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The slices of quaternionic Eilenberg–Mac Lane spectra

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We compute the slices and slice spectral sequence of integral suspensions of the equivariant Eilenberg–Mac Lane spectra $H\underline{\mathbb{Z}}$ for the group of equivariance Q_8. Along the way, we compute the Mackey functors $\underline{\pi}_{k\rho}H\underline{\mathbb{Z}}$.

55N91; 55P91, 55Q91

1 Introduction

Let G be a finite group. The G-equivariant slice filtration was first defined in the context of G-equivariant stable homotopy theory by Dugger in [6]; it came to prominence due to its role in the proof of the Kervaire invariant one problem by Hill, Hopkins and Ravenel [9]. The slice filtration is an analogue in the G-equivariant stable homotopy category of the classical Postnikov filtration of spectra. One can also define a G-equivariant Postnikov filtration; on passage to fixed points with respect to any subgroup $H \leq G$, this recovers the Postnikov filtration of the H-fixed point spectrum. However, there are many equivariant spectra which possess a periodicity with respect to suspension by a G-representation sphere, and this periodicity is not visible in the G-equivariant Postnikov filtration. The slice filtration was devised by Dugger in order to display this periodicity for the case of the C_2-spectrum $K\mathbb{R}$.

Since the groundbreaking work [9], a number of authors have calculated the slice filtration, as well as the associated slice spectral sequence, for G-spectra of interest. A few cases are understood for an arbitrary finite group G. If $\underline{M}$ is a G-Mackey functor, then the equivariant Eilenberg–Mac Lane spectrum $H_G\underline{M}$ is always a 0-slice [9] (in this article, we use the "regular" slice filtration, as introduced by Ullman in [22]). The slice filtrations of $\Sigma^1 H_G\underline{M}$ and $\Sigma^{-1} H_G\underline{M}$ were described in [22]. The slices of certain suspensions of equivariant Eilenberg–Mac Lane spectra were determined for G an odd cyclic p-group by Angeltveit [1], Hill, Hopkins and Ravenel [10], and Yarnall [25]; for dihedral groups of order $2p$, where p is odd, by Zou [27]; and for the Klein four-group by Guillou and Yarnall [7] and Slone [19]. A majority of the work to date on the slice filtration has focused on cyclic groups of equivariance, with only a few exceptions. We extend the literature by considering in this article the case of the nonabelian group $G = Q_8$.

The choice of the group Q_8 is not arbitrary, as we now explain. Some of the most far-reaching applications of the slice filtration and associated spectral sequence have come in the case of cyclic p-groups of equivariance. In addition to [9], this also includes Hill, Hopkins and Ravenel [11], Hill, Shi, Wang and Xu [12], Meier, Shi and Zeng [18], and Sulyma [20]. In particular, in [12] the authors use slice technology to understand a C_4-equivariant, height 4 Lubin–Tate theory at the prime 2. For each height n, there is a height n Lubin–Tate theory that comes equipped with a continuous action of the height n (profinite) Morava stabilizer group. The homotopy fixed points with respect to this action gives a model for the $K(n)$-local sphere, a central object of study in stable homotopy theory. However, the homotopy fixed points with respect to the entire stabilizer group are quite difficult to calculate. More approachable are the homotopy fixed points with respect to finite subgroups of the stabilizer group. At height 4, the Morava stabilizer group contains a C_4-subgroup (in fact a C_8), which gives the context for [12]. On the other hand, at height $2m$, where m is odd, the Morava stabilizer group contains a Q_8-subgroup. Therefore it is reasonable to expect that Q_8-equivariant slice techniques will eventually shed light on the $K(n)$-local sphere when $n = 2m$ and m is odd.

The focus of our article is the determination of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$, for $n \geq 0$. We list the slices in Section 6 and describe the associated spectral sequence in Section 8. We rely heavily on the computation of the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ given by the second author in [19]. The quotient map $Q_8 \to K_4$ allows us to gain insight into the Q_8-equivariant slices from the K_4-case, as we now explain in greater generality.

Given a normal subgroup $N \trianglelefteq G$, there are several constructions that will produce a G-spectrum from a G/N-spectrum. First is the ordinary pullback, or inflation, functor. If $q\colon G \to G/N$ is the quotient, then inflation is denoted $q^*\colon \mathbf{Sp}^{G/N} \to \mathbf{Sp}^G$; it is left adjoint to the N-fixed point functor. This inflation functor plays an important role. For instance $q^*(S^0_{G/N})$ is equivalent to S^0_G. However, from our point of view, this construction has two deficiencies. First, the ordinary inflation does not interact well with the slice filtration. Secondly, the inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module does not have a canonical $H_G\underline{\mathbb{Z}}$-module structure.

On the other hand, the "geometric inflation" functor [8, Definition 4.1; 16, Section II.9]

$$\phi_N^* \colon \mathbf{Sp}^{G/N} \to \mathbf{Sp}^G,$$

which is right adjoint to the geometric fixed points functor, interacts well with slices. Namely, if N is a normal subgroup of order d and X is a G/N-spectrum, then

$$\phi_N^* P_k^k(X) \simeq P_{dk}^{dk}(\phi_N^* X),$$

by [22, Corollary 4.5] (see also [8, Section 4.2]). However, in general the geometric inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module will not be an $H_G\underline{\mathbb{Z}}$-module.

The third variant is the $\underline{\mathbb{Z}}$-module inflation functor [26, Section 3.2]

$$\Psi_N^* \colon \mathrm{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \to \mathrm{Mod}_{H_G\underline{\mathbb{Z}}}.$$

By design, the $\underline{\mathbb{Z}}$-module inflation of an $H_{G/N}\underline{\mathbb{Z}}$-module has a canonical $H_G\underline{\mathbb{Z}}$-module structure, though in general this functor does not interact well with the slice filtration.

In some cases, these constructions agree. For instance, if N is a bottleneck subgroup (Definition 3.5) and the underlying spectrum of the G/N-spectrum X is contractible, then $q^*X \simeq \phi_N^* X$ (see the proof of Proposition 3.18). If X is furthermore an $H_{G/N}\underline{\mathbb{Z}}$-module, then the three inflation functors coincide on X (Proposition 3.18).

The above discussion applies to the slices of $\Sigma^n H_{G/N}\underline{\mathbb{Z}}$: all slices, except for the bottom slice, have trivial underlying spectrum. It follows that these inflate to give many of the slices of $\Sigma^n H_G\underline{\mathbb{Z}}$.

Our main result along these lines, Theorem 3.19, describes the higher slices of such an inflated $H_G\underline{\mathbb{Z}}$-module. In the case of $G = Q_8$, $N = Z(Q_8)$, and $G/N = Q_8/Z \cong K_4$, it gives the following:

Theorem 1.1 *Let $n \geq 0$. Then the nontrivial slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$, above level $2n$, are*

$$P_{2k}^{2k}(\Sigma^n H_{Q_8}\underline{\mathbb{Z}}) \simeq \Psi_Z^* P_k^k(\Sigma^n H_{K_4}\underline{\mathbb{Z}}) \simeq \phi_Z^* P_k^k(\Sigma^n H_{K_4}\underline{\mathbb{Z}})$$

for $k > n$. Furthermore,

$$P_n^{2k}(\Sigma^n H_{Q_8}\underline{\mathbb{Z}}) \simeq \Psi_Z^* P_n^k(\Sigma^n H_{K_4}\underline{\mathbb{Z}})$$

for $k \geq n$.

As the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ were determined by the second author in [19, Section 8], this immediately provides all of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ above level $2n$ when n is positive. In particular, the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ occur in slice dimension n to $4(n-4)$, and it follows that the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ occur in slice dimension n to $8(n-4)$.

In more detail, the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ are fully geometric (ie vanish at proper subgroups) in slice dimension greater than $2(n-3)$ and therefore occur only in slice dimensions a multiple of 4. It follows that the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ are fully geometric in slice dimension greater than $4(n-3)$ and only occur in slice dimensions a multiple of 8. See Proposition 6.10 for a more precise answer.

In slice dimensions from $n+1$ to $2(n-3)$, the second author showed that the K_4-spectrum $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ has some fully geometric slices (in dimensions a multiple of 4) as well as slices detected at the order two subgroups (in dimensions $4k$ if n is odd and $4k+2$ if n is even). It follows that the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$, in slice dimension from $2n+1$ to $4(n-3)$, occur only in slice dimensions a multiple of 4. See Proposition 6.11 for a more precise answer.

The remaining slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$, which occur in slice dimensions from n to $2n$, are then given by analyzing the slice tower of $\Psi_N^*(P_n^n H_K\underline{\mathbb{Z}})$. We perform this analysis in Section 6.1.

Though we have stated Theorem 1.1, as well as Theorem 3.19, for $n \geq 0$, a similar result holds for $n < 0$. The determination of the slices of $\Sigma^n H_{K_4}\underline{\mathbb{Z}}$ in [19] includes the case of $n < 0$, and so the methods in this article would similarly determine the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ for $n < 0$. See also Remark 3.20.

1.1 Notation

Throughout, whenever referencing the slice filtration, we will always mean the "regular" slice filtration of [22].

We will often write simply Q and K to denote the quaternion group Q_8 and Klein four group K_4, respectively. We write Z for the central subgroup of Q of order two generated by $z = -1$. We write

$$L = \langle i \rangle, \quad D = \langle k \rangle, \quad R = \langle j \rangle$$

for the normal, cyclic subgroups of Q of order 4. We also use the same names for the images of these subgroups in $Q/Z \cong K$. In other words, the subgroup lattices of Q_8 and K_4 are

$$\begin{tikzcd}[arrows=dash] & Q_8 \arrow[dl] \arrow[d] \arrow[dr] & \\ L \arrow[dr] & D \arrow[d] & R \arrow[dl] \\ & Z \arrow[d] & \\ & e & \end{tikzcd} \qquad \begin{tikzcd}[arrows=dash] & K_4 \arrow[dl] \arrow[d] \arrow[dr] & \\ L \arrow[dr] & D \arrow[d] & R \arrow[dl] \\ & e & \end{tikzcd}$$

Our nomenclature for the order 4 subgroups of Q_8 amounts to a choice of isomorphism $Q/Z \cong K$.

The sign representation of C_2 will be denoted σ, and we will write $\mathbb{Z}^\sigma$ for the corresponding C_2-module.

1.2 Organization

The paper is organized as follows. In Section 2, we review the representations of C_4, K_4, and Q_8, as well as Mackey functors over C_4 and K_4. Then in Section 3, we introduce three inflation functors from a quotient group G/N of some finite group G as well as several results that will aid in the calculation of the slices of $\Sigma^n H_{Q_8} H\underline{\mathbb{Z}}$. The relevant Q_8-Mackey functors and the homology of $\Sigma^{k\rho_{Q_8}} H_{Q_8}\underline{\mathbb{Z}}$ are found in Section 4. The slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ must restrict to the appropriate slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$; thus, we review this information in Section 5. We provide some slice towers and describe all slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 6. We then compute the homotopy Mackey functors of the slices of $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 7. Finally, we provide some examples of the slice spectral sequence for $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ and $\Sigma^n H_{Q_8}\underline{\mathbb{Z}}$ in Section 8.

1.3 Acknowledgements

The authors are very happy to thank Agnès Beaudry, Michael Geline, Cherry Ng, and Mingcong Zeng for a number of helpful discussions. The spectral sequence charts in Section 8 were created using Hood Chatham's `spectralsequences` package.

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2 Background

2.1 Background for C_4

The C_4-sign representation σ_{C_4} is the inflation $p^*\sigma_{C_2}$ of the C_2-sign representation along the surjection $C_4 \to C_2$. We will simply write σ for σ_{C_4}. Then the regular representation for C_4 splits as

$$\rho_{C_4} = 1 \oplus \sigma \oplus \lambda,$$

where λ is the irreducible 2-dimensional rotation representation of C_4. The $RO(C_4)$-graded homotopy Mackey functors of $H_{C_4}\underline{\mathbb{Z}}$ are given in [11]. More specifically, the homotopy Mackey functors of $\Sigma^{k\rho_{C_4}} H_{C_4}\underline{\mathbb{Z}}$, $\Sigma^{k\lambda} H_{C_4}\underline{\mathbb{Z}}$, and $\Sigma^{k\sigma} H_{C_4}\underline{\mathbb{Z}}$ are given in [11, Figures 3 and 6]. Some C_4-Mackey functors that will appear below are displayed in Table 1. All of these Mackey functors have trivial Weyl-group actions.

$\square = \underline{\mathbb{Z}}$	$\boxed{*} = \underline{\mathbb{Z}}^*$	$\underline{\mathbb{Z}}(2,1)$	$\circ = \underline{B}(2,0)$
$\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 1}\downarrow\;\uparrow{\scriptstyle 2} \\ \mathbb{Z} \\ {\scriptstyle 1}\downarrow\;\uparrow{\scriptstyle 2} \\ \mathbb{Z} \end{array}$	$\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 2}\downarrow\;\uparrow{\scriptstyle 1} \\ \mathbb{Z} \\ {\scriptstyle 2}\downarrow\;\uparrow{\scriptstyle 1} \\ \mathbb{Z} \end{array}$	$\begin{array}{c} \mathbb{Z} \\ {\scriptstyle 2}\downarrow\;\uparrow{\scriptstyle 1} \\ \mathbb{Z} \\ {\scriptstyle 1}\downarrow\;\uparrow{\scriptstyle 2} \\ \mathbb{Z} \end{array}$	$\begin{array}{c} \mathbb{Z}/4 \\ {\scriptstyle 1}\downarrow\;\uparrow{\scriptstyle 2} \\ \mathbb{Z}/2 \\ \\ 0 \end{array}$
$\bullet = \underline{g}$	$\bar{\bullet} = \phi^*\underline{f}$	$\blacklozenge = \phi^*\underline{\mathbb{F}_2}$	$\phi^*\underline{\mathbb{F}_2}^*$
$\begin{array}{c} \mathbb{F}_2 \\ \\ 0 \\ \\ 0 \end{array}$	$\begin{array}{c} 0 \\ \\ \mathbb{F}_2 \\ \\ 0 \end{array}$	$\begin{array}{c} \mathbb{F}_2 \\ \downarrow{\scriptstyle 1} \\ \mathbb{F}_2 \\ \\ 0 \end{array}$	$\begin{array}{c} \mathbb{F}_2 \\ {\scriptstyle 1}\uparrow \\ \mathbb{F}_2 \\ \\ 0 \end{array}$

Table 1: Some C_4-Mackey functors.

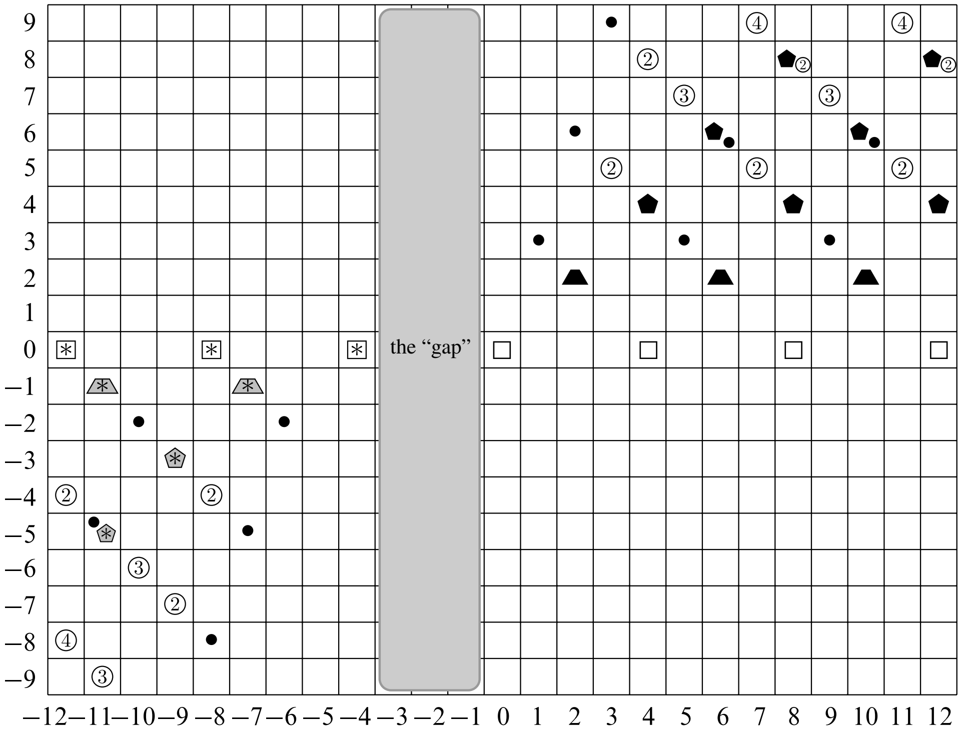


Figure 1: The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4}\underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4}\underline{\mathbb{Z}}$ appears in position $(k, 4n-k)$.

2.2 Background for K_4

The Klein 4-group $K_4 = C_2 \times C_2$ has three sign representations, obtained as the inflation along the three surjections $K_4 \to C_2$. We denote these three surjections by p_1, m, and p_2. Then the regular representation of K_4 splits as

$$\rho_{K_4} \cong 1 \oplus p_1^*\sigma \oplus m^*\sigma \oplus p_2^*\sigma.$$

Some K_4-Mackey functors that will appear below are displayed in Table 2.

