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The asymptotic behaviors of the colored Jones polynomials of the figure-eight knot, and an affine representation

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We study the asymptotic behavior of the N-dimensional colored Jones polynomial of the figure-eight knot evaluated at $\exp(\kappa + 2p\pi\sqrt{-1}/N)$, where $\kappa := \operatorname{arccosh}\left(\frac{3}{2}\right)$ and p is a positive integer. We can prove that it grows exponentially with growth rate determined by the Chern–Simons invariant of an affine representation from the fundamental group of the knot complement to the Lie group $\mathrm{SL}(2;\mathbb{C})$.

57K14; 57K10

1 Introduction

For a knot K in the three-dimensional sphere S^3 and a positive integer N, let $J_N(K;q)$ be the N-dimensional colored Jones polynomial associated with the N-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}(2;\mathbb{C})$, where we normalize it as $J_N(U;q) = 1$ for the unknot U, and when $N = 2$ it satisfies the following skein relation:

$$qJ_2(\diagup\!\!\!\!\diagdown;q) - q^{-1}J_2(\diagdown\!\!\!\!\diagup;q) = (q^{1/2} - q^{-1/2})J_2(\,)\;(\,;q).$$

If we replace q with $e^{2\pi\sqrt{-1}/N}$, we obtain a complex number $J_N(K;e^{2\pi\sqrt{-1}/N})$, which is known as the Kashaev invariant; see Kashaev [14], and J Murakami and the author [25]. The volume conjecture (see Kashaev [15], and J Murakami and the author [25]) states that the series $\{J_N(K;e^{2\pi\sqrt{-1}/N})\}_{N=1,2,3,\dots}$ grows exponentially with growth rate proportional to the simplicial volume $\operatorname{Vol}(K)$ of $S^3 \setminus K$. Here the simplicial volume is also known as the Gromov norm; see Gromov [7]. It coincides with the hyperbolic volume divided by the volume v_3 of the ideal regular hyperbolic tetrahedron if the knot is hyperbolic, that is, its complement $S^3 \setminus K$ possesses a complete hyperbolic structure with finite volume. If the knot is not hyperbolic, then the simplicial volume is the sum of the hyperbolic volumes of the hyperbolic pieces of $S^3 \setminus K$ after the Jaco–Shalen–Johannson decomposition; see Jaco and Shalen [12] and Johannson [13].

Conjecture 1.1 (volume conjecture) *For any knot K in S^3, we have*

$$\lim_{N\to\infty} \frac{\log|J_N(K;e^{2\pi\sqrt{-1}/N})|}{N} = \frac{v_3 \operatorname{Vol}(K)}{2\pi}.$$

The volume conjecture has been generalized in various ways. It can be complexified as follows; see J Murakami, Okamoto, Takata, Yokota, and the author [26]. Let $\mathscr{K} \subset S^3$ be a hyperbolic knot, and

$$\operatorname{cv}(\mathscr{K}) := \sqrt{-1}\operatorname{Vol}(\mathscr{K}) - 2\pi^2 \operatorname{CS}^{\mathrm{SO}(3)}(\mathscr{K})$$

be the complex volume of $S^3 \setminus \mathscr{K}$, where $\mathrm{CS}^{\mathrm{SO}(3)}(\mathscr{K}) \pmod{\pi^2}$ is the Chern–Simons invariant of the Levi-Civita connection of $S^3 \setminus \mathscr{K}$ associated with the complete hyperbolic structure.

Conjecture 1.2 (complexification of the volume conjecture) *For any hyperbolic knot $\mathscr{K}$ in S^3, we have*

$$\lim_{N\to\infty} \frac{\log J_N(\mathscr{K}; e^{2\pi\sqrt{-1}/N})}{N} = \frac{\mathrm{cv}(\mathscr{K})}{2\pi\sqrt{-1}}.$$

The volume conjecture and its complexification can be refined as follows (see Gukov [8], and also Gukov and H Murakami [9], Dimofte, Gukov, Lenells, and Zagier [5], and Ohtsuki [30]):

Conjecture 1.3 (refined volume conjecture) *Let $\mathscr{K} \subset S^3$ be a hyperbolic knot. Then we have the asymptotic equivalence*

$$J_N(\mathscr{K}; e^{2\pi\sqrt{-1}/N}) \underset{N\to\infty}{\sim} \left(\frac{T(\mathscr{K})}{2\sqrt{-1}}\right)^{1/2} N^{3/2} \exp\left(\frac{\mathrm{cv}(\mathscr{K})}{2\pi\sqrt{-1}} N\right),$$

where $F(N) \underset{N\to\infty}{\sim} G(N)$ means $\lim_{N\to\infty} F(N)/G(N) = 1$, and $T(\mathscr{K})$ is the adjoint (cohomological) Reidemeister torsion twisted by the holonomy representation $\rho_0 \colon \pi_1(S^3 \setminus \mathscr{K}) \to \mathrm{SL}(2;\mathbb{C})$.

The refined volume conjecture has been proved for the figure-eight knot (see Andersen and Hansen [1]) and hyperbolic knots with at most seven crossings; see Ohtsuki [30; 31] and Ohtsuki and Yokota [32].

We can also generalize the refined volume conjecture by replacing $2\pi\sqrt{-1}$ in $e^{2\pi\sqrt{-1}/N}$ with a complex number.

Let $\rho_u \colon \pi_1(S^3 \setminus \mathscr{K}) \to \mathrm{SL}(2;\mathbb{C})$ be an irreducible representation, which is a small deformation of the holonomy representation ρ_0. Then it defines an incomplete hyperbolic structure of $S^3 \setminus \mathscr{K}$. Up to conjugation, we can assume that ρ_u sends the meridian and preferred longitude of $\mathscr{K}$ to

$$\begin{pmatrix} e^{u/2} & * \\ 0 & e^{-u/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{v(u)/2} & * \\ 0 & e^{-v(u)/2} \end{pmatrix},$$

respectively. Then we can define the cohomological adjoint Reidemeister torsion $T_u(\mathscr{K})$ (see Porti [34]) and the Chern–Simons invariant $\mathrm{CS}_{u,v(u)}(\rho_u)$; see Kirk and Klassen [17].

The following conjecture was proposed by the author [23]; see also Gukov and Murakami [9] and Dimofte and Gukov [4].

Conjecture 1.4 (generalized volume conjecture) *For a hyperbolic knot $\mathscr{K}$, there exists a neighborhood $U \in \mathbb{C}$ of 0 such that if $u \in U \setminus \pi\sqrt{-1}\mathbb{Q}$, then we have the asymptotic equivalence*

$$J_N(\mathscr{K}; e^{(u+2\pi\sqrt{-1})/N}) \underset{N\to\infty}{\sim} \frac{\sqrt{-\pi}}{2\sinh(u/2)} T_u(\mathscr{K})^{1/2} \left(\frac{N}{u+2\pi\sqrt{-1}}\right)^{1/2} \exp\left(\frac{S_u(\mathscr{K})}{u+2\pi\sqrt{-1}} N\right),$$

where $S_u(\mathscr{K}) := \mathrm{CS}_{u,v(u)}(\rho_u) + u\pi\sqrt{-1} + \frac{1}{4}uv(u)$.

The generalized volume has been proved just for the figure-eight knot; see Yokota and the author [28]. The asymptotic equivalence in Conjecture 1.4 was also proved in the case where $0<u<\kappa:=\operatorname{arccosh}\bigl(\frac{3}{2}\bigr)$ by the author [23].

In the previous paper [24], the author proved the following theorem generalizing the result in [23]:

Theorem 1.5 *Let $\mathscr{E}$ be the figure-eight knot. For a real number u with $0<u<\kappa$ and a positive integer p, we have*

$$\begin{aligned}&J_N(\mathscr{E};e^{(u+2p\pi\sqrt{-1})/N})\\&\underset{N\to\infty}{\sim} J_p(\mathscr{E};e^{4N\pi^2/(u+2p\pi\sqrt{-1})})\frac{\sqrt{-\pi}}{2\sinh\bigl(\frac{1}{2}u\bigr)}T_u(\mathscr{E})^{1/2}\biggl(\frac{N}{u+2p\pi\sqrt{-1}}\biggr)^{1/2}\exp\biggl(\frac{S_u(\mathscr{E})}{u+2p\pi\sqrt{-1}}N\biggr).\end{aligned}$$

Note that in the case of the figure-eight knot, we have

$$\begin{aligned}T_u(\mathscr{E})&=\frac{2}{\sqrt{(2\cosh u+1)(2\cosh u-3)}},\\S_u(\mathscr{E})&=\mathrm{Li}_2(e^{-u-\varphi(u)})-\mathrm{Li}_2(e^{-u+\varphi(u)})+u(\varphi(u)+2\pi\sqrt{-1}),\end{aligned}$$

where we put

$$\varphi(u):=\log\bigl(\cosh u-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{(2\cosh u+1)(2\cosh u-3)}\bigr),$$

and $\mathrm{Li}_2(z):=-\int_0^z\log(1-x)/x\,dx$ is the dilogarithm function.

So it is impossible to extend Theorem 1.5 to the case where $u=\kappa$ because $T_u(\mathscr{E})$ is not defined. Topologically/geometrically speaking, the corresponding hyperbolic structure of the figure-eight knot complement collapses at $u=\kappa$.

On the other hand, for the figure-eight knot, we have the following theorems:

Theorem 1.6 (the author [21]) *If $\zeta\in\mathbb{C}$ satisfies the inequality $|\cosh\zeta-1|<\frac{1}{2}$ and $|\operatorname{Im}\zeta|<\frac{1}{3}\pi$, then*

$$\lim_{N\to\infty}J_N(\mathscr{E};e^{\zeta/N})=\frac{1}{\Delta(\mathscr{E};e^{\zeta})},$$

where $\Delta(K;t)$ is the Alexander polynomial of a knot K.

Theorem 1.7 (Hikami and the author [11]) *If $\zeta=\kappa$, then the colored Jones polynomial $J_N(\mathscr{E};e^{\kappa/N})$ grows polynomially. More precisely,*

$$J_N(\mathscr{E};e^{\kappa/N})\underset{N\to\infty}{\sim}\frac{\Gamma\bigl(\frac{1}{3}\bigr)}{3^{2/3}}\biggl(\frac{N}{\kappa}\biggr)^{2/3},$$

where $\Gamma(x)$ is the gamma function.

Here we will extend Theorem 1.5 to the case $u=\kappa$.

Theorem 1.8 *Let $\mathscr{E}$ be the figure-eight knot, and $\xi := \kappa + 2p\pi\sqrt{-1}$ with $\kappa := \operatorname{arccosh}\left(\frac{3}{2}\right)$ and p a positive integer. Then we have the asymptotic equivalence*

$$J_N(\mathscr{E}; e^{\xi/N}) \underset{N\to\infty}{\sim} J_p(\mathscr{E}; e^{4\pi^2 N/\xi}) \frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}} \left(\frac{N}{\xi}\right)^{2/3} \exp\left(\frac{S_\kappa(\mathscr{E})}{\xi}N\right),$$

where $S_\kappa(\mathscr{E}) := 2\kappa\pi\sqrt{-1}$, and we put $\xi^{1/3} := |\xi|^{1/3} e^{\arctan(2p\pi/\kappa)\sqrt{-1}/3}$.

As a corollary, we obtain a similar result for $J_N(\mathscr{E}; e^{\xi'/N})$ with $\xi' := -\kappa + 2p\pi\sqrt{-1}$.

Corollary 1.9 *We have*

$$J_N(\mathscr{E}; e^{\xi'/N}) \underset{N\to\infty}{\sim} J_p(\mathscr{E}; e^{4\pi^2 N/\xi'}) \frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}} \left(\frac{N}{\xi'}\right)^{2/3} \exp\left(\frac{S_{-\kappa}(\mathscr{E})}{\xi'}N\right),$$

where we put $S_{-\kappa}(\mathscr{E}) := -2\kappa\pi\sqrt{-1}$.

See Section 6 for a topological interpretation of $S_u(\mathscr{E})$ for $|u| \le \kappa$. It is defined to be

$$\mathrm{CS}_{u,v(u)}(\rho_u) + u\pi\sqrt{-1} + \frac{1}{4}uv(u),$$

where

$$\mathrm{CS}_{u,v(u)}(\rho_u)$$

is the Chern–Simons invariant of a nonabelian representation $\rho_u\colon \pi_1(S^3 \setminus \mathscr{E}) \to \mathrm{SL}(2;\mathbb{C})$.

Remark 1.10 Since the highest-degree term of the Laurent polynomial $J_p(\mathscr{E}; q)$ is $q^{p(p-1)}$, we have $J_p(\mathscr{E}; e^{4\pi^2 N/\xi}) \underset{N\to\infty}{\sim} e^{4p(p-1)\pi^2 N/\xi}$. So we also have

$$J_N(\mathscr{E}; e^{\xi/N}) \underset{N\to\infty}{\sim} \frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}} \left(\frac{N}{\xi}\right)^{2/3} \exp\left(\frac{4p^2\pi^2}{\xi}N\right),$$

because $2\kappa\pi\sqrt{-1}/\xi + 4p(p-1)\pi^2/\xi = 4p^2\pi^2/\xi + 2\pi\sqrt{-1}$. A similar result holds for ξ'.

There are two difficulties in proving Theorem 1.8.

The first one is that when we apply the saddle point method to the integral that approximates $J_N(\mathscr{E}; e^{\xi/N})$, the saddle point is of order two, that is, it looks like the saddle point of $\operatorname{Re} z^3$; see Figure 1.

To approximate the colored Jones polynomial by an integral as above, we use a quantum dilogarithm function, which converges to a function described by the dilogarithm function. However, the second difficulty is that our saddle point is on the boundary of the region of convergence. So we need to extend the domain of definition of the quantum dilogarithm slightly by using a functional identity.

The paper is organized as follows. In Section 2, we define the quantum dilogarithm and extend it as we require. In Section 3, we express the colored Jones polynomial as a sum of the terms described by the

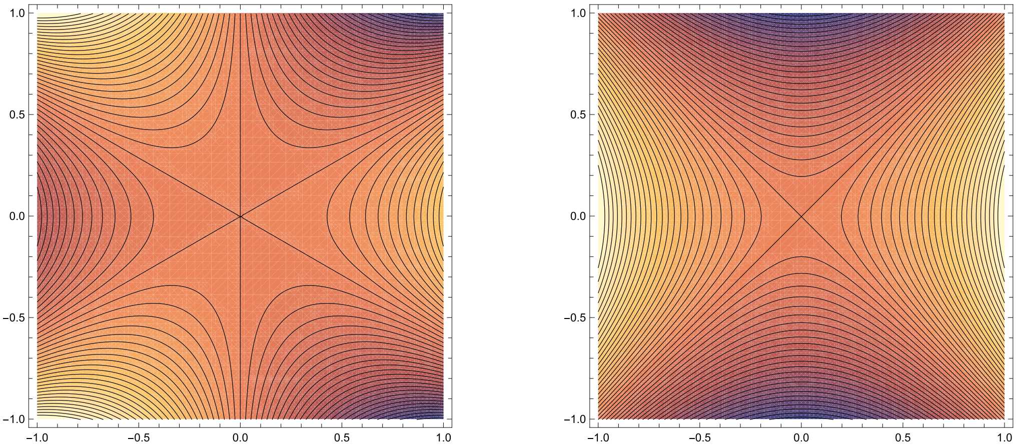

Figure 1: Contour plots of the functions $\operatorname{Re} z^3$ (left) and $\operatorname{Re} z^2$ (right) around their saddle points. The saddle point O of $\operatorname{Re} z^3$ is of order two, and that of $\operatorname{Re} z^2$ is of order one.

quantum dilogarithm. To approximate the sum by an integral, we use the Poisson summation formula in Section 4. Then in Section 5 we use the saddle point method to obtain the asymptotic formula, proving Theorem 1.8. Appendices A and B are devoted to proofs of the Poisson summation formula and the saddle point method, respectively. In Appendix C, we give some computer calculations about the asymptotic behavior of $J_N(\mathscr{S}; e^{(\pm\tilde{\kappa}+2\pi\sqrt{-1})/N})$ for the stevedore knot $\mathscr{S}$, where $\tilde{\kappa} := \log 2$. Since we know that $e^{\pm\kappa}$ ($e^{\pm\tilde{\kappa}}$, respectively) are zeros of the Alexander polynomial of the figure-eight knot (the stevedore knot, respectively), we try to generalize Theorem 1.8 to another knot in vain.

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2 Quantum dilogarithm

In this section, we fix a complex number γ with $\operatorname{Re}\gamma > 0$ and $\operatorname{Im}\gamma < 0$. We will introduce a quantum dilogarithm [6]. See also [1; 15; 30].

We put

$$T_N(z) := \frac{1}{4}\int_{\widehat{\mathbb{R}}} \frac{e^{(2z-1)x}}{x\sinh(x)\sinh(\gamma x/N)}\,dx \tag{2-1}$$

for an integer $N > |\gamma|/\pi$, where $\widehat{\mathbb{R}} := (-\infty,-1] \cup \{w \in \mathbb{C} \mid |w| = 1, \operatorname{Im} w \geq 0\} \cup [1,\infty)$ with orientation from $-\infty$ to ∞. Note that $\widehat{\mathbb{R}}$ avoids the poles of the integrand. We can prove that the integral above converges if $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1 + \operatorname{Re}\gamma/(2N)$.

Lemma 2.1 *The integral in the right side of (2-1) converges if* $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1 + \operatorname{Re}\gamma/(2N)$.

Proof First note that

$$\sinh(as) \underset{s\to\infty}{\sim} \tfrac{1}{2}e^{as} \quad\text{and}\quad \sinh(as) \underset{s\to-\infty}{\sim} -\tfrac{1}{2}e^{-as},$$

for a complex number a with $\operatorname{Re} a > 0$. So we have

$$\frac{e^{(2z-1)x}}{x\sinh(x)\sinh(\gamma x/N)} \underset{x\to\infty}{\sim} \frac{4}{x}\exp((2z-2-\gamma/N)x),$$
$$\frac{e^{(2z-1)x}}{x\sinh(x)\sinh(\gamma x/N)} \underset{x\to-\infty}{\sim} -\frac{4}{x}\exp((2z+\gamma/N)x),$$

since $\operatorname{Re}\gamma > 0$.

Therefore if $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1+\operatorname{Re}\gamma/(2N)$, the integral converges. □

Thus $T_N(z)$ is a holomorphic function in the region $\{z\in\mathbb{C} \mid -\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1+\operatorname{Re}\gamma/(2N)\}$.

We will study properties of $T_N(z)$, first introducing three related functions:

Definition 2.2 For a complex number z with $0 < \operatorname{Re} z < 1$, we put

$$\mathcal{L}_0(z) := \int_{\widehat{\mathbb{R}}} \frac{e^{(2z-1)x}}{\sinh(x)}\,dx,\quad \mathcal{L}_1(z) := -\frac{1}{2}\int_{\widehat{\mathbb{R}}} \frac{e^{(2z-1)x}}{x\sinh(x)}\,dx,\quad \mathcal{L}_2(z) := \frac{\pi\sqrt{-1}}{2}\int_{\widehat{\mathbb{R}}} \frac{e^{(2z-1)x}}{x^2\sinh(x)}\,dx.$$

In a similar way to the proof of Lemma 2.1, we can prove that the three integrals above converge if $0 < \operatorname{Re} z < 1$. The functions above can be expressed in terms of well-known functions.

Lemma 2.3 [27, Lemma 2.5] *We have the following formulas*:

$$\mathcal{L}_0(z) = \frac{-2\pi\sqrt{-1}}{1-e^{-2\pi\sqrt{-1}z}}, \tag{2-2}$$

$$\mathcal{L}_1(z) = \begin{cases} \log(1-e^{2\pi\sqrt{-1}z}) & \textit{if } \operatorname{Im} z \ge 0,\\ \pi\sqrt{-1}(2z-1)+\log(1-e^{-2\pi\sqrt{-1}z}) & \textit{if } \operatorname{Im} z < 0,\end{cases} \tag{2-3}$$

$$\mathcal{L}_2(z) = \begin{cases} \mathrm{Li}_2(e^{2\pi\sqrt{-1}z}) & \textit{if } \operatorname{Im} z \ge 0,\\ \pi^2\left(2z^2-2z+\frac{1}{3}\right)-\mathrm{Li}_2(e^{-2\pi\sqrt{-1}z}) & \textit{if } \operatorname{Im} z < 0.\end{cases} \tag{2-4}$$

Here the branch cuts of log *and* Li_2 *are* $(-\infty,0]$ *and* $[1,\infty)$, *respectively.*

The proof is similar to that of [27, Lemma 2.5], and so we omit it.

The function $\mathcal{L}_0(z)$ can be extended to the whole complex plane $\mathbb{C}$ except for integers. The functions $\mathcal{L}_1(z)$ and $\mathcal{L}_2(z)$ can be extended to holomorphic functions on $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$ as follows.

Definition 2.4 For a complex number z in $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$, we put

$$\mathcal{L}_1(z) = \begin{cases} \log(1-e^{2\pi\sqrt{-1}z}) & \text{if } \operatorname{Im} z \ge 0,\\ \pi\sqrt{-1}(2z-1)+\log(1-e^{-2\pi\sqrt{-1}z}) & \text{if } \operatorname{Im} z < 0,\end{cases} \tag{2-5}$$

$$\mathcal{L}_2(z) = \begin{cases} \mathrm{Li}_2(e^{2\pi\sqrt{-1}z}) & \text{if } \operatorname{Im} z \ge 0,\\ \pi^2\left(2z^2-2z+\frac{1}{3}\right)-\mathrm{Li}_2(e^{-2\pi\sqrt{-1}z}) & \text{if } \operatorname{Im} z < 0.\end{cases} \tag{2-6}$$

Lemma 2.5 *When* $\operatorname{Im} z < 0$, *the functions* $\mathcal{L}_1(z)$ *and* $\mathcal{L}_2(z)$ *can also be written as*

$$\mathcal{L}_1(z) = \log(1-e^{2\pi\sqrt{-1}z}) + 2\lfloor \operatorname{Re} z\rfloor \pi\sqrt{-1}, \quad \mathcal{L}_2(z) = \operatorname{Li}_2(e^{2\pi\sqrt{-1}z}) - 2\pi^2\lfloor \operatorname{Re} z\rfloor(\lfloor \operatorname{Re} z\rfloor - 2z + 1),$$

where $\lfloor x\rfloor$ *is the greatest integer that does not exceed* x.

Proof For $\mathcal{L}_1(z)$, we have

$$\begin{aligned}\log(1-e^{-2\pi\sqrt{-1}z}) &= \log[(1-e^{2\pi\sqrt{-1}z})e^{-2\pi\sqrt{-1}z+\pi\sqrt{-1}}] \\ &= \log(1-e^{2\pi\sqrt{-1}z}) - 2\pi\sqrt{-1}z + \pi\sqrt{-1} + 2\lfloor \operatorname{Re} z\rfloor \pi\sqrt{-1}.\end{aligned}$$

The last equality follows because

- if $0 < \operatorname{Re} z - \lfloor \operatorname{Re} z\rfloor < \frac{1}{2}$, then $-\pi < \arg(1-e^{2\pi\sqrt{-1}z}) < 0$,
- if $\frac{1}{2} \le \operatorname{Re} z - \lfloor \operatorname{Re} z\rfloor < 1$, then $0 \le \arg(1-e^{2\pi\sqrt{-1}z}) < \pi$,

and so the imaginary part of the rightmost side is between $-\pi$ and π. Thus we obtain $\mathcal{L}_1(z) = \log(1-e^{2\pi\sqrt{-1}z}) + 2\lfloor \operatorname{Re} z\rfloor \pi\sqrt{-1}$ from (2-3).

For $\mathcal{L}_2(z)$, from the well known formula

$$\operatorname{Li}_2(w^{-1}) = -\operatorname{Li}_2(w) - \tfrac{1}{6}\pi^2 - \tfrac{1}{2}(\log(-w))^2, \tag{2-7}$$

we have

$$\begin{aligned}\operatorname{Li}_2(e^{-2\pi\sqrt{-1}z}) &= -\operatorname{Li}_2(e^{2\pi\sqrt{-1}z}) - \tfrac{1}{6}\pi^2 - \tfrac{1}{2}(2\pi\sqrt{-1}z - (2\pi\lfloor \operatorname{Re} z\rfloor + \pi)\sqrt{-1})^2 \\ &= -\operatorname{Li}_2(e^{2\pi\sqrt{-1}z}) + \pi^2\big(2z^2 - 2z + \tfrac{1}{3}\big) + 2\pi^2\lfloor \operatorname{Re} z\rfloor^2 - 4\pi^2\lfloor \operatorname{Re} z\rfloor z + 2\pi^2\lfloor \operatorname{Re} z\rfloor,\end{aligned}$$

and the result follows. □

Corollary 2.6 *If* $\operatorname{Im} z < 0$, *then we have* $\mathcal{L}_1(z+1) - \mathcal{L}_1(z) = 2\pi\sqrt{-1}$ *and* $\mathcal{L}_2(z+1) - \mathcal{L}_2(z) = 4\pi^2 z$.

Lemma 2.7 *The derivatives of* $\mathcal{L}_i(z)$ *for* $i = 1, 2$ *are given as follows*:

$$\frac{d}{dz}\mathcal{L}_2(z) = -2\pi\sqrt{-1}\mathcal{L}_1(z), \tag{2-8}$$

$$\frac{d}{dz}\mathcal{L}_1(z) = -\mathcal{L}_0(z) = \frac{2\pi\sqrt{-1}}{1-e^{-2\pi\sqrt{-1}z}}. \tag{2-9}$$

Proof The first equality follows from the well-known equality $(d/dw)\operatorname{Li}_2(w) = -\log(1-w)/w$. The second one also follows easily. □

Now we will show three identities expressing the difference $T_N(z+a) - T_N(z)$ in terms of $\mathcal{L}_1$.

Lemma 2.8 *If* $|\operatorname{Re} z| < \operatorname{Re}\gamma/(2N)$, *then*

$$T_N(z) - T_N(z+1) = \mathcal{L}_1\Big(\frac{N}{\gamma}z + \frac{1}{2}\Big). \tag{2-10}$$

Remark 2.9 Since $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < \operatorname{Re}\gamma/(2N)$ and $1-\operatorname{Re}\gamma/(2N) < \operatorname{Re}(z+1) < 1+\operatorname{Re}\gamma/(2N)$, both z and $z+1$ are in the domain of T_N.

We will check that $Nz/\gamma+\frac{1}{2}$ is in $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$, the domain of $\mathcal{L}_1$.

If not, then $(N/\gamma)z+\frac{1}{2}=s$ for $s\le 0$ or $s\ge 1$. Putting $s':=s-\frac{1}{2}$, we have $z=(\gamma/N)s'$ with $|s'|\ge\frac{1}{2}$, which implies $|\operatorname{Re} z|\ge\operatorname{Re}\gamma/(2N)$, a contradiction.

Proof By definition, we have

$$T_N(z)-T_N(z+1)=\frac{1}{4}\int_{\widehat{\mathbb{R}}}\frac{e^{(2z-1)x}-e^{(2z+1)x}}{x\sinh x\sinh(\gamma x/N)}\,dx=-\frac{1}{2}\int_{\widehat{\mathbb{R}}}\frac{e^{2zx}}{x\sinh(\gamma x/N)}\,dx.$$

Then setting $y:=\gamma x/N$, this equals

$$-\frac{1}{2}\int_{\widehat{\mathbb{R}}'}\frac{e^{2Nzy/\gamma}}{y\sinh y}\,dy=-\frac{1}{2}\int_{\widehat{\mathbb{R}}}\frac{e^{2Nzy/\gamma}}{y\sinh y}\,dy=\mathcal{L}_1\Bigl(\frac{N}{\gamma}z+\frac{1}{2}\Bigr),$$

where $\widehat{\mathbb{R}}'$ is obtained from $\widehat{\mathbb{R}}$ by multiplying by γ/N. The last equality follows since there are no poles of $1/(y\sinh y)$, that is, integer multiples of $\pi\sqrt{-1}$ between $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{R}}'$. □

Lemma 2.10 *If* $0<\operatorname{Re} z<1$, *then*

$$T_N\Bigl(z-\frac{\gamma}{2N}\Bigr)-T_N\Bigl(z+\frac{\gamma}{2N}\Bigr)=\mathcal{L}_1(z). \tag{2-11}$$

Proof From the definition, we have

$$\begin{aligned}T_N\Bigl(z-\frac{\gamma}{2N}\Bigr)-T_N\Bigl(z+\frac{\gamma}{2N}\Bigr)&=\frac{1}{4}\int_{\widehat{\mathbb{R}}}\frac{e^{(2z-\gamma/N-1)x}-e^{(2z+\gamma/N-1)x}}{x\sinh x\sinh(\gamma x/N)}\,dx\\&=-\frac{1}{2}\int_{\widehat{\mathbb{R}}}\frac{e^{(2z-1)x}}{x\sinh x}\,dx=\mathcal{L}_1(z).\end{aligned}$$ □

The third one is a little tricky.

