AGT

Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:6 (2025) 3251–3270
DOI: 10.2140/agt.2025.25.3251
Published: 1 October 2025

A note on embeddings of 3-manifolds in symplectic 4-manifolds

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We show that any closed oriented 3-manifold can be topologically embedded in some simply connected closed symplectic 4-manifold and that it can be made a smooth embedding after one stabilization. As a corollary of the proof, we show that the homology cobordism group is generated by Stein fillable 3-manifolds. We also find obstructions on smooth embeddings: there exist 3-manifolds that cannot smoothly embed in a way that appropriately respects orientations in any symplectic manifold with a weakly convex boundary.

57K43; 53D05

1 Introduction

Embedding 3-manifolds into higher-dimensional spaces presents an intriguing challenge. Whitney's embedding theorem [31] establishes that every closed oriented 3-manifold smoothly embeds in $\mathbb{R}^6$. Hirsch [16] later refined this by demonstrating that any 3-manifold can be smoothly embedded in S^5. Concurrently, Lickorish [19] and Wallace [29] independently showed that every 3-manifold can be smoothly embedded in a 4-manifold. Moreover, a generalization of their approach reveals that any 3-manifold can be smoothly embedded in the connected sum of $S^2 \times S^2$ copies. Freedman's work [12] established that all integer homology 3-spheres can be topologically and locally flatly embedded in S^4. However, the Rokhlin invariant μ and Donaldson's diagonalization theorem [6] demonstrate that certain integer homology spheres cannot be smoothly embedded in S^4. This leads to the question: can a compact 4-manifold exist wherein all 3-manifolds embed? Shiomi [28] provided a negative answer to this inquiry. Aceto, Golla, and Larson [1] delved into embedding 3-manifolds in spin 4-manifolds. Among the diverse class of 4-manifolds, symplectic manifolds hold particular interest. In [10], Etnyre, Min, and the author conjectured the following.

Conjecture 1 *Every closed, oriented smooth 3-manifold smoothly embeds in a symplectic 4-manifold.*

For example, notice that if Y is obtained by doing integer surgery on a knot in S^3 then Y can have a smooth oriented embedding in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \#_2 \overline{\mathbb{CP}}^2$ depending on whether the surgery coefficient is odd or even. There does not seem to be an analogous result for links.

1.1 Embeddings in symplectic manifolds

While we cannot resolve the above conjecture, we can show the existence of topological embeddings and smooth embeddings after stabilization.

Theorem 1.1 *Given a closed, connected, oriented 3-manifold Y, there exists a simply connected symplectic closed 4-manifold X such that Y can be embedded topologically, locally flatly in X. This embedding can be made a smooth embedding after one stabilization, ie Y can smoothly embed in $X \# (S^2 \times S^2)$.*

The above Theorem 1.1 indicates that no obstructions are coming from the topological category; however, when smooth embeddings in symplectic manifolds do exist, they can imply interesting things about topology.

When considering the smooth embeddings of a 3-manifold Y into a smooth 4-manifold X, an ambiguity arises. As an oriented manifold, Y differs significantly from its orientation reversal $-Y$: for instance, the Poincaré homology sphere with a positive orientation bounds a negative-definite 4-manifold, whereas its counterpart with a negative orientation does not. Conversely, if Y smoothly embeds in X, then the boundary of a small neighborhood of Y is $-Y \sqcup Y$, implying that $-Y$ also smoothly embeds in X. However, this ambiguity can be resolved using cobordisms.

Definition 1.2 A smooth embedding of Y in a cobordism W from Y_0 to Y_1 is termed an *oriented cobordism embedding* if Y is either nonseparating or if Y separates W into $W_1 \sqcup W_2$ such that Y serves as an oriented manifold boundary component of W_1, with all other components of ∂W_1 (if any) belonging to Y_0.

Recall that a closed oriented rational homology sphere is called an *L-space* if its Heegaard Floer homology group is "as simple as possible", as specified in Section 2.3. Embeddings of L-spaces in symplectic manifolds are constrained as follows.

Theorem 1.3 *If an L-space Y smoothly embeds in a closed symplectic 4-manifold X, it must be separating. Furthermore, if $X = X_1 \cup_Y X_2$, then one of the X_i must be a negative-definite 4-manifold. Therefore, if an L-space Y does not bound a negative-definite 4-manifold, it cannot possess an oriented cobordism embedding in any symplectic 4-manifold with a weakly convex boundary.*

Remark 1.4 Ozsváth and Szabó [25] have proven the aforementioned result for separating L-spaces in closed symplectic manifolds. Therefore, the key contribution of the theorem above is to demonstrate that L-spaces cannot be embedded as nonseparating hypersurfaces in symplectic manifolds. Our proof of this result was inspired by Agol and Lin's work on hyperbolic 4-manifolds [2].

Remark 1.5 There exist numerous such L-spaces, including the Poincaré homology sphere with negative orientation and r-surgery for $r \in [9, 15)$ on the pretzel knot $P(-2, 3, 7)$ in S^3 [17; 18] (the latter L-spaces are hyperbolic). Prior research had already established that these manifolds are not Stein fillable [17; 18; 20]. Here, we have demonstrated that beyond not being weakly fillable, they cannot even possess a smooth oriented cobordism embedding in any weak filling of any 3-manifold.

Corollary 1.6 *If Y' admits a weakly fillable contact structure then any L-space Y which does not bound negative-definite 4-manifolds cannot have any smooth oriented cobordism embedding in $Y' \times I$.*

Theorem 1.3 gives rise to a very interesting question.

Question 1 *Does every L-space bound a definite 4-manifold?*

Remark 1.7 Notice that if Conjecture 1 is true then the above question has a positive answer.

In Section 4, following the proof of Theorem 1.3, we will explore a strategy outlined in Remark 4.1 aimed at providing a negative response to Question 1. Such a response would, in turn, furnish a counterexample to the conjecture positing that closed 3-manifolds can be smoothly embedded in symplectic 4-manifolds.

1.2 Cobordisms and symplectic structures

We call a closed oriented 3-manifold *Stein fillable* if it admits a Stein fillable contact structure, the definition of which is postponed until Section 2.2. Not every smooth filling X of a 3-manifold Y can be equipped with a Stein structure. To illustrate this, note that if a $\mathbb{Z}HS^3$ bounds a $\mathbb{Q}HB^4$ that is not a $\mathbb{Z}HB^4$, then every handle decomposition must possess a 3-handle. However, Eliashberg [7] demonstrated that any Stein decomposition cannot include a 3-handle. For instance, Fintushel and Stern [11] established that $\Sigma(2,3,7)$ does not bound a $\mathbb{Z}HB^4$ but does bound a $\mathbb{Q}HB^4$. Therefore, this rational ball cannot serve as a Stein filling of $\Sigma(2,3,7)$ or, indeed, any 3-manifold homology cobordant to $\Sigma(2,3,7)$ cannot possess a rational ball Stein filling. Moreover, certain 3-manifolds do not admit any Stein fillings; see Lisca [20]. Hence, two questions naturally arise:

Question 2 *Is every 3-manifold integral homology cobordant to a Stein fillable 3-manifold?*

Question 3 *If X is an oriented smooth compact 4-manifold with boundary Y, does there exist a 3-manifold Y' which is rational homology cobordant to Y and has a symplectic filling X' with the same algebraic topology as X?*

We will address these two questions in the following theorem. First, let's recall that we refer to an integral homology cobordism from Y_0 to Y_1 as a *$\mathbb{Z}$-ribbon cobordism* if this integer homology cobordism is achieved by attaching handles of index only 1 and 2 to $Y_0 \times [0,1]$ along $Y_0 \times 1$. We denote such a cobordism by saying Y_0 is *ribbon cobordant* to Y_1. It is important to note that this relation is a partial ordering on 3-manifolds and not necessarily a symmetric relation. We also refer to such a cobordism W as an *invertible cobordism* if there is a cobordism W' from Y_1 to Y_0 such that $W \cup W' = Y_0 \times I$. (Similarly, we can define $\mathbb{Q}$-ribbon homology cobordism.)

