AGT

Algebraic & Geometric Topology

Volume 25 (2025)

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Algebraic & Geometric Topology **25**:6 (2025) 3315–3340
DOI: 10.2140/agt.2025.25.3315
Published: 1 October 2025

The Curtis–Wellington spectral sequence through cohomology

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We study stable homotopy groups through unstable methods applied to their representing infinite loop space Q_0S^0, as pioneered by Curtis and Wellington. Using cohomology instead of homology, we find a width filtration whose subquotients are simple quotients of Dickson algebras, which thus gives a new filtration of stable homotopy groups. We make initial calculations and determine towers in the resulting width spectral sequence. We also make calculations related to the image of J, and prove that the J homomorphism induces a splitting of the indecomposables of the cohomology of Q_0S^0.

55Q40; 55P47, 55Q50

1 Introduction

Our project, first taken up by Curtis [8] and Wellington [28], is to study stable homotopy groups at the prime two through the unstable Adams spectral sequence for their representing space. We call this the Curtis–Wellington spectral sequence (CWSS), which to our knowledge has not been studied above the zero line for almost forty years.

1.1 Main results

Recall that Q_0S^0 is the zeroth component of the infinite loop space which represents stable homotopy. Barratt and Priddy [3] and Quillen showed that its cohomology is isomorphic to that of the infinite symmetric group. Nakaoka [24] computed the mod-two cohomology of the infinite symmetric group as polynomial. We call its algebra indecomposables, which are a module over the Steenrod algebra, the Nakaoka module $\mathfrak{N}$.

Recall as well that the Dickson algebras are rings of invariants $D_n = \mathbb{F}_2[x_1, \dots, x_n]^{\mathrm{Gl}_n(\mathbb{F}_2)}$, calculated by Dickson as polynomial on generators in degree $2^n - 2^l$. The Steenrod algebra action on the ambient polynomial algebras restricts to the Dickson algebras, which provide a rich and still not fully understood collection of (unstable) modules over the Steenrod algebra. Let D_n^o be the quotient of D_n by all perfect squares.

Theorem 1.1 *There is a width filtration* (see Definition 3.3 below) *of the Nakaoka module* $\mathfrak{N}$ *whose subquotients are isomorphic to* D_n^o *as unstable modules over the Steenrod algebra.*

This width filtration is related to composition length in the Dyer–Lashof algebra. Applying a reduction of Bousfield (see Theorem 3.1 below) we have the following.

Corollary 1.2 *There is a width spectral sequence with*

$$E_1^{s,t;n} = \mathrm{Ext}_{\mathfrak{U}}^{s,t}(\Sigma^{-1} D_n^o, \mathbb{F}_2)$$

and $d_r : \mathrm{Ext}^{s,t}(\Sigma^{-1} D_n^o) \to \mathrm{Ext}^{s+1,t}(\Sigma^{-1} D_{n+r}^o)$ *which converges to the* E_2 *page of the Curtis–Wellington spectral sequence.*

Accordingly, there is a filtration on stable homotopy. In Section 5 we prove that the first submodule in this filtration detects the image of J. Our calculations, made at the prime 2, show that this must be different than the chromatic filtration since the first filtration corresponds to only the image of J and no additional elements.

Our calculations rely on a presentation of the cohomology of $Q_0 S^0$ due to Giusti, Salvatore and Sinha. Through that presentation we can already manage by-hand calculations of the Curtis–Wellington spectral sequence more readily than by previous techniques. Moreover, filtration with Dickson algebra subquotients is particularly amenable to computer calculation, for which we thank Hood Chatham; see Figure 8.

It is elementary to eliminate the possibility of many differentials in the width spectral sequence, so these computer calculations provide a first picture of the E_2 page of the Curtis–Wellington spectral sequence as well. While, disappointingly, the E_2 page of the CWSS is much larger than the classical Adams spectral sequence, we have explored two accessible phenomena.

The first accessible phenomenon is the existence of towers of elements connected by multiplication by h_0. In Section 4 we make calculations to determine the locations of these towers in the resulting width spectral sequence.

Theorem 1.3 *There are infinite towers* T_{a_1} *in* $\mathrm{Ext}^{s,t}(\Sigma^{-1} D_1^o, \mathbb{F}_2)$ *only in degrees* $t - s = a_1 - 1$ *for* $a_1 = 4k - 1$, *where* k *is a positive integer.*

Theorem 1.4 *Let $n \geq 2$ be an integer. There are towers $T_{a_1,\dots,a_n}$ in $\mathrm{Ext}^{s,t}(\Sigma^{-1} D_n^o, \mathbb{F}_2)$ in degree $t-s = \left(\sum_{i=1}^n (2^n - 2^{n-i})a_i\right) - 1$ satisfying one of the following two conditions*:

(1) $a_n = 0$, a_{n-1} *is even*, *at least one other* a_i *is odd*, *and* $\sum_{i=1}^n (2^n - 2^{n-i})a_i = 4l$.

(2) a_i for $1 \leq i \leq n-2$ are all even, a_{n-1} and a_n are odd, and $\sum_{i=1}^n (2^n - 2^{n-i})a_i = 4k-1$.

These results agree with the locations of towers identified by Wellington in the E_2 page of the CWSS, and thus imply that there are no differentials between the towers internal to the width spectral sequence.

The lowest filtration is also accessible, and related to the image of J. In Section 5 we make calculations which show preliminarily that the image of J is compatible with the width filtration. In studying the image of J, we first noticed that the unstable Adams Ext chart for $H^*(BO, \mathbb{F}_2)$ is a shifted version of the unstable Adams Ext chart for the first quotient of the width filtration.

Proposition 1.5 $$\mathrm{Ext}_{\mathcal{U}}^{s,t}(\Sigma^{-1} D_1^o) \cong \mathrm{Ext}_{\mathcal{U}}^{s,t+1}(\Sigma^{-1}\,\mathrm{Ind}\,H^*(BO)).$$

While it seems this would be classical, we haven't found any treatment of this in the literature. This motivated us to study the image of the J map on cohomology, and the following.

Theorem 1.6 *The map induced by the image of J on cohomology induces a splitting of the Nakaoka module $\mathfrak{N}$.*

Corollary 1.7 *The algebraic map induced by the image of J on* Ext *induces a splitting of* $\mathrm{Ext}(\mathfrak{N})$.

We also speculate that the map on the E_2 page of the unstable Adams spectral sequence induced by the J map on cohomology agrees with the map on Ext induced by the map on cohomology. If this were the case, then the splitting of Theorem 1.6 would give rise to the standard splitting of homotopy by the image of J.

1.2 Background

Recall that the stable homotopy groups of spheres are isomorphic to the unstable homotopy groups of $Q_0 S^0$, the degree zero component of $\varinjlim \Omega^d S^d$. The unstable Adams spectral sequence for $Q_0 S^0$, which we choose to name the Curtis–Wellington spectral sequence after the only two people to study it globally, was introduced by Curtis [8]. He outlined some first calculations, noticing that Adams filtration lowered and that both the Hopf and Kervaire classes were in filtration zero, leading to the well-known and still open conjecture that these are the only classes to survive on the zero line. But Curtis made some fundamental errors, which Wellington corrected before going on to establish more global properties, including the classification of Bockstein towers (which in particular preclude any upper vanishing lines).

In order to make these calculations, Curtis and Wellington applied a deep connection between stable homotopy and symmetric groups. At the level of homology, this was noticed independently and simultaneously by Barratt and Priddy [3] and Quillen. Briefly, one models the classifying space for the n^{th} symmetric group as $\varinjlim_d \mathrm{Disks}_n(\mathbb{R}^d)$, a colimit over d of the space of n disks in $\mathbb{R}^d$. Then, given a set of n disks in $\mathbb{R}^d$, associate to it a collapse map from $S^d = \mathbb{R}^d \cup \infty$ to itself which sends the complement of the disks to the basepoint and each interior of a disk homeomorphically onto the $S^d \setminus \infty$. These maps from the space of disks to $\Omega^d S^d$ can be assembled to a map from the colimit. The Barratt–Priddy–Quillen theorem tells us that the resulting map from the classifying space for the infinite symmetric groups to $Q_0 S^0$ is an isomorphism in homology.

Wellington and Curtis then applied the known structure of the homology of $Q_0 S^0$. This homology is the symmetric algebra on the Kudo–Araki–Dyer–Lashof algebra; see Cohen, Lada and May [7]. While algebraic topologists are comfortable with "homological coalgebra", in this case one runs into difficulties because calculations of the homology coproduct as well as the Steenrod coaction (Nishida relations) require regular applications of Adem relations. Wellington had to filter carefully to make things at all tractable. Through nonexplicit methods, Nakaoka [23; 24] had previously shown that the cohomology of the infinite symmetric group and thus $Q_0 S^0$ is polynomial, generated in combinatorially interesting degrees. Finer control of that calculation, and in particular incorporation of the Steenrod algebra action, motivated many authors to study the cohomology of symmetric groups in more detail in the Eighties and Nineties; see for example Adem, Maginnis and Milgram [1], Adem and Milgram [2] and Feshbach [11]. Relatively recently, Giusti, Salvatore and Sinha [12] found a new Hopf ring presentation for the cohomology of symmetric groups, as algebras over the Steenrod algebra, yielding a "skyline diagram" presentation for the cohomology of $\mathcal{S}_\infty$ in the limit. We take their work as a starting point, and ultimately see that it makes the Curtis–Wellington spectral sequence much more accessible than the previous approach through homology.

Plan of the paper

In Section 2 we review work of Giusti, Salvatore and Sinha on the cohomology of symmetric groups. The unstable Adams spectral sequence is typically intractable, with a nonabelian Quillen homology defining its E_2 page. But in Section 3 we apply a standard result of Bousfield [4] in the special case that a cohomology ring is free, as is the case here, equating the E_2 page with Ext in the category of unstable modules over the Steenrod algebra of the desuspended indecomposables. Using the skyline diagram presentation of the cohomology of the infinite symmetric group, these indecomposables are manageable.

Indeed, we show that a filtration by skyline diagram width (which corresponds to composition length in the Kudo–Araki–Dyer–Lashof algebra) yields subquotients which are given by the Dickson algebras, modulo perfect squares. The resulting width spectral sequence is relatively tractable, allowing us, for example, to reveal an error, likely of transcription, in Wellington's Ext charts (at the 11 and 12 stems). We then share computer calculations which imply many more differentials than in the classical Adams spectral sequence, but regular phenomena as well.

