AGT

Algebraic & Geometric Topology

Volume 25 (2025)

# Nonuniform lattices of large systole containing a fixed 3-manifold group

Paige Hillen

msp

*Algebraic & Geometric Topology* **25**:5 (2025) 3089–3102
DOI: 10.2140/agt.2025.25.3089
Published: 17 September 2025

# Nonuniform lattices of large systole containing a fixed 3-manifold group

PAIGE HILLEN

Let $d \geq 2$ be a squarefree integer and $\mathbb{Q}(\sqrt{d})$ a totally real quadratic field over $\mathbb{Q}$. We show there exists an arithmetic lattice $\mathcal{L}$ in $\mathrm{SL}(8,\mathbb{R})$ with entries in the ring of integers of $\mathbb{Q}(\sqrt{d})$ and a sequence of lattices $\Lambda_n$ commensurable to $\mathcal{L}$ such that the systole of the locally symmetric finite volume manifold $\Lambda_n \backslash \mathrm{SL}(8,\mathbb{R})/\mathrm{SO}(8)$ goes to infinity as $n \to \infty$, yet every $\Lambda_n$ contains the same hyperbolic 3-manifold group $\Pi$, a finite index subgroup of the arithmetic hyperbolic 3-manifold vol3. Notably, such an example does not exist in rank one, so this is a feature unique to higher rank lattices.

57R19, 22E40

## 1 Introduction

The relationship between the *systolic genus*, the minimal genus surface subgroup, and the *systole*, the minimal length of a noncontractible closed geodesic, is notably different in higher rank. In 2012, Belolipetsky [1] (see also [2]) showed that in the hyperbolic setting, the systolic genus is bounded from below in terms of the systole. In contrast, Long and Reid [15] found a family of sequences of lattices in $\mathrm{SL}(3,\mathbb{R})$ (commensurable to an arbitrary nonuniform arithmetic lattice not commensurable to $\mathrm{SL}(3,\mathbb{Z})$) with systole going to infinity, yet with each containing a genus 3 surface subgroup. They found a similar result in the uniform case. Hence in higher rank, the systolic genus is not linked to the systole in the same way as in the hyperbolic setting.

Our result continues in this line of research. We expand Long and Reid's result to the existence of a fixed 3-manifold group in a sequence of commensurable lattices with arbitrarily large systole. More specifically, we find an infinite family of nonuniform arithmetic lattices in $\mathrm{SL}(8,\mathbb{R})$ each with a sequence of commensurable lattices whose systole goes to infinity, however every lattice in the sequence contains the same hyperbolic 3-manifold group. Our nonuniform arithmetic lattices are indexed by squarefree numbers $d \geq 2$: for each such $d$, we consider the integral special unitary group

$$\mathrm{SU}(I_8; \mathcal{O}_d, \tau) := \{A \in \mathrm{SL}(8, \mathcal{O}_d) : \tau(A)^\top A = I_8\} < \mathrm{SL}(8,\mathbb{R}),$$

where $I_8$ is the identity matrix in $\mathrm{SL}(8,\mathbb{R})$, $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$ and $\tau \in \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ is the nontrivial involution sending $\sqrt{d}$ to $-\sqrt{d}$.

© 2025 The Author, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

**Theorem 1.1** *Fix squarefree $d \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{L} := \mathrm{SU}(I_8; \mathcal{O}_d, \tau)$. There exists a sequence of nonuniform arithmetic lattices $\Lambda_n < \mathrm{SL}(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that*

$$\mathrm{sys}(\Lambda_n) \to \infty \quad \text{as } n \to \infty,$$

*yet every $\Lambda_n$ contains a fixed hyperbolic 3-manifold group $\Pi$, a finite index subgroup of* vol3.

In this setting, rank and determinant up to $\tau$-Hermitian square classify SU-equivalent $\tau$-Hermitian forms [12]. In turn, equivalent $\tau$-Hermitian forms yield commensurable integral special unitary groups. Hence $\mathrm{SU}(I_8; \mathcal{O}_d, \tau)$ is commensurable to $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ for any nondegenerate $\tau$-Hermitian form $J$ over $\mathbb{Q}(\sqrt{d})^8$ such that $\det J$ is a $\tau$-Hermitian square.

Critical to the proof is an 8-dimensional version of the family of discrete and faithful representations $\rho_t$ of the hyperbolic 3-manifold vol3 found by Cooper, Long, and Thistlethwaite [8] (see also [7]). We were unable to choose values of $t$ for which the entries of $\rho_t$ lie in a ring of integers. However, we are able to choose such specialized values of $t$ for a certain conjugate of $\rho_t \oplus \rho_t$. From this family, we obtain a sequence of representations of vol3 into lattices $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ for nondegenerate forms $J$. To ensure the systole is going to infinity, we then consider principal congruence subgroups of each $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ of level $p$ for an increasing sequence of primes $p$. We can no longer guarantee that vol3 is contained in a principal congruence subgroup, but by carefully choosing the primes $p$, we can guarantee the fundamental group of a certain fixed 320-sheeted cover of vol3 is still contained in each principal congruence subgroup.

It is worth remarking that our finite index subgroup $\Pi$ of vol3 contains surface subgroups. Therefore our result provides an alternative proof to Long and Reid's [15] result that systolic genus can be bounded in a sequence of commensurable higher rank lattices whose systole is diverging to infinity.

## 1.1 Structure of the paper

Section 2 gives necessary background regarding systoles and lattices. Section 3 introduces the definition of principal congruence subgroup and proves Proposition 3.3, regarding the growth of the systoles of certain sequences of principal congruence subgroups. Section 4 recalls the main result we aim to prove, and discusses the representation of the 3-manifold vol3 which is necessary for the proof. In Section 5, we give the proof of the main theorem. Section 6 provides examples of finding sequences of algebraic numbers which aid in the proof in Section 5. Section 7 makes a few observations about the result and possible future research directions. Finally, the appendix lists the two explicit representations of vol3 mentioned in Section 4.

## Acknowledgements

The author would like to thank Darren Long for his steady guidance and cautious optimism throughout the development of this work. Additionally, the author is grateful to the referee for helpful comments on a previous version of this paper.

# 2 Background

Consider the following measures of the geometry and topology of a space:

**Definition 2.1** The *systole* of a Riemannian manifold $M$, denoted $\operatorname{sys}(M)$, is the minimal length of a noncontractible closed geodesic in $M$. We will interchangeably refer to the systole of $M$ as the systole of $\pi_1(M)$.