Lemma 2.11 *If* $|\operatorname{Re} z|<\operatorname{Re}\gamma/N<1$, *then*

$$T_N\Bigl(z+1-\frac{\gamma}{2N}\Bigr)-T_N\Bigl(z+\frac{\gamma}{2N}\Bigr)=\begin{cases}\mathcal{L}_1(z)-\mathcal{L}_1((N/\gamma)z) & \textit{if } \operatorname{Re} z\ge 0 \textit{ and } z\ne 0,\\ \mathcal{L}_1(z+1)-\mathcal{L}_1((N/\gamma)z+1) & \textit{if } \operatorname{Re} z<0,\\ \log((\gamma/N)) & \textit{if } z=0.\end{cases} \tag{2-12}$$

Remark 2.12 If $|\operatorname{Re} z|<\operatorname{Re}\gamma/N<1$, then $1-3\operatorname{Re}\gamma/(2N)<\operatorname{Re}(z+1-\gamma/(2N))<1+\operatorname{Re}\gamma/(2N)$ and $-\operatorname{Re}\gamma/(2N)<\operatorname{Re}(z+\gamma/(2N))<3\operatorname{Re}\gamma/(2N)$, and so both $z+1-\gamma/(2N)$ and $z+\gamma/(2N)$ are in the domain of T_N.

We will check that the arguments in the right-hand side are in $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$, the domain of $\mathcal{L}_1$.

- Suppose $0\le\operatorname{Re} z$ and $z\ne 0$. Since $\operatorname{Re} z<\operatorname{Re}\gamma/N$, z is in the domain of $\mathcal{L}_1$ if N is sufficiently large. Suppose for a contradiction that $(N/\gamma)z$ is not in the domain of $\mathcal{L}_1$. Then $(N/\gamma)z\in(-\infty,0]\cup[1,\infty)$ and so $(N/\gamma)z=s$ for $s\ge 1$ or $s\le 0$. If $s\le 0$, then $\operatorname{Re} z=s\operatorname{Re}\gamma/N\le 0$ and so $z=s=0$, which is a contradiction. If $s\ge 1$, then $\operatorname{Re} z\ge s\operatorname{Re}\gamma/N\ge\operatorname{Re}\gamma/N$, which is also a contradiction.

- Suppose $\operatorname{Re} z < 0$. Then $z+1$ is in the domain of $\mathcal{L}_1$ because $1-\operatorname{Re}\gamma/N < \operatorname{Re}(z+1) < 1$. Suppose for a contradiction that $Nz/\gamma+1$ is not in the domain of $\mathcal{L}_1$. Then $(N/\gamma)z+1 = s$ for $s \le 0$ or $s \ge 1$. Thus $\operatorname{Re} z = \operatorname{Re}((s-1)\gamma/N)$ and so we have $\operatorname{Re} z \le -\operatorname{Re}\gamma/N$, which is impossible.

Thus the arguments in the right-hand side are in the domain of $\mathcal{L}_1$.

Proof We first assume that $\operatorname{Re} z > 0$. Then from Lemmas 2.10 and 2.8 we have

$$T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right)=\mathcal{L}_1(z),\quad T_N\left(z+1-\frac{\gamma}{2N}\right)-T_N\left(z-\frac{\gamma}{2N}\right)=-\mathcal{L}_1\left(\frac{N}{\gamma}z\right),$$

and the equality follows. Note that $-\operatorname{Re}\gamma/(2N) < \operatorname{Re}(z-\gamma/(2N)) < \operatorname{Re}\gamma/(2N)$ and so we can apply Lemma 2.8 to the second equality. Similarly, if $\operatorname{Re} z < 0$, we have

$$T_N\left(z+1-\frac{\gamma}{2N}\right)-T_N\left(z+1+\frac{\gamma}{2N}\right)=\mathcal{L}_1(z+1),\quad T_N\left(z+1+\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right)=-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right),$$

and the equality also holds.

When $\operatorname{Re} z = 0$, put $z := y\sqrt{-1}$ for $y \in \mathbb{R}\setminus\{0\}$ and consider the limit

$$\lim_{\varepsilon\to 0}\left(T_N\left(y\sqrt{-1}+1-\frac{\gamma}{2N}+\varepsilon\right)-T_N\left(y\sqrt{-1}+\frac{\gamma}{2N}+\varepsilon\right)\right).$$

Since T_N is a holomorphic function in $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1+\operatorname{Re}\gamma/(2N)$, the limit above coincides with the left-hand side of (2-10). From the result above, considering the limit from the right, we have

$$\begin{aligned}
&T_N\left(y\sqrt{-1}+1-\frac{\gamma}{2N}\right)-T_N\left(y\sqrt{-1}+\frac{\gamma}{2N}\right)\\
&\qquad=\lim_{\varepsilon\searrow 0}\left(T_N\left(y\sqrt{-1}+1-\frac{\gamma}{2N}+\varepsilon\right)-T_N\left(y\sqrt{-1}+\frac{\gamma}{2N}+\varepsilon\right)\right)\\
&\qquad=\lim_{\varepsilon\searrow 0}\left(\mathcal{L}_1(y\sqrt{-1}+\varepsilon)-\mathcal{L}_1\left(\frac{N}{\gamma}(y\sqrt{-1}+\varepsilon)\right)\right)\\
&\qquad=\mathcal{L}_1(y\sqrt{-1})-\mathcal{L}_1\left(\frac{N}{\gamma}(y\sqrt{-1})\right)
\end{aligned}$$

if $y \ne 0$, because we extend $\mathcal{L}_1(z)$ to $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$. Let us confirm that the limit from the left gives the same answer. We have

$$\lim_{\varepsilon\nearrow 0}\left(\mathcal{L}_1(y\sqrt{-1}+\varepsilon+1)-\mathcal{L}_1\left(\frac{N}{\gamma}(y\sqrt{-1}+\varepsilon)+1\right)\right)=\mathcal{L}_1(y\sqrt{-1}+1)-\mathcal{L}_1\left(\frac{N}{\gamma}y\sqrt{-1}+1\right),$$

which coincides with $\mathcal{L}_1(y\sqrt{-1})-\mathcal{L}_1((N/\gamma)(y\sqrt{-1}))$ if $y \ne 0$ from Lemma 2.5, noting that

$$\operatorname{Im}((N/\gamma)y\sqrt{-1}) = Ny\operatorname{Re}\gamma/|\gamma|^2$$

has the same sign as y.

Now, we consider the case where $z = 0$. Since $\operatorname{Im}\gamma < 0$, we have $\operatorname{Im}(N/\gamma)\varepsilon > 0$ for $\varepsilon > 0$. Thus

$$\lim_{\varepsilon\searrow 0}\left(\mathcal{L}_1(\varepsilon)-\mathcal{L}_1\left(\frac{N}{\gamma}\varepsilon\right)\right)=\lim_{\varepsilon\searrow 0}(\log(1-e^{2\pi\sqrt{-1}\varepsilon})-\log(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})). \tag{2-13}$$

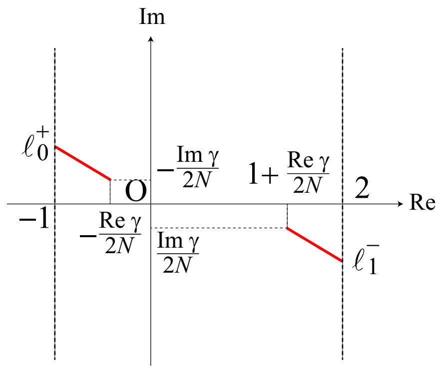

Figure 2: The region (2-14) is between the two thick dotted lines minus the two red lines ℓ_0^+ and ℓ_1^-.

Since we have that $\lim_{\varepsilon\searrow 0}\arg(1-e^{2\pi\sqrt{-1}\varepsilon})=-\frac{1}{2}\pi$, and $-\pi<\arg(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})<0$ because $\operatorname{Im}(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})<0$, (2-13) turns out to be

$$\lim_{\varepsilon\searrow 0}\log\frac{1-e^{2\pi\sqrt{-1}\varepsilon}}{1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma}}=\log\left(\frac{\gamma}{N}\right)$$

by l'Hôpital's rule,

Just for safety, we will check the other limit, $\lim_{\varepsilon\nearrow 0}(\mathcal{L}_1(\varepsilon+1)-\mathcal{L}_1((N/\gamma)\varepsilon+1))$. Since $\operatorname{Im}(N\varepsilon/\gamma+1)<0$ when $\varepsilon<0$, from Lemma 2.5

$$\begin{aligned}\lim_{\varepsilon\nearrow 0}\Bigl(\mathcal{L}_1(\varepsilon+1)-\mathcal{L}_1\Bigl(\frac{N}{\gamma}\varepsilon+1\Bigr)\Bigr)\\ &=\lim_{\varepsilon\nearrow 0}\Bigl(\log(1-e^{2\pi\sqrt{-1}\varepsilon})-\log(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})-2\pi\sqrt{-1}\Bigl\lfloor\operatorname{Re}\Bigl(\frac{N}{\gamma}\varepsilon\Bigr)+1\Bigr\rfloor\Bigr)\\ &=\lim_{\varepsilon\nearrow 0}\log\frac{1-e^{2\pi\sqrt{-1}\varepsilon}}{1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma}}=\log\left(\frac{\gamma}{N}\right),\end{aligned}$$

where the second equality follows since $\lim_{\varepsilon\nearrow 0}\arg(1-e^{2\pi\sqrt{-1}\varepsilon})=\frac{1}{2}\pi$, $0<\arg(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})<\pi$ because $\operatorname{Im}(1-e^{2N\varepsilon\pi\sqrt{-1}/\gamma})>0$, and $\lim_{\varepsilon\nearrow 0}\lfloor\operatorname{Re}(N\varepsilon/\gamma)+1\rfloor=0$. □

We use Lemma 2.8 to extend the function T_N to the region

$$\{z\in\mathbb{C}\mid -1<\operatorname{Re}z<2\}\setminus(\ell_0^+\cup\ell_1^-), \tag{2-14}$$

where

$$\ell_0^+:=\Bigl\{z\in\mathbb{C}\Bigm| z=s\gamma \text{ with } -\frac{1}{\operatorname{Re}\gamma}<s\le-\frac{1}{2N}\Bigr\},\quad \ell_1^-:=\Bigl\{z\in\mathbb{C}\Bigm| z=1+s\gamma \text{ with } \frac{1}{2N}\le s<\frac{1}{\operatorname{Re}\gamma}\Bigr\}.$$

See Figure 2. Note that T_N is already defined for z with $-\gamma/(2N)<\operatorname{Re}z<1+\gamma/(2N)$.

If $-1<\operatorname{Re}z\le-\operatorname{Re}\gamma/(2N)$, then we use (2-10) to define

$$T_N(z):=T_N(z+1)+\mathcal{L}_1\Bigl(\frac{N}{\gamma}z+\frac{1}{2}\Bigr), \tag{2-15}$$

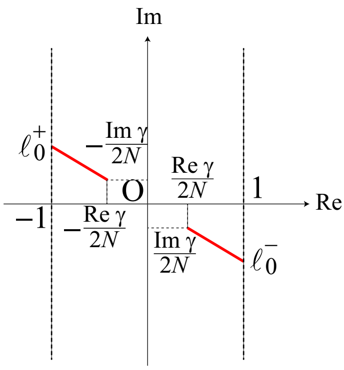

Figure 3: The region (2-17) is between the two thick dotted lines minus the two red lines ℓ_0^+ and ℓ_0^-.

noting that $z+1$ is in the domain of T_N. For the argument of $\mathcal{L}_1$, see Remark 2.13 below. Similarly, if $1+\mathrm{Re}\,\gamma/(2N) \le \mathrm{Re}\,z < 2$, we define

$$T_N(z) := T_N(z-1) - \mathcal{L}_1\Big(\frac{N}{\gamma}(z-1)+\frac{1}{2}\Big), \tag{2-16}$$

noting that $z-1$ is in the domain of T_N. For the argument of $\mathcal{L}_1$, see Remark 2.13 below.

Remark 2.13 Recall that $\mathcal{L}_1(z)$ is defined except for $z \in (-\infty,0] \cup [1,\infty)$. Therefore $Nz/\gamma+\frac{1}{2}$ and $N(z-1)/\gamma+\frac{1}{2}$ are in the domain of $\mathcal{L}_1$ unless

- $-1 < \mathrm{Re}\,z \le -\mathrm{Re}\,\gamma/(2N)$ and $(N/\gamma)z+\frac{1}{2} = s$ for $s \in (-\infty,0] \cup [1,\infty)$, or
- $1+\mathrm{Re}\,\gamma/(2N) \le \mathrm{Re}\,z < 2$ and $(N/\gamma)(z-1)+\frac{1}{2} = t$ for $t \in (-\infty,0) \cup (1,\infty)$.

This is equivalent to

- $-1 < \mathrm{Re}\,z \le -\mathrm{Re}\,\gamma/(2N)$ and $z = s'\gamma$ with $|s'| \ge 1/(2N)$, or
- $1+\mathrm{Re}\,\gamma/(2N) \le \mathrm{Re}\,z < 2$ and $z = 1+t'\gamma$ with $|t'| \ge 1/(2N)$.

Since $\mathrm{Re}\,\gamma > 0$, the condition above turns out to be $z \in \ell_0^+$ or $z \in \ell_1^-$.

We will also use $T_N(z)$ to denote the function extended by using (2-15) and (2-16). Then we have:

Lemma 2.14 *The function $T_N(z)$ extended as above also satisfies (2-10) for any z in the region*

$$\{z \in \mathbb{C} \mid -1 < \mathrm{Re}\,z < 1\} \setminus (\ell_0^+ \cup \ell_0^-) \tag{2-17}$$

with $\ell_0^- := \{z \in \mathbb{C} \mid z = s\gamma$ with $1/(2N) \le s < 1/\mathrm{Re}\,\gamma\}$; see Figure 3.

Remark 2.15 As in Remark 2.13, $Nz/\gamma+\frac{1}{2}$ is in the domain of $\mathcal{L}_1$ unless $z \in \ell_0^+ \cup \ell_0^-$.

Proof If $-\mathrm{Re}\,\gamma/(2N) < \mathrm{Re}\,z < \mathrm{Re}\,\gamma/(2N)$, then (2-10) is proved in Lemma 2.8. If $-1 < \mathrm{Re}\,z \le -\mathrm{Re}\,\gamma/(2N)$ and $\mathrm{Re}\,\gamma/(2N) \le \mathrm{Re}\,z < 1$, then we define T_N by (2-15) and (2-16), respectively, so that (2-10) holds. □

Lemma 2.16 *The function $T_N(z)$ defined as above is holomorphic in the region (2-14).*

Proof From (2-1), $T_N(z)$ is holomorphic in $\{z \in \mathbb{C} \mid -\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 1 + \operatorname{Re}\gamma/(2N)\}$. Therefore from the definition using (2-15) and (2-16), $T_N(z)$ is holomorphic in the disjoint strips

$$\left\{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < -\frac{\operatorname{Re}\gamma}{2N}\right\} \sqcup \left\{z \in \mathbb{C} \mid 1 + \frac{\operatorname{Re}\gamma}{2N} < \operatorname{Re} z < 2\right\}.$$

So we need to confirm that $T_N(z)$ is holomorphic for z with $\operatorname{Re} z = -\operatorname{Re}\gamma/(2N)$ or $1 + \operatorname{Re}\gamma/(2N)$.

Let B be an open disk centered at z (where $\operatorname{Re} z = -\operatorname{Re}\gamma/(2N)$) with radius less than $\operatorname{Re}\gamma/(2N)$. Then for $w \in B$ with $\operatorname{Re} w \le -\operatorname{Re}\gamma/(2N)$, we have

$$T_N(w) = T_N(w+1) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}w + \frac{1}{2}\Bigr)$$

from (2-15). On the other hand, for $w \in B$ with $\operatorname{Re} w > -\operatorname{Re}\gamma/(2N)$, $T_N(w)$ is defined by using (2-1). However, from Lemma 2.8, this coincides with $T_N(w+1) + \mathcal{L}_1\bigl((N/\gamma)w + \frac{1}{2}\bigr)$. Therefore T_N is holomorphic in this case.

Similarly, we can prove the holomorphicity of T_N for the other case. □

Let Ω be the region defined as

$$\Omega := \left\{z \in \mathbb{C} \;\middle|\; -1 + \frac{\operatorname{Re}\gamma}{2N} < \operatorname{Re} z < 2 - \frac{\operatorname{Re}\gamma}{2N}\right\} \setminus (\Delta_0^+ \cup \Delta_1^+), \tag{2-18}$$

where we put

$$\Delta_0^+ := \left\{z \in \mathbb{C} \;\middle|\; -1 + \frac{\operatorname{Re}\gamma}{2N} < \operatorname{Re} z \le 0,\ \operatorname{Im} z \ge 0, \text{ and } \operatorname{Im}\Bigl(\frac{z}{\gamma}\Bigr) \le 0\right\},$$

$$\Delta_1^- := \left\{z \in \mathbb{C} \;\middle|\; 1 \le \operatorname{Re} z < 2 - \frac{\operatorname{Re}\gamma}{2N},\ \operatorname{Im} z \le 0, \text{ and } \operatorname{Im}\Bigl(\frac{z-1}{\gamma}\Bigr) \ge 0\right\}.$$

See Figure 4. Note that Ω is contained in the region (2-14) because

$$\ell_0^+ \cap \left\{z \in \mathbb{C} \;\middle|\; -1 + \frac{\operatorname{Re}\gamma}{2N} < \operatorname{Re} z < 2 - \frac{\operatorname{Re}\gamma}{2N}\right\}$$

and

$$\ell_1^- \cap \left\{z \in \mathbb{C} \;\middle|\; -1 + \frac{\operatorname{Re}\gamma}{2N} < \operatorname{Re} z < 2 - \frac{\operatorname{Re}\gamma}{2N}\right\},$$

are on the upper side of Δ_0^+ and the lower side of Δ_1^-, respectively.

Lemma 2.17 *The function $T_N(z)$ extended by using (2-10) satisfies (2-11) for $z \in \Omega$.*

Remark 2.18 The left-hand side of (2-11) is defined for z such that $z \pm \gamma/(2N)$ is in the region (2-14), that is, $z \pm \gamma/(2N) \notin \ell_0^+ \cup \ell_1^-$. This is equivalent to saying that z is not on the two rays $\{s\gamma \in \mathbb{C} \mid s \le 0\} \cup \{1 + s\gamma \in \mathbb{C} \mid s \ge 0\}$. Note that the ray $\{s\gamma \in \mathbb{C} \mid s \le 0\}$ includes the upper edge of Δ_0^+, and that the ray $\{1 + s\gamma \in \mathbb{C} \mid s \ge 0\}$ includes the lower edge of Δ_1^-. The right-hand side of (2-11) is defined unless $z \in (-\infty, 0] \cup [1, \infty)$.

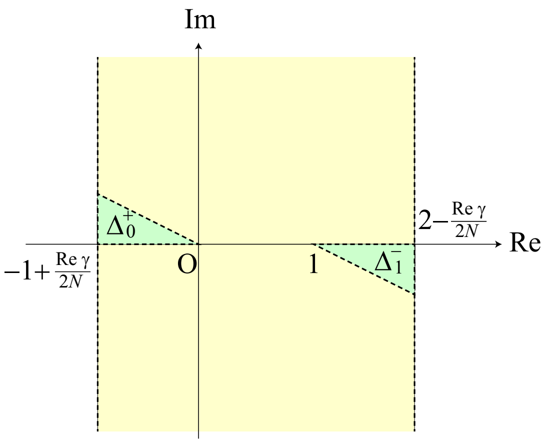

Figure 4: The yellow region is Ω. The green triangles are Δ_0^+ and Δ_1^-.

Proof We need to prove (2-11) for z with $-1+\operatorname{Re}\gamma/(2N) < \operatorname{Re} z \le 0$ or $1 \le \operatorname{Re} z < 2-\operatorname{Re}\gamma/(2N)$.

If $-\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < 0$, from (2-12), we have

$$\begin{aligned} T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right) &= T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+1-\frac{\gamma}{2N}\right)+\mathcal{L}_1(z+1)-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right) \\ &= \mathcal{L}_1\left(\frac{N}{\gamma}\left(z-\frac{\gamma}{2N}\right)+\frac{1}{2}\right)+\mathcal{L}_1(z+1)-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right) \\ &= \mathcal{L}_1\left(\frac{N}{\gamma}z\right)+\mathcal{L}_1(z+1)-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right), \end{aligned}$$

where we use Lemma 2.14 for $z-\gamma/(2N)$ at the second equality. If $\operatorname{Im} z \ge 0$, then $\operatorname{Im}(z/\gamma) > 0$ from (2-18). So $\mathcal{L}_1(z+1) = \mathcal{L}_1(z)$ and $\mathcal{L}_1(Nz/\gamma+1) = \mathcal{L}_1(Nz/\gamma)$ from (2-5), which implies (2-11). If $\operatorname{Im} z < 0$, then we have $\operatorname{Im}(Nz/\gamma+1) = (N/|\gamma|^2)(\operatorname{Re}\gamma \operatorname{Im} z - \operatorname{Im}\gamma \operatorname{Re} z) < 0$. So $\mathcal{L}_1(z+1) = \mathcal{L}_1(z)+2\pi\sqrt{-1}$ and $\mathcal{L}_1(Nz/\gamma+1) = \mathcal{L}_1(Nz/\gamma)+2\pi\sqrt{-1}$ from Corollary 2.6, proving (2-11).

If $\operatorname{Re} z = 0$, then noting that 0 is not included in Ω, similarly we have

$$\begin{aligned} T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right) &= T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+1-\frac{\gamma}{2N}\right)+\mathcal{L}_1(z)-\mathcal{L}_1\left(\frac{N}{\gamma}z\right) \\ &= \mathcal{L}_1\left(\frac{N}{\gamma}\left(z-\frac{\gamma}{2N}\right)+\frac{1}{2}\right)+\mathcal{L}_1(z)-\mathcal{L}_1\left(\frac{N}{\gamma}z\right) = \mathcal{L}_1(z). \end{aligned}$$

If $\operatorname{Re} z = -\operatorname{Re}\gamma/(2N)$, then $\operatorname{Re}(z+\gamma/(2N)) = 0$ and $-1 < \operatorname{Re}(z-\gamma/(2N)) = -\operatorname{Re}\gamma/N < -\operatorname{Re}\gamma/(2N)$. Therefore from (2-15) we have

$$\begin{aligned} T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right) &= T_N\left(z-\frac{\gamma}{2N}+1\right)-T_N\left(z+\frac{\gamma}{2N}\right)+\mathcal{L}_1\left(\frac{N}{\gamma}\left(z-\frac{\gamma}{2N}\right)+\frac{1}{2}\right) \\ &= T_N\left(z-\frac{\gamma}{2N}+1\right)-T_N\left(z+\frac{\gamma}{2N}\right)+\mathcal{L}_1\left(\frac{N}{\gamma}z\right) \\ &= \mathcal{L}_1(z+1)-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right)+\mathcal{L}_1\left(\frac{N}{\gamma}z\right), \end{aligned} \tag{2-19}$$

where the last equality follows from Lemma 2.11 since $\operatorname{Re} z < 0$. If $\operatorname{Im} z \ge 0$, then $\operatorname{Im}(z/\gamma) > 0$, and so (2-19) turns out to be $\mathcal{L}_1(z)$. If $\operatorname{Im} z < 0$, then $\operatorname{Im}(z/\gamma) < 0$. Therefore (2-19) equals

$$\log(1-e^{2\pi\sqrt{-1}z}) - 2\pi\sqrt{-1} = \mathcal{L}_1(z)$$

from Lemma 2.5 and Corollary 2.6.

We consider the case where $-1+\operatorname{Re}\gamma/(2N) < \operatorname{Re} z < -\operatorname{Re}\gamma/(2N)$. Note that $-1 < \operatorname{Re}(z \pm \gamma/(2N)) < 0$. Therefore from (2-15) we have

$$\begin{aligned} T_N\Bigl(z-\frac{\gamma}{2N}\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}\Bigr) &= T_N\Bigl(z-\frac{\gamma}{2N}+1\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}+1\Bigr) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}\Bigl(z-\frac{\gamma}{2N}\Bigr)+\frac{1}{2}\Bigr) - \mathcal{L}_1\Bigl(\frac{N}{\gamma}\Bigl(z+\frac{\gamma}{2N}\Bigr)+\frac{1}{2}\Bigr) \\ &= \mathcal{L}_1(z+1) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}z\Bigr) - \mathcal{L}_1\Bigl(\frac{N}{\gamma}z+1\Bigr), \end{aligned}$$

where we use (2-11) because $0 < \operatorname{Re}(z+1) < 1$. By the same reason as above, this equals $\mathcal{L}_1(z)$.

If $1 \le \operatorname{Re} z < 1+\operatorname{Re}\gamma/(2N)$, then from (2-16), we have

$$\begin{aligned} T_N\Bigl(z-\frac{\gamma}{2N}\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}\Bigr) &= T_N\Bigl(z-\frac{\gamma}{2N}\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}-1\Bigr) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}\Bigl(z+\frac{\gamma}{2N}-1\Bigr)+\frac{1}{2}\Bigr) \\ &= \mathcal{L}_1(z-1) - \mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)\Bigr) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)+1\Bigr), \end{aligned}$$

where we use (2-12) for $z-1$ at the second identity, noting $1 \notin \Omega$. If $\operatorname{Im} z \ge 0$, then $\operatorname{Im}((z-1)/\gamma) = (1/|\gamma|^2)(\operatorname{Im}\gamma(1-\operatorname{Re} z)+\operatorname{Re}\gamma\operatorname{Im} z) \ge 0$, so the last line equals $\mathcal{L}_1(z)$. If $\operatorname{Im} z < 0$, then $\operatorname{Im}((z-1)/\gamma) < 0$ from the definition of Δ_1^-, and so we have $\mathcal{L}_1(z-1) = \mathcal{L}_1(z) - 2\pi\sqrt{-1}$ and $\mathcal{L}_1(N(z-1)/\gamma+1) = \mathcal{L}_1(N(z-1)/\gamma) + 2\pi\sqrt{-1}$ from Corollary 2.6, which implies (2-11).

Lastly, we consider the case where $1+\operatorname{Re}\gamma/(2N) \le \operatorname{Re} z < 2-\operatorname{Re}\gamma/(2N)$. Since $1 \le \operatorname{Re}(z \pm \gamma/(2N)) < 2$, from (2-16), we have

$$\begin{aligned} T_N\Bigl(z-\frac{\gamma}{2N}\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}\Bigr) &= T_N\Bigl(z-\frac{\gamma}{2N}-1\Bigr) - T_N\Bigl(z+\frac{\gamma}{2N}-1\Bigr) - \mathcal{L}_1\Bigl(\frac{N}{\gamma}\Bigl(z-\frac{\gamma}{2N}-1\Bigr)+\frac{1}{2}\Bigr) \\ &\qquad + \mathcal{L}_1\Bigl(\frac{N}{\gamma}\Bigl(z+\frac{\gamma}{2N}-1\Bigr)+\frac{1}{2}\Bigr) \\ &= \mathcal{L}_1(z-1) - \mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)\Bigr) + \mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)+1\Bigr), \end{aligned} \tag{2-20}$$

using (2-11) at the last equality. If $\operatorname{Im} z \ge 0$, then $\operatorname{Im}((z-1)/\gamma) = (1/|\gamma|^2)(\operatorname{Im}\gamma(1-\operatorname{Re} z)+\operatorname{Re}\gamma\operatorname{Im} z) > 0$ since $\operatorname{Re} z > 1$. So (2-20) equals $\log(1-e^{2\pi\sqrt{-1}z}) = \mathcal{L}_1(z)$. If $\operatorname{Im} z < 0$, then $\operatorname{Im}((z-1)/\gamma) < 0$. Therefore (2-20) becomes

$$\log(1-e^{2\pi\sqrt{-1}z}) + 2\pi\sqrt{-1} = \mathcal{L}_1(z)$$

from Lemma 2.5. □

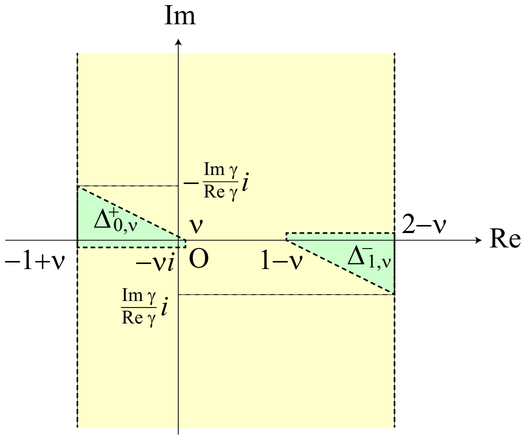

Figure 5: The yellow region is Ω^*_ν. The green trapezoids are $\Delta^+_{0,\nu}$ and $\Delta^-_{1,\nu}$.

Remark 2.19 Even if $z \in \operatorname{Int}\Delta^+_0 \cup \operatorname{Int}\Delta^-_1$, where Int means the interior, both sides of (2-11) are defined from Remark 2.18. However, if $z \in \operatorname{Int}\Delta^+_0$, then from the proof above,

$$T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right)=\mathcal{L}_1(z+1)+\mathcal{L}_1\left(\frac{N}{\gamma}z\right)-\mathcal{L}_1\left(\frac{N}{\gamma}z+1\right)=\mathcal{L}_1(z)-2\pi\sqrt{-1},$$

where the second equality follows from Corollary 2.6 since $\operatorname{Im} z > 0$ and $\operatorname{Im}(N/\gamma) < 0$. Similarly, if $z \in \operatorname{Int}\Delta^-_1$, we have

$$T_N\left(z-\frac{\gamma}{2N}\right)-T_N\left(z+\frac{\gamma}{2N}\right)=\mathcal{L}_1(z-1)-\mathcal{L}_1\left(\frac{N}{\gamma}(z-1)\right)+\mathcal{L}_1\left(\frac{N}{\gamma}(z-1)+1\right)=\mathcal{L}_1(z)-2\pi\sqrt{-1},$$

since $\operatorname{Im} z < 0$ and $\operatorname{Im}((N/\gamma)(z-1)) > 0$.