In particular, there are Bockstein (h_0) towers, which we classify in the width spectral sequence in Section 4. These occur in the same dimensions as Wellington identified, with considerably more effort, in the E_2

of the CWSS. Thus, there are no differentials in the width spectral sequence with h_0 inverted, and we conjecture no differentials in the width spectral sequence in general, a purely algebraic question. It would be interesting to understand differentials between these h_0 towers and the special roles the resulting classes in homotopy might play.

Based on a remarkable identification of the unstable Adams E_2 for BO and some preliminary calculations, in Section 5 we initiate the study of the J homomorphism. We calculate the induced map on cohomology of the indecomposables and prove that it induces a splitting of the Nakaoka module.

Acknowledgements

I would like to thank my advisor Dev Sinha for all of his help and guidance, Hal Sadofsky for many helpful conversations, Dan Isaksen for confirming interest in the aspects of the Curtis–Wellington spectral sequence which make up the bulk of this work, Haynes Miller for providing references, and Peter May for clarifying some of the history. I would also like to acknowledge the partial support of NSF grant DMS-2039316.

2 Review

Recall that $H^*(Q_0S^0, \mathbb{F}_2) \cong H^*(BS_\infty, \mathbb{F}_2)$. Recent calculations of cohomology of finite symmetric groups by taking all symmetric groups together and considering both cup product and a transfer or induction product gives a concise presentation.

Theorem 2.1 (GSS) *As a Hopf ring, $\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{F}_2)$ is generated by classes $\gamma_{l[n]} \in H^{n(2^l-1)}(B\mathcal{S}_{n2^l})$ along with unit classes on each component. The coproduct of $\gamma_{l[n]}$ is given by*

$$\Delta\gamma_{l[n]} = \sum_{i+j=n} \gamma_{l[i]} \otimes \gamma_{l[j]}.$$

Relations between transfer products of these generators are given by

$$\gamma_{l[n]} \odot \gamma_{l[m]} = \binom{n+m}{n}\gamma_{l[n+m]}.$$

Relations between cup products of generators are that cup products of generators on different components are zero.

These can be presented graphically, as "skyline diagrams". The generator $\gamma_{l[n]}$ is represented by a rectangular block of width $n \cdot 2^{l-1}$ and total area $n \cdot (2^l - 1)$, so that the area of the block corresponds to its degree in cohomology and we get a unique block type for each generator. Cup product is indicated by vertical stacking to make columns, whose placement next to each other denotes transfer product. We also draw in vertical dashed lines separating the block into n equal sections, for purposes of the coproduct. An example of such a diagram can be seen in Figure 1.

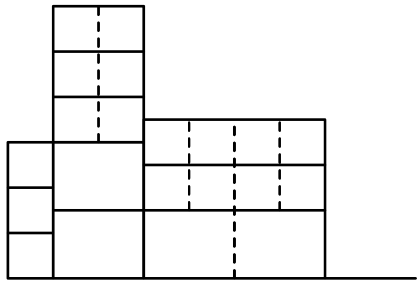

Figure 1: Skyline diagram for $\gamma_1^3 \odot \gamma_2\gamma_{1[2]}^3 \odot \gamma_{2[2]}\gamma_{1[4]}^2 \odot 1_4$.

The cohomology of the infinite symmetric group is the inverse limit

$$H^*(BS_\infty) = \varprojlim_n H^*(BS_n).$$

The maps $H^*(BS_n) \to H^*(BS_m)$ for $n > m$ take a diagram in $H^*(BS_n)$ which has a "tail" of width greater than or equal to $\frac{1}{2}(n-m)$ — that is, a $\odot$-product factor of 1_k with $k > \frac{1}{2}(n-m)$ — to a diagram in $H^*(BS_m)$ obtained by shortening its tail to make it the appropriate width to be an element of $H^*(BS_m)$. If a class is not such a transfer product with a sufficiently large unit class, it maps to zero. In Figure 1 the diagram has a tail of width 2 and is an element of $H^*(BS_{18})$. Shortening the tail once, it becomes $\gamma_1^3 \odot \gamma_2\gamma_{1[2]}^3 \odot \gamma_{2[2]}\gamma_{1[4]}^2 \odot 1_2 \in H^*(BS_{16})$. Shortening it again produces $\gamma_1^3 \odot \gamma_2\gamma_{1[2]}^3 \odot \gamma_{2[2]}\gamma_{1[4]}^2 \in H^*(BS_{14})$ after which point the tail can no longer be shortened and its image in $H^*(BS_{2k})$ is 0 for $k < 7$.

With restriction maps taking this form, the cohomology of $B\mathcal{S}_\infty$ could be viewed through such diagrams with "infinitely long tails", or in monomial form as "$\odot 1_\infty$". As they confer no additional information, we prefer to omit the tails altogether, only using them implicitly when we calculate using the Hopf ring product structures. Using this presentation, we next recall another basic result of Giusti, Salvatore and Sinha, refining a classical result of Nakaoka.

Theorem 2.2 *The cohomology of BS_∞ is a polynomial algebra. Minimal generators of $H^*(BS_\infty)$ as an algebra under cup product are represented graphically by single columns with at least one block type appearing an odd number of times. That is, generators as an algebra are products $\prod_i \gamma_{l[n_i]}^{a_i}$ of Hopf ring generators of the same width, ie $n_i 2^{l_i - 1}$ is equal for all i, with at least one a_i odd.*

These minimal generators form a basis for the indecomposables of the cohomology of BS_∞, which we call the Nakaoka module $\mathfrak{N}$.

The idea of the proof is to consider the product of single-column diagrams which contribute to a skyline diagram. This product results in a sum of diagrams, the widest of which is the original skyline diagram. This shows such products are algebraically independent, and a simple filtration argument shows the

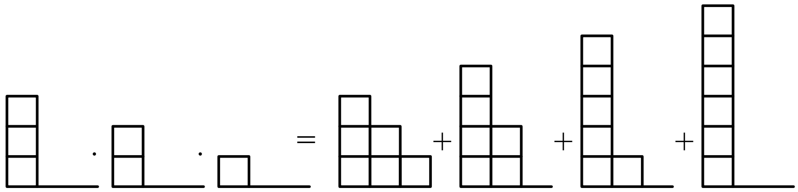

Figure 2: Skyline diagram illustrating $\gamma_1^3 \odot 1_\infty \cdot \gamma_1^2 \odot 1_\infty \cdot \gamma_1 \odot 1_\infty = \gamma_1^3 \odot \gamma_1^2 \odot \gamma_1 \odot 1_\infty +$ lower-width terms.

polynomial ring they generate exhausts the cohomology. For example, consider $\gamma_1^3 \odot \gamma_1^2 \odot \gamma_1 \odot 1_\infty \in H^*(BS_\infty)$, whose skyline diagram has three width-one columns of heights three, two and one. It is the highest-width term in the product of its three columns $\gamma_1^3 \odot 1_\infty$, $\gamma_1^2 \odot 1_\infty$ and $\gamma_1 \odot 1_\infty$, as illustrated in Figure 2.

In the same paper, Giusti, Salvatore and Sinha describe the Steenrod algebra action on the cohomology of symmetric groups in terms of the basis elements $\gamma_{l[2^k]}$. Because there are Cartan formulas for both cup and transfer product, this determines the Steenrod structure on the whole.

Definition 2.3 (i) A gathered block is a monomial of the form $\prod_i \gamma_{l_i[n_i]}^{d_i}$ where the product is the cup product and all the widths $2^{l_i} n_i$ are equal.

(ii) The algebraic degree of a gathered block is the total number of Hopf ring generators $(\gamma_{l_i[n_i]})$ cup-multiplied to make the gathered block.

(iii) The height of one of these skyline diagrams is the largest of the algebraic degrees of its constituent gathered blocks.

(iv) The effective scale of a gathered block, composed of $\gamma_{l[n]}$ cup-multiplied together, is the largest such l that occurs in the block. The effective scale of a gathered monomial is the minimum of the effective scales of its gathered blocks.

(v) A monomial is full-width as long as it is not a nontrivial transfer product of some monomial with a unit class 1_k for some positive integer k.

As an example of the definitions above, consider the monomial $\gamma_1^3 \odot \gamma_2 \gamma_{1[2]}^3 \odot \gamma_{2[2]} \gamma_{1[4]}^2 \odot 1_4 \in H^*(BS_{18})$, whose skyline diagram representation is shown in Figure 1. This monomial is made up of three gathered blocks: γ_1^3 of algebraic degree 3 and effective scale 1, $\gamma_2 \gamma_{1[2]}^3$ of algebraic degree 5 and effective scale 2, and $\gamma_{2[2]} \gamma_{1[4]}^2$ of algebraic degree 3 and effective scale 2. Since 5 is the largest degree and 1 is the smallest effective scale, this diagram has height 5 and effective scale 1. This monomial is not full-width since it is the transfer product of the full-width monomial $\gamma_1^3 \odot \gamma_2 \gamma_{1[2]}^3 \odot \gamma_{2[2]} \gamma_{1[4]}^2$ and the unit class 1_4.

Theorem 2.4 [12, Theorem 8.3] *The Steenrod square* $\mathrm{Sq}^i\, \gamma_{l[2^k]}$ *is the sum of all full-width monomials of total degree* $2^k(2^l - 1) + i$*, height* 1 *or* 2*, and effective scale at least* l*, with height* 2 *only allowed if the effective scale is* l*.*

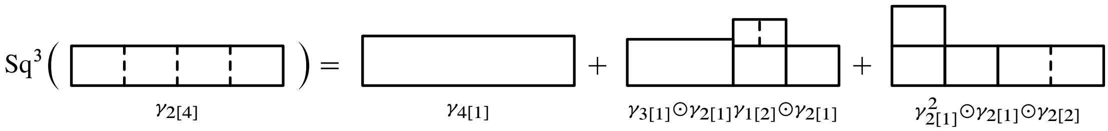

Figure 3: Skyline diagrams for the three summands of $\mathrm{Sq}^3(\gamma_{2[4]})$.