**Definition 2.2** Let $S_g$ denote the closed surface with genus $g \geq 2$. The minimal $g$ such that $\pi_1(S_g)$ injects into $\pi_1(M)$ is called the *systolic genus of $M$*, denoted $\operatorname{sysg}(M)$ [1].

For hyperbolic manifolds, the behavior of the systole puts some restrictions on the behavior of the systolic genus. This is clear for hyperbolic surfaces, since Besicovitch's inequality

$$\operatorname{sys}(S_g)^2 \leq 2\operatorname{area}(S_g),$$

combined with the Gauss–Bonnet theorem (for a hyperbolic metric on $S_g$),

$$\operatorname{area}(S_g) \leq 4\pi(g-1),$$

together show that a sequence of closed hyperbolic surfaces with systole going to infinity requires that the genera of the surfaces also tend toward infinity. In fact, Belolipetsky showed that this generalizes to higher dimensions in the following sense [1, Theorem 5.1]:

**Theorem 2.3** *Let $M_n = \mathbb{H}^m / \Gamma_n$ be a sequence of closed hyperbolic $m$-manifolds such that $\operatorname{sys}(\Gamma_n) \to \infty$ as $n \to \infty$. Then the systolic genus $\operatorname{sysg}(M_n)$ goes to infinity as well.*

Intuitively, if the manifolds are getting complicated enough for the systole to grow arbitrarily large, their topology must be getting complicated as well. However, once we leave the hyperbolic setting, this is no longer true. In particular, the presence of flats seems to allow enough space for systoles to grow, without the systolic genus increasing.

**Definition 2.4** A discrete subgroup $\Gamma < \mathrm{SL}(m,\mathbb{R})$ is a *lattice* if the quotient orbifold

$$M_\Gamma := \Gamma \backslash \mathrm{SL}(m,\mathbb{R}) / \mathrm{SO}(m)$$

has finite volume. Note that $M_\Gamma$ is a manifold if and only if $\Gamma$ is torsion-free. A lattice $\Gamma$ is *uniform* (or cocompact) if $M_\Gamma$ is compact. Otherwise, $\Gamma$ is *nonuniform*.

In the theory of lattices, passing to a finite index subgroup usually results in only minor differences. Since we often like to ignore these minor differences, we usually care about lattices up to commensurability.

**Definition 2.5** Lattices $\Gamma_1, \Gamma_2 < \mathrm{SL}(m,\mathbb{R})$ are *commensurable* if for some $g \in \mathrm{SL}(m,\mathbb{R})$,

$$[\Gamma_1 : \Gamma_1 \cap g\Gamma_2 g^{-1}] < \infty.$$

Equivalently, their manifolds $M_{\Gamma_1}$ and $M_{\Gamma_2}$ have a common finite-sheeted cover (ie $M_{\Gamma_1 \cap g\Gamma_2 g^{-1}}$).

We review a construction of a family of nonuniform arithmetic lattices in $\mathrm{SL}(m,\mathbb{R})$. See [16, Section 6.8] for more details.

**Construction 2.6** Fix a squarefree $d \in \mathbb{Z}_{>1}$ and $m \geq 3$. Then:

- $F = \mathbb{Q}(\sqrt{d})$ is a totally real algebraic number field.
- $\tau \in \mathrm{Gal}(F/\mathbb{Q})$ is the nontrivial involution sending $\sqrt{d}$ to $-\sqrt{d}$.
- The ring of integers of $F$ is

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2, 3 \pmod 4, \\ \mathbb{Z}\left[\frac{1}{2}(1+\sqrt{d})\right] & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Let $J \in M_{m\times m}(F)$ be a sesquisymmetric matrix with respect to $\tau$ (ie $J^\top = \tau(J)$). We view $J$ as a $\tau$-Hermitian form on the $F$-vector space $F^m$. The associated special unitary group is

$$\mathrm{SU}(J; F, \tau) = \{M \in \mathrm{SL}(m, F) : M^* J M = J\},$$

where $M^* := \tau(M)^\top$. The integer points of this group form the associated *integral special unitary group*

$$\mathrm{SU}(J; \mathcal{O}_d, \tau) = \{M \in \mathrm{SL}(m, \mathcal{O}_d) : M^* J M = J\}.$$

**Definition 2.7** $\tau$-Hermitian forms $J$ and $J'$ on $F^m$ are *SU-equivalent* if $J' = P^* J P$ for some change of basis matrix $P \in \mathrm{GL}(m, F)$.

Observe that

$$\mathrm{SU}(P^* J P; F, \tau) = P^{-1}(\mathrm{SU}(J; F, \tau))P.$$

The corresponding integral groups may not be conjugate, but $P \in \mathrm{GL}(m, F)$ is in the commensurator of both integral groups, so they are commensurable lattices. Our interest in commensurability classes of arithmetic lattices with entries in $\mathcal{O}_d$ leads to the question: when are two $\tau$-Hermitian forms over $F^m$ equivalent?

**Definition 2.8** Let $F = \mathbb{Q}(\sqrt{d})$ and $\tau \in \mathrm{Gal}(F/\mathbb{Q})$ be the involution sending $\sqrt{d}$ to $-\sqrt{d}$. A *$\tau$-Hermitian square* is an element $g \in F$ such that $g = \tau(h)h$ for some $h \in F$.

By Landherr [12] (see [14, Section 3]), an equivalence class of $\tau$-Hermitian forms on $F^m$ for $m \geq 3$ is uniquely determined by

- the rank of the form, and
- the discriminant of the form up to $\tau$-Hermitian square.[1]

**Proposition 2.9** *Suppose $J$ is a full rank $\tau$-Hermitian form on $F^m$. Then the group $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ as constructed above is a nonuniform arithmetic lattice. Moreover, $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ is commensurable to $\mathrm{SU}(J'; \mathcal{O}_d, \tau)$ for any full rank $\tau$-Hermitian form $J'$ such that $|\det J - \det J'|$ is a $\tau$-Hermitian square.*

[1] Since $(\sqrt{d})\tau(\sqrt{d}) = -d$, there is no signature in this setting.

