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Noémie Legout

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Calabi–Yau structure on the Chekanov–Eliashberg algebra of a Legendrian sphere

NOÉMIE LEGOUT

We prove that the Chekanov–Eliashberg algebra of a horizontally displaceable n-dimensional Legendrian sphere in the contactization of a Liouville manifold is an $(n+1)$-Calabi–Yau differential graded algebra. In particular it means that there is a quasi-isomorphism of DG bimodules between the diagonal bimodule and the inverse dualizing bimodule associated to the Chekanov–Eliashberg algebra. On some cyclic version of these bimodules, computing the Hochschild homology and cohomology of the Chekanov–Eliashberg algebra, we construct A_∞-operations and show that the Calabi–Yau isomorphism extends to a family of maps satisfying the A_∞-functor equations.

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1 Introduction

We consider n-dimensional Legendrian submanifolds in a contact manifold (Y,α) which is the contactization of a Liouville manifold (P,β). Among the numerous invariants of Legendrians up to Legendrian isotopy, many of them are derived from the famous Chekanov–Eliashberg algebra (C-E algebra) [8; 22], originally defined by Chekanov for Legendrian links in $\mathbb{R}^3$ and generalized to higher-dimensional Legendrians in the contactization of a Liouville domain by Ekholm, Etnyre and Sullivan [19; 20]. This algebra is a unital differential graded algebra (DGA) associated to a Legendrian Λ and generated by Reeb chords of Λ; see Section 4 for a brief recall of the definition. In this paper we take the coefficient ring to be the field $\mathbb{Z}_2$.

It is well known that linearized versions of the C-E algebra satisfy a particular type of duality, which was first proved by Sabloff [33] for Legendrian knots, and generalized to higher dimensions by Ekholm, Etnyre and Sabloff [18] and to the bilinearized case by Bourgeois and Chantraine [1]. We prove in this paper that under some assumptions the full C-E algebra, ie not linearized, also satisfies a similar type of duality. This duality is expressed in terms of differential graded bimodules (DG bimodules). Namely, if $\mathcal{A}$ denotes the C-E algebra of a n-dimensional Legendrian sphere, then $\mathcal{A}$ is a DG $\mathcal{A}$-bimodule called the *diagonal bimodule*, and the *inverse dualizing bimodule* of $\mathcal{A}$ is the DG $\mathcal{A}$-bimodule $\mathcal{A}^! := \mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A}\otimes\mathcal{A})$. We prove that there is a quasi-isomorphism

$$\mathcal{CY}\colon \mathcal{A} \xrightarrow{\simeq} \mathcal{A}^![-n-1] \tag{1}$$

satisfying $\mathcal{CY} \simeq \mathcal{CY}^![-n-1]$. Together with the fact that $\mathcal{A}$ is a *homologically smooth* differential graded algebra (see Section 2), this leads to the following result:

Theorem 1.1 *Let $\Lambda \subset Y$ be an horizontally displaceable Legendrian sphere. Then its C-E-algebra is a $(n+1)$-Calabi–Yau differential graded algebra.*

The definition of Calabi–Yau structure we use here is the one of Ginzburg [26, Definition 3.2.3] (see also Kontsevich and Soibelman [28]), although we have some opposite sign convention for degrees.

For the following reasons, it was reasonable to expect that the C-E algebra of a displaceable Legendrian sphere admits a Calabi–Yau structure. In his thesis, Ganatra [24] showed that the wrapped Fukaya category $\mathcal{W}$ of a Weinstein manifold is a noncompact Calabi–Yau category, where the Calabi–Yau structure is induced similarly as for (1) by an equivalence of A_∞-bimodules between the diagonal bimodule $\mathcal{W}$ and an analogue of the inverse dualizing bimodule in the A_∞-setting, $\mathcal{W}^!$. Now, observe that the wrapped Fukaya category of a Weinstein manifold X (of finite type) is generated by the Lagrangian cocores obtained by attaching critical handles to Legendrian spheres in the ideal contact boundary of X, as proved by Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [7], and Ganatra, Pardon and Shende [25]. A Calabi–Yau structure on the C-E algebra of these Legendrian spheres should then be possible to construct by using the surgery isomorphism of Bourgeois, Ekholm and Eliashberg [3] (see also Ekholm and Lekili [21] and Ekholm [17]). Recall that the surgery isomorphism gives an A_∞-quasi-isomorphism between the wrapped Floer cohomology of the cocores, ie the endomorphism groups of the generators of the wrapped Fukaya category, and the C-E algebra of the Legendrian attaching spheres. Note that X is taken here to be a subcritical Weinstein manifold. When X is the standard ball, the C-E algebra of a Legendrian in its boundary can be computed inside a contact Darboux ball where it is displaceable; see, for example, the work of Dimitroglou Rizell, Ekholm and Tonkonog [14].

This suggests that a Calabi–Yau structure on the C-E algebra of a Legendrian sphere could potentially be obtained from the one defined by Ganatra via the surgery isomorphism. However we will not adopt this method in this paper. Instead, we restrict ourselves to Legendrian spheres in the contactization of a Liouville manifold and introduce a version of the Rabinowitz Floer homology for Legendrians with coefficients in the free $\mathcal{A}$-bimodule $\mathcal{A}\otimes\mathcal{A}$ of rank 1, where $\mathcal{A}$ denotes the C-E algebra.

Rabinowitz Floer homology was originally defined as an homology theory for contact-type hypersurfaces by Cieliebak and Frauenfelder in [9]. A relative theory, Lagrangian Rabinowitz Floer homology, has then been introduced in [31] for exact Lagrangians in a Liouville manifold, and more recently both the nonrelative and relative theories have been generalized to the case of Liouville cobordisms admitting a filling by Cieliebak and Oancea in [10]. These previous theories were defined in the Hamiltonian setting and in [29] we introduced an SFT-type version of the Lagrangian Rabinowitz Floer homology for Lagrangians in a (trivial) Liouville cobordism. There, we used augmentations of the C-E algebras of Legendrians in the negative ends of Lagrangian cobordisms in order to define a complex over $\mathbb{Z}_2$. In this paper, we consider only the Rabinowitz complex of cylinders over a Legendrian submanifold and we don't use augmentations, so that we get a DG bimodule with coefficients in the C-E algebras of the Legendrians.

More precisely, the Rabinowitz DG bimodule considered here is generated by mixed chords of a 2-copy $\Lambda_0 \cup \Lambda_1$ of a Legendrian Λ, where Λ_1 is a small negative push-off of $\Lambda_0 := \Lambda$, and its differential is defined by a count of pseudoholomorphic discs with boundary on $\mathbb{R} \times (\Lambda_0 \cup \Lambda_1)$. The differential is lower triangular and so the Rabinowitz bimodule is the cone of a DG bimodule map $C_+(\Lambda_0, \Lambda_1) \to C_-(\Lambda_0, \Lambda_1)$, where C_+ is generated by chords from Λ_0 to Λ_1 (in bijective correspondence with chords of Λ) and C_- is generated by chords from Λ_1 to Λ_0 (in bijective correspondence with chords of Λ and critical points of a Morse function on Λ). For a Legendrian sphere Λ and its C-E algebra $\mathcal{A}$, the Rabinowitz bimodule can be described as the cone of a slightly different DG bimodule map

$$\mathrm{CY}\colon \widehat{C}_+(\Lambda_0, \Lambda_1) \to \check{C}_-(\Lambda_0, \Lambda_1),$$

which corresponds to a slight modification of the action filtration. Here $\widehat{C}_+(\Lambda_0, \Lambda_1)$ is generated by Reeb chords from Λ_0 to Λ_1 as well as the maximum of a given Morse function on Λ, and $\check{C}_-(\Lambda_0, \Lambda_1)$ is generated by chords from Λ_1 to Λ_0 and the minimum of the Morse function. Then, we show that $\widehat{C}_+(\Lambda_0, \Lambda_1)[-n-1]$ is quasi-isomorphic to $\mathcal{A}$ and $\check{C}_-(\Lambda_0, \Lambda_1)$ is quasi-isomorphic to $\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$. The invariance of the Rabinowitz homology up to Legendrian isotopies implies that if Λ is horizontally displaceable, then the complex is acyclic. The shifted map $\mathrm{CY}[-n-1]$ provides thus a Calabi–Yau quasi-isomorphism $\mathcal{A} \to \mathcal{A}^![-n-1]$.

In general, we expect that the Rabinowitz complex is acyclic for any Legendrian in the contact boundary of a subcritical Weinstein manifold. For example the periodic orbit version of Rabinowitz Floer homology and Rabinowitz Floer homology for fillable Legendrians both vanish there; see [10]. We expect also that the Calabi–Yau structure we construct here coincides with that constructed by Ganatra in [24], and plan to show this in future work with Asplund. One advantage with the perspective taken here compared to that of Ganatra is from the point of view of computability. The pseudoholomorphic discs that define the operations we consider can be computed using Ekholm's theory of gradient flow trees [15], while the operations defined by Hamiltonian perturbations seem to lack general techniques for computation.

