofiber sequence $P(V_{I,\ell})_+ \xrightarrow{\theta_\ell} P(V_{I,\ell+1})_+ \to S^{V_{I,\ell}}$. At degree ω_I we get

$$\cdots \to \widetilde{H}_G^{\omega_I}(S^{V_{I,\ell}}) \to \widetilde{H}_G^{\omega_I}(P(V_{I,\ell+1})_+) \xrightarrow{\theta_\ell^*} \widetilde{H}_G^{\omega_I}(P(V_{I,\ell})_+) \to \widetilde{H}_G^{\omega_I+1}(S^{V_{I,\ell}}) \to \cdots.$$

We claim θ_ℓ^* is an isomorphism. To see this we observe that as $i < d$ and d is a power of p, all the fixed-point dimensions of λ^{i-d} and λ^i are same. Hence all the fixed-point dimensions of $\omega_I - V_{I,\ell}$ are less than or equal to -2, so $H_G^{\omega_I - V_{I,\ell}}(\mathrm{pt}) = 0$ and $H_G^{\omega_I + 1 - V_{I,\ell}}(\mathrm{pt}) = 0$ for $\ell \geq 0$. As the restriction of χ^* to the orbit G/e is an isomorphism, we have

(7.9) $$\mathrm{res}_e^G(\Delta_{\omega_I}^{V_{I,k}}) = x^{|I|+1}.$$

In the same spirit, using the cofiber sequence

$$P(V_{I,\ell})_+ \hookrightarrow P(V_{I,\ell+1})_+ \xrightarrow{\chi} S^{V_{I,\ell}},$$

we may define the class $\Omega_{V_{I,\ell}}^{V_{I,\ell+1}} \in H_G^{V_{I,\ell}}(P(V_{I,\ell+1}))$ to be the image of $\chi^*(1)$ where $1 \in \widetilde{H}_G^{V_{I,\ell}}(S^{V_{I,\ell}}) \cong \mathbb{Z}$. As for $\Delta_{\omega_I}^{V_{I,k}}$, this lifts uniquely to define the class $\Omega_{V_{I,\ell}}^{V_{I,k}} \in H_G^{V_{I,\ell}}(P(V_{I,k}))$. We define $\Omega_{V_{I,k}}^{V_{I,k}} = 0$. Its restriction to the orbit G/e is

$$\mathrm{res}_e^G(\Omega_{V_{I,\ell}}^{V_{I,k}}) = x^{|I|+\ell+2}.$$

For $i \in I$, let $\tau_{i,k}$ (or simply τ_i) be the inclusion map $P(V_{I\setminus\{i\},k}) \hookrightarrow P(V_{I,k})$.

7.10 Proposition *For the map $\tau_{i,k}$ we get the following*:

(1) $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}$.

(2) $\tau_{i,k}^*(\Omega_{V_{I,\ell}}^{V_{I,k}}) = a_{\lambda^i} \Omega_{V_{I\setminus\{i\},\ell}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i} \Omega_{V_{I\setminus\{i\},\ell+1}}^{V_{I\setminus\{i\},k}}$ *for* $0 \le \ell < k$.

Proof We prove (1). The proof of (2) is analogous. We start with the representation $V_{I,0}$, and successively add $1_{\mathbb{C}}$ to reach $V_{I,k}$. First consider the following diagram:

(7.11)
$$\begin{array}{ccc}
P(V_{I\setminus\{i\},0} - \lambda^d)_+ & \hookrightarrow & P(V_{I,0} - \lambda^d)_+ \\
\downarrow & & \downarrow \\
P(V_{I\setminus\{i\},0})_+ & \xrightarrow{\tau_{i,0}} & P(V_{I,0})_+ \\
\downarrow & & \downarrow \\
S^{\omega_{I\setminus\{i\}}} & \xrightarrow{a_{\lambda^{i-d}}} & S^{\omega_I}
\end{array}$$

Since $i < d$, the p-adic valuation of $i-d$ is the same as i. Thus

(7.12)
$$a_{\lambda^{i-d}} = \Theta_{i,d} \cdot a_{\lambda^i}$$

by Proposition 2.13. At $\deg \omega_I$, the bottom commutative square gives us

$$\tau_{i,0}^*(\Delta_{\omega_I}^{V_{I,0}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},0}}.$$

In the next step, we add $1_{\mathbb{C}}$ to the representations in the middle row of the diagram (7.11). This fits into the following diagram:

(7.13)
$$\begin{array}{ccc}
P(V_{I\setminus\{i\},0})_+ & \xrightarrow{\tau_{i,0}} & P(V_{I,0})_+ \\
{\scriptstyle +1_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle +1_{\mathbb{C}}} \\
P(V_{I\setminus\{i\},1})_+ & \xrightarrow{\tau_{i,1}} & P(V_{I,1})_+ \\
\downarrow & & \downarrow \\
S^{V_{I\setminus\{i\},0}} & \longrightarrow & S^{V_{I,0}}
\end{array}$$

Since we have built $P(V_{I\setminus\{i\},0})$ by attaching cells in a proper order, the boundary maps in the cohomology long exact sequence induced by the left-hand cofibration sequence are trivial. Moreover, $H_G^\star(P(V_{I\setminus\{i\},0}))$ is free as an $H_G^\star(\mathrm{pt})$-module. So

$$H_G^{\omega_I}(P(V_{I\setminus\{i\},1}) \cong H_G^{\omega_I}(P(V_{I\setminus\{i\},0})) \oplus H_G^{\omega_I - V_{I\setminus\{i\},0}}(\mathrm{pt}).$$

