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Whitney tower concordance and knots in homology spheres

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A Levine (Forum Math. Sigma 4 (2016) art. id. e34) proved the surprising result that there exist knots in homology spheres which are not smoothly concordant to any knot in S^3, where one allows for concordances in homology cobordisms. Since then subsequent works due to Hom, Levine and Lidman (Duke Math. J. 171 (2022) 3089–3131) and Zhou (J. Topol. 14 (2021) 1369–1395) have strengthened this result showing that there are many knots in homology spheres which are not smoothly concordant to knots in S^3. In this paper we present evidence that the opposite is true topologically. We study the Whitney tower filtration of concordance (Ann. of Math. 157 (2003) 433–519) and prove that modulo any term in this filtration every knot (or link) in a homology sphere is equivalent to a knot (or link) in S^3. As an application we recover a result of the author (J. Topol. 13 (2020) 343–355), namely that the solvable filtration similarly fails to distinguish links in homology spheres from links in S^3.

57K10, 57N70

1 Introduction and statement of results

Two links K and J in S^3 are called (*topologically*) *concordant* if $K \times \{1\}$ and $J \times \{0\}$ cobound a disjoint union of locally flat embedded annuli in $S^3 \times [0, 1]$. A link is (*topologically*) *slice* if it bounds a disjoint union of locally flat embedded disks, called slice disks, in the 4-ball, B^4. A link is slice if and only if it is concordant to the unlink. The set of concordance classes of links is denoted by $\mathcal{C}$. If these embedded annuli and disks are smooth, then we call the resulting links smoothly concordant and smoothly slice, respectively. We denote by $\mathcal{C}_{\text{sm}}$ the set of links up to smooth concordance. The difference between $\mathcal{C}$ and $\mathcal{C}_{\text{sm}}$ has been central to the interplay between topological and smooth 4-manifolds. For instance, the existence of links which are topologically slice but not smoothly slice can be used to produce smooth 4-manifolds which are homeomorphic but not diffeomorphic to $\mathbb{R}^4$ [9]. In this paper we present evidence for a surprising difference between smooth and topological concordance of links in homology spheres.

Conjecture 1.1 Let $K \subseteq Y$ be a link in a homology sphere. There is a link J in S^3 and a homology cobordism W from Y to S^3 in which K and J cobound a disjoint union of locally flat embedded annuli. Moreover, W can be chosen to be simply connected.

This conjecture is false in the smooth setting, as proved in [14] with subsequent improvements in [11; 19]. Recall that a homology cobordism between two 3-manifolds X and Y is informally a 4-manifold bounded by $X \sqcup -Y$ which has the same homology as $X \times [0, 1]$. Homology cobordism of homology spheres was first studied in [10, Chapter 1], inspired by the study of h-cobordism of homotopy spheres of [12].

This conjecture fits into two extensions of concordance to the setting of links in homology spheres. The first is $\mathbb{Z}$-concordance. Two links K and J in homology spheres X and Y are called *$\mathbb{Z}$-concordant* if there is a homology cobordism from X to Y in which the components of K and J cobound disjoint locally flat embedded annuli. The $\mathbb{Z}$-concordance class of the unlink is the set of links whose components bound disjoint locally flat embedded disks in a homology ball. We denote by $\widehat{\mathcal{C}}_{\mathbb{Z}}$ the set of links in homology spheres up to $\mathbb{Z}$-concordance.

Suppose that W is the homology cobordism of the above paragraph. If $\pi_1(X)\to\pi_1(W)$ and $\pi_1(Y)\to\pi_1(W)$ are each surjective then W is called a *strong homology cobordism* and K and J are called *strongly concordant*. We denote by $\widehat{\mathcal{C}}$ the set of all links up to strong topological concordance. The strong concordance class of the unlink is the set of all links in homology spheres which bound disjoint unions of embedded disks in contractible 4-manifolds. There is an inclusion and a projection map

$$\mathcal{C} \xhookrightarrow{i} \widehat{\mathcal{C}} \xrightarrow{p} \widehat{\mathcal{C}_{\mathbb{Z}}}.$$

The injectivity of i follows from the 4-dimensional topological Poincaré conjecture [7]. Conjecture 1.1 can now be interpreted as saying that $\mathcal{C}\to\widehat{\mathcal{C}}$ is a surjection, and as a consequence is bijective.

In [5] the author presents evidence for Conjecture 1.1 in the language of the solvable filtration of Cochran, Orr and Teichner [3]. While this filtration has since received intense study and has been used to detect highly subtle structure in the study of link concordance, it is the weakest of three filtrations studied in that paper. The others are the symmetric Whitney tower filtration and the grope filtration. We focus on the Whitney tower filtration,

$$\mathcal{C} \supseteq \mathcal{W}_2 \supseteq \mathcal{W}_3 \supseteq \cdots.$$

A link in S^3 sits in $\mathcal{W}_h$ if it bounds a height $h-1$ symmetric Whitney tower in the 4-ball. While we give a brief discussion presently and a complete definition in Section 2, one can think of a Whitney tower as an immersed disk which comes close to admitting a regular homotopy to an embedded disk. Thus, if a link is in $\mathcal{W}_h$ then it is close to being slice. See, for example, [3, Definition 7.7; 4, Section 2.1]. A *Whitney tower*, T, is a 2-complex in a 4-manifold given by starting with an immersed disk, pairing up its double points with (possibly nonembedded) Whitney disks, as in Figure 1, left, pairing up the double points amongst these Whitney disks with Whitney disks as in Figure 1, middle, and so on. If any of these Whitney disks is embedded and has interior disjoint from all other surfaces in T, then the Whitney trick of Figure 1, right, can be used to remove these two intersection points. See, for example, [1, Section 1.2] for a discussion of the Whitney trick. If each round of Whitney disks added has interior disjoint from all of the previously added Whitney disks then T is called *symmetric* and the height of T records how many rounds of Whitney disks are added. For any $h\in\mathbb{N}$, $\mathcal{W}_h$ is the set of all links in homology spheres bounding immersed disks in contractible 4-manifolds that extend to height $h-1$ symmetric Whitney towers.

In [2, Definition 2.12], Cha extends $\mathcal{W}_h$ to an equivalence relation. We construct a similar equivalence relation for links in homology spheres as follows: Links in homology spheres are height h Whitney tower

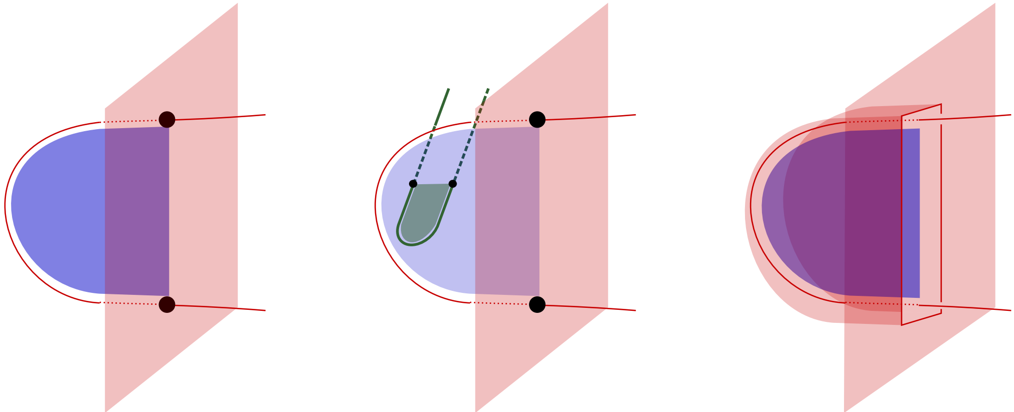

Figure 1: Left: A Whitney disk pairing up two points of intersection. Middle: A (portion of a) Whitney tower. Right: The Whitney moves canceling two points of intersection.

concordant if they cobound an immersed union of annuli in a strong homology cobordism that extends to a height $h-1$ Whitney tower. See Definition 2.5 for a more complete discussion. Our main result is that every link in a homology sphere is height h Whitney tower concordant to a link in S^3.