For a real number $0 < \nu < \frac{1}{2}$ and a positive real number M, we put

(2-21) $$\Omega^*_\nu := \{z \in \mathbb{C} \mid -1+\nu \le \operatorname{Re} z \le 2-\nu, |\operatorname{Im} z| \le M\} \setminus (\Delta^+_{0,\nu} \cup \Delta^-_{1,\nu}),$$

where we put

$$\Delta^+_{0,\nu} := \left\{z \in \mathbb{C} \;\middle|\; -1+\nu \le \operatorname{Re} z < \nu, \operatorname{Im} z > -\nu, \text{ and } \operatorname{Im}\left(\frac{z-\nu}{\gamma}\right) < 0\right\},$$
$$\Delta^-_{1,\nu} := \left\{z \in \mathbb{C} \;\middle|\; 1-\nu < \operatorname{Re} z \le 2-\nu, \operatorname{Im} z < \nu, \text{ and } \operatorname{Im}\left(\frac{z-1+\nu}{\gamma}\right) > 0\right\}.$$

Note that $\Omega^*_\nu \subset \Omega$ if $N > \operatorname{Re}\gamma/(2\nu)$. Note also that $\Delta^+_{0,\nu} \cap \Delta^-_{1,\nu} = \varnothing$ since $\nu < \frac{1}{2}$; see Figure 5.

We can prove that $T_N(z)$ uniformly converges to $N/(2\pi\sqrt{-1}\gamma)\mathcal{L}_2(z)$ in Ω^*_ν. To do that, we prepare several lemmas.

Lemma 2.20 *Let ν and M be positive real numbers with $0 < \nu < \frac{1}{2}$. Then*

$$\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z-\frac{\gamma}{2N}\right)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z+\frac{\gamma}{2N}\right)=\mathcal{L}_1(z)+O(N^{-2})$$

as $N \to \infty$ for z in the region

$$\{z \in \mathbb{C} \mid -1+\nu \le \operatorname{Re} z \le 2-\nu,\ |\operatorname{Im} z| \le M\} \setminus (\square_\nu^- \cup \square_\nu^+), \tag{2-22}$$

where

$$\square_\nu^- := \{z \in \mathbb{C} \mid -1+\nu \le \operatorname{Re} z \le \nu, |\operatorname{Im} z| \le \nu\}, \quad \square_\nu^+ := \{z \in \mathbb{C} \mid 1-\nu \le \operatorname{Re} z \le 2-\nu, |\operatorname{Im} z| \le \nu\}.$$

This means that there exists a constant $c > 0$ that does not depend on z such that

$$\left| \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z - \frac{\gamma}{2N}\right) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z + \frac{\gamma}{2N}\right) - \mathcal{L}_1(z)\right| < \frac{c}{N^2}$$

for sufficiently large N.

Proof Note that if z is in the region (2-22), then $z \pm \gamma/(2N)$ is also in the same region, assuming that N is large enough. Note also that $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic there.

Since

$$\mathcal{L}_2'(z) = -2\pi\sqrt{-1}\mathcal{L}_1(z), \quad \mathcal{L}_2''(z) = \frac{4\pi^2}{1-e^{-2\pi\sqrt{-1}z}} \quad \text{and} \quad \mathcal{L}_2^{(3)}(z) = 2\pi^3\sqrt{-1}\csc^2(\pi z)$$

($\csc x = 1/\sin x$ is the cosecant of x, as you may know) from Lemma 2.7, we have

$$\begin{aligned}\mathcal{L}_2\left(z \pm \frac{\gamma}{2N}\right) = \mathcal{L}_2(z) \mp 2\pi\sqrt{-1}\mathcal{L}_1(z)\frac{\gamma}{2N} + \frac{2\pi^2}{1-e^{-2\pi\sqrt{-1}z}}\frac{\gamma^2}{4N^2}& \\ \pm \frac{\pi^3\sqrt{-1}}{3\sin^2(\pi z)}\frac{\gamma^3}{8N^3} + \sum_{j=4}^{\infty}\frac{2\pi^3\sqrt{-1}}{j!}\frac{d^{j-3}\csc^2(\pi z)}{dz^{j-3}}\left(\pm\frac{\gamma}{2N}\right)^j&\end{aligned}$$

if N is large enough that $z \pm \gamma/(2N)$ is contained in the region (2-22). So

$$\begin{aligned}\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z - \frac{\gamma}{2N}\right) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\left(z + \frac{\gamma}{2N}\right)& \\ = \mathcal{L}_1(z) - \sum_{k=1}^{\infty}\frac{\pi^2}{(2k+1)!}\frac{d^{2k-2}\csc^2(\pi z)}{dz^{2k-2}}\left(\frac{\gamma}{2N}\right)^{2k}&.\end{aligned} \tag{2-23}$$

From Lemma 2.21 below, we have

$$\sin^{2k}(\pi z)\frac{d^{2k-2}\csc^2(\pi z)}{dz^{2k-2}} = 2\pi^{2k-2}\sum_{j=0}^{k-1}a_{2k-2,2j}\cos(2j\pi z)$$

with $a_{2k-2,2j} > 0$ for $j = 0, 1, \dots, k-1$ and $\sum_{j=0}^{k-1}a_{2k-2,2j} = \frac{1}{2}(2k-1)!$. Letting L be the maximum of $|\cos(z)|$ in the closure of (2-22), we have

$$\begin{aligned}\left|\sin^{2k}(\pi z)\frac{\pi^2}{(2k+1)!}\frac{d^{2k-2}\csc^2(\pi z)}{dz^{2k-2}}\left(\frac{\gamma}{2N}\right)^{2k}\right| &\le \frac{\pi^2}{(2k+1)!}2\pi^{2k-2}Lk\frac{(2k-1)!}{2}\left(\frac{|\gamma|}{2N}\right)^{2k} \\ &= \frac{L}{2(2k+1)}\left(\frac{\pi|\gamma|}{2N}\right)^{2k}.\end{aligned}$$

Let l be the minimum of $|\sin(\pi z)|$ in the closure of the region (2-22). Since the closure is compact and does not contain the zeros of $\sin(\pi z)$, we conclude that $l > 0$. So

$$N^2\Bigl|\sum_{k=1}^{\infty}\frac{\pi^2}{(2k+1)!}\frac{d^{2k-2}\csc^2(\pi z)}{dz^{2k-2}}\Bigl(\frac{\gamma}{2N}\Bigr)^{2k}\Bigr| < \frac{L\pi^2|\gamma|^2}{8l^2}\sum_{k=1}^{\infty}\frac{1}{2k+1}\Bigl(\frac{\pi|\gamma|}{2lN}\Bigr)^{2k-2},$$

which converges if $N > \pi|\gamma|/(2l)$.

Therefore the right-hand side of (2-23) turns out to be $\mathcal{L}_1(z) + O(N^{-2})$, completing the proof. □

Lemma 2.21 *Let m be a positive integer. The m^{th} derivative of $\csc^2(\pi z)$ can be expressed as*

$$\frac{d^m\csc^2(\pi z)}{dz^m} = 2(-\pi)^m\csc^{m+2}(\pi z)P_m(z),$$

where $P_m(z)$ is of the form

$$P_m(z) = \sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} a_{m,j}\cos(j\pi z)$$

with

- (i) $a_{m,j} > 0$ *for* $0 \le j \le m$ *and* $j \equiv m \pmod 2$,
- (ii) $\sum_{0\le j\le m,\, j\equiv m \pmod 2} a_{m,j} = \frac{1}{2}(m+1)!$, *and*
- (iii) $a_{m,m} = 1$.

Proof First of all, recall that $\csc'(x) = -\cos(x)\csc^2(x)$.

We proceed by induction on m.

For $m = 1$, since $(d/dz)\csc^2(\pi z) = 2\csc(\pi z)(-\pi\cos(\pi z)\csc^2(\pi z)) = -2\pi\csc^3(\pi z)\cos(\pi z)$, we have $P_1(z) = \cos(\pi z)$, which agrees with (i)–(iii).

Suppose that the lemma is true for m. We calculate the $(m+1)^{\text{st}}$ derivative by using the inductive hypothesis for $P_m(z)$. We have

$$\begin{aligned}
&\frac{d^{m+1}\csc^2(\pi z)}{dz^{m+1}}\\
&\quad= 2(-\pi)^m\frac{d}{dz}(\csc^{m+2}(\pi z)P_m(z))\\
&\quad= 2(-\pi)^m\bigl((m+2)\csc^{m+1}(\pi z)(-\pi\cos(\pi z)\csc^2(\pi z))P_m(z) + \csc^{m+2}(\pi z)P'_m(z)\bigr)\\
&\quad= 2(-\pi)^m\csc^{m+3}(\pi z)[-(m+2)\pi\cos(\pi z)P_m(z) + \sin(\pi z)P'_m(z)]\\
&\quad= 2(-\pi)^m\csc^{m+3}(\pi z)\\
&\qquad\quad\cdot\Bigl[-(m+2)\pi\cos(\pi z)\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} a_{m,j}\cos(j\pi z) - \sin(\pi z)\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} j\pi a_{m,j}\sin(j\pi z)\Bigr]\\
&\quad= 2(-\pi)^{m+1}\csc^{m+3}(\pi z)\\
&\qquad\quad\cdot\Bigl[(m+2)\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} a_{m,j}\cos(\pi z)\cos(j\pi z) + \sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} ja_{m,j}\sin(\pi z)\sin(j\pi z)\Bigr].
\end{aligned}$$

Now we will calculate the terms inside the square brackets. We write $x := \pi z$. From the product–sum identities, we have

$$\sin(x)\sin(jx) = \tfrac{1}{2}\cos((j-1)x) - \tfrac{1}{2}\cos((j+1)x),$$
$$\cos(x)\cos(jx) = \tfrac{1}{2}\cos((j-1)x) + \tfrac{1}{2}\cos((j+1)x).$$

So we have

$$\begin{aligned}
&(m+2)\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} a_{m,j}\cos(x)\cos(jx) + \sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} j a_{m,j}\sin(x)\sin(jx) \\
&= \frac{1}{2}\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} (m+2)a_{m,j}\bigl(\cos((j-1)x)+\cos((j+1)x)\bigr) \\
&\qquad\qquad + \frac{1}{2}\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} j a_{m,j}\bigl(\cos((j-1)x)-\cos((j+1)x)\bigr) \\
&= \frac{1}{2}\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} ((m+j+2)a_{m,j}\cos((j-1)x) + (m-j+2)a_{m,j}\cos((j+1)x)) \\
&= \frac{1}{2}\sum_{\substack{-1\le k\le m-1\\ k\equiv m+1 \pmod 2}} (m+k+3)a_{m,k+1}\cos(kx) + \frac{1}{2}\sum_{\substack{1\le k\le m+1\\ k\equiv m+1 \pmod 2}} (m-k+3)a_{m,k-1}\cos(kx) \\
&= \begin{cases} \frac{1}{2}\displaystyle\sum_{\substack{0\le k\le m-1\\ k\equiv m+1 \pmod 2}} ((m+k+3)a_{m,k+1}+(m-k+3)a_{m,k-1})\cos(kx) \\ \qquad\qquad + a_{m,m}\cos((m+1)x) & \text{if } m \text{ is odd,} \\ \frac{1}{2}\displaystyle\sum_{\substack{0\le k\le m-1\\ k\equiv m+1 \pmod 2}} ((m+k+3)a_{m,k+1}+(m-k+3)a_{m,k-1})\cos(kx) \\ \qquad\qquad + a_{m,m}\cos((m+1)x) + \frac{1}{2}(m+2)a_{m,0}\cos(x) & \text{if } m \text{ is even.} \end{cases}
\end{aligned} \tag{2-24}$$

Therefore we obtain the following recursive formula for $a_{m,k}$:

$$2a_{m+1,k} = \begin{cases} (m+k+3)a_{m,k+1} + (m-k+3)a_{m,k-1} & \text{if } k \neq 1, \\ (m+k+3)a_{m,2} + 2(m-k+3)a_{m,0} & \text{if } k = 1. \end{cases}$$

Note that this also holds for $k=0$ and $k=m+1$ by putting $a_{m,-1} = a_{m,m+2} = 0$. Then, (i) follows since $m-k+3 \ge 3$, (iii) follows since $a_{m+1,m+1} = 1$, and (ii) follows since the sum of the coefficients in the third expression of (2-24) equals

$$\frac{1}{2}\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} ((m+j+2)a_{m,j} + (m-j+2)a_{m,j}) = (m+2)\sum_{\substack{0\le j\le m\\ j\equiv m \pmod 2}} a_{m,j}. \qquad \square$$

For a real number $\nu > 0$, we define the region

$$\bowtie_\nu := \left\{ z\in\mathbb{C} \;\middle|\; \operatorname{Im} z \ge 0, \operatorname{Im}\left(\frac{z-\nu}{\gamma}\right) \le 0 \right\} \cup \left\{ z\in\mathbb{C} \;\middle|\; \operatorname{Im} z \le 0, \operatorname{Im}\left(\frac{z+\nu}{\gamma}\right) \ge 0 \right\}.$$

See Figure 6.

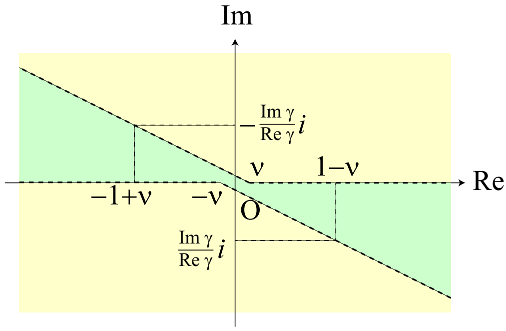

Figure 6: The green region is $\bowtie_\nu$.

Lemma 2.22 *There exist positive real numbers c and ε such that*

$$\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z+1)-\mathcal{L}_1\left(\frac{N}{\gamma}z+\frac{1}{2}\right)\right|<ce^{-\varepsilon N}$$

for any z in the region $\mathbb{C}\setminus\bowtie_\nu$ if N is sufficiently large.

Remark 2.23 The left-hand side is defined unless $\operatorname{Im} z=0$ or $z=s\gamma$ $(|s|\geq 1/(2N))$. Therefore if $z\notin\bowtie_\nu$, then the left-hand side is defined.

Proof Note that $\operatorname{Im} z\neq 0$ if $z\notin\bowtie_\nu$.

First, suppose that $\operatorname{Im} z>0$.

Since $\mathcal{L}_2(z)=\mathcal{L}_2(z+1)$ from (2-4), we will prove that $|\mathcal{L}_1\bigl((N/\gamma)z+\frac{1}{2}\bigr)|<ce^{-\varepsilon N}$ for some $c>0$ and $\varepsilon>0$. Note that $\operatorname{Im}(z/\gamma)>\operatorname{Im}(\nu/\gamma)=-\nu\operatorname{Im}\gamma/|\gamma|^2>0$ because $z\notin\bowtie_\nu$. So

$$\mathcal{L}_1\left(\frac{N}{\gamma}z+\frac{1}{2}\right)=\log(1+e^{2N\pi\sqrt{-1}z/\gamma})$$

from (2-3). Now since $\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}x^k/k$ for $|x|<1$, one has

$$|\log(1+e^a)|\leq\sum_{k=1}^{\infty}\frac{e^{k\operatorname{Re} a}}{k}<\sum_{k=1}^{\infty}e^{k\operatorname{Re} a}=\frac{e^{\operatorname{Re} a}}{1-e^{\operatorname{Re} a}}$$

if $\operatorname{Re} a<0$. Since $\operatorname{Re}(2N\pi\sqrt{-1}z/\gamma)=-2N\pi\operatorname{Im}(z/\gamma)<2N\pi\nu\operatorname{Im}\gamma/|\gamma|^2<0$, we have

$$|\log(1+e^{2N\pi\sqrt{-1}z/\gamma})|<\frac{e^{2N\pi\nu\operatorname{Im}\gamma/|\gamma|^2}}{1-e^{2N\pi\nu\operatorname{Im}\gamma/|\gamma|^2}}<ce^{-\varepsilon N}$$

where we put $\varepsilon:=-2\pi\nu\operatorname{Im}\gamma/|\gamma|^2>0$ and $c:=1/(1-e^{-\varepsilon})>0$.

Next, suppose that $\operatorname{Im} z<0$.

From Corollary 2.6, we have

$$\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z+1)=\frac{2N\pi\sqrt{-1}z}{\gamma}.$$

Since $z \notin \bowtie_\nu$, we have $\operatorname{Im}(z/\gamma) < -\operatorname{Im}(\nu/\gamma) = \nu \operatorname{Im}\gamma/|\gamma|^2 < 0$. Thus from (2-3) we obtain

$$\mathcal{L}_1\Big(\frac{N}{\gamma}z+\frac{1}{2}\Big) = \log(1+e^{-2N\pi\sqrt{-1}z/\gamma}) + \frac{2N\pi\sqrt{-1}z}{\gamma}.$$

Since $\operatorname{Re}(-2N\pi\sqrt{-1}z/\gamma) = 2N\pi\operatorname{Im}(z/\gamma) < 2N\pi\nu\operatorname{Im}\gamma/|\gamma|^2$, we finally have

$$\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z+1) - \mathcal{L}_1\Big(\frac{N}{\gamma}z+\frac{1}{2}\Big)\right| = |\log(1+e^{-2N\pi\sqrt{-1}z/\gamma})| < ce^{-\varepsilon N}$$

as above, completing the proof. □

The following lemma is similar to [24, Lemma 2.4] and the proof is omitted.

Lemma 2.24 *Let ν and M be positive real numbers. Then there exists a constant $c>0$ such that*

$$\left|T_N(z) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| = \frac{c}{N}$$

for z in the region $\{z\in\mathbb{C} \mid \nu \le \operatorname{Re} z \le 1-\nu,\ |\operatorname{Im} z| \le M\}$ if N is sufficiently large, where c does not depend on z.

Now we can prove the following proposition:

Proposition 2.25 *Suppose that $\nu < \frac{1}{4}$. We have*

$$T_N(z) = \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z) + O(N^{-1})$$

as $N\to\infty$ in the region Ω^*_ν.

Proof We need to prove the proposition for z with $-1+\nu \le \operatorname{Re} z < \nu$ or $1-\nu < \operatorname{Re} z \le 2-\nu$.

If $z \in \Omega^*_\nu$ and $-1+\nu \le \operatorname{Re} z < -\nu$, we use (2-15). We have

$$\begin{aligned}
&\left|T_N(z) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| \\
&\quad = \left|T_N(z+1) + \mathcal{L}_1\Big(\frac{N}{\gamma}z+\frac{1}{2}\Big) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| \\
&\quad \le \left|T_N(z+1) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z+1)\right| + \left|\mathcal{L}_1\Big(\frac{N}{\gamma}z+\frac{1}{2}\Big) - \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z) + \frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z+1)\right| \\
&\quad = O(1/N),
\end{aligned}$$

where we apply Lemmas 2.24 and 2.22, noting that we can apply Lemma 2.22 because $z \notin \bowtie_\nu$.

Similarly, if $z \in \Omega^*_\nu$ and $1+\nu < \operatorname{Re} z \le 2-\nu$, using (2-16), we have

$$\begin{aligned}
&\left|T_N(z)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| \\
&\quad=\left|T_N(z-1)-\mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)+\frac{1}{2}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| \\
&\quad\le\left|T_N(z-1)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z-1)\right|+\left|-\mathcal{L}_1\Bigl(\frac{N}{\gamma}(z-1)+\frac{1}{2}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)+\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z-1)\right| \\
&\quad=O(1/N),
\end{aligned}$$

noting that we can apply Lemma 2.22 because $z-1 \notin \bowtie_\nu$.

If $z \in \Omega^*_\nu$ and $-\nu \le \operatorname{Re} z < \nu$, we put $m := \lfloor 2N\nu/\operatorname{Re}\gamma \rfloor + 1$. From Lemma 2.17 we have

$$\begin{aligned}
T_N(z) &= T_N\Bigl(z+\frac{\gamma}{N}\Bigr)+\mathcal{L}_1\Bigl(z+\frac{\gamma}{2N}\Bigr) = T_N\Bigl(z+\frac{2\gamma}{N}\Bigr)+\mathcal{L}_1\Bigl(z+\frac{3\gamma}{2N}\Bigr)+\mathcal{L}_1\Bigl(z+\frac{\gamma}{2N}\Bigr) \\
&= \cdots = T_N\Bigl(z+\frac{m\gamma}{N}\Bigr)+\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr).
\end{aligned}$$

Now since $m \le 2N\nu/\operatorname{Re}\gamma + 1 < m+1$, we have $\nu < \operatorname{Re}(z+m\gamma/N) < 3\nu + \operatorname{Re}\gamma/N < 1-\nu$ if $N > \operatorname{Re}\gamma/(1-4\nu)$, and so we can apply Lemma 2.24 to $z+m\gamma/N$. We have

$$\begin{aligned}
&\left|T_N(z)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)\right| \\
&\qquad=\left|T_N\Bigl(z+\frac{m\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)+\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr)\right| \\
&\qquad\le\left|T_N\Bigl(z+\frac{m\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{m\gamma}{N}\Bigr)\right| \\
&\qquad\qquad+\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{m\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)+\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr)\right| \\
&\qquad=\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{m\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)+\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr)\right|+O(1/N).
\end{aligned}$$

Since

$$\mathcal{L}_2\Bigl(z+\frac{m\gamma}{N}\Bigr)-\mathcal{L}_2(z)=\sum_{j=1}^{m}\Bigl(\mathcal{L}_2\Bigl(z+\frac{j\gamma}{N}\Bigr)-\mathcal{L}_2\Bigl(z+\frac{(j-1)\gamma}{N}\Bigr)\Bigr),$$

we have

$$\begin{aligned}
&\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{m\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2(z)+\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr)\right| \\
&\quad\le\sum_{j=1}^{m}\left|\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{j\gamma}{N}\Bigr)-\frac{N}{2\pi\sqrt{-1}\gamma}\mathcal{L}_2\Bigl(z+\frac{(j-1)\gamma}{N}\Bigr)+\mathcal{L}_1\Bigl(z+\frac{(2j-1)\gamma}{2N}\Bigr)\right|.
\end{aligned}\tag{2-25}$$

We use Lemma 2.20 to conclude that each summand of the right-hand side of (2-25) is less than c/N^2 for $c>0$. Note that c is independent of j. Since $m=\lfloor 2N\nu/\operatorname{Re}\gamma\rfloor+1$, the right-hand side of (2-25) is less than

$$\frac{mc}{N^2}\le\Bigl(\frac{2N\nu}{\operatorname{Re}\gamma}+1\Bigr)\frac{c}{N^2}\le\frac{c'}{N}$$

if we put $c':=(2c\nu/\operatorname{Re}\gamma+1)$.

If $1-\nu<\operatorname{Re}z\le 1+\nu$, from Lemma 2.17 we have

$$\begin{aligned}T_N(z)&=T_N\Bigl(z-\frac{\gamma}{N}\Bigr)-\mathcal{L}_1\Bigl(z-\frac{\gamma}{2N}\Bigr)=T_N\Bigl(z-\frac{2\gamma}{N}\Bigr)-\mathcal{L}_1\Bigl(z-\frac{3\gamma}{2N}\Bigr)-\mathcal{L}_1\Bigl(z-\frac{\gamma}{2N}\Bigr)\\&=\cdots=T_N\Bigl(z-\frac{m\gamma}{N}\Bigr)-\sum_{j=1}^{m}\mathcal{L}_1\Bigl(z-\frac{(2j-1)\gamma}{2N}\Bigr),\end{aligned}$$

where we put $m:=\lfloor 2N\nu/\operatorname{Re}\gamma\rfloor+1$ as before. Since $\nu<\operatorname{Re}(z-m\gamma/N)<1-\nu$ as before, we can prove the proposition similarly. □

3 The colored Jones polynomial

In this section, we show several results following [24].

First of all, we recall the following formula due to Habiro [10, page 36, (1)] and Le [18, 1.2.2 Example, page 129] (see also [20, Theorem 5.1]):

$$\text{(3-1)}\qquad \begin{aligned}J_N(\mathscr{E};q)&=\sum_{k=0}^{N-1}\prod_{l=1}^{k}(q^{(N-l)/2}-q^{-(N-l)/2})(q^{(N+l)/2}-q^{-(N+l)/2})\\&=\sum_{k=0}^{N-1}q^{-kN}\prod_{l=1}^{k}(1-q^{N-l})(1-q^{N+l}).\end{aligned}$$

For a positive integer p, we put $\xi:=\kappa+2p\pi\sqrt{-1}$, where $\kappa:=\operatorname{arccosh}\bigl(\frac{3}{2}\bigr)$. We will study the asymptotic behavior of

$$J_N(\mathscr{E};e^{\xi/N})=\sum_{k=0}^{N-1}\prod_{l=1}^{k}e^{-k\xi}(1-e^{(N-l)\xi/N})(1-e^{(N+l)\xi/N})$$

as $N\to\infty$.

We can express $J_N(\mathscr{E};e^{\xi/N})$ in terms of T_N, putting $\gamma:=\xi/(2\pi\sqrt{-1})$, similarly to [24, Section 3, (3.2)]. We have

$$\text{(3-2)}\qquad J_N(\mathscr{E};e^{\xi/N})=(1-e^{-4pN\pi^2/\xi})\sum_{m=0}^{p-1}\Biggl(\beta_{p,m}\sum_{mN/p<k\le(m+1)N/p}\exp\Bigl(Nf_N\Bigl(\frac{2k+1}{2N}-\frac{m}{\gamma}\Bigr)\Bigr)\Biggr)$$

since $2\sinh\bigl(\frac{1}{2}\kappa\bigr)=1$, where we put

$$\text{(3-3)}\qquad \beta_{p,m}:=e^{-4mpN\pi^2/\xi}\prod_{j=1}^{m}(1-e^{4(p-j)N\pi^2/\xi})(1-e^{4(p+j)N\pi^2/\xi}),$$

$$\text{(3-4)}\qquad f_N(z):=\frac{1}{N}T_N(\gamma(1-z)-p+1)-\frac{1}{N}T_N(\gamma(1+z)-p)-\kappa z-\frac{2p\pi\sqrt{-1}}{\gamma}.$$

Lemma 3.1 *The function f_N is defined in the region*

$$\Theta_0 := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{1}{2N} \right\} \setminus (\underline{\nabla}_0^+ \cup \underline{\nabla}_0^- \cup \overline{\nabla}_0^+ \cup \overline{\nabla}_0^-),$$

where we put

$$\underline{\nabla}_0^+ := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le 0,\ \operatorname{Re}(\xi z) \le \kappa,\ \operatorname{Im} z \le -\frac{2p\pi\kappa}{|\xi|^2} \right\},$$

$$\underline{\nabla}_0^- := \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{1}{2N},\ \operatorname{Re}(\xi z) \ge \kappa,\ \operatorname{Im} z \ge \frac{2(1-p)\pi\kappa}{|\xi|^2} \right\},$$

$$\overline{\nabla}_0^+ := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le 0,\ \operatorname{Re}(\xi z) \le -\kappa,\ \operatorname{Im} z \le \frac{2p\pi\kappa}{|\xi|^2} \right\},$$

$$\overline{\nabla}_0^- := \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{1}{2N},\ \operatorname{Re}(\xi z) \ge -\kappa,\ \operatorname{Im} z \ge \frac{2(p+1)\pi\kappa}{|\xi|^2} \right\}.$$

See Figure 7, where we put

$$\overline{K} := \left\{ z \in \mathbb{C} \mid z = \frac{2\pi\sqrt{-1}}{\xi} t - 1,\ t \in \mathbb{R} \right\} = \{ z \in \mathbb{C} \mid \operatorname{Re}(\xi z) = -\kappa \},$$

$$\underline{K} := \left\{ z \in \mathbb{C} \mid z = \frac{2\pi\sqrt{-1}}{\xi} t + 1,\ t \in \mathbb{R} \right\} = \{ z \in \mathbb{C} \mid \operatorname{Re}(\xi z) = \kappa \},$$

$$L_s := \{ z \in \mathbb{C} \mid \operatorname{Im}(\xi z) = 2s\pi \}.$$

Proof Recall that the function T_N is defined in Ω; see (2-18).