Theorem 1.8 *Given any closed oriented* 3*-manifold* Y *there exists a Stein fillable* 3*-manifold* Y' *and a* $\mathbb{Z}$*-ribbon invertible homology cobordism* W *from* Y *to* Y' *which is obtained from* $Y \times [0,1]$ *by attaching a single pair of algebraically canceling* 1*- and* 2*-handle. Moreover if* X *is an oriented compact* 4*-manifold with connected boundary* $\partial X = Y$*, then*:

- (i) *If* $b_1(X) = 0$*, there exists a Stein* 4*-manifold* X' *with boundary* $\partial X' = Y'$ *such that there is a* $\mathbb{Q}$*-ribbon homology cobordism from* Y *to* Y' *and* $b_2(X) = b_2(X')$*.*
- (ii) *If* $\partial X = Y$ *is a* $\mathbb{Q}HS^3$*, there exists a Stein* 4*-manifold* X' *with boundary* $\partial X' = Y'$ *such that the intersection form of* X *is isomorphic to the intersection form of* X' *and there is a* $\mathbb{Q}$*-ribbon homology cobordism from* Y *to* Y'*.*

Remark 1.9 Yasui has pointed out to the author that, although they do not specifically discuss cobordisms, the first part of this result, namely constructing W from Y to Y', without the restriction of only needing a single 1- and 2-handle pair, can be proven by assembling various results from [5]. One of the primary techniques for proving this result involved utilizing embedded cork and cork twists, which theoretically can transform non-Stein manifolds into Stein ones. These ideas had been previously employed in the works of Akbulut and Matveyev [4] and Akbulut and Yasui [5]. The nontrivial aspect of the aforementioned result lies in how to handle the 3-handles in such cobordisms.

In low-dimensional topology, the investigation of the integer homology cobordism group $\Theta^3_{\mathbb{Z}}$ and rational homology cobordism group $\Theta^3_{\mathbb{Q}}$ holds special significance. The aforementioned result provides a new generator for these groups.

Corollary 1.10 *The homology cobordism groups* $\Theta^3_{\mathbb{Z}}$ *and* $\Theta^3_{\mathbb{Q}}$ *are generated by Stein fillable* 3*-manifolds.*

Remark 1.11 It remains unknown whether $\Theta^3_{\mathbb{Q}}$ is generated by L-spaces. Nozaki, Sato and Taniguchi [21] demonstrated that $\Sigma(2,3,11) \#_2 (-\Sigma(2,3,5))$ does not bound a definite 4-manifold. If we can discover an L-space Y that is rationally cobordant to $\Sigma(2,3,11)$, then $Y \#_2 (-\Sigma(2,3,5))$ cannot bound a definite 4-manifold. As $Y \#_2 (-\Sigma(2,3,5))$ is itself an L-space, Theorem 1.3 asserts that this manifold cannot be smoothly embedded in any symplectic 4-manifold. Conversely, if all 3-manifolds can be embedded in some symplectic 4-manifold, then $\Theta^3\mathbb{Q}$ is not generated by L-spaces. Thus, we arrive at the following question:

Question 4 *Is* $\Sigma(2,3,11)$ *rationally cobordant to some* L*-space*?

For a closed oriented 3-manifold Y, $H_3(Y;\mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. Thus, a map $f\colon Y_0 \to Y_1$ induces a homomorphism on the top-dimensional homology group, $f\colon \mathbb{Z} \to \mathbb{Z}$. The degree of f is denoted by $f(1) \in \mathbb{Z}$. It has been established, through the work of Ruberman [27], that all 3-manifolds are the target of a degree one map from a hyperbolic 3-manifold. By employing a similar line of reasoning, we can assert the following:

Corollary 1.12 *Given any 3-manifold Y there exists a Stein fillable 3-manifold Y' and a degree one map $f: Y' \to Y$.*

The distinction between smooth and topological embeddings serves as a tool for detecting exotic structures on compact manifolds. If we encounter two homeomorphic 4-manifolds such that a 3-manifold embeds smoothly in one but not the other, then they are not diffeomorphic; they form an exotic pair. Corollary 1.13, which we will present shortly, was initially demonstrated by Akbulut [3] and subsequently proven by many others. However, we will offer an alternative proof stemming from the study of embeddings of 3-manifolds into 4-manifolds.

Corollary 1.13 *There exist compact 4-manifolds with boundary X and X' such that $b_2(X) = b_2(X') = 1$ that are homeomorphic but not diffeomorphic.*

Acknowledgements The author expresses gratitude to Chris Gerig, Robert Gompf, Kyle Hayden, Tye Lidman, Hyunki Min, Maggie Miller, Lisa Piccirillo, Danny Ruberman, Oğuz Şavk, and Bülent Tosun for their insightful discussions and feedback on the first draft. Special thanks are extended to Jen Hom and Tom Mark for their assistance with aspects of Heegaard Floer homology. The author acknowledges Kouchi Yasui for bringing the paper [5] to their attention and for engaging in helpful discussions. Marco Golla's inquiry, which led to Theorem 1.8, is appreciated, along with his thorough review of earlier drafts and constructive comments. Lastly, the author expresses gratitude to their advisor John Etnyre for constant guidance and support, as well as to the referees for their valuable feedback.

2 Background

2.1 Contact geometry

Recall that a (*coorientable*) *contact structure* ξ on an oriented 3-manifold Y is the kernel of 1-form $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha$ is nondegenerate. Geometrically a contact structure on a 3-manifold is a distribution of 2-plane fields on the manifold that is not tangent to any embedded surface in the manifold. Darboux's theorem says that every contact 3-manifold (Y, ξ) is locally contactomorphic to $(\mathbb{R}^3, \xi_{\text{std}} = \ker(dz - y\,dx))$. All orientable 3-manifolds admit contact structures. A knot $L \subset (Y, \xi)$ is called *Legendrian* if at every point of L the tangent line to L lies in the contact plane at that point. A Legendrian knot L in a contact manifold (Y, ξ) has a standard neighborhood N and a framing fr_ξ given by the contact planes. If L is null-homologous then fr_ξ relative to the Seifert framing is the *Thurston–Bennequin* invariant of L. Suppose one does $fr_\xi - 1$ surgery on L by removing N and gluing back a solid torus to effect the desired surgery. In that case, there is a unique way to extend $\xi|_{Y-N}$ over the surgery torus so that it is tight on the surgery torus. The resulting contact manifold is said to be obtained from (Y, ξ) by *Legendrian surgery* on L.

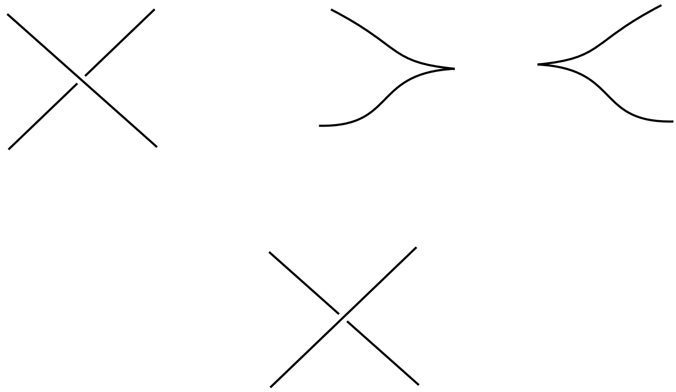

Figure 1: The top row indicates the correct crossing and the cusps in the front projection. The bottom picture crossing will not occur in the front projection diagram.