**Proof** That $\mathrm{SU}(J;\mathcal{O}_d,\tau)$ is an arithmetic lattice follows from [16, Proposition 6.8.14]. The same proposition tells us our lattice is nonuniform if and only if there exists a nonzero $x \in F^n$ such that $x^* J x = 0$.

By the classification above, $\mathrm{SU}(J;\mathcal{O}_d,\tau)$ is commensurable to $\mathrm{SU}(\mathrm{diag}(1,-1,-\det J,1\dots,1);\mathcal{O}_d,\tau)$. For $x=[1,1,0,\dots,0]$, clearly

$$x^*\,\mathrm{diag}(1,-1,-\det J,1,\dots,1)x = 0.$$

Thus $\mathrm{SU}(J;\mathcal{O}_d,\tau)$ is nonuniform. □

# 3 Systolic growth

We find a way to control the systole of certain lattices commensurable to those built by Construction 2.6. There is a 1-1 correspondence between closed geodesics in $M_\Gamma$ and $\Gamma$-conjugacy classes of semisimple elements in $\Gamma$. The length of the geodesic corresponding to the conjugacy class of a semisimple $\gamma \in \Gamma$ is proportional to the translation length of $\gamma$ on the geodesic it leaves invariant in $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$. Let $l(\gamma)$ denote this length. Hence

$$\mathrm{sys}(\Gamma) = \inf\{l(\gamma) : \gamma \in \Gamma \text{ is semisimple}\}.$$

The translation lengths are then bounded from below in terms of the trace [13, Theorem 3.1]:

**Theorem 3.1** (trace-length bounds) *Let $\gamma \in \mathrm{SL}(m,\mathbb{R})$ be semisimple with $|\mathrm{tr}(\gamma)| \geq 1$. Then*

$$l(\gamma) \geq \sqrt{2}\,\mathrm{arccosh}\left(\max\left\{1, \frac{|\mathrm{tr}(\gamma)|}{m}\right\}\right).$$

Since $\lim_{x\to\infty}\mathrm{arccosh}(x)=\infty$, we can control the lower bound for the systole of $M_\Gamma$ by controlling the lower bound for the traces of semisimple elements in $\Gamma$. Our tool for controlling the lower bound of the trace is principal congruence subgroups.

**Definition 3.2** Let $p$ be a rational prime and $\Gamma < \mathrm{SL}(m,\mathbb{R})$ an arithmetic lattice with entries in a ring of integers $\mathcal{O}$. Then

$$\Gamma^{(p)} = \mathrm{Ker}\big(\pi_p : \Gamma \to \mathrm{SL}(m,\mathcal{O}/(p))\big)$$

is the *principal congruence subgroup of* $\Gamma$ *of level* $p$, where $\pi_p$ is projection modulo $(p)$.

The subgroup $\Gamma^{(p)}$ is a normal subgroup of finite index in $\Gamma$. We will need the following proposition, whose proof uses ideas from the proof of Theorem 5.1 and Corollary 5.2 in [13].

**Proposition 3.3** *Fix squarefree $d \in \mathbb{Z}_{\geq 2}$. Let $F=\mathbb{Q}(\sqrt{d})$, $\mathcal{O}_d$ be the ring of integers of $F$, and $\tau$ be the nontrivial Galois automorphism of $F$ over $\mathbb{Q}$. Suppose $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of $\tau$-Hermitian forms over $F^m$. For each $n$, let*

$$\Gamma_n = \mathrm{SU}(J_n;\mathcal{O}_d,\tau).$$

*If* $\{p_n\}_{n\in\mathbb{N}}$ *is a sequence of rational primes diverging to* $\infty$*, then*

$$\operatorname{sys}(\Gamma_n^{(p_n)}) \to \infty,$$

*where* $\Gamma_n^{(p_n)}$ *denotes the principal congruence subgroup of* $\Gamma_n$ *of level* $p_n$*.*

**Proof** Let $k \in \mathbb{R}$. Since $\lim_{x\to\infty} \operatorname{arccosh}(x) = \infty$, there exists $M \geq 2$ such that

$$\frac{2\sqrt{2}}{m} \operatorname{arccosh}(M-1) \geq k.$$

Since $p_n \to \infty$, there exists $N \in \mathbb{N}$ such that $p_n > mM$ for all $n \geq N$. Fix $n \geq N$ and suppose that $\gamma \in \Gamma_n^{(p_n)}$ is semisimple. Then for each power $q$ of $\gamma$, $\gamma^q$ is a semisimple element of $\Gamma_n^{(p_n)}$. Hence $\operatorname{tr}(\gamma^q) \equiv m \pmod{p_n}$, so

$$\operatorname{tr}(\gamma^q) = p_n \alpha_q + m$$

for some $\alpha_q \in \mathcal{O}_d$. By the argument in the second paragraph[2] of the proof of Theorem 5.1 in [13], there exists an integer $q$, $|q| \leq \frac{1}{2}m$, such that $\operatorname{tr}(\gamma^q) \neq m$. Set $\alpha := \alpha_q$. Since $\operatorname{tr}(\gamma^q) \neq m$, we have that $\alpha \neq 0$. Hence

$$|\operatorname{tr}(\gamma^q)| > m(M|\alpha| - 1) \quad \text{and} \quad |\tau(\operatorname{tr}(\gamma^q))| > m(M|\tau(\alpha)| - 1).$$

For any $\alpha \in \mathcal{O}_d$, $\max(|\alpha|, |\tau(\alpha)|) \geq 1$. Thus

$$\max\left\{\frac{|\operatorname{tr}(\gamma^q)|}{m}, \frac{|\tau(\operatorname{tr}(\gamma^q))|}{m}\right\} > M - 1.$$

By definition of the special unitary group, $\gamma^* = J_n \gamma^{-1} J_n^{-1}$. Hence

$$\tau(\operatorname{tr}(\gamma^q)) = \operatorname{tr}((\gamma^q)^*) = \operatorname{tr}(\gamma^{-q}).$$

Since $l(\gamma^q) = l(\gamma^{-q})$, Theorem 3.1 implies

$$l(\gamma^q) \geq \sqrt{2} \operatorname{arccosh}\left(\max\left\{1, \frac{|\operatorname{tr}(\gamma^q)|}{m}, \frac{|\tau(\operatorname{tr}(\gamma^q))|}{m}\right\}\right).$$

Since arccosh is increasing on $[1, \infty)$ and $M \geq 2$,

$$l(\gamma^q) \geq \sqrt{2} \operatorname{arccosh}(M-1).$$

Since $l(\gamma^q) = |q| l(\gamma)$ and $|q| \leq \frac{1}{2}m$,

$$l(\gamma) \geq \frac{2\sqrt{2}}{m} \operatorname{arccosh}(M-1) \geq k.$$

Hence,

$$\operatorname{sys}(\Gamma_n^{(p_n)}) \geq k.$$

Since $k$ is arbitrary, this completes the proof. □

[2] The argument in the aforementioned paragraph does not use that $\Gamma$ is derived from a central simple algebra, only that $\gamma$ is a semisimple element and Newton's identities to obtain a formula for the characteristic polynomial of $\gamma$ in terms of the trace of powers of $\gamma$.