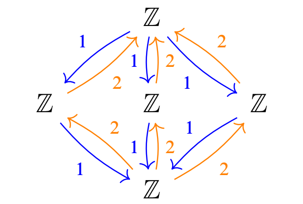
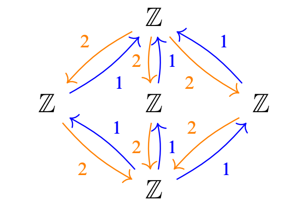
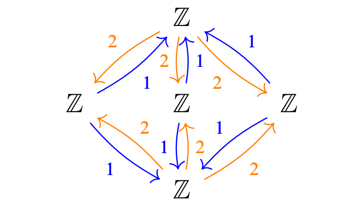
The homotopy Mackey functors of $\Sigma^{n\rho} H_K\underline{\mathbb{Z}}$ were computed in [19, Section 9]. They are displayed in Figure 1. The homotopy Mackey functors of $\Sigma^{n\rho} H_K\underline{\mathbb{F}_2}$ were computed in [7, Section 7]. They are displayed in Figure 2.

2.3 Background for Q_8

The regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where $\mathbb{H}$ is the 4-dimensional irreducible Q_8-representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient.

$\square = \underline{\mathbb{Z}}$	$\boxed{*} = \underline{\mathbb{Z}}^*$	$\underline{\mathbb{Z}}(2,1)$
		
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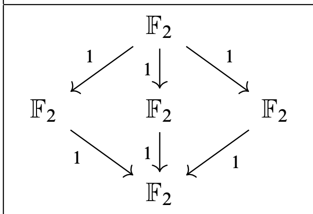	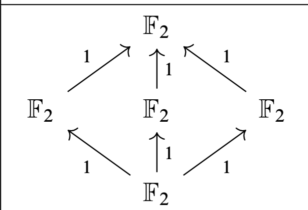	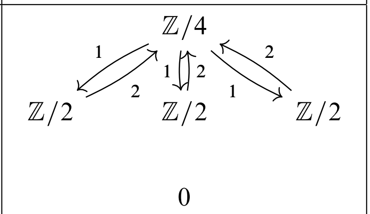
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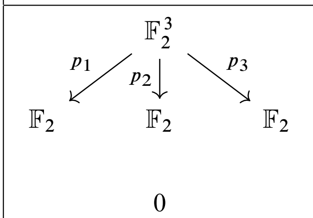
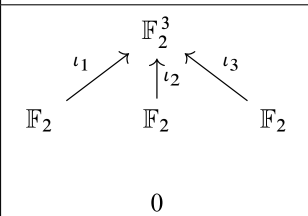
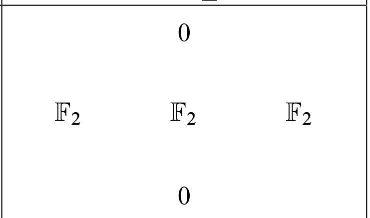
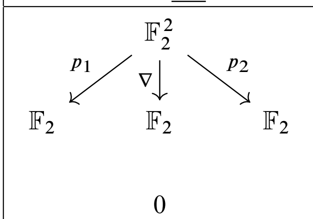
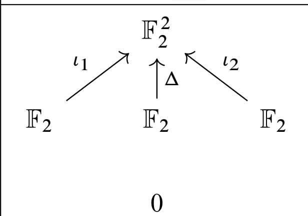
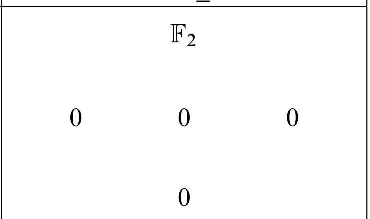
		
$\blacktriangle = \underline{mg}$	$\circledast = \underline{mg}^*$	$\bullet = \underline{g}$
		
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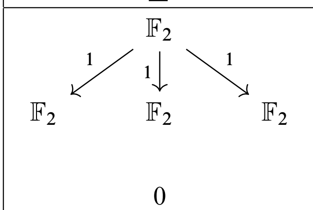	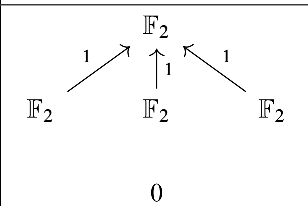	
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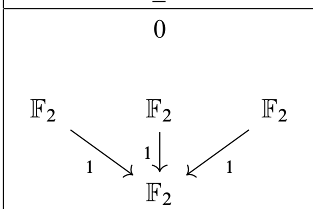
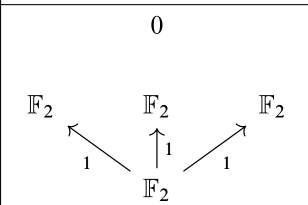
		

Table 2: Some K_4-Mackey functors.

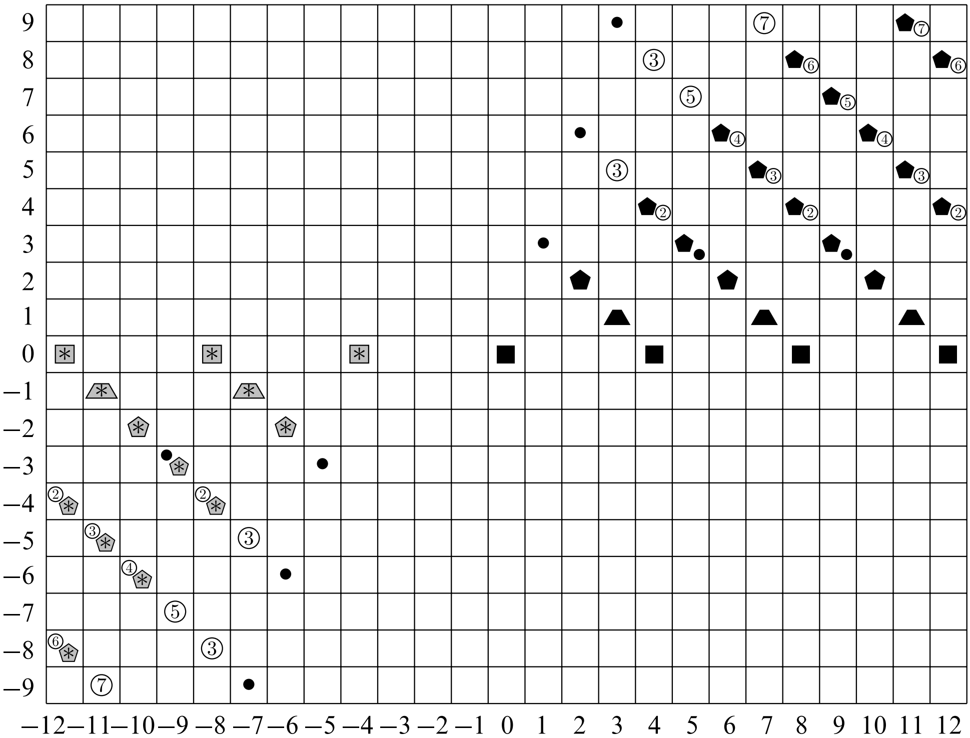


Figure 2: The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_{K_4}\underline{\mathbb{F}_2}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_{K_4}\underline{\mathbb{F}_2}$ appears in position $(k, 4n-k)$.

Denoting by C_4 any of the subgroups L, D, or R of Q_8, we have that

$$\downarrow^{Q_8}_{C_4} \rho_K = 2 + 2\sigma \quad \text{and} \quad \downarrow^{Q_8}_{C_4} \mathbb{H} = 2\lambda.$$

3 Inflation functors

3.1 Inflation and the projection formula

Let $N \trianglelefteq G$ be a normal subgroup and $q\colon G \to G/N$ the quotient map. Recall that there is an induced adjunction

$$\mathbf{Sp}^{G/N} \underset{(-)^N}{\overset{q^*}{\rightleftarrows}} \mathbf{Sp}^G$$

where the pullback functor q^*, called inflation, is strong symmetric monoidal. We will also need a description of the N-fixed points of an Eilenberg–Mac Lane G-spectrum. First note that there is a functor

(3.1) $$\mathbf{Mack}(G) \xrightarrow{q_*} \mathbf{Mack}(G/N)$$

given by

$$q_*(\underline{M})(\overline{H}) = \underline{M}(H),$$

where $\overline{H} = H/N \leq G/N$ whenever $N \leq H$. The functor q_* is denoted by $\beta^!$ in [21, Lemma 5.4]. Then the homotopy Mackey functors of the N-fixed points of a G-spectrum X are given by

(3.2) $$\underline{\pi}_n(X^N) \cong q_* \underline{\pi}_n(X).$$

In the case of an Eilenberg–Mac Lane spectrum this yields an equivalence

$$(H_G \underline{M})^N \simeq H_{G/N}(q_* \underline{M}).$$

The following result will be quite useful.

Proposition 3.3 (projection formula [14, Lemma 2.13; 2, Proposition 2.15]) *Let $N \trianglelefteq G$ be a normal subgroup and $q\colon G \to G/N$ be the quotient map. Then for $X \in \mathbf{Sp}^{G/N}$ and $Y \in \mathbf{Sp}^G$, there is a natural equivalence of G/N-spectra*

$$(q^* X \wedge Y)^N \simeq X \wedge Y^N.$$

We will frequently employ this in the case that $X = S^V$ for some G/N-representation V and $Y = H_G \underline{M}$ for some G-Mackey functor $\underline{M}$. Then the projection formula reads

(3.4) $$(S^{q^* V} \wedge H_G \underline{M})^N \simeq S^V \wedge H_{G/N}(q_* \underline{M}).$$

See also [26, Corollary 5.8].

3.2 Geometric fixed points

For a normal subgroup $N \trianglelefteq G$, we define the family of subgroups $\mathscr{F}[N]$ of G to consist of those subgroups that do not contain N. Recall that the N-geometric fixed points spectrum of a G-spectrum is defined as

$$\Phi^N(X) = (\widetilde{E}\mathscr{F}[N] \wedge X)^N.$$

This notation is simultaneously used to denote the resulting G/N-spectrum as well as the underlying spectrum. The N-geometric fixed points has a right adjoint, given by the geometric inflation functor

$$\phi_N^*(Z) = \widetilde{E}\mathscr{F}[N] \wedge q^* Z.$$

To sum up, we have an adjunction

$$\mathbf{Sp}^G \underset{\phi_N^*}{\overset{\Phi^N}{\rightleftarrows}} \mathbf{Sp}^{G/N}.$$

3.3 Bottleneck subgroups

The subgroup $Z \trianglelefteq Q$ plays an important role in this article. The primary reason is that it satisfies the following property.

Definition 3.5 We say that $N \trianglelefteq G$ is a *bottleneck* subgroup if it is a nontrivial, proper subgroup such that, for any subgroup $H \leq G$, either H contains N or N contains H.

We now demonstrate that bottleneck subgroups only occur in cyclic p-groups or quaternion groups. The following argument was sketched to us by Mike Geline.

Proposition 3.6 *Let $N \trianglelefteq G$ be a bottleneck subgroup of G. Then N is cyclic, and G is either a cyclic p-group or a generalized quaternion group.*

Proof We will refer to a subgroup $H \le G$ which neither contains N nor is contained in N as "adjacent" to N. The assumption that N is a bottleneck subgroup means precisely that G has no subgroups that are adjacent to N. To see that N must be cyclic, note that if g is not in N, then $N \le \langle g \rangle$, which implies that N is cyclic.

We next observe that G is necessarily a p-group. This is because if N is contained in some Sylow p-subgroup, then any Sylow q-subgroup, for a different prime q, would be adjacent. It follows that N contains all of the Sylow subgroups and therefore is all of G.

Next, we recall [5, Theorem 4.3] that for a p-group G, the group contains a unique subgroup of order p if and only if G is either cyclic or generalized quaternion. So we will argue that G contains a unique subgroup of order p. The first step is to note that G cannot contain a subgroup isomorphic to $C_p \times C_p$. This is because such a subgroup would necessarily contain N. This would imply that $N \cong C_p$, and then N would have a complement in $C_p \times C_p$, which would be a subgroup adjacent to N in G.

Finally, note that the center $Z(G)$ contains a subgroup of order p. If G has another subgroup of order p, these two would generate a $C_p \times C_p$, contradicting the previous step. □

Remark 3.7 It follows from Proposition 3.6 that if $N \trianglelefteq G$ is a bottleneck subgroup, then G/N is either a cyclic p-group or a dihedral 2-group.

If $N \trianglelefteq G$ is a bottleneck subgroup, then geometric fixed points with respect to G can be computed in terms of geometric fixed points with respect to the quotient group G/N.

Proposition 3.8 *Let $N \trianglelefteq G$ be a bottleneck subgroup. Then $\Phi^G X \simeq \Phi^{G/N} X^N$ for any $X \in \mathbf{Sp}^G$.*

Proof If $N \trianglelefteq G$ is a bottleneck subgroup, then $q^* \widetilde{E}\mathcal{P}_{G/N} \simeq \widetilde{E}\mathcal{P}_G$. Thus

$$\Phi^G X = (\widetilde{E}\mathcal{P}_G \wedge X)^G \simeq ((q^* \widetilde{E}\mathcal{P}_{G/N} \wedge X)^N)^{G/N}.$$

By the projection formula (Proposition 3.3), this is equivalent to

$$(\widetilde{E}\mathcal{P}_{G/N} \wedge X^N)^{G/N} = \Phi^{G/N} X^N.$$

□

Proposition 3.8 also follows from the more general [15, Proposition 9].

3.4 Inflation for $\underline{\mathbb{Z}}$-modules

Given a surjection $q\colon G \to G/N$, the inflation functor

$$\phi_N^*\colon \mathbf{Mack}(G/N) \to \mathbf{Mack}(G)$$

does not send $\underline{\mathbb{Z}}$-modules for G/N to $\underline{\mathbb{Z}}$-modules for G. We now describe a modified inflation functor that exists at the level of $\underline{\mathbb{Z}}$-modules. This functor previously appeared in [26, Section 3.2] and [3, Section 3.10].

Definition 3.9 Let $\mathcal{B}\mathbb{Z}_G \subset \mathrm{Mod}_{\mathbb{Z}[G]}$ denote the full subcategory of permutation G-modules. Recall [26, Proposition 2.15] that $\underline{\mathbb{Z}}_G$-modules correspond to additive functors $\mathcal{B}\mathbb{Z}_G^{op} \to \mathrm{Ab}$. Then the $\underline{\mathbb{Z}}$-module inflation functor

$$\Psi_N^*\colon \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}} \to \mathrm{Mod}_{\underline{\mathbb{Z}}_G}$$

is defined to be the left Kan extension along the inflation functor $\mathcal{B}\mathbb{Z}_{G/N} \to \mathcal{B}\mathbb{Z}_G$.

The following is an immediate corollary of the definition as a left Kan extension.

Proposition 3.10 *The functor Ψ_N^* is left adjoint to the functor $q_*\colon \mathrm{Mod}_{\underline{\mathbb{Z}}_G} \to \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, defined as in (3.1).*

Proposition 3.11 [3, (3.11)] *For $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, the $\underline{\mathbb{Z}}_G$-module $\Psi_N^*(\underline{M})$ satisfies*

(1) *$q_*(\Psi_N^*(\underline{M}))$ is $\underline{M}$, and*
(2) *$\downarrow_N^G (\Psi_N^*(\underline{M}))$ is the constant Mackey functor at $\underline{M}(e)$.*

Note that Proposition 3.11 completely describes $\Psi_N^*(\underline{M})$ if N is a bottleneck subgroup. The following result states that $\underline{\mathbb{Z}}$-module inflation agrees with ordinary inflation on geometric Mackey functors.

Proposition 3.12 *Let $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$, and let $N \trianglelefteq G$ be a bottleneck subgroup. If $\underline{M}(e) = 0$, then $\Psi_N^*\underline{M} \cong \phi_N^*\underline{M}$.*

Proof This follows immediately from Proposition 3.11. □

Remark 3.13 Note that Proposition 3.12 is not true without the bottleneck hypothesis. For instance, if $N = C_3 \trianglelefteq \Sigma_3$, then $\downarrow_{C_2}^{\Sigma_3} (\Psi_{C_3}^*\underline{M}) \cong \underline{M}$. In particular, it is not true that $\Psi_{C_3}^*\underline{M}$ is concentrated over $N = C_3$ when $\underline{M}(e) = 0$.

We now discuss the extension to equivariant spectra.

Proposition 3.14 *The N-fixed points functor*

$$(-)^N\colon \mathrm{Mod}_{H_G\underline{\mathbb{Z}}} \to \mathrm{Mod}_{H_{G/N}\underline{\mathbb{Z}}}$$

for $H\underline{\mathbb{Z}}$-modules has a left adjoint

$$\Psi_N^*\colon \mathrm{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \to \mathrm{Mod}_{H_G\underline{\mathbb{Z}}}.$$

If $N \trianglelefteq G$ is a bottleneck subgroup, then the spectrum-level functor Ψ_N^* extends the functor Ψ_N^* of Definition 3.9, in the sense that

$$\Psi_N^* H_{G/N}\underline{M} \simeq H_G(\Psi_N^*\underline{M}) \tag{3.15}$$

for $\underline{M}$ in $\mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$.

Proof For an $H_{G/N}\underline{\mathbb{Z}}$-module X, the inflation q^*X is canonically a module over $q^*H_{G/N}\underline{\mathbb{Z}}$. We then define the spectrum-level functor Ψ_N^* by the formula

$$\Psi_N^* X = H\underline{\mathbb{Z}} \wedge_{q^* H\underline{\mathbb{Z}}} (q^* X).$$

We leave it to the reader to verify that this is indeed left adjoint to the N-fixed points functor.