By taking bimodule tensor products of both bimodules $\widehat{C}_+(\Lambda_0, \Lambda_1)$ and $\check{C}_-(\Lambda_0, \Lambda_1)$ with the diagonal bimodule $\mathcal{A}$ one gets complexes which we denote by $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0, \Lambda_1)$ and $\check{C}_-^{\mathrm{cyc}}(\Lambda_0, \Lambda_1)$ and which compute the Hochschild homology and cohomology of $\mathcal{A}$, respectively. On these complexes we construct higher-order maps, ie for any $(d+1)$-copy $\Lambda_0 \cup \cdots \cup \Lambda_d$ we construct maps

$$\widehat{\mathfrak{m}}_d : \widehat{C}_+^{\text{cyc}}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \widehat{C}_+^{\text{cyc}}(\Lambda_0, \Lambda_1) \to \widehat{C}_+^{\text{cyc}}(\Lambda_0, \Lambda_d),$$
$$\check{\mathfrak{m}}_d : \check{C}_-^{\text{cyc}}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \check{C}_-^{\text{cyc}}(\Lambda_0, \Lambda_1) \to \check{C}_-^{\text{cyc}}(\Lambda_0, \Lambda_d)$$

satisfying the A_∞-equations; see Section 7. The maps $\check{\mathfrak{m}}_d$ are defined by a count of pseudoholomorphic discs with boundary on $\mathbb{R} \times (\Lambda_0 \cup \cdots \cup \Lambda_d)$, with d negative asymptotics which are inputs and one positive asymptotic which is the output. These maps are well known: they compute the A_∞-structure of the augmentation category $\mathcal{A}\text{ug}_-(\Lambda)$ of Bourgeois and Chantraine [1] with a formal unit added (corresponding to the minimum of the Morse function here). One the other side, the maps $\widehat{\mathfrak{m}}_d$ are defined by a count of certain 2-level pseudoholomorphic buildings appearing in the boundary of the compactification of 1-dimensional moduli spaces. In this sense it can be seen as a *secondary-type* product. Finally, we show that the map CY induced on the cyclic complexes extends to a family of maps

$$\text{CY}_d : \widehat{C}_+^{\text{cyc}}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \widehat{C}_+^{\text{cyc}}(\Lambda_0, \Lambda_1) \to \check{C}_-^{\text{cyc}}(\Lambda_0, \Lambda_d)$$

satisfying the A_∞-functor equations.

Observe that the map CY gives an isomorphism between the Hochschild homology and cohomology of $\mathcal{A}$, which after generalizing our definition of the Rabinowitz complex to Legendrians in more general contact manifolds, would recover the quasi-isomorphism between Hochschild homology and cohomology for the wrapped Fukaya category in [24]. Moreover we presumably recover the relation between the different product structures as constructed by Bourgeois, Ekholm and Eliashberg in [2].

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2 Background on DG bimodules

Let $(\mathcal{A}_0, \partial_{\mathcal{A}_0})$, $(\mathcal{A}_1, \partial_{\mathcal{A}_1})$ be unital differential graded algebras (DGAs) over $\mathbb{Z}_2$ (we restrict to $\mathbb{Z}_2$ for simplicity here but it is not strictly necessary). A *DG* $\mathcal{A}_1$*-*$\mathcal{A}_0$*-bimodule* is a graded $\mathcal{A}_1$-$\mathcal{A}_0$-bimodule $\mathcal{B}$ endowed with a degree-1 differential $\partial_{\mathcal{B}}$ such that

$$\partial_{\mathcal{B}}(\boldsymbol{\alpha}_1 b \boldsymbol{\alpha}_0) = \partial_{\mathcal{A}_1}(\boldsymbol{\alpha}_1) b \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \partial_{\mathcal{B}}(b) \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 b \partial_{\mathcal{A}_0}(\boldsymbol{\alpha}_0).$$

If $(\mathcal{A}_0, \partial_{\mathcal{A}_0}) = (\mathcal{A}_1, \partial_{\mathcal{A}_1}) = (\mathcal{A}, \partial_{\mathcal{A}})$, we write simply $\mathcal{A}$-bimodule instead of $\mathcal{A}$-$\mathcal{A}$-bimodule.

Example 2.1 (a) A DGA $(\mathcal{A}, \partial_{\mathcal{A}})$ is itself a DG $\mathcal{A}$-bimodule, called the *diagonal bimodule*.

(b) The tensor product of a DGA $\mathcal{A}$ with itself over $\mathbb{Z}_2$ is also a DG $\mathcal{A}$-bimodule, with differential given by $\partial_{\mathcal{A}\otimes\mathcal{A}}(a_1 \otimes a_0) = \partial_{\mathcal{A}}(a_1) \otimes a_0 + a_1 \otimes \partial_{\mathcal{A}}(a_0)$. This bimodule carries two different bimodule structures, the *outer* and the *inner* bimodule structure:

$$\mu^{\text{out}}_{\mathcal{A}\otimes\mathcal{A}} \colon \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad \boldsymbol{\alpha} \otimes (a \otimes a') \otimes \boldsymbol{\alpha}' \mapsto \boldsymbol{\alpha} a \otimes a' \boldsymbol{\alpha}',$$
$$\mu^{\text{in}}_{\mathcal{A}\otimes\mathcal{A}} \colon \mathcal{A} \otimes (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad \boldsymbol{\alpha} \otimes (a \otimes a') \otimes \boldsymbol{\alpha}' \mapsto a\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}' a'.$$

(c) Given two DG $\mathcal{A}_1$-$\mathcal{A}_0$-bimodules $\mathcal{B}$ and $\mathcal{C}$, the set $\operatorname{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\mathcal{B}, \mathcal{C})$ of bimodule maps is a chain complex whose differential is given by $D(\phi) = \phi \circ \partial_{\mathcal{B}} + \partial_{\mathcal{C}} \circ \phi$.

In this paper, we write $-\otimes-$ instead of $-\otimes_{\mathbb{Z}_2}-$ for the tensor product over $\mathbb{Z}_2$. A *DG morphism* of $\mathcal{A}_1$-$\mathcal{A}_0$-bimodules from $\mathcal{B}$ to $\mathcal{C}$ is a degree-0 element ϕ in $\operatorname{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\mathcal{B}, \mathcal{C})$ which commutes with the differentials of $\mathcal{B}$ and $\mathcal{C}$; in other words, it is a degree-0 cycle in $(\operatorname{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\mathcal{B}, \mathcal{C}), D)$. A *quasi-isomorphism* of DG bimodules is a DG morphism which induces an isomorphism in homology.

Following [23], we recall that a DG $\mathcal{A}_1$-$\mathcal{A}_0$-bimodule $(\mathcal{B}, \partial_{\mathcal{B}})$ is *free* if it is isomorphic to $\mathcal{A}_1 \otimes V \otimes \mathcal{A}_0$ where V is a $\mathbb{Z}_2$-vector space generated by cycles. In this case a *free generating set* for $\mathcal{B}$ is a basis of V. Given a DGA $\mathcal{A}$ the diagonal bimodule is not a free bimodule while $\mathcal{A} \otimes \mathcal{A}$ endowed with either the outer or the inner structure is free, generated by $1 \otimes 1$. A DG bimodule $\mathcal{B}$ is called *semifree* if there is a filtration

$$\{0\} \subset F_0\mathcal{B} \subset F_1\mathcal{B} \subset \cdots \subset \mathcal{B}$$

such that $F_i\mathcal{B}$ is a DG subbimodule for all $i \geq 0$, $\bigcup F_i\mathcal{B} = \mathcal{B}$, and $F_0\mathcal{B}$ and $F_{i+1}\mathcal{B}/F_i\mathcal{B}$ are free bimodules. We say that a semifree DG bimodule $\mathcal{B}$ as above is of *finite rank* if there is a $k \in \mathbb{N}$ such that $F_k\mathcal{B} = \mathcal{B}$. A *semifree resolution* of a DG bimodule $\mathcal{B}$ is a semifree DG bimodule $R_{\mathcal{B}}$ together with a quasi-isomorphism of DG bimodules $R_{\mathcal{B}} \to \mathcal{B}$.

In the category of DG $\mathcal{A}_1$-$\mathcal{A}_0$-bimodules, $\operatorname{RHom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(-, \mathcal{C})$ denotes the right derived functor of the functor $\operatorname{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(-, \mathcal{C})$. Let $R_{\mathcal{B}}$ be any semifree resolution of the $\mathcal{A}_1$-$\mathcal{A}_0$-bimodule $\mathcal{B}$; we have by definition that $\operatorname{RHom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\mathcal{B}, \mathcal{C}) = \operatorname{Hom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(R_{\mathcal{B}}, \mathcal{C})$, which is well defined up to quasi-isomorphism. Given a DGA $\mathcal{A}$, the *inverse dualizing bimodule*, denoted by $\mathcal{A}^!$, is $\operatorname{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$. The elements of $\operatorname{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ are bimodule morphisms from the diagonal bimodule $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A} \otimes \mathcal{A}$ is endowed with the inner bimodule structure. Then, $\operatorname{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ is a DG $\mathcal{A}$-bimodule with a bimodule structure induced by the outer bimodule structure on $\mathcal{A} \otimes \mathcal{A}$, ie for $\phi \in \operatorname{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ which can be written as $\phi = \phi_l \otimes \phi_r$ and for any $\boldsymbol{v}, \boldsymbol{a}, \boldsymbol{w} \in \mathcal{A}$ we have

$$(\boldsymbol{v} \cdot \phi \cdot \boldsymbol{w})(\boldsymbol{a}) = \boldsymbol{v}\phi_l(\boldsymbol{a}) \otimes \phi_r(\boldsymbol{a})\boldsymbol{w}.$$

If $\phi \colon \mathcal{B} \to \mathcal{C}$ is a DG morphism of $\mathcal{A}$-bimodules in the derived sense, then there is an induced morphism $\phi^! \colon \mathcal{C}^! \to \mathcal{B}^!$ given by $\phi^!(c^!) = c^! \circ \phi$.