Further, $H_G^{\omega_I - V_{I\setminus\{i\},0}}(\mathrm{pt}) \cong \mathbb{Z}$ generated by the class u_{λ^i}. So the diagram (7.13) implies

$$\tau_{i,1}^*(\Delta_{\omega_I}^{V_{I,1}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},1}} + c \cdot u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},1}}$$

for some $c \in \mathbb{Z}$. We claim that $c = 1$. For this, consider the diagram

$$\begin{array}{ccc}
P(V_{I\setminus\{i\},1})_+ & \xrightarrow{\tau_{i,1}} & P(V_{I,1})_+ \\
\big\downarrow{\scriptstyle +1_{\mathbb{C}}} & & \big\downarrow{\scriptstyle +1_{\mathbb{C}}} \\
P(V_{I\setminus\{i\},2})_+ & \xrightarrow{\tau_{i,2}} & P(V_{I,2})_+ \\
\big\downarrow & & \big\downarrow \\
S^{V_{I\setminus\{i\},1}} & \longrightarrow & S^{V_{I,1}}
\end{array}$$

All the fixed-point dimensions of $\omega_I - V_{I\setminus\{i\},1}$ are less than or equal to -2. Hence $H_G^{\omega_I - V_{I\setminus\{i\},1}}(\mathrm{pt}) = 0$. Consequently, $H_G^{\omega_I}(P(V_{I\setminus\{i\},2})) \cong H_G^{\omega_I}(P(V_{I\setminus\{i\},1}))$. Furthermore, $H_G^{\omega_I - V_{I,1}}(\mathrm{pt}) = 0$. As a result,

$$\tau_{i,2}^*(\Delta_{\omega_I}^{V_{I,2}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},2}} + c \cdot u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},2}}.$$

Repeating this process, we obtain

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = \Theta_{i,d} \cdot a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + c \cdot u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

The map $\tau_{i,k}^*$ at the orbit G/e is an isomorphism. Moreover, by (7.9),

$$\mathrm{res}_e^G(\Delta_{\omega_I}^{V_{I,k}}) = x^{|I|+1} = \mathrm{res}_e^G(\Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}).$$

So c must be 1, otherwise, we get a contradiction by restricting to the orbit G/e. □

In the case $d = p^m$, the following simplification occurs as $\frac{p^m}{i} a_{\lambda^i} = 0$ by (3.2).

7.14 Proposition *For the map $\tau_{i,k}$, we get*

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = a_{\lambda^i} \Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i} \Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

If we work with $\underline{\mathbb{Z}/p}$-coefficients, then $a_{\lambda^{i-d}} = a_{\lambda^i}$ as $\Theta_{i,d} \equiv 1 \pmod p$ by Proposition 2.13. This simplification leads to the following.

7.15 Proposition *In $\mathbb{Z}/p$-coefficients*

$$\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}}) = a_{\lambda^i}\,\Delta_{\omega_{I\setminus\{i\}}}^{V_{I\setminus\{i\},k}} + u_{\lambda^i}\,\Omega_{V_{I\setminus\{i\},0}}^{V_{I\setminus\{i\},k}}.$$

The value of $\tau_{i,k}^*(\Omega_{V_{I,\ell}}^{V_{I,k}})$ remains the same as in Proposition 7.10.

7.16 Remark For the group C_p, we have $P(V_{\varnothing,k}) = P(\mathbb{C}^{k+2})$. Hence the class $\Delta_{\omega_\varnothing}^{V_{\varnothing,k}}$ is the class $x \in H_G^2(P(\mathbb{C}^{k+2}))$, and the class $\Omega_{V_{\varnothing,\ell}}^{V_{\varnothing,k}}$ is the class $x^{\ell+2} \in H_G^{2\ell+4}(P(\mathbb{C}^{k+2}))$.

7.17 Proposition *In the case when $I = \varnothing$, we obtain*

(1) $\tau_{d,k}^*(\Delta_{\omega_\varnothing}^{V_{\varnothing,k}}) = u_{\lambda^d}\cdot x$,

(2) $\tau_{d,k}^*(\Omega_{V_{\varnothing,\ell}}^{V_{\varnothing,k}}) = a_{\lambda^d}\cdot x^{\ell+1} + u_{\lambda^d}\cdot x^{\ell+2}$.

Proof Recall that $V_{\varnothing,k} = 1_{\mathbb{C}} + \lambda^d + k\cdot 1_{\mathbb{C}}$. The cofiber sequence

$$P(1_{\mathbb{C}})_+ \xrightarrow{\tau_{d,0}} P(1_{\mathbb{C}} + \lambda^d)_+ \to S^{\omega_\varnothing}$$

implies $\tau_{d,0}^*(\Delta_{\omega_\varnothing}^{V_{\varnothing,0}}) = 0$. The rest of the proof is quite similar to Proposition 7.10. So we describe it briefly. In the next step, we have

$$\begin{array}{ccc}
P(1_{\mathbb{C}})_+ & \xhookrightarrow{\tau_{d,0}} & P(V_{\varnothing,0})_+ \\
\big\downarrow & & \big\downarrow \\
P(2\cdot 1_{\mathbb{C}})_+ & \xhookrightarrow{\tau_{d,1}} & P(V_{\varnothing,1})_+ \\
\big\downarrow & & \big\downarrow \\
S^2 & \longrightarrow & S^{V_{\varnothing,0}}
\end{array}$$