# 4 Result

**Theorem 1.1** *Fix squarefree $d \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{L} = \mathrm{SU}(I_8; \mathcal{O}_d, \tau)$. There exists a sequence of nonuniform arithmetic lattices $\Lambda_n < \mathrm{SL}(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that*

$$\mathrm{sys}(\Lambda_n) \to \infty \quad \text{as } n \to \infty,$$

*yet every $\Lambda_n$ contains a fixed hyperbolic 3-manifold group $\Pi$, a finite index subgroup of* vol3.

**Remark** The 3-manifold group $\Pi$ contains surface groups, so each $\Lambda_n$ contains a fixed surface group. In particular, the systolic genus of these lattices is bounded from above for all $n$.

The manifold vol3 is an arithmetic hyperbolic 3-manifold with the third lowest volume in the census. The fundamental group of vol3, which we will also refer to as vol3, has presentation

$$\mathrm{vol3} = \langle a, b \mid aabbABAbb; aBaBabaaab \rangle,$$

where $A = a^{-1}$ and $B = b^{-1}$. The hyperbolic representation of vol3 into $\mathrm{SO}(3,1)$ admits discrete and faithful deformations in $\mathrm{SL}(4, \mathbb{R})$. An explicit one-parameter family of these deformations was found by Cooper, Long, and Thistlethwaite [7]. The group vol3 covers an orbifold, denoted $\mathrm{vol3}/\langle u \rangle$, commensurable to $\mathcal{L}$ which has a simpler representation

$$\rho_t \colon \mathrm{vol3}/\langle u \rangle \to \mathrm{SL}(4, \mathbb{Q}(t, \sqrt{t^2-1}, \sqrt{t^2+2})) \quad \text{for } t \geq 1.$$

The representation in [7] uses parameter $v$ instead of $t$, with the substitution $v = 2t$. Two elements, denoted here $u$ and $c$, generate $\mathrm{vol3}/\langle u \rangle$, and the image of these elements under $\rho_t$ are listed in the appendix, as well as in the accompanying Mathematica file [10]. To recover the manifold group, one can use the relations $a = u^2 c$ and $b = (aua)^{-1} u$. In practice, we work with the orbifold group when interacting with the explicit matrix representation.

The hyperbolic representation at $t = 1$ and for real values $t \geq 1$ is the holonomy of a real projective structure on vol3, and is thus discrete and faithful. It is not clear how to specialize $t$ to ensure the entries of the image all lie in a ring of integers over some field. As luck would have it, we found 16 specific elements $\{1, g_1, \dots, g_{15}\} \in \mathrm{vol3}/\langle u \rangle$ such that:

- $\mathcal{B}_v = \{\rho_v(1), \rho_v(g_1), \dots, \rho_v(g_{15})\}$ is a basis for the vector space $M_{4\times 4}(\mathbb{R})$.
- The left regular representation of $\mathrm{vol3}/\langle u \rangle$ with respect to the basis $\mathcal{B}_v$ yields a representation

$$\eta_t \colon \mathrm{vol3}/\langle u \rangle \to \mathrm{SL}(16, \mathbb{Q}(t, \sqrt{t^2-1})).$$

- Both $\eta_t(u)$ and $\eta_t(c)$ have entries in $\mathbb{Z}[t, \sqrt{t^2-1}]$.

In an attempt to find an integral representation of smaller dimension, we study invariant subspaces of this 16-dimensional representation. By considering eigenspaces of elements in the centralizer of $\eta_t(\mathrm{vol3}/\langle u \rangle)$, we were able to find an 8-dimensional invariant subspace, whose corresponding representation has entries in $\mathbb{Z}[t, \sqrt{t^2-1}]$. Let

$$\omega_t : \mathrm{vol3}/\langle u\rangle \to \mathrm{SL}(8, \mathbb{Z}[t, \sqrt{t^2-1}])$$

denote this representation. The matrices $\omega_t(u)$ and $\omega_t(c)$ are listed in the appendix. This representation is conjugate to $\rho_t \oplus \rho_t$; hence is faithful. Computations confirming $\omega_t$ is conjugate to $\rho_t \oplus \rho_t$, the 16-dimensional representation $\eta_t$, as well as the explicit 8-dimensional subspace mentioned above can all be found in the accompanying Mathematica file [10].

# 5 Proof

For the remainder of this paper, fix squarefree $d \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{L} := \mathrm{SU}(I_8; \mathcal{O}_d, \tau)$. Interpreting $t$ as transcendental, let $\tau \in \mathrm{Gal}(\mathbb{Q}(t, \sqrt{t^2-1})/\mathbb{Q}(t))$ be the involution sending $\sqrt{t^2-1}$ to $-\sqrt{t^2-1}$. For a $\tau$-Hermitian form $J_t \in (\mathbb{Q}(t, \sqrt{t^2-1}))^m$, let

$$\mathrm{SU}\big(J_t; \mathbb{Q}(t, \sqrt{t^2-1}), \tau\big) := \big\{M \in \mathrm{SL}(m, \mathbb{Q}(t)) : M^* J_t M = J_t\big\}$$

for $M^* := \tau(M)^\top$.

We start the proof of Theorem 1.1 by finding a sequence of arithmetic lattices commensurable to $\mathcal{L}$ which contain vol3.

**Lemma 5.1** *There exists a family of Hermitian forms $J_t \in \mathrm{SL}(8, \mathbb{Q}(t))$ such that*:

- *For $t \geq 1$, $J_t$ is full rank with $\det J_t$ equal to a square in $\mathbb{Q}(t)$.*
- $\omega_t(\mathrm{vol3}) < \mathrm{SU}(J_t; \mathbb{Q}(t, \sqrt{t^2-1}), \tau)$.