To see that (3.15) holds, we show first that this holds on the indecomposable projective $\underline{\mathbb{Z}}_{G/N}$-modules. These are of the form $\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}$, and the diagram of commuting adjoint functors

$$\begin{tikzcd}
\mathrm{Mod}_{H_{G/N}\underline{\mathbb{Z}}} \arrow[r, shift left, "\Psi_N^*"] \arrow[d, shift left, "\downarrow_{K/N}^{G/N}"] & \mathrm{Mod}_{H_G\underline{\mathbb{Z}}} \arrow[l, shift left, "(-)^N"] \arrow[d, shift left, "\downarrow_K^G"] \\
\mathrm{Mod}_{H_{K/N}\underline{\mathbb{Z}}} \arrow[r, shift left, "\Psi_N^*"] \arrow[u, shift left, "\uparrow_{K/N}^{G/N}"] & \mathrm{Mod}_{H_K\underline{\mathbb{Z}}} \arrow[l, shift left, "(-)^N"] \arrow[u, shift left, "\uparrow_K^G"]
\end{tikzcd}$$

shows that

$$\Psi_N^*\big(H_{G/N} \uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}\big) \simeq \uparrow_K^G \Psi_N^*(H_{K/N}\underline{\mathbb{Z}}) \simeq \uparrow_K^G H_K\underline{\mathbb{Z}} \simeq H_G \uparrow_K^G \underline{\mathbb{Z}} \simeq H_G\Psi_N^*\big(\uparrow_{K/N}^{G/N} \underline{\mathbb{Z}}\big).$$

Since the functor $\Psi_N^* \colon \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}} \to \mathrm{Mod}_{\underline{\mathbb{Z}}_G}$ is exact [26, Lemma 3.14], it follows that if $\mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$ has finite global projective dimension, then (3.15) will hold for any $\underline{\mathbb{Z}}_{G/N}$-module $\underline{M}$. By [4, Theorem 1.7], this is the case precisely when G/N is as described in Remark 3.7. □

Example 3.16 Let $X \in \mathbf{Sp}^{G/N}$ and $\underline{M} \in \mathbf{Mack}(G/N)$, with $\underline{M}(e) = 0$. Again assume that N is a bottleneck subgroup. Then Propositions 3.12 and 3.14 give that

$$\Psi_N^*(X \wedge H_{G/N}\underline{M}) \simeq q^*(X) \wedge \Psi_N^*(H_{G/N}\underline{M}) \simeq q^*(X) \wedge \phi_N^* H_{G/N}\underline{M} \simeq \phi_N^*(X \wedge H_{G/N}\underline{M}).$$

We will employ this equivalence when X is a representation sphere.

Proposition 3.17 *Let $N \trianglelefteq G$ be a bottleneck subgroup. Then for any G/N-representation V and $\underline{\mathbb{Z}}_{G/N}$-module $\underline{L}$, we have*

$$\underline{\pi}_n(\Psi_N^*\Sigma^V H_{G/N}\underline{L}) \cong \Psi_N^*\underline{\pi}_n(\Sigma^V H_{G/N}\underline{L}).$$

Proof Let us write $X = \Psi_N^*\Sigma^V H_{G/N}\underline{L} \simeq \Sigma^{q^*V} H_G\Psi_N^*\underline{L}$. Since N is a bottleneck subgroup, it is enough to describe $\downarrow_N^G \underline{\pi}_n X$ and $q_*\underline{\pi}_n X$. Now

$$\downarrow_N^G \underline{\pi}_n X \cong \underline{\pi}_n \downarrow_N^G X = \underline{\pi}_n \Sigma^{\dim V} H_N \underline{L}(N/N).$$

This is a constant Mackey functor. On the other hand, by (3.2) and (3.4), we have

$$q_*\underline{\pi}_n X \cong \underline{\pi}_n(X^N) \cong \underline{\pi}_n(\Sigma^V H_{G/N}\underline{L}).$$

By Proposition 3.11, this agrees with $\Psi_N^*\underline{\pi}_n(\Sigma^V H_{G/N}\underline{L})$. □

More generally, we have an extension of Proposition 3.12 to $H\underline{\mathbb{Z}}$-modules:

Proposition 3.18 *Let $X \in \mathrm{Mod}_{H\underline{\mathbb{Z}}_{G/N}}$ and let $N \trianglelefteq G$ be a bottleneck subgroup. If the underlying spectrum $\downarrow_e^{G/N} X$ is contractible, then $\Psi_N^*(X) \simeq \phi_N^* X$.*

Proof If the underlying spectrum of X is contractible, then $X \simeq \widetilde{E}(G/N) \wedge X$. The assumption that N is a bottleneck subgroup implies that $E(G/N) = q^*(E(G/N))$ is the universal space for the family of subgroups of N, so $\widetilde{E}(G/N) \wedge \widetilde{E}\mathcal{F}[N] \simeq \widetilde{E}(G/N)$, and it follows that

$$q^* X \simeq \widetilde{E}(G/N) \wedge q^* X \simeq \widetilde{E}(G/N) \wedge \phi_N^*(X) \simeq \phi_N^* X.$$

Now

$$\Psi_N^*(X) = H_G\underline{\mathbb{Z}} \wedge_{q^* H_{G/N}\underline{\mathbb{Z}}} q^*(X) \simeq H_G\underline{\mathbb{Z}} \wedge_{q^* H_{G/N}\underline{\mathbb{Z}}} (\widetilde{E}(G/N) \wedge q^*(X)).$$

Since $\widetilde{E}(G/N)$ is smash idempotent, this can be rewritten as

$$\Psi_N^*(X) \simeq \widetilde{E}(G/N) \wedge H_G\underline{\mathbb{Z}} \wedge_{\widetilde{E}(G/N) \wedge q^* H_{G/N}\underline{\mathbb{Z}}} \widetilde{E}(G/N) \wedge q^*(X).$$

It remains only to show that

$$\widetilde{E}(G/N) \wedge H_G\underline{\mathbb{Z}} \simeq \widetilde{E}(G/N) \wedge q^* H_{G/N}\underline{\mathbb{Z}}.$$

Both sides restrict trivially to an N-equivariant spectrum, so it suffices to show an equivalence on Φ^H, where H properly contains N. Without loss of generality, we may suppose that $H = G$. Since $\Phi^G(\widetilde{E}(G/N)) \simeq S^0$, it suffices to show that

$$\Phi^G H_G\underline{\mathbb{Z}} \simeq \Phi^G q^* H_{G/N}\underline{\mathbb{Z}}.$$

According to Proposition 3.8, the left side is $\Phi^{G/N} H_{G/N}\underline{\mathbb{Z}}$. Similarly, Proposition 3.8 and the projection formula (Proposition 3.3) show that the right side is

$$\Phi^G q^* H_{G/N}\underline{\mathbb{Z}} \simeq \Phi^{G/N}\big(H_{G/N}\underline{\mathbb{Z}} \wedge (S_G^0)^N\big) \simeq \Phi^{G/N} H_{G/N}\underline{\mathbb{Z}} \wedge \Phi^{G/N}(S_G^0)^N \simeq \Phi^{G/N} H_{G/N}\underline{\mathbb{Z}}. \quad \square$$

We now establish our main result, which partially describes the slice filtration of inflated $H\underline{\mathbb{Z}}$-modules. Recall that $P_n^t X = \mathrm{fib}\, P^t X \to P^{n-1} X$ denotes the slice approximation of X that contains the slices of X in dimensions n to t.

Theorem 3.19 *Let $n \geq 0$ and let $N \trianglelefteq G$ be a bottleneck subgroup of order p, a prime. Let $\underline{M} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$ such that $P_n^n \Sigma^n H_{G/N}\underline{M}$ is of the form $\Sigma^V H_{G/N}\underline{L}$, for some G/N-representation V and $\underline{L} \in \mathrm{Mod}_{\underline{\mathbb{Z}}_{G/N}}$. Then the nontrivial slices of the Eilenberg–Mac Lane G-spectrum $\Sigma^n H_G(\Psi_N^* \underline{M})$, above level pn, are*

$$P_{pk}^{pk}\big(\Sigma^n H_G(\Psi_N^*\underline{M})\big) \simeq \Psi_N^* P_k^k(\Sigma^n H_{G/N}\underline{M}) \simeq \phi_N^* P_k^k(\Sigma^n H_{G/N}\underline{M})$$

for $k > n$. Furthermore,

$$P_n^{pk}\big(\Sigma^n H_G(\Psi_N^*\underline{M})\big) \simeq \Psi_N^* P_n^k(\Sigma^n H_{G/N}\underline{M})$$

for $k \geq n$.

Proof Applying the functor Ψ_N^* to the slice tower for $\Sigma^n H_{G/N}\underline{M}$ produces a tower of fibrations whose layers are $\Psi_N^* P_k^k(\Sigma^n H_{G/N}\underline{M})$ for $k \geq n$. We wish to show that this is a partial slice tower for

$\Sigma^n H_G(\Psi_N^* \underline{M})$. For $k > n$, the k-slice $P_k^k(\Sigma^n H_{G/N} \underline{M})$ has trivial underlying spectrum. It follows from Proposition 3.18 that

$$\Psi_N^* P_k^k(\Sigma^n H_{G/N} \underline{M}) \simeq \phi_N^* P_k^k(\Sigma^n H_{G/N} \underline{M})$$

for $k > n$. As the geometric inflation of a k-slice, this is a pk-slice.

It remains to show that

$$\Psi_N^* P_n^n(\Sigma^n H_{G/N} \underline{M}) \simeq \Psi_N^* \Sigma^V H_{G/N} \underline{L} \simeq \Sigma^V H_G \Psi_N^* \underline{L}$$

has no slices above level pn. First, note that the restriction of $\Sigma^V H_G \Psi_N^* \underline{L}$ to N is the N-spectrum $\Sigma^n H_N \underline{L(N)}$, where $\underline{L(N)}$ is being considered as a constant N-Mackey functor at the value $\underline{L}(G/N)$. It follows that this N-spectrum has no slices above dimension $|N| \cdot n = pn$. Therefore, to show that $\Sigma^V H_G \Psi_N^* \underline{L}$ is less than or equal to pn, it suffices to show that

$$[G_+ \wedge_H S^{k\rho_H + r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G = 0$$

for any $N < H \leq G$ and integers $r \geq 0$ and k such that $k|H| > pn$. Without loss of generality we consider the case $H = G$.

Denote by U a complement of $\rho_{G/N}$ in ρ_G, so that

$$\rho_G \cong \rho_{G/N} \oplus U.$$

We then have a cofiber sequence

$$S(kU)_+ \wedge S^{k\rho_{G/N}} \to S^{k\rho_{G/N}} \to S^{k\rho_G}$$

and a resulting exact sequence

$$\begin{aligned}[][\Sigma^1 S(kU)_+ \wedge S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G &\to [S^{k\rho_G + r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G \\ &\to [S^{k\rho_{G/N}+r}, \Sigma^V H_G \Psi_N^* \underline{L}]^G = 0.\end{aligned}$$

We must show that the left term vanishes. Note that the G-action on $S(kU)$ is free, since N is order p. Then the desired vanishing follows from the fact that $\Sigma^1 S(kU)_+ \wedge S^{k\rho_{G/N} - V}$ is G-connected, since $\dim k\rho_{G/N} > \dim V = n$. ◻

Remark 3.20 While we have chosen to state Theorem 3.19 in the case of $n \geq 0$, an analogous result holds for negative suspensions. There, we have

$$P_{-pk}^{-n}(\Sigma^{-n} H_G(\Psi_N^* \underline{M})) \simeq \Psi_N^* P_{-k}^{-n}(\Sigma^{-n} H_{G/N} \underline{M})$$

for $k \geq n > 0$. That $\Psi_N^* P_{-n}^{-n}(\Sigma^{-n} H_{G/N} \underline{M}) \simeq \Sigma^{-V} H_G \Psi_N^* \underline{L}$ is at least $-pn$ can be seen from Hill and Yarnall's characterization of slice connectivity in terms of geometric fixed points [13, Theorem 2.5].

4 Q_8-Mackey functors and Bredon homology

We display a number of the Q_8-Mackey functors that will be relevant in Table 3. In these Lewis diagrams, we are using the subgroup lattice of Q_8 as displayed in Section 1.1. We will also often abuse notation and

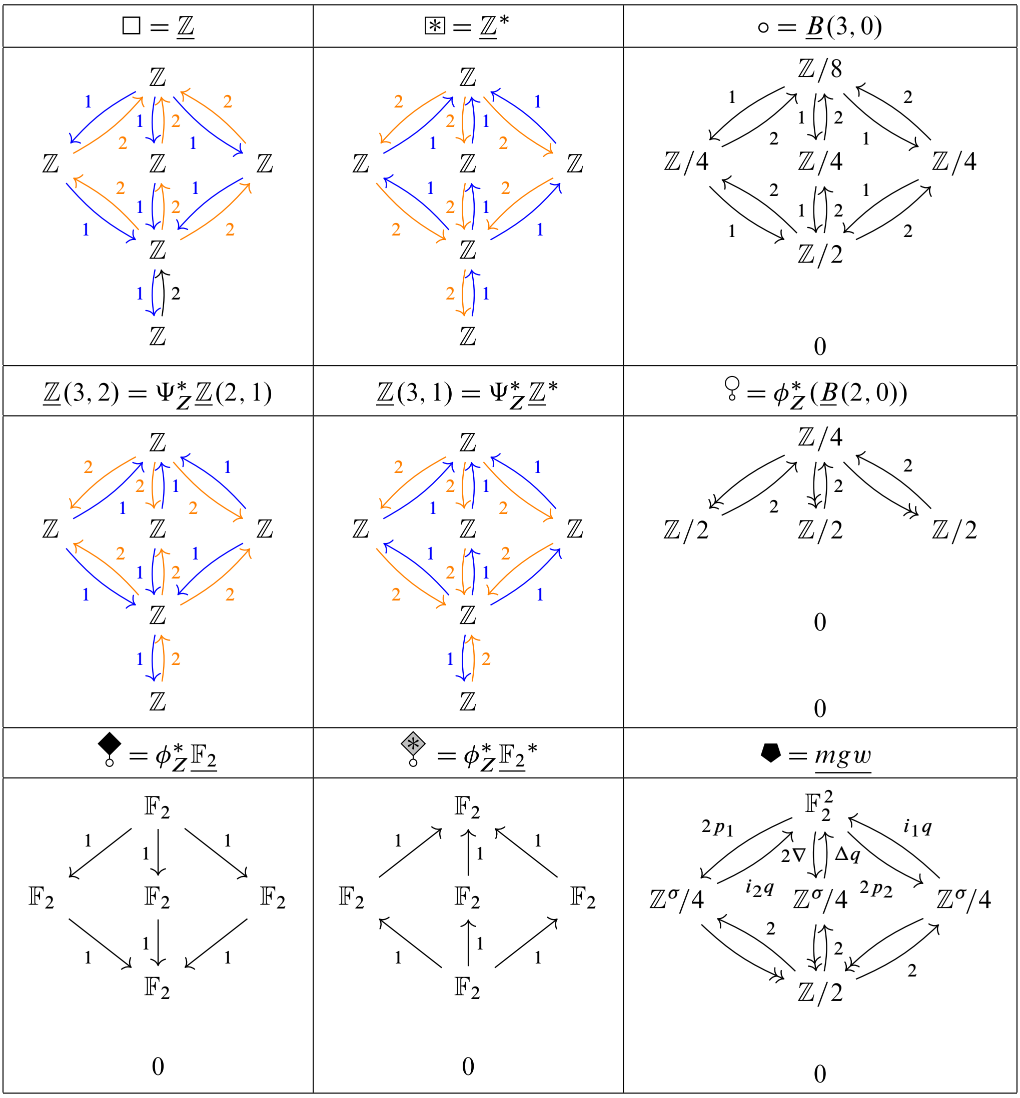


Table 3: Some Q_8-Mackey functors.

write the name for a K_4-Mackey functor, such as $\underline{m}$ or $\underline{mg}$, to denote the resulting inflated Q_8-Mackey functor. We will only write the symbol ϕ_Z^* when it is necessary to resolve an ambiguity, for instance between $\phi_Z^*\underline{\mathbb{F}_2}$ and $\underline{\mathbb{F}_2}$.

In [10, Section 2.1], the authors introduce "forms of $\underline{\mathbb{Z}}$" Mackey functors $\underline{\mathbb{Z}}(i,j)$, where $i \geq j \geq 0$, in the case of $G = C_{p^n}$. From our point of view, Q_8 behaves very similarly to C_8, and we similarly write $\underline{\mathbb{Z}}(i,j)$ for the Mackey functor that looks like $\underline{\mathbb{Z}}^*$ between the subgroups of order 2^i and 2^j and looks like $\underline{\mathbb{Z}}$ outside of this range. We will at times follow [10] in denoting by $\underline{B}(i,j)$ the cokernel of $\underline{\mathbb{Z}}(i,j) \hookrightarrow \underline{\mathbb{Z}}$, although we will often instead use the descriptions given in Proposition 4.1.