Remark 2.2 The semifree resolution mentioned above in order to compute $\mathrm{RHom}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\mathcal{B},\mathcal{C})$ can be taken to be the bar resolution of $\mathcal{B}$; however this one is not of finite rank. In this paper we will be interested in the inverse dualizing bimodule $\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})$ of the Chekanov–Eliashberg DGA (see Section 4) of a closed Legendrian submanifold. In this case, the diagonal bimodule $\mathcal{A}$ admits a finite-rank semifree resolution; see Section 6.4.

The purpose of this paper is to prove that under some hypothesis the Chekanov–Eliashberg algebra of a Legendrian sphere is a Calabi–Yau DGA. Let us recall the necessary definition.

Definition 2.3 [28] Let $(\mathcal{A},\partial_{\mathcal{A}})$ be a DGA.

(a) A DG $\mathcal{A}$-bimodule is *perfect* if it is quasi-isomorphic to a direct summand of a finite-dimensional semifree DG bimodule.

(b) The DGA $\mathcal{A}$ is *homologically smooth* if it is perfect as a DG $\mathcal{A}$-bimodule.

Definition 2.4 [26] A homologically smooth DGA $\mathcal{A}$ is called *d-Calabi–Yau* if there is a quasi-isomorphism of DG $\mathcal{A}$-bimodules

$$\phi\colon\mathcal{A}\to\mathcal{A}^![-d]$$

such that $\phi\simeq\phi^![-d]$.

Notation 2.5 The convention we use for shifts in this paper is that if $|a|$ is the degree of an element of a DG bimodule $\mathcal{B}$, then the same element viewed in $\mathcal{B}[d]$ has degree $|a|+d$. Then, for a DG bimodule morphism $f\colon\mathcal{B}\to\mathcal{C}$ we denote by $f[d]\colon\mathcal{B}[d]\to\mathcal{C}[d]$ the shifted map.

3 Moduli spaces

We will be working in the same setting as in [29] and refer to Sections 2.2–2.6 of the mentioned paper for more details about the moduli spaces of pseudoholomorphic discs we consider here. Throughout this paper, when we consider a Legendrian submanifold of Y we always assume that it is closed and nondegenerate in the sense that it admits a finite number of Reeb chords which are isolated and correspond to transverse intersection points of the Lagrangian projection on P.

Let $\Lambda\subset Y$ be a Legendrian and denote by $\mathcal{R}(\Lambda)$ the set of Reeb chords of Λ. For any chord $\gamma\in\mathcal{R}(\Lambda)$ we denote by $\mathrm{CZ}(\gamma)$ its Conley–Zehnder index; see [19].

Remark 3.1 When Λ is not connected, there are additional choices (as, for example, choices of paths between the various connected components) to make in order to define the Conley–Zehnder index of a Reeb chord connecting two distinct connected components; see [13]. The index of a chord depends on these additional choices but for any two chords from a connected component to another, the difference in index is independent of the choices. Moreover, we assume in this paper that the Legendrians we consider always have Maslov number 0 and that the first Chern class of P vanishes. In this case we get a well-defined $\mathbb{Z}$-valued Conley–Zehnder index (after the potentially additional choices discussed above).

Let J_P be an almost complex structure on (P,β) which is compatible with $d\beta$ and cylindrical outside of a compact set in the cylindrical end of P. We call such a structure *admissible*. Then, we denote by J the almost complex structure on $(\mathbb{R}\times Y, d(e^t\alpha))$ which is the cylindrical lift of J_P, ie the unique cylindrical almost complex structure J on $\mathbb{R}\times Y$ such that the projection $\pi_P:\mathbb{R}\times P\times\mathbb{R}\to P$ is (J,J_P)-holomorphic. Given Reeb chords $\gamma,\gamma_1,\dots,\gamma_d$ of Λ we denote by

$$\widehat{\mathcal{M}}_\Lambda(\gamma;\gamma_1,\dots,\gamma_d)$$

the moduli space of J-holomorphic discs with boundary on $\mathbb{R}\times\Lambda$ having a positive asymptotic at γ and negative asymptotics at $\gamma_1,\dots,\gamma_d$. As the boundary condition for these discs is cylindrical as well as the almost complex structure, there is an action of $\mathbb{R}$ by translation on these moduli spaces. We denote by

$$\mathcal{M}_\Lambda(\gamma;\gamma_1,\dots,\gamma_d) =: \widehat{\mathcal{M}}_\Lambda(\gamma;\gamma_1,\dots,\gamma_d)/\mathbb{R}$$

the quotient by this action.

Let $\Lambda_0,\dots,\Lambda_d\subset Y$ be $d+1$ Legendrian submanifolds such that the link $\Lambda_0\cup\dots\cup\Lambda_d$ is nondegenerate. For any $0\le i\ne j\le d$, we denote by $\mathcal{R}(\Lambda_i,\Lambda_j)$ the set of Reeb chords from Λ_j to Λ_i. Such chords are called *mixed* while chords in $\mathcal{R}(\Lambda_i)$ are called *pure*. Let $\gamma_{0d}\in\mathcal{R}(\Lambda_0,\Lambda_d)$, $(\gamma_1,\dots,\gamma_d)$ be a d-tuple of Reeb chords such that $\gamma_i\in\mathcal{R}(\Lambda_{i-1},\Lambda_i)\cup\mathcal{R}(\Lambda_i,\Lambda_{i-1})$, and $\boldsymbol{\delta}_i$ for $0\le i\le d$ words of Reeb chords of Λ_i. We denote by

$$\mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{0d};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d)$$

the quotient by the action of $\mathbb{R}$ of the moduli space of pseudoholomorphic discs satisfying the following:

- The boundary of the discs lie on the ordered $(d+1)$-tuple of Lagrangians $\mathbb{R}\times\Lambda_0,\dots,\mathbb{R}\times\Lambda_d$ when following the boundary counterclockwise.
- The discs in this moduli space are positively asymptotic to the Reeb chord γ_{0d}, positively or negatively asymptotic to the Reeb chords γ_i, and negatively asymptotic to the words of Reeb chords $\boldsymbol{\delta}_i$.

Similarly, for a chord $\gamma_{d0}\in\mathcal{R}(\Lambda_d,\Lambda_0)$, and other chords as above, we denote by

$$\mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{d0};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d)$$

the quotient by the action of $\mathbb{R}$ of the moduli space of pseudoholomorphic discs with the same boundary conditions as above, negatively asymptotic to the Reeb chord γ_{d0}, and with the same asymptotic conditions as above for the other punctures.

Remark 3.2 In the notation we employ, the first asymptotic γ, γ_{0d} or γ_{d0} will always be the output of a map defined by a count of rigid discs in the corresponding moduli spaces. When the Lagrangian boundary conditions for the pseudoholomorphic discs is the ordered $(d+1)$-tuple $(\mathbb{R}\times\Lambda_0,\dots,\mathbb{R}\times\Lambda_d)$, knowing if the first asymptotic is a positive or negative asymptotic is enough to determine if the other mixed asymptotics are positive or negative, according to their direction. More precisely, if a chord γ_i is in $\mathcal{R}(\Lambda_i,\Lambda_{i-1})$ then it will always be a positive asymptotic while if it is in $\mathcal{R}(\Lambda_{i-1},\Lambda_i)$ it will always be a negative asymptotic.

By [13], if the almost complex structure J is the cylindrical lift of an admissible almost complex structure on P which is regular (that is, such that the pseudoholomorphic discs $\pi_P \circ u$ are transversely cut out, for any pseudoholomorphic disc u in any of the moduli spaces described above), then J is regular meaning that the moduli spaces we described above are transversely cut out. The necessary transversality results for moduli spaces of pseudoholomorphic discs in P with boundary on $\pi_P(\Lambda)$ are carried out in [19]. When transversality holds, these moduli spaces are thus smooth manifolds which can moreover be compactified in the sense of Gromov.