Since $\gamma = \xi/(2\pi\sqrt{-1}) = (\kappa + 2p\pi\sqrt{-1})/(2\pi\sqrt{-1})$, we have $\operatorname{Re}\gamma = p$, $\operatorname{Im}\gamma = -\kappa/(2\pi)$, and

$$\operatorname{Re}(\gamma(1 \pm z)) = p \pm \frac{\operatorname{Im}(\xi z)}{2\pi}, \quad \operatorname{Im}(\gamma(1 \pm z)) = -\frac{\kappa}{2\pi} \mp \frac{\operatorname{Re}(\xi z)}{2\pi}.$$

Therefore

$$-1 + \frac{p}{2N} < \operatorname{Re}(\gamma(1-z) - p + 1) < 2 - \frac{p}{2N} \iff -\frac{1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{1}{2N}$$
$$\iff -1 + \frac{p}{2N} < \operatorname{Re}(\gamma(1+z) - p) < 2 - \frac{p}{2N}.$$

We can also see that the condition $\gamma(1-z) - p + 1 \in \Delta_0^+$ is equivalent to $z \in \underline{\nabla}_0^-$, that the condition $\gamma(1-z) - p + 1 \in \Delta_1^-$ is equivalent to $z \in \underline{\nabla}_0^+$, that the condition $\gamma(1+z) - p \in \Delta_0^+$ is equivalent to $z \in \overline{\nabla}_0^+$, and that the condition $\gamma(1+z) - p \in \Delta_1^-$ is equivalent to $z \in \overline{\nabla}_0^-$. □

We would like to approximate $f_N(z)$ by using $\mathcal{L}_2$. From Proposition 2.25 and (3-4), the series of functions $\{f_N(z)\}$ converges uniformly to

$$F(z) := \frac{1}{\xi}\left(\mathcal{L}_2\left(\frac{\xi(1-z)}{2\pi\sqrt{-1}} - p + 1 \right) - \mathcal{L}_2\left(\frac{\xi(1+z)}{2\pi\sqrt{-1}} - p \right) \right) - \kappa z + \frac{4p\pi^2}{\xi}$$

in the region

$$\left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{\nu}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} \le \frac{2}{p} - \frac{\nu}{p},\ |\operatorname{Re}(\xi z)| \le 2M\pi - \kappa \right\} \setminus (\underline{\nabla}_{0,\nu}^+ \cup \underline{\nabla}_{0,\nu}^- \cup \overline{\nabla}_{0,\nu}^+ \cup \overline{\nabla}_{0,\nu}^-), \tag{3-5}$$

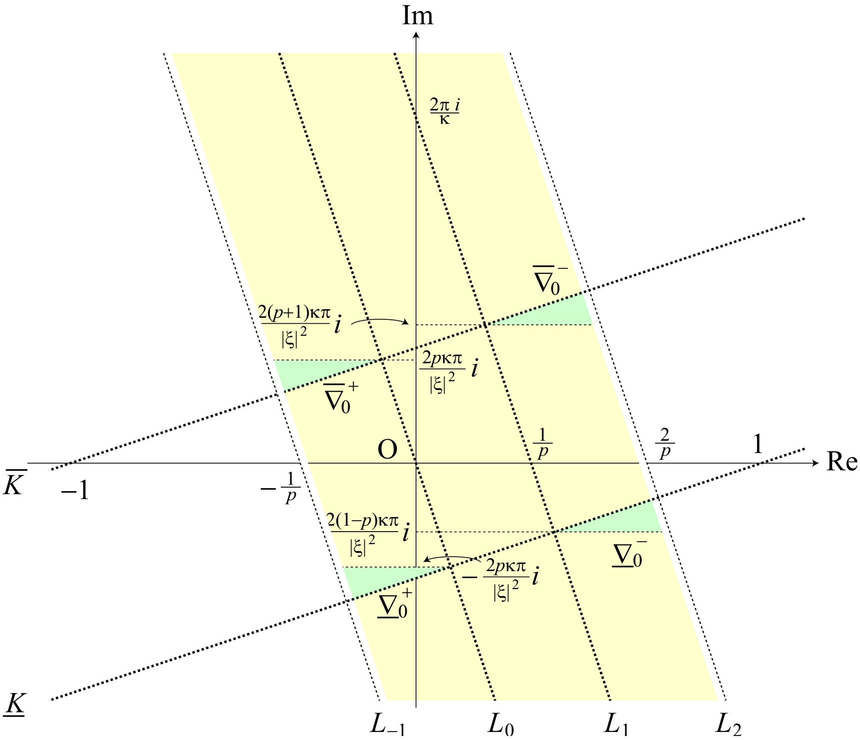

Figure 7: The function f_N is defined in the yellow region Θ_0. The green triangles are $\underline{\nabla}_0^+$, $\underline{\nabla}_0^-$, $\overline{\nabla}_0^+$, and $\overline{\nabla}_0^-$.

where we put

$$\underline{\nabla}_{0,\nu}^+ := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{\nu}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{\nu}{p}, \operatorname{Re}(\xi z) < \kappa + 2\pi\nu, \operatorname{Im} z < -\frac{2(p-\nu)\pi\kappa}{|\xi|^2} \right\},$$

$$\underline{\nabla}_{0,\nu}^- := \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{p} - \frac{\nu}{p} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le \frac{2}{p} - \frac{\nu}{p}, \operatorname{Re}(\xi z) > \kappa - 2\pi\nu, \operatorname{Im} z > \frac{2(1-p-\nu)\pi\kappa}{|\xi|^2} \right\},$$

$$\overline{\nabla}_{0,\nu}^+ := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{\nu}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{\nu}{p}, \operatorname{Re}(\xi z) < -\kappa + 2\pi\nu, \operatorname{Im} z < \frac{2(p+\nu)\pi\kappa}{|\xi|^2} \right\},$$

$$\overline{\nabla}_{0,\nu}^- := \left\{ z \in \mathbb{C} \;\middle|\; \frac{1}{p} - \frac{\nu}{p} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le \frac{2}{p} - \frac{\nu}{p}, \operatorname{Re}(\xi z) > -\kappa - 2\pi\nu, \operatorname{Im} z > \frac{2(p+1-\nu)\pi\kappa}{|\xi|^2} \right\}.$$

Lemma 3.2 *The series of functions $\{f_N(z)\}$ uniformly converges to $F(z)$ in the region (3-5).*

Proof In a way similar to the proof of Lemma 3.1, we have

$$\begin{aligned} -1+\nu < \operatorname{Re}(\gamma(1-z)-p+1) < 2-\nu &\iff -\frac{1}{p} + \frac{\nu}{p} < \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{\nu}{p} \\ &\iff -1+\nu < \operatorname{Re}(\gamma(1+z)-p) < 2-\nu, \\ |\operatorname{Im}(\gamma(1 \pm z))| \le M &\iff \kappa - 2M\pi \le \mp\operatorname{Re}(\xi z) \le \kappa + 2M\pi, \end{aligned}$$

and

$$\gamma(1-z)-p+1 \in \Delta^{+}_{0,\nu} \iff z \in \underline{\nabla}^{-}_{0,\nu}, \quad \gamma(1-z)-p+1 \in \Delta^{-}_{1,\nu} \iff z \in \underline{\nabla}^{+}_{0,\nu},$$
$$\gamma(1+z)-p \in \Delta^{+}_{0,\nu} \iff z \in \overline{\nabla}^{+}_{0,\nu}, \qquad \gamma(1+z)-p \in \Delta^{-}_{1,\nu} \iff z \in \overline{\nabla}^{-}_{0,\nu}.$$

Then the lemma follows from Proposition 2.25. □

We can express $F(z)$ in terms of Li_2 for certain cases.

Lemma 3.3 *If z is in between $\overline{K}$ and $\underline{K}$, or between L_0 and L_1, then we have*

$$F(z)=\frac{1}{\xi}\mathrm{Li}_2(e^{-\xi(1+z)})-\frac{1}{\xi}\mathrm{Li}_2(e^{-\xi(1-z)})+\kappa z-2\pi\sqrt{-1}. \tag{3-6}$$

Moreover, if z is between L_0 and L_1, we also have

$$F(z)=\frac{1}{\xi}\mathrm{Li}_2(e^{\xi(1-z)})-\frac{1}{\xi}\mathrm{Li}_2(e^{\xi(1+z)})-\kappa z+\frac{4p\pi^2}{\xi}. \tag{3-7}$$

Proof Since $\mathrm{Im}(\xi(1\pm z)/(2\pi\sqrt{-1}))=(-1/2\pi)(\kappa\pm\mathrm{Re}(\xi z))$, we see that $\mathrm{Im}(\xi(1+z)/(2\pi\sqrt{-1}))<0$ and $\mathrm{Im}(\xi(1-z)/(2\pi\sqrt{-1}))<0$ if z is between $\overline{K}$ and $\underline{K}$. Thus, in this case, we have (3-6) from (2-6).

Next, we consider the case where z is between L_0 and L_1, that is, where $0<\mathrm{Im}(\xi z)<2\pi$.

We have $\mathrm{Re}(\xi(1-z)/(2\pi\sqrt{-1}))-p+1=1-\mathrm{Im}(\xi z)/(2\pi)$ and $\mathrm{Re}(\xi(1+z)/(2\pi\sqrt{-1}))-p=\mathrm{Im}(\xi z)/(2\pi)$, both of which are between 0 and 1. So, from Lemmas 2.3 and 2.5, we have (3-7).

Now we will show that (3-6) also holds in this case.

From (2-7), we have

$$\begin{aligned}\mathrm{Li}_2(e^{\xi(1-z)})&=-\mathrm{Li}_2(e^{-\xi(1-z)})-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(\log(-e^{-\xi(1-z)}))^2\\&=-\mathrm{Li}_2(e^{-\xi(1-z)})-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(-\xi(1-z)+(2p-1)\pi\sqrt{-1})^2\end{aligned}$$

since $\mathrm{Im}\,\xi(1-z)=2p\pi-\mathrm{Im}(\xi z)$, which is between $2(p-1)\pi$ and $2p\pi$ when $0<\mathrm{Im}(\xi z)<2\pi$, that is, when z is between L_0 and L_1. Similarly,

$$\begin{aligned}\mathrm{Li}_2(e^{\xi(1+z)})&=-\mathrm{Li}_2(e^{-\xi(1+z)})-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(\log(-e^{-\xi(1+z)}))^2\\&=-\mathrm{Li}_2(e^{-\xi(1+z)})-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(-\xi(1+z)+(2p+1)\pi\sqrt{-1})^2\end{aligned}$$

since $\mathrm{Im}\,\xi(1+z)=2p\pi+\mathrm{Im}(\xi z)$, which is between $2p\pi$ and $2(p+1)\pi$. Thus, from (3-7), we obtain (3-6), completing the proof. □

The derivatives of $F(z)$ are given as follows from Lemma 2.7:

$$F'(z)=\mathcal{L}_1\left(\frac{\xi(1-z)}{2\pi\sqrt{-1}}-p+1\right)+\mathcal{L}_1\left(\frac{\xi(1+z)}{2\pi\sqrt{-1}}-p\right)-\kappa, \tag{3-8}$$

$$F''(z)=\frac{\xi(e^{-\xi z}-e^{\xi z})}{3-e^{\xi z}-e^{-\xi z}}, \tag{3-9}$$

$$F^{(3)}(z)=\frac{\xi^2(4-3(e^{\xi z}+e^{-\xi z}))}{(3-e^{\xi z}-e^{-\xi z})^2}. \tag{3-10}$$

If z is between $\overline{K}$ and $\underline{K}$, or between L_0 and L_1, we have

$$F'(z) = \log(1 - e^{-\kappa-\xi z}) + \log(1 - e^{-\kappa+\xi z}) + \kappa = \log(3 - e^{\xi z} - e^{-\xi z}) \tag{3-11}$$

from Lemma 3.3, where the second equality follows from the same reason as [24, (4.2)].

Put $\sigma_0 := 2\pi\sqrt{-1}/\xi = (2\pi/|\xi|^2)(2p\pi + \kappa\sqrt{-1})$. Since $\operatorname{Re}(\xi\sigma_0) = 0$ and $\operatorname{Im}(\xi\sigma_0) = 2\pi$, we conclude that σ_0 is on L_1 and between $\overline{K}$ and $\underline{K}$. From (3-6), (3-11), (3-9), and (3-10) we have

$$F(\sigma_0) = \frac{4p\pi^2}{\xi}, \quad F'(\sigma_0) = 0, \quad F''(\sigma_0) = 0, \quad F^{(3)}(\sigma_0) = -2\xi^2. \tag{3-12}$$

4 The Poisson summation formula

In (3-2), we put $\varphi_{m,N}(z) := f_N(z - 2m\pi\sqrt{-1}/\xi)$ for $m = 0, 1, 2, \dots, p-1$ so that

$$J_N(\mathscr{E}; e^{\xi/N}) = (1 - e^{-4pN\pi^2/\xi}) \sum_{m=0}^{p-1} \biggl(\beta_{p,m} \sum_{mN/p < k \le (m+1)N/p} \exp\Bigl(N\varphi_{m,N}\Bigl(\frac{2k+1}{2N}\Bigr)\Bigr)\biggr). \tag{4-1}$$

Note that the function $\varphi_{m,N}(z)$ is defined in the region

$$\Theta_m := \left\{ z \in \mathbb{C} \;\middle|\; -\frac{1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{2}{p} - \frac{1}{2N} \right\} \setminus (\underline{\nabla}^+_m \cup \underline{\nabla}^-_m \cup \overline{\nabla}^+_m \cup \overline{\nabla}^-_m)$$

from Lemma 3.1, where we put

$$\begin{aligned}
\underline{\nabla}^+_m &:= \left\{ z \in \mathbb{C} \;\middle|\; \frac{m-1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le \frac{m}{p}, \operatorname{Re}(\xi z) \le \kappa, \operatorname{Im} z \le \frac{2(m-p)\pi\kappa}{|\xi|^2} \right\}, \\
\underline{\nabla}^-_m &:= \left\{ z \in \mathbb{C} \;\middle|\; \frac{m+1}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{m+2}{p} - \frac{1}{2N}, \operatorname{Re}(\xi z) \ge \kappa, \operatorname{Im} z \ge \frac{2(m-p+1)\pi\kappa}{|\xi|^2} \right\}, \\
\overline{\nabla}^+_m &:= \left\{ z \in \mathbb{C} \;\middle|\; \frac{m-1}{p} + \frac{1}{2N} < \frac{\operatorname{Im}(\xi z)}{2p\pi} \le \frac{m}{p}, \operatorname{Re}(\xi z) \le -\kappa, \operatorname{Im} z \le \frac{2(m+p)\pi\kappa}{|\xi|^2} \right\}, \\
\overline{\nabla}^-_m &:= \left\{ z \in \mathbb{C} \;\middle|\; \frac{m+1}{p} \le \frac{\operatorname{Im}(\xi z)}{2p\pi} < \frac{m+2}{p} - \frac{1}{2N}, \operatorname{Re}(\xi z) \ge -\kappa, \operatorname{Im} z \ge \frac{2(m+p+1)\pi\kappa}{|\xi|^2} \right\}.
\end{aligned}$$

We would like to show that the sum

$$\sum_{mN/p < k \le (m+1)N/p} \exp\Bigl(N\varphi_{m,N}\Bigl(\frac{2k+1}{2N}\Bigr)\Bigr)$$

is approximated by the integral

$$N \int_{m/p}^{(m+1)/p} e^{N\varphi_{m,N}(z)}\, dz.$$

To do that, we use the following proposition, known as the Poisson summation formula:

Proposition 4.1 *Let a and b be real numbers with $a < b$, and $\{\psi_N(z)\}_{N=1,2,3,\ldots}$ be a series of holomorphic functions in a domain $D \subset \mathbb{C}$ containing the closed interval $[a,b]$. We assume that $\psi_N(z)$ uniformly converges to a holomorphic function $\psi(z)$ in D. We also assume that $\operatorname{Re}\psi(a) < 0$ and $\operatorname{Re}\psi(b) < 0$.*

Putting $R_+ := \{z \in D \mid \operatorname{Im} z \ge 0, \operatorname{Re}\psi(z) < 2\pi \operatorname{Im} z\}$ and $R_- := \{z \in D \mid \operatorname{Im} z \le 0, \operatorname{Re}\psi(z) < -2\pi \operatorname{Im} z\}$, we also assume that there are paths $C_\pm$ connecting a and b such that $C_\pm \subset R_\pm$ and that $C_\pm$ is homotopic to $[a,b]$ in D with a and b fixed.

Then we have
$$\frac{1}{N}\sum_{a \le k/N \le b} e^{N\psi_N(k/N)} = \int_a^b e^{N\psi_N(z)}\,dz + O(e^{-\varepsilon N})$$
for some $\varepsilon > 0$ independent of N.

A proof, which is essentially the same as that of [30, Proposition 4.2], is given in Appendix A.

From Lemma 3.2, the series of functions $\{\varphi_{m,N}(z)\}$ uniformly converges to $\Phi_m(z) := F(z - 2m\pi\sqrt{-1}/\xi)$ in the region $\Theta^*_{m,\nu}$ defined as

$$\begin{aligned}(4\text{-}2)\quad \Theta^*_{m,\nu} := \{z \in \mathbb{C} \mid 2(m-1+\nu)\pi \le \operatorname{Im}(\xi z) \le 2(m+2-\nu)\pi,\ |\operatorname{Re}(\xi z)| \le 2M\pi - \kappa\}\\ \setminus (\underline{\nabla}^+_{m,\nu} \cup \underline{\nabla}^-_{m,\nu} \cup \overline{\nabla}^+_{m,\nu} \cup \overline{\nabla}^-_{m,\nu}),\end{aligned}$$

where we put

$$\underline{\nabla}^+_{m,\nu} := \{z \in \mathbb{C} \mid 2(m-1+\nu)\pi \le \operatorname{Im}(\xi z) < 2(m+\nu)\pi,\ \operatorname{Re}(\xi z) < \kappa + 2\pi\nu,\ \operatorname{Im} z < 2(\nu - p + m)\pi\kappa/|\xi|^2\},$$
$$\underline{\nabla}^-_{m,\nu} := \{z \in \mathbb{C} \mid 2(m+1-\nu)\pi < \operatorname{Im}(\xi z) \le 2(m+2-\nu)\pi,\ \operatorname{Re}(\xi z) \ge \kappa - 2\pi\nu,\ \operatorname{Im} z > 2(1-p+m-\nu)\pi\kappa/|\xi|^2\},$$
$$\overline{\nabla}^+_{m,\nu} := \{z \in \mathbb{C} \mid 2(m-1+\nu)\pi \le \operatorname{Im}(\xi z) < 2(m+\nu)\pi,\ \operatorname{Re}(\xi z) < -\kappa + 2\pi\nu,\ \operatorname{Im} z < 2(\nu + p + m)\pi\kappa/|\xi|^2\},$$
$$\overline{\nabla}^-_{m,\nu} := \{z \in \mathbb{C} \mid 2(m+1-\nu)\pi < \operatorname{Im}(\xi z) \le 2(m+2-\nu)\pi,\ \operatorname{Re}(\xi z) > -\kappa - 2\pi\nu,\ \operatorname{Im} z > 2(p+m+1-\nu)\pi\kappa/|\xi|^2\},$$

and we always assume that N is sufficiently large. From (3-8)–(3-10), we have

$$\Phi'_m(z) = \mathcal{L}_1\!\left(\frac{\xi(1-z)}{2\pi\sqrt{-1}} + m - p + 1\right) + \mathcal{L}_1\!\left(\frac{\xi(1+z)}{2\pi\sqrt{-1}} - m - p\right) - \kappa, \tag{4-3}$$

$$\Phi''_m(z) = \frac{\xi(e^{-\xi z} - e^{\xi z})}{3 - e^{\xi z} - e^{-\xi z}}, \tag{4-4}$$

$$\Phi^{(3)}_m(z) = \frac{\xi^2(4 - 3(e^{\xi z} + e^{-\xi z}))}{(3 - e^{\xi z} - e^{-\xi z})^2}. \tag{4-5}$$

Since $z - 2m\pi\sqrt{-1}/\xi$ is between L_0 and L_1 ($\overline{K}$ and $\underline{K}$, respectively) if and only if z is between L_m and L_{m+1} ($\overline{K}$ and $\underline{K}$, respectively), from (3-11) we have

$$\Phi'_m(z) = \log(3 - e^{\xi z} - e^{-\xi z}) \tag{4-6}$$

when z is between $\overline{K}$ and $\underline{K}$, or L_m and L_{m+1}.

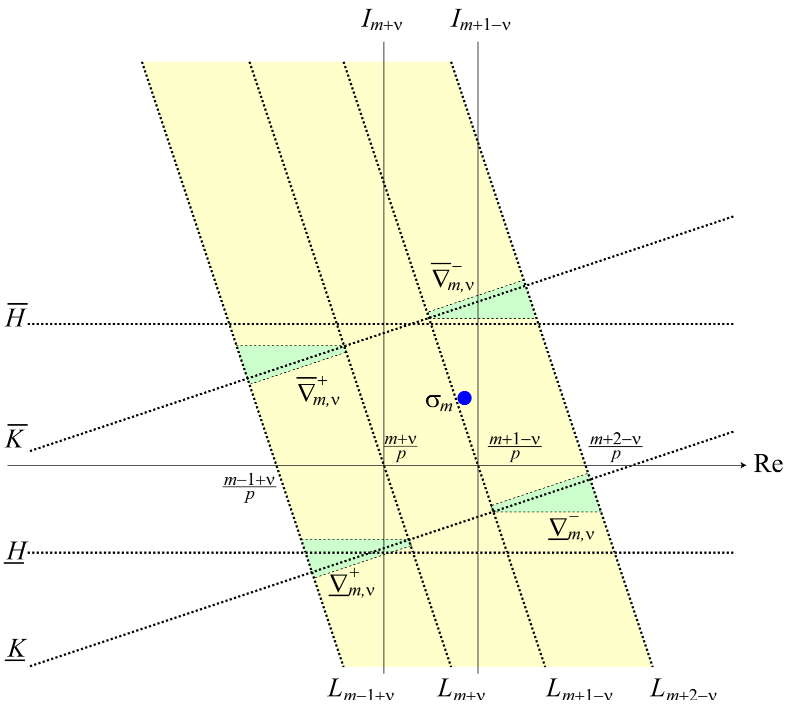

Figure 8: The yellow region is $\Theta^*_{m,\nu}$. The blue point is σ_m. The green trapezoids are $\underline{\nabla}^+_{m,\nu}$, $\underline{\nabla}^-_{m,\nu}$, $\overline{\nabla}^+_{m,\nu}$, and $\overline{\nabla}^-_{m,\nu}$.

We also put $\sigma_m := \sigma_0 + 2m\pi\sqrt{-1}/\xi = 2(m+1)\pi\sqrt{-1}/\xi = (2(m+1)\pi/|\xi|^2)(2p\pi+\kappa\sqrt{-1})$ so that

$$\Phi_m(\sigma_m) = \frac{4p\pi^2}{\xi}, \quad \Phi'_m(\sigma_m) = 0, \quad \Phi''_m(\sigma_m) = 0, \quad \Phi^{(3)}_m(\sigma_m) = -2\xi^2 \tag{4-7}$$

from (3-12). Since $\operatorname{Re}(\xi\sigma_m) = 0$ and $\operatorname{Im}(\xi\sigma_m) = 2(m+1)\pi$, we see that σ_m is between $\overline{K}$ and $\underline{K}$ and on the line L_{m+1}; see Figure 8.

Let I_s be the vertical line $\operatorname{Re} z = s/p$ for $s \in \mathbb{R}$.

For a small number $\chi > 0$, let $\Xi_{m,\chi}$ be the pentagonal region defined as

$$\Xi_{m,\chi} := \left\{ z \in \mathbb{C} \;\middle|\; \frac{m-\chi}{p} < \operatorname{Re} z < \frac{m+1+\chi}{p}, -\frac{2(m+1)\kappa\pi}{|\xi|^2} < \operatorname{Im} z < \frac{(p+m)\kappa}{2p^2\pi}, \right. \\ \left. \operatorname{Im}(\xi z) + \frac{(2\chi+1)|\xi|^2}{2(m+1)\kappa} \operatorname{Im} z > 2(m-\chi)\pi \right\}$$

when $m < p-1$, and

$$\Xi_{p-1,\chi} := \left\{ z \in \mathbb{C} \;\middle|\; \frac{p-1-\chi}{p} < \operatorname{Re} z < \frac{p+\chi}{p}, -\frac{2p\kappa\pi}{|\xi|^2} < \operatorname{Im} z < \frac{(2p-1)\kappa}{2p^2\pi}, \right. \\ \left. \operatorname{Im}(\xi z) + \frac{(2\chi+1)|\xi|^2}{2p\kappa} \operatorname{Im} z > 2(p-1-\chi)\pi \right\} \setminus \diamond_\nu,$$

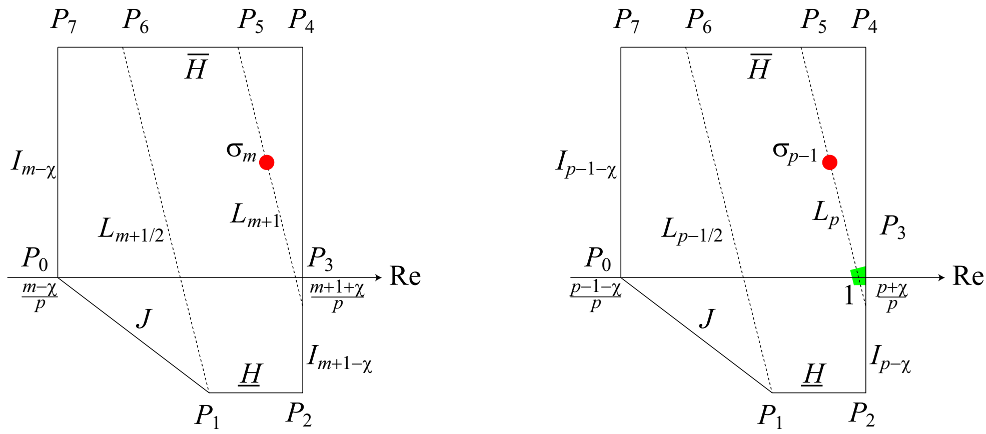

Figure 9: The region $\Xi_{m,\chi}$ when $m < p-1$ (left) and the region $\Xi_{p-1,\chi}$ (right), where the green quadrilateral indicates $\diamond_\nu$. Precisely speaking, the points P_0 and P_7 should be a little more to the right than indicated, and the points P_2, P_3, and P_4 should be a little more to the left than indicated.

where we put

$$\diamond_\nu := \{z \in \mathbb{C} \mid \operatorname{Re} z < 1 + \chi/p\} \cap \underline{\nabla}^-_{p-1,\nu}.$$

Note that $\Xi_{m,\chi}$ $(m < p-1)$ is surrounded by $I_{m-\chi}$, $I_{m+1+\chi}$, $\overline{H}$, $\underline{H}$, and J, where $\overline{H}$ and $\underline{H}$ are the horizontal lines $\operatorname{Im} z = (p+m)\kappa/(2p^2\pi)$ and $\operatorname{Im} z = -\operatorname{Im}\sigma_m$, respectively, and J is the line connecting $(m-\chi)/p$ and $L_{m+1/2} \cap \underline{H}$, which is given as

$$J := \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Im}(\xi z) + \frac{(2\chi+1)|\xi|^2}{2(m+1)\kappa} \operatorname{Im} z = 2(m-\chi)\pi \right\}. \tag{4-8}$$

See Figure 9, left. Figure 9, right, indicates $\Xi_{p-1,\chi}$, where $\diamond_\nu$ is indicated by the green quadrilateral. Note that it is a neighborhood of the point 1.

Lemma 4.2 *If $\nu > 0$ is sufficiently small, then we can choose $\chi > 0$ so that $\Xi_{m,\chi}$ is included in $\Theta^*_{m,\nu}$ for $m = 0, 1, 2, \dots, p-1$.*

Proof First, $\Xi_{m,\chi}$ is in the rectangle surrounded by $I_{m-\chi}$, $I_{m+1+\chi}$, $\overline{H}$, and $\underline{H}$, with bottom left vertex

$$v_1 := (m-\chi)/p - (2(m+1)\kappa\pi/|\xi|^2)\sqrt{-1}$$

and top right vertex

$$v_2 := (m+1+\chi)/p + ((p+m)\kappa/2p^2\pi)\sqrt{-1}.$$

The vertices v_1 and v_2 are on the lines $L_{\operatorname{Im}(\xi v_1)/(2\pi)}$ and v_2 $L_{\operatorname{Im}(\xi v_2)/(2\pi)}$, respectively. Since

$$\frac{\operatorname{Im}(\xi v_1)}{2\pi} - (m-1+\nu) = (1-\chi-\nu) - \frac{(m+1)\kappa^2}{|\xi|^2}$$

and

$$(m+2-\nu)-\frac{\operatorname{Im}(\xi v_2)}{2\pi}=(1-\chi-\nu)-\frac{(p+m)\kappa^2}{4p^2\pi^2},$$

if

$$\nu+\chi<\min\left\{1-(m+1)\left(\frac{\kappa}{|\xi|}\right)^2,1-(p+m)\left(\frac{\kappa}{2p\pi}\right)^2\right\}=1-(p+m)\left(\frac{\kappa}{2p\pi}\right)^2, \tag{4-9}$$

then $\Xi_{m,\chi}$ is between the lines $L_{m-1+\nu}$ and $L_{m+2-\nu}$ for $m=0,1,2,\dots,p-1$.

So it remains to show that $\Xi_{m,\chi}$ excludes $\overline{\nabla}^{+}_{m,\nu}$, $\overline{\nabla}^{-}_{m,\nu}$, $\underline{\nabla}^{+}_{m,\nu}$, and $\underline{\nabla}^{-}_{m,\nu}$.