A Legendrian knot L in $(\mathbb{R}^3, \xi_{\text{std}})$ projects to a closed curve γ in the xz-plane which also known as *front projection* of L. The curve γ uniquely determines the Legendrian knot L which can be reconstructed by setting $y(t)$ as the slope of $\gamma(t)$. Thus, at a crossing of the projection, the most negatively sloped curve always stays at the front. There are two types of cusps singularity possible when $dz/dx = 0$ which are called left cusps and right cusps. See Figure 1.

Looking at an oriented front projection one can compute the Thurston–Bennequin invariant of a Legendrian knot $\text{tb}(K) = \text{writhe}(K) - \#\{\text{left cusps}\}$. For more details, we refer to [13; 14].

A contact 3-manifold (Y, ξ) is called *overtwisted* if there exists a Legendrian unknot with Thurston–Bennequin number 0, otherwise it is called *tight*.

2.2 Symplectic fillings, cobordisms and caps

We recall that a compact symplectic manifold (X, ω) is a *strong symplectic filling* of (Y, ξ) if $\partial X = Y$ and there is a vector field v defined near ∂X such that the Lie derivative of ω satisfies $\mathscr{L}_v \omega = \omega$, v points out of X and $\iota_v \omega$ is a contact form for ξ. Moreover, (X, ω) is a strong symplectic cap for (Y, ξ) if it satisfies all the properties above, except $\partial X = -Y$, and v points into X. We also say (X, ω) is a *weak filling* of (Y, ξ) if $\partial X = Y$ and $\omega|_\xi > 0$ (here all our contact structures are cooriented). Similarly, (X, ω) is a weak cap of (Y, ξ) if $\partial X = -Y$ and $\omega|_\xi > 0$. We shall say that (Y, ξ) is (strongly or weakly) *semifillable* if there is a connected (strong or weak) filling (X, ω) whose boundary is a disjoint union of (Y, ξ) with an arbitrary nonempty contact manifold.

A *symplectic cobordism* from the contact manifold (Y_-, ξ_-) to (Y_+, ξ_+) is a compact symplectic manifold (W, ω) with boundary $-Y_- \cup Y_+$ where Y_- is a *concave* boundary component and Y_+ is *convex*; this means that there is a vector field v near ∂W which points transversally inwards at Y_- and transversally outwards at Y_+, and $\mathscr{L}_v \omega = \omega$. The first result we will need concerns when symplectic cobordisms can be glued together.

Lemma 2.1 [8] *Let (X_i, ω_i) be a symplectic cobordism from (Y_i^-, ξ_i^-) to (Y_i^+, ξ_i^+), for $i = 1, 2$, and (Y_1^+, ξ_1^+) is contactomorphic to (Y_2^-, ξ_2^-). Then we can construct a symplectic cobordism (X, ω) from (Y_1^-, ξ_1^-) to (Y_2^+, ξ_2^+) such that X is diffeomorphic to $X_1 \cup_{Y_1^+} X_2$.*

Now recall that a *Stein domain* is a triple (X, J, ψ) where J is a complex structure on X and $\psi : X \to \mathbb{R}$ is a proper plurisubharmonic function, that is a smooth function such that $\psi^{-1}(-\infty, c]$ is compact for all $c \in \mathbb{R}$ and $\omega_\psi(v, w) = -d(d\psi \circ J)(v, w)$ is a symplectic form. A closed contact manifold (Y, ξ) is called Stein fillable if there exists a Stein manifold (X, J, ψ) such that ψ is bounded from below, M is an inverse image of a regular value of ψ and $\xi = \ker(-d\psi \circ J)$. We have the following characterization of Stein 4-manifolds.

Theorem 2.2 (Elaishberg [7] and Gompf [14]) *A 4-manifold is a Stein domain if and only if it has a handle decomposition with 0-handles, 1-handles and 2-handles, and the 2-handles are attached along Legendrian knots in $\#S^1 \times S^2$ with framing one less than the contact framing.*

Another way to build cobordisms is by Weinstein handle attachment [30]. One may attach a 0-, 1-, or 2-handle to the convex end of a symplectic cobordism to get a new symplectic cobordism with the new convex end described as follows. For a 0-handle attachment, one merely forms the disjoint union with a standard 4-ball and so the new convex boundary will be the old boundary disjoint union with the standard contact structure on S^3. For a 1-handle attachment, the convex boundary undergoes a, possibly internal, connected sum. A 2-handle is attached along a Legendrian knot L with framing one less than the contact framing, and the convex boundary undergoes a Legendrian surgery.

Theorem 2.3 *Given a contact 3-manifold (Y, ξ) let W be a part of its symplectization, that is*

$$(W = Y \times [0, 1], \omega = d(\alpha e^t)).$$

Let L be a Legendrian knot in (Y, ξ) where we think of Y as $Y \times \{1\}$. If W' is obtained from W by attaching a 2-handle along L with framing one less than the contact framing, then the upper boundary (Y', ξ') is still a convex boundary. Moreover, if the 2-handle is attached to a Stein filling (respectively strong, weak filling) of (Y, ξ) then the resultant manifold would be a Stein filling (respectively strong, weak filling) of $(Y' \xi')$.

The theorem for Stein fillings was proven by Eliashberg [7], for strong fillings by Weinstein [30], and was first stated for weak fillings by Etnyre and Honda [9].

Starting with a Stein filling (respectively strong, weak filling) of (Y, ξ) one can construct a symplectic closed manifold by capping it off. Various people have studied concave caps on contact manifolds but for our purpose, we need the result of Etnyre, Min, and the author [10].

Theorem 2.4 *If (W, ω) is weak filling of (Y, ξ) then there exists a closed symplectic 4-manifold (X, ω') in which (W, ω) symplectically embeds such that the complement of W in X is simply connected and has $b_2^+ > 0$.*

2.3 Heegaard Floer homology

Recall that Heegaard Floer homology is an Abelian group associated to a 3-manifold Y, equipped with a Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. These homology groups are invariant of the pair $(Y, \mathfrak{t})$ and are denoted by $HF^\infty(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U, U^{-1}]$ module; $HF^+(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U^{-1}]$ module; $HF^-(Y, \mathfrak{t})$, which is a graded $\mathbb{Z}[U]$ module. These invariants fit into a long exact sequence

$$\cdots \longrightarrow HF^-(Y, \mathfrak{t}) \xrightarrow{\iota} HF^\infty(Y, \mathfrak{t}) \xrightarrow{\pi} HF^+(Y, \mathfrak{t}) \xrightarrow{\delta} \cdots .$$

Recall that associated to this long exact sequence there is another 3-manifold invariant

$$HF^+_{\mathrm{red}}(Y, \mathfrak{t}) = \mathrm{Coker}(\pi) \cong \mathrm{Ker}(\iota) = HF^-_{\mathrm{red}}(Y, \mathfrak{t}).$$

The isomorphism in the middle is induced by the coboundary map. Recall that $d(Y, \mathfrak{t})$ is the minimum grading of the torsion-free elements in the image of $\pi\colon HF^\infty(Y, \mathfrak{t}) \to HF^+(Y, \mathfrak{t})$. For more details, readers are referred to [23; 24].

Now recall that an *L-space* Y is a rational homology 3-sphere whose Heegaard Floer homology is as simple as possible, that is $HF^+_{\mathrm{red}}(Y, \mathfrak{t}) = 0$ for all Spin^c structures $\mathfrak{t} \in \mathrm{Spin}^c(Y)$.