The dimension of a moduli space can be expressed in terms of the Conley–Zehnder indices of its asymptotics. For moduli spaces with only pure asymptotics, we have

$$\dim \mathcal{M}_\Lambda(\gamma;\gamma_1,\dots,\gamma_d) = (\mathrm{CZ}(\gamma)-1) - \sum_{i=1}^{d}(\mathrm{CZ}(\gamma_i)-1)-1.$$

Then, for a word of pure Reeb chords $\boldsymbol{\delta} = \delta_1 \dots \delta_k$ let us denote by $\ell(\boldsymbol{\delta}) = k$ its length and define $\mathrm{CZ}(\boldsymbol{\delta}) := \sum_{i=1}^k \mathrm{CZ}(\delta_i)$. Moreover, let us denote by j^+ the number of positive mixed Reeb chord asymptotics among $\{\gamma_1,\dots,\gamma_d\}$. We have

$$\begin{aligned}\dim \mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{0d};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d) = (\mathrm{CZ}(\gamma_{0d})-1) + \sum_{\gamma_i\in\mathcal{R}(\Lambda_i,\Lambda_{i-1})}(\mathrm{CZ}(\gamma_i)-1) \\ - \sum_{\gamma_i\in\mathcal{R}(\Lambda_{i-1},\Lambda_i)}(\mathrm{CZ}(\gamma_i)-1) - \sum(\mathrm{CZ}(\boldsymbol{\delta}_i)-\ell(\boldsymbol{\delta}_i)) + (2-n)j^+ - 1\end{aligned}$$

and

$$\begin{aligned}\dim \mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{d0};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d) = -(\mathrm{CZ}(\gamma_{d0})-1) + \sum_{\gamma_i\in\mathcal{R}(\Lambda_i,\Lambda_{i-1})}(\mathrm{CZ}(\gamma_i)-1) \\ - \sum_{\gamma_i\in\mathcal{R}(\Lambda_{i-1},\Lambda_i)}(\mathrm{CZ}(\gamma_i)-1) - \sum(\mathrm{CZ}(\boldsymbol{\delta}_i)-\ell(\boldsymbol{\delta}_i)) + (2-n)(j^+-1) - 1.\end{aligned}$$

We refer to [5, Section 4.3] for the computation of these dimensions. In the following we add an exponent to indicate the dimension of the moduli space, ie $\mathcal{M}^i_\Lambda(\gamma;\gamma_1,\dots,\gamma_d)$ denotes an i-dimensional moduli space. We call *rigid* the pseudoholomorphic discs in a 0-dimensional moduli space.

We define the *action* $\mathfrak{a}(\gamma_{ij})$ of a Reeb chord γ_{ij} by $\mathfrak{a}(\gamma_{ij}) = \int_{\gamma_{ij}}\alpha$ if $i > j$, and $\mathfrak{a}(\gamma_{ij}) = -\int_{\gamma_{ij}}\alpha$ if $i \le j$. By positivity of energy for pseudoholomorphic discs in the moduli spaces defined above, we have the following. If the moduli space $\mathcal{M}_\Lambda(\gamma;\gamma_1,\dots,\gamma_d)$ is not empty then the action of the asymptotics satisfies

$$-\mathfrak{a}(\gamma) + \sum_{i=1}^{d}\mathfrak{a}(\gamma_i) \ge 0.$$

If $\mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{0d};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d)$ is not empty then we have

$$-\mathfrak{a}(\gamma_{0d}) + \sum_{\gamma_i\in\mathcal{R}(\Lambda_{i-1},\Lambda_i)}\mathfrak{a}(\gamma_i) + \sum_{\gamma_i\in\mathcal{R}(\Lambda_i,\Lambda_{i-1})}\mathfrak{a}(\gamma_i) + \sum_{j=0}^{d}\mathfrak{a}(\boldsymbol{\delta}_j) \ge 0.$$

Finally if $\mathcal{M}_{\Lambda_{0\dots d}}(\gamma_{d0};\boldsymbol{\delta}_0,\gamma_1,\boldsymbol{\delta}_1,\dots,\gamma_d,\boldsymbol{\delta}_d)$ is nonempty then we have

$$-\mathfrak{a}(\gamma_{d0})+\sum_{\gamma_i\in\mathcal{R}(\Lambda_{i-1},\Lambda_i)}\mathfrak{a}(\gamma_i)+\sum_{\gamma_i\in\mathcal{R}(\Lambda_i,\Lambda_{i-1})}\mathfrak{a}(\gamma_i)+\sum_{j=0}^{d}\mathfrak{a}(\boldsymbol{\delta}_j)\geq 0.$$

In particular, the definition of action we choose here implies that the maps which will be defined later by a count of pseudoholomorphic discs in the moduli spaces above will be action decreasing.

4 The Chekanov–Eliashberg DGA

In this section we briefly recall the definition of the Chekanov–Eliashberg algebra (C-E algebra) of a Legendrian originally defined in [8; 22] and refer to [8; 19; 20] for details.

Let $\Lambda\subset Y$ be a Legendrian and denote by $C(\Lambda)$ the $\mathbb{Z}_2$-module generated by Reeb chords of Λ. The Chekanov–Eliashberg DGA of Λ, denoted by $\mathcal{A}(\Lambda)$, is the tensor algebra of $C(\Lambda)$ over $\mathbb{Z}_2$:

$$\mathcal{A}(\Lambda)=\bigoplus_{i\geq 0}C(\Lambda)^{\otimes i}$$

with $C(\Lambda)^{\otimes 0}=\mathbb{Z}_2$. The grading of a Reeb chord $\gamma\in\mathcal{R}(\Lambda)$ is given by $|\gamma|_{\mathcal{A}}=1-\mathrm{CZ}(\gamma)$, and we extend it to the whole algebra $\mathcal{A}(\Lambda)$ by

$$|\gamma_1\dots\gamma_d|_{\mathcal{A}}=|\gamma_1|_{\mathcal{A}}+\dots+|\gamma_d|_{\mathcal{A}}.$$

Remark 4.1 In this paper we use a slightly unconventional grading for the C-E algebra. This way the differential will be a map of degree 1 instead of a map of degree -1. We make this choice in order for the Rabinowitz DG bimodule defined in the next section to have a differential of degree 1, as it generalizes (in the sense that we don't use augmentations of the C-E-algebra) the Rabinowitz complex defined in [29] whose differential has degree 1.

The differential $\partial_{\mathcal{A}}$ on $\mathcal{A}$ is given on Reeb chords by

$$\partial_{\mathcal{A}}(\gamma)=\sum_{d\geq 0}\sum_{\gamma_1,\dots,\gamma_d}\#\mathcal{M}^0_{\Lambda}(\gamma;\gamma_1,\dots,\gamma_d)\cdot\gamma_1\dots\gamma_d$$

and extends to the whole algebra $\mathcal{A}$ by Leibniz rule, that is,

$$\partial_{\mathcal{A}}(\gamma_1\gamma_2)=\partial_{\mathcal{A}}(\gamma_1)\gamma_2+\gamma_1\partial_{\mathcal{A}}(\gamma_2).$$

Remark 4.2 In the original definition of the C-E algebra of a Legendrian, the differential is defined by a count of pseudoholomorphic discs with boundary on the projection $\pi_P(\Lambda)$, for $\pi_P:\mathbb{R}\times P\to P$. Dimitroglou Rizell proved in [13] that the differential can equivalently be defined by a count of pseudoholomorphic discs with boundary on $\mathbb{R}\times\Lambda\subset\mathbb{R}\times Y$. We use this latter perspective in this paper.

Theorem 4.3 [8; 19; 20] *The map $\partial_{\mathcal{A}}$ is a degree-1 map which satisfies $\partial^2_{\mathcal{A}}=0$.*

The homology of the complex $(\mathcal{A}(\Lambda),\partial_{\mathcal{A}})$ is called the *Legendrian contact homology* of Λ.

5 The Rabinowitz DG bimodule

Let $\Lambda_0, \Lambda_1 \subset Y$ be Legendrian submanifolds and $\mathcal{A}(\Lambda_0)$, $\mathcal{A}(\Lambda_1)$ denote their C-E DGAs. We denote by $C(\Lambda_0, \Lambda_1)$ the graded $\mathbb{Z}_2$-module generated by chords in $\mathcal{R}(\Lambda_0, \Lambda_1)$, and graded with the Conley–Zehnder index. We further denote by $C_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$ the $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodule generated by chords in $\mathcal{R}(\Lambda_0, \Lambda_1)$, ie elements of $C_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$ are linear combinations of words $\boldsymbol{v}_1\gamma_{01}\boldsymbol{v}_0$ where $\boldsymbol{v}_i$ are words of Reeb chords of Λ_i for $i = 0, 1$ and $\gamma_{01} \in \mathcal{R}(\Lambda_0, \Lambda_1)$. The degree of $\boldsymbol{v}_1\gamma_{01}\boldsymbol{v}_0$ is given by $\mathrm{CZ}(\gamma_{01}) + |\boldsymbol{v}_0|_{\mathcal{A}_0} + |\boldsymbol{v}_1|_{\mathcal{A}_1}$. Analogously denote by $C_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_1, \Lambda_0)$ the $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodule generated by chords in $\mathcal{R}(\Lambda_1, \Lambda_0)$.

The Rabinowitz DG bimodule $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$ is a DG $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodule which is defined as follows. The underlying graded bimodule has two different types of generators:

$$\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1) = C_+(\Lambda_0, \Lambda_1) \oplus C_-(\Lambda_0, \Lambda_1),$$

where

- $C_+(\Lambda_0, \Lambda_1)$ is the $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodule whose elements are the same as in $C_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_1, \Lambda_0)$ but the grading of a mixed chord γ_{10} is modified by taking the negative and adding n, that is, $|\gamma_{10}|_{\mathrm{RFC}(\Lambda_0,\Lambda_1)} = n - \mathrm{CZ}(\gamma_{10})$,
- $C_-(\Lambda_0, \Lambda_1) = C_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$.