*Moreover, there exists a sequence $t_n \to \infty$ such that*

$$\omega_{t_n}(\mathrm{vol3}) < \mathrm{SU}(J_{t_n}; \mathcal{O}_d, \tau) \quad \text{for all } n \in \mathbb{N}.$$

*By Proposition 2.9, the $\mathrm{SU}(J_{t_n}; \mathcal{O}_d, \tau)$ are commensurable to $\mathcal{L}$ for $n \gg 0$.*

**Proof** The first part of the lemma follows from a computation: we solve for $J = J_t \in \mathrm{GL}(8, \mathbb{R})$ such that $\omega_t(u)^* J \omega_t(u) = J$ and $\omega_t(c)^* J \omega_t(c) = J$. By replacing $J$ with $J + J^*$, we can ensure that $J$ is sesquisymmetric. There are four free variables in the solution for $J$, and by making a choice of numerical value for each free variable,[3] we obtain a $\tau$-Hermitian form $J$ which is full rank (for all but finitely many choices of $t$). Indeed,

$$\det J_t = \frac{16(3-4t^2)^4}{(1-4t^2)^2},$$

which is a square in $\mathbb{Q}(t)$. For $t \geq 1$, $\det J_t$ is nonzero, so $J_t$ is full rank. The matrix $J_t$ can be found in the accompanying Mathematica file [10].

[3] Even leaving all four free variables in $J$ as unknowns, the determinant of $J$ is a square in $\mathbb{Q}(t)$. Thus choosing values in $\mathbb{Q}(t)$ for the free variables does not change the resulting commensurability class of the lattice.

To guarantee $\omega_t(\mathrm{vol3})$ lies in an *integral* special unitary group, it is necessary for the entries of $\omega_t(a)$ and $\omega_t(b)$ to lie in $\mathcal{O}_d$. It is sufficient to choose $t \in \mathbb{N}$ so that $\sqrt{t^2-1} \in \mathcal{O}_d$. Equivalently, we need to find infinitely many integral solutions $(t, y)$ to Pell's equation

$$t^2 - dy^2 = 1.$$

It is well known that for any positive nonsquare $d \in \mathbb{Z}$, Pell's equation has a fundamental solution $(t_1, y_1) \in \mathbb{N}^2$ and the other solutions are exactly the integers $(t_n, y_n)$ such that

$$u^n = t_n + y_n\sqrt{d}$$

for $u = t_1 + y_1\sqrt{d}$. Therefore, for this sequence $\{t_n\}_{n=1}^{\infty}$,

$$\omega_{t_n}(\mathrm{vol3}) < \mathrm{SU}(J_{t_n}; \mathcal{O}_d, \tau)$$

for all $n \in \mathbb{N}$. We can write $t_n = \frac{1}{2}(u^n + u^{-n})$, so the sequence $t_n$ goes to infinity as $n \to \infty$. $\square$

Let $\Gamma_n := \mathrm{SU}(J_{t_n}; \mathcal{O}_d, \tau)$ for the sequence $\{t_n\}_{n\in\mathbb{N}}$ from Lemma 5.1. Our next goal is to find finite index subgroups of $\Gamma_n$ whose systole goes to infinity as $n \to \infty$. This is accomplished by considering the principal congruence subgroups described in Section 3. We will let

$$\Lambda_n := \Gamma_n^{(p_n)}$$

for carefully chosen primes $p_n \to \infty$. By Proposition 3.3, $\mathrm{sys}(\Lambda_n) \to \infty$ as $n \to \infty$.

We will choose primes $p_n$ so that $\omega_{t_n}(\Pi) < \Lambda_n$ for a finite index subgroup $\Pi < \mathrm{vol3}$. More specifically, set

$$\Pi = \mathrm{Ker}\big(\omega_0 \colon \mathrm{vol3} \to \mathrm{SL}(8, \mathbb{Z}[i])\big).$$

At $t = 0$, one can check computationally that $|\omega_0(\mathrm{vol3})| = 320$, so $\Pi$ is indeed finite index in vol3.

Consider the following diagram:

$$\begin{array}{ccccc}
\Pi & \hookrightarrow & \cdots\cdots\cdots & \dashrightarrow & \mathrm{Ker}\big(\pi_p \colon \mathrm{SU}(J_t, \mathcal{O}_d, \tau) \to \mathrm{SL}(8, \mathcal{O}_d/(p))\big) \\
\downarrow & & & & \downarrow \\
\mathrm{vol3} & & \xrightarrow{\quad\omega_t\quad} & & \mathrm{SU}(J_t, \mathcal{O}_d, \tau) \\
\downarrow{\scriptstyle \omega_0} & & \circlearrowright & & \downarrow{\scriptstyle \pi_p} \\
\mathrm{SL}(8, \mathbb{Z}[i]) & \xrightarrow{\pi_p} & \mathrm{SL}(8, \mathbb{Z}[i]/(p)) & \xrightarrow{\quad f_*\quad} & \mathrm{SL}(8, \mathcal{O}_d/(p))
\end{array}$$

The primes we choose will divide $t$. Since our choice of $t$ solves Pell's equation, we have $(pk)^2 - 1 = dy^2$ for integers $k$ and $d$. Thus

$$f \colon \mathbb{Z}[i]/(p) \to \mathcal{O}_d/(p),$$

induced by sending $\bar{1}$ to $\bar{1}$ and $\bar{\imath}$ to $\overline{y\sqrt{d}}$, is a ring homomorphism. This induces a group homomorphism

$$f_* \colon \mathrm{SL}(8, \mathbb{Z}[i]/(p)) \to \mathrm{SL}(8, \mathcal{O}_d/(p)).$$

Our purpose in constructing this diagram is in the observation that, as long as the diagram commutes, we can guarantee that $\omega_t(\Pi) < \mathrm{Ker}(\pi_p)$. In fact, by inspecting the representation (see the appendix), $p$ dividing $t$ is sufficient to guarantee that the diagram commutes.