Theorem 1.2 *Let $h \in \mathbb{N}$ and K be a link in a homology sphere. There is some link in S^3 which is height h Whitney tower concordant to K.*

This theorem should be interpreted as saying that any link in a homology sphere cobounds with a link in S^3 an immersed union of annuli which is close to admitting a sequence of Whitney moves to a disjoint union of locally flat embedded annuli. Thus, any link in any homology sphere is close (from the perspective of Whitney towers) to being strongly concordant to a link in S^3.

As mentioned previously, there are two related filtrations of knot concordance, the solvable filtration $\mathcal{C} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots$ and the grope filtration $\mathcal{C} \supseteq \mathcal{G}_2 \supseteq \mathcal{G}_3 \supseteq \cdots$ [3, Definitions 7.9 and 8.5]. The symmetric Whitney tower filtration gives stronger approximations to sliceness than the solvable filtration in that $\mathcal{W}_{h+2} \subseteq \mathcal{F}_h$ [3, Theorem 8.12]. In [5, Theorem 1.2] the author defines a sequence of equivalence relations called n-solvable concordance and proves that modulo n-solvable concordance every link in a homology sphere is equivalent to a link in S^3. In Section 5 we show that if links are height $h+2$ Whitney tower concordant then they are h-solvably concordant.

Proposition 5.4 *Let K and J be links in homology spheres. If K and J are height $h+2$ Whitney tower concordant, then K and J are h-solvably concordant.*

We recover the main result of [5] as a consequence.

Corollary 1.3 [5, Theorem 1.2] *Let $n \in \mathbb{N}$ and K be a link in a homology sphere. There is some link J in S^3 which is n-solvably concordant to K.*

The Grope filtration is at least as strong as the Whitney tower filtration, in that $\mathcal{G}_n \subseteq \mathcal{W}_n$ [17, Corollary 2], and is stronger than the solvable filtration in that $\mathcal{G}_n \subsetneq \mathcal{F}_{n+2}$ [15, Corollary 6.8]. We take a moment to explain what new idea allows our work to proceed that was not present in [5] and what prevents us from working in $\mathcal{G}_h$, the strongest of the three filtrations of [3]. The key tool in [5] used in the construction of the needed solvable cobordisms are n-solvable surgery curves [5, Definition 2.7]. These provide surgery instructions reducing a link in a homology sphere to a link in S^3 while preserving the n-solvable concordance class. Playing a similar role in this paper is a tool called the relative Whitney trick [6]. This tool attempts to use the double points in an immersed surface bounded by a link L to guide a homotopy of L to a new link bounding an embedded surface. The object used to guide this trick, called a relative Whitney tower, also allows for the construction of needed Whitney towers. Relative Whitney towers are detailed in Section 3.

The grope filtration is defined by setting $\mathcal{G}_h$ to be those links in S^3 which bound height h symmetric gropes in the 4-ball. This naturally extends to links in homology spheres by allowing gropes to sit in homology balls (or contractible 4-manifolds). It further extends to an equivalence relation by looking for annular gropes in a (strong) homology cobordism. See [2, Definition 2.14]. The obstacle that prevents us from working in this stronger setting is the lack of a suitable analogue to n-solvable surgery curves or relative Whitney disks. We close with a question.

Question 1.4 Does the grope filtration also fail to detect a difference between knots in homology spheres and knots in S^3? More precisely, for any $h \in \mathbb{N}$ and any link K in a homology sphere, is there is a link J in S^3 such that the components of K and J cobound a symmetric annular grope of height h in a (strong) homology cobordism?

Outline of paper

In Section 2 we give the formal definition of symmetric Whitney tower concordance and its relationship with the symmetric Whitney tower filtration of [3]. In Section 3 we translate the notion of a relative Whitney tower from [6] to the setting of symmetric relative Whitney towers and explain how use them to construct symmetric Whitney towers. In Section 4 we construct symmetric relative Whitney towers and complete the proof of Theorem 1.2. In Section 5 we recall the notion of solvable concordance and prove that Whitney tower concordance implies solvable concordance.

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2 Symmetric Whitney towers

In order to make sense of Whitney disks and Whitney towers in a topological 4-manifold one must have notions like immersion, transversality, and tubular neighborhood. A careful exposition on these topics appears in [16, Section 3]. We give a very brief summary here. A continuous function $f: S \to W$ from a surface to a 4-manifold is called a *generic immersion* if it locally is a smooth generic immersion, meaning that there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of W with homeomorphisms $\phi_\alpha: U_\alpha \to \mathbb{R}^4$ satisfying that for all α, the composition $\phi_\alpha \circ f|_{f^{-1}[U_\alpha]}: f^{-1}[U_\alpha] \to \mathbb{R}^4$ is a smooth generic immersion. Generic immersions admit linear normal bundles and so have tubular neighborhoods homeomorphic to a self plumbing of the total space of a vector bundle over S [8, Section 9.3]. From here on all immersions are assumed to be generic and all immersed surfaces are assumed to be transverse to each other and to the boundary of W.

Suppose that $p, q \in f[S]$ are double points of f with opposite sign. Then $f^{-1}\{p\} = \{p_1, p_2\}$ and $f^{-1}\{q\} = \{q_1, q_2\}$. For $i = 1, 2$, let α_i be a smoothly embedded arc in S running from p_i to q_i. Assume that other than its endpoints, $f(\alpha_i)$ is disjoint from all double points of f and that α_1 and α_2 are disjoint. The simple closed curve $\alpha := f(\alpha_1) * f(\overline{\alpha_2}) \subseteq W$ is called a *Whitney circle*. (Here $*$ indicates concatenation and $\overline{\alpha_2}$ is the reverse of α_2.) Assuming that α is nullhomotopic in W, there is an immersed disk $\Delta \subseteq W$ transverse to S bounded by α.

Being an immersed disk, Δ has a trivial normal bundle, N, admitting a unique trivialization $\psi: N \to D^2 \times \mathbb{R}^2$. In order for Δ to be called a Whitney disk, the restriction $\psi|_\alpha$ of ψ to $\alpha = \partial\Delta$ must agree with a trivialization coming from $f: S \to W$. See [2, Section 2]. In order to be explicit, we construct this trivialization. Let $U_1, U_2 \subseteq S$ be disjoint disk neighborhoods of α_1 and α_2 respectively. By taking U_1 and U_2 to be small enough, we arrange that $f|_{U_i}: U_i \to W$ is an embedding for $i = 1, 2$ and that $f[U_1] \cap f[U_2] = \{p_1, p_2\}$. At any point $x \in f(\alpha_i)$ let $\vec{u}_i$ be the tangent to $f(U_i)$ normal to α_i and $\vec{v}_i$ be normal to both Δ and $f(U_i)$. Since the points of intersection p and q have opposite signs, we may arrange that at p and q, $\vec{u}_1 = \vec{v}_2$ and $\vec{u}_2 = \vec{v}_1$. These vectors can be chosen to vary continuously and so give another trivialization τ of $N|_\alpha$. If $\tau = \psi|_\alpha$ then Δ is a *Whitney disk* for S pairing p and q.