At degree $\omega_\varnothing$, $H_G^{\omega_\varnothing-2}(\mathrm{pt}) \cong \mathbb{Z}\{u_{\lambda^d}\}$, and $H_G^{\omega_\varnothing - V_{\varnothing,0}}(\mathrm{pt}) = 0$. Using restriction to the orbit G/e, we may conclude that $\tau_{d,1}^*(\Delta_{\omega_\varnothing}^{V_{\varnothing,k}}) = \tau_{d,k}^*(\Delta_{\omega_\varnothing}^{V_{\varnothing,k}}) = u_{\lambda^d}\cdot x$. □

The following is a direct calculation.

7.18 Proposition *Let $\mathcal{I} = \{i_1,\dots,i_r\}$. Then*

$$\tau_d\tau_{i_1}\cdots\tau_{i_r}(\Omega_{V_{I,t}}^{V_{I,k}}) = x^{t+1}(a_{\lambda^d} + u_{\lambda^d}\cdot x)\prod_{s=1}^{r}(a_{\lambda^{i_s}} + x u_{\lambda^{i_s}}).$$

Proof Let $\mathcal{I} = \{i_1,\dots,i_r\}$. Propositions 7.10 and 7.17 give

$$\tau_d\tau_{i_1}\cdots\tau_{i_r}(\Omega_{V_{I,t}}^{V_{I,k}}) = \tau_d\Big[\sum_{\ell=t}^{t+r}\Big(\Omega_{V_{\varnothing,\ell}}^{V_{\varnothing,k}}\sum_{\{j_1,\dots,j_{\ell-t}\}\subseteq\mathcal{I}} u_{\lambda^{j_1}}\cdots u_{\lambda^{j_{\ell-t}}}a_{\lambda^{j_{\ell-t+1}}}\cdots a_{\lambda^{j_r}}\Big)\Big].$$

Applying τ_d to $\Omega_{V_{\varnothing,\ell}}^{V_{\varnothing,k}}$, we get

$$\begin{aligned}\tau_d\tau_{i_1}\cdots\tau_{i_r}(\Omega_{V_{I,t}}^{V_{I,k}}) &= \sum_{\ell=t}^{t+r}\Big((a_{\lambda^d}\cdot x^{\ell+1}+u_{\lambda^d}\cdot x^{\ell+2})\sum_{\{j_1,\dots,j_{\ell-t}\}\subseteq\mathcal{I}} u_{\lambda^{j_1}}\cdots u_{\lambda^{j_{\ell-t}}}a_{\lambda^{j_{\ell-t+1}}}\cdots a_{\lambda^{j_r}}\Big)\\ &= x^{t+1}(a_{\lambda^d}+u_{\lambda^d}\cdot x)\sum_{\ell=t}^{t+r}x^{\ell-t}\Big(\sum_{\{j_1,\dots,j_{\ell-t}\}\subseteq\mathcal{I}} u_{\lambda^{j_1}}\cdots u_{\lambda^{j_{\ell-t}}}a_{\lambda^{j_{\ell-t+1}}}\cdots a_{\lambda^{j_r}}\Big)\\ &= x^{t+1}(a_{\lambda^d}+u_{\lambda^d}\cdot x)\sum_{\ell=0}^{r}x^{\ell}\Big(\sum_{\{j_1,\dots,j_{\ell}\}\subseteq\mathcal{I}} u_{\lambda^{j_1}}\cdots u_{\lambda^{j_{\ell}}}a_{\lambda^{j_{\ell+1}}}\cdots a_{\lambda^{j_r}}\Big).\end{aligned}$$

This easily factorizes to imply the result. □

Now we are in a position to determine $q_0^*(\alpha_{\phi_d})$.

7.19 Proposition *The image of α_{ϕ_d} under the map q_0^* is*

$$q_0^*(\alpha_{\phi_d})=\sum_{i=0}^{d-1}\Big[\Big(\prod_{j=1}^{i}\Theta_{j,d}a_{\lambda^j}\Big)u_{\lambda^{i+1}}x\prod_{s=i+2}^{d}(a_{\lambda^s}+u_{\lambda^s}x)\Big].$$

Proof Consider the map $i\colon P(V_{\underline{d-1},d})\to P(\mathcal{U})$ given by inclusion. We claim $i^*(\alpha_{\phi_d})=\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}$. The reason is as follows: both the classes $\alpha_{\phi_d}^{W_d}$ and $\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},0}}$ were defined to be $\chi^*(1)$ using the cofiber sequence $P(W_{d-1})_+\hookrightarrow P(W_d)_+\xrightarrow{\chi} S^{\phi_d}$ (see Sections 4.11 and 7.8). So these two classes are the same. Then we extended these classes through a chain of isomorphisms by successively adding the representation λ^i (resp $1_{\mathbb{C}}$) to define the class α_{ϕ_d} (resp $\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}$). In the end, we have $P(V_{\underline{d-1},d})\hookrightarrow P(\mathcal{U})$, so $i^*(\alpha_{\phi_d})=\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}$.