A cobordism between two 3-manifolds induces a map on Heegaard Floer homology. More precisely, if W is a cobordism from Y_0 to Y_1 and $\mathfrak{s}$ is a Spin^c structure in W whose restriction on Y_i is denoted by $\mathfrak{s}_i$ for $i = 0, 1$, then there is a map $F^\circ_{W,\mathfrak{s}}\colon HF^\circ(Y_0, \mathfrak{s}_0) \to HF^\circ(Y_1, \mathfrak{s}_2)$, where $\circ = +, -$ or ∞.

Theorem 2.5 (Ozsváth and Szabó [26]) *If W is a cobordism between Y_0 and Y_1, and $\mathfrak{s}$ is a Spin^c structure on W whose restriction to Y_i is denoted by $\mathfrak{s}_i$ for $i = 0, 1$ then we have*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & HF^-(Y_0, \mathfrak{s}_0) & \xrightarrow{\iota_0} & HF^\infty(Y_0, \mathfrak{s}_0) & \xrightarrow{\pi_0} & HF^+(Y_0, \mathfrak{s}_0) & \xrightarrow{\delta_0} & \cdots \\
 & & \downarrow F^-_{W,\mathfrak{s}} & & \downarrow F^\infty_{W,\mathfrak{s}} & & \downarrow F^+_{W,\mathfrak{s}} & & \\
\cdots & \longrightarrow & HF^-(Y_1, \mathfrak{s}_1) & \xrightarrow{\iota_1} & HF^\infty(Y_1, \mathfrak{s}_1) & \xrightarrow{\pi_1} & HF^+(Y_1, \mathfrak{s}_1) & \xrightarrow{\delta_1} & \cdots
\end{array}$$

where the vertical maps are uniquely determined up to an overall sign, and all the squares are commutative.

The *composition law* states that if W_0 is a cobordism from Y_0 to Y_1 and W_1 is a cobordism from Y_1 to Y_2, and if $\mathfrak{s}_i$ is the Spin^c structure on W_i for $i = 0, 1$, then the relationship between composition of $F_{W_0,\mathfrak{s}_0}$ with $F_{W_1,\mathfrak{s}_1}$ and the maps induced by the composite cobordism $W = W_0 \cup_{Y_1} W_1$ is

$$F^\circ_{W_1,\mathfrak{s}_1} \circ F^\circ_{W_0,\mathfrak{s}_0} = \sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(W) \,|\, \mathfrak{s}|W_i = \mathfrak{s}_i, i=0,1\}} \pm F^\circ_{W,\mathfrak{s}}.$$

2.4 Closed 4-manifold invariants

There is a variant of the cobordism invariant which is defined for cobordism with $b_2^+(W) > 1$. The following lemma was proved by Ozsváth and Szabó [26]

Lemma 2.6 *Let W be a cobordism between Y_0 and Y_1 with $b_2^+(W) > 0$. Then the induced cobordism map $F^{\infty}_{W,\mathfrak{s}}$ vanishes for all* Spin^c *structures on W.*

If we have a cobordism W with $b_2^+(W) > 1$, then we can cut W along a 3-manifold N, which divides W into two cobordisms, W_0 and W_1, both of which have $b_2^+(W_i) > 0$, in such a way that the map induced by the restriction

$$\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(W_0) \times \mathrm{Spin}^c(W_1)$$

is injective. Such a cut N is called an *admissible cut*.

Remark 2.7 Notice that if in a cobordism W with $b_2^+(W) > 1$ we find a separating rational homology 3-sphere N such that both the pieces have $b_2^+ > 0$, then N is an admissible cut.

If $\mathfrak{s}$ is a Spin^c structure on W whose restriction to W_i is $\mathfrak{s}_i$ and the induced Spin^c structures on 3-manifolds Y_0, Y_1 and N is $\mathfrak{t}_0$, $\mathfrak{t}_1$ and $\mathfrak{t}$, then

$$F^-_{W_0,\mathfrak{s}_0}\colon HF^-(Y_0,\mathfrak{t}_0) \to HF^-(N,\mathfrak{t})$$

factors through the inclusion $HF^-_{\mathrm{red}}(N,\mathfrak{t}) \to HF^-(N,\mathfrak{t})$, and

$$F^+_{W_1,\mathfrak{s}_1}\colon HF^+(N,\mathfrak{t}) \to HF^+(Y_1,\mathfrak{t}_1)$$

factors through the projection $HF^+(N,\mathfrak{t}) \to HF^+_{\mathrm{red}}(N,\mathfrak{t})$. And thus by using the identification of $HF^+_{\mathrm{red}}(N,\mathfrak{t}) \cong HF^-_{\mathrm{red}}(N,\mathfrak{t})$ in the middle, we can define the *mixed invariant* as a map

$$F^{\mathrm{mix}}_{W,\mathfrak{s}}\colon HF^-(Y_0,\mathfrak{t}_0) \to HF^+(Y_1,\mathfrak{t}_1).$$

Remark 2.8 It is also proven in [26] that F^{mix} does not depend on the choice of the admissible cut.

From the discussion above one immediately sees the following result.

Lemma 2.9 *If an admissible cut N of W is an L-space then $F^{\mathrm{mix}}_{W,\mathfrak{s}}$ vanishes.*

Theorem 2.10 (Ozsváth and Szabó [25]) *If (X,ω) is a closed, symplectic manifold with $b_2^+(X) > 1$, then for the canonical* Spin^c *structure $\mathfrak{k}$ corresponding to the symplectic form, $F^{\mathrm{mix}}_{X,\mathfrak{k}}$ is nonvanishing. Here we think of X as a cobordism from S^3 to S^3 by taking out two 4-balls.*

Remark 2.11 The above discussion implies that an L-space cannot be an admissible cut for a closed symplectic 4-manifold with $b_2^+ > 1$.

3 Topological embedding of 3-manifolds in symplectic 4-manifolds

Now we will begin the proof of topological embedding of 3-manifolds into symplectic 4-manifolds.

Proof of Theorem 1.1 We will topologically embed a 3-manifold Y into a symplectic manifold X in three steps. In the fourth step, we will show that the embedding is smooth after a single stabilization with $S^2 \times S^2$. We start with a Kirby picture, consisting of only a 0-handle and 2-handles, for a 4-manifold whose boundary is Y.

Step 1 Stein modification of the Kirby picture.

Let $K_1, \dots, K_m$ be the attaching spheres for the 2-handles. We can Legendrian realize the K_i so that each K_i intersects a fixed Darboux ball B in a horizontal arc. We can blow up meridians to each K_i so that the framing on K_i is less than $\mathrm{tb}(K_i) - 2$. All the blown-up unknots can be gathered in the Darboux ball as shown in the top left of Figure 2. Blow up one more unknot as indicated in the upper right of Figure 2 and notice that the resulting link L can be Legendrian realized as in the bottom diagram of Figure 2. Let K be the unknot with framing -1 in the figure. We now stabilize the components of $L - K$ so that the Thurston–Bennequin invariant of each component is one larger than its surgery coefficients. Legendrian surgery on $L - K$ together with -1 surgery on K gives our manifold Y. (Notice that to realize L each K_i might need to be stabilized one extra time as shown in the figure. This is why we arranged the surgery coefficients to be less than $\mathrm{tb}(K_i) - 2$.) So the manifold W_0 obtained by attaching Stein 2-handles to $L - K$ is a Stein manifold and we denote the boundary by Y_0. Now attaching a 2-handle to W_0 along K with framing -1 gives a 4-manifold W with boundary Y.

Step 2 Attach a cork and apply a cork twist.