These Mackey functors fit together in exact sequences as follows, using notation from Table 3:

Proposition 4.1 *There are exact sequences of Mackey functors*

(1) $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$,
(2) $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{B}(2,0)$,
(3) $\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^*$,
(4) $\underline{\mathbb{Z}}(2,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{m}$,
(5) $\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}_2}$,
(6) $\underline{\mathbb{Z}}^* \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{B}(3,0)$,
(7) $\underline{mg} \hookrightarrow \underline{mgw} \twoheadrightarrow \underline{w}$.

4.1 $RO(Q_8)$-graded Mackey functor $\underline{\mathbb{Z}}$-homology of a point

We will now compute the homology of $S^{k\rho_Q}$, with coefficients in $\underline{\mathbb{Z}}$, as a Mackey functor. The starting point is that the regular representation of Q splits as

$$\rho_Q \cong \mathbb{H} \oplus \rho_K,$$

where $\mathbb{H}$ is the 4-dimensional irreducible Q-representation given by the action of the unit quaternions on the algebra of quaternions and ρ_K is the regular representation of K, inflated to Q along the quotient. We begin by computing the homology of $S^{k\mathbb{H}}$. See also [17, Section 2] for an alternative viewpoint.

First, Proposition 3.3 and [19, Proposition 9.1] combine to yield the following.

Proposition 4.2 *For $k \geq 0$, the nontrivial homotopy Mackey functors of $\Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(\Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 4k, \\ \underline{mg} & \text{if } n = 4k-2, \\ \underline{g}^{\frac{1}{2}(4k-n-1)} & \text{if } n \in [2k, 4k-3] \text{ and } n \text{ is odd}, \\ \underline{g}^{\frac{1}{2}(4k-n-4)} \oplus \phi_{LDR}^* \underline{\mathbb{F}_2} & \text{if } n \in [2k, 4k-3] \text{ and } n \text{ is even}, \\ \underline{g}^{n-k+1} & \text{if } n \in [k, 2k-1]. \end{cases}$$

Next, we employ the cofiber sequence

$$S(\mathbb{H})_+ \to S^0 \to S^{\mathbb{H}} \tag{4.3}$$

to obtain the homology of S^{ρ_Q} from that of S^{ρ_K}.

Proposition 4.4 *The nontrivial homotopy Mackey functors of $S(\mathbb{H})_+ \wedge H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(S(\mathbb{H})_+ \wedge H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 3, \\ \underline{mgw} & \text{if } n = 1, \\ \underline{\mathbb{Z}}^* & \text{if } n = 0. \end{cases}$$

Proof Since the action of Q on $S(\mathbb{H})$ is free, we can write down an equivariant cell structure using only free cells. Viewing $S(\mathbb{H})$ as the one-point compactification of $\mathbb{R}^3$, there is a straight-forward cell structure

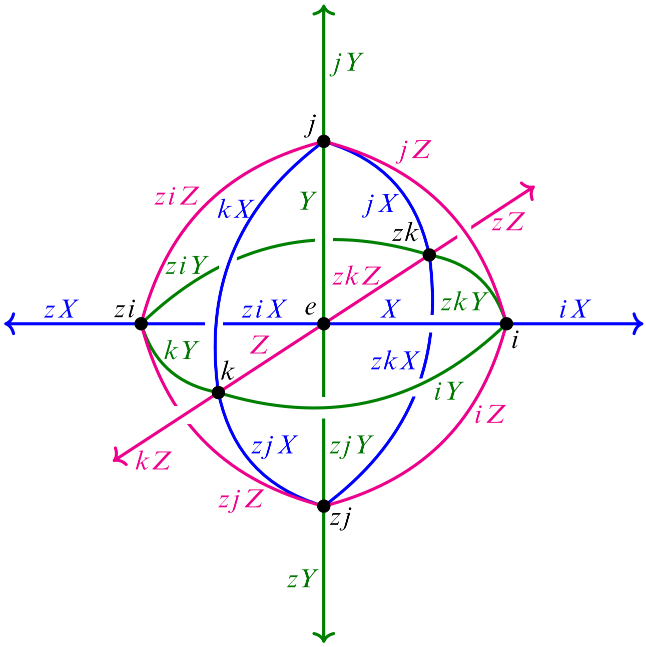


Figure 3: The 1-skeleton of $S(\mathbb{H})$.

in which the subgroups L, D, and R act freely on the x-, y-, and z-axes, respectively. We display the 1-skeleton in Figure 3, and the cell structure is described by the following complex of $\mathbb{Z}[Q]$-modules:

$$\mathbb{Z}[Q]^2 \xrightarrow{\begin{pmatrix} e & j \\ -e & -i \\ e & k \\ -e & -e \end{pmatrix}} \mathbb{Z}[Q]^4 \xrightarrow{\begin{pmatrix} k & e & e & k \\ -e & -e & i & i \\ e & -j & -e & j \end{pmatrix}} \mathbb{Z}[Q]^3 \xrightarrow{(i-e\ \ j-e\ \ k-e)} \mathbb{Z}[Q].$$

This yields an associated complex of induced Mackey functors

$$\underline{\mathbb{Z}[Q]}^2 \to \underline{\mathbb{Z}[Q]}^4 \to \underline{\mathbb{Z}[Q]}^3 \to \underline{\mathbb{Z}[Q]}$$

leading to the claimed homology Mackey functors. □

Remark 4.5 A smaller chain complex for computing the homology of $S(\mathbb{H})$ is given by

$$\mathbb{Z}[Q] \xrightarrow{\binom{i-e}{e-k}} \mathbb{Z}[Q]^2 \xrightarrow{\begin{pmatrix} e+i & e+k \\ -e-j & -e+i \end{pmatrix}} \mathbb{Z}[Q]^2 \xrightarrow{(i-e\ \ j-e)} \mathbb{Z}[Q].$$

We gave a less efficient chain complex in the proof of Proposition 4.4 for geometric reasons.

Using (4.3), this immediately yields the following.

Corollary 4.6 *The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}} H_Q \underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(\Sigma^{\mathbb{H}} H_Q \underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 4, \\ \underline{mgw} & \text{if } n = 2, \\ \underline{B}(3,0) & \text{if } n = 0. \end{cases}$$

We will use this to compute the homology of S^{ρ_Q}, using the following periodicity result.

Proposition 4.7 [23, Proposition 4.1] *For any orientable representation V of dimension d and free Q-space X, the orientation $u_V \in \mathrm{H}_d(S^V; \underline{\mathbb{Z}})$ induces an equivalence*

$$\Sigma^d X_+ \wedge H_Q\underline{\mathbb{Z}} \simeq \Sigma^V X_+ \wedge H_Q\underline{\mathbb{Z}}.$$

We now compute the homology of S^{ρ_Q}.

Proposition 4.8 *The nontrivial homotopy Mackey functors of $\Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(\Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 8, \\ \underline{mgw} & \text{if } n = 6, \\ \underline{B}(3,0) & \text{if } n = 4, \\ \underline{mg} & \text{if } n = 2, \\ \underline{g} & \text{if } n = 1. \end{cases}$$

Proof The representation ρ_K is orientable. For example, using the basis $\{1, i, j, k\}$ for $\rho_K = \mathbb{R}[K]$, the matrix $\rho_K(i)$ is given by

$$\rho_K(i) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which has determinant equal to 1. By Proposition 4.7, we have

$$\underline{\pi}_n\big(S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}}\big) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 7, \\ \underline{mgw} & \text{if } n = 5, \\ \underline{\mathbb{Z}}^* & \text{if } n = 4. \end{cases}$$

The result then follows from the cofiber sequence

$$S(\mathbb{H})_+ \wedge \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}. \qquad \square$$

Corollary 4.6 generalizes as follows.

Proposition 4.9 *The nontrivial homotopy Mackey functors of $\Sigma^{k\mathbb{H}} H_Q\underline{\mathbb{Z}}$, for $k > 0$ are*

$$\underline{\pi}_n(\Sigma^{k\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 4k, \\ \underline{mgw} & \text{if } 0 < n < 4k \text{ and } n \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } 0 \le n < 4k \text{ and } n \equiv 0 \pmod 4. \end{cases}$$

Proof This follows by induction, using the cofiber sequence

$$S(\mathbb{H})_+ \wedge S^{(k-1)\mathbb{H}} \to S^{(k-1)\mathbb{H}} \to S^{k\mathbb{H}}$$

and Proposition 4.7. The latter applies since $\mathbb{H}$, and therefore also $(k-1)\mathbb{H}$, is orientable. $\square$

Combining this with the cofiber sequence

$$S(k\mathbb{H})_+ \wedge \Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}} \to \Sigma^{k\rho_K} H_Q\underline{\mathbb{Z}} \to \Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$$

and Proposition 4.7 gives the following result.

Proposition 4.10 *The nontrivial homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$, for $k > 0$, are*

$$\underline{\pi}_n(\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n = 8k, \\ \underline{mgw} & \text{if } 4k < n < 8k \text{ and } n \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } 4k \le n < 8k \text{ and } n \equiv 0 \pmod 4, \\ \phi_Z^* \underline{\pi}_n(\Sigma^{k\rho_K} H_K\underline{\mathbb{Z}}) & \text{if } n < 4k, \end{cases}$$

where the latter Mackey functors are listed in Proposition 4.2.

The homotopy Mackey functors of $\Sigma^{k\rho_Q} H_Q\underline{\mathbb{Z}}$ are displayed in Figure 4. When k is negative, the computation follows the same strategy. The initial input, which can again be computed using the chain complex given in Proposition 4.4, is that

$$\text{(4.11)} \qquad \underline{\mathrm{H}}^n(S(\mathbb{H}); \underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}\big(F(S(\mathbb{H})_+, H_Q\underline{\mathbb{Z}})\big) \cong \begin{cases} \underline{\mathbb{Z}}^* & \text{if } n = 3, \\ \underline{mgw} & \text{if } n = 2, \\ \underline{\mathbb{Z}} & \text{if } n = 0. \end{cases}$$

Using this and [19, Proposition 9.2] leads to the following answer.

Proposition 4.12 *The nontrivial homotopy Mackey functors of $\Sigma^{-k\rho_Q} H_Q\underline{\mathbb{Z}}$, for $k > 0$, are*

$$\underline{\pi}_{-n}(\Sigma^{-k\rho_Q} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^* & \text{if } n = 8k, \\ \underline{mgw} & \text{if } n \in [4k, 8k] \text{ and } n \equiv 3 \pmod 4, \\ \underline{B}(3,0) & \text{if } n \in [4k+5, 8k] \text{ and } n \equiv 1 \pmod 4, \\ \phi_Z^* \underline{B}(2,0) & \text{if } n = 4k+1, \\ \underline{mg}^* & \text{if } n = 4k-1, \\ \underline{g}^{\frac{4k-n}{2}} & \text{if } n \in [2k+4, 4k-2] \text{ and } n \equiv 0 \pmod 2, \\ \underline{g}^{\frac{4k-n-3}{2}} \oplus \phi_{LDR}^* \underline{\mathbb{F}_2}^* & \text{if } n \in [2k+3, 4k-2] \text{ and } n \equiv 1 \pmod 2, \\ \underline{g}^{n-k-3} & \text{if } n \in [k+4, 2k+2]. \end{cases}$$

Remark 4.13 The "gap theorem" [9, Proposition 3.20] predicts that the groups $\pi_n^Q \Sigma^{-k\rho} H_Q\underline{\mathbb{Z}}$ vanish for $k \ge 0$ and $n \in [-3, -1]$, as indicated in Figure 4. Actually, for $k \ge 2$ the argument there proves more. It tells us that for $k \ge 2$, the cohomology groups $\mathrm{H}_Q^n(S^{k\rho}; \underline{M})$ vanish for positive $n \le k+1$. This is equivalent to saying that $\pi_{-n}^Q \Sigma^{-k\rho} H_Q\underline{M}$ vanishes, with the same conditions on k and n.

4.2 Additional homology calculations

We will also need the following auxiliary calculations in Section 6.

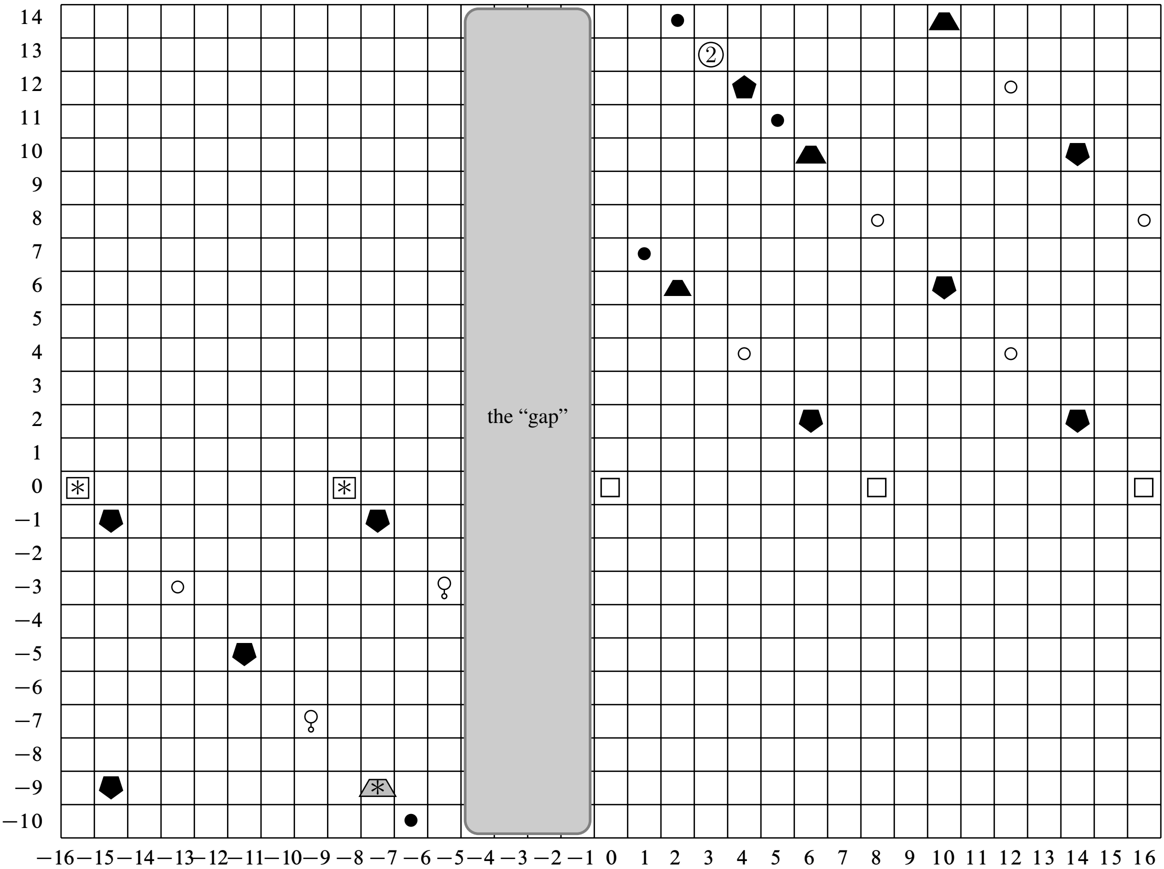


Figure 4: The homotopy Mackey functors of $\bigvee_n \Sigma^{n\rho} H_Q\underline{\mathbb{Z}}$. The Mackey functor $\underline{\pi}_k \Sigma^{n\rho} H_Q\underline{\mathbb{Z}}$ appears in position $(k, 8n-k)$.