Definition 2.1 [2, Definition 2.5; 3, Definition 7.7] Let S be a framed immersed surface in a 4-manifold. A *symmetric Whitney tower* T of height $h \in \mathbb{N}$ based on S is a sequence $T_0, T_1, \dots, T_h$ of immersed surfaces such that

- $T_0 = S$,
- for $j \geq 1$, T_j is a collection of transverse framed immersed Whitney disks with disjoint boundaries pairing up all of the double points of T_{j-1}, and
- for $j > i$, T_j has interior disjoint from T_i.

We call T_i the i^{th} *stage of* T, and Whitney disks in T_i are called *height* i. If such a T exists then we say that S *extends to* T.

There is a more general notion simply called a Whitney tower. For a survey the reader is directed to [18]. All Whitney towers in this paper are symmetric. As a consequence any time we refer a Whitney tower T it should be understood that T is symmetric, even if we omit the word.

Remark 2.2 There is a discrepancy between the notion of height in [3] and [2]. In [3, Definition 7.7] the immersed surface S is called height 1, while in [2, Definition 2.5] it is height 0. The definition above follows the conventions of [2, Definition 2.5].

Remark 2.3 As is explored in [2, Remark 2.19], it is a consequence of results of Freedman and Quin on smoothing theory [8, Theorems 8.1A and 8.2A] that we could do equally well writing the definition above in the smooth setting. Indeed, suppose that $T \subseteq W$ is a Whitney tower in a 4-manifold. If W does not admit a smooth structure, then remove a point $p \notin T$ from W and appeal to [8, Theorems 8.2A] to see that $W \setminus \{p\}$ has a smooth structure. Briefly [8, Theorem 8.1 A] says that after changing T by a regular homotopy, which preserves the height of T, we may arrange that $T \subseteq W \setminus \{p\}$ is smooth. In the remainder of this paragraph, we fill in some of the details. We start with the lowest stages of T and work our way up. Indeed, suppose that we have already found a Whitney tower T of height h with the same boundary as T and for which $T_0 \cup \cdots \cup T_{i-1}$ is smoothly immersed. Let $S_i = T_i \setminus \nu(T_0 \cup \cdots \cup T_{i-1})$ be the (topologically) immersed surface in the smooth manifold $W \setminus (\{p\} \cup \nu(T_0 \cup \cdots \cup T_{i-1}))$ given by removing a neighborhood of the lower order stages. It has a tubular neighborhood $\nu(S_i)$ homeomorphic to a self plumbing of a 2-dimensional vector bundle over a surface (a disjoint union of disks if $i > 0$). Let X be such a self plumbing, $X_0 \subseteq X$ be its zero section and $h\colon (X, X_0) \to (\nu(S_i), S_i)$ be a homeomorphism. Being an open submanifold of a smooth manifold, $\nu(S_i) \subseteq W \setminus (\{p\} \cup \nu(T_0 \cup \cdots \cup T_{i-1}))$ is smooth. We now have that $h\colon X \to \nu(T_i)$ is a homeomorphism of smooth 4-manifolds and that $X_0 \subseteq X$ is a smooth 2-complex. By [8, Theorem 8.1A], we may change $X_0 \subset \nu(T)$ by a smooth regular homotopy and an ambient isotopy in X to get new 2-complex X_0' for which h is smooth on a small neighborhood of X_0'. Since regular homotopy consists of the Whitney move along embedded Whitney disks and its inverse, we may modify T by replacing S_i by $h[X_0']$ and arrive at a Whitney tower of the same height with $T_0, \dots, T_i$ all smooth.

Definition 2.4 Let K be a link in a homology sphere Y and $\mathcal{B}$ be a contractible 4-manifold bounded by Y. We say that $K \in \mathcal{W}_h$ if there is a framed immersed union of disks in $\mathcal{B}$ bounded by K, with framing extending the 0-framing on K and which extends to a height $h-1$ Whitney tower.

This definition inspires an equivalence relation on the set of links in homology spheres. Instead of building a Whitney tower starting with a union of immersed disks, start with immersed annuli.

Definition 2.5 (compare with [2, Definition 2.12]) Let $h \in \mathbb{N}$, K and J be n-component links in homology spheres X and Y, and W be a strong homology cobordism from X to Y. For $i = 1, \dots, n$ let A_i be an immersed annulus with $\partial A_i = K_i \cup -J_i$ and over which the 0-framing on $K_i \cup -J_i$ extends.

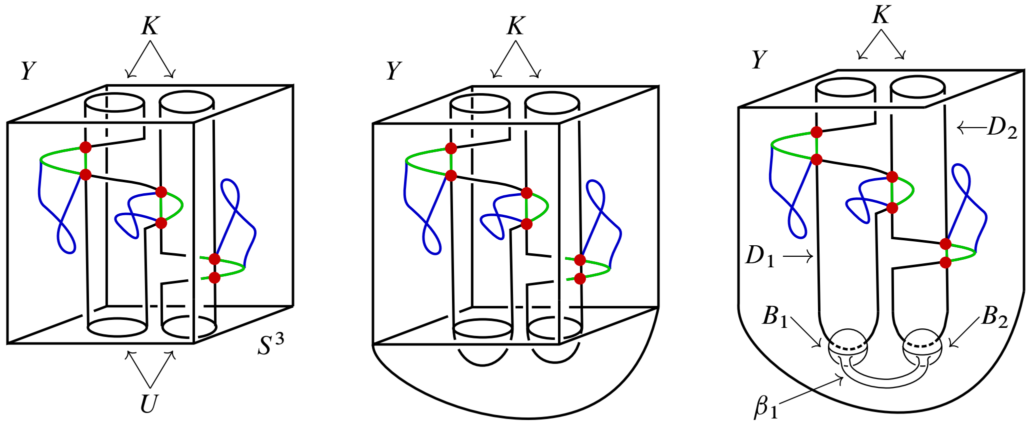

Figure 2: Left: An immersed union of annuli A bounded by $K \subseteq Y$ and $U \subseteq S^3$ extending to a height 1 Whitney tower. The curves in green represent Whitney circles. Middle: Capping A with disks bounded by U produces an immersed union of disks extending to a height 1 Whitney tower. Right: An immersed union of disks $D_1 \cup D_2$ bounded by a link K together with a pair of small 4-balls B_1 and B_2. Tube B_1 and B_2 together along the arc β_1 to get a single 4-ball B. Remove the interior of B to arrive at a Whitney tower concordance from K to the unlink.

If $A = A_1 \cup \cdots \cup A_n$ extends to a height $h-1$ symmetric Whitney tower T, then we say that K is *height h Whitney tower concordant* to J and write $K \sim_h J$. We call T a *height h Whitney tower concordance*.

In the above definition $-J$ indicates the reverse of J sitting in the orientation reverse of Y. We close this section by proving that the definitions of $\mathcal{W}_h$ and $\sim_h$ are compatible, in the sense that a link is in $\mathcal{W}_h$ if and only if it is $\sim_h$-equivalent to the unlink.

Proposition 2.6 *Let K be a link in a homology sphere and $h \in \mathbb{N}$. Then K is height h Whitney tower concordant to the unlink in S^3 if and only if $K \in \mathcal{W}_h$.*

Proof Suppose that $K \subseteq Y$ is a link in a homology sphere and $U \subseteq S^3$ is the unlink. Let W be a strong homology cobordism from Y to S^3 and $A \subseteq W$ be an immersed union of annuli bounded by K and U. Furthermore, assume that the 0-framing on $K \cup U$ extends over A and A extends to a height $h-1$ symmetric Whitney tower, T. A schematic appears in Figure 2, left. Let $\mathcal{B}$ be the contractible 4-manifold resulting from capping the S^3-boundary component of W with a 4-ball. It follows from the fact that W is a strong homology cobordism that $\mathcal{B}$ is a simply connected homology ball, and so is contractible. Let $D \subseteq B^4$ be a disjoint union of smoothly embedded disks bounded by U. Notice that A and D both induce the 0-framing on U so that $A \cup D \subseteq \mathcal{B}$ is an immersed union of disks in a contractible 4-manifold over which the 0-framing of K extends and which forms the base surface of a height $h-1$ symmetric Whitney tower, $T \cup D$. Thus, $K \in \mathcal{W}_h$. A schematic of $A \cup D \subseteq \mathcal{B}$ appears in Figure 2, middle.