For example, Figure 3 illustrates the three summands of $\mathrm{Sq}^3(\gamma_{2[4]})$:

$$\mathrm{Sq}^3(\gamma_{2[4]}) = \gamma_{4[1]} + \gamma_{3[1]} \odot \gamma_{2[1]}\gamma_{1[2]} \odot \gamma_{2[1]} + \gamma_{2[1]}^2 \odot \gamma_{2[1]} \odot \gamma_{2[2]}.$$

It is straightforward to use Cartan formulas to calculate Steenrod action on $\mathfrak{N}$, the indecomposables. This gives a refinement of Nakaoka's work, which only determined this module additively, and is a much more accessible presentation than of the homology primitives [7; 19]. Indeed, much of Wellington's work on the CWSS is devoted to calculations with these primitives. These calculations are generally simplified or become immaterial through this cohomology approach.

3 Width spectral sequence

In this section, we equate the Curtis–Wellington E_2 with an explicit Ext group in the category of unstable modules over the Steenrod algebra, at which point there is an immediate filtration to develop. Recall that work of [6] and others proves that, for simply connected X with $\pi_*(X)$ of finite type, there is an unstable analog to the Adams spectral sequence with $E_2 \cong \mathrm{Ext}^{s,t}_{\mathcal{U}\mathcal{A}}(H^*(X), \mathbb{F}_2)$ converging to $\pi_*(X)$.

There have been relatively few computations made of the unstable Adams spectral sequence, with some explicit calculations of Curtis and Mahowald [10; 9] and Miller's proof of the Sullivan conjecture [22] being spectacular exceptions. A main roadblock is that the Ext groups which occur, which we call $\mathrm{Ext}_{\mathcal{U}\mathcal{A}}$ for the category of unstable algebras over the Steenrod algebra, are not Ext groups in the usual sense of derived homomorphisms in an abelian category. While Goerss [13] established that they are a "nonabelian" derived Hom, in the sense of Quillen, in the category of simplicial algebras over $\mathcal{A}$, this has not to our knowledge been used in any way for calculations.

To make calculations, one can hope for equivalent Ext calculations in abelian categories. The category of unstable modules over the Steenrod algebra, $\mathcal{U}$, is abelian and there is a free unstable algebra functor $\mathcal{U} \to \mathcal{U}\mathcal{A}$. However, the cohomology of a space is very rarely in the image of this functor, even if it is free. If the cohomology is free only as an algebra, there is still an alternative form of reduction. Let A be an augmented algebra with $\bar{A}$ its augmentation ideal. We let $\mathrm{Ind}\, A$ denote the algebra indecomposables $\bar{A}/(\bar{A} \cdot \bar{A})$. Note that $\Sigma^{-1}\,\mathrm{Ind}\, A$ is naturally in $\mathcal{U}$. The following was originally stated by Bousfield [4] and follows from the composite functor spectral sequence constructed by Miller [22].

Theorem 3.1 *Let P be an unstable algebra over $\mathcal{A}$ that is free as an algebra on* Ind P. *Then*

$$\mathrm{Ext}^{s,t}_{\mathcal{U}\mathcal{A}}(P, k) \cong \mathrm{Ext}^{s,t-1}_{\mathcal{U}}(\Sigma^{-1}\mathrm{Ind}\, P, k).$$

Recall that we refer to Ind $H^*(Q_0S^0, \mathbb{F}_2)$ as the Nakaoka module, $\mathfrak{N}$. Thus, Theorem 3.1 applies to give the following reduction to a calculation in the abelian category $\mathcal{U}$.

Corollary 3.2 *The E_2 term of the Curtis–Wellington spectral sequence is isomorphic to* $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1}\mathfrak{N})$.

Our graphical skyline diagram presentation, in particular of the indecomposables as stated in Theorem 2.2, points immediately to a filtration of the Nakaoka module.

Definition 3.3 Let F_n be the submodule of $\mathfrak{N}$ of elements of width less than or equal to 2^{n-1}.

This is a submodule as unstable Steenrod modules by the formula from Theorem 2.4. Moreover, F_n/F_{n-1} will be spanned by single columns of width exactly 2^{n-1} with at least one block type appearing an odd number of times.

Proposition 3.4 *The quotient F_n/F_{n-1} is isomorphic to D_n^o, where D_n^o is the quotient of D_n by all perfect squares (just the squares themselves, not the ideal generated by squares).*

Proof Let V_n be the transitive elementary abelian 2-subgroup of S_{2^n}, $(\mathbb{Z}_2)^n$. Corollary 7.6 of [12] follows an argument of Milgram to show the restriction of $\gamma_{l[2^k]}$ with $l+k=n$ to V_n is the Dickson class $d_{k,l}$. Since F_n is the submodule of elements of width less than or equal to 2^{n-1} and F_{n-1} is the submodule of elements of width less than or equal to 2^{n-2}, F_n/F_{n-1} will be spanned by single columns of width exactly 2^{n-1} with at least one block type appearing an odd number of times. The blocks of width 2^{n-1} are $\gamma_{i[2^{n-i}]}$ for i an integer between 1 and n; thus, when restricted to V_n, we get Dickson classes $d_{i,n-i}$, all the elements of D_n. Since at least one block type must appear an odd number of times, the only Dickson algebra elements we cannot get are those where each $d_{i,n-i}$ has an even power, which leaves us with the quotient of D_n by just the perfect squares. □

This filtration allows us to consider only full-width terms in the image of the Steenrod action, substantially simplifying calculations. Assembling the long exact sequences in cohomology associated to the short exact sequences

$$0 \to F_{n-1} \to F_n \to D_n^o \to 0$$

from the filtration described above produces a trigraded spectral sequence converging to the E_2 page of the Curtis–Wellington spectral sequence for stable homotopy.

Theorem 3.5 *The spectral sequence associated to the width filtration has*

$$E_1^{s,t;n} = \mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1}D_n^o, \mathbb{F}_2)$$

and $d_r : \mathrm{Ext}^{s,t}(\Sigma^{-1}D_n^o) \to \mathrm{Ext}^{s+1,t}(\Sigma^{-1}D_{n+r}^o)$. *It converges to* $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1}\mathfrak{N})$, *and thus the E_2 of the Curtis–Wellington spectral sequence.*

Using the well-known Steenrod structure on Dickson algebras, as presented for example in [15], we have been able to make hand calculations out to the 17 stem. Hood Chatham kindly produced an Ext chart

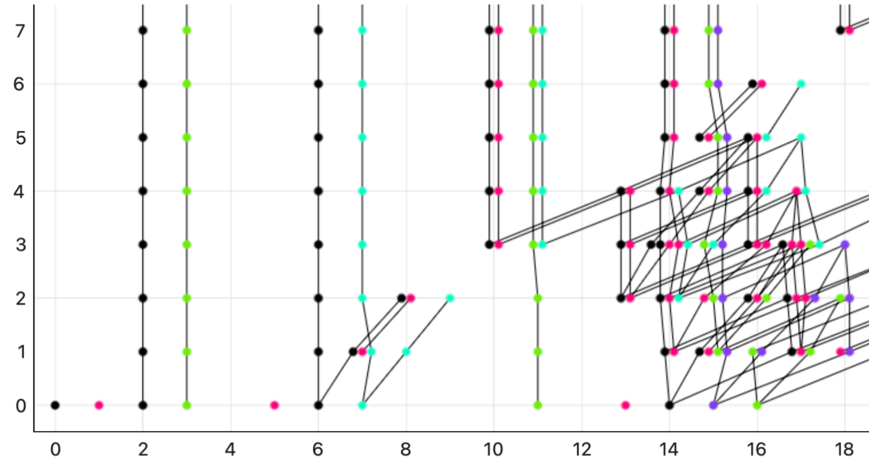

Figure 4: The E_1 page of the width spectral sequence, with width filtration encoded by color: black corresponds to D_1, red to D_2, green to D_3, teal to D_4, and purple to D_5.

illustrating the first page of this spectral sequence, out to the forty-five stem. We share a clip here in Figure 4 and the full chart in the appendix. In this chart, the black connecting lines indicate the partial action of $\mathrm{Ext}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ described in slightly more detail at the start of Section 4. Along the zero line in this width spectral sequence, we see the Steenrod indecomposables of the Dickson algebras, which have been of interest, for example in work of Hung and Peterson [16], and are far from understood. This zero line receives the Hurewicz map for Q_0S^0, as studied by Lannes and Zarati [18].

Wellington made similar computations for $\mathrm{Ext}(H_*Q_0S^0)$ at the prime 2, including charts out to the 17 stem, which we reproduce with a minor correction in Figures 5 and 6. Comparing Wellington's results to ours, they mainly agree, but our calculations reveal an error in the 11 and 12 stem in the Ext chart. The original chart had a class in bidegree $(12, 4)$ and a d_2 differential to the class in degree $(11, 6)$.

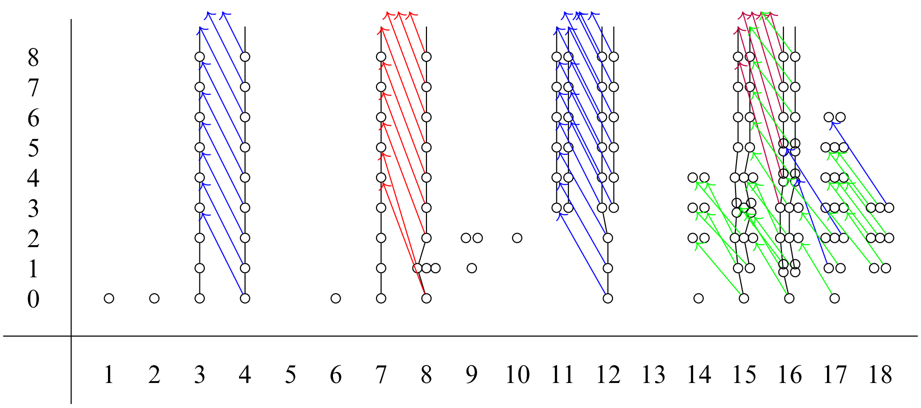

Figure 5: Part of [28, Table 13] (with correction in the 12 stem), depicting the E_2 page and differentials of the CWSS.