This similar technique of using cork-twist to produce Stein manifold has been previously used in the work of Akbulut and Yasui [5] and Akbulut and Matveyev [4]. We adopt this method in our setup by constructing a manifold W_2 by modifying the surgery presentation that is adding the 1- and 2-handle shown in Figure 3, where the 2-handle links K as indicated and is otherwise disjoint from L (by abuse of language we will call this operation attaching a Mazur cork). Said another way, we can build a cobordism W_1 by attaching the 1- and 2-handle to $Y \times [0,1]$ along $Y \times \{1\}$. The manifold W_2 is now simply $W \cup W_1$ with ∂W glued to $-Y \subset W_1$ and W_1 is a cobordism from Y to some manifold Y'.

We can apply a cork-twist by interchanging the 1-handle and the 0-framed 2-handle. The cork twist does not change the boundary 3-manifold Y'. After the cork twists the knot K passes over the 1-handle geometrically once and thus they cancel each other. After this handle cancellation the knot K in the original picture, Figure 2, is replaced by -1 framed knot in the third picture of Figure 3. Notice that this new knot can be realized by a Legendrian knot that has Thurston–Bennequin invariant $+1$ and thus -1 smooth surgery on this knot can be realized as Legendrian surgery on stabilization of the knot. So we get a Stein filling of the boundary.

Step 3 Construct a simply connected closed symplectic 4-manifold.

Also, notice that the W_1 deformation retracts onto Y and W is a 2-handlebody, so in particular W_2' is simply connected. So in Step 2 when we do the cork-twist the new manifold W_2' is still homeomorphic to

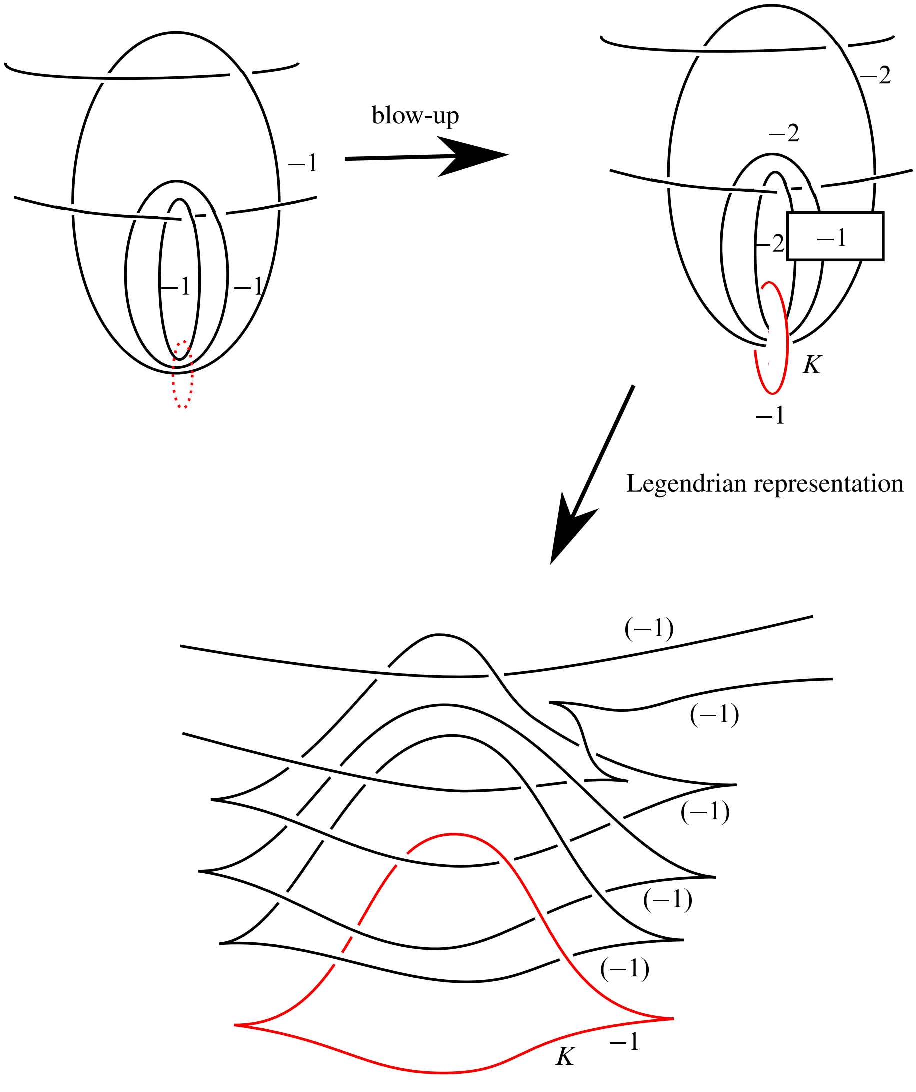

Figure 2: Converting a Kirby picture of the 3-manifold by blowing up such that removing the red knot yields a framed link representing a Stein handle diagram. Here (-1) is measured relative to the contact framing.

the compact manifold W_2 by a result of Freedman [12] and thus the original 3-manifold has a topological, locally flat embedding in W_2'. Now we use the simply connected cap constructed in Theorem 2.4 to cap off the upper boundary W_2' to get a closed simply connected symplectic 4-manifold X into which the 3-manifold Y topologically, locally flatly embeds.

Step 4 Smooth embedding after one stabilization.

We can stabilize X by adding a Hopf link to W_2' and using the same cap (of course this stabilized 4-manifold is no longer symplectic). We can now handle slide one of the components C of the Hopf link

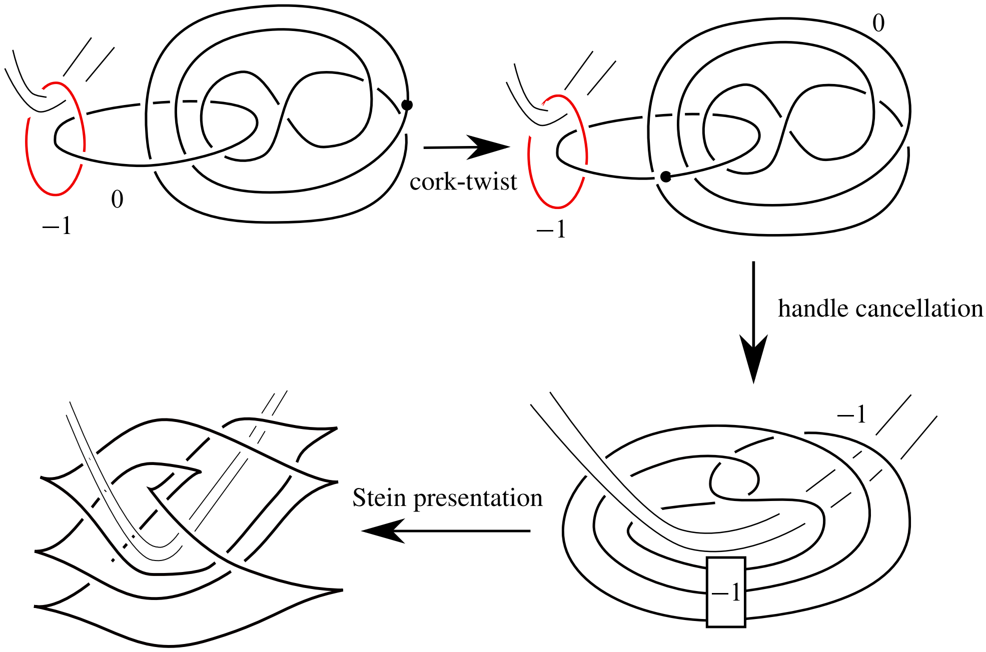

Figure 3: The maximal Thurston–Benequin number of the black knot in the bottom picture is $+1$. So it is a Stein 2-handle attachment.

over the 0-framed knot in the Mazur cork as indicated at the top of Figure 4. Using the 0-framed meridian to C we can untangle C from the 2-handle in the Mazur cork as shown in the middle of Figure 4. We can further slide C over the 0-framed meridian to turn C into a meridian of the 1-handle. Thus the 1-handle can be canceled with C, leaving the bottom picture in Figure 4. Thus we see a smooth embedding of Y into $W_2' \# S^2 \times S^2$ and thus into $X \# S^2 \times S^2$. □

4 Embedding L-spaces in symplectic 4-manifolds

We now prove that smooth embeddings of L-spaces in symplectic 4-manifolds are always separating.