Conversely, suppose that $\mathcal{B}$ is a contractible 4-manifold bounded by Y, that $D \subseteq \mathcal{B}$ is a framed immersed union of disks bounded by the 0-framing of K and that D extends to a height $h-1$ symmetric Whitney tower, T in $\mathcal{B}$. For any i, let K_i be the i^{th} component of K and $D_i \subseteq D$ be the immersed disk bounded by K_i. Pick a point p_i interior to D_i, disjoint from all of the Whitney disks in T and a small closed 4-ball neighborhood $B_i \subseteq \mathcal{B}$ of p_i. By taking B_i small enough, we arrange that $\partial B_i \cap D_i$ is an unknot and that $B_i \cap D_i = B_i \cap T$ is an embedded disk bounded by this unknot. For $i = 1, \dots, n-1$, let β_i be an embedded arc in $\mathcal{B}$ running from B_i to B_{i+1}. A dimensionality argument allows us to arrange that β_i is disjoint from T. Let $B \subseteq \mathcal{B}$ be the 4-ball given by tubing together $B_1, \dots, B_n$ along $\beta_1, \dots, \beta_{n-1}$. See Figure 2, right, for a schematic.

We now see a strong homology cobordism $\mathcal{B} \setminus \text{int}(B)$ from Y to S^3, in which $A = D \setminus \text{int}(B)$ is an immersed union of annuli bounded by K and U. Here $\text{int}(B)$ indicates the interior of B. That W is a strong homology cobordism follows from the fact that $\mathcal{B}$ is contractible. We restrict the framing on D to $D \cap B$. Since $D \cap B$ is a collection of slice disks for U, this framing restricts to the 0-framing on U. Thus, the 0-framing on $K \cup U$ extends over A. Finally A is the base surface for a height $h-1$ symmetric Whitney tower, $T \setminus \text{int}(B)$. Thus, K is height h symmetric Whitney tower concordant to the unlink in S^3. □

3 Symmetric relative Whitney towers

Symmetric Whitney towers are a restriction of a larger class of immersed 2-complexes in 4-manifolds, simply called Whitney towers. In [6, Section 5, immediately before Remark 5.3] the author along with Orson and Park define a variation of a Whitney tower called a relative Whitney tower. Relative Whitney towers have two important properties. First, they exist much more readily than actual Whitney towers [6, Lemma 5.4]. Second, they can be used to guide a homotopy of an immersed surface which changes the boundary in a controlled way to a new immersed surface which extends to a Whitney tower [6, Lemma 5.5]. In order to prove Theorem 1.2 we extend these two results to the setting of symmetric Whitney towers.

Let $\psi\colon S \to W$ be an immersion of a surface (possibly with corners) into a 4-manifold. Assume that ψ is transverse to the boundary so that $\psi^{-1}[\partial W] \subseteq \partial S$ is a disjoint union of arcs and circles. Let p be a double point in $\psi(S)$ with preimage $\psi^{-1}\{p\} = \{p_1, p_2\}$. Let $\alpha_1, \alpha_2\colon [0,1] \hookrightarrow S$ be disjoint embedded arcs in S running from p_1 and p_2 to points q_1 and q_2 interior to $\psi^{-1}(\partial W)$, α_1 and α_2 interior to S except for the endpoints at q_1 and q_2. Let α_3 be an arc in ∂W running from $\psi(q_1)$ to $\psi(q_2)$, which is otherwise disjoint from $\psi[S]$. The concatenation $\psi(\alpha_1) * \alpha_3 * \psi(\overline{\alpha_2})$ is a simple closed curve in W. If this curve bounds an immersed disk Δ transverse to S then Δ is called a *relative Whitney disk* associated to the double point p. The arc α_3 is called the *relative Whitney arc* of Δ. A schematic appears in Figure 3.

The reader will notice that while the definition of a Whitney disk requires some discussion of trivialization of normal bundles, the definition of a relative Whitney disk does not. This is because a trivialization of a vector bundle on an arc on the boundary of a of a disk automatically extends to a trivialization over the disk. Thus, if N is the normal bundle of Δ and $N|_{\psi(\overline{\alpha_2}) * \psi(\alpha_1)}$ is its restriction to $\psi(\overline{\alpha_2}) * \psi(\alpha_1)$ then any trivialization of $N|_{\psi(\overline{\alpha_2}) * \psi(\alpha_1)}$ automatically extends over N. We construct a trivialization of

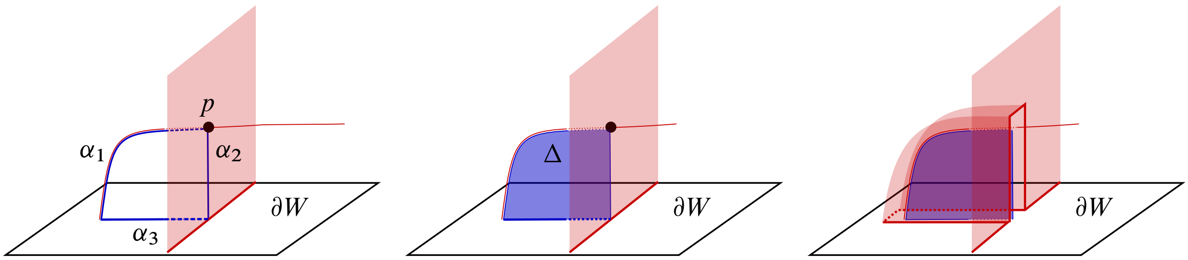

Figure 3: Left: A double point p in an immersed surface together with arcs α_1 and α_2 running from p to points in $\partial W \cap S$ and a third arc α_3 connecting the endpoints of α_1 and α_2. Middle: $\alpha_1 * \alpha_3 * \overline{\alpha_2}$ bounding an immersed disk, called a relative Whitney disk. Right: The relative Whitney trick along Δ removes the double point at p.

$N|_{\psi(\overline{\alpha_2})*\psi(\alpha_1)}$ as follows. At any point in $\psi(\alpha_i)$ let $\vec{u}_i$ be a tangent vector to $\psi(S)$ which is normal to $\psi(\alpha_i)$ and let $\vec{v}_i$ be normal to Δ and to $\psi(S)$. These may be chosen to vary continuously and so that at the double point p, $\vec{u}_1 = \vec{v}_2$ and $\vec{v}_2 = \vec{u}_1$. These two normal vectors together give a trivialization of $N|_{\psi(\overline{\alpha_2})*\psi(\alpha_1)}$ which we extend to give a framing of Δ. Notice that this imposes a framing on the relative Whitney arc, α_3. See [6, Section 2.2] for a more detailed discussion.

Definition 3.1 Let S be an immersed surface in a 4-manifold. A *symmetric relative Whitney tower* of height $h \in \mathbb{N}$ based on S is a sequence $T_0, T_1, \dots, T_h$ of immersed surfaces such that

- $T_0 = S$,
- for $j \geq 1$, T_j is a collection of transverse relative Whitney disks with disjoint boundaries consisting of one relative Whitney disk associated to each double point in T_{j-1}, and
- for $j > i$, T_j has interior disjoint from T_i.

We call T_i the i^{th} stage of T, and relative Whitney disks in T_i are called height i.