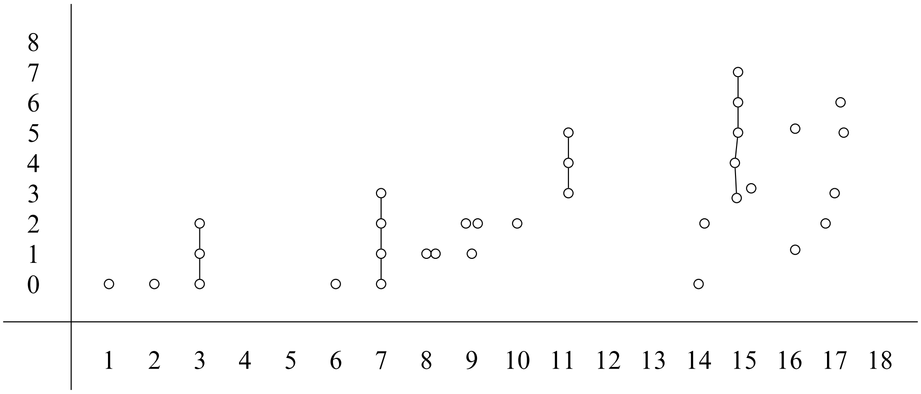

Figure 6: E_∞ page of the Curtis–Wellington spectral sequence up through the 17 stem.

Instead, our hand calculations show that there is a class in degree $(12,3)$, which appears in the computer calculations in degree $(11,3)$ after the desuspension. Since we know that the CWSS must converge to stable homotopy, we know that there must be a d_3 differential instead of the d_2 differential. Also note that, with the exception of the class just discussed, which we believe to be just a minor transcription error, every class on the width spectral sequence E_1 is still present on the E_2 page of the CWSS and thus survives the width spectral sequence.

The differentials of the spectral sequence of Theorem 3.5, namely

$$d_r\colon \mathrm{Ext}^{s,t}(D_n^o) \to \mathrm{Ext}^{s+1,t}(D_{n+r}^o),$$

fix topological degree (t), increase cobar length by 1 (s), and increase filtration by r. In the charts, they will be moving one unit left, one unit up, and in our representation of the third (width) grading by color move between different colors so that, if the source is in D_n^o, the target is in D_{n+r}^o.

4 h_0 towers

One of the immediate differences between the Adams spectral sequence and Curtis–Wellington spectral sequence is the presence of infinite h_0 towers. We classify these in the width spectral sequence.

We begin by defining a partial action of $\mathrm{Ext}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2)$. In the Yoneda approach to Ext, an element of $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$ is an equivalence class of extensions of length s from $\mathbb{F}_2$ to $\Sigma^t\mathbb{F}_2$, namely

$$0 \to \Sigma^t\mathbb{F}_2 \to E_1 \to \cdots \to E_s \to \mathbb{F}_2 \to 0,$$

where the E_i are $\mathscr{A}$-modules. An element of $\mathrm{Ext}_{\mathfrak{U}}^{p,q}(\Sigma^{-1}D_n^o,\mathbb{F}_2)$ is an extension of length p from $\Sigma^{-1}D_n^o$ to $\Sigma^q\mathbb{F}_2$,

$$0 \to \Sigma^q\mathbb{F}_2 \to F_1 \to \cdots \to F_p \to \Sigma^{-1}D_n^o \to 0,$$

where the F_i are unstable $\mathscr{A}$-modules and the maps $F_i \to F_{i+1}$ are multiplication by elements of $\mathscr{A}$ of excess less than or equal to the degree of the generator of F_{i+1}.

Definition 4.1 Define a partial action of $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ on $\mathrm{Ext}^{p,q}_{\mathcal{U}}(\Sigma^{-1}D^o_n,\mathbb{F}_2)$, defined when $t-s+1\le q$, by suspending a representative of the equivalence class of the stable extension q times and concatenating it on the left with a representative of the equivalence class of the unstable extension to give an extension which defines a representative of an element of $\mathrm{Ext}^{s+p,t+q}_{\mathcal{U}}(\Sigma^{-1}D^o_n,\mathbb{F}_2)$:

$$0\to\Sigma^{t+q}\mathbb{F}_2\to\Sigma^qE_1\to\cdots\to\Sigma^qE_s\to\Sigma^q\mathbb{F}_2\to F_1\to\cdots\to F_p\to\Sigma^{-1}D^o_n\to 0.$$

We next recall the Λ-algebra [6; 5], which gives an explicit though computationally involved way to compute some Ext groups over the Steenrod algebra. The Λ-algebra is the graded associative differential algebra with unit over $\mathbb{F}_2$ with

(1) a generator λ_i of degree i for each $i\ge 0$;

(2) for each $i,k\ge 0$, a relation
$$\lambda_i\lambda_{2i+1+k}=\sum_{j\ge 0}\binom{k-1-j}{j}\lambda_{i+k-j}\lambda_{2i+1+j};$$

(3) a differential ∂ given by
$$\partial(\lambda_i)=\sum_{j\ge 1}\binom{i-j}{j}\lambda_{i-j}\lambda_{j-1}.$$

Note that $\Lambda=\bigoplus_{s\ge 0}\Lambda^s$, where Λ^s is generated by monomials λ_I of length s.

To define complexes using the Λ-algebra, we need our action to be on the right. Let $\mathcal{U}_{\mathcal{R}}$ denote the category of unstable right $\mathcal{A}$-modules and continue to denote by $\mathcal{U}$ the category of unstable left $\mathcal{A}$-modules. For M of finite type, $\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2,M)\cong\mathrm{Ext}^{s,t}_{\mathcal{U}}(M^*,\mathbb{F}_2)$.

The Λ-algebra gives one method to approach calculation of Ext groups; in particular, from [5],

$$\mathrm{Ext}^{s,t}_{\mathcal{U}_{\mathcal{R}}}(\mathbb{F}_2,M)\cong H^s(V(M))_{t-s},$$

where $V(M)$ is the chain complex

$$M\to M\mathbin{\hat{\otimes}}\Lambda^1\to M\mathbin{\hat{\otimes}}\Lambda^2\to\cdots,$$

where $M\mathbin{\hat{\otimes}}\Lambda$ is the subspace of $M\otimes\Lambda$ generated by all $x\otimes\alpha$ with $x\in M_n$ and $\alpha\in\Lambda(n)$, the subspace of Λ generated by allowable λ_I with $i_1<n$ or I empty.

Bousfield [4] defines the following tower complex as a quotient of the chain complex $V(M)$.

Definition 4.2 Let
$$T^s(M)=\begin{cases}f\otimes(\lambda_0)^s & \text{if } s=0,1,\\ M\otimes(\lambda_0)^s\oplus\sum_{k>0}M_{2k}\otimes\lambda_{2k-1}(\lambda_0)^{s-1} & \text{if } s>1,\end{cases}$$
with
$$\delta(x\otimes\lambda_{2k-1}(\lambda_0)^{s-1})=0$$
and
$$\delta(x\otimes(\lambda_0)^s)=\begin{cases}x\cdot\mathrm{Sq}^1\otimes(\lambda_0)^{s+1}+x\cdot\mathrm{Sq}^{2k}\otimes\lambda_{2k-1}(\lambda_0)^s & \text{if } s>0, x\in M_{4k},\\ x\cdot\mathrm{Sq}^1\otimes(\lambda_0)^{s+1} & \text{otherwise.}\end{cases}$$

Remark 2.4 of [4] notes that the towers in $H^*(T(M))$ correspond with those in $H^*(V(M))$ and thus also with the towers in $\mathrm{Ext}^{s,t}_{U_R}(\mathbb{F}_2, M)$. Applying these tower detectors to $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1} D^o_n, \mathbb{F}_2)$, we conclude the following two theorems.

Theorem 4.3 *There are infinite towers T_{a_1} in $\mathrm{Ext}^{s,t}(\Sigma^{-1} D^o_1, \mathbb{F}_2)$ only in degrees $t - s = a_1 - 1$ for $a_1 = 4k - 1$, where k is a positive integer.*

Theorem 4.4 *Let $n \geq 2$ be an integer. There are towers $T_{a_1,\dots,a_n}$ in $\mathrm{Ext}^{s,t}(\Sigma^{-1} D^o_n, \mathbb{F}_2)$ in degree $t - s = \left(\sum_{i=1}^n (2^n - 2^{n-i}) a_i\right) - 1$ satisfying one of the following two conditions*:

1. *$a_n = 0$, a_{n-1} is even, at least one other a_i is odd, and $\sum_{i=1}^n (2^n - 2^{n-i}) a_i = 4l$.*
2. *a_i for $1 \leq i \leq n-2$ are all even, a_{n-1} and a_n are odd, and $\sum_{i=1}^n (2^n - 2^{n-i}) a_i = 4k - 1$.*

Wellington [28] also used these tower detecting complexes, but of course with his homology approach. Recall that the homology of $Q_0 S^0$ is free under the product induced by loop sum on classes Q^I where I is admissible. Wellington proves there are towers in dimensions $4k - 1$ and $4k$ generated by Q^I either in degree $4k$ with excess 0 and some odd index, or Q^I in degree $4k - 1$ with final index odd and all others even. Our calculations agree with Wellington's in that the towers in $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1} D^o_n, \mathbb{F}_2)$ for each n correspond with his generated by Q^I with $l(I) = n$. Indeed, in degree $(4k - 1) - 1$, we have towers $T_{a_1,\dots,a_n}$ where the a_i for $1 \leq i \leq n-2$ are all even, a_{n-1} and a_n are odd, and $\sum_{i=1}^n (2^n - 2^{n-i}) a_i = 4k - 1$. In other words, we have a tower corresponding to each integer solution of

$$(2^{n-1} - 2^{n-2}) b_1 + \cdots + (2^{n-1} - 1) b_{n-1} + (2^n - 1) c_n + (2^n - 1) = k$$

where $a_i = 2b_i$ for $1 \leq i \leq n-2$, $a_{n-1} = 2b_{n-1} + 1$ and $a_n = 4c_n + 3$. Each of these corresponds to the tower generated by Q^I with $I = (s_1, s_2, \dots, s_n)$ where $s_n = 2k + 1 + 2c_n$ and the other s_i can be computed inductively from right to left with the formula

$$s_i = \frac{1}{2^{(n-1)-i}} \left(k + 1 + a_n + \sum_{j=i}^{n-1} 2^{(n-1)-j} b_j \right) \quad \text{for } 2 \leq i \leq n-1,$$

and finally

$$s_1 = 4k - 1 - \sum_{j=2}^{n} s_j.$$

In degree $4k - 1$, we found towers $T_{a_1,\dots,a_n}$ where $a_n = 0$, a_{n-1} is even, at least one other a_i is odd, and $\sum_{i=1}^n (2^n - 2^{n-i}) a_i = 4k$. Thus, we have a tower for each integer solution of

$$(2^{n-2} - 2^{n-3}) a_1 + \cdots + (2^{n-2} - 1) a_{n-2} + (2^{n-1} - 1) b_{n-1} = k,$$

where $a_{n-1} = 2b_{n-1}$. Each of these corresponds to the tower generated by Q^I with $I = (s_1, s_2, \dots, s_n)$ where we once again find each term in the index working from right to left. First, $s_n = 2k$ and $s_{n-1} = k + b_{n-1}$, then inductively we compute

$$s_i = \frac{1}{2^{(n-1)-i}} \left(k + b_{n-1} + \sum_{j=i}^{n-2} 2^{(n-2)-j} a_j \right) \quad \text{for } 2 \leq i \leq n-1,$$

and finally

$$s_1 = 4k - \sum_{j=2}^{n} s_j.$$

This agreement between our towers in the width spectral sequence and Wellington's imply there are no differentials in the width filtration spectral sequence with h_0 inverted. The only thing that we can definitively say independent of Wellington's results is that there are no possible width spectral sequence differentials until the 60 stem. Indeed, since the differentials in the width spectral sequence increase width filtration, the earliest possible differential would be from some tower from D_3 in the $4k$ stem to some tower from D_4 in the $4k-1$ stem, the first of which is in the 59 stem. Between the fact that these results indicate there are no differentials between towers and that some differentials can be eliminated by hand calculations in low degrees, we wonder whether there are any differentials in the width spectral sequence at all, a purely algebraic question.