Proposition 4.14 *The nontrivial homotopy Mackey functors of $\Sigma^{\rho_K - \mathbb{H}} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_n(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } n = 1, \\ \underline{\mathbb{Z}}^* & \text{if } n = 0. \end{cases}$$

Proof The fiber sequence

$$\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \to F(S(\mathbb{H})_+, \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}}) \simeq \Sigma^4 F(S(\mathbb{H})_+, H_Q\underline{\mathbb{Z}})$$

yields an isomorphism $\underline{\pi}_0(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}^*$ and shows that the homotopy vanishes for n outside of $[0, 2]$. Given that the restriction to any C_4, which is the C_4-spectrum $\Sigma^{2+2\sigma-2\lambda} H_{C_4}\underline{\mathbb{Z}}$, has a trivial $\underline{\pi}_2$ [26, Theorem 6.10], the long exact sequence further shows that $\underline{\pi}_2$ vanishes as well, and it implies that we have an extension

$$\underline{w} \hookrightarrow \underline{\pi}_1(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \twoheadrightarrow \underline{g}.$$

It remains to show this is not the split extension. The fiber sequence

$$\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda} H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{1+p_1^*\sigma+p_2^*\sigma-\mathbb{H}} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}$$

shows that $\underline{\pi}_1(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}})$ injects into

$$\underline{\pi}_0\big(\uparrow_D^Q \Sigma^{1+2\sigma-2\lambda} H_{C_4}\underline{\mathbb{Z}}\big) \cong \uparrow_D^Q \phi_{C_2}^* \underline{\mathbb{F}_2}.$$

It follows that $\underline{\pi}_1(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \phi_Z^*\underline{\mathbb{F}_2}$. □

Proposition 4.15 *The nontrivial homotopy Mackey functors of $\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)$ are*

$$\underline{\pi}_n\big(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)\big) \cong \begin{cases} \underline{w} & \text{if } n=1, \\ \underline{\mathbb{Z}}^* & \text{if } n=0. \end{cases}$$

Proof The short exact sequence

$$\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$$

gives rise to a cofiber sequence

$$\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2) \to \Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_K-\mathbb{H}} H_Q\underline{g} \simeq \Sigma^1 H_Q\underline{g}.$$

Using a naturality square, the second map factors as

$$\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}} \to \Sigma^{\rho_K} H_Q\underline{\mathbb{Z}} \to \Sigma^1 H_Q\underline{g},$$

where the first map is an epimorphism on $\underline{\pi}_1$ by the proof of Proposition 4.14 and the second is an isomorphism on $\underline{\pi}_1$. The conclusion follows. □

Proposition 4.16 *The nontrivial homotopy Mackey functors of $\Sigma^{\mathbb{H}-\rho_K} H_Q\underline{\mathbb{Z}}(2,0)$ are*

$$\underline{\pi}_n\big(\Sigma^{\mathbb{H}-\rho_K} H_Q\underline{\mathbb{Z}}(2,0)\big) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n=0, \\ \underline{w}^* & \text{if } n=-2. \end{cases}$$

Proof This follows from Proposition 4.15 by duality. In more detail, Proposition 4.15 gives a fiber sequence

$$\Sigma^1 H_Q\underline{w} \to \Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2) \to H_Q\underline{\mathbb{Z}}^*.$$

Applying Anderson duality (see [19, Section 2.2]) gives a fiber sequence

$$I(\Sigma^1 H_Q\underline{w}) \leftarrow I\big(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)\big) \leftarrow I(H_Q\underline{\mathbb{Z}}^*),$$

or in other words

$$\Sigma^{-1} I(H_Q\underline{w}) \leftarrow \Sigma^{\mathbb{H}-\rho_K} H_Q\underline{\mathbb{Z}}(2,0) \leftarrow H_Q\underline{\mathbb{Z}}.$$

But as the Mackey functor $\underline{w}$ is torsion, the Anderson dual is the desuspension of the Brown–Comenetz dual. In other words, $I(H_Q\underline{w}) \simeq \Sigma^{-1} I_{\mathbb{Q}/\mathbb{Z}} H_Q\underline{w} \simeq \Sigma^{-1} H_Q\underline{w}^*$. □

5 Review of the C_4-slices of $\Sigma^n H\mathbb{Z}$

In this section, we review the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ from [24]. Note that the slices as listed in [24] are written using the classical slice filtration, whereas we use the regular slice filtration. The only difference is a suspension by one. The Mackey functors that appear here were introduced in Table 1.

According to [24, Section 4.2], the C_4-spectrum $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is an n-slice for $0 \le n \le 4$. For $n \ge 5$, $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ has a nontrivial slice tower. Yarnall's method for determining these slice towers is to splice together suspensions of the cofiber sequences

$$\Sigma^{-1} H_{C_4}\underline{g} \to \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{2\sigma} H_{C_4}\underline{\mathbb{Z}},$$
$$\Sigma^{-1} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}^* \to \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda} H_{C_4}\underline{\mathbb{Z}}(2,1),$$
$$\Sigma^{-1} H_{C_4}\underline{B}(2,0) \to \Sigma^2 H_{C_4}\underline{\mathbb{Z}} \to \Sigma^{\lambda} H_{C_4}\underline{\mathbb{Z}},$$

in combination with the equivalences

$$\Sigma^2 H_{C_4}\underline{\mathbb{Z}} \simeq \Sigma^{2\sigma} H_{C_4}\underline{\mathbb{Z}}(2,1),$$
$$\Sigma^{-1} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}^* \simeq \Sigma^{-\sigma} H_{C_4}\phi^*_{C_2}\underline{f} \simeq \Sigma^{1-2\sigma} H_{C_4}\phi^*_{C_2}\underline{\mathbb{F}_2}.$$

We first review these slices for odd n.

Proposition 5.1 [24, Theorem 4.2.6] *Let $n \ge 5$ be odd. The bottom slice of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is*

$$P^n_n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-5}{4}\rho+4+\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 1 \pmod 8, \\ \Sigma^{\frac{n-3}{4}\rho+3} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 3 \pmod 8, \\ \Sigma^{\frac{n-5}{4}\rho+3+2\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 5 \pmod 8, \\ \Sigma^{\frac{n-3}{4}\rho+2+\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Proposition 5.2 [24, Lemma 4.2.5] *Let $n \ge 5$ be odd. The nontrivial $4k$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{4k}_{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{k\rho} H_{C_4}\underline{B}(2,0) & \text{if } 4k \in [n+1, 2(n-3)] \text{ and } k \text{ is even}, \\ \Sigma^{k\rho} H_{C_4}\phi^*\underline{f} & \text{if } 4k \in [n+1, 2(n-3)] \text{ and } k \text{ is odd}, \\ \Sigma^{k\rho} H_{C_4}\underline{g} & \text{if } 4k \in [2(n-1), 4(n-3)] \text{ and } k \text{ is even}. \end{cases}$$

The $4k$-slices can also be read off of [11, Figure 3]. When n is odd, these are the only nontrivial slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$.

We now recall the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ for even n.

Proposition 5.3 [24, Theorem 4.2.9] *Let $n \ge 6$ be even. The bottom slice of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ is*

$$P^n_n(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{n-4}{4}\rho+3+\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 0 \pmod 8, \\ \Sigma^{\frac{n-6}{4}\rho+3+3\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 2 \pmod 8, \\ \Sigma^{\frac{n-4}{4}\rho+4} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 4 \pmod 8, \\ \Sigma^{\frac{n-6}{4}\rho+4+2\sigma} H_{C_4}\underline{\mathbb{Z}} & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

Proposition 5.4 [24, Lemma 4.2.7] *Let $n \ge 6$ be even. The nontrivial $4k$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{4k}_{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^k H_{C_4}\underline{g}, \quad k \text{ odd}$$

for $4k$ in the range $[n+2, 4n-12]$.

Again, the $4k$-slices can also be read off of [11, Figure 3].

Proposition 5.5 [24, Theorem 4.2.9] *Let $n \geq 6$ be even. The $4k+2$-slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ are*

$$P^{8k+2}_{8k+2}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{1+2k\rho} H\phi^*\underline{\mathbb{F}_2},$$
$$P^{8k+6}_{8k+6}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Sigma^{3+2k\rho} H\phi^*\underline{\mathbb{F}_2},$$

for $8k+2$ or $8k+6$ in the range $[n+2, 2n-6]$.

We may also view these slices through the perspective of the $\underline{\mathbb{Z}}$-module inflation functor. By Theorem 3.19,

$$\Psi^*_{C_2} : \mathrm{Mod}_{H_{C_2}\underline{\mathbb{Z}}} \to \mathrm{Mod}_{H_{C_4}\underline{\mathbb{Z}}}$$

will provide all slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ above level $2n$. Let $r \equiv n \pmod 4$ with $3 \leq r \leq 6$. It follows from [19, Proposition 3.5] that the slices of $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$ in level at least $2n+2r-4$ are

$$P^{4k}_{4k}(\Sigma^n H_{C_4}\underline{\mathbb{Z}}) \simeq \Psi^*_{C_2}\Sigma^k H_{C_2}\underline{g} \simeq \Sigma^k H_{C_4}\underline{g}$$

for $4k \in [2n+2r-4, 4(n-3)]$. The rest of the slices then follow from determining the slices of

$$\Psi^*_{C_2} P^n_n \Sigma^n H_{C_2}\underline{\mathbb{Z}} = \Psi^*_{C_2}\Sigma^{\frac{n-r}{2}\rho_{C_2}+r} H_{C_2}\underline{\mathbb{Z}} \simeq \Sigma^{\frac{n+r}{2}+\frac{n-r}{2}\sigma} H_{C_4}\underline{\mathbb{Z}}.$$

The slice tower for this C_4-spectrum can be found by splicing together the cofiber sequences listed at the start of this section.

6 Q_8-slices

The slices of $\Sigma^n H_K\underline{\mathbb{Z}}$ were determined by the second author in [19, Section 8]. As stated in Theorem 3.19, it follows that the $\underline{\mathbb{Z}}$-module inflation functor

$$\Psi^*_Z : \mathrm{Mod}_{H_K\underline{\mathbb{Z}}} \to \mathrm{Mod}_{H_Q\underline{\mathbb{Z}}}$$

of Proposition 3.14 will produce all slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ in degree larger than $2n$, as the inflation of the slices of $\Sigma^n H_K\underline{\mathbb{Z}}$ above degree n.

The remaining slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ will be given as the slices of $\Psi^*_Z\big(P^n_n(\Sigma^n H_K\underline{\mathbb{Z}})\big)$. By [19, Proposition 8.5], these are of the form

$$\Psi^*_Z(\Sigma^{r+j\rho_K} H_K\underline{\mathbb{Z}}) \simeq \Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}}, \tag{6.1}$$

where $r \in \{3,4,5\}$, if $n \not\equiv 2 \pmod 4$. In the case $n \equiv 2 \pmod 4$, the same result states that this is

$$\Psi^*_Z\big(\Sigma^{2+j\rho_K} H_K\underline{\mathbb{Z}}(1,0)\big) \simeq \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1).$$

But the cofiber sequence (Proposition 4.1)

$$\Sigma^{1+j\rho_K} H_Q\underline{m} \to \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \to \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}} \tag{6.2}$$

reduces the computation of slices of $\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1)$ to the question of the slice tower for $\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}$, given that $\Sigma^{1+j\rho_K} H_Q\underline{m} \simeq \phi^*_Z(\Sigma^{1+j\rho_K} H_K\underline{m})$ is an $8j+4$-slice [19, Proposition 5.7]. We determine the slices of $\Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}}$, for $r \in \{2,\dots,5\}$ in Section 6.1.

6.1 Slice towers for $\Sigma^{r+j\rho_K} H_Q\underline{\mathbb{Z}}$

The K_4-spectrum $\Sigma^{r+j\rho_K} H_K\underline{\mathbb{Z}}$ is an n-slice for $r \in \{2,\dots,5\}$ [19, Proposition 7.1]. However, the inflation of this to Q_8 is no longer a slice. We here determine the slice towers of these inflations. Throughout, we will implicitly use Proposition 6.7, which does not rely on the following material.

6.1.1 $\boldsymbol{r=2}$ First, we observe that $\Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 6-slice. To see this we first note that it restricts to a 6-slice at every proper subgroup by Proposition 5.3. It therefore remains only to show that it does not have any $8k$-slices for $k \geq 1$. This is equivalent to showing that $\underline{\pi}_{-2}(\Sigma^{\rho_K - k\rho_Q} H_Q\underline{\mathbb{Z}})$ vanishes for $k \geq 1$. In the case $k=1$, (4.11) shows that $\Sigma^{-\mathbb{H}} H_Q\underline{\mathbb{Z}}$ is (-3)-truncated, in the sense that it has no homotopy Mackey functors above dimension -3. This remains true after further desuspending by copies of ρ_Q.

Next, the tower for $\Sigma^{2+2\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd}
P^{14}_{14} = \Sigma^{-1+2\rho_Q} H_Q\underline{w}^* \arrow[r] & \Sigma^{2+2\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{12}_{12} = \Sigma^{1+\rho_Q} H_Q\underline{m} \arrow[r] & \Sigma^{2+\rho_Q} H_Q\underline{\mathbb{Z}}(2,0) \arrow[d] \\
& P^{10}_{10} = \Sigma^{2+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)
\end{tikzcd}$$

This uses the computation (see Proposition 4.16)

$$\underline{\pi}_n\big(\Sigma^{\mathbb{H}-\rho_K} H_Q\underline{\mathbb{Z}}(2,0)\big) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } n=0, \\ \underline{w}^* & \text{if } n=-2, \end{cases}$$

to produce the first cofiber sequence.

Finally, for $j \geq 3$, the tower may be obtained by recursively using

$$\begin{tikzcd}
P^{8j-2}_{8j-2} = \Sigma^{-1+j\rho_Q} H_Q\underline{w}^* \arrow[r] & \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{8j-4}_{8j-4} = \Sigma^{1+(j-1)\rho_Q} H_Q\underline{m} \arrow[r] & \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}(2,0) \arrow[d] \\
P^{8j-6}_{8j-6} = \Sigma^{1+(j-1)\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) \arrow[d] \\
& \Sigma^{2+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}
\end{tikzcd}$$

We have proved the following result.

Proposition 6.3 *Let $j \geq 1$. The bottom slice of $\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{2+4j}_{2+4j}(\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{1+\rho_K+\frac{j-1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } j \text{ is odd,} \\ \Sigma^{2+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } j \text{ is even.} \end{cases}$$

6.1.2 $r = 3$ By (4.11), the cohomology of $S^{\mathbb{H}}$ is given by

$$\underline{\widetilde{\mathrm{H}}}^n(S^{\mathbb{H}};\underline{\mathbb{Z}}) \cong \underline{\pi}_{-n}(\Sigma^{-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}}^* & \text{if } n = 4, \\ \underline{mgw} & \text{if } n = 3. \end{cases}$$

Suspending by $3 + \rho_Q$ leads to the cofiber sequence

$$\begin{tikzcd} P_8^8 = \Sigma^{\rho_Q} H_Q\underline{mgw} \arrow[r] & \Sigma^{3+\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\ & P_7^7 = \Sigma^{\rho_Q - 1} H_Q\underline{\mathbb{Z}}^* \end{tikzcd}$$

The tower for $\Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}}$, where $j \geq 2$, is then given recursively by

$$\begin{tikzcd} P_{8j}^{8j} = \Sigma^{j\rho_Q} H_Q\underline{mgw} \arrow[r] & \Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\ & \Sigma^{(j-1)\rho_K+\rho_Q-1} H_Q\underline{\mathbb{Z}}^* \arrow[d, equal] \\ P_{8j-4}^{8j-4} = \Sigma^{2+(j-1)\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{3+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) \arrow[d] \\ & \Sigma^{3+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}} \end{tikzcd}$$

The last cofiber sequence arises from Proposition 4.1. We have proved the following result.

Proposition 6.4 *Let $j \geq 1$. The bottom slice of $\Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P_{3+4j}^{3+4j}(\Sigma^{3+j\rho_K} H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{-1+\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } j \text{ is odd}, \\ \Sigma^{3+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } j \text{ is even}. \end{cases}$$

6.1.3 $r = 4$ The tower for $\Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd} P_{12}^{12} = \Sigma^{\rho_Q+1} H_Q\underline{mg} \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\ P_{10}^{10} = \Sigma^{\rho_Q+1}\underline{w} \arrow[r] & \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\ & P_8^8 = \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}^* \end{tikzcd}$$

This uses the short exact sequence (Proposition 4.1)

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}}(3,2) \twoheadrightarrow \underline{m}^*,$$

the equivalence $\Sigma^{\rho_K} H_K\underline{m}^* \simeq \Sigma^2 H_K\underline{mg}$ [7, Proposition 4.8], and the computation (see Proposition 4.15)

$$\underline{\pi}_n\big(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}(3,2)\big) \cong \begin{cases} \underline{w} & \text{if } n = 1, \\ \underline{\mathbb{Z}}^* & \text{if } n = 0. \end{cases}$$

The tower for $\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}$, where $j \ge 2$, may then be obtained recursively from

$$\begin{tikzcd}
P^{8j+4}_{8j+4} = \Sigma^{1+j\rho_Q} H_Q\underline{mg} \arrow[r] & \Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{8j+2}_{8j+2} = \Sigma^{1+j\rho_Q}\underline{w} \arrow[r] & \Sigma^{(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\
 & \Sigma^{(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}^* \arrow[d, equal] \\
P^{8j-2}_{8j-2} = \Sigma^{3+(j-1)\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{4+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) \arrow[d] \\
 & \Sigma^{4+(j-2)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}
\end{tikzcd}$$

Proposition 6.5 *Let $j \ge 1$. The bottom slice of $\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{4+4j}_{4+4j}(\Sigma^{4+j\rho_K} H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } j \text{ is odd,} \\ \Sigma^{4+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } j \text{ is even.} \end{cases}$$

6.1.4 $\boldsymbol{r = 5}$ Here, we start with the slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$, as this is not a slice. The short exact sequence

$$\underline{\mathbb{Z}}(3,1) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^*\underline{B}(2,0)$$

gives rise to a cofiber sequence

$$P^8_8 = \Sigma^{\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \to \Sigma^5 H_Q\underline{\mathbb{Z}} \simeq \Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \to \Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}.$$

Now the argument showing that $\Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 6-slice, given above in Section 6.1.1, also applies to show that $\Sigma^{1+\rho_K} H_Q\underline{\mathbb{Z}}$ is a 5-slice. Thus, this cofiber sequence is the slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$.

Next, the tower for $\Sigma^{5+\rho_K} H_Q\underline{\mathbb{Z}}$ is given by

$$\begin{tikzcd}
P^{16}_{16} = \Sigma^{2\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \arrow[r] & \Sigma^{5+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{1+2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{12}_{12} = \Sigma^{2+\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{1+2\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
 & P^9_9 = \Sigma^{1+\rho_Q} H_Q\underline{\mathbb{Z}}^*
\end{tikzcd}$$

where the bottom cofiber sequence arises from the computation (Proposition 4.14)

$$\underline{\pi}_n(\Sigma^{\rho_K-\mathbb{H}} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \phi_Z^*\underline{\mathbb{F}_2} & \text{if } n = 1, \\ \underline{\mathbb{Z}}^* & \text{if } n = 0. \end{cases}$$

The tower for $\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}$, where $j \geq 2$, may then be obtained recursively from

$$\begin{tikzcd}
P^{8j+8}_{8j+8} = \Sigma^{(j+1)\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) \arrow[r] & \Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d, equal] \\
 & \Sigma^{1+(j+1)\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{8j+4}_{8j+4} = \Sigma^{2+j\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{1+(j+1)\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{8j}_{8j} = \Sigma^{j\rho_Q} H_Q\underline{B}(3,0) \arrow[r] & \Sigma^{1+(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}^* \arrow[d] \\
 & \Sigma^{1+(j-1)\rho_K+\rho_Q} H_Q\underline{\mathbb{Z}}
\end{tikzcd}$$

Proposition 6.6 *Let $j \geq 1$. The bottom slice of $\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}$ is*

$$P^{5+4j}_{5+4j}(\Sigma^{5+j\rho_K} H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{1+\frac{j+1}{2}\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } j \text{ is odd,} \\ \Sigma^{1+\rho_K+\frac{j}{2}\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } j \text{ is even.} \end{cases}$$

6.2 Slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$

In this section, we describe all slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ for $n \geq 0$.