To determine $q_0^*(\alpha_{\phi_d})$, it is enough to work out $q_0^*(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}})$. For this, we successively remove all the nontrivial representations from $V_{\underline{d-1},d}$. Now

$$q_0^*(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}})=\tau_d\cdots\tau_2\tau_1(\Delta_{\omega_{\underline{d-1}}}^{V_{\underline{d-1},d}}).$$

Applying Proposition 7.15, this becomes

$$\tau_d\cdots\tau_2(\Theta_{1,d}\cdot a_\lambda\Delta_{\omega_{\underline{d-1}\setminus\{1\}}}^{V_{\underline{d-1}\setminus\{1\},d}})+\tau_d\cdots\tau_2(u_\lambda\Omega_{V_{\underline{d-1}\setminus\{1\},0}}^{V_{\underline{d-1}\setminus\{1\},d}}). \tag{7.20}$$

Let $z_s=a_{\lambda^s}+u_{\lambda^s}x$. The second term can be simplified by Proposition 7.18 to

$$u_\lambda x\prod_{s=2}^{d}(a_{\lambda^s}+u_{\lambda^s}x)=u_\lambda x\prod_{s=2}^{d}z_s.$$

Now applying τ_2 in (7.20) and repeating the above procedure, we get

$$\tau_d\cdots\tau_3(\Theta_{1,d}\cdot a_\lambda\Theta_{2,d}\cdot a_{\lambda^2}\Delta_{\omega_{\underline{d-1}\setminus\{1,2\}}}^{V_{\underline{d-1}\setminus\{1,2\},d}})+u_\lambda x\prod_{s=2}^{d}z_s+\Theta_{1,d}\cdot a_\lambda u_{\lambda^2}x\prod_{s=3}^{d}z_s.$$

Repeating this process up to τ_d we obtain the description of $q_0^*(\alpha_{\phi_d})$. □

When $d = p^m$, in the expression of $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}})$, the numbers $\Theta_{i,d}$ become 1 (see Propositions 7.14 and 7.10). Using Proposition 7.14 we obtain the following simplification.

7.21 Proposition *The image of* $\alpha_{\phi_{p^m}}$ *under the map* q_0^* *is*

$$q_0^*(\alpha_{\phi_{p^m}}) = \prod_{i=1}^{p^m} (a_{\lambda^i} + x u_{\lambda^i}) - \prod_{i=1}^{p^m} a_{\lambda^i}.$$

Proof Since $\frac{p^m}{i} a_{\lambda^i} = 0$ by (3.2), from Proposition 7.19 we get

$$\begin{aligned} q_0^*(\alpha_{\phi_{p^m}}) &= \sum_{i=0}^{p^m-1} \Big[\Big(\prod_{j=1}^{i} a_{\lambda^j} \Big) u_{\lambda^{i+1}} x \prod_{s=i+2}^{p^m} (a_{\lambda^s} + u_{\lambda^s} x) \Big] \\ &= \sum_{i=0}^{p^m-1} \Big[\Big(\prod_{j=1}^{i} a_{\lambda^j} \Big) u_{\lambda^{i+1}} x \prod_{s=i+2}^{p^m} z_s \Big] \\ &= \sum_{i=0}^{p^m-1} \Big[\Big(\prod_{j=1}^{i} a_{\lambda^j} \Big) \prod_{s=i+1}^{p^m} z_s - \Big(\prod_{j=1}^{i+1} a_{\lambda^j} \Big) \prod_{s=i+2}^{p^m} z_s \Big] \\ &= \prod_{i=1}^{p^m} z_s - \prod_{i=1}^{p^m} a_{\lambda^i}, \end{aligned}$$

where $z_s = a_{\lambda^s} + u_{\lambda^s} x$. □

Either taken in $\underline{\mathbb{Z}/p}$-coefficient or in the case of $d = p^m$, the expression of $\tau_{i,k}^*(\Delta_{\omega_I}^{V_{I,k}})$ is the same (see Propositions 7.14 and 7.15). As a result we may proceed as in Proposition 7.21 to obtain the following.

7.22 Proposition *With* $\underline{\mathbb{Z}/p}$*-coefficients,*

$$q_0^*(\alpha_{\phi_d}) = \prod_{i=1}^{d} (a_{\lambda^i} + x u_{\lambda^i}) - \prod_{i=1}^{d} a_{\lambda^i}.$$

Once we have the expressions for q_0 on the multiplicative generators, we relate them to obtain the relations in the cohomology ring. The following proposition states that q_0^* is injective, which means that the image of q_0^* may be used to detect relations.

7.23 Proposition *For every* $j \geq 1$, *the map* q_0^* *is injective at the degree* $\zeta_{p^j} = \lambda + \cdots + \lambda^{p^j-1} + \lambda^{p^{j-1}}$.