Proof of Theorem 4.3 Let d_1 be the generator of D_1. Then elements of D_1^o are d_1^{2i+1} in degree $2i$ after desuspension. Take $\{x_1, x_3, \dots, x_{2i+1}, \dots\}$ as a basis for the linear dual $(\Sigma^{-1} D_1^o)^*$, where x_{2i+1} is dual to d_1^{2i+1}, so each x_{2i+1} is in degree $2i$ in the desuspension. Consider $M = (\Sigma^{-1} D_1^o)^*$ as an unstable right $\mathscr{A}$-module by defining the linear map $x_i \cdot \mathrm{Sq}^k$ as

$$(x_i \cdot \mathrm{Sq}^k)(y) = x_i(\mathrm{Sq}^k\, y) \quad \text{for } y \in D_1^o.$$

We can use Bousfield's tower detector to determine where there are towers in $\mathrm{Ext}^{s,t}_{\mathscr{U}_{\mathscr{R}}}(\mathbb{F}_2, M)$. As defined in Definition 4.2, $T^s(M)$ is constructed so that in degree 2 and above the next degree is constructed from the previous by multiplying by λ_0 on the right and the differential from degree 1 onward is the same as the previous degree but with an extra factor of λ_0 on the right. Thus, it is sufficient to compute $H^2(T(M))$ to determine where the towers are.

We can see in Definition 4.2 that the differential δ only involves Sq^1 and Sq^{2k} and thus in this case is determined by the fact that

$$x_i \cdot \mathrm{Sq}^1 = 0 \quad \text{and} \quad x_{4k+1} \cdot \mathrm{Sq}^{2k} = x_{2k+1}.$$

Elements of $H^2(T(M))$ take three forms. First, all $x_{4k+1} \otimes (\lambda_0)^2$ are not cycles since $\delta(x_{4k+1} \otimes (\lambda_0)^2) = x_{2k+1} \otimes \lambda_{2k-1}(\lambda_0)^2$. Second, all $x_{2k+1} \otimes \lambda_{2k-1}\lambda_0$ are cycles, but also boundaries hit by $x_{4k+1} \otimes \lambda_0$. Finally, all $x_{4k-1} \otimes (\lambda_0)^2$ are cycles and cannot be boundaries since the image of $T^1(M)$ is only elements of the form $x_{2k+1} \otimes \lambda_{2k-1}\lambda_0$. Thus, we get a tower for $x_{4k-1} \otimes (\lambda_0)^s$ in degree $4k-2$ for each positive integer k. □

Proof of Theorem 4.4 We know that D_n is generated by n elements, one in each degree $2^n - 2^i$ for $0 \le i \le n-1$. Let these generators be represented by $d_{2^n - 2^i}$. Then an arbitrary basis element of D_n^o is of the form $d_{2^n-2^{n-1}}^{a_1} \cdots d_{2^n-2^0}^{a_n}$ with at least one a_i odd. Let $x_{a_1,\dots,a_n} \in (D_n^o)^*$ denote the linear dual of $d_{2^n-2^{n-1}}^{a_1} \cdots d_{2^n-2^0}^{a_n}$ in degree $\sum_{i=1}^n a_i(2^n - 2^{n-i})$.

Working now in $(\Sigma^{-1}D_n^o)^*$, $x_{a_1,\dots,a_n} \in (D_n^o)^*$ will now be in degree $\sum_{i=1}^n a_i(2^n - 2^{n-i}) - 1$. Let $M = (\Sigma^{-1}D_n^o)^*$ and consider the tower detecting complex $T(M)$. As described in the proof of Theorem 4.3, it is sufficient to calculate $H^2(T(M))$ to determine the location of the towers.

Since the differential δ of Definition 4.2 only uses Sq^1 and Sq^{2k},we need only understand the right action of Sq^1 on an arbitrary $x_{a_1,\dots,a_n}$ and the right action of Sq^{2k} on $x_{a_1,\dots,a_n}$ with $\sum_{i=1}^n a_i(2^n-2^{n-i}) = 4k+1$. These are given by the formulas

$$x_{a_1,\dots,a_n} \cdot \mathrm{Sq}^1 = \begin{cases} x_{a_1,\dots,a_{n-2},a_{n-1}+1,a_n-1} & \text{if } a_{n-1} \text{ even, } a_n \geq 1 \text{, at least one } a_i \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and, if $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k + 1$,

$$x_{a_1,\dots,a_n} \cdot \mathrm{Sq}^{2k} =$$
$$\begin{cases} x_{(a_1+2)/2,a_2/2,\dots,a_{n-1}/2,(a_n-1)/2} & \text{if } a_n \text{ odd, } a_i \text{ even,} \\ x_{a_1/2,\dots,a_{j-1}/2,(a_j-1)/2,(a_{j+1}+2)/2,a_{j+2}/2,\dots,a_{n-1}/2,(a_n-1)/2} & \text{if } a_j, a_n \text{ odd, } a_i \text{ even for } i \neq j, n, \\ x_{a_1/2,\dots,a_{n-2}/2,(a_{n-1}-1)/2,(a_n+1)/2} & \text{if } a_{n-1}, a_n \text{ odd, } a_i \text{ even for } i < n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where throughout i and j are between 1 and $n-1$ inclusively.

With these formulas in hand, we analyze $H^2(T(M))$. Our strategy will be to first characterize all cycles in $H^2(T(M))$ and then go through each type of cycle to determine which are boundaries. All of those that are not boundaries will correspond to our tower generators. There are four types of cycles:

(a) $x_{a_1,\dots,a_n} \otimes (\lambda_0)^2$ with $\sum_{i=1}^n a_i(2^n - 2^{n-i}) \neq 4k + 1$ and a_{n-1} odd;

(b) $x_{a_1,\dots,a_n} \otimes (\lambda_0)^2$ with a_{n-1} even and $a_n = 0$;

(c) $x_{a_1,\dots,a_n} \otimes (\lambda_0)^2$ with $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 4k + 1$ and a_n, a_{n-1} and some other a_i with $1 \leq i < n-1$ odd;

(d) $x_{a_1,\dots,a_n} \otimes \lambda_{2k-1}\lambda_0$ with $\sum_{i=1}^n a_i(2^n - 2^{n-i}) = 2k + 1$.

To see that these are all of the cycles, we look at our formulas for the action of Sq^1 and Sq^{2k}. As long as the degree of $x_{a_i,\dots,a_n}$ is not a multiple of four, the differential on $x_{a_i,\dots,a_n} \otimes (\lambda_0)^2$ only involves Sq^1. So, if $\sum_{i=1}^n a_i(2^n - 2^{n-i}) - 1 \neq 4k$, $x_{a_i,\dots,a_n} \otimes (\lambda_0)^2$ is a cycle when $x_{a_i,\dots,a_n} \cdot \mathrm{Sq}^1 = 0$. This means either a_{n-1} is odd and we get cycles of type (a), $a_n = 0$ and a_{n-1} is even and we get cycles of type (b), or all a_i are even, but then $x_{a_i,\dots,a_n}$ is not an element of $(D_1^o)^*$.

If $x_{a_i,\dots,a_n}$ is in degree $4k$, the differential on $x_{a_i,\dots,a_n} \otimes (\lambda_0)^2$ involves both Sq^1 and Sq^{2k}. Then $\sum_{i=1}^n a_i(2^n - 2^{n-i}) - 1 = 4k$, which means that $a_n > 0$ and a_{n-1} is even. For both Sq^1 and Sq^{2k} to act trivially, a_{n-1} must be odd and some other a_i for $1 \leq i < n-1$ is also odd to give the cycles of type (c).

Finally, we have the elements of the form $x_{a_i,\dots,a_n} \otimes \lambda_{2k-1}(\lambda_0)^{s-1}$, which are all cycles by definition and give us the cycles in class (d).

Beginning with cycles of class (a), we want to determine which are also boundaries. For cycles in (a), we will split into three cases based on the value of $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) \bmod 4$.

If $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k-1$ and a_{n-1} is odd, then a_n is odd and

$$\delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$$

as long as one of $a_1,\dots,a_{n-2}$ is odd. Thus, we see that the only cycles not hit by boundaries are $x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ with a_{n-1}, a_n odd and a_i even for $1 \le i \le n-2$.

If $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k-2$ and a_{n-1} is odd, then a_n is even and

$$\delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$$

and we see that all these cycles are also boundaries.

If $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k$ and a_{n-1} is odd with $a_n \ge 2$, then $x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ is the boundary of

$$\begin{cases} \delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0 \\ \qquad\qquad + x_{a_1+2,a_2,\dots,a_{n-2},a_{n-1},a_n-1} \otimes \lambda_0) & \text{if } a_{n-1} \text{ odd, all other } a_i \text{ even,} \\ \delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0 \\ \qquad + x_{a_1,\dots,a_{j-1},a_j-1,a_{j+1}+2,a_{j+2},\dots,a_{n-2},a_{n-1},a_n+1} \otimes \lambda_0) & \text{if } a_j,\ a_{n-1} \text{ odd, all other } a_i \text{ even,} \\ \delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) & \text{if } a_{n-1} \text{ odd, at least two other } a_i \text{ odd,} \end{cases}$$

and we see that all these cycles are also boundaries. This now covers all cases for our class (a) cycles.