Proof of Theorem 1.3 Suppose Y is an L-space that smoothly embeds in a closed symplectic 4-manifold. We begin by showing it is separating. To this end, we assume it is nonseparating. Let X_1 be the compact manifold obtained from X by cutting along Y. Notice that $\partial X_1 = Y \sqcup -Y$, so we can glue two copies, X_1^1 and X_1^2, of X_1 along their boundaries to get a closed manifold X'. As constructed, X' is a double cover of X so, in particular, we can lift the symplectic form using the covering map, and thus X' is symplectic. Let N be a neighborhood of an arc in X_1^1 connecting its boundary components. Set $X_1' = \overline{X_1^1 - N}$ and $X_2' = X_1^2 \cup N$. Clearly $\partial X_i' = Y \# -Y$ which is an L-space. As X is symplectic and Y is a rational homology sphere, by using the Mayer–Vietoris sequence we can see that $b_2^+(X_i') = b_2^+(X) > 0$ (since X is symplectic, the cohomological element corresponds to the symplectic form produces an element of b_2^+). So $Y \# -Y$ is an admissible cut for a symplectic manifold X' which contradicts Remark 2.11.

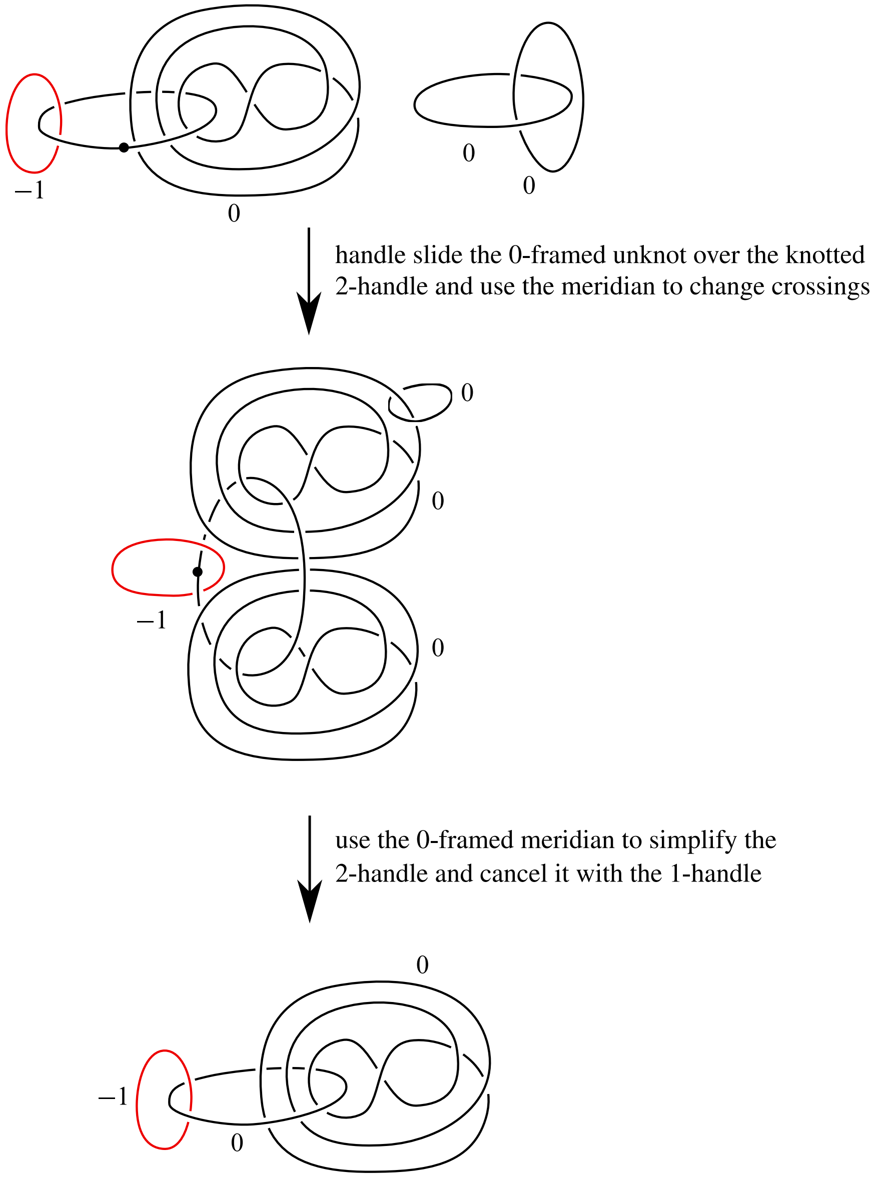

Figure 4: In this picture we are describing the Kirby moves of how connected summing with $S^2 \times S^2$ helps to cancel the 1-handle of the cork.

Now when Y embeds in X in a separating manner then one of the components of $X - Y$ must have $b_2^+ = 0$ or we will get the same contradiction as before.

Now let Y be an L-space that does not bound a negative-definite 4-manifold. If Y is embedded in any symplectic 4-manifold with weakly convex boundary W then it has to be separating since otherwise we can cap off with a concave cap to get a closed symplectic manifold where Y is nonseparating which contradicts the above discussion. So Y has to be separated. When we cap off the upper boundary of

W by a cap with $b_2^+ > 0$, since Y does not bound a negative-definite 4-manifold, both the sides of Y have $b_2^+ > 0$. In particular, Y is an admissible cut for a symplectic 4 manifold with $b_2^+ > 1$ which is a contradiction by Remark 2.11. □

Remark 4.1 We now discuss a strategy to show the negative answer to Question 1 about L-spaces bounding definite 4-manifolds. Before that, notice all lens spaces bound both positive-definite and negative-definite 4-manifolds, because every lens space can be thought of as the boundary of a negative plumbed manifold and $-L(p,q) = L(p, p-q)$. One can obstruct rational homology spheres Y with $H_1(Y,\mathbb{Z}) \neq 0$ from bounding negative-definite manifolds by using the technique developed by Owens and Strle [22] where in Theorem 2 they proved that if the maximum value of the d-invariant of Y is smaller than $1/4$ (with some more algebraic conditions) then Y cannot bound a negative-definite 4 manifold. Now $d(Y,\mathfrak{s}) = -d(-Y,\mathfrak{s})$, so if we find an L-space Y with $H_1(Y,\mathbb{Z}) \neq 0$, for which the absolute differences between d-invariants for different Spin^c structures are very small then that could be used to obstruct it from bounding positive-definite and negative-definite 4-manifolds (as both Y and $-Y$ cannot bound negative-definite 4-manifolds). So we can ask:

Question 5 *For every $n \in \mathbb{N}$ does there exist an L-space which is not a $\mathbb{Z}HS^3$ and whose d-invariant values are in $(-1/n, 1/n)$?*

Proof of Corollary 1.6 Let Y' admit a weakly fillable contact structure and Y be an L-space that does not bound a negative-definite 4-manifold. If Y has an oriented cobordism embedding in $Y' \times [0,1]$, then since Y' is weakly fillable Y has an oriented cobordism embedding in a symplectic 4 manifold with weakly convex boundary, contradicting Theorem 1.3. □

We now show the existence of exotic manifolds with boundary and $b_2 = 1$ using the ideas above.