Proof Recall from the additive decomposition of Theorem 4.7 that

$$H_G^\star(P(\mathcal{U})) \cong \bigoplus_{i \geq 0} H_G^{\star-\phi_i}(\mathrm{pt})\{\alpha_{\phi_i}\}.$$

The notation here means that α_{ϕ_i} generates the factor $H_G^{\star-\phi_i}(\mathrm{pt})$ of the free $H_G^\star(\mathrm{pt})$-module $H_G^\star(P(\mathcal{U}))$. Therefore, $H_G^{\zeta_{p^j}}(P(\mathcal{U})) \cong \bigoplus_{i=0}^{p^j} H_G^{\zeta_{p^j}-\phi_i}(\mathrm{pt})\{\alpha_{\phi_i}\}$. The higher-degree generators ($\alpha_{\phi_{p^j+1}}$ onwards) cannot appear as by Proposition 3.10 the group $H_G^{\zeta_{p^j}-\phi_{p^j}+\ell}(\mathrm{pt})$ is zero for $\ell \geq 1$. For $i \leq p^j - 1$, we use Theorem 3.1 to note that the element $a_{\zeta_{p^j}-\phi_i} := a_{\lambda^{i+1}} \cdots a_{\lambda^{p^j-1}} a_{\lambda^{p^{j-1}}}$ generates the group $H_G^{\zeta_{p^j}-\phi_i}(\mathrm{pt}) \cong \mathbb{Z}/p^{m-j+1}$ (look at the discussion following Theorem 3.1). Therefore the element $a_{\zeta_{p^j}-\phi_i}\alpha_{\phi_i} \in H_G^{\zeta_{p^j}}(P(\mathcal{U}))$ also has order p^{m-j+1}. Hence,

$$H_G^{\zeta_{p^j}}(P(\mathcal{U}) \cong \mathbb{Z} \oplus (\mathbb{Z}/p^{m-j+1})^{\oplus p^j}.$$

Similarly,

$$H_G^{\star}(P(\infty 1_{\mathbb{C}})) \cong \bigoplus_{i \geq 0} H_G^{\star-2i}(\mathrm{pt})\{x^i\}.$$

So $H_G^{\zeta_{p^j}}(P(\infty 1_{\mathbb{C}})) \cong \bigoplus_{i=0}^{p^j} H_G^{\zeta_{p^j}-2i}(\mathrm{pt})\{x^i\}$. By the discussion following Theorem 3.1, $H_G^{\zeta_{p^j}-2i}(\mathrm{pt}) \cong \mathbb{Z}/p^{t_{j,i}}$ for $i < p^j$, where

$$t_{j,i} = \max\bigl[m - \max\{v_p(s_1), \ldots, v_p(s_{p^j-i})\} \mid s_1, \ldots, s_{p^j-i} \in \{1, \ldots, p^j - 1, p^{j-1}\}\bigr].$$

This tells us

$$H_G^{\zeta_{p^j}}(P(\infty 1_{\mathbb{C}})) \cong \mathbb{Z} \oplus \mathbb{Z}/p^{t_{j,p^j-1}} \oplus \cdots \oplus \mathbb{Z}/p^{t_{j,0}}.$$

We observe that the element

$$u_{\phi_i} a_{\zeta_{p^j}-\phi_i} := u_\lambda u_{\lambda^2} \cdots u_{\lambda^i} a_{\lambda^{i+1}} \cdots a_{\lambda^{p^j-1}} a_{\lambda^{p^{j-1}}} \in H_G^{\zeta_{p^j}-2i}(\mathrm{pt})$$

has order p^{m-j+1}. So $a_{\zeta_{p^j}-\phi_i} u_{\phi_i} x^i \in H_G^{\zeta_{p^j}}(P(\infty 1_{\mathbb{C}}))$ is also of order p^{m-j+1} in $\mathbb{Z}/t_{j,i}$. Since $\mathrm{res}_e^G(\alpha_{\phi_i}) = x^i$, we have $\mathrm{res}_e^G(q_0^*(\alpha_{\phi_i})) = x^i$. This implies $q_0^*(\alpha_{\phi_i}) = u_{\phi_i} x^i + \sum_{l=0}^{i-1} c_l x^l$, for some coefficients c_l. Thus

$$q_0^*(a_{\zeta_{p^j}-\phi_i}\alpha_{\phi_i}) = a_{\zeta_{p^j}-\phi_i} q_0^*(\alpha_{\phi_i}) = a_{\zeta_{p^j}-\phi_i} u_{\phi_i} x^i + a_{\zeta_{p^j}-\phi_i} \sum_{l=0}^{i-1} c_l x^l.$$

Therefore q_0^* as a map

$$\mathbb{Z} \oplus (\mathbb{Z}/p^{m-j+1})^{\oplus p^j} \to \mathbb{Z} \oplus \mathbb{Z}/p^{t_{j,p^j-1}} \oplus \cdots \oplus \mathbb{Z}/p^{t_{j,0}}$$

is a lower triangular matrix of the form

$$\begin{pmatrix} 1 & & & \\ * & k_{j,p^j-1} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & k_{j,0} \end{pmatrix},$$

where $k_{j,i} = p^{t_{j,i}-m+j-1}$. Hence q_0^* is injective at the degree ζ_{p^j}. □

7.24 Relations for complex projective spaces If $d = p^m$, then $a_{\lambda^d} = 0$ and $u_{\lambda^d} = 1$, so Proposition 7.21 simplifies to $q_0^*(\alpha_{\phi_n}) = \prod_{i=1}^{n-1} x(a_{\lambda^i} + x u_{\lambda^i})$. For the group C_p, using Proposition 2.13, this further reduces to $q_0^*(\alpha_{\phi_p}) = x \prod_{i=1}^{p-1}(i a_\lambda + u_\lambda x)$. Using the fact that $q_0^*(\alpha_{\phi_1}) = u_\lambda x$ from Proposition 7.7,

$$q_0^*\Bigl(u_\lambda \alpha_{\phi_p} - \alpha_{\phi_1} \prod_{i=1}^{p-1} (i a_\lambda + \alpha_{\phi_1})\Bigr) = 0.$$