Turning to our class (b) cycles, we see that none of these are boundaries. Indeed, they would need to be in the image of some $x_{b_1,\dots,b_n} \otimes (\lambda_0)$ with $x_{b_1,\dots,b_n} \cdot \mathrm{Sq}^1 = x_{a_1,\dots,a_n}$. However, Sq^1 changes the parity of each of the last two indices and is only nonzero if b_{n-1} is even, but then its image a_{n-1} must be odd, a contradiction.

Next, all class (c) cycles are boundaries hit by

$$\delta(x_{a_1,\dots,a_{n-2},a_{n-1}-1,a_n+1} \otimes \lambda_0) = x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2.$$

Thus, all class (c) cycles are boundaries as well.

Finally, for class (d), if $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 2k+1$, then

$$\delta(x_{2a_1,\dots,2a_{n-2},2a_{n-1}+1,2a_n-1} \otimes \lambda_0) = x_{a_1,\dots,a_n} \otimes \lambda_{2k-1}\lambda_0.$$

So these too are all boundaries.

This leaves us with only those classes of type (a) corresponding to $x_{a_1,\dots,a_{n-2},a_{n-1},a_n} \otimes (\lambda_0)^2$ with a_{n-1}, a_n odd and a_i even for $1 \le i \le n-2$, and all cycles from class (b) with $x_{a_1,\dots,a_n} \otimes (\lambda_0)^2$ with a_{n-1} even and $a_n = 0$ such that $\sum_{i=1}^{n} a_i(2^n - 2^{n-i}) = 4k$ in $H^2(T(M))$. □

While these calculations immediately give towers in the width filtration spectral sequence, we do not perform the analysis to determine which survive to the E_2 of the CWSS, instead citing agreement with Wellington's towers in the E_2 of the CWSS. His argument is substantially more involved than what was required above, so we would like to find a self-contained argument at some point.

It would be interesting to investigate the elements of homotopy which correspond to these towers, which must of course all support or receive differentials. Would the resulting finite towers at E_∞ be exceptional in any way?

5 First analysis of J map

We start with a fun observation. The Bousfield result [4] equating the E_2 of the unstable Adams spectral sequence with an Ext in unstable modules can be applied to $\mathrm{Ext}^{s,t}_{\mathcal{U}\mathcal{A}}(H^*(BO),\mathbb{F}_2)$, as of course $H^*(BO)$ is polynomial on the Stiefel–Whitney classes. We get

$$\mathrm{Ext}^{s,t}_{\mathcal{U}\mathcal{A}}(H^*(BO),\mathbb{F}_2) \cong \mathrm{Ext}^{s,t-1}_{\mathcal{U}}(\Sigma^{-1}\operatorname{Ind} H^*(BO),\mathbb{F}_2).$$

Up to decomposables, $\mathrm{Sq}^i(w_j) = \binom{j-1}{i} w_{j+i}$, where w_j is the j^{th} Stiefel Whitney class [17; 27; 29]. Preliminary calculations for $\mathrm{Ext}_{\mathcal{U}}$ led to the observation that its Ext chart looked like $\mathrm{Ext}_{\mathcal{U}}(\Sigma^{-1}D_1^o)$, but shifted to the right. Recall that D_1^o is the cohomology of $\mathbb{R}P^\infty$, modulo squares. This then led us to the following isomorphism between familiar modules, which we found surprising.

Proposition 5.1 $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Sigma^{-1}D_1^o) \cong \mathrm{Ext}^{s,t+1}_{\mathcal{U}}(\Sigma^{-1}\operatorname{Ind} H^*(BO)).$

This hints that the lowest degree in the width filtration yields the image of J, an idea that we expand on later in this section. Before proving this proposition, we recall notation from [26]. Define $\Phi\colon \mathcal{U}\to\mathcal{U}$ on $M\in\mathcal{U}$ at the prime 2 by

$$(\Phi M)^n \cong M^{n/2}, \quad \mathrm{Sq}^i\,\Phi x = \Phi\,\mathrm{Sq}^{i/2}\,x,$$

where $M^{n/2}$ is trivial if $n/2$ is not an integer. Define $\lambda_M\colon \Phi M\to M$ by $\Phi x\mapsto \mathrm{Sq}^{|x|}\,x$. Define the functor Ω and its first (and only nontrivial) left derived function Ω_1 by

$$\ker\lambda_M = \Sigma\Omega_1 M \quad\text{and}\quad \operatorname{coker}\lambda_M = \Sigma\Omega M.$$

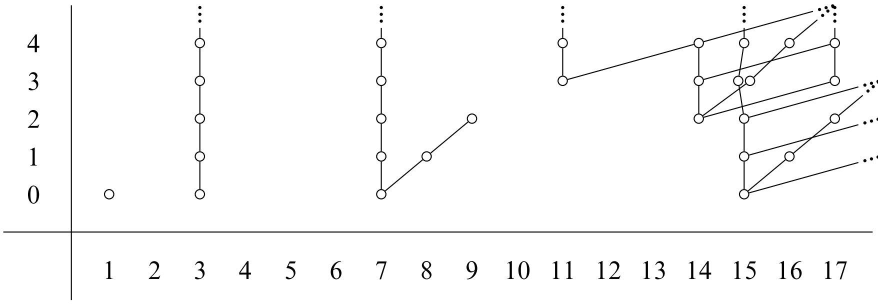

Figure 7: Calculations of $\mathrm{Ext}_{\mathcal{U}}(BO,\mathbb{F}_2)$.

Proposition 5.2 *If $M \in \mathcal{U}$ and λ_M is injective, then* $\mathrm{Ext}^{s,t}_{\mathcal{U}}(\Omega M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t+1}(M, \mathbb{F}_2)$.

Proof We know that

$$0 \to \ker(\lambda_M) \to \Phi(M) \to M \to \mathrm{coker}(\lambda_M) \to 0$$

is an exact sequence where $\ker(\lambda_M) \cong \Sigma\Omega_1 M$ and $\mathrm{coker}(\lambda_M) \cong \Sigma\Omega M$. Since λ_M is injective, $\ker(\lambda_M) = 0$, so $\Omega_i M = 0$ for all $i > 0$. Let $P_\bullet \to M$ be a free resolution of M. Then $\Omega P_\bullet \to \Omega M$ is a free resolution of ΩM. Thus, $\mathrm{Ext}_{\mathcal{U}}(\Omega M, \mathbb{F}_2) \cong H_s(\mathrm{Hom}^t(\Omega P_\bullet, \mathbb{F}_2))$. Since Ω is left adjoint to Σ, we get

$$H_s(\mathrm{Hom}^t(\Omega P_\bullet, \mathbb{F}_2)) \cong H_s(\mathrm{Hom}^t(P_\bullet, \Sigma\mathbb{F}_2)) \cong H_s(\mathrm{Hom}^{t+1}(P_\bullet, \mathbb{F}_2)) \cong \mathrm{Ext}^{s,t+1}_{\mathcal{U}}(M, \mathbb{F}_2). \qquad \square$$

Proof of Proposition 5.1 The result will follow from Proposition 5.2 if we can show that λ_M is injective for $M = \Sigma^{-1}\,\mathrm{Ind}\, H^*(BO)$ and $\mathrm{coker}\,\lambda_M \cong \Sigma\Sigma^{-1} D_1^o$.

For any $i > 0$, $w_i \in M$, w_i has degree $i-1$ and

$$\lambda_M(w_i) = \mathrm{Sq}_0(w_i) = \mathrm{Sq}^{i-1}(w_i) = \binom{i-1}{i-1} w_{i+i-1} = w_{2i-1},$$

so λ_M is clearly injective. We can also see that the image of λ_M will be the span of w_{2i-1} for $i \geq 1$. Then $\mathrm{coker}\,\lambda_M$ will be $w_{2i} + \mathrm{im}(\lambda_M)$ in degrees $2i-1$. The map $w_{2i} + \mathrm{im}(\lambda_M) \mapsto d_1^{2i-1}$ then defines an isomorphism with $D_1^o \cong \Sigma\Sigma^{-1} D_1^o$, so we conclude that $\Sigma\Sigma^{-1} D_1^o \cong \mathrm{coker}\,\lambda_M$. Thus, applying Proposition 5.2, $\mathrm{Ext}^{s,t}(\Sigma^{-1} D_1^o) \cong \mathrm{Ext}^{s,t+1}(\Sigma^{-1}\,\mathrm{Ind}\, H^*(BO))$. $\square$

The suggestive calculation above for BO could tell us about the delooping of the image of J. But our calculations in this paper are based on the cohomology of QS^0 and not its delooping. For QS^0 itself, the first calculations above are consistent with the following.

Theorem 5.3 *The map induced by the image of J on cohomology induces a splitting of the Nakaoka module $\mathfrak{N}$.*

Corollary 5.4 *The algebraic map induced by the image of J on* Ext *induces a splitting of* $\mathrm{Ext}(\mathfrak{N})$.

To make our calculations of both the map induced by the image of J on cohomology and the induced map on Ext, we begin by studying the map ind on homology. We will use the basis for the homology of SO presented by Hatcher [14],

$$H_*(\mathrm{SO}, \mathbb{F}_2) \cong E[e^1, \dots, e^n, \dots],$$

and the basis for $H_*(Q_1 S^0)$ in Theorem 3.44 of Madsen and Milgram [20],

$$H_*(Q_1 S^0, \mathbb{F}_2) \cong E(g_1, \dots, g_i, \dots) \otimes P(g_I),$$

where $I = (i_1, \dots, i_m)$ and runs over all sequences of integers $0 \leq i_1 \leq \cdots \leq i_m$ and $i_1 = 0$ implies $m = 2$ and $i_2 > 0$. We can translate between this basis and the Kudo–Araki–Dyer–Lashof basis of admissible q_I.

Proposition 5.5 [21, Theorem 3.9] *Let ι be the class of the identity map $(S^\infty, *) \to (S^\infty, *)$ in $Q_1 S^0$, let p be any prime, and let I be any admissible sequence of length j. Define*

$$s(Q_I(\iota)) = Q_I(\iota) * (\bar{\iota}) * \cdots * (\bar{\iota}),$$

where there are p^j copies of $\bar{\iota}$, the class of a point in $Q_{-1} S^0$. Then

$$H_*(Q_0 S^0, \mathbb{Z}_p) = \Lambda\{s(Q_I(\iota))\}$$

as an algebra.