Proposition 6.7 *The Q_8-spectrum $\Sigma^n H_Q\underline{\mathbb{Z}}$ is an n-slice for $0 \leq n \leq 4$.*

Proof Since this is true after restricting to any C_4 (see Section 5), any higher slices would necessarily be geometric and therefore occur in slice dimension at least 8. But we can show directly that $\Sigma^n H_Q\underline{\mathbb{Z}} < 8$ if $n \in [0,4]$. This follows from the vanishing of $\pi_{\rho_Q}\Sigma^n H_Q\underline{\mathbb{Z}} \cong \pi_{-n}\Sigma^{-\rho_Q} H_Q\underline{\mathbb{Z}}$ as displayed in Figure 4. □

It remains to determine the slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ when $n \geq 5$. Note that Theorem 3.19 applies by [19, Proposition 8.5]. We first describe the bottom slice.

Proposition 6.8 (the n-slice) *For $n \geq 5$, write $n = 8k + r$, where $r \in [5,12]$. Then the n-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is*

$$P^n_n(\Sigma^n H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{1+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } r = 5, \\ \Sigma^{2+\rho_K+k\rho_Q} H_Q\underline{\mathbb{Z}}(3,2) & \text{if } r = 6, \\ \Sigma^{-1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } r = 7, \\ \Sigma^{(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } r = 8, \\ \Sigma^{1+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}^* & \text{if } r = 9, \\ \Sigma^{2+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}}(1,0) & \text{if } r = 10, \\ \Sigma^{3+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } r = 11, \\ \Sigma^{4+(k+1)\rho_Q} H_Q\underline{\mathbb{Z}} & \text{if } r = 12. \end{cases}$$

Proof By Theorem 3.19, the n-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is the n-slice of the $\underline{\mathbb{Z}}$-module inflation of the n-slice of $\Sigma^n H_K\underline{\mathbb{Z}}$. By [19, Proposition 8.5], writing $n = 4j + r_4$ with $r_4 \in \{2,3,4,5\}$, we have

$$\Psi_Z^* P_n^n(\Sigma^n H_{K_4}\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) & \text{if } n \equiv 2 \pmod 4, \\ \Sigma^{r_4+j\rho_K} H_Q\underline{\mathbb{Z}} & \text{otherwise.} \end{cases}$$

If $n \not\equiv 2 \pmod 4$, the slice tower was given in Section 6.1. For the case of $n \equiv 2$, since $\Sigma^{1+j\rho_K} H_Q\underline{m} \simeq \phi_Z^*(\Sigma^{1+j\rho_K} H_K\underline{m})$ is an $8j+4$-slice [19, Proposition 5.7], the cofiber sequence (Proposition 4.1)

$$\Sigma^{1+j\rho_K} H_Q\underline{m} \to \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \to \Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}, \tag{6.9}$$

combines with the work of Section 6.1.1 to show that

$$P_n^n\big(\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}(2,1)\big) \simeq P_n^n(\Sigma^{2+j\rho_K} H_Q\underline{\mathbb{Z}}).$$

The latter is given in Proposition 6.3. □

Proposition 6.10 (the $8k$-slices) *For $n \ge 5$ and $8k > n$, the $8k$-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is*

$$P_{8k}^{8k}(\Sigma^n H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^k H_Q\underline{g}^{n-k-3} & \text{if } 8k \in [4n-8, 8n-32], \\ \Sigma^{k\rho_Q} H_Q\underline{g}^{\frac{4k-n}{2}} & \text{if } 8k \in [2n+4, 4n-16] \text{ and } n \equiv 0 \pmod 2, \\ \Sigma^{k\rho_Q} H_Q\underline{g}^{\frac{4k-n-3}{2}} \oplus \phi_{LDR}^*\underline{\mathbb{F}_2}^* & \text{if } 8k \in [2n+4, 4n-12] \text{ and } n \equiv 1 \pmod 2, \\ \Sigma^{k\rho_Q} H_Q\underline{mg}^* & \text{if } 8k = 2n+2, \\ \Sigma^{k\rho_Q} H_Q\phi_Z^*\underline{B}(2,0) & \text{if } 8k = 2n-2, \\ \Sigma^{k\rho_Q} H_Q\underline{B}(3,0) & \text{if } 8k \in [n+3, 2n-10] \text{ and } n \equiv 1 \pmod 4, \\ \Sigma^{k\rho_Q} H_Q\underline{mgw} & \text{if } 8k \in [n+1, 2n] \text{ and } n \equiv 3 \pmod 4. \end{cases}$$

Proof This is a translation of Proposition 4.12. Alternatively, the slices above dimension $2n$ follow from Theorem 3.19 and [19, Proposition 8.6]. The slices in dimensions $2n$ and lower follow from the towers computed in Section 6.1. □

Proposition 6.11 (the $8k+4$-slices) *For $n \ge 5$ and $8k+4 > n$, the $8k+4$-slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$ are*

$$P_{8k+4}^{8k+4}(\Sigma^n H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{3+k\rho_Q} H_Q\phi_{LDR}^*\underline{\mathbb{F}_2} & \text{if } 8k+4 \in [2n+4, 4n-12] \text{ and } n \text{ is even}, \\ \Sigma^{2+k\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} & \text{if } 8k+4 \in [n+1, 2n-4] \text{ and } n \text{ is odd}, \\ \Sigma^{1+k\rho_Q} H_Q\underline{m} & \text{if } 8k+4 \in [n+2, 2n] \text{ and } n \equiv 2 \pmod 4, \\ \Sigma^{1+k\rho_Q} H_Q\underline{mg} & \text{if } 8k+4 \in [n+4, 2n-4] \text{ and } n \equiv 0 \pmod 4. \end{cases}$$

Proof The first case follows from [19, Proposition 8.7]. The remaining cases follow from (6.9) and Section 6.1. □

Proposition 6.12 (the $4k+2$-slices) *Let $n \ge 5$. If n is odd, then $\Sigma^n H_Q\underline{\mathbb{Z}}$ has no nontrivial $4k+2$-slices if $4k+2 > n$. If n is even and $8k+2 > n$, then the $8k+2$-slice of $\Sigma^n H_Q\underline{\mathbb{Z}}$ is nontrivial only if*

$8k+2 \in [n+1, 2n]$, *in which case the slice is*

$$P^{8k+2}_{8k+2}(\Sigma^n H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{1+k\rho_Q} H_Q\underline{w} & \text{if } n \equiv 0 \pmod 4, \\ \Sigma^{1+k\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2} & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

Similarly, if n is even and $8k-2 > n$, the $8k-2$-slice is nontrivial only if $8k-2 \in [n+1, 2n]$, in which case the slice is

$$P^{8k-2}_{8k-2}(\Sigma^n H_Q\underline{\mathbb{Z}}) \simeq \begin{cases} \Sigma^{-1+k\rho_Q} H_Q\phi_Z^*\underline{\mathbb{F}_2}^* & \text{if } n \equiv 0 \pmod 4), \\ \Sigma^{-1+k\rho_Q} H_Q\underline{w}^* & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

Proof According to [19], the K_4-spectrum $\Sigma^n H_K\underline{\mathbb{Z}}$ does not have any nontrivial slices in odd dimensions, except for the n-slice. By Theorem 3.19, this implies that $\Sigma^n H_Q\underline{\mathbb{Z}}$ does not have any $4k+2$-slices above dimension $2n$. The slices in dimensions below $2n$ are given in Section 6.1, using (6.1). $\square$

6.3 Slice towers for $\Sigma^n H_Q\underline{\mathbb{Z}}$

By Proposition 6.7, $\Sigma^n H_Q\underline{\mathbb{Z}}$ is an n-slice for $n \in \{0, \ldots, 4\}$. The slice tower for $\Sigma^5 H_Q\underline{\mathbb{Z}}$ was given in Section 6.1.4. We now display a few more examples of slice towers.

Example 6.13 The slice tower for $\Sigma^6 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P^{16}_{16} = \Sigma^2 H_Q\underline{g} \arrow[r] & \Sigma^6 H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{12}_{12} = \Sigma^{1+\rho} H_Q\underline{m} \arrow[r] & \Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
 & P^6_6 = \Sigma^{2+\rho_K} H_Q\underline{\mathbb{Z}}
\end{tikzcd}$$

This follows immediately from combining [19, Example 8.2], (6.9), and Section 6.1.1.

Example 6.14 The slice tower for $\Sigma^7 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P^{24}_{24} = \Sigma^3 H_Q\underline{g} \arrow[r] & \Sigma^7 H_Q\underline{\mathbb{Z}} \arrow[d] \\
P^{16}_{16} = \Sigma^{2+\rho_Q} H_Q\underline{m} \arrow[r] & \Sigma^{3+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
P^8_8 = \Sigma^{\rho_Q} H_Q\underline{mgw} \arrow[r] & \Sigma^{3+\rho_K} H_Q\underline{\mathbb{Z}} \arrow[d] \\
 & P^7_7 = \Sigma^{\rho_Q-1} H_{Q_8}\underline{\mathbb{Z}}^*
\end{tikzcd}$$

This follows immediately from combining [19, Example 8.3] and Section 6.1.2.

Example 6.15 The slices, but not the slice tower, for $\Sigma^8 H_K\underline{\mathbb{Z}}$ were determined in [19, Section 8]. Let us denote by F the fiber of the map $H_Q\underline{\mathbb{Z}} \to H_Q\phi_{LDR}\underline{\mathbb{F}_2}$ induced by the map of Q_8-Mackey functors $\underline{\mathbb{Z}} \to \phi_{LDR}\underline{\mathbb{F}_2}$ that is surjective at L, D, and R. Then the nontrivial homotopy Mackey functors of F are $\underline{\pi}_0(F) \simeq \underline{\mathbb{Z}}(2,1)$ and $\underline{\pi}_{-1}(F) \cong \underline{g}^2$. The slice tower for $\Sigma^8 H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd}
P^{32}_{32} = \Sigma^4 H_Q\underline{g} \arrow[r] & \Sigma^8 H_Q\underline{\mathbb{Z}} \simeq \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{24}_{24} = \Sigma^3 H_Q\underline{g}^2 \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}}(2,1) \arrow[d] \\
P^{20}_{20} = \Sigma^{3+\rho_Q} H_Q\phi^*_{LDR}\underline{\mathbb{F}_2} \arrow[r] & \Sigma^{4+\rho_K} F \arrow[d] \\
P^{12}_{12} = \Sigma^{1+\rho_Q} H_Q\underline{mg} \arrow[r] & \Sigma^{4+\rho_K} H_Q\underline{\mathbb{Z}} \simeq \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,1) \arrow[d] \\
P^{10}_{10} = \Sigma^{\rho_Q+1}\underline{w} \arrow[r] & \Sigma^{2\rho_K} H_Q\underline{\mathbb{Z}}(3,2) \arrow[d] \\
 & P^8_8 = \Sigma^{\rho_Q} H_Q\underline{\mathbb{Z}}^*
\end{tikzcd}$$

where the bottom of the tower comes from Section 6.1.3.

7 Homology calculations

In Section 6, we described the slices of $\Sigma^n H_Q\underline{\mathbb{Z}}$. In Section 8 below, we will give the corresponding slice spectral sequences. The E_2-pages of those spectral sequences are given by the homotopy Mackey functors of the slices. We describe those homotopy Mackey functors here.

7.1 The *n*-slice

We start with the n-slices in the order listed in Proposition 6.8. The homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}$ were calculated in Proposition 4.10. We use the same methods to determine the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}$.

Proposition 7.1 *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K+j\rho_Q} H_Q\underline{\mathbb{Z}}) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } i = 8j+4, \\ \underline{mgw} & \text{if } i \in [4j+4, 8j+3] \text{ and } i \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } i \in [4j+4, 8j+3] \text{ and } i \equiv 0 \pmod 4, \\ \phi_Z^*\underline{\pi}_i(\Sigma^{(j+1)\rho_K} H_K\underline{\mathbb{Z}}) & \text{if } i \in [j+1, 4j+3]. \end{cases}$$

See Proposition 4.2 or Figure 1 for the homotopy Mackey functors of $\Sigma^{(j+1)\rho_K} H_K\underline{\mathbb{Z}}$.

We may now use Proposition 7.1 and the exact sequence $\underline{\mathbb{Z}}(3,2) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \underline{g}$ to get the homotopy Mackey functors of $\Sigma^{\rho_K + j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)$.

Proposition 7.2 *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{\rho_K + j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K + j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } i = 8j+4, \\ \underline{mgw} & \text{if } i \in [4j+4, 8j+3] \text{ and } i \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } i \in [4j+4, 8j+3] \text{ and } i \equiv 0 \pmod 4, \\ \phi_Z^* \underline{\pi}_i(\Sigma^{(j+1)\rho_K} H_K\underline{\mathbb{Z}}) & \text{if } i \in [j+2, 4j+3]. \end{cases}$$

The key point here is that the homotopy Mackey functors of $\Sigma^{\rho_K + j\rho_Q} H_Q\underline{\mathbb{Z}}(3,2)$ are the same as that of $\Sigma^{\rho_K + j\rho_Q} H_Q\underline{\mathbb{Z}}$, except that the $\underline{g}$ in degree $j+1$ has been removed.

In Proposition 4.12 we list the homotopy Mackey functors of $\Sigma^{-j\rho_Q} H_Q\underline{\mathbb{Z}}$. Anderson duality then provides us with the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*$.

Proposition 7.3 *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}^*) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } i = 8j, \\ \underline{mgw} & \text{if } i \in [4j+1, 8j-1] \text{ and } i \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } i \in [4j+1, 8j-1] \text{ and } i \equiv 0 \pmod 4, \\ \phi_Z^* \underline{B}(2,0) & \text{if } i = 4j, \\ \phi_Z^* \underline{\pi}_{i-4}(\Sigma^{(j-1)\rho_K} H_K\underline{\mathbb{Z}}) & \text{if } i \in [j+3, 4j-1]. \end{cases}$$

Finally, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)$ follow from the exact sequence

$$\underline{\mathbb{Z}}(1,0) \hookrightarrow \underline{\mathbb{Z}} \twoheadrightarrow \phi_Z^* \underline{\mathbb{F}_2}.$$

Proposition 7.4 *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{\mathbb{Z}}(1,0)) \cong \begin{cases} \underline{\mathbb{Z}} & \text{if } i = 8j, \\ \underline{mgw} & \text{if } i \in [4j+1, 8j-2] \text{ and } i \equiv 2 \pmod 4, \\ \underline{B}(3,0) & \text{if } i \in [4j+1, 8j-2] \text{ and } i \equiv 0 \pmod 4, \\ \phi_Z^* \underline{B}(2,0) & \text{if } i = 4j, \\ \phi_Z^* \underline{\pi}_i(\Sigma^{j\rho_K} H_K\underline{\mathbb{Z}}) & \text{if } i \in [j, 4j-1]. \end{cases}$$

7.2 The $8k$-slices

We now move on to the $8k$-slices.

Proposition 7.5 *For $j = 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\phi_Z^* \underline{B}(2,0)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_Q} H_Q\phi_Z^* \underline{B}(2,0)) \cong \begin{cases} \underline{mg} & \text{if } i = 2, \\ \underline{g} & \text{if } i = 1. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0)) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j+2, 2j-1], \\ \underline{g}^2 & \text{if } i = j+1, \\ \underline{g} & \text{if } i = j. \end{cases}$$

Proof Because $\phi_Z^* \underline{B}(2,0)$ is a pullback,

$$\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{B}(2,0) \simeq \Sigma^{j\rho_K} H_Q \phi_Z^* \underline{B}(2,0).$$

The exact sequence of K-Mackey functors $\underline{m}^* \to \underline{B}(2,0) \to \underline{g}$ provides us with

$$\Sigma^{j\rho_K} H_K \underline{m}^* \to \Sigma^{j\rho_K} H_K \underline{B}(2,0) \to \Sigma^{j\rho_K} H_K \underline{g}.$$

The conclusion follows from [7, Propositions 4.8 and 7.4] and the resulting long exact sequence in homotopy. □

We may again use this strategy of reducing the calculations from Q to K for determining the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$.