The differential on $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$ is given by a lower triangular 2×2-matrix

$$\mathfrak{m}_1 = \begin{pmatrix} \boldsymbol{\Delta}_1^{++} & 0 \\ \boldsymbol{b}_1^{-+} & \boldsymbol{b}_1^{--} \end{pmatrix}$$

for which we describe the components on generators of $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0, \Lambda_1)$. The map $\boldsymbol{\Delta}_1^{++} : C_+(\Lambda_0, \Lambda_1) \to C_+(\Lambda_0, \Lambda_1)$ on generators is given by the Legendrian contact homology differential of $\Lambda_0 \cup \Lambda_1$ restricted to mixed chords from Λ_0 to Λ_1, namely for a chord $\gamma_{10} \in \mathcal{R}(\Lambda_1, \Lambda_0)$ we have

$$\boldsymbol{\Delta}_1^{++}(\gamma_{10}) = \sum_{\beta_{10},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0_{\Lambda_{01}}(\beta_{10}; \boldsymbol{\delta}_0, \gamma_{10}, \boldsymbol{\delta}_1) \cdot \boldsymbol{\delta}_1\, \beta_{10}\, \boldsymbol{\delta}_0,$$

where the sum is over all mixed chords $\beta_{01} \in \mathcal{R}(\Lambda_0, \Lambda_1)$ and all words of pure Reeb chords $\boldsymbol{\delta}_i$ of Λ_i for $i = 0, 1$. Then, we extend it to bimodule elements $\boldsymbol{v}_1\gamma_{10}\boldsymbol{v}_0$ by

$$\boldsymbol{\Delta}_1^{++}(\boldsymbol{v}_1\gamma_{10}\boldsymbol{v}_0) = \partial_{\mathcal{A}(\Lambda_1)}(\boldsymbol{v}_1)\gamma_{10}\boldsymbol{v}_0 + \boldsymbol{v}_1\boldsymbol{\Delta}_1^{++}(\gamma_{10})\boldsymbol{v}_0 + \boldsymbol{v}_1\gamma_{10}\partial_{\mathcal{A}(\Lambda_0)}(\boldsymbol{v}_0)$$

and then by linearity. The component $\boldsymbol{b}_1^{--} : C_-(\Lambda_0, \Lambda_1) \to C_-(\Lambda_0, \Lambda_1)$ is given by the restriction to mixed chords from Λ_1 to Λ_0 of the Legendrian contact cohomology differential:

$$\boldsymbol{b}_1^{--}(\gamma_{01}) = \sum_{\beta_{01},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01}; \boldsymbol{\delta}_0, \gamma_{01}, \boldsymbol{\delta}_1) \cdot \boldsymbol{\delta}_1\, \beta_{01}\, \boldsymbol{\delta}_0\,.$$

We extend it to bimodule elements $\boldsymbol{v}_1\gamma_{01}\boldsymbol{v}_0$ by

$$\boldsymbol{b}_1^{--}(\boldsymbol{v}_1\gamma_{01}\boldsymbol{v}_0) = \partial_{\mathcal{A}(\Lambda_1)}(\boldsymbol{v}_1)\gamma_{01}\boldsymbol{v}_0 + \boldsymbol{v}_1\, \boldsymbol{b}_1^{--}(\gamma_{01})\boldsymbol{v}_0 + \boldsymbol{v}_1\gamma_{01}\partial_{\mathcal{A}(\Lambda_0)}(\boldsymbol{v}_0).$$

Finally, the component $\boldsymbol{b}_1^{-+}: C_+(\Lambda_0,\Lambda_1) \to C_-(\Lambda_0,\Lambda_1)$ is the *banana map* defined on generators by

$$\boldsymbol{b}_1^{-+}(\gamma_{10}) = \sum_{\beta_{01},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\cdot \boldsymbol{\delta}_1\,\beta_{01}\,\boldsymbol{\delta}_0$$

and we extend it to bimodule elements $\boldsymbol{v}_1\gamma_{10}\boldsymbol{v}_0$ by

$$\boldsymbol{b}_1^{-+}(\boldsymbol{v}_1\gamma_{10}\boldsymbol{v}_0) = \boldsymbol{v}_1\,\boldsymbol{b}_1^{-+}(\gamma_{10})\boldsymbol{v}_0.$$

Proposition 5.1 *We have*:

(a) $\mathfrak{m}_1$ *is a degree*-1 *map.*

(b) $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0,\Lambda_1)$ *is a DG* $\mathcal{A}(\Lambda_1)$*-*$\mathcal{A}(\Lambda_0)$*-bimodule*; *ie* $\mathfrak{m}_1^2 = 0$.

Proof (a) This is obtained by the dimension formula of the moduli spaces; see Section 3.

(b) This is obtained by considering the algebraic contributions of pseudoholomorphic buildings arising in the boundary of 1-dimensional moduli spaces of the following type:

- $\mathcal{M}^1_{\Lambda_{01}}(\beta_{10};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)$: strips with a positive asymptotic at γ_{10}, a negative asymptotic at β_{10} and negative pure asymptotics at the words $\boldsymbol{\delta}_0$ and $\boldsymbol{\delta}_1$.
- $\mathcal{M}^1_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1)$: strips with a positive asymptotic at β_{10} and negative asymptotic at γ_{10}.
- $\mathcal{M}^1_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)$: bananas with positive asymptotics at β_{10} and γ_{10}. □

Equivalently we have that $(C_+(\Lambda_0,\Lambda_1),\boldsymbol{\Delta}_1^{++})$ and $(C_-(\Lambda_0,\Lambda_1),\boldsymbol{b}_1^{--})$ are DG $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodules, and $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0,\Lambda_1)$ is the cone of the DG bimodule map $\boldsymbol{b}_1^{-+}: C_+(\Lambda_0,\Lambda_1)[1] \to C_-(\Lambda_0,\Lambda_1)$.

Notation 5.2 We suppress the exponents "$++$", "$-+$" and "$--$" indicating the output and input of each component of the differential, and define $\boldsymbol{\Delta}_1 := \boldsymbol{\Delta}_1^{++}$ and $\boldsymbol{b}_1 := \boldsymbol{b}_1^{-+} + \boldsymbol{b}_1^{--}$. We will write explicitly, when needed, if we consider some restriction of the map $\boldsymbol{b}_1$.

Remark 5.3 If the C-E DGAs $\mathcal{A}(\Lambda_0)$ and $\mathcal{A}(\Lambda_1)$ admit augmentations (see [8]) ε_0 and ε_1, respectively, over $\mathbb{Z}_2$, we can use them to turn the bimodule coefficients into elements of $\mathbb{Z}_2$ and thus get a $\mathbb{Z}_2$-module $\mathrm{RFC}(\Lambda_0,\Lambda_1)$. This latter module corresponds to the one defined in [29] in the case where the pair of Lagrangian cobordisms is a pair of trivial cylinders. Moreover, if Λ_1 is a copy of Λ_0, this complex is isomorphic to the complex of the 2-copy described in [18].

We recall now a sufficient condition for the Rabinowitz complex to be acyclic.

Definition 5.4 A pair of Legendrians (Λ_0,Λ_1) in $Y = P\times\mathbb{R}$ is said to be *horizontally displaceable* if there is a Hamiltonian isotopy φ_t of P such that $\Pi_P(\Lambda_0)\cap\varphi_1(\Pi_P(\Lambda_1)) = \varnothing$, where $\Pi_P: Y\to P$ is the projection.

Theorem 5.5 [29] *If* (Λ_0,Λ_1) *is horizontally displaceable*, *then* $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0,\Lambda_1)$ *is acyclic.*

Remark 5.6 In [29], Theorem 5.5 is proved for the $\mathbb{Z}_2$-module $\mathrm{RFC}(\Lambda_0,\Lambda_1)$ but the proof extends directly to the bimodule case.

Assume that Λ_1 is a perturbation of Λ_0 by a small negative Morse function; see Section 6.1 below. If the perturbation is sufficiently small then by invariance of the C-E algebra [20] (and the fact that the almost complex structure on $\mathbb{R}\times Y$ is the cylindrical lift of an admissible almost complex structure on P; see [13]), the DGAs $\mathcal{A}(\Lambda_0)$ and $\mathcal{A}(\Lambda_1)$ are canonically isomorphic in the sense that there is a canonical identification of Reeb chords of Λ_0 with Reeb chords of Λ_1 such that the differentials coincide under this identification. In this case, we denote by $(\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1),\mathfrak{m}_1)$ the Rabinowitz bimodule, where $\mathcal{A}:=\mathcal{A}(\Lambda_0)=\mathcal{A}(\Lambda_1)$. It is a DG $\mathcal{A}$-bimodule.

6 The Calabi–Yau structure

The goal of this section is to prove the following theorem:

Theorem 6.1 *The C-E algebra $\mathcal{A}(\Lambda)$ of a horizontally displaceable Legendrian sphere $\Lambda\subset Y$ is an $(n+1)$-Calabi–Yau differential graded algebra.*

In order to prove the theorem, we want to find a quasi-isomorphism $\mathcal{A}\to\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})$ of DG $\mathcal{A}$-bimodules. We will start by describing DG bimodules $\widehat{C}_+(\Lambda_0,\Lambda_1)$ and $\check{C}_-(\Lambda_0,\Lambda_1)$ which are, respectively, a quotient bimodule and a subbimodule of $\mathrm{RFC}_{\mathcal{A}_1\text{-}\mathcal{A}_0}(\Lambda_0,\Lambda_1)$ for a 2-copy $\Lambda_0\cup\Lambda_1$ of Λ. We show then that these bimodules are quasi-isomorphic to $\mathcal{A}$ and $\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})$, respectively (with some degree shifts). Finally, we define a DG bimodule morphism $\mathrm{CY}\colon\widehat{C}_+(\Lambda_0,\Lambda_1)\to\check{C}_-(\Lambda_0,\Lambda_1)$ and show that the cone of this morphism is quasi-isomorphic to the DG bimodule $\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$. By acyclicity of $\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$, CY is a quasi-isomorphism and thus we get the sought quasi-isomorphism.