An important application of relative Whitney disks is the so-called relative Whitney trick, schematically described in Figure 3, right, and described in detail in [6, Subsection 2.2]. Another is the *partial relative Whitney trick* of Figure 4, right, [6, Proof of Lemma 5.5]. We give the properties of the partial relative Whitney trick. Let p be a double point in an immersed surface A. Let Δ be a relative Whitney disk associated with p and $\alpha \subseteq \partial W$ be its relative Whitney arc. The partial relative Whitney trick

- changes A by a homotopy which is constant away from a small neighborhood of α,
- introduces exactly one new double point q to A,
- replaces Δ with a Whitney disk pairing the intersection points p and q, and
- changes ∂A by a homotopy pushing an arc in ∂A along the framed relative Whitney arc over a meridian of ∂A, as in Figure 4, middle.

Let T be a symmetric relative Whitney tower. By performing the partial relative Whitney move on all of the relative Whitney disks of T one gets a symmetric Whitney tower.

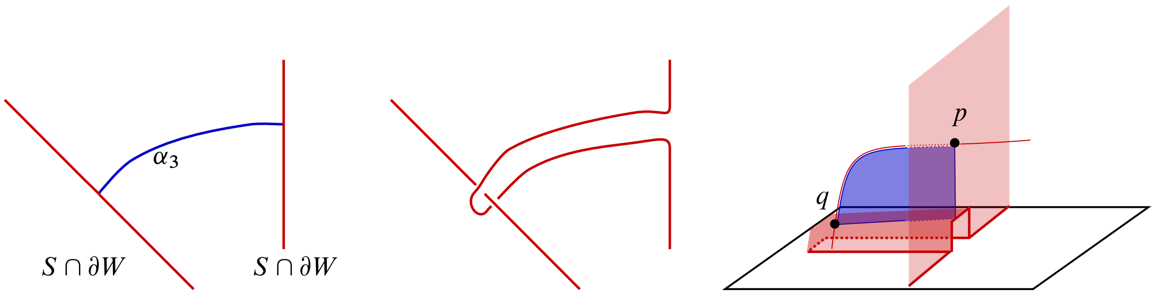

Figure 4: Left: Two arcs in $S \cap \partial W$ connected by the relative Whitney arc α_3 of a relative Whitney disk. Middle: The relative Whitney move changes $S \cap \partial W$ by sliding along α_3 over a meridian. Right: The partial relative Whitney trick introduces a new point of intersection and replaces a relative Whitney disk with a Whitney disk.

Lemma 3.2 (compare to [6, Lemma 5.5]) *Let W be a 4-manifold and T be a height h symmetric relative Whitney tower in W. If Y is a connected submanifold of ∂W and contains all of the relative Whitney arcs of T, then there exists a symmetric relative Whitney tower, T' of height h such that the height 0 surfaces of T and T' differ by a homotopy which is constant outside of a small neighborhood of Y.*

Proof As the proof is identical to that of [6, Lemma 5.5], we give only a summary. Start with a symmetric relative Whitney tower T of height h. Perform the partial relative Whitney move along each of the height h relative Whitney disks in T. These moves change T_{h-1} by a homotopy to T'_{h-1} (a new collection of relative Whitney disks if $h-1>0$). We now have $T_0 \cup T_1 \cup \cdots \cup T_{h-2} \cup T'_{h-1}$ is a height $h-1$ relative Whitney tower and T'_{h-1} extends to a height 1 symmetric Whitney tower.

Now do the same along all height $h-1$ relative Whitney disks, so that $T_0 \cup T_1 \cup \cdots \cup T_{h-3} \cup T'_{h-2}$ is a height $h-2$ relative Whitney tower and T'_{h-2} extends to a height 2 Whitney tower. Iterate until you have performed the relative Whitney trick along every relative Whitney disk. What results is a height 0 relative Whitney tower (ie an immersed surface) T'_0 which extends to a height h Whitney tower. Moreover T'_0 differs from T_0 by a homotopy supported in a small neighborhood of the relative Whitney arcs of T. As Y contains all of the relative Whitney arcs, this completes the proof. □

4 Constructing symmetric relative Whitney towers: proof of Theorem 1.2

Let S be an immersed surface and Δ be a Whitney disk pairing two double points of S. If Δ is embedded and has interior disjoint from S then the Whitney trick can be used to remove these points of intersection. The *finger move* is an inverse to the Whitney trick. Let $S \subseteq W$ be an immersed surface in a 4-manifold. Let β be framed arc running from a point interior to S to another point interior to S. A schematic for the result of changing S by a finger move along β is depicted in Figure 5. In the result of the finger move

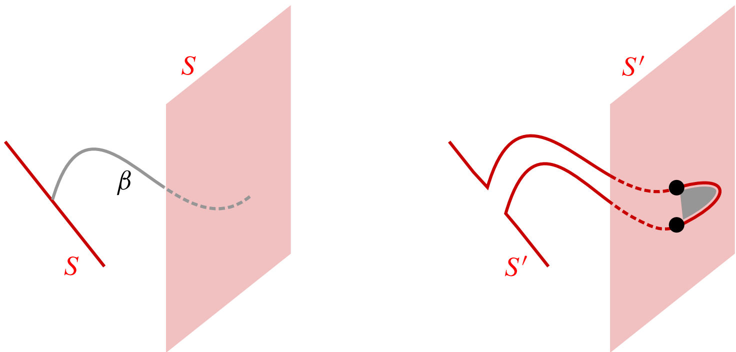

Figure 5: Left: An arc $\beta \subseteq W$ between two points in immersed surface $S \subseteq W$. Right: The result of the finger move β along with an embedded Whitney disk which undoes the finger move.

there are two new intersection points paired by an embedded Whitney disk. Performing the Whitney move along this disk results in a surface isotopic to S. See also [1, Section 11.2].

Lemma 3.2 gives a path to the proof of Theorem 1.2. Let K be a link in a homology sphere Y. By [8, Corollary 9.3] Y bounds a simply connected homology ball. By removing a small open 4-ball from its interior one gets a simply connected homology cobordism W from Y to S^3. Since W is simply connected, there is a free homotopy (a union of immersed annuli) in W from K to the unlink U in S^3. Call this immersed union of annuli A. If we can arrange that A extends to a height h symmetric relative Whitney tower, then we will be able to use Lemma 3.2 to replace it with a Whitney tower concordance from K to some link in S^3. The first step is arranging that $\pi_1(S^3 \setminus \nu(U)) \twoheadrightarrow \pi_1(W \setminus \nu(A))$ is surjective.

Lemma 4.1 *Let W be a 4-manifold with boundary. Let S be an immersed surface in W. Let $Y \subseteq \partial W$ be a connected 3-manifold. Assume that every component of S has nonempty intersection with Y and that the inclusion induced map $\pi_1(Y) \twoheadrightarrow \pi_1(W)$ is onto.*

There is an immersed surface S' in W satisfying that the inclusion induced map $\pi_1(Y \setminus \nu(S')) \twoheadrightarrow \pi_1(W \setminus \nu(S'))$ is onto and that S' differs from S by sequence of finger moves. In particular $\partial S = \partial S'$.

Proof Let W, S and Y be as in the lemma. Pick a base point y in $Y \setminus \nu(S)$. Let $\{g_1, \dots, g_k\}$ be a generating set for $\pi_1(W \setminus \nu(S), y)$. Since $\pi_1(Y, y) \twoheadrightarrow \pi_1(W, y)$ is surjective, for each i, there is some $g_i' \in \pi_1(Y, y)$ and an immersed disk E_i bounded by $g_i * \overline{g_i'}$. A dimensionality argument allows us to arrange that g_i' is disjoint from ∂S. If E_i is disjoint from S, then g_i is already in the image of $\pi_1(Y \setminus \nu(S)) \to \pi_1(W \setminus \nu(S))$. Otherwise, let p be a transverse point of intersection between E_i and S. Let α_1 and α_2 be arcs in S and E_i from p to points in Y. Since $\pi_1(Y, y) \twoheadrightarrow \pi_1(W, y)$, there is an arc α_3 in Y so that $\alpha_1 * \alpha_3 * \overline{\alpha_2}$ is nullhomotopic and so bounds a relative Whitney disk, Δ associated with p.