Proposition 7.6 *For $j = 1$ the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are*

$$\underline{\pi}_i(\Sigma^{\rho_K} H_K \underline{B}(3,0)) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4, \\ \underline{mg} & \text{if } i = 2, \\ \underline{g} & \text{if } i = 1. \end{cases}$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{B}(3,0)) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4j, \\ \underline{mg} & \text{if } i = 4j-1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-2-i} & \text{if } i \in [2j+2, 4j-2], \\ \underline{g}^{2(k-2)+1} & \text{if } i = 2j+1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{2(j-3)+1} & \text{if } i = 2j, \\ \underline{g}^{2(i-j-1)} & \text{if } i \in [j+3, 2j-1], \\ \underline{g}^{i-j+1} & \text{if } i \in [j, j+2]. \end{cases}$$

Proof Because the underlying spectrum of $H_Q \underline{B}(3,0)$ is contractible,

$$\Sigma^{\rho_Q} H_Q \underline{B}(3,0) \simeq \Sigma^{\rho_K} H_Q \underline{B}(3,0).$$

Now, we may consider $\underline{B}(3,0)$ as a pullback $\phi_Z^* B := \underline{B}(3,0)$; thus the calculation is reduced to one of K-Mackey functors. The sequence of K-Mackey functors $\underline{\mathbb{Z}}^* \xrightarrow{2} \underline{\mathbb{Z}} \to \underline{B}$ provides us with

$$\Sigma^{j\rho_K} H_K \underline{\mathbb{Z}}^* \to \Sigma^{j\rho_K} H_K \underline{\mathbb{Z}} \to \Sigma^{j\rho_K} H_K \underline{B}.$$

Except for $i = 4j-2$, the result follows from the associated long exact sequence in homotopy. In degree $4j-2$ we have an extension

$$\underline{mg} \to \underline{\pi}_{4j-2}(\Sigma^{j\rho_K} H\underline{B}) \to \underline{g}.$$

We need to show this is not the split extension. This follows from the exact sequence $\underline{B}(2,0) \to \underline{B} \to \underline{\mathbb{F}_2}$ of K-Mackey functors. □

Proposition 7.7 *For $j=1$ and $j=2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{mgw}$ are*

$$\underline{\pi}_i(\Sigma^{\rho_Q} H_Q\underline{mgw}) \cong \begin{cases} \phi_Z^*\underline{\mathbb{F}_2} & \text{if } i=4, \\ \phi_Z^*\underline{B}(2,0) & \text{if } i=2. \end{cases}$$

and

$$\underline{\pi}_i(\Sigma^{2\rho_Q} H_Q\underline{mgw}) \cong \begin{cases} \phi_Z^*\underline{\mathbb{F}_2} & \text{if } i=8, \\ \underline{mg} & \text{if } i=7, \\ \phi_{LDR}\underline{\mathbb{F}_2} & \text{if } i=6, \\ \underline{g} & \text{if } i=5, \\ \underline{mg} & \text{if } i=4, \\ \underline{g} & \text{if } i=3. \end{cases}$$

For $j \geq 3$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{mgw}$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q\underline{mgw}) \cong \begin{cases} \phi_Z^*\underline{\mathbb{F}_2} & \text{if } i=4j, \\ \underline{mg} & \text{if } i=4j-1, \\ \phi_{LDR}\underline{\mathbb{F}_2} \oplus \underline{g}^{4j-i-2} & \text{if } i \in [2j+2, 4j-2], \\ \underline{g}^{2j-3} & \text{if } i=2j+1, \\ \underline{g}^{2j-5} \oplus \phi_{LDR}\underline{\mathbb{F}_2} & \text{if } i=2j, \\ \underline{g}^{2(i-j)-2} & \text{if } i \in [j+2, 2j-1], \\ \underline{g} & \text{if } i=j+1. \end{cases}$$

Proof We first deal with the case $j=1$. The short exact sequence of Mackey functors

$$\underline{w}^* \hookrightarrow \underline{mgw} \twoheadrightarrow \underline{mg}^*$$

combines with Propositions 7.17 and 7.9 to show that the only nontrivial Mackey functors are $\phi_Z^*\underline{\mathbb{F}_2}$ in degree 4 and an extension of $\underline{m}$ by $\underline{g}$ in degree 2. It remains to see that this extension is $\phi_Z^*\underline{B}(2,0)$. According to Proposition 4.12, the Postnikov tower for $\Sigma^{-\rho_Q} H_Q\underline{\mathbb{Z}}$ is

$$\begin{tikzcd} \Sigma^{-5} H_Q\phi_Z^*\underline{B}(2,0) \arrow[r] & \Sigma^{-\rho_Q} H_Q\underline{\mathbb{Z}} \arrow[d] \\ \Sigma^{-7} H_Q\underline{mgw} \arrow[r] & X \arrow[d] \\ & \Sigma^{-8} H_Q\underline{\mathbb{Z}}^* \end{tikzcd}$$

Desuspending this diagram once by ρ_Q gives a tower for computing the homotopy Mackey functors of $\Sigma^{-2\rho_Q} H_Q\underline{\mathbb{Z}}$. The homotopy Mackey functors for $\Sigma^{-8-\rho_Q} H_Q\underline{\mathbb{Z}}^*$ and $\Sigma^{-5\rho_Q} H_Q\Psi^*\underline{B}(2,0)$ follow, using Anderson duality, from Propositions 4.10 and 7.5. Long exact sequences in homotopy then imply that

$$\underline{\pi}_{-9}(\Sigma^{-7-\rho_Q} H_Q\underline{mgw}) \cong \phi_Z^*\underline{B}(2,0).$$

Dualizing gives that $\underline{\pi}_2(\Sigma^{\rho_Q} H_Q\underline{mgw})$ is $\phi_Z^*\underline{B}(2,0)$.

We now have a fiber sequence

$$\Sigma^4 H_Q\phi_Z^*\underline{\mathbb{F}_2} \to \Sigma^{\rho_Q} H_Q\underline{mgw} \to \Sigma^2 H_Q\phi_Z^*\underline{B}(2,0). \tag{7.8}$$

Suspending this sequence by ρ_Q immediately gives the homotopy Mackey functors of $\Sigma^{2\rho_Q} H_Q\underline{mgw}$. The same is true in the case $j=3$, except that we have an extension

$$\underline{g} \hookrightarrow \underline{\pi}_6\Sigma^{3\rho_Q} H_Q\underline{mgw} \twoheadrightarrow \phi_{LDR}\underline{\mathbb{F}_2}.$$

We claim that, more generally, any extension of $\underline{\mathbb{Z}}$-modules

$$\underline{g}^m \hookrightarrow \underline{E} \twoheadrightarrow \phi_{LDR}\underline{\mathbb{F}_2}$$

is necessarily the split extension. To see this, first note that $\phi_{LDR}\underline{\mathbb{F}_2}$ is, by definition, the direct sum $\phi_L^*\underline{\mathbb{F}_2} \oplus \phi_D^*\underline{\mathbb{F}_2} \oplus \phi_R^*\underline{\mathbb{F}_2}$. It therefore suffices to show that the only $\underline{\mathbb{Z}}$-module extension of $\phi_L^*\underline{\mathbb{F}_2}$ by $\underline{g}^m$ is the split extension. Since any such extension will vanish at the subgroups D and R, the $\underline{\mathbb{Z}}$-module structure forces the value at Q to be 2-torsion and therefore equal to $\underline{\mathbb{F}_2}^{m+1}$. Since there is a nontrivial restriction to the subgroup L, the $\underline{\mathbb{Z}}$-module structure forces the transfer from L to vanish. Thus the extension must be the split extension.

The suspension by $(j-1)\rho_Q$ of (7.8) gives the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q\underline{mgw}$ in degrees $2j+1$ and higher. Now we argue by induction that the Mackey functors for $\Sigma^{j\rho_Q} H_Q\underline{mgw}$ are as claimed, for $j \geq 3$. For instance, since the bottom Mackey functor is

$$\underline{\pi}_j(\Sigma^{(j-1)\rho_Q} H_Q\underline{mgw}) \cong \underline{g},$$

we see by decomposing $\Sigma^{(j-1)\rho_Q} H_Q\underline{mgw}$ using the Postnikov tower that

$$\underline{\pi}_{j+1}(\Sigma^{j\rho_Q} H_Q\underline{mgw}) \cong \underline{g}.$$

The values of the Mackey functors $\underline{\pi}_i$, for $i \leq 2j-2$, follow in a similar way. The values

$$\underline{\pi}_{2j-2}(\Sigma^{(j-1)\rho_Q} H_Q\underline{mgw}) \cong \underline{g}^{2j-7} \oplus \phi_{LDR}\underline{\mathbb{F}_2},$$

and

$$\underline{\pi}_{2j-1}(\Sigma^{(j-1)\rho_Q} H_Q\underline{mgw}) \cong \underline{g}^{2j-5}$$

give that

$$\underline{\pi}_{2j-1}(\Sigma^{j\rho_Q} H_Q \underline{mgw}) \cong \underline{g}^{2j-4}$$

and that we have an extension of $\underline{\mathbb{Z}}$-modules

$$\underline{g}^{2j-5} \hookrightarrow \underline{\pi}_{2j}(\Sigma^{j\rho_Q} H_Q \underline{\mathbb{Z}}) \twoheadrightarrow \phi_{LDR}\underline{\mathbb{F}_2}.$$

By the argument given above, this must be the split extension. □

The homotopy Mackey functors for the remaining $8k$-slices follow from [19, Propositions 9.5 and 9.8].

Proposition 7.9 [7, Proposition 4.8; 19, Proposition 9.5] *We have the equivalence*

$$\Sigma^{\rho_Q} H_Q \underline{mg}^* \simeq \Sigma^2 H_Q \underline{m}.$$

For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{mg}^*$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{mg}^*) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j+2, 2j-1], \\ \underline{g} & \text{if } i = j+1. \end{cases}$$

Proposition 7.10 [19, Proposition 9.8] *We have equivalences*

$$\Sigma^{j\rho_Q} H\phi^*_{LDR}\underline{\mathbb{F}_2}^* \simeq \begin{cases} \Sigma^2 H\phi^*_{LDR}\underline{f} & \text{if } j = 1, \\ \Sigma^4 H\phi^*_{LDR}\underline{\mathbb{F}_2} & \text{if } j = 2. \end{cases}$$

Then for $j \geq 3$, the nontrivial homotopy Mackey functors of $\Sigma^{j\rho_Q} H\phi^*_{LDR}\underline{\mathbb{F}_2}^*$ are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi^*_{LDR}\underline{\mathbb{F}_2}^*) = \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j+2, 2j-1]. \end{cases}$$

7.3 The $8k+4$-slices

Similarly, the homotopy Mackey functors of the $8k+4$-slices follow from [19, Proposition 9.8] and [7, Corollary 7.2, Propositions 7.3 and 7.4].

Proposition 7.11 [7, Proposition 3.6] *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi^*_{LDR}\underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi^*_{LDR}\underline{\mathbb{F}_2}) \cong \begin{cases} \phi^*_{LDR}\underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j, 2j-1]. \end{cases}$$

Proposition 7.12 [7, Corollary 7.2] *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4j, \\ \underline{mg} & \text{if } i = 4j-1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-2-i} & \text{if } i \in [2j, 4j-2], \\ \underline{g}^{2(i-j)+1} & \text{if } i \in [j, 2j-1]. \end{cases}$$

Proposition 7.13 [7, Proposition 7.3] *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{m}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{m}) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j+1, 2j-1], \\ \underline{g} & \text{if } i = j. \end{cases}$$

Proposition 7.14 [7, Proposition 7.4] *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{mg}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{mg}) \cong \begin{cases} \phi_{LDR}^* \underline{\mathbb{F}_2} & \text{if } i = 2j, \\ \underline{g}^3 & \text{if } i \in [j+1, 2j-1], \\ \underline{g}^2 & \text{if } i = j. \end{cases}$$

7.4 The $4k+2$-slices

The homotopy Mackey functors of the $4k+2$-slice $\Sigma^{1+k\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}$ are given in Proposition 7.12. The homotopy Mackey functors of the remaining $4k+2$-slices are as follows.

Proposition 7.15 [7, Proposition 4.8, Corollary 7.2] *We have the equivalence $\Sigma^{\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^* \simeq \Sigma^4 H_Q \phi_Z^* \underline{\mathbb{F}_2}$. For $j \geq 2$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \phi_Z^* \underline{\mathbb{F}_2}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4j, \\ \underline{mg} & \text{if } i = 4j-1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-2-i} & \text{if } i \in [2j+2, 4j-2], \\ \underline{g}^{2(i-j)-5} & \text{if } i \in [j+3, 2j+1]. \end{cases}$$

Finally, we have the homotopy of $\Sigma^{j\rho_Q} H_Q \underline{w}$ and $\Sigma^{j\rho_Q} H_Q \underline{w}^*$.

Proposition 7.16 *For $j \geq 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4j, \\ \underline{mg} & \text{if } i = 4j-1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-2-i} & \text{if } i \in [2j, 4j-2], \\ \underline{g}^{2(i-j)+1} & \text{if } i \in [j+1, 2j-1]. \end{cases}$$

Proof The underlying spectrum of $\Sigma^{j\rho_Q} H_Q \underline{w}$ is contractible; thus,

$$\Sigma^{j\rho_Q} H_Q \underline{w} \simeq \Sigma^{j\rho_K} H_Q \underline{w}.$$

Then, because $\underline{w}$ is a pullback over Z, the calculation is essentially K-equivariant. Consider the short exact sequence of K-Mackey functors $\underline{w} \to \underline{\mathbb{F}_2} \to \underline{g}$ and the corresponding cofiber sequence $\Sigma^{j\rho_K} H_K \underline{w} \to \Sigma^{j\rho_K} H_K \underline{\mathbb{F}_2} \to \Sigma^{j\rho_K} H_K \underline{g}$. The statement follows immediately from the resulting long exact sequence in homotopy. □

Proposition 7.17 *For $j = 1$, the homotopy Mackey functors of $\Sigma^{j\rho_Q} H_Q \underline{w}^*$ are*

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4, \\ \underline{g} & \text{if } i = 2. \end{cases}$$

For $j \geq 2$, they are

$$\underline{\pi}_i(\Sigma^{j\rho_Q} H_Q \underline{w}^*) \cong \begin{cases} \phi_Z^* \underline{\mathbb{F}_2} & \text{if } i = 4j, \\ \underline{mg} & \text{if } i = 4j-1, \\ \phi_{LDR}^* \underline{\mathbb{F}_2} \oplus \underline{g}^{4j-2-i} & \text{if } i \in [2j+2, 4j-2], \\ \underline{g}^{2(i-j)-5} & \text{if } i \in [j+3, 2j+1], \\ \underline{g} & \text{if } i = j+1. \end{cases}$$

Proof The proof is the same as that in Proposition 7.16, except that we start with the exact sequence of K-Mackey functors $\underline{g} \to \underline{\mathbb{F}_2}^* \to \underline{w}^*$. □

8 Slice spectral sequences

Here we include the slice spectral sequences for $\Sigma^n H_Q \underline{\mathbb{Z}}$ for several values of n between 5 and 15. See Figures 5–9. In some cases, we use the restriction to the C_4-subgroups to determine some of the slice differentials.

The grading is the same as that in [9, Section 4.4.2]. The Mackey functor $\underline{E}_2^{t-n,t}$ is $\underline{\pi}_n P_t^t(X)$. We also follow the Adams convention, where $\underline{\pi}_n P_t^t(X)$ has coordinates $(n, t-n)$ and the differential

$$d_r \colon \underline{E}_r^{s,t} \to \underline{E}_r^{s+r,t+r-1}$$

points left one and up r.

$\square = \underline{\mathbb{Z}}$	◆ $= \phi_Z^* \underline{\mathbb{F}_2}$	⬟ $= \phi_{LDR}^* \underline{\mathbb{F}_2}$	⬢ $= \underline{mgw}$
$\circ = \underline{B}(3,0)$	♀ $= \phi_Z^* \underline{B}(2,0)$	▰ $= \underline{mg}$	ⓝ $= \underline{g}^n$

Table 4: Symbols for Q-Mackey functors.

$\square = \underline{\mathbb{Z}}$	$\blacklozenge = \phi^*_{C_2}\underline{\mathbb{F}_2}$	$\circ = \underline{B}(2,0)$	$\bullet = \underline{g}$

Table 5: Symbols for C_4-Mackey functors.

The Q-Mackey functors that appear in these spectral sequences are listed in Table 4. We also display some companion C_4-slice spectral sequences, and the C_4-Mackey functors that appear are listed in Table 5.

Example 8.1 In the spectral sequences for $\Sigma^5 H_Q\underline{\mathbb{Z}}$, $\Sigma^6 H_Q\underline{\mathbb{Z}}$, and $\Sigma^7 H_Q\underline{\mathbb{Z}}$, because we must be left with

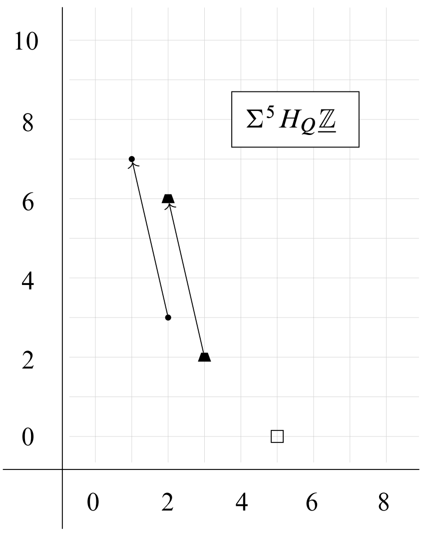
$$\underline{\pi}_n(P^n_n \Sigma^n H_Q\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}},$$

all differentials are forced.