6.1 The 2-copy $\Lambda_0\cup\Lambda_1$

In all of Section 6 we will assume that $\Lambda\subset Y$ is an n-dimensional Legendrian sphere. Let $f\colon\Lambda\to\mathbb{R}$ be a $\mathcal{C}^1$-small negative Morse function with exactly one maximum and one minimum, such that the norm of f is much smaller than the length of the shortest Reeb chord of Λ. Let Λ_1 denote the 1-jet of f in a standard neighborhood of Λ (identified with a neighborhood of the 0-section in $J^1(\Lambda)$) and define $\Lambda_0:=\Lambda$. Then we say that $\Lambda_0\cup\Lambda_1$ is a 2-copy of Λ.

Each Reeb chord γ of Λ gives rise to two mixed Reeb chords of $\Lambda_0\cup\Lambda_1$: $\gamma_{01}\in\mathcal{R}(\Lambda_0,\Lambda_1)$ and $\gamma_{10}\in\mathcal{R}(\Lambda_1,\Lambda_0)$. The choice of capping paths in order to define the Conley–Zehnder index of mixed chords is made in such a way that $\mathrm{CZ}(\gamma_0)=\mathrm{CZ}(\gamma_{01})=\mathrm{CZ}(\gamma_{10})$. The Legendrian $\Lambda_0\cup\Lambda_1$ admits two additional mixed Reeb chords from Λ_1 to Λ_0 corresponding to the critical points of f and called *Morse chords*. We denote by y_{01} the Reeb chord corresponding to the maximum of f and by x_{01} the one corresponding to the minimum. Note that $\ell(y_{01})<\ell(x_{01})$ because the function f is negative, and so we will call y_{01} the *minimum Morse Reeb chord* and x_{01} the *maximum Morse Reeb chord*. Note that $\mathrm{CZ}(y_{01})=0$ while $\mathrm{CZ}(x_{01})=n$. Finally, we denote by $C^l_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$ the $\mathcal{A}(\Lambda_1)$-$\mathcal{A}(\Lambda_0)$-bimodule (or $\mathcal{A}$-bimodule for short, as the algebras $\mathcal{A}(\Lambda_0)$ and $\mathcal{A}(\Lambda_1)$ are canonically identified) generated by *long chords* from Λ_1 to Λ_0, ie chords from Λ_1 to Λ_0 which are not Morse chords.

6.2 The bimodules $\widehat{C}_+(\Lambda_0, \Lambda_1)$ and $\check{C}_-(\Lambda_0, \Lambda_1)$

The DG $\mathcal{A}$-bimodule $(\widehat{C}_+(\Lambda_0, \Lambda_1), \widehat{\mathfrak{m}}_1)$ is defined as follows. The underlying graded bimodule is generated by the positive action generators of the Rabinowitz bimodule and the maximum Morse chord:

$$\widehat{C}_+(\Lambda_0, \Lambda_1) = C_+(\Lambda_0, \Lambda_1)[1] \oplus \langle x_{01} \rangle_{\mathcal{A}\text{-}\mathcal{A}}[1],$$

where $\langle x_{01} \rangle_{\mathcal{A}\text{-}\mathcal{A}}$ denotes the $\mathcal{A}$-subbimodule of $C_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0, \Lambda_1)$ generated by x_{01}. So the chord x_{01} in $\langle x_{01} \rangle_{\mathcal{A}\text{-}\mathcal{A}}[1]$ has degree $n+1 = \mathrm{CZ}(x_{01}) + 1$. The differential $\widehat{\mathfrak{m}}_1$ is given on generators by

$$\widehat{\mathfrak{m}}_1(\gamma_{10}) = \boldsymbol{\Delta}_1(\gamma_{10}) + \gamma_1 x_{01} + x_{01}\gamma_0, \quad \widehat{\mathfrak{m}}_1(x_{01}) = 0,$$

where γ_i denotes the pure chord of Λ_i corresponding to γ_{10}.

Proposition 6.2 *The bimodule $(\widehat{C}_+(\Lambda_0, \Lambda_1), \widehat{\mathfrak{m}}_1)$ is a semifree DG bimodule.*

We state here a lemma that we will use repeatedly in several proofs.

Lemma 6.3 [18] *For a 2-copy $\Lambda_0 \cup \Lambda_1$ of Λ_0 we have:*

(a) *For every Reeb chord $\gamma_{01} \in C^l(\Lambda_0, \Lambda_1)$, there are exactly two rigid pseudoholomorphic strips with positive asymptotic at γ_{01} and negative asymptotic at the minimum Reeb chord y_{01}. Moreover, each of these strips has exactly one pure negative chord asymptotic which is the chord γ_0 for one strip and γ_1 for the other (where γ_i denotes the pure Reeb chord of Λ_i corresponding to γ_{01}).*

(b) *For every Reeb chord $\gamma_{10} \in \mathcal{R}(\Lambda_1, \Lambda_0)$, there are exactly two rigid pseudoholomorphic discs with boundary on $\mathbb{R} \times (\Lambda_0 \cup \Lambda_1)$ which are bananas with positive asymptotics at γ_{10} and at the maximum Reeb chord x_{01}. Moreover, each of these bananas has exactly one pure negative chord asymptotic which is the chord γ_0 for one banana and γ_1 for the other.*

(c) *The count of rigid pseudoholomorphic strips with boundary on $\mathbb{R} \times (\Lambda_0 \cup \Lambda_1)$ admitting a positive puncture at the maximum x_{01} and a negative puncture at a chord β_{01}, vanishes.*

Proof By [18, Theorem 3.6], rigid pseudoholomorphic strips with boundary on $\mathbb{R} \times (\Lambda_0 \cup \Lambda_1)$, a positive asymptotic at γ_{01} and a negative asymptotic at the minimum Reeb chord y_{01} (which corresponds to the maximum of the function f) correspond to rigid *generalized discs* which consist of a pseudoholomorphic disc with boundary on $\mathbb{R} \times \Lambda_0$ with a positive asymptotic at γ_0 and a negative gradient flow line of f from the maximum critical point to a point on the boundary of the disc. By rigidity, the pseudoholomorphic disc must be a constant strip $\mathbb{R} \times \gamma_0$ and then there are two ways the flow line can be attached to it (either on $\mathbb{R} \times \{\text{starting point of } \gamma_0\}$, or on $\mathbb{R} \times \{\text{ending point of } \gamma_0\}$). These give the two strips in (a).

For (b), any rigid banana with positive asymptotics at γ_{10} and at the maximum Reeb chord x_{01} corresponds to a rigid generalized discs consisting of a pseudoholomorphic disc with boundary on $\mathbb{R} \times \Lambda_0$ with a positive asymptotic at γ_0 and a negative gradient flow line of f flowing from a point on the boundary of the disc to the minimum critical point (remember x_{01} corresponds to the minimum of f). By rigidity again the disc must be constant and the two possible ways the flow line can be attached to it provides the two bananas in (b).

For (c), by action reasons the chord β_{01} must be a Morse chord and the strip has no negative pure Reeb chord asymptotics. Thus, [18, Theorem 3.6] tells us that such discs are in bijective correspondence with negative gradient flow lines of f from the critical point corresponding to β_{01} to the minimum. But in our case f has only two critical points so there are either no flow line (for degree reasons in the case $n \geq 2$), or exactly two flow lines (from the maximum to the minimum). □

Proof of Proposition 6.2 We compute $(\widehat{\mathfrak{m}}_1)^2(\gamma_{10}) = \widehat{\mathfrak{m}}_1(\mathbf{\Delta}_1(\gamma_{10})) + \partial_{\mathcal{A}}(\gamma_1)x_{01} + x_{01}\partial_{\mathcal{A}}(\gamma_0)$. For any term $\boldsymbol{\delta}_1\,\beta_{10}\,\boldsymbol{\delta}_0$ appearing in $\mathbf{\Delta}_1(\gamma_{10})$, we have

$$\begin{aligned}\widehat{\mathfrak{m}}_1(\boldsymbol{\delta}_1\,\beta_{10}\,\boldsymbol{\delta}_0) &= \partial_{\mathcal{A}}(\boldsymbol{\delta}_1)\beta_{10}\,\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1(\mathbf{\Delta}_1(\beta_{10}) + \beta_1 x_{01} + x_{01}\beta_0)\,\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1\,\beta_{10}\partial_{\mathcal{A}}(\boldsymbol{\delta}_0)\\ &= \partial_{\mathcal{A}}(\boldsymbol{\delta}_1)\beta_{10}\,\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1\,\mathbf{\Delta}_1(\beta_{10})\,\boldsymbol{\delta}_0 + \boldsymbol{\delta}_1\,\beta_{10}\partial_{\mathcal{A}}(\boldsymbol{\delta}_0) + \boldsymbol{\delta}_1(\beta_1 x_{01} + x_{01}\beta_0)\,\boldsymbol{\delta}_0\\ &= \mathbf{\Delta}_1(\boldsymbol{\delta}_1\,\beta_{10}\,\boldsymbol{\delta}_0) + \boldsymbol{\delta}_1(\beta_1 x_{01} + x_{01}\beta_0)\,\boldsymbol{\delta}_0\,.\end{aligned}$$

So we have

$$(\widehat{\mathfrak{m}}_1)^2(\gamma_{10}) = (\mathbf{\Delta}_1)^2(\gamma_{10}) + \partial_{\mathcal{A}}(\gamma_1)x_{01} + x_{01}\partial_{\mathcal{A}}(\gamma_0) + \sum_{\boldsymbol{\delta}_1\,\beta_{10}\,\boldsymbol{\delta}_0\in\mathbf{\Delta}_1(\gamma_{10})} \boldsymbol{\delta}_1(\beta_1 x_{01} + x_{01}\beta_0)\,\boldsymbol{\delta}_0\,.$$

We know that $(\mathbf{\Delta}_1)^2(\gamma_{10}) = 0$ and it remains to understand why the last three terms vanish.