**Lemma 5.2** *Let $\{t_n\}$ be as in Lemma 5.1. Possibly passing to a subsequence of $t_n$, there exists a sequence of primes $p_n \to \infty$ with each $p_n$ dividing $t_n$. Hence,*

$$\omega_{t_n}(\Pi) < \Gamma_n^{(p_n)} \quad \text{for all } n \in \mathbb{N}.$$

**Proof** For each $n$, we would like a prime $p_n$ dividing $t_n$, but not dividing $t_m$ for $m < n$. The latter condition will ensure a subsequence of $p_n$ is strictly increasing. Some results from number theory will help us accomplish this. We start with the following definition from [3]:

**Definition 5.3** A pair of algebraic integers $(\alpha, \beta)$ is called a *Lucas pair* if $\alpha + \beta$ and $\alpha\beta$ are nonzero coprime rational integers and $\alpha/\beta$ is not a root of unity.

By [5; 3, Theorem A], if $(\alpha, \beta)$ is a Lucas pair, then for $n \gg 0$ the $n^{\text{th}}$ term of the sequence

$$S_n := \alpha^n + \beta^n$$

has a primitive prime divisor, ie a prime $p_n$ such that $p_n$ divides $S_n$ but not $S_m$ for $m < n$.

**Claim** *Let $u = t_1 + y_1\sqrt{d}$ be as in the proof of Lemma 5.1. Then $(u, 1/u)$ is a Lucas pair.*

Observe that

$$2t_n = u^n + \left(\frac{1}{u}\right)^n.$$

Therefore, pending the claim above, the sequence $2t_n$ has a corresponding sequence of primitive prime divisors $p_n$ for $n \gg 0$. Passing to a subsequence, we may assume the $p_n$ are strictly increasing, each prime $p_n \neq 2$. Hence each $p_n$ divides $t_n$ and $p_n \to \infty$. Hence, the diagram above commutes and therefore

$$\omega_{t_n}(\Pi) < \Gamma_n^{(p_n)}.$$

**Proof of claim** Since $u \in \mathbb{Z}[\sqrt{d}] \subseteq \mathcal{O}_d$, we know that $u$ is an algebraic integer. Since $t_1^2 - dy_1^2 = 1$,

$$\frac{1}{u} = t_1 - y_1\sqrt{d}.$$

Hence $1/u$ is also an algebraic integer. Further, $u + 1/u = 2t_1$ and $u(1/u) = 1$, so $u + 1/u$ and $u(1/u)$ are nonzero coprime rational integers. Moreover $u/(1/u) = u^2$ is not a root of unity. Hence $(u, 1/u)$ is a Lucas pair. □

Now we put these pieces together to prove the main theorem:

**Theorem 1.1** *Fix squarefree $d \in \mathbb{Z}_{\geq 2}$ and let $\mathcal{L} := \mathrm{SU}(I_8; \mathcal{O}_d, \tau)$. There exists a sequence of nonuniform arithmetic lattices $\Lambda_n < \mathrm{SL}(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that*

$$\mathrm{sys}(\Lambda_n) \to \infty \quad \text{as } n \to \infty,$$

*yet every $\Lambda_n$ contains a fixed hyperbolic 3-manifold group $\Pi$, a finite index subgroup of* vol3.

| $n$ | $u^n$ | $t_n$ | $p_n$ |
|---|---|---|---|
| 1 | $2+\sqrt{3}$ | 2 | 2 |
| 2 | $7+4\sqrt{3}$ | 7 | 7 |
| 3 | $26+15\sqrt{3}$ | 26 | 13 |
| 4 | $97+56\sqrt{3}$ | 97 | 97 |
| 5 | $362+209\sqrt{5}$ | 362 | 181 |

| $n$ | $u^n$ | $t_n$ | $p_n$ |
|---|---|---|---|
| 1 | $9+4\sqrt{5}$ | 9 | 3 |
| 2 | $161+72\sqrt{5}$ | 161 | 7 |
| 3 | $2889+1292\sqrt{5}$ | 2889 | 107 |
| 4 | $51841+23184\sqrt{5}$ | 51841 | 1103 |
| 5 | $930249+416020\sqrt{5}$ | 930249 | 2521 |

Table 1

**Proof** For each $n \in \mathbb{N}$, let $\Gamma_n := \mathrm{SU}(J_{t_n}; \mathcal{O}_d, \tau)$ for forms $J_{t_n}$ from Lemma 5.1 and let $\Lambda_n := \Gamma_n^{(p_n)}$ be the principal congruence subgroup of $\Gamma_n$ of level $p_n$ for primes $p_n$ from Lemma 5.2. By Lemma 5.1, each $\Gamma_n$ is a nonuniform lattice commensurable to $\mathcal{L}$. Since $\Lambda_n$ is a finite index subgroup of $\Gamma_n$, each $\Lambda_n$ is also a nonuniform lattice commensurable to $\mathcal{L}$. By Lemma 5.2, $\omega_{t_n}(\Pi) < \Lambda_n$ and by Proposition 3.3, $\mathrm{sys}(\Lambda_n) \to \infty$ as $n \to \infty$. □

# 6 Examples

**Example 6.1** We describe how to find the first few terms of the sequence $(t_n, p_n)$ in the case that $d = 3$. Since $3 \not\equiv 1 \pmod 4$, we have $\mathcal{O}_3 = \mathbb{Z}[\sqrt{3}]$.

To guarantee that the entries of $\omega_t(u)$ and $\omega_t(c)$ lie in $\mathcal{O}_3$, we solve for $t$ in Pell's equation

$$t^2 - 3y^2 = 1.$$

The fundamental solution is $(t_1, y_1) = (2, 1)$. Let $u = 2 + \sqrt{3}$. Powers of $u$ allow us to find all the other solutions; see Table 1 (left). To ensure that the diagram commutes and consequently that $\omega_t(\Pi) < \Gamma_n^{(p)}$, we need to find corresponding primes $p_n$ such that $t_n = 0 \pmod{p_n}$. Observe that each prime does not divide any of the previous values of $t_n$.

**Example 6.2** Let $d = 5$. Since $5 \equiv 1 \pmod 4$, we have that $\mathcal{O}_5 = \mathbb{Z}\big[\frac{1}{2}(1+\sqrt{5})\big]$. We first solve Pell's equation

$$t^2 - 5y^2 = 1.$$

The fundamental solution is $(t_1, y_1) = (9, 4)$. Following the same process as above, we find the first five terms of the sequences $t_n$ and $p_n$, as shown in Table 1 (right).