We would like to use the relative Whitney trick to eliminate the point of intersection at p, but Δ might be neither embedded nor disjoint from S. Let r be a transverse point of intersection in $\Delta \cap S$. Find an arc β

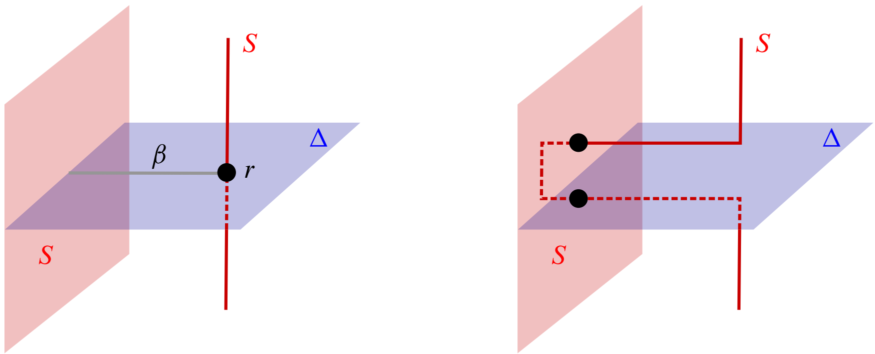

Figure 6: Left: A point q in the intersection of a surface S with a Whitney disk Δ associated to a point in $E_i \cap S$ together with an arc $\beta \subseteq \Delta$ from r to a point in $S \cap \partial\Delta$. Right: A finger move reduces the number of points in $S \cap \Delta$ at a cost of two new double points in S.

in Δ from r to a point interior to $\alpha_1 \subseteq S$. As in Figure 6 we modify S by a finger move along β in order to reduce the number of points in $\Delta \cap S$ by 1. This move adds two new double points to S. As is briefly explained in [13, Chapter XII Section 2], the finger move changes $\pi_1(W - \nu(S))$ by killing a commutator of two meridians of S. In particular, $g_1, \dots, g_n$ is still a generating set for $\pi_1(W - \nu(S))$. Iterating, we arrange that $\Delta \cap S = \varnothing$. Finally, we modify E_i by the relative Whitney trick along Δ. This eliminates the point of intersection $p \in S \cap E_i$. It may add double points to E_i and will change g_i'. Regardless, we now have that g_i is in the image of $\pi_1(Y \setminus \nu(S)) \to \pi_1(W \setminus \nu(S), y)$. By following the steps in this paragraph for every $i = 1, \dots, n$ we arrange that $\pi_1(Y \setminus \nu(S)) \twoheadrightarrow \pi_1(W \setminus \nu(S), y)$ is surjective. Notice that this process changes S by some finger moves and does not change ∂S. □

For any link K in any homology sphere Y we can find a simply connected homology cobordism W and an immersed union of annuli $A \subseteq W$ bounded by K and the unlink in S^3. Appealing to Lemma 4.1, we arrange that $\pi_1(S^3 \setminus \nu(U)) \twoheadrightarrow \pi_1(W \setminus \nu(A))$. An iterative application of the following lemma allows us to extend A to a symmetric relative Whitney tower of arbitrary height. We delay its proof until the end of the section.

Lemma 4.2 (compare to [6, Lemma 5.4]) *Let W be a 4-manifold with boundary. Let $T \subseteq W$ be a symmetric relative Whitney tower of height h. Let $Y \subseteq \partial W$ be a connected 3-manifold which contains all of the relative Whitney arcs of T. Assume that every component of T_0 has nonempty intersection with Y and that the inclusion induced map $\pi_1(Y \setminus \nu(T)) \twoheadrightarrow \pi_1(W \setminus \nu(T))$ is onto.*

Then T extends to a symmetric relative Whitney tower T' of height $h+1$ such that Y contains all of the relative Whitney arcs of T' and such that the inclusion induced map $\pi_1(Y \setminus \nu(T')) \twoheadrightarrow \pi_1(W \setminus \nu(T'))$ is onto.

Between Lemmas 4.1, 4.2, and 3.2 we can prove Theorem 1.2. We recall its statement.

Theorem 1.2 *Let $h \in \mathbb{N}$ and K be a link in a homology sphere. There is some link in S^3 which is height h Whitney tower concordant to K.*

Figure 7: Left: A double point in a framed surface S. Right: A pushoff S^+ intersects the original surface in two points with the same sign as that double point.

Proof Let K be a link in a homology sphere X. By [8, Corollary 9.3] X bounds a simply connected homology ball. Removing an open 4-ball from the interior, we see a simply connected homology cobordism W from X to S^3.

Let A be an immersed union of annuli bounded K and the unlink in S^3. This union of annuli exists since W is simply connected. By Lemma 4.1 we arrange that $\pi_1(S^3 \setminus \nu(A)) \twoheadrightarrow \pi_1(W \setminus \nu(A))$ is surjective. Since a height 0 symmetric relative Whitney tower is just an immersed surface, the assumptions of Lemma 4.2 are satisfied by $T = A$, $Y = S^3$, and $h = 0$. The tower produced by Lemma 4.2 satisfies all of the assumptions of Lemma 4.2 and so we can appeal to the lemma again. Iterating, we see that A extends to a height $h-1$ symmetric relative Whitney tower in W and that all of its relative Whitney arcs are contained in $S^3 \subseteq W$. Finally, by Lemma 3.2 there is a height $h-1$ symmetric Whitney tower whose base surface is an annulus bounded by K together with some link J in S^3.

It remains only to arrange that there is a trivialization of the normal bundle of A which restricts to the 0-framings on K and J. We do so one component at a time. First extend the 0-framing on K_i over A_i (the component of A bounded by $K_i \cup -J_i$). This can be done since an annulus deformation retracts to its either boundary component. This restricts to $-J_i$ to give the f_i-framing for some $f_i \in \mathbb{Z}$. Let $F_i \subseteq Y$ be a Seifert surface for K_i. Cap the Y-boundary component of W with a contractible 4-manifold. Call the result $\mathcal{B}$. Since $\mathcal{B}$ is a contractible 4-manifold bounded by S^3, $\mathcal{B}$ is homeomorphic to the 4-ball by the topological 4-dimensional Poincaré conjecture. Since the 0-framing of K_i extends over both A_i and F_i, $S_i = A_i \cap F_i$ is a framed surface in the 4-ball which restricts to the f_i-framing of $-J_i$. Then a framed pushoff S_i^+ is bounded by $(-J_i)^+$, the f_i-framed pushoff of $-J_i$.

Each double point of S_i results in two points of intersection with the same sign between S_i and S_i^+, as in Figure 7. We may as well assume that $h-1 \geq 1$, so that all of the self intersections of S_i are paired by Whitney disks, and in particular the signed count of the number of intersections between S_i and S_i^+ is zero. Finally since linking numbers can be computed as the signed count of the number of intersections between immersed surfaces in the 4-ball,

$$f_i = \mathrm{lk}(-J_i, (-J_i)^+) = S_i \cdot S_i^+ = 0.$$

Therefore, the 0-framings on K and J each extend over A, and A extends to a height $h-1$ Whitney tower. Thus, $K \sim_h J$ completing the proof □

Finally, we prove Lemma 4.2 by appealing to Lemma 4.1.