The boundary of the compactification of 1-dimensional (after dividing by translation) moduli spaces of bananas with positive asymptotics at γ_{10} and x_{01}, and pure negative Reeb chord asymptotics consists of broken discs which are 2-level buildings connected by a Reeb chord. This Reeb chord is either a mixed chord or a pure chord. In the first case, using Lemma 6.3(c), the only (nonvanishing) possibility is that the upper level of the building is a disc contributing to $\mathbf{\Delta}_1(\gamma_{10})$ with a positive asymptotic at γ_{10}, a negative asymptotic at a chord β_{10} and negative pure Reeb chord asymptotics; and the lower level is a rigid banana with positive asymptotic at β_{10} and x_{01}. By Lemma 6.3(b) there are two such bananas and so we get two buildings contributing to $\boldsymbol{\delta}_1(\beta_1 x_{01} + x_{01}\beta_0)\,\boldsymbol{\delta}_0$, for any $\boldsymbol{\delta}_1\,\beta_{10}\,\boldsymbol{\delta}_0 \in \mathbf{\Delta}_1(\gamma_{10})$. In the second case, the top level of the building is a rigid banana with positive asymptotics at γ_{10} and x_{01}. By Lemma 6.3(b) the pure connecting Reeb chord is either γ_0 or γ_1. The lower level of the building is thus a disk contributing to $\partial_{\mathcal{A}}(\gamma_i)$, for $i = 0, 1$ depending on which one is the connecting pure Reeb chord. This implies that $\partial_{\mathcal{A}}(\gamma_1)x_{01} + x_{01}\partial_{\mathcal{A}}(\gamma_0) = 0$.

The semifree property of $\widehat{C}_+(\Lambda_0, \Lambda_1)$ is deduced directly from the action filtration on chords. Indeed, denote by γ_{10}^j for $1 \leq j \leq k$ the generators of $\widehat{C}_+(\Lambda_0, \Lambda_1)$ which are long chords. Up to relabeling, there are real numbers $\ell_0, \ell_1, \dots, \ell_k$ such that

$$\mathfrak{a}(x_{01}) < \ell_0 < 0 < \mathfrak{a}(\gamma_{10}^1) < \ell_1 < \cdots < \ell_{k-1} < \mathfrak{a}(\gamma_{10}^k) < \ell_k.$$

The $\mathbb{Z}_2$-vector spaces $F_j\widehat{C}_+ = \{\gamma \in C_+(\Lambda_0,\Lambda_1)[1] \oplus \langle x_{01}\rangle_{\mathbb{Z}_2}[1] : \mathfrak{a}(\gamma) < \ell_j\}$, for $0 \le j \le k$, produce a filtration

$$\mathcal{A} \otimes F_0\widehat{C}_+ \otimes \mathcal{A} \subset \mathcal{A} \otimes F_1\widehat{C}_+ \otimes \mathcal{A} \subset \cdots \subset \mathcal{A} \otimes F_k\widehat{C}_+ \otimes \mathcal{A} = \widehat{C}_+(\Lambda_0,\Lambda_1)$$

giving that $\widehat{C}_+(\Lambda_0,\Lambda_1)$ is semifree and of finite rank. □

Let us now describe the $\mathcal{A}$-bimodule $(\check{C}_-(\Lambda_0,\Lambda_1), \check{\mathfrak{m}}_1)$. The underlying graded bimodule is

$$\check{C}_-(\Lambda_0,\Lambda_1) = C^{l}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1) \oplus \langle y_{01}\rangle_{\mathcal{A}\text{-}\mathcal{A}}$$

and as a $\mathcal{A}$-subbimodule of $C_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$ the elements are graded with the Conley–Zehnder index. The differential $\check{\mathfrak{m}}_1$ on generators is given by the restriction to $\check{C}_-(\Lambda_0,\Lambda_1)$ of the map $\boldsymbol{b}_1^{--}$ which was defined in Section 5. By Lemma 6.3(a) we have

$$\check{\mathfrak{m}}_1(y_{01}) = \boldsymbol{b}_1^{--}(y_{01}) = \sum_{\gamma \in \mathcal{R}(\Lambda)} \gamma_1\gamma_{01} + \gamma_{01}\gamma_0.$$

Proposition 6.4 *The bimodule* $(\check{C}_-(\Lambda_0,\Lambda_1), \check{\mathfrak{m}}_1)$ *is a semifree DG bimodule.*

Proof Consider the boundary of the compactification of moduli spaces of strips with a positive and a negative mixed asymptotics at a generator of $\check{C}_-(\Lambda_0,\Lambda_1)$. This boundary consists of 2-level buildings connected by either a mixed chord or a pure chord. If the connecting mixed chord is x_{01}, then by Lemma 6.3(c) the building will either not exist or arise in pair so its contribution vanishes algebraically. The other cases are exactly those contributing to $(\check{\mathfrak{m}}_1)^2$.

The argument showing that $\check{C}_-(\Lambda_0,\Lambda_1)$ is semifree is analogous to what was done in the proof of Proposition 6.2. □

6.3 Alternative cone description of the Rabinowitz complex of a sphere

We will define a DG bimodule map

$$\mathrm{CY}\colon \widehat{C}_+(\Lambda_0,\Lambda_1) \to \check{C}_-(\Lambda_0,\Lambda_1) \tag{2}$$

and prove that the cone of CY is quasi-isomorphic to the DG bimodule $\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$. For γ_{10}, x_{01} generators of $\widehat{C}_+(\Lambda_0,\Lambda_1)$, we set

$$\mathrm{CY}(\gamma_{10}) = \sum_{\substack{\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\cup\{y_{01}\}\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1}} \#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\cdot\boldsymbol{\delta}_1\,\beta_{01}\,\boldsymbol{\delta}_0,$$

$$\mathrm{CY}(x_{01}) = \sum_{\substack{\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1}} \#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1)\cdot\boldsymbol{\delta}_1\,\beta_{01}\,\boldsymbol{\delta}_0\,.$$

See Figure 1. In the last equation β_{01} can never be the minimum Morse chord y_{01} for energy reasons. Observe also that as ungraded maps the map CY is given on generators of $\widehat{C}_+(\Lambda_0,\Lambda_1)$ by the component of the map $\boldsymbol{b}_1$ taking values in $\check{C}_-(\Lambda_0,\Lambda_1)$.

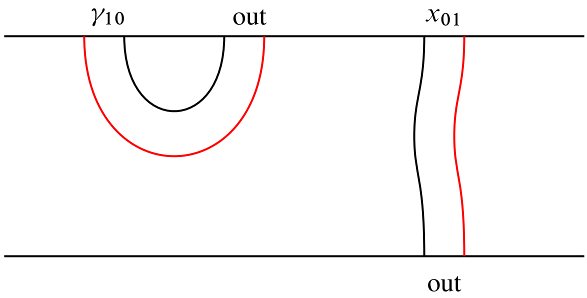

Figure 1: Pseudoholomorphic discs contributing to the map CY.