Example 8.2 For $\Sigma^8 H_Q\underline{\mathbb{Z}}$, the pattern of differentials emanating from the Mackey functor

$$\underline{\pi}_6(P^8_8 \Sigma^8 H_Q\underline{\mathbb{Z}})$$

is forced; no other pattern of differentials wipes out all classes in this region. The shorter differentials clearing out the smaller region are then similarly forced.



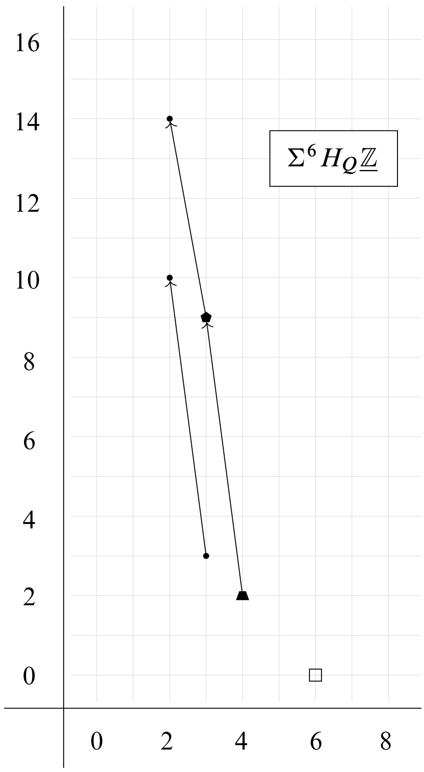


Figure 5

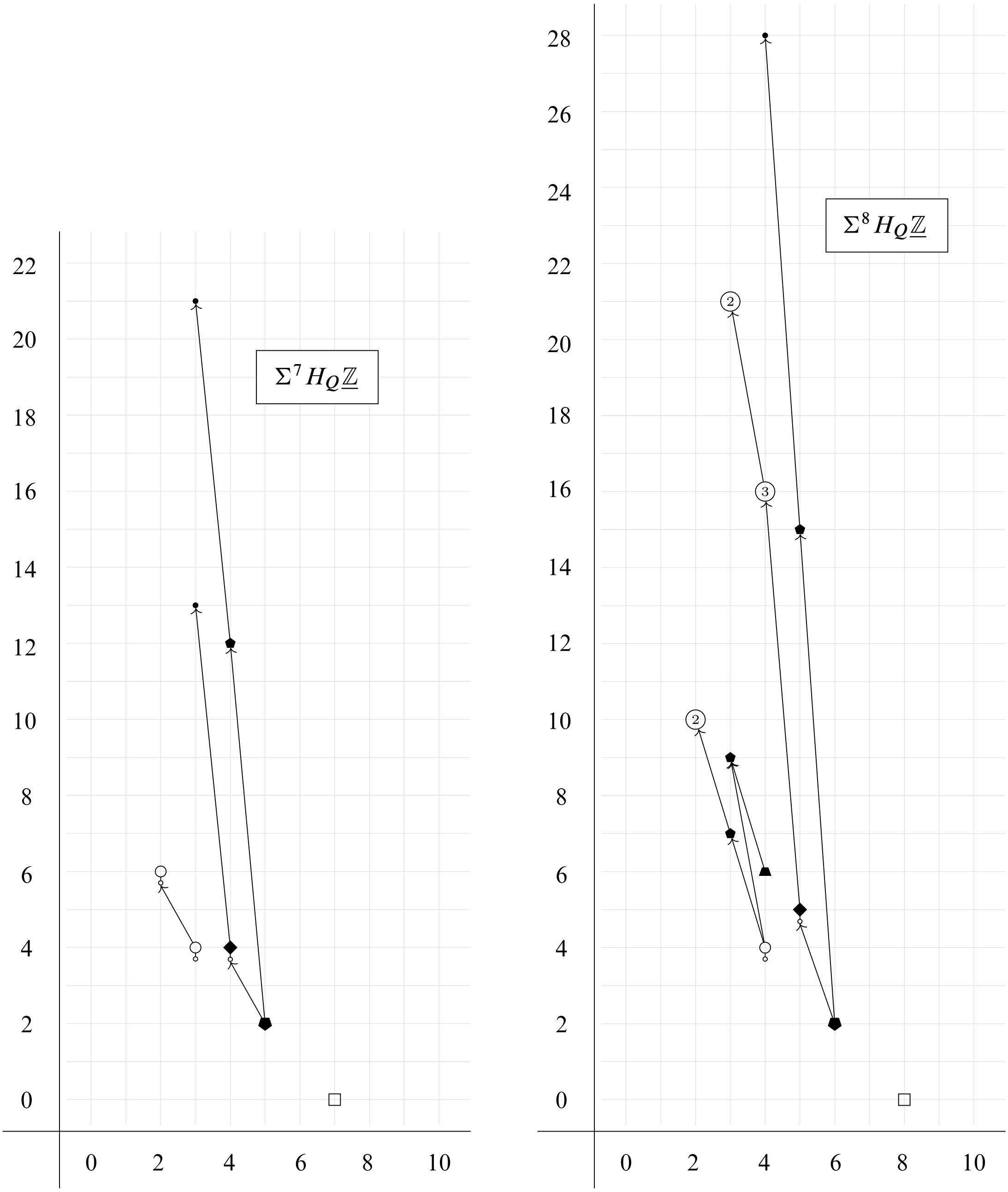


Figure 6

Example 8.3 In the cases of $\Sigma^n H_Q\underline{\mathbb{Z}}$ for $n = 10$, 12, and 15, we also display the corresponding slice spectral sequence for $\Sigma^n H_{C_4}\underline{\mathbb{Z}}$, where we use C_4 to indiscriminately refer to any of the subgroups $L, D, R \leq Q$. The slice differentials in the C_4-case force many of the slice differentials for the Q-equivariant spectra.

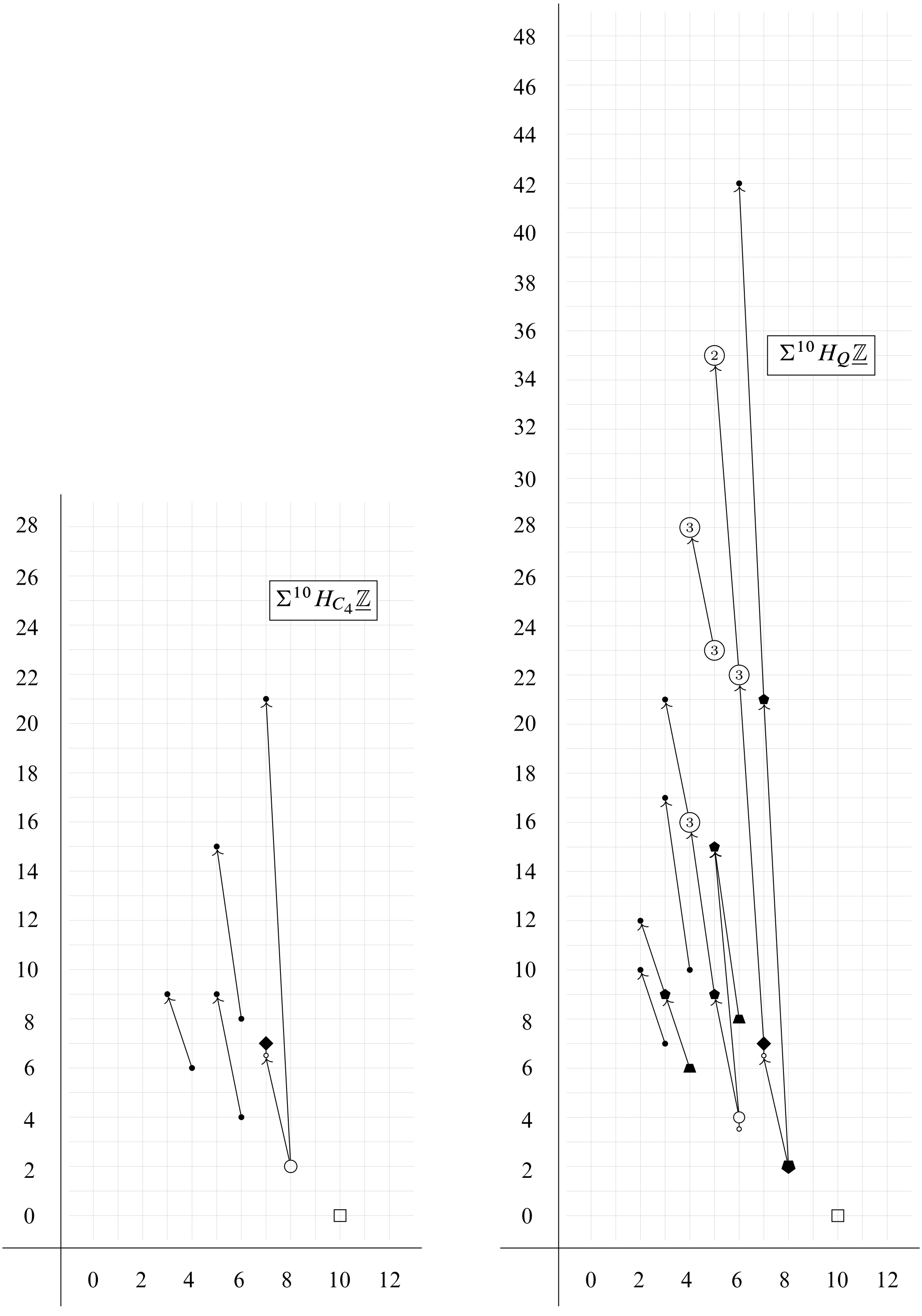
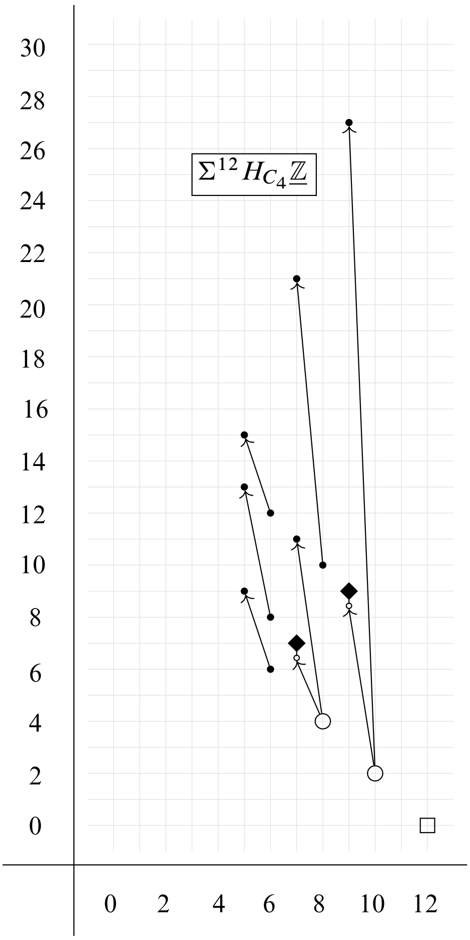

$\Sigma^{10} H_{C_4}\underline{\mathbb{Z}}$
$\Sigma^{10} H_{Q}\underline{\mathbb{Z}}$
3
3
3
2
3
0 2 4 6 8 10 12
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48

Figure 7


$\Sigma^{12} H_{C_4}\underline{\mathbb{Z}}$
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30

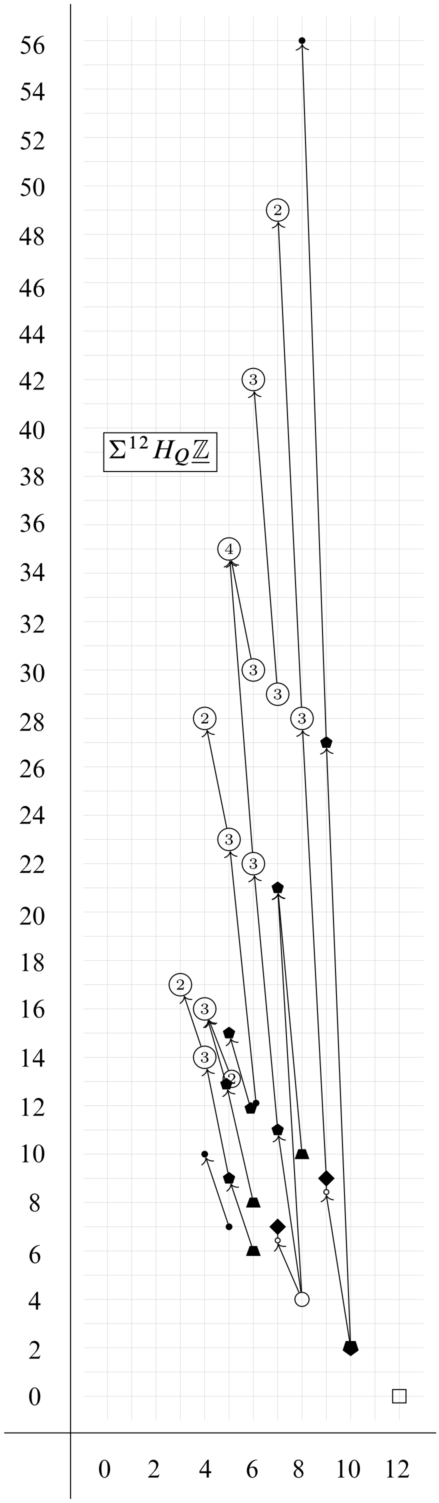

$\Sigma^{12} H_{Q}\underline{\mathbb{Z}}$
0
2
4
6
8
10
12
14
16
18
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Figure 8

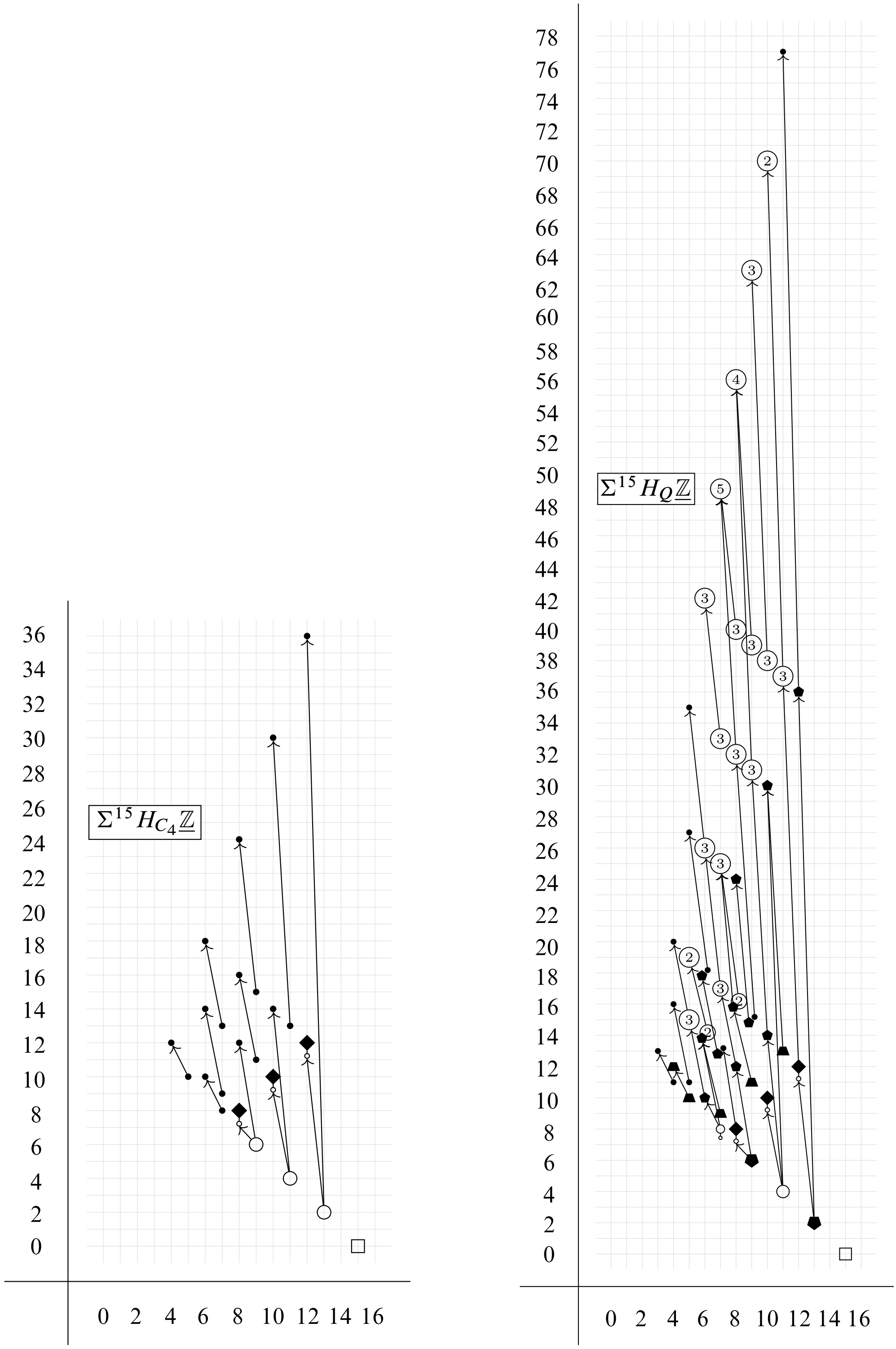
$\Sigma^{15} H_{C_4}\underline{\mathbb{Z}}$
$\Sigma^{15} H_Q\underline{\mathbb{Z}}$

Figure 9

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