Proof of Lemma 4.2 Let W, T, and Y be as in the statement. Recall that T_h is the height h part of T. Let $p^1, p^2, \dots, p^k$ be the double points in T_h. Let $\psi\colon S \to W$ is the immersion of a surface parametrizing T_h. For any $j = 1, \dots, k$, let $\psi^{-1}\{p^j\} = \{p_1^j, p_2^j\}$. There are embedded arcs α_1^j and α_2^j in S running from p_1^j and p_2^j to points q_1^j and q_2^j interior to $\psi^{-1}(Y)$. We will further arrange that $\alpha_1^1, \alpha_2^1, \dots, \alpha_1^k, \alpha_2^k$ are all disjoint from each other. Such arcs exist when $h = 0$ because Y has nonempty intersection with each component of T_0 and when $h > 0$ because Y contains every relative Whitney arc. Since $\pi_1(Y \setminus \nu(T)) \twoheadrightarrow \pi_1(W \setminus \nu(T))$, for $j = 1, \dots, k$, there is some arc $\alpha_3^j \subseteq Y$ running between $\psi(q_1^j)$ and $\psi(q_2^j)$ so that $\psi(\alpha_1^j) * \alpha_3^j * \psi(\overline{\alpha_2^j})$ bounds an immersed disk Δ^j with interior disjoint from T. Notice that $T \cup (\Delta^1 \cup \dots \cup \Delta^k)$ is now a height $h+1$ relative Whitney tower.

The assumptions of Lemma 4.1 are satisfied by $W' = W \setminus \nu(T)$, $Y' = Y \setminus \nu(T) \subseteq \partial W'$ and the immersed surface $S' = \Delta^1 \cup \dots \cup \Delta^k \subseteq W'$. Thus, we can change $\Delta^1 \cup \dots \cup \Delta^k$ by a sequence of finger moves in $W \setminus \nu(T)$ so that $\pi_1(Y \setminus \nu(T \cup \Delta^1 \cup \dots \cup \Delta^k)) \twoheadrightarrow \pi_1(W \setminus \nu(T \cup \Delta^1 \cup \dots \cup \Delta^k))$ is surjective, completing the proof. □

5 Whitney tower concordance and solvable concordance

For any link $K \subseteq S^3$, if $K \in \mathcal{W}_{h+2}$ then K is h-solvable [3, Theorem 8.12]. In this section we show that if two links in homology spheres are height $h+2$-Whitney tower concordant then they are h-solvably concordant. We begin by recalling the definition of h-solvable concordance.

Definition 5.1 [5, Definition 2.3] Let $K \subseteq X$ and $J \subseteq Y$ be links in homology spheres. We say that K is *n-solvably concordant* to J if there exists a cobordism W from X to Y in which the components of K and J cobound a disjoint collection of embedded locally flat annuli $A \subseteq W$ such that:

(1) $H_1(W) = 0$, so that W is an H_1-cobordism.
(2) There exist locally flat embedded closed oriented surfaces in W with trivial normal bundles $L_1, D_1, \dots, L_k, D_k$ all disjoint from C and from each other except that for each $i = 1, \dots, k$, the surfaces L_i and D_i intersect transversely in a single point.
(3) $\{[L_1], [D_1], \dots, [L_k], [D_k]\}$ generates $H_2(W)$.
(4) For all $i = 1, \dots, k$, the images of $\pi_1(L_i) \to \pi_1(E(A))$ and $\pi_1(D_i) \to \pi_1(E(A))$ are both contained in $\pi_1(E(A))^{(n)}$.

Here $E(A) = W \setminus \nu(A)$ is the exterior of A and $\pi_1(E(A))^{(n)}$ refers to the *derived series* of the group $\pi_1(E(A))$. It is defined recursively as

$$\pi_1(E(A))^{(0)} = \pi_1(E(A)) \quad \text{and} \quad \pi_1(E(A))^{(n+1)} = [\pi_1(E(A))^{(n)}, \pi_1(E(A))^{(n)}].$$

Figure 8: A pair of double points of opposite sign in an immersed surface may be canceled at the cost of increasing the genus of the surface by 1.

Remark 5.2 Similar to Remark 2.3, one gets an equivalent definition by requiring these surfaces to be smooth, either with regard to a smooth structure on $W \setminus \nu(A)$, if one exists, or a smooth structure on $W \setminus (\nu(A) \cup \{p\})$. Indeed by [8, Theorem 8.1A] $L_1 \cup D_1 \cup \cdots \cup L_k \cup D_k$ admits a regular homotopy inside of a small neighborhood to a smoothly immersed surface. Since a regular homotopy preserves intersection numbers, L_i still has algebraically zero self intersections, and so by increasing the genus as in Figure 8 we can arrange that the geometric intersections between $L_1 \cup D_1 \cup \cdots \cup L_k \cup D_k$ are preserved. As this occurs in a small neighborhood of $L_1 \cup D_1 \cup \cdots \cup L_k \cup D_k$, the image of $\pi_1(L_i) \to \pi_1(E(A))$ and $\pi_1(D_i) \to \pi_1(E(A))$ is still contained in $\pi_1(E(A))^{(n)}$.

Remark 5.3 The remaining result of this paper, Proposition 5.4, requires the construction of an h-solvable cobordism from a height $h+2$ symmetric Whitney tower concordance. We will need a strategy to demonstrate that certain elements of a fundamental group sit deep in the derived series of some fundamental group. We will use the following geometric translation of derived series of the fundamental group in the language of gropes. A height 1 symmetric grope is a compact oriented surface. A height $h+1$ symmetric grope consists of a surface together with a symplectic basis $\{a_1, b_1, \dots, a_g, b_g\}$ and $2g$ many height h gropes bounded by $a_1, b_1, \dots, a_g, b_g$. Generally in the literature, these are taken to be embedded in a 4-manifold and have some framing condition, but these conditions are not relevant to our work. For any 4-manifold X, a straightforward inductive argument reveals that $\gamma \in \pi_1(X)^{(h)}$ if and only if γ bounds an immersed height h grope in X.

Proposition 5.4 *Let K and J be links in homology spheres. If K and J are height $h+2$ Whitney tower concordant, then K and J are h-solvably concordant.*

Proof Let K and J be links in homology spheres X and Y, respectively. Let W be a homology cobordism from X to Y in which the components of K and J cobound an immersed union of annuli A which extends to a height $h+1$ symmetric Whitney tower T.

Let $\Delta_1, \dots, \Delta_k$ be a complete list of all of the height $h+1$ Whitney disks in T_{h+1}. For every $j = 1, \dots, k$, let γ_j be the framed curve given by pushing $\partial \Delta_j$ slightly into the interior of Δ_j. Let W' be the result of modifying W by surgery along $\gamma_1, \dots, \gamma_k$.

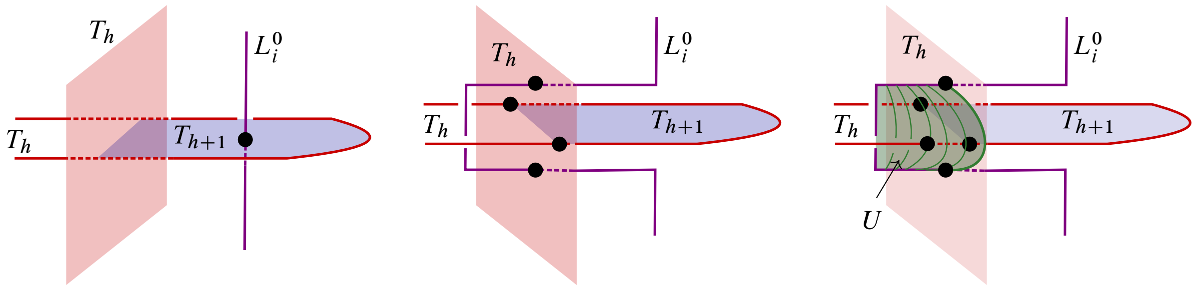

Figure 9: Left: L_i^0 intersects T_{h+1} transversely in a single point. Middle: After a finger move L_i^0 (purple) intersects T_h (red) in two points with opposite sign. Right: An embedded Whitney disk U pairing these two points of intersection and which intersects T_h transversely in one point.