# 7 Further questions and observations

(1) **Thin 3-manifold group** Our 8-dimensional representation $\omega$ is conjugate to two copies of the original representation $\rho$. By the proof of Theorem 2.1 in [9], $\rho_t(\mathrm{vol3})$ is Zariski dense in $\mathrm{SL}(4, \mathbb{R})$. Therefore the Zariski closure of $\omega_t(\mathrm{vol3})$ in $\mathrm{SL}(8, \mathbb{R})$ is a subgroup isomorphic to $\mathrm{SL}(4, \mathbb{R})$. We also observe that for any of our chosen values of $t$, $\omega_t(\Pi)$ is infinite index in $\mathrm{SU}(J_t; \mathcal{O}_d, \tau)$. To see this,

observe that $\mathrm{SU}(J_t;\mathcal{O}_d,\tau)$ is an irreducible arithmetic lattice of $\mathbb{R}$-rank $\geq 2$. Thus, by Margulis's normal subgroup theorem, if $\omega_t(\mathrm{vol3})$ were finite index in $\mathrm{SU}(J_t;\mathcal{O}_d,\tau)$, then the abelianization of $\omega_t(\mathrm{vol3})$ would be finite. However, vol3 is an arithmetic hyperbolic 3-manifold with positive first Betti number. Thus vol3 must have infinite virtual first Betti number, so the abelianization cannot be finite [6]. It would be interesting to know whether $\omega_t(\mathrm{vol3})$ is infinite index in the intersection of $\mathrm{SU}(J_t;\mathcal{O}_d,\tau)$ and $\mathrm{Zcl}(\omega_t(\mathrm{vol3}))\cong \mathrm{SL}(4,\mathbb{R})$. If so, then $\omega_t(\mathrm{vol3})$ is a thin 3-manifold group.

(2) **Non-Hitchin surface subgroups** At $v=2$ the image $\rho_2(\mathrm{vol3})$ lies in $\mathrm{SO}(3,1)$, so every nontrivial semisimple element has a pair of eigenvalues with the same modulus. Thus $\rho_2$ is not Borel Anosov (for example, by the characterization of Anosov in Theorem 4.3 of [11]). The restriction to any subgroup is also not Borel Anosov. Hence for any value of $v\geq 2$ and any surface subgroup $H$, $\rho_v(H)$ is a discrete and faithful representation in $\mathrm{SL}(4,\mathbb{R})$ which is not on the Hitchin component. This is notable, since many examples of interesting representations of surface subgroups in the literature are in the Hitchin component. In particular, if one can find a surface subgroup of $\rho_v(\mathrm{vol3})$ which is Borel Anosov for some value of $v>2$, then it would be an example of a non-Hitchin Borel Anosov representation, answering a question posed by Canary [4, Question 50.6].

# Appendix

## A.1 The original 4-dimensional representation of vol3

The hyperbolic 3-manifold vol3 has presentation

$$\mathrm{vol3}=\langle a,b\mid aabbABAbb;aBaBabaaab\rangle,$$

where $A=a^{-1}$ and $B=b^{-1}$. The orbifold group $\mathrm{vol3}/\langle u\rangle$ which contains vol3 as a subgroup is generated by $u$ and $c$, of order 4 and 2 respectively. To recover vol3, we have $a=u^2c$ and $b=(aua)^{-1}u$. A conjugate of the image of $u$ and $c$ under the original 4-dimensional representation from Section 2.4 of [8] with the substitution $t=\frac{1}{2}v$ is

$$\rho_t(u)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\sqrt{\frac{t^2-1}{2+t^2}}&1\\0&0&-\frac{1+2t^2}{2+t^2}&-\sqrt{\frac{t^2-1}{2+t^2}}\end{pmatrix},$$

$$\rho_t(c)=\begin{pmatrix}\frac{1}{2}(t+\sqrt{2+t^2})&0&\frac{1}{2}(1-t^2-t\sqrt{2+t^2})&0\\0&\frac{1}{2}(t-\sqrt{2+t^2})&0&\frac{1}{2}(-1+t^2-t\sqrt{2+t^2})\\1&0&\frac{1}{2}(-t-\sqrt{2+t^2})&0\\0&-1&0&\frac{1}{2}(-t+\sqrt{2+t^2})\end{pmatrix}.$$

The exact representation from [8] and the matrix which conjugates it to $\rho_t$ listed above can be found in the accompanying Mathematica file [10].

Let $\tau \in \mathrm{Gal}(\mathbb{Q}(t, \sqrt{t^2-1})/\mathbb{Q}(t))$ sending $\sqrt{t^2-1}$ to $-\sqrt{t^2-1}$. There is one $\tau$-Hermitian form (up to scalar multiples) which both generators $\rho_t(u)$ and $\rho_t(c)$ preserve, that is

$$M_t = \begin{pmatrix} -\frac{2\sqrt{2+t^2}}{2t+t^3-\sqrt{2+t^2}+t^2\sqrt{2+t^2}} & 0 & 0 & 0 \\ 0 & \frac{2(2+t^2)}{(1+2t^2)(1-t^2+t\sqrt{2+t^2})} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2+t^2}{1+2t^2} \end{pmatrix}.$$

For $t = 1$, we have $\sqrt{t^2-1} = 0$, so the involution $\tau$ is trivial. Since $M_1$ is a diagonal matrix with signature $(3, 1)$, it is clear that $\rho_1$ lies in $\mathrm{SO}(3, 1)$.

## A.2 The explicit 8-dimensional representation of vol3

Our 8-dimensional representation which is integral for values of $t$ solving Pell's equation is generated by

$$\omega_t(u) = \begin{pmatrix} 1 & 0 & 0 & 2t & -2t & 2t & 0 & 0 \\ 0 & 1 & 0 & t-\sqrt{t^2-1} & -t+\sqrt{t^2-1} & t-\sqrt{t^2-1} & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -t+\sqrt{t^2-1} & -t-\sqrt{t^2-1} \\ 0 & 0 & 1 & -1 & 0 & 1 & -t+\sqrt{t^2-1} & -t-\sqrt{t^2-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\omega_t(c) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}.$$

An explicit computation found in the Mathematica file [10] shows this representation is conjugate to $\rho_t \oplus \rho_t$. One can also find the $\tau$-Hermitian form $J_t$ which both $\omega_t(u)$ and $\omega_t(c)$ preserve in said file.