- The real part of the bottom right corner of $\overline{\nabla}^{+}_{m,\nu}$ is $(\kappa(2\pi\nu-\kappa)+4(m+\nu)p\pi^2)/|\xi|^2$, which is smaller than $(m-\chi)/p$ if

$$2p\pi(\kappa+2p\pi)\nu+|\xi|^2\chi<(p+m)\kappa^2. \tag{4-10}$$

So the trapezoid $\overline{\nabla}^{+}_{m,\nu}$ is to the right of $I_{m-\chi}$ if (4-10) holds.

- The difference between the imaginary parts of the bottom line of $\overline{\nabla}^{-}_{m,\nu}$ and $\overline{H}$ is

$$\frac{2(p+m+1-\nu)\kappa\pi}{|\xi|^2}-\frac{(p+m)\kappa}{2p^2\pi}=\frac{\kappa(4p^2(1-\nu)\pi^2-(p+m)\kappa^2)}{2p^2\pi|\xi|^2},$$

which is positive if

$$\nu<1-(p+m)\left(\frac{\kappa}{2p\pi}\right)^2. \tag{4-11}$$

So we conclude that $\overline{\nabla}^{-}_{m,\nu}$ is outside of $\Xi_{m,\chi}$ if (4-11) holds.

- To obtain a condition ensuring that $\underline{\nabla}^{+}_{m,\nu}$ is below J, it is enough to find a condition ensuring that the top right corner z_0 of the trapezoid is below J, since $L_{m+\nu}$ is steeper than J. Since $\operatorname{Im} z_0=2(\nu-p+m)\kappa\pi/|\xi|^2$ and z_0 is on $L_{m+\nu}$, the condition is

$$2(m+\nu)\pi+\frac{(2\chi+1)|\xi|^2}{2(m+1)\kappa}\frac{2(\nu-p+m)\kappa\pi}{|\xi|^2}<2(m-\chi)\pi$$

from (4-8). Therefore, if

$$2\nu\chi+(2m+3)\nu+2(2m-p+1)\chi<p-m, \tag{4-12}$$

the trapezoid $\underline{\nabla}^{+}_{m,\nu}$ is out of $\Xi_{m,\chi}$.

- The real part of the top left corner of $\underline{\nabla}^{-}_{m,\nu}$ is $(\kappa(\kappa-2\pi\nu)+4(m+1-\nu)p\pi^2)/|\xi|^2$, which is bigger than $(m+1+\chi)/p$ if

$$2p\pi(\kappa+2p\pi)\nu+|\xi|^2\chi<(p-m-1)\kappa^2. \tag{4-13}$$

So the trapezoid $\underline{\nabla}^{-}_{m,\nu}$ is outside of $\Xi_{m,\chi}$ if (4-13) holds.

From (4-10)–(4-13), we conclude that if $m < p-1$ and

$$\nu < \min\left\{\frac{(p+m)\kappa}{2p\pi(\kappa+2p\pi)}, 1-(p+m)\left(\frac{\kappa}{2p\pi}\right)^2, \frac{p-m}{2m+3}, \frac{(p-m-1)\kappa^2}{2p\pi(\kappa+2p\pi)}\right\},$$

then we can choose $\chi > 0$ so that $\Xi_{m,\chi}$ is included in $\Theta^*_{m,\nu}$.

If $m = p-1$, then the real part of the top left corner of $\underline{\nabla}^-_{p-1,\nu}$ is $1-2\pi\nu(\kappa+2p\pi)/|\xi|^2$, which is slightly to the left of $1+\chi/p$. Its imaginary part is $2\pi\nu(2p\pi-\kappa)/|\xi|^2$, which is slightly above the real axis. The bottom left corner of $\underline{\nabla}^-_{p-1,\nu}$ is $1-4p\nu\pi^2/|\xi|^2$, which is slightly smaller than 1. Its imaginary part is $--2\nu\kappa\pi/|\xi|^2$, which is below the real axis. So if we exclude $\underline{\nabla}^-_{p-1,\nu}$, the rest is included in $\Theta^*_{p-1,\nu}$; see Figure 9, right. □

We will show that the assumption of Proposition 4.1 holds for the function $\psi_N(z) := \varphi_{m,N}(z) - \varphi_{m,N}(\sigma_m)$, the domain $D := \Xi_{m,\chi}$, and the numbers $a := m/p$ and $b := (m+1)/p$, with small $\chi > 0$. Note that the series of functions $\{\psi_N(z)\} := \{\varphi_{m,N}(z) - \varphi_{m,N}(\sigma_m)\}$ uniformly converges to $\psi(z) := \Phi_m(z) - \Phi_m(\sigma_m)$ in $\Xi_{m,\chi}$ for sufficiently small $\chi > 0$.

From now we will study properties of $\Phi_m(z)$ in the region $\Xi_{m,\chi}$ as if $\chi = 0$, taking care of the case where $\chi > 0$ if necessary.

Let $P_0, P_1, \dots, P_7$ be points defined as follows, which are already indicated in Figure 9:

$$P_0 := I_m \cap \text{real axis}, \quad P_1 := L_{m+1/2} \cap \underline{H}, \quad P_2 := I_{m+1} \cap \underline{H}, \quad P_3 := I_{m+1} \cap \text{real axis},$$
$$P_4 := I_{m+1} \cap \overline{H}, \quad P_5 := L_{m+1} \cap \overline{H}, \quad P_6 := L_{m+1/2} \cap \overline{H}, \quad P_7 := I_m \cap \overline{H}.$$

Their coordinates are given as follows:

$$P_0 := \frac{m}{p}, \qquad P_1 := \frac{m+\frac{1}{2}}{p} + \frac{\operatorname{Im}\sigma_m}{2p\pi}\bar{\xi}, \qquad P_2 := \frac{m+1}{p} - \operatorname{Im}\sigma_m\sqrt{-1},$$
$$P_3 := \frac{m+1}{p}, \qquad P_4 := \frac{m+1}{p} + \frac{(p+m)\kappa}{2p^2\pi}\sqrt{-1}, \qquad P_5 := \frac{m+1}{p} - \frac{(p+m)\kappa}{4p^3\pi^2}\bar{\xi},$$
$$P_6 := \frac{m+\frac{1}{2}}{p} - \frac{(p+m)\kappa}{4p^3\pi^2}\bar{\xi}, \qquad P_7 := \frac{m}{p} + \frac{(p+m)\kappa}{2p^2\pi}\sqrt{-1}.$$

Lemma 4.3 *We have the following inequalities*:

$$\operatorname{Re} P_6 < \operatorname{Re} P_1 < \operatorname{Re} P_5.$$

Proof It is clear that $\operatorname{Re} P_6 < \operatorname{Re} P_1$, and so we will show the other inequality.

Since $\operatorname{Im}\sigma_m = 2(m+1)\kappa\pi/|\xi|^2$ and $\kappa > 1$, we have

$$\begin{aligned}\operatorname{Re} P_5 - \operatorname{Re} P_1 &= \frac{m+1}{p} - \frac{(p+m)\kappa^2}{4p^3\pi^2} - \left(\frac{2m+1}{2p} + \frac{(m+1)\kappa^2}{p|\xi|^2}\right)\\ &> \frac{1}{2p} - \frac{p+m}{4p^3\pi^2} - \frac{m+1}{4p^3\pi^2} \ge \frac{1}{2p} - \frac{3p-1}{4p^3\pi^2} > \frac{1}{2p} - \frac{3}{4p^2\pi^2} > 0,\end{aligned}$$

proving the inequality $\operatorname{Re} P_5 > \operatorname{Re} P_1$. □

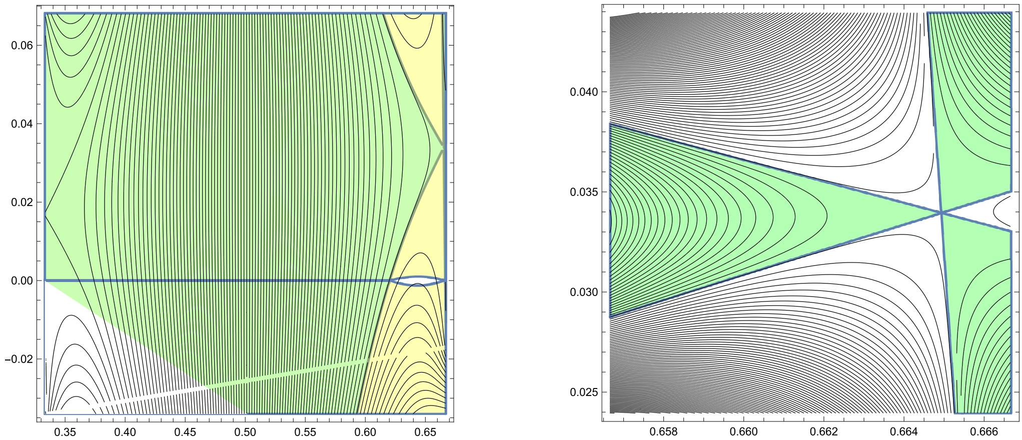

Figure 10: The left picture shows a contour plot of $\operatorname{Re}\Phi_1(z)$ in $\Xi_{1,0}$ for $p=3$, where $R_\pm$ are indicated by yellow and green and W_1^- is indicated by dark green. The right picture shows a contour plot of $\operatorname{Re}\Phi_1(z)$ in a neighborhood of σ_1, where W_1^- is indicated by green.

We put

$$W_m^+ := \{z \in \Xi_{m,\chi} \mid \operatorname{Re}\Phi_m(z) > \operatorname{Re}\Phi_m(\sigma_m)\}, \quad W_m^- := \{z \in \Xi_{m,\chi} \mid \operatorname{Re}\Phi_m(z) < \operatorname{Re}\Phi_m(\sigma_m)\}$$

for $m = 0, 1, 2, \dots, p-1$. Recall that in this case, $R_\pm$ in Proposition 4.1 becomes

$$R_+ := \{z \in \Xi_{m,\chi} \mid \operatorname{Im} z \ge 0, \operatorname{Re}\Phi_m(z) - \operatorname{Re}\Phi_m(\sigma) < 2\pi \operatorname{Im} z\},$$
$$R_- := \{z \in \Xi_{m,\chi} \mid \operatorname{Im} z \le 0, \operatorname{Re}\Phi_m(z) - \operatorname{Re}\Phi_m(\sigma) < -2\pi \operatorname{Im} z\}.$$

In fact, we will show the following lemma, whose proof will be given later.

Lemma 4.4 *The following hold for $m = 0, 1, \dots, p-2$:*

(i) *The points m/p and $(m+1)/p$ are in W_m^-.*

(ii) *There is a path C_+ in R_+ connecting m/p and $(m+1)/p$.*

(iii) *There is a path C_- in R_- connecting m/p and $(m+1)/p$.*

When $m = p-1$, there exists $\delta > 0$ such that the following hold:

(i′) *The points $1 - 1/p$ and $1-\delta$ are in W_{p-1}^-.*

(ii′) *There is a path C_+ in R_+ connecting $1-1/p$ and $1-\delta$.*

(iii′) *There is a path C_- in R_- connecting $1-1/p$ and $1-\delta$.*

Note that since $\Xi_{m,\chi}$ is simply connected, both C_+ and C_- are homotopic to the segment $[m/p, (m+1)/p]$ ($[1-1/p, 1-\delta]$ if $m = p-1$) in $\Xi_{m,\chi}$ keeping the boundary points fixed.

See Figure 10.

To prove the lemma above, we study the behavior of $\operatorname{Re}\Phi_m$ in $\Xi_{m,0}$ more precisely.

We divide $\Xi_{m,0}$ into six parts by the three lines L_{m+1}, $L_{m+1/2}$, and K_σ, where we put

$$K_\sigma : \operatorname{Re}(\xi z) = 0.$$

We can see that σ_m is just the intersection of L_{m+1} and K_σ.

We also introduce the four points

$$P_{34} := I_{m+1} \cap K_\sigma, \quad P_{70} := I_m \cap K_\sigma,$$

with coordinates

$$P_{34} := \frac{(m+1)\bar{\xi}\sqrt{-1}}{2p^2\pi} = \frac{m+1}{p} + \frac{(m+1)\kappa}{2p^2\pi}\sqrt{-1}, \quad P_{70} := \frac{m\bar{\xi}\sqrt{-1}}{2p^2\pi} = \frac{m}{p} + \frac{m\kappa}{2p^2\pi}\sqrt{-1}.$$

Note that P_{34} is between P_3 and P_4 (when $p=1$, P_{34} coincides with P_4), and that P_{70} is between P_7 and P_0 (when $m=0$, P_{70} coincides with P_0).

As in the proof of Lemma 5.2 in [24], we can prove the following lemma:

Lemma 4.5 *Write $z = x + y\sqrt{-1}$ for $z \in \Xi_{m,\chi}$ with $x, y \in \mathbb{R}$. Then we have:*

- $(\partial \operatorname{Re}\Phi_m/\partial y)(z) > 0$ *if and only if*
  - $\operatorname{Re}(\xi z) > 0$ *and* $2k\pi < \operatorname{Im}(\xi z) < (2k+1)\pi$ *for some integer* k, *or*
  - $\operatorname{Re}(\xi z) < 0$ *and* $(2l-1)\pi < \operatorname{Im}(\xi z) < 2l\pi$ *for some integer* l.
- $(\partial \operatorname{Re}\Phi_m/\partial y)(z) < 0$ *if and only if*
  - $\operatorname{Re}(\xi z) < 0$ *and* $2k\pi < \operatorname{Im}(\xi z) < (2k+1)\pi$ *for some integer* k, *or*
  - *or* $\operatorname{Re}(\xi z) > 0$ *and* $(2l-1)\pi < \operatorname{Im}(\xi z) < 2l\pi$ *for some integer* l.

See Figure 11.

Proof From (4-6), we have

$$\frac{\partial \operatorname{Re}\Phi_m(z)}{\partial y} = -\arg(3 - 2\cosh(\xi z)).$$

The right-hand side is positive (negative, respectively) if and only if $\operatorname{Im}(3-2\cosh(\xi z))$ is negative (positive, respectively). Since $\operatorname{Im}(3-2\cosh(\xi z)) = -2\sinh(\operatorname{Re}(\xi z))\sin(\operatorname{Im}(\xi z))$, $\partial \operatorname{Re}\Phi_m(z)/\partial y$ is positive (negative, respectively) if and only if $\operatorname{Re}(\xi z) > 0$ and $2k\pi < \operatorname{Im}(\xi z) < (2k+1)\pi$ for some integer k, or $\operatorname{Re}(\xi z) < 0$ and $(2l-1)\pi < \operatorname{Im}(\xi z) < 2l\pi$ for some integer l ($\operatorname{Re}(\xi z) < 0$ and $2k\pi < \operatorname{Im}(\xi z) < (2k+1)\pi$ for some integer k, or $\operatorname{Re}(\xi z) > 0$ and $(2l-1)\pi < \operatorname{Im}(\xi z) < 2l\pi$ for some integer l, respectively). □

Lemma 4.6 *Let z be a point on the segment $\overline{P_{70}P_{34}}$. If $z \neq \sigma_m$ is between σ_m and P_{70}, then $z \in W_m^-$. Moreover, if $z \neq \sigma_m$ is between σ_m and P_{34}, then $z \in W_m^+$.*

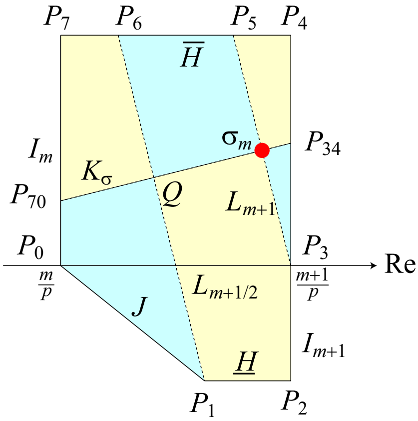

Figure 11: In the cyan (yellow, respectively) region, $\operatorname{Re}\Phi_m(z)$ is increasing (decreasing, respectively) with respect to $\operatorname{Im} z$.

Proof The segment $\overline{P_{70}P_{34}} \subset K_\sigma$ is parametrized as $(2\pi\sqrt{-1}/\xi)t$ for $m + m\kappa^2/(4p^2\pi^2) \le t \le m+1+(m+1)\kappa^2/(4p^2\pi^2)$. From (4-6) we have

$$\frac{d}{dt}\operatorname{Re}\Phi_m\left(\frac{2\pi\sqrt{-1}}{\xi}t\right) = \operatorname{Re}\left(\frac{2\pi\sqrt{-1}}{\xi}\log\left(3-2\cosh(2\pi\sqrt{-1}t)\right)\right) = \frac{4p\pi^2}{|\xi|^2}\log(3-2\cos 2\pi t) \ge 0,$$

and the equality holds only when $t = m+1$, that is, $z = \sigma_m$. So we conclude that if $z \in \overline{P_{70}\sigma_m}\setminus\{\sigma_m\}$, then $\operatorname{Re}\Phi_m(z) < \operatorname{Re}\Phi_m(\sigma_m)$, and that if $z \in \overline{\sigma_m P_{34}}\setminus\{\sigma_m\}$, then $\operatorname{Re}\Phi_m(z) > \operatorname{Re}\Phi_m(\sigma_m)$, as required. $\square$

We can prove a similar result for $\overline{P_3P_5}$.

Lemma 4.7 *Let z be a point on the segment $\overline{P_3P_5}$. If $z \ne \sigma_m$ is between σ_m and P_3, then $z \in W_m^-$. Moreover, if $z \ne \sigma_m$ is between σ_m and P_5, then $z \in W_m^+$.*

Proof A point on $\overline{P_3P_5}$ is parametrized as $(m+1)/p - ((p+m)\kappa/(4p^3\pi^2))\bar{\xi}t$ for $0 \le t \le 1$. From (4-6) we have

$$\begin{aligned}\frac{d}{dt}\operatorname{Re}\Phi_m\left(\frac{m+1}{p} - \frac{(p+m)\kappa}{4p^3\pi^2}\bar{\xi}t\right) &= -\operatorname{Re}\left[\frac{(p+m)\kappa}{4p^3\pi^2}\bar{\xi}\log\left(3-2\cosh\left(\frac{(m+1)\xi}{p} - \frac{(p+m)\kappa}{4p^3\pi^2}|\xi|^2t\right)\right)\right] \\ &= \frac{-(p+m)\kappa^2}{4p^3\pi^2}\log\left(3-2\cosh\left(\frac{(m+1)\kappa}{p} - \frac{(p+m)\kappa}{4p^3\pi^2}|\xi|^2t\right)\right) \ge 0,\end{aligned}$$

where the equality holds when $t = 4(m+1)p^2\pi^2/((p+m)|\xi|^2)$, which shows that $\operatorname{Re}\Phi_m(z) < \operatorname{Re}\Phi_m(\sigma_m)$ if $z \in \overline{P_3\sigma_m}\setminus\{\sigma_m\}$ and that $\operatorname{Re}\Phi_m(z) > \operatorname{Re}\Phi_m(\sigma_m)$ if $z \in \overline{\sigma_m P_5}\setminus\{\sigma_m\}$, completing the proof. $\square$

So far we have found two directions $\overrightarrow{\sigma_m P_{70}}$ and $\overrightarrow{\sigma_m P_3}$ that go down valleys, and two directions $\overrightarrow{\sigma_m P_{34}}$ and $\overrightarrow{\sigma_m P_5}$ that go up hills. Since the function $\Phi_m(z)$ is of the form $\Phi_m(\sigma_m) - \frac{1}{3}\xi^2z^3 + \cdots$ from (4-7), that is, σ_m is a saddle point of order two, there should be another pair of valley and hill.

Lemma 4.8 *Let G be the line segment in $\Xi_{m,0}$ that bisects the angle $\angle P_{34}\sigma_m P_5$. If $z \in G \setminus \{\sigma_m\}$ is on the same side of P_{34} and P_5, then $z \in W_m^-$. If $z \in G \setminus \{\sigma_m\}$ is on the opposite side of P_{34} and P_5, and close enough to σ_m, then $z \in W_m^+$.*

Proof Since the vector $\overrightarrow{\sigma_m P_{34}}$ has the same direction as $\sqrt{-1}/\xi$ and the vector $\overrightarrow{\sigma_m P_5}$ has the same direction as $-1/\xi$, the bisector is parametrized as $\sigma_m + (\sqrt{-1}-1)t/\xi$ with $t \in \mathbb{R}$. Note that if $t > 0$, it goes to the top right, and that if $t < 0$, it goes to the bottom left.

From (4-6), we have

$$\frac{d}{dt}\operatorname{Re}\Phi_m\Big(\sigma_m + \frac{\sqrt{-1}-1}{\xi}t\Big) = \operatorname{Re}\Big(\frac{\sqrt{-1}-1}{\xi}\log(3-2\cosh((\sqrt{-1}-1)t))\Big),$$

and so $(d/dt)\operatorname{Re}\Phi_m\big(\sigma_m + ((\sqrt{-1}-1)/\xi)t\big) = 0$ when $t = 0$. Thus, it is sufficient to show that the second derivative of $\operatorname{Re}\Phi_m(\sigma_m + (\sqrt{-1}-1)t/\xi)$ is positive when $t < 0$ and $|t|$ is small, and that it is negative when $t > 0$ and $\sigma_m + (\sqrt{-1}-1)t/\xi \in \Xi_{m,0}$.

From (4-4), we have

$$\begin{aligned}\frac{d^2}{dt^2}\operatorname{Re}\Phi_m\bigg(\sigma_m + \frac{\sqrt{-1}-1}{\xi}t\bigg) &= \operatorname{Re}\bigg(\frac{(\sqrt{-1}-1)^2}{\xi^2}\frac{\xi(e^{-(\sqrt{-1}-1)t}-e^{(\sqrt{-1}-1)t})}{3-e^{(\sqrt{-1}-1)t}-e^{-(\sqrt{-1}-1)t}}\bigg)\\ &= \frac{1}{|\xi|^2}\operatorname{Re}((-4p\pi-2\kappa\sqrt{-1})\lambda(t)) = \frac{1}{|\xi|^2}(-4p\pi\operatorname{Re}\lambda(t)+2\kappa\operatorname{Im}\lambda(t)),\end{aligned}$$

where we put

$$\lambda(t) := \frac{e^{-(\sqrt{-1}-1)t} - e^{(\sqrt{-1}-1)t}}{3 - e^{(\sqrt{-1}-1)t} - e^{-(\sqrt{-1}-1)t}}.$$

We have

$$\begin{aligned}\lambda(t) &= \frac{2\sinh t\cos t - 2\sqrt{-1}\cosh t\sin t}{3 - 2\cosh t\cos t + 2\sqrt{-1}\sinh t\sin t}\\ &= \frac{2(\sinh t\cos t - \sqrt{-1}\cosh t\sin t)}{(3-2\cosh t\cos t)^2 + 4\sinh^2 t\sin^2 t}(3 - 2\cosh t\cos t - 2\sqrt{-1}\sinh t\sin t)\\ &= \frac{2\sinh t(3\cos t - 2\cosh t) + 2\sqrt{-1}\sin t(2\cos t - 3\cosh t)}{(3-2\cosh t\cos t)^2 + 4\sinh^2 t\sin^2 t}.\end{aligned}$$

Therefore if t is negative and $|t|$ is small enough, then $\operatorname{Re}\lambda(t) < 0$ and $\operatorname{Im}\lambda(t) > 0$, and so in this case $(d^2/dt^2)\operatorname{Re}\Phi_m\big(\sigma_m + ((\sqrt{-1}-1)/\xi)t\big) > 0$.

Next, we consider the case where $t > 0$.

Since $\operatorname{Re}(\sigma_m + (\sqrt{-1}-1)t/\xi) = (1/|\xi|^2)(4(m+1)p\pi^2 + (2p\pi-\kappa)t)$, a point in G that is between σ_m and I_{m+1} is parametrized as $\sigma_m + (\sqrt{-1}-1)t/\xi$ with $0 < t < (m+1)\kappa^2/(p(2p\pi-\kappa))$. Since $(m+1)\kappa^2/(p(2p\pi-\kappa)) \le \kappa^2/(2p\pi-\kappa) \le \kappa^2/(2\pi-\kappa)$, it is sufficient to prove

$$(d^2/dt^2)\operatorname{Re}\Phi_m(\sigma_m + ((\sqrt{-1}-1)/\xi)t) < 0$$

for $0 < t < \kappa^2/(2\pi-\kappa)$.

Since $3\cos(\kappa^2/(2\pi-\kappa)) - 2\cosh(\kappa^2/(2\pi-\kappa)) = 0.924\ldots$, and the function $3\cos t - 2\cosh t$ is monotonically decreasing when $t > 0$, we see that $\operatorname{Re}\lambda(t) > 0$ for $0 < t < \kappa^2/(2\pi-\kappa)$. We can easily see that $\operatorname{Im}\lambda(t) < 0$ for $t > 0$, and so we conclude that $(d^2/dt^2)\operatorname{Re}\Phi(\sigma_m + ((\sqrt{-1}-1)/\xi)t) < 0$. □

Remark 4.9 The imaginary part of the intersection of G with I_{m+1} is $(m+1)\kappa/(p(2p\pi-\kappa))$, which is smaller than the imaginary part of $\overline{H}$ when $p > 1$. This is because

$$\begin{aligned}\frac{(p+m)\kappa}{2p^2\pi} - \frac{(m+1)\kappa}{p(2p\pi-\kappa)} &= \frac{(2p\pi(p-1)-(p+m)\kappa)\kappa}{2p^2\pi(2p\pi-\kappa)} \\ &> \frac{(2p\pi(p-1)-(2p-1))\kappa}{2p^2\pi(2p\pi-\kappa)} = \frac{((2p\pi-1)(p-1)-p)\kappa}{2p^2\pi(2p\pi-\kappa)},\end{aligned}$$

which is positive when $p > 1$, where we use the inequalities $\kappa < 1$ and $m \le p-1$. So G intersects with the segment $\overline{P_4P_{34}}$.

If $p = 1$, G intersects with the segment $\overline{P_4P_5}$.

Note that G does not intersect with $L_{m+1/2}$ in $\Xi_{m,0}$. This is because the intersection between G and $L_{m+1/2}$ is $(\pi + (2m+1)\pi\sqrt{-1})/\xi$, whose imaginary part is less than $-\operatorname{Im}\sigma_m$.

There are more line segments that are included in W_m^-.

Lemma 4.10 *The line segments $\overline{P_6P_1}$, $\overline{P_0P_{70}}$, and $\overline{P_0P_1}$ are in W_m^-.*

Proof A point on the segment $\overline{P_6P_1}$ is parametrized as

$$\frac{m+\frac{1}{2}}{p} + \frac{\bar{\xi}}{2p\pi}t \quad \text{where} \quad -\frac{(p+m)\kappa}{2p^2\pi} \le t \le \operatorname{Im}\sigma_m.$$

We have

$$\begin{aligned}\frac{d}{dt}\operatorname{Re}\Phi_m\left(\frac{m+\frac{1}{2}}{p} + \frac{\bar{\xi}}{2p\pi}t\right) &= \operatorname{Re}\left(\frac{\bar{\xi}}{2p\pi}\log\left(3 - 2\cosh\left(\frac{m+\frac{1}{2}}{p}\xi + \frac{|\xi|^2 t}{2p\pi}\right)\right)\right) \\ &= \frac{\kappa}{2p\pi}\log\left(3 + 2\cosh\left(\frac{(m+\frac{1}{2})\kappa}{p} + \frac{|\xi|^2 t}{2p\pi}\right)\right) > 0.\end{aligned}$$

From Lemma 4.11 below we know that $\operatorname{Re}\Phi_m(P_1) < \operatorname{Re}\Phi_m(\sigma_m)$. It follows that $\overline{P_6P_1} \subset W_m^-$.

From Lemma 4.5, $\operatorname{Re}\Phi_m(z)$ is increasing with respect to $\operatorname{Im} z$ in the quadrilateral $P_{70}P_0P_1Q$, where Q is the crossing between K_σ and $L_{m+1/2}$. Since the upper segments $\overline{P_{70}Q}$ and $\overline{QP_1}$ are in W_m^-, so are the lower segments $\overline{P_{70}P_0}$ and $\overline{P_0P_1}$. □

Lemma 4.11 *The point P_1 is in W_m^-.*

Proof The following proof is similar to that of [24, Lemma 5.3].

Since P_1 is on $L_{m+1/2}$, we have $\operatorname{Im}(\xi(P_1 - 2m\pi\sqrt{-1}/\xi)) = \pi$ and so $P_1 - 2m\pi\sqrt{-1}/\xi$ is on $L_{1/2}$. So from (3-6) we have

$$\xi\Phi_m(P_1) - \xi\Phi_m(\sigma_m) = \operatorname{Li}_2(-e^{-\kappa-(4m+3)\kappa/(2p)}) - \operatorname{Li}_2(-e^{-\kappa+(4m+3)\kappa/(2p)}) + \frac{(4m+3)\kappa^2}{2p} - \kappa\pi\sqrt{-1}.$$

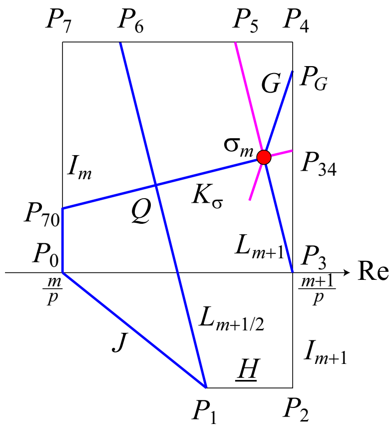

Figure 12: The blue (magenta, respectively) lines are included in W_m^- (W_m^+, respectively).