In our application, $p = 2$. As discussed in the proof of [21, Theorem C], the product of these generators with ι (shifting to the 1-component) give the homology generators as described by

$$g_i \otimes 1 \mapsto q_i(\iota) * \bar{\iota}, \quad 1 \otimes g_{0,i} \mapsto q_i^2(\iota) * \bar{\iota}^3, \quad 1 \otimes g_I \mapsto q_I(\iota) * \bar{\iota}^{2^m - 1}.$$

The homology map induced by the J homomorphism follows from [21, Corollary 1.5], giving that $J_*\colon H_*(\mathrm{SO}) \to H_*(Q_1 S^0)$ is defined by $e_i \mapsto g_i \otimes 1$, which is $q_i(\iota) * \bar{\iota}$.

We can use this and the pairings between homology and cohomology for both SO and $Q_1 S^0$ to understand the induced map $J^*\colon \Sigma^{-1} \operatorname{Ind} H^*(Q_1 S^0) \to \Sigma^{-1} \operatorname{Ind} H^*(\mathrm{SO})$. For cohomology, use the basis from Hatcher [14],

$$H^*(\mathrm{SO}, \mathbb{F}_2) \cong \bigotimes_{i \text{ odd}} \mathbb{F}_2[\beta_i],$$

where β_i is the linear dual to e^i. To work out the pairing between homology and cohomology for SO, we inductively use the Hopf algebra structure.

In general, the induced map on cohomology is given by the following.

Proposition 5.6 *The map on cohomology indecomposables induced by the J homomorphism is*

$$\gamma_{1[1]}^{2k+1} \mapsto \beta_{2k+1}, \quad \gamma_{1[2]}^{i} \gamma_{2[1]} \mapsto \beta_{2i+3}$$

and all other indecomposables map to 0.

To prove this we make some preliminary calculations.

Lemma 5.7 *Odd-degree elements of $\mathfrak{N}$ pair trivially with any nontrivial product of two or more elements in $H_*(Q_1 S^0)$.*

Proof Let $x \in \mathfrak{N}$ be an indecomposable of odd degree and width 2^k. Any indecomposable of width 2^k in odd degree must contain at least one block of full width; thus, $\psi(x) = x \otimes 1 + 1 \otimes x$. Let q_{I_1} and q_{I_2} be nontrivial elements of $H_*(Q_1 S^0)$. Then

$$\langle x, q_{I_1} * q_{I_2} \rangle = \langle \psi(x), q_{I_1} \otimes q_{I_2} \rangle = 0,$$

using the product and coproduct structure of the homology and cohomology. □

Lemma 5.8 $$\beta_{2k+1}^{*}=e^{2k+1}+\sum_{a+b=2k+1}e^{a}\wedge e^{b}.$$

Proof We begin with the identity $\beta_{2k+1}=(e^{2k+1})^{*}$, since β_{2k+1} is primitive and pairs nontrivially with only e^{2k+1}. Thus, $\left\langle \beta_{2k+1}, e^{2k+1}+\sum_{a+b=2k+1}e^{a}\wedge e^{b}\right\rangle =1$.

We would like to show that $\left\langle x, e^{2k+1}+\sum_{a+b=2k+1}e^{a}\wedge e^{b}\right\rangle =0$ for any other $x\in H^{*}(Q_1S^0)$. We begin by writing x as the sum of products of primitives

$$x=\sum_{j}\prod_{i}\beta_{2a_{i,j}+1}^{2^{b_{i,j}}}.$$

By linearity and degree arguments, it suffices to consider products of the form $x=\prod_{i=1}^{n}\beta_{2a_i+1}^{2^{b_i}}$ where $\deg x=\sum_i(2a_i+1)\cdot 2^{b_i}=2k+1$. Then $\langle x, e^{a}\wedge e^{b}\rangle =1$ if there are an odd number of ways to split up the n terms in the product into two products of total degrees a and b; otherwise, $\langle x, e^{a}\wedge e^{b}\rangle =0$.

The number of ways to split the n terms into two groups is

$$\begin{cases}\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n}{i} & \text{if } n \text{ is odd},\\ \sum_{i=1}^{n/2-1}\binom{n}{i}+\binom{n}{n/2} & \text{if } n \text{ is even}.\end{cases}$$

In either case, this is equivalent to

$$\frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}-1=\frac{2^n}{2}-1=2^{n-1}-1,$$

which is odd for $n\geq 2$. Certainly not all of these pairs will be distinct, but they will cancel each other out in pairs, leaving an odd number of possibilities. Thus, $\langle x, e^{a}\wedge e^{b}\rangle =1$. We can also calculate that $\langle x, e^{2k+1}\rangle =1$, which means that

$$\left\langle x, e^{2k+1}+\sum_{a+b=2k+1}e^{a}\wedge e^{b}\right\rangle =1+1=0$$

and β_{2k+1} is the only term in $H^{*}(Q_1S^0)$ that pairs nontrivially with $e^{2k+1}+\sum_{a+b=2k+1}e^{a}\wedge e^{b}$. □

In the proofs that follow, we use the notation $[x^n]f(x)$ to denote the coefficient of x^n in $f(x)$.

Lemma 5.9 $$\sum_{i=0}^{m}\binom{n}{i}\equiv_2\binom{n-1}{m}.$$

Proof We have

$$\sum_{i=0}^{m}\binom{n}{i}=[x^m]\frac{(1+x)^n}{1-x}\equiv_2[x^m]\frac{(1+x)^n}{1+x}=[x^m](1+x)^{n-1}=\binom{n-1}{m}.$$ □

Lemma 5.10 *We have*

$$\binom{b-1}{k-b} = [x^{k-b}]F_k(x),$$

where

$$F_k(x) = \frac{C(-x)^{-k} - (-xC(-x))^k}{\sqrt{1-4x}} \equiv_2 C(x)^{-k} + (xC(x))^k$$

and $C(x)$ is the generating function for the Catalan numbers

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Lemma 5.11 *Fix $k > 0$. For $\lceil \frac{1}{3}(2k+1) \rceil \le j \le k$,*

$$\binom{j-1}{2j-k-1} + \sum_{b=\lceil (k+1)/2 \rceil}^{\lfloor (2k-j)/2 \rfloor} \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof We know that $\binom{b-1}{2b-k-1} = \binom{b-1}{k-b}$. Applying Lemma 5.10, we get

$$\binom{b-1}{k-b} = [x^{k-b}]F_k(x),$$

and

$$\binom{j-b-1}{2k-j-2b} = [x^{2j-(2k+1)+b}]F_{3j-2k-1}(x).$$

Given the range on j, we know that $3j-2k-1 \le k-1$. Thus, $k > 3j-2j-1$, so

$$\begin{aligned} \sum_b \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} &= \sum_b ([x^{k-b}]F_k(x))([x^{2j-(2k+1)+b}]F_{3j-2k-1}) \\ &= [x^{2j-k-1}]F_k(x) \cdot F_{3j-2k-1}(x), \end{aligned}$$

which is the coefficient of x^{2j-k-1} in $F_k(x) \cdot F_{3j-2k-1}(x)$. Expanding out this product, we get

$$\begin{aligned} &F_k(x) \cdot F_{3j-2k-1}(x) \\ &\quad = (C(x)^{-k} + x^k C(x)^k)(C(x)^{-3j+2k+1} + x^{3j-2k-1}C(x)^{3j-2k-1}) \\ &\quad = C(x)^{-3j+k+1} + x^k C(x)^{-3j+3k+1} + x^{3j-2k-1}C(x)^{3j-3k-1} + x^{3j-k-1}C(x)^{3j-k-1} \\ &\quad = C(x)^{-(3j-k-1)} + x^{3j-2k-1}(C(x)^{-(3k-3j+1)} + x^{3k-3j+1}C(x)^{3k-3j+1}) + (xC(x))^{3j-k-1} \\ &\quad = F_{3j-k-1} + x^{3j-2k-1}F_{3k-3j+1}. \end{aligned}$$

Then the x^{2j-k-1} coordinate is $[x^{2j-k-1}]F_{3j-k-1} + [x^{k-j}]F_{3k-3j+1} = \binom{j-1}{2j-k-1} + \binom{2k-2j}{k-j}$. Putting everything together, we have

$$\begin{aligned} \binom{j-1}{2j-k-1} + \sum_{b=\lceil (k+1)/2 \rceil}^{\lfloor (2k-j)/2 \rfloor} \binom{b-1}{2b-k-1}\binom{j-b-1}{2k-j-2b} &= \binom{j-1}{2j-k-1} + \binom{j-1}{2j-k-1} + \binom{2k-2j}{k-j} \\ &= \binom{2(k-j)}{k-j}, \end{aligned}$$

which is 1 if and only if $j = k$. □

Now we are ready to prove Proposition 5.6.

Proof of Proposition 5.6 Since $H^*(\mathrm{SO})$ has indecomposables in only odd degrees, we only need to understand the map on odd-degree elements in $\mathfrak{N}$ and thus only need to keep track of homology classes that are dual to odd-degree indecomposables. Then, by Lemma 5.7, we can ignore any products of two or more classes in $H_*(Q_1S^0)$ as we work out the maps. Additionally, for each odd degree there is exactly one indecomposable in $H^*(\mathrm{SO})$, namely β_{2k+1}. Then Lemma 5.8 tells us β_{2k+1} is a term in $(e^{2k+1})^*$ and $(e^a \wedge e^b)^*$ but not in the linear dual of any products of three or more terms. So it suffices to look at only products of one and two e^i in $H_*(\mathrm{SO})$ since products of three or more are dual only to decomposables.