# References


[1] **M Belolipetsky**, *On* 2*-systoles of hyperbolic* 3*-manifolds*, Geom. Funct. Anal. 23 (2013) 813–827 MR

[2] **M Belolipetsky**, **C Dória**, *Free subgroups of* 3*-manifold groups*, Groups Geom. Dyn. 14 (2020) 243–254 MR

[3] **Y Bilu**, **G Hanrot**, **P M Voutier**, *Existence of primitive divisors of Lucas and Lehmer numbers*, J. Reine Angew. Math. 539 (2001) 75–122 MR

[4] **R D Canary**, *Anosov representations: informal lecture notes* (2021) Available at https://dept.math.lsa.umich.edu/~canary/Anosovlecnotes.pdf

[5] **R D Carmichael**, *On the numerical factors of the arithmetic forms* $\alpha^n \pm \beta^n$, Ann. of Math. 15 (1913) 30–48 MR

[6] **D Cooper**, **D D Long**, **A W Reid**, *On the virtual Betti numbers of arithmetic hyperbolic* 3*-manifolds*, Geom. Topol. 11 (2007) 2265–2276 MR

[7] **D Cooper**, **D Long**, **M Thistlethwaite**, *Computing varieties of representations of hyperbolic* 3*-manifolds into* $\mathrm{SL}(4,\mathbb{R})$, Exp. Math. 15 (2006) 291–305 MR

[8] **D Cooper**, **D D Long**, **M B Thistlethwaite**, *Flexing closed hyperbolic manifolds*, Geom. Topol. 11 (2007) 2413–2440 MR

[9] **D Cooper**, **D D Long**, **M Thistlethwaite**, *Constructing non-congruence subgroups of flexible hyperbolic* 3*-manifold groups*, Proc. Amer. Math. Soc. 137 (2009) 3943–3949 MR

[10] **P Hillen**, vol3 *in lattices Mathematica file*, software (2024) Available at https://github.com/pkhillen/Vol3-in-Lattices-Mathematica-Files

[11] **F Kassel**, *Geometric structures and representations of discrete groups*, from "Proceedings of the International Congress of Mathematicians, II", World Sci., Hackensack, NJ (2018) 1133–1170 MR

[12] **W Landherr**, *Äquivalenz Hermitescher Formen über einem beliebigen algebraischen Zahlkörper*, Abh. Math. Sem. Univ. Hamburg 11 (1935) 245–248 MR

[13] **S Lapan**, **B Linowitz**, **J S Meyer**, *Systole inequalities up congruence towers for arithmetic locally symmetric spaces*, Comm. Anal. Geom. 31 (2023) 847–878 MR

[14] **D W Lewis**, *The isometry classification of Hermitian forms over division algebras*, Linear Algebra Appl. 43 (1982) 245–272 MR

[15] **D D Long**, **A W Reid**, *Sequences of high rank lattices with large systole containing a fixed genus surface group*, New York J. Math. 25 (2019) 145–155 MR

[16] **D W Morris**, *Introduction to arithmetic groups*, Deductive Press, Lethbridge, AB (2015) MR

*Department of Mathematics, University of Wisconsin, Madison*
*Madison, WI, United States*

hillen2@wisc.edu

https://sites.google.com/view/paigehillen

Received: 22 March 2024 Revised: 29 June 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY
msp.org/agt

**EDITORS**

PRINCIPAL ACADEMIC EDITORS

| John Etnyre | Kathryn Hess |
|---|---|
| etnyre@math.gatech.edu | kathryn.hess@epfl.ch |
| Georgia Institute of Technology | École Polytechnique Fédérale de Lausanne |

BOARD OF EDITORS

| | | | |
|---|---|---|---|
| Julie Bergner | University of Virginia<br>jeb2md@eservices.virginia.edu | Thomas Koberda | University of Virginia<br>thomas.koberda@virginia.edu |
| Steven Boyer | Université du Québec à Montréal<br>cohf@math.rochester.edu | Markus Land | LMU München<br>markus.land@math.lmu.de |
| Tara E Brendle | University of Glasgow<br>tara.brendle@glasgow.ac.uk | Christine Lescop | Université Joseph Fourier<br>lescop@ujf-grenoble.fr |
| Indira Chatterji | CNRS & Univ. Côte d'Azur (Nice)<br>indira.chatterji@math.cnrs.fr | Norihiko Minami | Yamato University<br>minami.norihiko@yamato-u.ac.jp |
| Octav Cornea | Université' de Montreal<br>cornea@dms.umontreal.ca | Andrés Navas | Universidad de Santiago de Chile<br>andres.navas@usach.cl |
| Alexander Dranishnikov | University of Florida<br>dranish@math.ufl.edu | Robert Oliver | Université Paris 13<br>bobol@math.univ-paris13.fr |
| Tobias Ekholm | Uppsala University, Sweden<br>tobias.ekholm@math.uu.se | Jessica S Purcell | Monash University<br>jessica.purcell@monash.edu |
| Mario Eudave-Muñoz | Univ. Nacional Autónoma de México<br>mario@matem.unam.mx | Birgit Richter | Universität Hamburg<br>birgit.richter@uni-hamburg.de |
| David Futer | Temple University<br>dfuter@temple.edu | Jérôme Scherer | École Polytech. Féd. de Lausanne<br>jerome.scherer@epfl.ch |
| John Greenlees | University of Warwick<br>john.greenlees@warwick.ac.uk | Vesna Stojanoska | Univ. of Illinois at Urbana-Champaign<br>vesna@illinois.edu |
| Matthew Hedden | Michigan State University<br>mhedden@math.msu.edu | Zoltán Szabó | Princeton University<br>szabo@math.princeton.edu |
| Kristen Hendricks | Rutgers University<br>kristen.hendricks@rutgers.edu | Maggy Tomova | University of Iowa<br>maggy-tomova@uiowa.edu |
| Hans-Werner Henn | Université Louis Pasteur<br>henn@math.u-strasbg.fr | Daniel T Wise | McGill University, Canada<br>daniel.wise@mcgill.ca |
| Daniel Isaksen | Wayne State University<br>isaksen@math.wayne.edu | Lior Yanovski | Hebrew University of Jerusalem<br>lior.yanovski@gmail.com |

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/
© 2025 Mathematical Sciences Publishers

# ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 5 (pages 2527–3144) 2025