Its real part is

$$\mathrm{Li}_2(-e^{-\kappa-(4m+3)\kappa/(2p)}) - \mathrm{Li}_2(-e^{-\kappa+(4m+3)\kappa/(2p)}) + \frac{(4m+3)\kappa^2}{2p}$$

and its imaginary part is $-\kappa\pi$.

Therefore we have

$$\begin{aligned}(4\text{-}14)\quad & \frac{|\xi|^2}{\kappa}(\mathrm{Re}\,\Phi_m(P_1) - \mathrm{Re}\,\Phi_m(\sigma_m)) \\ &= \mathrm{Re}(\xi\Phi_m(P_1) - \xi\Phi_m(\sigma_m)) + \frac{2p\pi}{\kappa}\,\mathrm{Im}(\xi\Phi_m(P_1) - \xi\Phi_m(\sigma_m)) \\ &= \mathrm{Li}_2(-e^{-\kappa-(4m+3)\kappa/(2p)}) - \mathrm{Li}_2(-e^{-\kappa+(4m+3)\kappa/(2p)}) + \frac{(4m+3)\kappa^2}{2p} - 2p\pi^2,\end{aligned}$$

which is increasing with respect to m, fixing p. When $m = p-1$, (4-14) equals

$$\mathrm{Li}_2(-e^{-\kappa-(4p-1)\kappa/(2p)}) - \mathrm{Li}_2(-e^{-\kappa+(4p-1)\kappa/(2p)}) + \frac{(4p-1)\kappa^2}{2p} - 2p\pi^2. \tag{4-15}$$

Its derivative with respect to p is

$$\frac{\kappa}{2p^2}\log\Bigl(3 + 2\cosh\Bigl(\kappa\bigl(2 - \frac{1}{2p}\bigr)\Bigr)\Bigr) - 2\pi^2,$$

which is less than $\log(3 + 2\cosh(2\kappa)) - 2\pi^2 = \log(10) - 2\pi^2 < 0$. Since (4-15) equals $-17.2195\ldots$ when $p = 1$, we conclude that (4-14) is negative, proving the lemma. □

Remark 4.12 One can also show that the polygonal line $\overline{P_{70}P_7P_6}$ is in W_m^-.

The results in Lemmas 4.6–4.8 and 4.10 are summarized in Figure 12.

Proof of Lemma 4.4 First, suppose that $m < p-1$.

(i) Since $m/p = P_0$ and $(m+1)/p = P_3$, it follows from Figure 12 that these points are in W_m^-.

(ii) Consider the polygonal line $C_+ := \overline{P_0 P_{70} \sigma_m P_3}$. From Figure 12, it is in W_m^- and in the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$. So it is contained in R_+.

(iii) From Figure 12, the line segment J is in W_m^- and in the lower half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}$. This implies that $J \subset R_-$.

We will show that the segments $\overline{P_1 P_2}$ and $\overline{P_2 P_3}$ are also in R_-.

We first show that $\overline{P_1 P_2} \subset R_-$, that is, $\operatorname{Re} \Phi_m(z) - \operatorname{Re} \Phi_m(\sigma_m) < -2\pi \operatorname{Im} z$, if $z \in \overline{P_1 P_2}$. From the proof of Lemma 4.5, $-\pi < \partial \operatorname{Re} \Phi_m(z)/\partial y < 0$ if $z = x + y\sqrt{-1}$ is in the pentagonal region $Q P_1 P_2 P_3 \sigma_m$. We also know that if $z \in \overline{Q \sigma_m P_3}$, then $\operatorname{Re} \Phi_m(z) - \operatorname{Re} \Phi_m(\sigma_m) \leq 0$. Since the difference between the imaginary part of the point on $\overline{Q \sigma_m P_3}$ and that of the point on $\overline{P_1 P_2}$ is less than or equal to $2 \operatorname{Im} \sigma_m$, it follows that for $z \in \overline{P_1 P_2}$, we have $\operatorname{Re} \Phi_m(z) - \operatorname{Re} \Phi_m(\sigma_m) < \pi(2 \operatorname{Im} \sigma_m) = -2\pi \operatorname{Im} z$.

Next we show $\overline{P_2 P_3} \subset R_-$. Consider $r(y) := \operatorname{Re} \Phi_m((m+1)/p + y\sqrt{-1}) - \operatorname{Re} \Phi_m(\sigma_m) + 2\pi y$. Since $(d/dy) r(y) = (\partial/\partial y) \operatorname{Re} \Phi_m((m+1)/p + y\sqrt{-1}) + 2\pi > 0$ and $r(0) < 0$ from the argument above, we conclude that $r(y) < 0$ if $y \geq -\operatorname{Im} \sigma_m$. So if $z \in \overline{P_2 P_3}$, then $z \in R_-$.

Therefore, we can put $C_- := \overline{P_0 P_1 P_2 P_3} \subset R_-$.

Next, we consider the case where $m = p - 1$. Here we can push P_3 slightly to the left to avoid $\diamond_\nu$. Accordingly, we move the segments $\overline{\sigma_m P_3}$ and $\overline{P_2 P_3}$ slightly. $\square$

Therefore we can apply Proposition 4.1 to the series of functions $\psi_N(z) = \varphi_{m,N}(z) - \varphi_{m,N}(\sigma_m)$. We conclude that

$$\frac{1}{N} e^{-N\varphi_{m,N}(\sigma_m)} \sum_{m/p \leq k/N \leq (m+1)/p} e^{N\varphi_{m,N}(k/N)} = e^{-N\varphi_{m,N}(\sigma_m)} \int_{m/p}^{(m+1)/p} e^{N\varphi_{m,N}(z)}\, dz + O(e^{-\varepsilon_m N}) \tag{4-16}$$

for $\varepsilon_m > 0$ if $m < p - 1$, and

$$\frac{1}{N} e^{-N\varphi_{p-1,N}(\sigma_m)} \sum_{(p-1)/p \leq k/N \leq 1-\delta} e^{N\varphi_{p-1,N}(k/N)} = e^{-N\varphi_{p-1,N}(\sigma_{p-1})} \int_{(p-1)/p}^{1-\delta} e^{N\varphi_{p-1,N}(z)}\, dz + O(e^{-\varepsilon_{p-1} N}) \tag{4-17}$$

for $\varepsilon_{p-1} > 0$.

5 The saddle point method of order two

We would like to know the asymptotic behavior of the integrals appearing in the right-hand sides of (4-16) and (4-17) by using the saddle point method of order two.

To describe it, let us consider a holomorphic function $\eta(z)$ in a domain $D \ni O$ with $\eta(0) = \eta'(0) = \eta''(0) = 0$ and $\eta^{(3)}(0) \neq 0$, where O is the origin of the complex plane. Write $\eta^{(3)}(0) = 6re^{\theta\sqrt{-1}}$ with $r > 0$ and $-\pi < \theta \leq \pi$. Then $\eta(z)$ is of the form $\eta(z) = re^{\theta\sqrt{-1}}z^3 g(z)$, where $g(z)$ is holomorphic with $g(0) = 1$. The origin is called a saddle point of $\operatorname{Re}\eta(z)$ of order two. We put $V := \{z \in D \mid \operatorname{Re}\eta(z) < 0\}$.

There exists a small disk $\widehat{D} \subset D$ centered at O, where we can define a cubic root $g^{1/3}(z)$ of $g(z)$ such that $g^{1/3}(0) = 1$. Put $G(z) := zg^{1/3}(z)$ in $\widehat{D} \subset D$. We can choose $\widehat{D}$ so that G gives a bijection from $\widehat{D}$ to $E := G(\widehat{D})$ from the inverse function theorem because $G'(0) = 1$. Since $re^{\theta\sqrt{-1}}G(z)^3 = \eta(z)$, the function G also gives a bijection from the region $V \cap \widehat{D}$ to the region $U := \{w \in E \mid \operatorname{Re}(re^{\theta\sqrt{-1}}w^3) < 0\}$.

The region U splits into the three connected components (valleys) U_1, U_2, and U_3. Therefore the region $V \cap \widehat{D}$ also splits into three valleys $V_k := G^{-1}(U_k)$, for $k = 1, 2, 3$, of $\operatorname{Re}\eta(z)$.

Remark 5.1 Since $G'(0) = 1$, and U_k contains the ray $\{w \in E \mid w = se^{((2k-1)\pi-\theta)\sqrt{-1}/3}, s > 0\}$ as a bisector, V_k also contains a segment $\{z \in \widehat{D} \mid z = te^{((2k-1)\pi-\theta)\sqrt{-1}/3}$ for $t > 0$ small$\}$.

The following is the statement of the saddle point method of order two:

Proposition 5.2 *Let $\eta(z)$ be a holomorphic function in a domain $D \ni O$ with $\eta(0) = \eta'(0) = \eta''(0) = 0$ and $\eta^{(3)}(0) \neq 0$. Write $\eta^{(3)}(0) = 6re^{\theta\sqrt{-1}}$ with $r > 0$ and $-\pi < \theta \leq \pi$. Put $V := \{z \in D \mid \operatorname{Re}\eta(z) < 0\}$ and define V_k for $k = 1, 2, 3$ as above. Let $C \subset D$ be a path from a to b with $a, b \in V$.*

We assume that there exist paths $P_k \subset V \cup \{O\}$ from a to O and $P_{k+1} \subset V \cup \{O\}$ from O to b such that

(i) $(P_k \cap \widehat{D}) \setminus \{O\} \subset V_k$,

(ii) $(P_{k+1} \cap \widehat{D}) \setminus \{O\} \subset V_{k+1}$, *and*

(iii) *the path $P_k \cup P_{k+1}$ is homotopic to C in D keeping a and b fixed*,

where $\widehat{D} \in O$ is a disk as above.

Let $\{h_N(z)\}$ be a series of holomorphic functions in D that uniformly converges to a holomorphic function $h(z)$ with $h(0) \neq 0$. We also assume that $|h_N(z)|$ is bounded irrelevant to z or N. Then

$$\int_C h_N(z)e^{N\eta(z)}\,dz = \frac{h(0)\Gamma(1/3)\sqrt{-1}}{\sqrt{3}r^{1/3}N^{1/3}}\omega^k e^{-\theta\sqrt{-1}/3}(1 + O(N^{-1/3})) \tag{5-1}$$

as $N \to \infty$, where $\omega := e^{2\pi\sqrt{-1}/3}$.

The proposition may be well known to experts, but we give a proof in Appendix B because the author is not an expert and could not find appropriate references.

We will apply Proposition 5.2 to

- $\eta(z) := \Phi_m(z + \sigma_m) - \Phi_m(\sigma_m)$,
- $D := \{z \in \mathbb{C} \mid z + \sigma_m \in \Xi_{m,\chi}\}$,

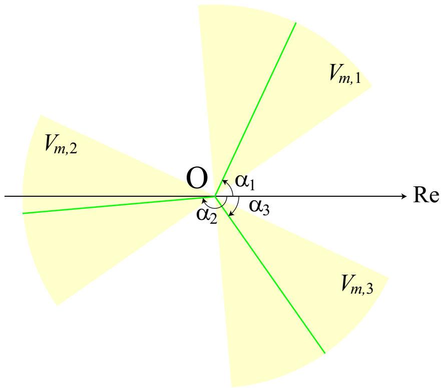

Figure 13: The yellow regions indicates the valleys $V_{m,1}$, $V_{m,2}$, and $V_{m,3}$.

- $h_N(z) := \exp[N(\varphi_{m,N}(z+\sigma_m) - \Phi_m(z+\sigma_m))]$, and
- $C := [m/p - \sigma_m, (m+1)/p - \sigma_m]$ for $m < p-1$, and $C := [(p-1)/p, 1-\delta]$ for $m = p-1$, where δ is a positive small number (see Lemma 4.4).

Note that $\eta(0) = \eta'(0) = \eta''(0) = 0$, $\eta^{(3)}(0) = -2\xi^2 \neq 0$, $h(z) := \lim_{N\to\infty} h_N(z) = 1$, and that V is equal to the region $\{z \in \mathbb{C} \mid z + \sigma_m \in W_m^-\}$.

Since $\eta(z) = -\frac{1}{3}\xi^2 z^3 + \cdots$, we can define a holomorphic function $g(z) := -3\eta(z)/(\xi^2 z^3)$ so that $g(0) = 1$. Put $G(z) := zg^{1/3}(z)$ as above. Let $\widehat{D} \subset D$ be a small disk centered at 0 such that the function $G(z)$ is a bijection. Then the region V splits into three valleys $V_{m,1}$, $V_{m,2}$, and $V_{m,3}$. From Remark 5.1, the argument of the bisector of $V_{m,k}$ is given by $(2k-1)\frac{1}{3}\pi - \frac{1}{3}\theta \pmod{2\pi}$ for $k = 1, 2, 3$, where $\theta := \arg(-2\xi^2) = -\pi + 2\arctan(2p\pi/\kappa)$. So the valley $V_{m,k}$ is approximated by the small sector

$$\{z \in \mathbb{C} \mid z = te^{\tau\sqrt{-1}} \text{ and } |\tau - \alpha_k| < \tfrac{1}{6}\pi \text{ for } t > 0 \text{ small}\},$$

where we put

$$\text{(5-2)}\quad \alpha_1 := -\tfrac{2}{3}\arctan(2p\pi/\kappa) + \tfrac{2}{3}\pi, \quad \alpha_2 := -\tfrac{2}{3}\arctan(2p\pi/\kappa) - \tfrac{2}{3}\pi, \quad \alpha_3 := -\tfrac{2}{3}\arctan(2p\pi/\kappa).$$

Note that since $\frac{1}{4}\pi < \arctan(2p\pi/\kappa) < \frac{1}{2}\pi$, we have $\frac{1}{3}\pi < \alpha_1 < \frac{1}{2}\pi$, $-\pi < \alpha_2 < -\frac{5}{6}\pi$, and $-\frac{1}{3}\pi < \alpha_3 < -\frac{1}{6}\pi$; see Figure 13.

Remark 5.3 Denote by P_G the intersection between G and the boundary of $\Xi_{m,0}$, as in Figure 12. Note that $P_G \subset I_{m+1}$ if $m < p-1$ and $P_G \subset \overline{H}$ if $p = 1$ from Remark 4.9. The arguments of $\overrightarrow{\sigma_m P_G}$, $\overrightarrow{\sigma_m P_{70}}$, and $\overrightarrow{\sigma_m P_3}$ are

$$\text{(5-3)}\qquad \beta_1 := -\arctan(2p\pi/\kappa) + \tfrac{3}{4}\pi, \quad \beta_2 := -\arctan(2p\pi/\kappa) - \tfrac{1}{2}\pi, \quad \beta_3 := -\arctan(2p\pi/\kappa),$$

respectively, because the vector $\overrightarrow{\sigma_m P_{70}}$ has the same direction as $-\sqrt{-1}/\xi$, the vector $\overrightarrow{\sigma_m P_3}$ has the same direction as $1/\xi$, and G is their bisection.

Since $\frac{1}{4}\pi < \arg(2p\pi/\kappa) < \frac{1}{2}\pi$, we can see

$$\alpha_1 - \beta_1 = -\tfrac{1}{12}\pi + \tfrac{1}{3}\arctan(2p\pi/\kappa), \quad \beta_2 - \alpha_2 = \tfrac{1}{6}\pi - \tfrac{1}{3}\arctan(2p\pi/\kappa), \quad \alpha_3 - \beta_3 = \tfrac{1}{3}\arctan(2p\pi/\kappa),$$

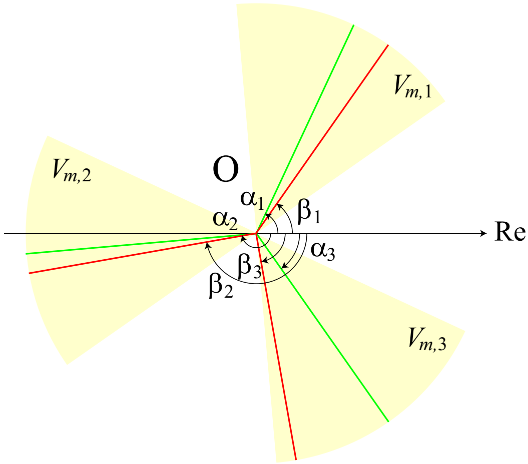

Figure 14: The yellow regions indicate the valleys.

and

$$\alpha_2 < \beta_2 < \beta_3 < \alpha_3 < \beta_1 < \alpha_1,$$

where α_k for $k = 1, 2, 3$ are given in (5-2). We also conclude that $|\alpha_k - \beta_k| < \frac{1}{6}\pi$, that is, $\overline{\sigma_m P_G}$ is in the valley $V_{m,1}$, $\overline{\sigma_m P_{50}}$ is in the valley $V_{m,2}$, and $\overline{\sigma_m P_3}$ is in the valley $V_{m,3}$; see Figure 14.

We need to show that the assumption of Proposition 5.2 holds, that is, we will show the following lemma:

Lemma 5.4 *First suppose that $m = 0, 1, 2, \dots, p-2$. If a disk $\widetilde{D} \subset \Xi_{m,\chi}$ centered at σ_m is small enough, then the following hold:*

(i) *There exists a path $\rho_2 \subset W_m^- \cup \{\sigma_m\}$ connecting m/p and σ_m such that $(\rho_2 \cap \widetilde{D}) \setminus \{\sigma_m\} \subset V_{m,2}$.*

(ii) *There exists a path $\rho_3 \subset W_m^- \cup \{\sigma_m\}$ connecting σ_m and $(m+1)/p$ such that $(\rho_3 \cap \widetilde{D}) \setminus \{\sigma_m\} \subset V_{m,3}$.*

Next, suppose that $m = p-1$. If a disk $\widetilde{D} \subset \Xi_{p-1,\chi}$ centered at σ_{p-1} is small enough, then the following hold:

(i′) *There exists a path $\rho_2 \subset W_{p-1}^- \cup \{\sigma_{p-1}\}$ connecting $1-1/p$ and σ_{p-1} such that $(\rho_2 \cap \widetilde{D}) \setminus \{\sigma_{p-1}\} \subset V_{p-1,2}$.*

(ii′) *There exists a path $\rho_3 \subset W_m^\nu \cup \{\sigma_{p-1}\}$ connecting σ_{p-1} and $1-\delta$ such that $(\rho_3 \cap \widehat{D}) \setminus \{\sigma_{p-1}\} \subset V_{p-1,3}$,*

Again, since $\Xi_{m,\chi}$ is simply connected, the path $\rho_2 \cup \rho_3$ is homotopic to the interval $[m/p, (m+1)/p]$ ($[1-1/p, 1-\delta]$, respectively) if $m < p-1$ ($m = p-1$, respectively).

Proof The proof is essentially the same for both cases $m < p-1$ and $m = p-1$.

(i) The path $\rho_2 := \overline{P_0 P_{70} \sigma_m}$ is a required one for $m = 0, 1, 2, \dots, p-1$.

(ii) When $m < p-1$, consider the path $\rho_3 := \overline{\sigma_m P_3}$, and when $m = p-1$ push it a little more to the left near the point 1. □

If $m < p-1$, we apply Proposition 5.2 to

$$\eta(z) = \Phi_m(z+\sigma_m) - \Phi_m(\sigma_m), \qquad h_N(z) = \exp[N(\varphi_{m,N}(z+\sigma_m) - \Phi_m(z+\sigma_m))],$$
$$C := [m/p - \sigma_m, (m+1)/p - \sigma_m], \qquad k = 2.$$

Noting that $h_N(z)$ converges to 1 and $\eta^{(3)}(0) = -2\xi^2 = 2|\xi|^2 e^{\theta\sqrt{-1}}$ with $\theta = -\pi + 2\arctan(2p\pi/\kappa)$ from the argument above, we have

$$\begin{aligned}\int_{m/p}^{(m+1)/p} e^{N(\varphi_{m,N}(z)-\Phi_m(\sigma_m))}\,dz &= \int_C e^{N(\varphi_{m,N}(z+\sigma_m)-\Phi_m(\sigma_m))}\,dz = \int_C h_N(z) e^{N\eta(z)}\,dz \\ &= \frac{\Gamma\bigl(\frac{1}{3}\bigr)\sqrt{-1}}{\sqrt{3}\bigl(\frac{1}{3}|\xi|^2\bigr)^{1/3} N^{1/3}} \omega^2 e^{\pi\sqrt{-1}/3 - 2\arctan(2p\pi/\kappa)\sqrt{-1}/3}(1+O(N^{-1/3})) \\ &= \frac{\Gamma\bigl(\frac{1}{3}\bigr)\sqrt{-1}}{3^{1/6}|\xi|^{2/3}N^{1/3}} e^{-(\pi+2\arctan(2p\pi/\kappa))\sqrt{-1}/3}(1+O(N^{-1/3}))\end{aligned}$$

as $N \to \infty$. Similarly, if $m = p-1$, putting $C := [1-1/p-\sigma_m, 1-\sigma_m-\delta]$, we have

$$\int_{1-1/p}^{1-\delta} e^{N(\varphi_{p-1,N}(z)-\Phi_{p-1}(\sigma_{p-1}))}\,dz = \frac{\Gamma\bigl(\frac{1}{3}\bigr)\sqrt{-1}}{3^{1/6}|\xi|^{2/3}N^{1/3}} e^{-(\pi+2\arctan(2p\pi/\kappa))\sqrt{-1}/3}(1+O(N^{-1/3}))$$

as $N \to \infty$. Since $\Phi_m(\sigma_m) = 4p\pi^2/\xi$ from (3-12), we conclude

$$\int_{m/p}^{(m+1)/p} e^{N\varphi_{m,N}(z)}\,dz = \frac{\Gamma\bigl(\frac{1}{3}\bigr)\sqrt{-1}}{3^{1/6}|\xi|^{2/3}N^{1/3}} e^{-(\pi+2\arctan(2p\pi/\kappa))\sqrt{-1}/3} e^{4p\pi^2 N/\xi}(1+O(N^{-1/3}))$$

if $m < p-1$, and

$$\int_{1-1/p}^{1-\delta} e^{N\varphi_{p-1,N}(z)}\,dz = \frac{\Gamma\bigl(\frac{1}{3}\bigr)\sqrt{-1}}{3^{1/6}|\xi|^{2/3}N^{1/3}} e^{-(\pi+2\arctan(2p\pi/\kappa))\sqrt{-1}/3} e^{4p\pi^2 N/\xi}(1+O(N^{-1/3})).$$

Since $\varphi_{m,N}(\sigma_m) = f_N(\sigma_0)$ converges to $F(\sigma_0) = 4p\pi^2/\xi$ as $N \to \infty$ from (3-12), together with (4-16) and (4-17), we finally have

$$\sum_{m/p \le k/N \le (m+1)/p} e^{N\varphi_{m,N}(k/N)} = \frac{\Gamma\bigl(\frac{1}{3}\bigr)e^{\pi\sqrt{-1}/6}}{3^{1/6}} \Bigl(\frac{N}{\xi}\Bigr)^{2/3} e^{4p\pi^2 N/\xi}(1+O(N^{-1/3})) \tag{5-4}$$

if $m < p-1$, and

$$\sum_{1-1/p \le k/N \le 1-\delta} e^{N\varphi_{p-1,N}(k/N)} = \frac{\Gamma\bigl(\frac{1}{3}\bigr)e^{\pi\sqrt{-1}/6}}{3^{1/6}} \Bigl(\frac{N}{\xi}\Bigr)^{2/3} e^{4p\pi^2 N/\xi}(1+O(N^{-1/3})) \tag{5-5}$$

because $\mathrm{Re}(4p\pi^2/\xi) > 0$, where we define $\xi^{2/3}$ to be $|\xi|^{2/3} e^{2\arctan(2p\pi/\kappa)\sqrt{-1}/3}$.

It remains to obtain the asymptotic behavior of $\sum_{1-1/p\le k/N<1} e^{N\varphi_{p-1,N}(k/N)}$ instead of the sum for $1-1/p \le k/N \le 1-\delta$. To do that, we need to estimate the sum $\sum_{1-\delta<k/N<1} e^{N\varphi_{p-1,N}(k/N)}$. We use the following lemma, which corresponds to [24, Lemma 6.1].

Lemma 5.5 *For any ε, there exists $\delta'>0$ such that*

$$\operatorname{Re}\varphi_{p-1,N}\left(\frac{2k+1}{2N}\right) < \operatorname{Re}\Phi_{p-1}(\sigma_{p-1})-\varepsilon$$

for sufficiently large N, if $1-\delta'<k/N<1$.

Since a proof is similar to that of [24, Lemma 6.1], we omit it.

From Lemma 5.5, we conclude that

$$\sum_{1-\delta<k/N<1} \exp\left(N\varphi_{p-1,N}\left(\frac{2k+1}{2N}\right)\right)$$

is of order $O(e^{N(\operatorname{Re}\Phi_{p-1}(\sigma_{p-1})-\varepsilon)})$ if $\delta'<\delta$. Since $\Phi_{p-1}(\sigma_{p-1}) = 4p\pi^2\sqrt{-1}/\xi$ from (4-7), we have

$$\sum_{1-1/p\le k/N<1} e^{N\Phi_{p-1,N}(k/N)} = \frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}}\left(\frac{N}{\xi}\right)^{2/3} e^{4p\pi^2N/\xi}(1+O(N^{-1/3}))$$

from (5-5). Together with (4-1) and (5-4), we have

$$J_N(\mathscr{E};e^{\xi/N}) = (1-e^{-4pN\pi^2/\xi})\frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}}\left(\frac{N}{\xi}\right)^{2/3} e^{4p\pi^2N/\xi}\left(\sum_{m=0}^{p-1}\beta_{p,m}\right)(1+O(N^{-1/3})). \tag{5-6}$$

Now from (3-3) and (3-1), the sum in the parentheses is just $J_p(\mathscr{E};e^{4\pi^2N/\xi})$. Therefore we finally have

$$J_N(\mathscr{E};e^{\xi/N}) = J_p(\mathscr{E};e^{4\pi^2N/\xi})\frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}}\left(\frac{N}{\xi}\right)^{2/3}\exp\left(\frac{2\kappa\pi\sqrt{-1}}{\xi}N\right)(1+O(N^{-1/3})),$$

where we replace $e^{4p\pi^2N/\xi}$ with $e^{(4p\pi^2N)/\xi+2N\pi\sqrt{-1}} = e^{2N\kappa\pi\sqrt{-1}/\xi}$ on purpose; see Section 6. Note that we choose the argument of $\xi^{2/3}$ as $\frac{2}{3}\arctan(2p\pi/\kappa)$, which is between $\frac{1}{6}\pi$ and $\frac{1}{3}\pi$.

Proof of Corollary 1.9 Since the figure-eight knot is amphicheiral, that is, it is equivalent to its mirror image, we have $J_N(\mathscr{E};q^{-1}) = J_N(\mathscr{E};q)$. It follows that $J_N(\mathscr{E};e^{\xi'/N}) = J_N(\mathscr{E};e^{-\xi'/N}) = J_N(\mathscr{E};e^{\bar{\xi}/N}) = \overline{J_N(\mathscr{E};e^{\xi/N})}$, where $\bar{\xi}$ is the complex conjugate. So we obtain

$$\begin{aligned} J_N(\mathscr{E};e^{\xi'/N}) &\underset{N\to\infty}{\sim} \overline{J_p(\mathscr{E};e^{4\pi^2N/\xi})}\frac{\Gamma\left(\frac{1}{3}\right)e^{-\pi\sqrt{-1}/6}}{3^{1/6}}\left(\frac{N}{\bar{\xi}}\right)^{2/3}\exp\left(\frac{-2\kappa\pi\sqrt{-1}}{\bar{\xi}}N\right) \\ &= J_p(\mathscr{E};e^{4\pi^2N/\xi'})\frac{\Gamma\left(\frac{1}{3}\right)e^{\pi\sqrt{-1}/6}}{3^{1/6}}\left(\frac{N}{\xi'}\right)^{2/3}\exp\left(\frac{S_{-\kappa}(\mathscr{E})}{\xi'}N\right), \end{aligned}$$

where $(\xi')^{1/3} := |\xi'|^{1/3}e^{-\arctan(2p\pi/\kappa)\sqrt{-1}/3}e^{-\pi\sqrt{-1}/3}$. The last equality follows since $e^{-2\kappa\pi\sqrt{-1}N/\bar{\xi}} = e^{2\kappa\pi\sqrt{-1}N/\xi'} = e^{2\kappa\pi\sqrt{-1}N/\xi'+4N\pi\sqrt{-1}} = e^{(-2\kappa\pi\sqrt{-1}-8pN\pi^2)N/\xi'} = e^{(S_{-\kappa}(\mathscr{E})-8pN\pi^2)/\xi'}$ and the Chern–Simons invariant is defined modulo an integer multiple of π^2 (see Section 6). □

6 The Chern–Simons invariant

In this section, we show a relation between $S_\kappa(E)=2\kappa\pi\sqrt{-1}$ appearing in Theorem 1.8 and the Chern–Simons invariant. For the definition of the Chern–Simons invariant of a representation from the fundamental group of a three-manifold with toric boundary to $\mathrm{SL}(2;\mathbb{C})$, we refer the readers to [17].