Proof of Corollary 1.13 Start with B^4 and attach a 2-handle h along the pretzel knot $K = P(-2,3,7)$ to get a 4-manifold W' with $S^3_9(K)$ (which is an L-space as mentioned in Remark 1.5) as its boundary. Attach a cork as in Step 2 of Theorem 1.1 and get W with $b_2^+(W) = 1$. After a cork-twist, we can see that the 2-handle h now passing over the 1-handle of the cork and this will increase the contact framing of h by one as in Figure 7; thus the resulting manifold W' will be Stein by Theorem 2.2. Before the cork-twist, we had a smooth embedding of $S^3_9(K)$ in W. But by Theorem 1.3 $S^3_9(K)$ cannot embed smoothly in W' so they are exotic pairs. □

5 Constructing ribbon cobordism

We now turn to Theorem 1.8 that given any 3-manifold Y there is a simple invertible $\mathbb{Z}$-ribbon homology cobordism to a Stein fillable manifold Y'. Moreover, that says that given a compact 4-manifold with some specific conditions, one can construct a Stein 4-manifold with the same algebraic topology but a different boundary.

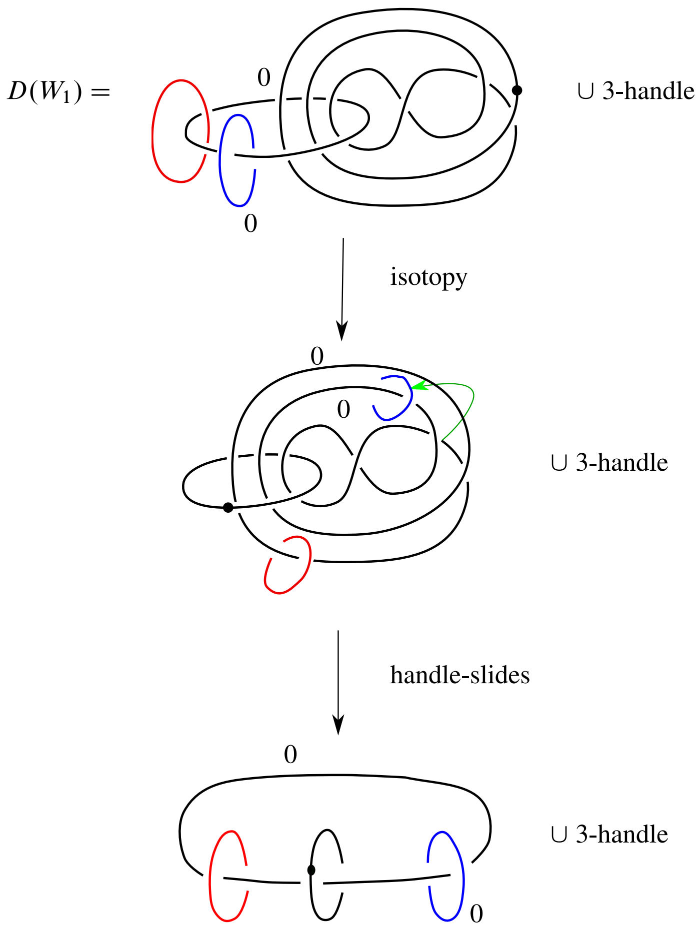

Figure 5: After isotopy, in the second picture, the 0-framed blue 2-handle will help to resolve the crossings of the black 2-handle so that it can cancel the 1-handle. And after that, the 3-handle will cancel the 0-framed unknotted blue 2-handle.

Proof of Theorem 1.8 The cobordism W_1 from Step 2 is the desired ribbon $\mathbb{Z}$-homology invertible cobordism which is attaching a cork along the red knot at the top of Figure 3.

To see that it is invertible, let us consider $D(W_1)$ to be the double of W_1 along Y', that is glue an upside-down copy of W_1 on top of W_1 along the boundary. If h is the 2-handle in W_1, then $D(W_1)$ is formed by attaching a 2- and 3-handle to W_1, with the 2-handle attached to a 0-framed meridian to h [15, Section 4.2]. Check Figure 5. Now by changing crossings on the attaching circle of h using the 0-framed meridian, we can arrange that h passes over the 1-handle geometrically once. Thus they cancel each other. And after the cancellation, the 0-framed meridian h will cancel the 3-handle. And thus the resultant manifold $D(W_1) = Y \times I$.

(i) Let X be a compact oriented 4-manifold with boundary Y and $b_1(X) = 0$. Turning a handle structure on X upside down, we can think of X as a cobordism from $-Y$ to $\varnothing$; this is indicated in Figure 6.

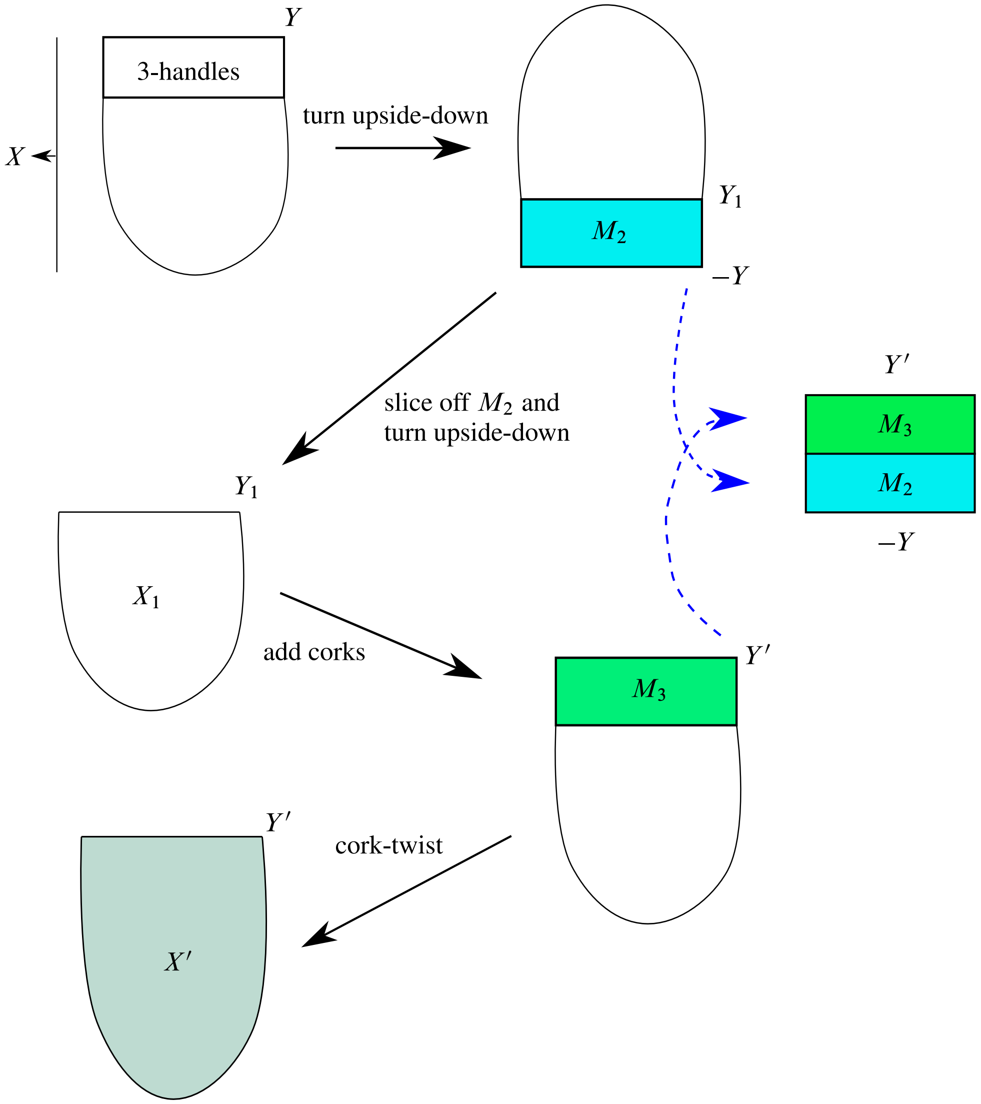

Figure 6: A schematic of the construction of a ribbon cobordism from Y to a Stein fillable Y'.