First we exhibit a disjoint union of embedded annuli in W' cobounded by the components of K and the components of J. Let $T' \subseteq W'$ be the height $h+1$ Whitney tower given starting with T and replacing each $\Delta_j \subseteq T_{h+1}$ with the embedded disk glued in along γ_j. Notice that all of the height $h+1$ Whitney disks in T' are embedded and have interiors disjoint from all other surfaces in T'. Thus, we may use the Whitney trick to eliminate all of the intersections in T'_h. Next use these embedded height h Whitney disks to eliminate intersections in T'_{h-1}. Iterate until T'_0 is embedded. We now have that in W' the components of K and J cobound an embedded union of disjoint annuli, which we will call A. Observe that A is contained in a small neighborhood of T'. The remainder of the proof amounts to showing that the conditions of Definition 5.1 are satisfied.

As modifying a 4-manifold by surgery along nullhomologous simple closed curves does not change first homology, $H_1(W') = 0$. Therefore W' is an H_1-cobordism. Each time we perform surgery we increase the rank of second homology by 2. Let L_i^0 be a meridional 2-sphere for γ_i (an embedded sphere bounding a 3-ball in W which intersects γ_i transversely in a single point). Let $D_i^0 \subseteq W'$ be the immersed sphere given by capping the disk in Δ_i bounded by γ_i with the disk glued to the surgery curve γ. We now have that $\{[L_1^0], [D_1^0], \dots, [L_k^0], [D_k^0]\}$ forms a basis for $H_2(W')$. Being the boundary of an embedded 3-sphere in W, L_i^0 has trivial normal bundle. Since the framing on γ_i used to perform surgery was the framing induced by Δ_i, D_i^0 has trivial normal bundle.

Our next goal is to replace L_i^0 with an embedded surface homologous to L_i^0 which extends to a height $h+1$ grope in the exterior of T'. Currently, L_i^0 intersects T'_{h+1} transversely in a single point. By performing a finger move we isotope L_i^0 to be disjoint from T'_{h+1} and to instead intersect T'_h in two points of opposite sign. As in Figure 9, these two points of intersection are paired with an embedded Whitney disk U disjoint from T_{h+1} whose interior intersects T_h transversely in a single point. As in Figure 10 we tube L_i^0 to itself along the arc $\partial U \cap T_h$ to get an embedded genus-1 surface L_i disjoint from T. Also in Figure 10 we see a pair of disks U and V bounded by a symplectic basis for L_i. These disks each intersect T_h transversely in a single point and $L_i \cup U \cup V$ is a height 2 grope.

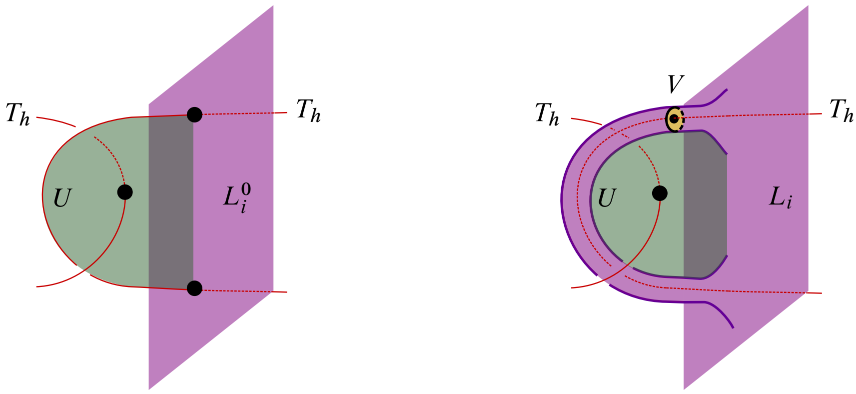

Figure 10: Left: Two points of intersection between L_i^0 and T_h which are paired by an embedded Whitney disk U intersecting T_h in one point. Right: Tubing produces a surface of genus 1 with a symplectic basis bounding embedded disks U and V each of which intersects T_h in a single point.

Next we apply exactly the same steps as the preceding paragraph to U and V this time instead of L^0. First, modify U and V by a finger move so that they are disjoint from T_h and each intersect T_{h-1} in two points paired with a Whitney disk. Then add a tube to each to replace U and V by genus-1 surfaces which admit symplectic bases bounding disks that each intersect T_{h-1} in a single point. The genus-1 surface L_i now forms the height 1 part of a height 3 grope G which is disjoint from T' except that the height 3 part of G consists of disks each intersecting the T'_{h-1} in a single point.

For any $k \in \{1, \dots, h+2\}$ we iterate this construction to produce a height k grope G_i with height 1 surface L_i, whose height k part consists of disjoint embedded disks each intersecting T'_{h+2-k} transversely in a single point, and which is otherwise disjoint from T'. Setting $k = h+2$ and dropping the disks that make up the height $h+2$ part of G, we see that L_i forms the height 1 part of a height $h+1$ grope in the exterior of T'. It follows that this grope is disjoint from A since A is contained in a neighborhood of T'. As the sphere L_i^0 has trivial normal bundle so does L_i. As L_i forms the height 1 part of a height $h+1$ grope disjoint from A, there is a generating set for $\pi_1(L_i)$ consisting of curves that bound height h gropes disjoint from A. We conclude using Remark 5.3 the image of $\pi_1(L_i) \to \pi_1(E(A))$ is contained in $\pi_1(E(A))^{(h)}$.

Next consider D_i^0, the immersed sphere formed from the union of Δ_i with the disk attached to γ_i. As the framing on γ_i used to perform surgery is the same as that coming from the normal bundle of Δ_i, D_i^0 has trivial normal bundle. These spheres are not embedded, nor are they disjoint from each other, but they are disjoint from T' and they are disjoint from $L_1 \cup \cdots \cup L_k$ except that D_i^0 intersects L_i transversely in a single point.

For each $i = 1, \dots, k$ and each point of self intersection of D_i^0, tube D_i^0 to a pushed off copy of L_i. This increases the genus of D_i^0 by 1, and reduces the number of self intersections by 1. The homology class $[D_i^0]$ is replaced by $[D_i^0] \pm [L_i]$ depending on the sign of the intersection point removed. In particular

$\{[L_1],[D_1^0],\dots,[L_k],[D_k^0]\}$ still generates $H_2(W')$. Since the result of tubing together framed surfaces is still framed, D_i^0 is still framed. By making this replacement at each point of self intersection, we arrange that D_i^0 is embedded. Similarly, we eliminate all points of intersection between any D_i^0 and D_j^0 by tubing D_i^0 into a pushed off copy of L_j. Let $D_1,\dots,D_k$ be the resulting surfaces. Each D_i consists of a union of pushoffs of $L_1,\dots,L_k$ together with a planar surface. As a consequence, the image of $\pi_1(D_i)\to\pi_1(E(A))$ is contained in the normal closure of the subgroup generated by the images of $\pi_1(L_1),\dots\pi_1(L_k)$. As we have already explained, this is contained in $\pi_1(E(A))^{(h)}$.

Thus, $L_1,D_1,\dots,L_k,D_k$ satisfy conditions (2), (3), and (4) of Definition 5.1. Therefore K and J are h-solvably concordant, completing the proof. □

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