Let W be the three-manifold obtained from S^3 by removing the open tubular neighborhood of a knot $K\subset S^3$. We denote by $X(W)$ the $\mathrm{SL}(2;\mathbb{C})$ character variety, that is, the set of characters of representations from $\pi_1(W)$ to $\mathrm{SL}(2;\mathbb{C})$. Let $E(\partial W)$ be the quotient space $(\mathrm{Hom}(\pi_1(\partial W),\mathbb{C})\times\mathbb{C}^\times)/G$, where $\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}$ and $G:=\langle x,y,b\mid xy=yx,\ bxbx=byby=b^2=1\rangle$ acts on $\mathrm{Hom}(\pi_1(\partial W),\mathbb{C})\times\mathbb{C}^\times$ as follows:

$$x\cdot(\alpha,\beta;z):=\bigl(\alpha+\tfrac{1}{2},\beta;z\exp(-4\pi\sqrt{-1}\beta)\bigr),\quad y\cdot(\alpha,\beta;z):=\bigl(\alpha,\beta+\tfrac{1}{2};z\exp(4\pi\sqrt{-1}\alpha)\bigr),$$
$$b\cdot(\alpha,\beta):=(-\alpha,-\beta;z). \tag{6-1}$$

Here we fix a generator $(\mu^*,\lambda^*)\in\mathrm{Hom}(\pi_1(\partial W);\mathbb{C})\cong\mathbb{C}^2$ for a meridian μ (the homotopy class of the loop that goes around K) and a preferred longitude λ (the homotopy class of the loop that goes along K so that its linking number with K is zero). Then the projection $p\colon E(\partial W)\to X(\partial W)$ sending $[\alpha,\beta;z]$ to $[\alpha,\beta]$ becomes a $\mathbb{C}^\times$-bundle, where the square brackets mean the equivalence class.

The $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant of W defines a lift $\mathrm{cs}_W\colon X(W)\to E(\partial W)$ of $X(W)\xrightarrow{i^*}X(\partial W)$, that is, $p\circ c_W=i^*$ holds, where i^* is induced by the inclusion map $i\colon\partial W\to W$:

$$\begin{tikzcd} & E(\partial W) \arrow[d, "p"] \\ X(W) \arrow[ur, "\mathrm{cs}_W"] \arrow[r, "i^*"] & X(\partial W) \end{tikzcd}$$

For a representation ρ, we have $\mathrm{cs}_W([\rho])=[u/(4\pi\sqrt{-1}),v/(4\pi\sqrt{-1});\exp((2/(\pi\sqrt{-1}))\,\mathrm{CS}_{u,v}(\rho))]$ if

$$\rho(\mu)=\begin{pmatrix}e^{u/2}&*\\0&e^{-u/2}\end{pmatrix}\quad\text{and}\quad\rho(\lambda)=\begin{pmatrix}e^{v/2}&*\\0&e^{-v/2}\end{pmatrix},$$

up to conjugation, where $[\rho]\in X(W)$ means the equivalence class, and $\mathrm{CS}_{u,v}(\rho)$ is the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant of ρ associated with (u,v). Note that $\mathrm{CS}_{u,v}(\rho)$ is defined modulo π^2, and depends on the choice of branches of logarithms of $e^{u/2}$ and $e^{v/2}$.

Now, we calculate the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant of the figure-eight knot. See also [29, Section 5.2] for calculation about the figure-eight knot complement.

By using generators x and y as indicated in Figure 15, the fundamental group $G_{\mathscr{E}}:=\pi_1(S^3\setminus\mathscr{E})$ has a presentation $\langle x,y\mid\omega x=y\omega\rangle$, where $\omega:=xy^{-1}x^{-1}y$. We choose (the homotopy class of) x as the meridian μ, and (the homotopy class of) l depicted in Figure 15 as the preferred longitude λ. The loop l presents the element $x\omega^{-1}\overleftarrow{\omega}^{-1}x^{-1}\in G_{\mathscr{E}}$, where $\overleftarrow{\omega}:=yx^{-1}y^{-1}x$ is the word obtained from ω by

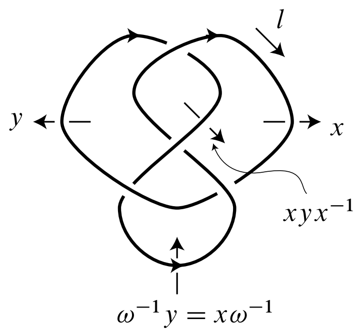

Figure 15: The figure-eight knot $\mathscr{E}$ and generators of $G_{\mathscr{E}} := \pi_1(S^3 \setminus \mathscr{E})$.

reading backward. Due to [36] (see also [22, Section 3]), for a real number u with $0 \le u \le \kappa$ we consider the nonabelian representation $\rho_u : G_{\mathscr{E}} \to \mathrm{SL}(2;\mathbb{C})$ sending x and y to

$$\begin{pmatrix} e^{u/2} & 1 \\ 0 & e^{-u/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{u/2} & 0 \\ d & e^{-u/2} \end{pmatrix},$$

respectively, where d is given as

$$d := \tfrac{3}{2} - \cosh u + \tfrac{1}{2}\sqrt{(2\cosh(u)+1)(2\cosh(u)-3)}.$$

The preferred longitude is sent to

$$\begin{pmatrix} e^{v(u)/2} & * \\ 0 & e^{-v(u)/2} \end{pmatrix},$$

where

$$v(u) := 2\log(\cosh(2u) - \cosh(u) - 1 - \sinh(u)\sqrt{(2\cosh(u)+1)(2\cosh(u)-3)}) + 2\pi\sqrt{-1}.$$

Here we add $2\pi\sqrt{-1}$ so that $v(0) = 0$.

It is well known [39] that when $u = 0$, the irreducible representation ρ_0 induces a complete hyperbolic structure in $S^3 \setminus \mathscr{E}$, and when $0 < u < \kappa$, ρ_u is irreducible and induces an incomplete hyperbolic structure. When $u = \kappa$, the representation ρ_κ becomes reducible (and nonabelian), and the hyperbolic structure collapses. In fact, in this case, both x and y are sent to upper triangular matrices, and so every element of $G_{\mathscr{E}}$ is sent to an upper triangular matrix, which means that ρ_κ is reducible. This kind of reducible and nonabelian representation is called affine, and corresponds to the zeroes of the Alexander polynomial; see [3; 16, Exercise 11.2; 35; 40, 2.4.3. Corollary].

Now, we calculate the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant $\mathrm{CS}_{\kappa,v(\kappa)}(\rho_\kappa)$ associated with $(\kappa, v(\kappa)) = (\kappa, 2\pi\sqrt{-1})$; see [17] for details.

Since the Chern–Simons invariant of a representation is determined by its character, and ρ_κ shares the same character (trace) with the abelian representation $\rho_\kappa^{\mathrm{abel}}$ sending $\mu := x$ to the diagonal matrix

$$\begin{pmatrix} e^{\kappa/2} & 0 \\ 0 & e^{-\kappa/2} \end{pmatrix}$$

and $\lambda := l$ to the identity matrix, it can be easily seen that $\mathrm{cs}_W(\rho_\kappa^{\mathrm{abel}}) = [\kappa/(4\pi\sqrt{-1}), 0; 1]$, where we put $W := S^3 \setminus N(\mathscr{E})$ with $N(\mathscr{E})$ is the open tubular neighborhood of $\mathscr{E}$ in S^3. Since we have

$$\left[\frac{\kappa}{4\pi\sqrt{-1}}, 0; 1\right] = \left[\frac{\kappa}{4\pi\sqrt{-1}}, \frac{1}{2}; e^{\kappa}\right]$$

from (6-1), we conclude that $\mathrm{CS}_{\kappa,2\pi\sqrt{-1}}(\rho_\kappa) = \frac{1}{2}\kappa\pi\sqrt{-1}$. Note that here we change the pair $(\kappa, 0)$ to $(\kappa, 2\pi\sqrt{-1})$.

As in [23], if we define

$$S_u(\mathscr{E}) := \mathrm{CS}_{u,v(u)}(\rho_u) + \pi\sqrt{-1}u + \tfrac{1}{4}uv(u) \tag{6-2}$$

for $0 \le u \le \kappa$, then $S_\kappa(\mathscr{E}) = 2\kappa\pi\sqrt{-1}$ when $(u, v(u)) = (\kappa, 2\pi\sqrt{-1})$.

Similarly, $\mathrm{CS}_{-\kappa,2\pi\sqrt{-1}}(\rho_{-\kappa}) = -\frac{1}{2}\kappa\pi\sqrt{-1}$, and $S_{-u}(\mathscr{E}) = -2\kappa\pi\sqrt{-1}$.

Appendix A Proof of the Poisson summation formula

In this appendix, we give a proof of the Poisson summation formula following [30, Proposition 4.2].

Proof of Proposition 4.1 Let $\varepsilon > 0$ be small enough that

$$\operatorname{Re}\psi(a) < -\varepsilon, \qquad \operatorname{Re}\psi(b) < -\varepsilon,$$
$$\operatorname{Re}\psi(z) - 2\pi\operatorname{Im} z < -\varepsilon \quad \text{if } z \in C_+, \qquad \operatorname{Re}\psi(z) + 2\pi\operatorname{Im} z < -\varepsilon \quad \text{if } z \in C_-.$$

Then for sufficiently large N, the following also hold:

(i) $\operatorname{Re}\psi_N(a) < -\varepsilon$,

(ii) $\operatorname{Re}\psi_N(b) < -\varepsilon$,

(iii) $\operatorname{Re}\psi_N(z) - 2\pi\operatorname{Im} z < -\varepsilon$ if $z \in C_+$,

(iv) $\operatorname{Re}\psi_N(z) + 2\pi\operatorname{Im} z < -\varepsilon$ if $z \in C_-$.

Moreover, there exists $\delta > 0$ such that $\operatorname{Re}\psi_N(t) < -\varepsilon$ if $t \in [a, a+\delta]$ or $t \in [b-\delta, b]$ from (i) and (ii) for such N.

Let $\beta\colon \mathbb{R} \to [0, 1]$ be a C^∞-function such that

$$\beta(t) = \begin{cases} 1 & \text{if } t \in [a+\delta, b-\delta], \\ 0 & \text{if } t < a \text{ or } t > b. \end{cases}$$

We also assume that $\beta(t)$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$, that is, $\sup_{x\in\mathbb{R}}|x^m f^{(n)}(x)| < \infty$ for any nonnegative integers m and n. Put $\Psi_N(x) := \beta(x/N)e^{N\psi_N(x/N)}$.

We have

$$\left|\sum_{a\le k/N<a+\delta} e^{N\psi_N(k/N)}\right| \le \sum_{a\le k/N<a+\delta} e^{N\operatorname{Re}\psi_N(k/N)} < \delta N e^{-\varepsilon N}, \tag{A-1}$$

where the second inequality follows since $\operatorname{Re}\psi_N(k/N) < -\varepsilon$ when $a \le k/N \le a+\delta$. Similarly we have

(A-2) $$\Bigl|\sum_{b-\delta\le k/N<b} e^{N\psi_N(k/N)}\Bigr| < \delta N e^{-\varepsilon N}.$$

We also have

(A-3) $$\Bigl|\sum_{k/N<a+\delta} \Psi_N(k)\Bigr| \le \sum_{a\le k/N<a+\delta} \beta(k/N)e^{N\operatorname{Re}\psi_N(k/N)} < \delta N e^{-\varepsilon N}$$

and

(A-4) $$\Bigl|\sum_{k/N>b-\delta} \Psi_N(k)\Bigr| < \delta N e^{-\varepsilon N}.$$

Since $\Psi_N(k) = e^{N\psi_N(k/N)}$ if $a+\delta \le k/N \le b-\delta$, we have

(A-5) $$\begin{aligned}\Bigl|\sum_{k\in\mathbb{Z}} \Psi_N(k) - \sum_{a\le k/N\le b} e^{N\psi_N(k/N)}\Bigr| &\le \Bigl|\sum_{k/N<a+\delta}\Psi_N(k)\Bigr| + \Bigl|\sum_{a\le k/N<a+\delta} e^{N\psi_N(k/N)}\Bigr| + \Bigl|\sum_{b-\delta<k/N\le b}\Psi_N(k)\Bigr| \\ &\qquad + \Bigl|\sum_{k/N>b-\delta} e^{N\psi_N(k/N)}\Bigr| \\ &< 4\delta N e^{-\varepsilon N}\end{aligned}$$

from (A-1)–(A-4).

Since $\Psi_N(t)$ is also in $\mathcal{S}(\mathbb{R})$, we can apply the Poisson summation formula (see eg [38, Theorem 3.1]):

(A-6) $$\sum_{k\in\mathbb{Z}} \Psi_N(k) = \sum_{l\in\mathbb{Z}} \widehat{\Psi}_N(l),$$

where $\widehat{\Psi}_N$ is the Fourier transform of Ψ_N, that is, $\widehat{\Psi}_N(l) := \int_{-\infty}^{\infty} \Psi_N(t)e^{-2l\pi\sqrt{-1}t}\,dt$.

Putting $s := t/N$,

(A-7) $$\widehat{\Psi}_N(l) = N\int_{-\infty}^{\infty} \beta(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds.$$

From the properties of $\beta(s)$, we have

(A-8) $$\begin{aligned}\Bigl|\frac{1}{N}\widehat{\Psi}_N(0) - \int_a^b e^{N\psi_N(s)}\,ds\Bigr| &\le \Bigl|\int_a^{a+\delta}(\beta(s)-1)e^{N\psi_N(s)}\,ds\Bigr| + \Bigl|\int_{b-\delta}^{b}(\beta(s)-1)e^{N\psi_N(s)}\,ds\Bigr| \\ &\le \int_a^{a+\delta}(1-\beta(s))e^{N\operatorname{Re}\psi_N(s)}\,ds + \int_{b-\delta}^{b}(1-\beta(s))e^{N\operatorname{Re}\psi_N(s)}\,ds \\ &< 2\delta e^{-\varepsilon N}.\end{aligned}$$

Therefore

$$\text{(A-9)}\quad \begin{aligned}
&\Big|\frac{1}{N}\sum_{a\le k/N\le b} e^{N\psi_N(k/N)}-\int_a^b e^{N\psi_N(s)}\,ds\Big| \\
&\le\Big|\frac{1}{N}\sum_{a\le k/N\le b} e^{N\psi_N(k/N)}-\frac{1}{N}\sum_{l\in\mathbb{Z}}\widehat{\Psi}_N(l)\Big|+\Big|\frac{1}{N}\sum_{l\in\mathbb{Z}}\widehat{\Psi}_N(l)-\int_a^b e^{N\psi_N(s)}\,ds\Big| \\
&\le\Big|\frac{1}{N}\sum_{a\le k/N\le b} e^{N\psi_N(k/N)}-\frac{1}{N}\sum_{k\in\mathbb{Z}}\Psi_N(k)\Big|+\Big|\frac{1}{N}\widehat{\Psi}_N(0)-\int_a^b e^{N\psi_N(s)}\,ds\Big|+\frac{1}{N}\sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}}|\widehat{\Psi}_N(l)| \\
&<\frac{1}{N}\sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}}|\widehat{\Psi}_N(l)|+6\delta e^{-\varepsilon N},
\end{aligned}$$

where the first inequality follows from (A-6), and the second from (A-5) and (A-8). Next we calculate $\widehat{\Psi}_N(l)$ for $l\ne 0$. Integrating the right-hand side of (A-7) by parts twice, we have

$$\begin{aligned}
\widehat{\Psi}_N(l)&=\frac{1}{2l\pi\sqrt{-1}}\int_{-\infty}^{\infty}\frac{d}{ds}(\beta(s)e^{N\psi_N(s)})e^{-2l\pi\sqrt{-1}Ns}\,ds \\
&=-\frac{1}{4l^2\pi^2N}\int_{-\infty}^{\infty}\frac{d^2}{ds^2}(\beta(s)e^{N\psi_N(s)})e^{-2l\pi\sqrt{-1}Ns}\,ds.
\end{aligned}$$

Putting

$$\begin{aligned}
B_N(s)&:=\beta''(s)+2N\beta'(s)\psi'_N(s)+N\beta(s)\psi''_N(s)+N^2\beta(s)(\psi'(s))^2, \\
\widetilde{B}_N(s)&:=N\psi''_N(s)+N^2(\psi'_N(s))^2,
\end{aligned}$$

we have

$$\begin{aligned}
&-4l^2\pi^2N\widehat{\Psi}_N(l) \\
&\quad=\int_a^b B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds \\
&\quad=\int_{a+\delta}^{b-\delta}\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds+\int_a^{a+\delta}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds \\
&\qquad\qquad+\int_{b-\delta}^{b}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds \\
&\quad=\int_a^b\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds-\int_a^{a+\delta}\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds \\
&\qquad\qquad-\int_{b-\delta}^{b}\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds+\int_a^{a+\delta}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds \\
&\qquad\qquad+\int_{b-\delta}^{b}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds,
\end{aligned}$$

where the second equality follows because $B_N(s)=\widetilde{B}_N(s)$ when $s\in[a+\delta,b-\delta]$. So we have

$$\text{(A-10)}\quad \begin{aligned}
&\Big|4l^2\pi^2N\widehat{\Psi}_N(l)+\int_a^b\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\Big| \\
&\le\Big|\int_a^{a+\delta}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\Big|+\Big|\int_{b-\delta}^{b}B_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\Big| \\
&\qquad+\Big|\int_a^{a+\delta}\widetilde{B}_N(s)e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\Big|+\Big|\int_{b-\delta}^{b}\widetilde{B}_N(s)e^{N(\psi_N(s)-2k\pi\sqrt{-1}s)}\,ds\Big|.
\end{aligned}$$

Since $\operatorname{Re}\psi_N(s) < -\varepsilon$ if $a \le s \le a+\delta$, we have

$$\left|\int_a^{a+\delta} B_N(s) e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\right| \le \int_a^{a+\delta} |B_N(s)| e^{N\operatorname{Re}\psi_N(s)}\,ds < \delta e^{-\varepsilon N} \max_{s\in[a,a+\delta]} |B_N(s)| \le K_a N^2 e^{-\varepsilon N}, \tag{A-11}$$

where we put

$$K_a := \max_{s\in[a,a+\delta]}|\beta''(s)| + \max_{s\in[a,a+\delta]}|2\beta'(s)\psi_N'(s)| + \max_{s\in[a,a+\delta]}|\beta(s)\psi_N''(x)| + \max_{s\in[a,a+\delta]}|\beta(s)(\psi_N'(s))^2|$$
$$\ge \max_{s\in[a,a+\delta]}\left|\frac{\beta''(s)}{N^2} + \frac{2\beta'(s)\psi_N'(s)}{N} + \frac{\beta(s)\psi_N''(s)}{N} + \beta(s)(\psi_N'(s))^2\right| = \max_{s\in[a,a+\delta]}|B_N(s)|\frac{1}{N^2}.$$

Similarly, putting

$$K_b := \max_{s\in[b-\delta,b]}|\beta''(s)| + \max_{s\in[b-\delta,b]}|2\beta'(s)\psi_N'(s)| + \max_{s\in[b-\delta,b]}|\beta(s)\psi_N''(x)| + \max_{s\in[b-\delta,b]}|\beta(s)(\psi_N'(s))^2|,$$
$$\widetilde{K}_a := \max_{s\in[a,a+\delta]}|\psi_N''(s)| + \max_{s\in[a,a+\delta]}|(\psi_N'(s))^2|, \quad \widetilde{K}_b := \max_{s\in[b-\delta,b]}|\psi_N''(s)| + \max_{s\in[b-\delta,b]}|(\psi_N'(s))^2|,$$

we have

$$\left|\int_{b-\delta}^{b} B_N(s) e^{N(\psi_N(s)-2k\pi\sqrt{-1}s)}\,ds\right| < K_b N^2 e^{-\varepsilon N}, \tag{A-12}$$

$$\left|\int_{a}^{a+\delta} \widetilde{B}_N(s) e^{N(\psi_N(s)-2k\pi\sqrt{-1}s)}\,ds\right| < \widetilde{K}_a N^2 e^{-\varepsilon N}, \tag{A-13}$$

$$\left|\int_{b-\delta}^{b} \widetilde{B}_N(s) e^{N(\psi_N(s)-2k\pi\sqrt{-1}s)}\,ds\right| < \widetilde{K}_b N^2 e^{-\varepsilon N}. \tag{A-14}$$

Therefore

$$|\widehat{\Psi}_N(l)| < \frac{1}{4l^2\pi^2 N}\left|\int_a^b \widetilde{B}_N(s) e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds\right| + \frac{KN}{4l^2\pi^2}e^{-\varepsilon N} \tag{A-15}$$

from (A-11)–(A-14), where we put $K := K_a + K_b + \widetilde{K}_a + \widetilde{K}_b$.

To evaluate $\int_a^b \widetilde{B}_N(s) e^{N(\psi_N(s)-2l\pi\sqrt{-1}s)}\,ds$, we consider the paths $C_\pm \subset R_\pm$. Note that $\widetilde{B}_N$ is defined in D.

By replacing the path $[a,b]$ with $C_\pm$, we have

$$\begin{aligned}\left|\int_a^b \widetilde{B}_N(s) e^{N(\psi_N(s)-2\pi\sqrt{-1}ls)}\,ds\right| &= \left|\int_{C_\pm} \widetilde{B}_N(z) e^{N(\psi_N(z)-2\pi\sqrt{-1}lz)}\,dz\right| \\ &\le \int_{C_\pm} |\widetilde{B}_N(z)| e^{N(\operatorname{Re}\psi_N(z)+2l\pi\operatorname{Im} z)}\,|dz| \\ &\le \max_{z\in C_\pm}|\widetilde{B}_N(z)| \int_{C_\pm} e^{N(\operatorname{Re}\psi_N(z)+2l\pi\operatorname{Im} z)}\,|dz| \\ &\le K_\pm N^2 \int_{C_\pm} e^{N(\operatorname{Re}\psi_N(z)+2l\pi\operatorname{Im} z)}\,|dz|,\end{aligned} \tag{A-16}$$

where we put

$$K_\pm := \max_{z\in C_\pm}|\psi_N''(z)| + \max_{z\in C_\pm}|(\psi_N'(z))^2| \ge \max_{z\in C_\pm}\left|\frac{\psi_N''(z)}{N} + (\psi_N'(z))^2\right| = \max_{z\in C_\pm}|\widetilde{B}_N(z)|\frac{1}{N^2}.$$

If $l \ge 1$, we use C_-. Since $C_- \subset R_-$, we have $\operatorname{Im} z \le 0$ and $\operatorname{Re}\psi_N(z) + 2\pi \operatorname{Im} z < -\varepsilon$ from (iv). So from (A-16), we have

$$\left|\int_a^b \widetilde{B}_N(s) e^{N(\psi_N(s) - 2\pi\sqrt{-1}ls)}\,ds\right| < \widetilde{K}_- N^2 e^{-\varepsilon N}, \tag{A-17}$$

where $\widetilde{K}_- := K_-(\text{length of } C_-)$.

Similarly, if $l \le -1$, putting $\widetilde{K}_+ := K_+(\text{length of } C_+)$, we have

$$\left|\int_a^b \widetilde{B}_N(s) e^{N(\psi_N(s) - 2\pi\sqrt{-1}ls)}\,ds\right| < \widetilde{K}_+ N^2 e^{-\varepsilon N} \tag{A-18}$$

from (iii).

Therefore, from (A-15)–(A-18), we have

$$\begin{aligned}\left|\sum_{l\in\mathbb{Z},l\neq 0} \hat{\Psi}_N(l)\right| &< \sum_{l=1}^{\infty}\left(\frac{\widetilde{K}_- N}{4l^2\pi^2}e^{-\varepsilon N} + \frac{KN}{4l^2\pi^2}e^{-\varepsilon N}\right) + \sum_{l=1}^{\infty}\left(\frac{\widetilde{K}_+ N}{4l^2\pi^2}e^{-\varepsilon N} + \frac{KN}{4l^2\pi^2}e^{-\varepsilon N}\right) \\ &= \left(\frac{\widetilde{K}_-}{24} + \frac{\widetilde{K}_+}{24} + \frac{K}{12}\right)Ne^{-\varepsilon N},\end{aligned}$$

since $\sum_{l=1}^{\infty} 1/l^2 = \frac{1}{6}\pi^2$.

From (A-9), we finally have

$$\left|\frac{1}{N}\sum_{a\le k/N\le b} e^{N\psi(k/N)} - \int_a^b e^{N\psi_N(s)}\,ds\right| < \left(6\delta + \frac{\widetilde{K}_-}{24} + \frac{\widetilde{K}_+}{24} + \frac{K}{12}\right)e^{-\varepsilon N},$$

proving the proposition. □

Appendix B Proof of the saddle point method of order two

In this appendix, we give a proof of Proposition 5.2.

Let $c := re^{\theta\sqrt{-1}}$ be a complex number with $r > 0$ and $-\pi < \theta \le \pi$, and put $U := \{z \in \mathbb{C} \mid \operatorname{Re}(cz^3) < 0\}$. If we write $z := se^{\tau\sqrt{-1}}$ with $s > 0$ and $\tau \in \mathbb{R}$, then since $cz^3 = rs^3 e^{(\theta+3\tau)\sqrt{-1}}$, the region U has three connected components U_k for $k = 1, 2, 3$:

$$U_k := \left\{w \in \mathbb{C} \mid w = se^{\tau\sqrt{-1}},\ s > 0,\ |\tau + \tfrac{1}{3}\theta - (2k-1)\tfrac{1}{3}\pi| < \tfrac{1}{6}\pi\right\}. \tag{B-1}$$

Note that U_k for $k = 1, 2, 3$ is obtained from U_{k-1} by the $\frac{2}{3}\pi$-rotation around the origin O, where U_0 means U_3. The origin O is a saddle point of order two for the function $\operatorname{Re}(cz^3)$, and the regions U_k are called valleys.

First of all, we study the asymptotic behavior of the integral $\int_C h_N(z)e^{Ncz^3}\,dz$ as $N \to \infty$, where C is a path starting at the origin and going into a valley, and $h_N(z)$ is a holomorphic function depending on N. The next lemma follows from the techniques described in [42, II.4]:

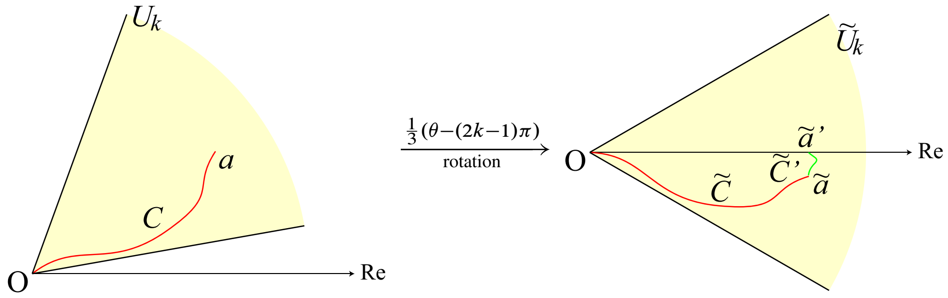

Figure 16: The yellow regions are U_k and $\widetilde{U}_k$, the red curves are C and $\widetilde{C}$, and the green curve is $\widetilde{C}'$.

Lemma B.1 *Let D be an open bounded region in $\mathbb{C}$ containing O, $h_N(z)$ be a holomorphic function in D depending on a positive integer N, and U_k be as above. We assume that $h_N(z)$ uniformly converges to a holomorphic function $h(z)$ with $h(0) \neq 0$ and that $|h_N(z)|$ is bounded irrelevant to z or N. We also assume that $U_k \cap D$ is connected and simply connected for each k. For a point $a \in U_k \cap D$, let $C \subset (U_k \cap D) \cup \{O\}$ be a path from O to a. Then we have*

$$\int_C h_N(z)e^{Ncz^3}\,dz = \frac{e^{((2k-1)\pi-\theta)\sqrt{-1}/3}h(0)\Gamma\left(\frac{1}{3}\right)}{3r^{1/3}N^{1/3}}(1+O(N^{-1/3}))$$

as $N \to \infty$, where $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt$ is the gamma function.

Proof Let $\widetilde{U}_k$ be the region obtained from U_k by the $\frac{1}{3}(\theta-(2k-1)\pi)$-rotation around O, that is,

$$\widetilde{U}_k := \left\{w \in \mathbb{C} \mid w = se^{\tau\sqrt{-1}},\ s > 0,\ |\tau| < \tfrac{1}{6}\pi\right\}. \tag{B-2}$$

The same rotation sends D to $\widetilde{D}$, C to $\widetilde{C} \subset (\widetilde{U}_k \cap \widetilde{D}) \cup \{O\}$, and a to $\tilde{a} := e^{(\theta-(2k-1)\pi)\sqrt{-1}/3}a \in \widetilde{U}_k \cap \widetilde{D}$; see Figure 16.

Putting

$$w := e^{(\theta-(2k-1)\pi)\sqrt{-1}/3}z$$

and

$$\tilde{h}_N(w) := h_N(e^{((2k-1)\pi-\theta)\sqrt{-1}/3}w),$$

we have

$$\int_C h_N(z)e^{Ncz^3}\,dz = e^{((2k-1)\pi-\theta)\sqrt{-1}/3}\int_{\widetilde{C}} \tilde{h}_N(w)e^{-Nrw^3}\,dw. \tag{B-3}$$

Since $\widetilde{U}_k \cap \widetilde{D}$ is connected, we can choose $\tilde{a}' > 0$ in $\mathbb{R} \cap \widetilde{U}_k \cap \widetilde{D}$ and connect $\tilde{a}$ to $\tilde{a}'$ by a path $\widetilde{C}' \subset \widetilde{U}_k \cap \widetilde{D}$. Now the function $\tilde{h}_N$ is defined in $\widetilde{D}$, and we will extend $\tilde{h}_N|_{\widetilde{U}_k \cap \widetilde{D} \cap \mathbb{R}}$ to a C^∞ function $h^*_N(t)$ for any $t \geq 0$. Here we assume the following:

(i) $h^*_N(t)$ is bounded.