Moreover, Proposition 7.23 tells us that q_0^* is injective. So the relation we obtain for C_p is

$$u_\lambda \alpha_{\phi_p} - \alpha_{\phi_1} \prod_{i=1}^{p-1} (i a_\lambda + \alpha_{\phi_1}).$$

For general C_{p^m} of order $n = p^m$, there are m relations of the form

$$u_{\lambda^{p^{i-1}} - \lambda^{p^i}} \alpha_{\phi_{p^i}} = \alpha_{\phi_{p^{i-1}}}^p + \text{lower-order terms},$$

for $1 \le i \le m$. In fact, the proof of Proposition 7.3 implies that the coefficients of the lower-order terms are expressible as a sum of monomials with coefficients that are linear combinations of products of a_{λ^j}. The naive idea is to apply q_0^* to such an equation to determine all the coefficients. However, the expression in Proposition 7.19 does not directly yield a simple closed relation. We are able to obtain a simple expression after mapping to $\mathbb{Z}/p$-coefficients.

The first observation when we look at $\mathbb{Z}/p$-coefficients is that q_0^* is no longer injective. For, in the proof of Proposition 7.23, the diagonal entries in the lower triangular matrix, other than at the top corner, turn out to be $0 \pmod{p}$. We use the formula for q_0^* and that it is injective with $\mathbb{Z}$-coefficients. Let $\mathcal{R}_d$ denote the algebra

$$\mathbb{Z}[u_{\lambda^i}, a_{\lambda^j}, u_{\lambda^{p^{d-1}} - \lambda^{p^d}}]/I,$$

where I is the ideal generated by the relations (3.2), those in Proposition 2.11 and

$$u_{\lambda^{p^{d-1}} - \lambda^{p^d}} u_{\lambda^{p^d}} = u_{\lambda^{p^{d-1}}}, \quad u_{\lambda^{p^{d-1}} - \lambda^{p^d}} a_{\lambda^{p^d}} = p a_{\lambda^{p^{d-1}}},$$

which maps to $\pi_{-\star} H\underline{\mathbb{Z}} = H_G^\star(\text{pt})$. The algebra $\mathcal{R}_d$ contains the classes u_{λ^i} and a_{λ^j} but they are not required to satisfy (3.3). Form the algebraic q_0^* map

$$Q_0 \colon \mathcal{R}_d[\alpha_{\phi_{p^j}} \mid 0 \le j \le m] \to \mathcal{R}_d[x]$$

given by the formula in Proposition 7.19. In the absence of (3.3) in $\mathcal{R}_d$, the lower triangular matrix in the proof of Proposition 7.23 gets replaced by one where the diagonal entries are inclusions of the corresponding summand. This becomes injective even after tensoring with $\mathbb{Z}/p$. The algebra $\mathcal{R}_d[\alpha_{\phi_{p^j}} \mid 0 \le j \le m]$ is denoted by $\mathcal{R}_d(P(\mathcal{U}))$.

We thus work with the diagram

$$\begin{array}{ccc} \mathcal{R}_d(P(\mathcal{U})) & \xrightarrow{\nu_p} & \mathbb{Z}/p \otimes \mathcal{R}_d(P(\mathcal{U})) \\ \big\downarrow Q_0 & & \big\downarrow Q_0 \\ \mathcal{R}_d[x] & \xrightarrow{\nu_p} & \mathbb{Z}/p \otimes \mathcal{R}_d[x] \end{array}$$

and seek relations χ which map to 0 in $\mathbb{Z}/p \otimes \mathcal{R}_d[x]$. It follows that $\chi \equiv 0 \pmod p$ in $\mathcal{R}_d(P(\mathcal{U}))$ and thus gives a relation in $H^\star_G(P(\mathcal{U}); \underline{\mathbb{Z}/p})$. We note from Proposition 2.13 that

$$a_{\lambda^{kp^{r-1}+i}} = (1 + kp^{r-1} \cdot i^{-1}) a_{\lambda^i}, \quad \text{and hence, } a_{\lambda^{kp^{r-1}+i}} \equiv a_{\lambda^i} \pmod p \text{ for } i < p^{r-1}. \tag{7.25}$$

The following is a consequence of the identity $\prod_{i=1}^{p-1}(x+i) \equiv x^{p-1} - 1 \pmod p$.

7.26 Lemma *There is a relation*

$$\prod_{i=1}^{p-1}(i a_{\lambda^{p^{r-1}}} + x u_{\lambda^{p^{r-1}}}) = (x u_{\lambda^{p^{r-1}}})^{p-1} - (a_{\lambda^{p^{r-1}}})^{p-1}.$$