Notice that in this upside-down cobordism, all the 1-handles of X are converted into 3-handles and all the 3-handles become 1-handles. In the upside-down X, 1-handles are attached onto $-Y \times [0,1]$ along $-Y \times \{1\}$, let us call this cobordism M_1. Notice that $b_1(X) = 0$ so the homology long exact sequence of the pair (X, Y) implies that there exists a minimal set of 2-handles such that if we attach those on top of M_1, and let us call it M_2, then $H_1(M_2, Y; \mathbb{Q})) = 0$. (Here by minimum we mean that if we take any 2-handle out from the set then $H_1(M_2, Y; \mathbb{Q}) \neq 0$.) Since we consider a minimal set of 2-handles for this construction, we have $H_2(M_2, Y; \mathbb{Q}) = 0$ as well because in this case the number of 1-handles of M_2 is the same as the number of 2-handles. Thus M_2 is a rational ribbon cobordism from Y to say Y_1 which is the top boundary of M_2; see the top right of Figure 6. Consider X_1 to be the handlebody

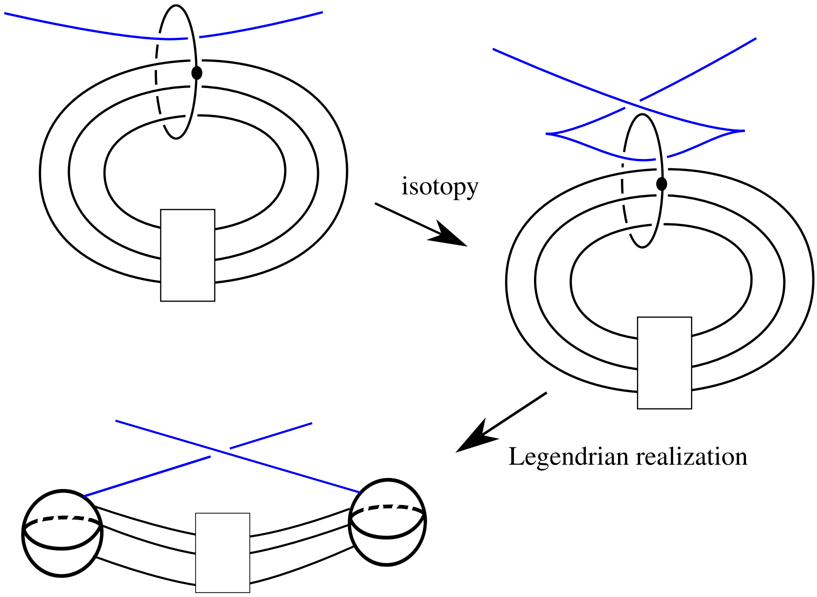

Figure 7: The contact framing of the blue knot increased by $+1$ after a cork-twist.

obtained from X by taking out M_2 and turning what remains upside-down; this is indicated in the third picture Figure 6. Thus X_1 only has 1- and 2-handles with boundary Y_1. If this is Stein then we are done. If not then that implies it has some 2-handles whose smooth framing is bigger than that of the contact framing minus 1 of the attaching circle in $\#S^1 \times S^2$ (in this case we can think of the top boundary Y_1 is obtained after attaching 2-handles on the boundary of 1-handlebody which is connected sum of $S^1 \times S^2$). To fix this framing issue, we repeatedly apply the Step 2 of the proof of Theorem 1.1. That is we attach a cork as in Figure 3 (where the red curve there is the handle that needs its Thurston–Bennequin invariant increased). We then do a cork twist that exchanges the 1-and 2-handles. We claim this increases the contact framing of the original attaching sphere of the 2-handle by 1. To see this, notice that if a knot passes over 1-handle then in the front projection diagram of a knot we are actually deleting two consecutive right and left cusps by connecting them through a 1-handle, and thus we are increasing the contact framing. See Figure 7. However, this process does not change the smooth surgery coefficient. Let us consider the cobordism X_2 obtained by attaching a suitable number of corks to X_1 so that the manifold X_2' obtained by applying the cork twists is Stein. The manifolds X_2 and X_2' are homeomorphic as the cork-twist homeomorphism can always be extended as homeomorphism on the 4-manifold by the result of Freedman [12]. Observe that $b_2(X_2') = b_2(X_1) = b_2(X)$. Let Y' be the top boundary of X_2'. Then there is a homology ribbon cobordism M_3 from Y_1 to Y' which is given by attaching the above corks to the top of X_1; see the fourth picture in Figure 6. Glue this cobordism on top of M_2 to get our desired ribbon rational homology cobordism $M = M_2 \cup M_3$ from Y to Y' with Y' Stein fillable.

(ii) Let X be a compact manifold with connected boundary Y which is a $\mathbb{Q}HS^3$; then we consider a handle decomposition of $X = X_0 \cup X_1 \cup X_2 \cup X_3$ where X_i contains handles of index i. Consider the minimum set of 1-handles which generate the free part of $(H_1(X;\mathbb{Q}))$. Let $\overline{X}$ be the manifold obtained from X by doing surgery on those 1-handles. (In Kirby calculus this is equivalent to replacing those dotted 1-handles with 0-framed unknotted 2-handles.) We will now show that this surgery operation does not change the b_2 (or more precisely the intersection form). As Y is a $\mathbb{Q}HS^3$, $H_1(\overline{X};\mathbb{Q}) = 0 = H_3(\overline{X};\mathbb{Q})$. But we are not doing anything with the 3-handles of X, so the only way the third homology of $\overline{X}$ vanishes with $\mathbb{Q}$ coefficients is if the 3-handles cancel the 2-handles in homology. And thus from cellular homology, we can see that $b_2(X) = b_2(\overline{X})$. Also, notice that the above surgery does not change the nontorsion elements of $H^2(X;\mathbb{Z})$ so they have the same intersection form. Now apply the proof of Theorem 1.8 on $\overline{X}$ and we get the desired Stein manifold X' with boundary Y'. □

We now prove that the 3-dimensional homology cobordism group is generated by Stein fillable manifolds

Proof of Corollary 1.10 Theorem 1.8 provides a homology cobordism from any manifold to a Stein manifold. Thus the homology cobordism groups are generated by Stein manifolds. □

We now prove the existence of a degree 1 map from a Stein fillable 3-manifold to a given 3-manifold.

Proof of Corollary 1.12 As we noticed previously Y' smoothly embeds in $Y \times I$ in a separating way. So, restriction to Y' of the projection map from $Y \times I$ onto Y will induce a degree 1 map from $Y' \to Y$. □

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Received: 12 March 2022 Revised: 15 March 2024

Algebraic & Geometric Topology

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 6 (pages 3145–3787) 2025