The homology map induced by the J homomorphism is $J_*\colon H_*(\mathrm{SO}) \to H_*(Q_1S^0)$, which is defined by $e_i \mapsto g_i \otimes 1 = q_i(\iota) * \bar{\iota}$ in the Kudo–Araki–Dyer–Lashof basis. It follows that

$$e^a \wedge e^b \mapsto (q_a(\iota) * \bar{\iota}) \circ (q_b(\iota) * \bar{\iota}) = \sum_{l=(a-b)/2}^{a/2} \binom{a-l}{a-2l} q_{b+2l-a,a-l} + \text{products}.$$

We consider all $a+b = 2k+1$. Then $(q_{2k+1-2j,j})^* \mapsto \sum (e^a \wedge e^b)^*$, where the sum runs over all $e^a \wedge e^b$ for which $q_{2k+1-2j,j}$ was one of the terms in the image under J_*. Summing over all $a+b = 2k+1$, the coefficient for admissible $q_{2k+1-2j,j}$ will be

$$\sum_{i=0}^{2j-k-1} \binom{j}{i} + \sum_{b=\lceil (k+1)/2 \rceil}^{\lfloor (2k-j)/2 \rfloor} \biggl(\sum_{l=0}^{2b-k-1} \binom{b}{l} \biggr) \binom{j-b-1}{2j-(2k+1)+b},$$

which simplifies, by Lemma 5.9, mod 2 to

$$\binom{j-1}{2j-k-1} + \sum_{b=\lceil (k+1)/2 \rceil}^{\lfloor (2k-j)/2 \rfloor} \binom{b-1}{2b-k-1} \binom{j-b-1}{2k-j-2b}.$$

By Lemma 5.11, we see that $(q_{1,k})^*$ contains an odd number of $(e^a \wedge e^b)^*$ in its image and thus β_{2k+1} is one of the terms in the image. For all other $q_{2k+1-2j,j}$, there are an even number of $(e^a \wedge e^b)^*$ in the image and thus β_{2k+1} does not appear as one of the terms in the linear dual, which leaves only decomposables. So only $J^*((q_{1,k})^*)$ is nonzero up to decomposables.

Next, for $2a + 3b = 2k+1$,

$$\langle \gamma_{1[2]}^a \gamma_{2[1]}^b, q_{1,k} \rangle = \begin{cases} 1 & \text{if } a = 1, b = k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\gamma_{1[2]}^{k-1}\gamma_{2[1]}$ appears as a term in $(q_{1,k})^*$ and is not a term in any other $(q_{i,j})^*$. So

$$J^*(\gamma_{1[2]}^{k-1}\gamma_{2[1]}) = J^*((q_{1,k})^*) + \sum J^*((q_{i,j})^*) = \beta_{2k+1} + 0.$$

Any other $\gamma_{1[2]}^a \gamma_{2[1]}^b$ when expressed as a sum of the $(q_{i,j})^*$ will have only terms with $i > 1$ and thus J^* will contain only decomposables. □

For an example of how this works in degree 15 ($k = 7$), see Section A.2.

Theorem 5.3 on the splitting of the Nakaoka module follows from the map on cohomology given in Proposition 5.6.

Proof of Theorem 5.3 At the level of unstable modules, we have the inclusion map $\Sigma^{-1} D_1^o \to \Sigma^{-1}\mathfrak{N}$ and $J^*\colon \Sigma^{-1}\mathfrak{N} \to \Sigma^{-1}$ Ind $H^*(\mathrm{SO})$, whose composition induces an isomorphism between the image of the first filtration D_1^o in $\mathfrak{N}$ and the indecomposables of $H^*(\mathrm{SO})$. □

5.1 Comparison of width filtration and chromatic filtration

The connection established between the lowest degree in the width filtration and the image of J invites a discussion of how the width filtration compares with the chromatic filtration. The chromatic filtration is a decreasing filtration while the width filtration is an increasing filtration. However, we can still compare the subquotients. The first subquotient in the width spectral sequence is D_1^o, which is exactly the image of J in cohomology, as proved in the previous section. The first subquotient in the chromatic filtration consists of the v_1-periodic elements, which at the prime 2 includes both the image of J and additional classes from the α family in degrees $8k+1$ and $8k+2$ [25]. In the width filtration, the first of these classes shows up in the D_3^o subquotient. So the width filtration differs from the chromatic filtration, and could faithfully rather than "approximately" extend the image of J to a full filtration.

Appendix

A.1 The width spectral sequence

The chart in Figure 8 depicts the E_1 page of the width spectral sequence as an $\mathbb{F}_2$-vector space. There is also a partial action of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$, as described in Section 4, with the actions of h_0, h_1 and h_2 indicated with black connecting lines.

A.2 J^* in degree 15

We will illustrate what is happening in the proof of Proposition 5.6 in degree 15 ($k=7$). Using Lemma 5.7 to ignore nontrivial products in $H_*(Q_1S^0)$ and Lemma 5.8 to ignore products of three or more terms in $H_*(\mathrm{SO})$, the relevant homology maps are

$$J_*(e^{15}) = q_{15}$$

and, for $a+b=15$,

$$J_*(e^a \wedge e^b) = \sum_{l=\lceil (a-b)/2\rceil}^{a/2} \binom{a-l}{a-2l} q_{b+2l-a,a-l} + \text{products}.$$

We conclude that $J^*(q_{15}^*) = J^*(\gamma_{1[1]}^{15}) = (e^{15})^* = \beta_{15}$ and $J^*((q_{15-2j,j})^*) = \sum (e^a \wedge e^b)^*$, where the sum runs over all $e^a \wedge e^b$ for which $q_{15-2j,j}$ was one of the terms in the image under J_*. Thus, summing over the image of J_* for all $a+b=15$ and looking at the coefficient of $q_{15-2j,j}$ mod 2 will tell us whether $q_{15-2j,j}$ is in the image of an odd or even number of $e^a \wedge e^b$. For $q_{15-2j,j}$ to be admissible,

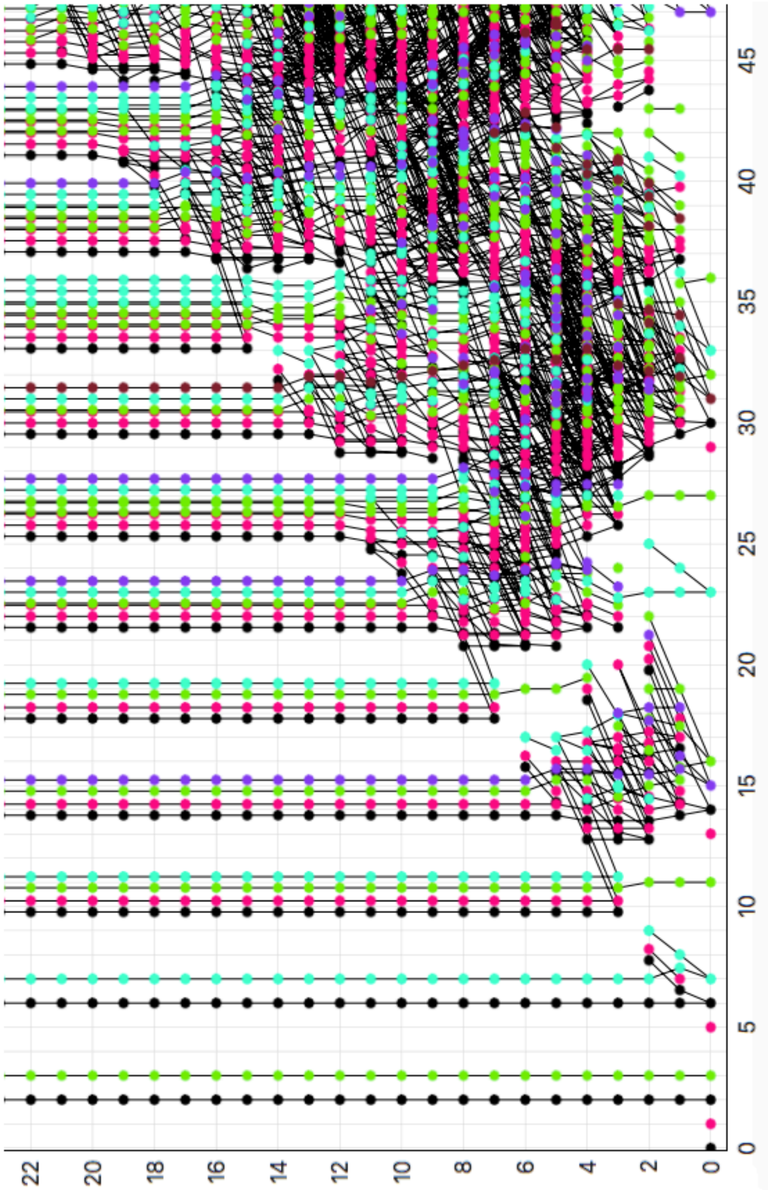

Figure 8: E_1 page of the width spectral sequence. Black corresponds to D_1, red to D_2, green to D_3, teal to D_4, purple to D_5, and brown to D_6.

we need $5 \leq j \leq 7$. We then calculate the coefficient for $q_{5,5}$ is

$$\sum_{i=0}^{2} \binom{5}{i} + \sum_{b=4}^{4} \left(\sum_{l=0}^{2b-8} \binom{b}{l} \right) \binom{5-b-1}{-5+b},$$

which by Lemma 5.9 is

$$\binom{4}{2} + \sum_{b=4}^{4} \binom{b-1}{2b-8} \binom{5-b-1}{-5+b} = 0.$$

Repeating the same process for $q_{3,6}$, the coefficient is equivalent mod 2 to

$$\binom{6}{4}+\sum_{b=4}^{4}\binom{b-1}{2b-8}\binom{6-b-1}{-3+b}=1+1=0.$$

And finally, for $q_{1,7}$, the coefficient is equivalent mod 2 to

$$\binom{6}{6}+\sum_{b=4}^{3}\binom{b-1}{2b-8}\binom{7-b-1}{-1+b}=1+0=1.$$

Thus, $q_{1,7}$ is the only element appearing an even number of times as the image of the $e^a \wedge e^b$ and thus only $J^*((q_{1,7})^*)$ is nonzero up to decomposables.

Next, we know that

$$(q_{1,7})^* = \gamma_{1[2]}^6\gamma_{2[1]} + \gamma_{1[2]}^3\gamma_{2[1]}^3 + \gamma_{2[1]}^5, \quad (q_{3,6})^* = \gamma_{1[2]}^3\gamma_{2[1]}^3 + \gamma_{2[1]}^5 \quad \text{and} \quad (q_{5,5})^* = \gamma_{2[1]}^5,$$

so

$$\begin{aligned} J^*(\gamma_{2[1]}^5) &= J^*(q_{5,5}^*) = 0, \\ J^*(\gamma_{1[2]}^3\gamma_{2[1]}^3) &= J^*((q_{3,6})^* + (q_{5,5}^*)) = 0+0, \\ J^*(\gamma_{1[2]}^6\gamma_{2[1]}) &= J^*((q_{1,7})^* + (q_{3,6})^* + (q_{5,5}^*)) = \beta_{15}, \end{aligned}$$

all up to decomposables.

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Received: 12 October 2022 Revised: 14 June 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 6 (pages 3145–3787) 2025