We write $P(z, w) = (z-w)^{p-1} - w^{p-1}$ and define the notation

$$\begin{aligned}
\mathcal{B}_r &= \prod_{i=1}^{p^r-1}(a_{\lambda^i} + u_{\lambda^i} x) \in \mathcal{R}_d[x], \\
\mathcal{T}_r &= \alpha_{\phi_{p^r}} + \prod_{i=1}^{p^r} a_{\lambda^i} \in \mathcal{R}_d(P(\mathcal{U})), \\
\mathbb{T}_r &= Q_0(\mathcal{T}_r) = \mathcal{B}_r \cdot (x u_{\lambda^{p^r}} + a_{\lambda^{p^r}}) \pmod p \quad \text{by Proposition 7.22,} && (7.27)\\
\mathbb{A}_0 &= P(\mathbb{T}_0, a_\lambda), \quad \text{and inductively,} \quad \mathbb{A}_j = P\Big(\mathbb{T}_j, a_{\lambda^{p^j}} \prod_{i=0}^{j-1} \mathbb{A}_i\Big), \\
\mathcal{A}_0 &= P(\mathcal{T}_0, a_\lambda), \quad \text{and} \quad \mathcal{A}_j = P\Big(\mathcal{T}_j, a_{\lambda^{p^j}} \prod_{i=0}^{j-1} \mathcal{A}_i\Big), \qquad \text{so that } Q_0(\mathcal{A}_j) = \mathbb{A}_j.
\end{aligned}$$

We now have the following relation with $\mathbb{Z}/p$ coefficients:

$$\begin{aligned}
\mathcal{B}_r &= \prod_{i=1}^{p^r-1}(a_{\lambda^i} + u_{\lambda^i} x) \\
&= \prod_{i=1,\, p^{r-1} \nmid i}^{p^r-1}(a_{\lambda^i} + u_{\lambda^i} x) \prod_{j=1}^{p-1}(a_{\lambda^{jp^{r-1}}} + u_{\lambda^{jp^{r-1}}} x) \\
&= \mathcal{B}_{r-1}^p \prod_{j=1}^{p-1}(j a_{\lambda^{p^{r-1}}} + u_{\lambda^{p^{r-1}}} x) \\
&= \mathcal{B}_{r-1}^p ((x u_{\lambda^{p^{r-1}}})^{p-1} - a_{\lambda^{p^{r-1}}}^{p-1}) \\
&= \mathcal{B}_{r-1}((\mathbb{T}_{r-1} - \mathcal{B}_{r-1} a_{\lambda^{p^{r-1}}})^{p-1} - (\mathcal{B}_{r-1} a_{\lambda^{p^{r-1}}})^{p-1}) \\
&= \mathcal{B}_{r-1} P(\mathbb{T}_{r-1}, a_{\lambda^{p^{r-1}}} \mathcal{B}_{r-1}),
\end{aligned}$$

where the third equality is by (7.25) and Proposition 2.13. From the expression, it inductively follows that

$$\mathcal{B}_r = \prod_{i=0}^{r-1} \mathbb{A}_r. \tag{7.28}$$

We finally obtain the following.

7.29 Proposition *With $\mathbb{Z}/p$-coefficients, the class $\alpha_{\phi_{p^r}}$ satisfies*

$$u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}} = \mathcal{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1}\mathcal{T}_{r-1}\Big(\prod_{i=0}^{r-2}\mathcal{A}_i\Big)^{p-1}.$$

Proof As observed above, it suffices to prove with $\mathbb{Z}/p$-coefficients that

$$Q_0(u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}}) = \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1}\mathbb{T}_{r-1}\Big(\prod_{i=0}^{r-2}\mathbb{A}_i\Big)^{p-1} = \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1}\mathbb{T}_{r-1}\mathcal{B}_{r-1}^{p-1}.$$

We verify

$$\begin{aligned}
Q_0(u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}}) &= u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\Big(\prod_{j=1}^{p^r}(xu_{\lambda^j}+a_{\lambda^j}) - \prod_{j=1}^{p^r}a_{\lambda^j}\Big) \\
&= \mathcal{B}_r x u_{\lambda^{p^{r-1}}} \\
&= \mathcal{B}_{r-1}^p((xu_{\lambda^{p^{r-1}}})^p - a_{\lambda^{p^{r-1}}}^{p-1}xu_{\lambda^{p^{r-1}}}) \\
&= \mathbb{T}_{r-1}^p - \mathcal{B}_{r-1}^p(a_{\lambda^{p^{r-1}}}^p + a_{\lambda^{p^{r-1}}}^{p-1}xu_{\lambda^{p^{r-1}}}) \\
&= \mathbb{T}_{r-1}^p - a_{\lambda^{p^{r-1}}}^{p-1}\mathbb{T}_{r-1}\mathcal{B}_{r-1}^{p-1}.
\end{aligned}$$

□

We now summarize the computation in the following theorem.

7.30 Theorem *As cohomology rings,*

$$H_G^\star(B_G S^1; \underline{\mathbb{Z}/p}) \cong H_G^\star(\text{pt}; \underline{\mathbb{Z}/p})[\alpha_{\phi_0}, \dots, \alpha_{\phi_m}]/(\rho_1, \dots, \rho_m).$$

The relations ρ_r are described by

$$\rho_r = u_{\lambda^{p^{r-1}}-\lambda^{p^r}}\alpha_{\phi_{p^r}} - \mathcal{T}_{r-1}^p + a_{\lambda^{p^{r-1}}}^{p-1}\mathcal{T}_{r-1}\Big(\prod_{i=0}^{r-2}\mathcal{A}_i\Big)^{p-1},$$

where $\mathcal{T}_j$ and $\mathcal{A}_j$ are defined in (7.27).