(ii) $h^*_N(t)$ converges uniformly to a C^∞ function $h^*(t)$.

(iii) $h^*_N(t) = \tilde{h}_N(t)$ and $h^*(t) = \tilde{h}(t) := h(e^{((2k-1)\pi-\theta)\sqrt{-1}/3}t)$ for $t \in \widetilde{U}_k \cap \widetilde{D} \cap \mathbb{R}$.

Then since $\widetilde{U}_k \cap \widetilde{D}$ is simply connected, by Cauchy's theorem we have

$$\int_{\widetilde{C}} \tilde{h}_N(w)e^{-Nrw^3}\,dw = I_1 - I_2 - I_3, \tag{B-4}$$

where we put

$$I_1 := \int_0^\infty h_N^*(w)e^{-Nrw^3}\,dw, \quad I_2 := \int_{\tilde{a}'}^\infty h_N^*(w)e^{-Nrw^3}\,dw, \quad I_3 := \int_{\widetilde{C}'} \tilde{h}_N(w)e^{-Nrw^3}\,dw.$$

We use Watson's lemma [41] to evaluate I_1. Putting $t := w^3$, we have

$$I_1 = \int_0^\infty h_N^*(t^{1/3})\frac{1}{3t^{2/3}}e^{-Nrt}\,dt.$$

Since $h_N^*(s)$ uniformly converges to an analytic function $h^*(s)$ in $\widetilde{D} \cap \mathbb{R}$, we conclude that

$$h_N^*(s) = h^*(s) + \frac{g_N(s)}{N}$$

with $|g_N(s)| < c$, where c is a constant independent of s. Since $h^*(0) = h(0)$, $h_N^*(s)$ is of the form

$$h_N^*(s) = h(0) + \frac{g_N(s)}{N} + \sum_{j=1}^\infty b_j s^j$$

near 0, where $b_j := (1/j!)(d^j/ds^j)h(0)$. So we have

$$h_N^*(t^{1/3})\frac{1}{3t^{2/3}} = \tfrac{1}{3}h(0)t^{-2/3} + \frac{g_N(t^{1/3})}{3t^{2/3}N} + \sum_{j=1}^\infty \tfrac{1}{3}b_j t^{(j-2)/3}.$$

Since $|g_N(s)| < c$,

$$\left|\int_0^\infty \frac{g_N(t^{1/3})}{3t^{2/3}N}e^{-Nrt}\,dt\right| < \frac{c}{3N}\int_0^\infty t^{-2/3}e^{-Nrt}\,dt = \frac{c\Gamma\left(\frac{1}{3}\right)}{3r^{1/3}N^{4/3}}.$$

Therefore from Watson's lemma [41, page 133] (see also [42, page 20]), we have

$$I_1 = \frac{h(0)\Gamma\left(\frac{1}{3}\right)}{3(rN)^{1/3}} + \sum_{j=1}^\infty \tfrac{1}{3}b_j\Gamma\left(\tfrac{1}{3}(j+1)\right)(rN)^{-(j+1)/3} + O(N^{-4/3}) = \frac{h(0)\Gamma\left(\frac{1}{3}\right)}{3(rN)^{1/3}} + O(N^{-2/3}) \tag{B-5}$$

as $N \to \infty$.

As for I_2, since $|h_N^*(w)| < M$ if $w \in \mathbb{R}$ for some $M > 0$, we have

$$|I_2| \le \int_{\tilde{a}'}^\infty |h_N^*(w)|e^{-rN\tilde{a}'^2 w}\,dw = \frac{Me^{-\tilde{a}'^3 rN}}{\tilde{a}'^2 rN} < M_1 e^{-\varepsilon_1 N} \tag{B-6}$$

if $N > 1$, where we put $M_1 := M/(r\tilde{a}'^2)$ and $\varepsilon_1 := r\tilde{a}'^3 > 0$.

As for I_3, we note that if $w \in \widetilde{C}' \subset \widetilde{U}_k$, then $\operatorname{Re} w^3 > \varepsilon_2$ for some $\varepsilon_2 > 0$, since $|\arg(w^3)| < \frac{1}{2}\pi$ from (B-2). So

$$|I_3| < \max_{w \in \widetilde{C}'} |\tilde{h}_N(w)| \int_{\widetilde{C}'} e^{-Nr\varepsilon_2}\,dw \le M_2 e^{-r\varepsilon_2 N}, \tag{B-7}$$

where we put $M_2 := \max_{w \in \widetilde{C}'} |\tilde{h}_N(w)|(\text{length of } \widetilde{C}')$.

From (B-4), (B-6), and (B-7), we have

$$\left| \int_{\widetilde{C}} \tilde{h}_N(z) e^{-Nrw^3}\, dw - I_1 \right| \le |I_2| + |I_3| = O(e^{-\varepsilon_3 N}),$$

with $\varepsilon_3 := \min\{\varepsilon_1, r\varepsilon_2\}$. Therefore from (B-3) and (B-5) we finally have

$$\int_C h_N(z) e^{Ncz^3}\, dz = \frac{e^{((2k-1)\pi-\theta)\sqrt{-1}/3} h(0)\Gamma\left(\frac{1}{3}\right)}{3r^{1/3}N^{1/3}}(1 + O(N^{-1/3})). \qquad \square$$

Corollary B.2 *Let $c := re^{\theta\sqrt{-1}}$, D, $h_N(z)$, $h(z)$, and U_k be as in Lemma B.1. Let $C \subset D$ be a path from $a_k \in U_k \cap D$ to $a_{k+1} \in U_{k+1} \cap D$, where U_4 means U_1. We also assume that there exist paths C_k from a_k to O and C_{k+1} from O to a_{k+1} with the following properties:*

(i) $C_k \setminus \{O\} \subset U_k \cap D$.

(ii) $C_{k+1} \setminus \{O\} \subset U_{k+1} \cap D$.

(iii) *The path $C_k \cup C_{k+1}$ is homotopic to C in D keeping a_k and a_{k+1} fixed.*

Then

$$\int_C h_N(z) e^{Ncz^3}\, dz = \frac{h(0)\Gamma\left(\frac{1}{3}\right)}{\sqrt{3}r^{1/3}N^{1/3}} \sqrt{-1}\omega^k e^{-\theta\sqrt{-1}/3}(1 + O(N^{-1/3})),$$

where we put $\omega := e^{2\pi\sqrt{-1}/3}$.

Proof By Cauchy's theorem, $\int_C h_N(z) e^{Ncz^3}\, dz = \int_{C_k \cup C_{k+1}} h_N(z) e^{Ncz^3}\, dz$. Then from Lemma B.1 we have

$$\begin{aligned}\int_{C_k \cup C_{k+1}} h_N(z) e^{Ncz^3}\, dz &= \frac{e^{-\theta\sqrt{-1}/3} h(0)\Gamma\left(\frac{1}{3}\right)}{3r^{1/3}N^{1/3}} (e^{(2k+1)\pi\sqrt{-1}/3} - e^{(2k-1)\pi\sqrt{-1}/3})(1 + O(N^{-1/3})) \\ &= \frac{e^{-\theta\sqrt{-1}/3} h(0)\Gamma\left(\frac{1}{3}\right)}{\sqrt{3}r^{1/3}N^{1/3}} \sqrt{-1}\omega^k (1 + O(N^{-1/3})),\end{aligned}$$

completing the proof. $\square$

Proof of Proposition 5.2 We use Cauchy's theorem to study the integral $\int_{P_k \cup P_{k+1}} h_N(z) e^{N\eta(z)}\, dz$. Since any point on P_k or P_{k+1} outside $\widehat{D}$ satisfies the inequality $\operatorname{Re} \eta(z) < -\varepsilon$ for some $\varepsilon > 0$, the integrals along P_k and P_{k+1} outside $\widehat{D}$ are of order $O(e^{-\varepsilon N})$. So it is enough to show that the integral $\int_P h_N(z) e^{N\eta(z)}\, dz$ equals the right-hand side of (5-1), where we put $P := (-P_k \cup P_{k+1}) \cap \widehat{D}$.

Define the function G so that $\eta(z) = re^{\theta\sqrt{-1}} G(z)^3$ and G is a bijection from $\widehat{D}$ to $E := G(\widehat{D})$, as described at the beginning of Section 5. Let $\widehat{P}$ be the image of P by G, and a_k and a_{k+1} be the endpoints of $\widehat{P}$ with $a_k \in V_k$ and $a_{k+1} \in V_{k+1}$. Putting $w := G(z)$ and $c := re^{\theta\sqrt{-1}}$, we have

$$\int_P h_N(z) e^{N\eta(z)}\, dz = \int_{\widehat{P}} \gamma_N(w) e^{Ncw^3}\, dw,$$

since $\eta(z)=re^{\theta\sqrt{-1}}G(z)^3$, where $\gamma_N(w):=h_N(G^{-1}(w))(dG^{-1}(w)/dw)$. Since $(d/dz)G(0)=1$ and $\gamma_N(w)$ converges to $\gamma(w):=h(G^{-1}(w))(dG^{-1}(w)/dw)$, we have $\gamma(0)=h(0)$. So from Corollary B.2, we conclude

$$\int_{\hat{P}} h_N(z)e^{N\eta(z)}\,dz=\frac{h(0)\Gamma\left(\frac{1}{3}\right)}{\sqrt{3}r^{1/3}N^{1/3}}\sqrt{-1}\omega^k e^{-\theta\sqrt{-1}/3}(1+O(N^{-1/3})),$$

completing the proof. □

Appendix C Some computer calculations on the stevedore knot

Theorem 1.8 says that the colored Jones polynomial of the figure-eight knot $\mathscr{E}$ evaluated at $(2\pi\sqrt{-1}+\kappa)/N$ grows exponentially with growth rate determined by the Chern–Simons invariant of an affine representation associated with the pair $(\kappa,2\pi\sqrt{-1})$, where $e^{\kappa}=\frac{1}{2}(3+\sqrt{5})$ is a zero of the Alexander polynomial $\Delta(\mathscr{E};t)=-t+3-t^{-1}$. Corollary 1.9 says that the same is true for $-\kappa$.

In this appendix, we use the computer programs Mathematica and PARI/GP [33] to study the asymptotic behavior of $J_N(\mathscr{S};e^{(2\pi\sqrt{-1}\pm\tilde{\kappa})/N})$ for the stevedore knot $\mathscr{S}$ with $\tilde{\kappa}:=\log 2$, expecting a similar asymptotic behavior as $\mathscr{E}$. Note that $e^{\pm\tilde{\kappa}}=2^{\pm 1}$ annihilates the Alexander polynomial $\Delta(\mathscr{S};t):=-2t+5-2t^{-1}$ of $\mathscr{S}$.

The stevedore knot $\mathscr{S}$ is the mirror image of the 6_1 knot in Rolfsen's book [37] (see also the knot atlas [2]) as depicted in Figure 17. Note that in KnotInfo [19] it is denoted by 6_1.

Due to [20, Theorem 5.1], we obtain

$$J_N(\mathscr{S};q)=\sum_{k=0}^{N-1}q^{-k(N+k+1)}\prod_{a=1}^{k}((1-q^{N+a})(1-q^{N-a}))\sum_{l=0}^{k}q^{l(k+1)}\frac{\prod_{b=l+1}^{k}(1-q^b)}{\prod_{c=1}^{k-l}(1-q^c)}.$$

Put $J_N^{\pm}:=J_N(\mathscr{S};e^{(2\pi\sqrt{-1}\pm\tilde{\kappa})/N})$. By using PARI/GP [33], we calculate $(2\pi\sqrt{-1}\pm\tilde{\kappa})\log(J_{N+1}^{\pm}/J_N^{\pm})$ for $N=2,3,4,\dots,200$, and plot them by using Mathematica in Figures 18 and 19. The plots indicate that J_N^{+} grows like $\exp\bigl((S_+/(2\pi\sqrt{-1}+\tilde{\kappa}))N\bigr)$(polynomial in N) with

$$S_+:=-6.485+5.697\sqrt{-1}, \tag{C-1}$$

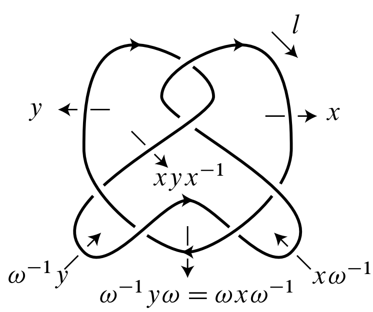

Figure 17: The stevedore knot.

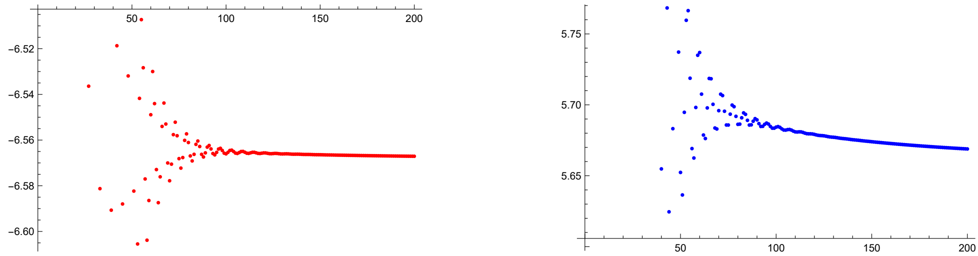

Figure 18: Plots of the real (left) and imaginary (right) parts of $(2\pi\sqrt{-1}+\tilde{\kappa})\log(J^+_{N+1}/J^+_N)$ with $N = 2, 3, 4, \dots, 200$.

and that J^-_N grows like $\exp\bigl((S_-/(2\pi\sqrt{-1}-\tilde{\kappa}))N\bigr)$(polynomial in N) with

$$S_- := -0.06880 + 8.747\sqrt{-1}. \tag{C-2}$$

Here we use Mathematica again to find the constants $S_\pm$ such that $S_\pm + c_{\pm,1}N^{-1} + c_{\pm,2}N^{-2}$ best fits the data. Note that the constants $S_\pm$ are defined modulo integral multiples of $2\pi\sqrt{-1}(2\pi\sqrt{-1}\pm\tilde{\kappa})$, and that they may also be defined modulo integral multiples of π^2 because of the definition of the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant (see Section 6).

From Theorem 1.8, we expect that $S_\pm = \pm 2\tilde{\kappa}\pi\sqrt{-1}$. However, since $\pm 2\tilde{\kappa}\pi\sqrt{-1} = \pm 4.355\sqrt{-1}$, neither S_+ nor S_- fits with $\pm 2\tilde{\kappa}\pi\sqrt{-1}$ even modulo $2\pi\sqrt{-1}(\pm 2\tilde{\kappa} + 2\pi\sqrt{-1})$ or π^2.

Now let us seek for other interpretations of $S_\pm$.

Let x and y be elements in the fundamental group $G_{\mathscr{S}} := \pi_1(S^3 \setminus \mathscr{S})$ as indicated in Figure 17. Then the group $G_{\mathscr{S}}$ has the presentation

$$G_{\mathscr{S}} = \langle x, y \mid \omega^2 x = y\omega^2\rangle,$$

where we put $\omega := xy^{-1}x^{-1}y$ as in the case of the figure-eight knot. The preferred longitude l is given as $x^3\omega^{-2}\overleftarrow{\omega}^{-2}x^{-3}$, where $\overleftarrow{\omega} := yx^{-1}y^{-1}x$, as before.

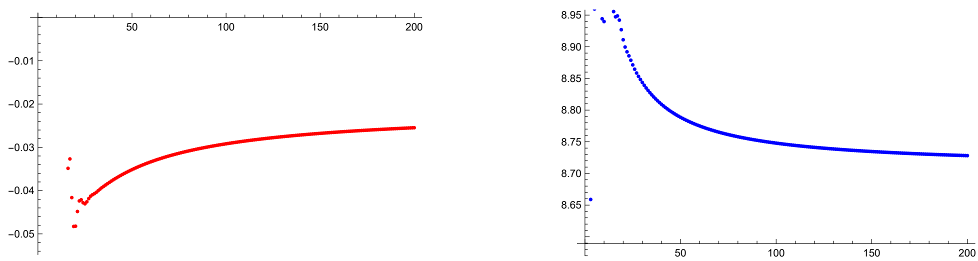

Figure 19: Plots of the real (left) and imaginary (right) parts of $(2\pi\sqrt{-1}-\tilde{\kappa})\log(J^-_{N+1}/J^-_N)$ with $N = 2, 3, 4, \dots, 200$.

Let $\rho\colon G_{\mathscr{S}} \to \mathrm{SL}(2;\mathbb{C})$ be a nonabelian representation. Due to R Riley, it is of the form

$$\rho(x) = \begin{pmatrix} m^{1/2} & 1 \\ 0 & m^{-1/2} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m^{1/2} & 0 \\ d & m^{-1/2} \end{pmatrix}$$

up to conjugation, for some $m \neq 0$ and d.

From the relation $\omega^2 x = y\omega^2$, d and m should satisfy the following equation, known as Riley's equation:

$$d^4 + (2(m+m^{-1})-5)d^3 + ((m^2+m^{-2})-6(m+m^{-1})+13)d^2 \\ -(m^2+m^{-2}-7(m+m^{-1})+14)d - (2(m+m^{-1})-5) = 0.$$

We call the left-hand side of this equation the Riley polynomial.

If $(m,d) = (1, 0.1049 + 1.552\sqrt{-1})$, then ρ is the holonomy representation of $G_{\mathscr{S}}$ and defines the complete hyperbolic structure of $S^3 \setminus \mathscr{S}$. If $(m,d) = (2,0)$ or $\left(\frac{1}{2}, 0\right)$, then ρ gives an affine representation.

Let us consider irreducible representations corresponding to $1 \le m \le 2$.

The Riley polynomial is a quartic equation with respect to d, and there are four solutions, $d_1(m)$, $d_2(m)$, $d_3(m)$, and $d_4(m)$. To describe them we introduce the following functions. Let $D(m)$ be the discriminant of the Riley polynomial with respect to d, that is,

$$D(m) := 5(m^6+m^{-6}) - 32(m^5+m^{-5}) + 56(m^4+m^{-4}) - 118(m^3+m^{-3}) + 124(m^2+m^{-2}) \\ +32(m+m^{-1}) + 123.$$

We also put

$$A(m) := 4B(m)C(m)^{-1/3} + 4C(m)^{1/3} + 3(2(m+m^{-1})-5)^2 - 8(m^2+m^{-2}-6(m+m^{-1})+13),$$

where

$$B(m) := m^4+m^{-4} - 6(m^3+m^{-3}) + 5(m^2+m^{-2}) + 3(m+m^{-1}) + 9,$$
$$C(m) := \tfrac{3}{2}\sqrt{3}\sqrt{-D(m)} + m^6+m^{-6} - 9(m^5+m^{-5}) + 21(m^4+m^{-4}) - \tfrac{9}{2}(m^3+m^{-3}) + 6(m^2+m^{-2}) \\ -27(m+m^{-1}) - \tfrac{31}{2}.$$

We also put

$$J_{\pm}(m) := \pm 3\sqrt{3}(2(m+m^{-1})+1)A(m)^{-1/2} - 2B(m)C(m)^{-1/3} - 2C(m)^{1/3} \\ -8(m^2+m^{-2}-6(m+m^{-1})+13) + 3(2(m+m^{-1})-5)^2.$$

Now define the following four functions for $1 \le m \le 2$:

$$d_1(m) := \tfrac{-1}{12}\left(6(m+m^{-1}) - 15 + \sqrt{3}\sqrt{A(m)} + \sqrt{6}\sqrt{J_-(m)}\right),$$
$$d_2(m) := \tfrac{-1}{12}\left(6(m+m^{-1}) - 15 + \sqrt{3}\sqrt{A(m)} - \sqrt{6}\sqrt{J_-(m)}\right),$$
$$d_3(m) := \tfrac{-1}{12}\left(6(m+m^{-1}) - 15 - \sqrt{3}\sqrt{A(m)} + \sqrt{6}\sqrt{J_+(m)}\right),$$
$$d_4(m) := \tfrac{-1}{12}\left(6(m+m^{-1}) - 15 - \sqrt{3}\sqrt{A(m)} - \sqrt{6}\sqrt{J_+(m)}\right).$$

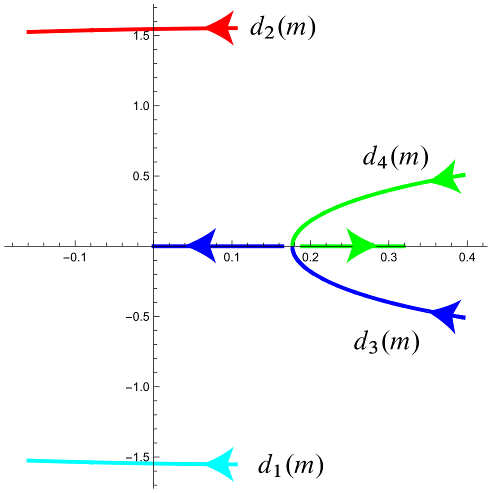

Figure 20: The cyan, red, blue, and green curves indicate $d_1(m)$, $d_2(m)$, $d_3(m)$, and $d_4(m)$, respectively. The arrows indicate the directions of increase with respect to m.

Note that

- $A(m)$, $B(m)$, $D(m)$, $J_+(m)$, and $J_-(m)$ are in $\mathbb{R}$,
- $A(m) > 0$, $B(m) > 0$, and $J_-(m) < 0$ for $1 \le m \le 2$,
- $D(m) > 0$ for $1 \le m < m_0$, $D(m_0) = 0$, and $D(m) < 0$ for $m_0 < m \le 2$, where $m_0 = 1.950$ is the unique solution to the equation $D(m) = 0$ between 1 and 2,
- $J_+(m) < 0$ for $1 \le m < m_0$, $J_+(m_0) = 0$, and $J_+(m) > 0$ for $m_0 < m \le 2$,
- $\operatorname{Im} C(m) = 0$ and $\operatorname{Re} C(m) > 0$ for $m_0 \le m \le 2$, and $\operatorname{Im} C(m) > 0$ for $1 \le m < m_0$,

which are checked by Mathematica (the author does not have proofs).

We plot, by using Mathematica, the complex-valued functions $d_i(m)$ (where $i = 1, 2, 3, 4$) for $1 \le m \le 2$ on the complex plane as in Figure 20. The following facts are also suggested by Mathematica (see Figure 20):

- $d_2(m) = \overline{d_1(m)}$ and $d_4(m) = \overline{d_3(m)}$ for $1 \le m \le 2$.
- $d_2(1) = 0.1049 + 1.552\sqrt{-1}$ and $d_2(2) = -0.1595 + 1.525\sqrt{-1}$.
- $d_3(m) \in \mathbb{R}$ and $d_4(m) \in \mathbb{R}$ for $m_0 \le m \le 2$.
- $d_3(m_0) = d_4(m_0) = 0.1770$, $d_3(2) = 0$, and $d_4(2) = 0.3189$.
- $d_3(1) = 0.3951 - 0.5068\sqrt{-1}$.

Therefore, for each i, $d_i(m)$ gives an irreducible representation $\rho_m\colon G_{\mathscr{S}} \to \mathrm{SL}(2;\mathbb{C})$ except for $d_3(2)$, and if $m \ne m_0$ they are mutually distinct.

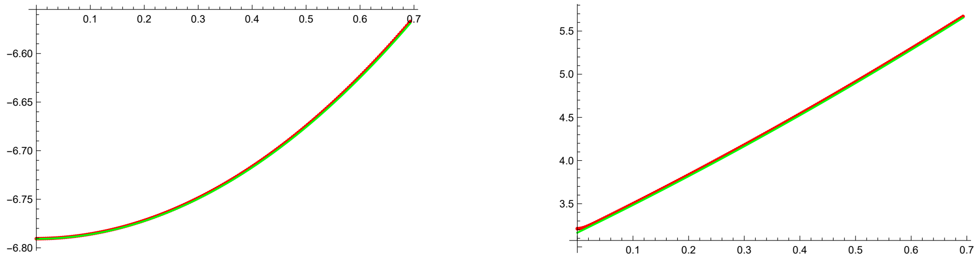

Figure 21: The left picture shows the plots of the real parts of $(2\pi\sqrt{-1}+u)\log(J_{201}(u)/J_{200}(u))$ (red) and $S_u(\mathscr{S})$ (green) for $0 \le u \le \log 2$, and the right picture shows the plots of the imaginary parts of $(u+2\pi\sqrt{-1})\log(J_{201}(u)/J_{200}(u))$ (red) and $S_u(\mathscr{S})$ (green) for $0 \le u \le \log 2$, where we put $J_N(u) := J_N(\mathscr{S}; e^{(u+2\pi\sqrt{-1})/N})$ and we use PARI/GP and Mathematica.

If we write $\rho_{d_i(m)}$ for the irreducible representation corresponding to $d_i(m)$, then we have the following:

- $\rho_{d_i(1)}$ is a parabolic representation for $i = 1, 2, 3, 4$.
- $\rho_{d_3(2)}$ is an affine representation since $d_3(2) = 0$.
- $\rho_{d_2(1)}$ is the holonomy representation, and $\rho_{d_1(1)}$ gives the holonomy representation for the mirror image of $\mathscr{S}$, because

$$\rho_{d_2(1)}(l) = \begin{pmatrix} -1 & -1.827-2.565\sqrt{-1} \\ 0 & -1 \end{pmatrix}, \quad \rho_{d_1(1)}(l) = \begin{pmatrix} -1 & -1.827+2.565\sqrt{-1} \\ 0 & -1 \end{pmatrix}.$$

Let $\lambda(m)$ be the $(1,1)$-entry of $\rho_{d_2(m)}(l)$, and put $v(u) := 2\log\lambda(e^{u/2})$, where we choose the logarithm branch so that $v(0) = 0$. Then the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant of $\rho_{d_2(e^{u/2})}$ associated with $(u, v(u))$ is given as

$$\mathrm{CS}_{u,v(u)}(\rho_{d_2(e^{u/2})}) = \mathrm{cv}(S^3 \setminus \mathscr{S}) + \frac{1}{2}\int_0^u v(s)\,ds - \frac{1}{4}uv(u),$$

where $\mathrm{cv}(S^3 \setminus \mathscr{S}) = -6.791 + 3.164\sqrt{-1}$ is the complex volume, which is defined to be

$$\sqrt{-1}\,\mathrm{Vol}(S^3 \setminus \mathscr{S}) - 2\pi^2\,\mathrm{CS}^{\mathrm{SO}(3)}(S^3 \setminus \mathscr{S}) \pmod{\pi^2},$$

with $\mathrm{CS}^{\mathrm{SO}(3)}$ the $\mathrm{SO}(3)$ Chern–Simons invariant of the Levi-Civita connection. Here the complex volume and $\mathrm{SO}(3)$ Chern–Simons invariant are taken from KnotInfo, where $\mathrm{Vol}(S^3 \setminus \mathscr{S}) = 3.163963229$ and $\mathrm{CS}^{\mathrm{SO}(3)}(S^3 \setminus \mathscr{S}) = 0.155977017$. Observe that $-0.155977017\pi^2 + \pi^2 = 6.79074$. Note that $\mathrm{cv}(S^3 \setminus \mathscr{S})$ coincides with the $\mathrm{SL}(2;\mathbb{C})$ Chern–Simons invariant $\mathrm{CS}_{(0,0)}(\rho_2(1))$; see [29, Chapter 5].

Putting

$$S_u(\mathscr{S}) := \mathrm{CS}_{u,v(u)}(\rho_{d_2(e^{u/2})}) + u\pi\sqrt{-1} + \tfrac{1}{4}uv(u), \tag{C-3}$$

the graphs depicted in Figure 21 indicate that

$$J_N(\mathscr{S}; e^{(u+2\pi\sqrt{-1})/N}) \underset{N\to\infty}{\sim} (\text{polynomial in } N)\exp\left(\frac{S_u(\mathscr{S})}{u+2\pi\sqrt{-1}}N\right)$$

for $0 \le u \le \tilde{\kappa} = \log 2$. When $u = \tilde{\kappa}$, Mathematica calculates $S_{\tilde{\kappa}}(\mathscr{S}) = -6.569 + 5.653\sqrt{-1}$, which is close to S_+ appearing in (C-1).

Note that the case $u = \tilde{\kappa}$ does not correspond to an affine representation. This also suggests that for $0 < u \le \tilde{\kappa}$ the representation $d_2(e^{u/2})$ induces an incomplete hyperbolic structure of $S^3 \setminus \mathscr{S}$, but the author does not know whether it is correct or not. The author does not know either any topological/geometric interpretation about the asymptotic behavior of $J_N(\mathscr{S}; e^{(u+2\pi\sqrt{-1})/N})$ for $u < 0$.

Compare this with Theorem 1.8 and Corollary 1.9, where $S_{\pm\kappa}(\mathscr{E}) = \pm 2\kappa\pi\sqrt{-1}$ are the Chern–Simons invariants of affine representations, which correspond to the fact that when $u = \pm\kappa$ the hyperbolic structure collapses.

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