7.31 Ring structure of B_GSU(2) As in the complex case, we get the multiplicative generators $\beta_{2\phi_d}$ of $H_G^\star(P(\mathcal{U}_\mathbb{H}))$ at the degrees $2\phi_d$ for divisors d of n. The construction is the same as the previous construction of the class α_{ϕ_d}. Consider the representation

$$W_d := 1_\mathbb{H} + \psi^1 + \psi^2 + \cdots + \psi^d.$$

We have the cofiber sequence

$$P(W_{d-1})_+ \to P(W_d)_+ \to S^{\lambda^{-d}\otimes_\mathbb{C} W_{d-1}}.$$

At degree $2\phi_d$ the associated long exact sequence is

$$\cdots \to \widetilde{H}^{2\phi_d-1}(P(W_{d-1})_+) \to \widetilde{H}^{2\phi_d}(S^{2\phi_d}) \to \widetilde{H}^{2\phi_d}(P(W_d)_+) \to \widetilde{H}^{2\phi_d}(P(W_{d-1})_+) \to \cdots$$

since $\lambda^{-d}\otimes_{\mathbb{C}}\sum_{i=0}^{d-1}(\lambda^i+\lambda^{-i})=\sum_{i=0}^{d-1}2\lambda^{i-d}=2\phi_d$. We define the classes $\beta_{2\phi_d}$ to be the image of 1 in $\widetilde{H}^{2\phi_d}(S^{2\phi_d})\cong\mathbb{Z}$. By induction, we extend $\beta_{2\phi_d}$ to get the generator of $B_G\mathrm{SU}(2)$ at degree $2\phi_d$. We use the notation $\mathcal{L}_j$ for $\bigl(\beta_{2\phi_{p^j}}+\prod_{i=1}^{p^j}(a_{\lambda^i})^2\bigr)$. As in the complex case with $\underline{\mathbb{Z}/p}$-coefficients we have the following.

7.32 Theorem *As cohomology rings,*

$$H_G^\star(B_GS^3;\underline{\mathbb{Z}/p})\cong H_G^\star(\mathrm{pt};\underline{\mathbb{Z}/p})[\beta_{2\phi_0},\dots,\beta_{2\phi_m}]/(\mu_1,\dots,\mu_m).$$

The relations μ_r are described by

$$\mu_r=(u_{\lambda^{p^{r-1}}-\lambda^{p^r}})^2\beta_{2\phi_{p^r}}-\mathcal{L}_{r-1}^p+a_{\lambda^{p^{r-1}}}^{2(p-1)}\mathcal{L}_{r-1}\Bigl(\prod_{i=0}^{r-2}\mathcal{C}_i\Bigr)^{p-1},$$

where $\mathcal{C}_i$ is inductively defined as $\mathcal{C}_0=P(\mathcal{L}_0,a_\lambda^2)$, and $\mathcal{C}_j=P\bigl(\mathcal{L}_j,a_{\lambda^{p^j}}^2\prod_{i=0}^{j-1}\mathcal{C}_i\bigr)$.

We conclude this section with the ring structure computation of $\mathbb{C}P_\tau^\infty$.

7.33 Ring structure for $\mathbb{C}P_\tau^\infty$ Recall the cofiber sequence from Theorem 4.8:

$$\mathbb{C}P_\tau^{n-1}{}_+\hookrightarrow\mathbb{C}P^n_{\tau+}\xrightarrow{\chi}S^{n+n\sigma}.$$

This implies the long exact sequence

$$\cdots\to\widetilde{H}_{C_2}^{n+n\sigma}(S^{n+n\sigma})\xrightarrow{\chi^*}\widetilde{H}_{C_2}^{n+n\sigma}(\mathbb{C}P^n_{\tau+})\to\widetilde{H}_{C_2}^{n+n\sigma}(\mathbb{C}P_\tau^{n-1}{}_+)\to\cdots.$$

Observe that $\chi^*(1)$ is nonzero where $1\in\widetilde{H}_{C_2}^{n+n\sigma}(S^{n+n\sigma})\cong\mathbb{Z}$. Let $\epsilon_{n+n\sigma}\in\widetilde{H}_{C_2}^{n+n\sigma}(\mathbb{C}P^n_{\tau+})$ be the element $\chi^*(1)$. As the restriction of χ^* to the orbit C_2/e is an isomorphism, we have $\mathrm{res}_e^{C_2}(\epsilon_{n+n\sigma})=x^n\in\widetilde{H}^{2n}(\mathbb{C}P^n)$. We claim $H_{C_2}^\star(\mathbb{C}P^\infty_\tau{}_+)$ is the polynomial ring $H_{C_2}^\star(\mathrm{pt})[\epsilon_{1+\sigma}]$. This follows from the fact that $\underline{H}_{C_2}^{n+n\sigma}(\mathbb{C}P^\infty_\tau{}_+)\cong\underline{\mathbb{Z}}$ and $\mathrm{res}_e^{C_2}(\epsilon_{n+n\sigma})=\mathrm{res}_e^{C_2}(\epsilon_{n+n\sigma}^n)$. Hence $\epsilon_{n+n\sigma}=\epsilon_{1+\sigma}^n$. Therefore, we have the following.

7.34 Theorem *We have an isomorphism of cohomology rings*

$$H_{C_2}^\star(\mathbb{C}P_\tau^\infty)\cong H_{C_2}^\star(\mathrm{pt})[\epsilon_{1+\sigma}].$$

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