Lemma 6.5 *The map* CY *is a DG bimodule map.*

Proof We compute first

$$\mathrm{CY}\circ\widehat{\mathfrak{m}}_1(x_{01})+\check{\mathfrak{m}}_1\circ\mathrm{CY}(x_{01})=\check{\mathfrak{m}}_1\circ\mathrm{CY}(x_{01})=(\boldsymbol{b}_1^{--})^2(x_{01})=0,$$

where the second and third equality hold simply by definition. Then, we have

$$\begin{aligned}\mathrm{CY}\circ\widehat{\mathfrak{m}}_1(\gamma_{10})+\check{\mathfrak{m}}_1\circ\mathrm{CY}(\gamma_{10}) &= \mathrm{CY}(\boldsymbol{\Delta}_1(\gamma_{10})+\gamma_1x_{01}+x_{01}\gamma_0)+\check{\mathfrak{m}}_1(\mathrm{CY}(\gamma_{10}))\\ &= \mathrm{CY}(\boldsymbol{\Delta}_1(\gamma_{10}))+\gamma_1\mathrm{CY}(x_{01})+\mathrm{CY}(x_{01})\gamma_0+\check{\mathfrak{m}}_1(\mathrm{CY}(\gamma_{10}))\\ &= \mathrm{CY}\Big(\sum\#\mathcal{M}^0(\beta_{10};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\boldsymbol{\delta}_1\beta_{10}\boldsymbol{\delta}_0\Big)+\gamma_1\mathrm{CY}(x_{01})+\mathrm{CY}(x_{01})\gamma_0\\ &\qquad +\widehat{\mathfrak{m}}_1\Big(\sum\#\mathcal{M}^0(\xi_{01};\boldsymbol{\sigma}_0,\gamma_{10},\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_1\xi_{01}\boldsymbol{\sigma}_0\Big)\end{aligned}$$

$$= \sum\#\mathcal{M}^0(\nu_{01},\boldsymbol{\delta}_0',\beta_{10},\boldsymbol{\delta}_1')\#\mathcal{M}^0(\beta_{10};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\nu_{01}\boldsymbol{\delta}_0'\boldsymbol{\delta}_0 \tag{3}$$

$$+\gamma_1\mathrm{CY}(x_{01})+\mathrm{CY}(x_{01})\gamma_0 \tag{4}$$

$$+\sum\#\mathcal{M}^0(\xi_{01};\boldsymbol{\sigma}_0,\gamma_{10},\boldsymbol{\sigma}_1)(\partial_{\mathcal{A}}(\boldsymbol{\sigma}_1)\xi_{01}\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_1\xi_{01}\partial_{\mathcal{A}}(\boldsymbol{\sigma}_0)) \tag{5}$$

$$+\sum\#\mathcal{M}^0(\nu_{01};\boldsymbol{\sigma}_0',\xi_{01},\boldsymbol{\sigma}_1')\#\mathcal{M}^0(\xi_{01};\boldsymbol{\sigma}_0,\gamma_{10},\boldsymbol{\sigma}_1)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1'\nu_{01}\boldsymbol{\sigma}_0'\boldsymbol{\sigma}_0. \tag{6}$$

The terms of this last equation correspond actually to the algebraic contributions of the pseudoholomorphic buildings appearing in the boundary of the compactification of moduli spaces of bananas with two positive asymptotics, one at γ_{10} and another at a generator of $\check{C}_-(\Lambda_0,\Lambda_1)$, and some negative pure Reeb chord asymptotics; see Figure 2. □

The cone of CY, $\mathrm{Cone}(\mathrm{CY})=\widehat{C}_+(\Lambda_0,\Lambda_1)[-1]\oplus\check{C}_-(\Lambda_0,\Lambda_1)$, is a DG $\mathcal{A}$-bimodule, and we prove that it is quasi-isomorphic to the Rabinowitz bimodule $\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$. This is almost trivial, indeed consider the bimodule map

$$\nu\colon \mathrm{Cone}(\mathrm{CY})\to\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$$

sending each mixed chord of the 2-copy $\Lambda_0\cup\Lambda_1$ to itself.

Proposition 6.6 *ν is a quasi-isomorphism of DG bimodules.*

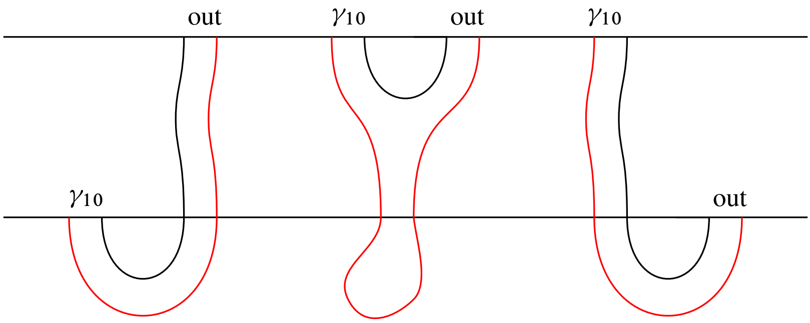

Figure 2: Types of pseudoholomorphic buildings in the boundary of moduli spaces of bananas with a positive asymptotic at γ_{10}. When the connecting Reeb chords between the two components of the leftmost building is a long chord or y_{01}, then it contributes algebraically to (6); and if it is the chord x_{01}, then it contributes algebraically to (4). The middle building schematizes the contributions of type (5) and the rightmost building contributes to (3).

Proof We have

$$\begin{aligned}\nu\circ\mathfrak{m}_1^{\mathrm{Cone(CY)}}(\gamma_{10})+\mathfrak{m}_1\circ\nu(\gamma_{10}) &= \nu(\widehat{\mathfrak{m}}_1(\gamma_{10})+\mathrm{CY}(\gamma_{10}))+\mathfrak{m}_1(\gamma_{10})\\ &= \nu(\boldsymbol{\Delta}_1(\gamma_{10})+\gamma_1x_{01}+x_{01}\gamma_0+\mathrm{CY}(\gamma_{10}))+\boldsymbol{\Delta}_1(\gamma_{10})+\boldsymbol{b}_1(\gamma_{10})\\ &= \boldsymbol{\Delta}_1(\gamma_{10})+\gamma_1x_{01}+x_{01}\gamma_0+\mathrm{CY}(\gamma_{10})+\boldsymbol{\Delta}_1(\gamma_{10})+\boldsymbol{b}_1(\gamma_{10}).\end{aligned}$$

Observe that $\gamma_1x_{01}+x_{01}\gamma_0+\mathrm{CY}(\gamma_{10})=\boldsymbol{b}_1(\gamma_{10})$ because $\mathrm{CY}(\gamma_{10})$ is defined by a count of bananas with positive asymptotics at γ_{10} and at a chord in $\mathcal{R}^l(\Lambda_0,\Lambda_1)$, and the two other terms correspond to bananas with positive asymptotics at γ_{10} and x_{10}. So the sum above vanishes. Then the relation $\nu\circ\mathfrak{m}_1^{\mathrm{Cone(CY)}}+\mathfrak{m}_1\circ\nu=0$ when the input is x_{01} or any generator of $\check{C}_-(\Lambda_0,\Lambda_1)$ follows directly from the definition of the maps. □

6.4 The Calabi–Yau isomorphism

In this subsection, we assume that Λ_0 is horizontally displaceable, which implies that the complex $\mathrm{RFC}_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_0,\Lambda_1)$ is acyclic. Hence it implies that the complex Cone(CY) defined in the previous section is also acyclic and thus that the degree-0 map $\mathrm{CY}\colon\widehat{C}_+(\Lambda_0,\Lambda_1)\to\check{C}_-(\Lambda_0,\Lambda_1)$ is a quasi-isomorphism of DG bimodules. We will show that this quasi-isomorphism is a Calabi–Yau isomorphism, by showing that there are quasi-isomorphisms of DG bimodules $\mathcal{A}\simeq\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]$ and $\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})\simeq\check{C}_-(\Lambda_0,\Lambda_1)$.

We define a bimodule map $F\colon\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]\to\mathcal{A}$ by

$$F(\boldsymbol{v}_1\gamma_{10}\boldsymbol{v}_0)=0,\qquad F(\boldsymbol{v}_1x_{01}\boldsymbol{v}_0)=\boldsymbol{v}_1\boldsymbol{v}_0$$

for $\boldsymbol{v}_i$ words of Reeb chords of Λ_i which on the right-hand side are canonically identified with words of Reeb chords of Λ_0.

Proposition 6.7 *The map F is a quasi-isomorphism of DG bimodules.*

Proof The fact that F is a degree-0 chain map follows directly from the definition. It is thus a DG bimodule morphism and we check that it is a quasi-isomorphism by proving that its cone is acyclic. For a word of Reeb chords $\boldsymbol{w}_1 = w^1 \dots w^k$ of Λ_1, write

$$\overline{\boldsymbol{w}}_1 = \sum_{j=1}^{k} w^1 \dots w^{j-1} w^j_{10} w^{j+1} \dots w^k \in C_{\mathcal{A}\text{-}\mathcal{A}}(\Lambda_1, \Lambda_0).$$

Consider the DG bimodule $\mathrm{Cone}(F)$, whose differential we denote by $\partial^{\mathrm{Cone}(F)}$. We define a degree -1 $\mathbb{Z}_2$-linear map $h\colon \mathrm{Cone}(F) \to \mathrm{Cone}(F)$ by

$$h(\boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0) = 0, \quad h(\boldsymbol{w}_1 x_{01} \boldsymbol{v}_0) = \overline{\boldsymbol{w}}_1 \boldsymbol{v}_0, \quad h(a^1 \dots a^k) = x_{01} a^1 \dots a^k$$

and show that h defines a homotopy between the identity map and the zero map on $\mathrm{Cone}(F)$, ie that

$$\partial^{\mathrm{Cone}(F)} \circ h + h \circ \partial^{\mathrm{Cone}(F)} = \mathrm{id}_{\mathrm{Cone}(F)}\,.$$

We check this relation for the three different types of elements in $\mathrm{Cone}(F)$. First

$$\begin{aligned}(\partial^{\mathrm{Cone}(F)} \circ h + h \circ \partial^{\mathrm{Cone}(F)})(\boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0) &= 0 + h \circ \widehat{\mathfrak{m}}_1(\boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0) + h \circ F(\boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0) \\ &= h(\partial_{\mathcal{A}}(\boldsymbol{w}_1)\gamma_{10}\boldsymbol{v}_0 + \boldsymbol{w}_1 \gamma_{10} \partial_{\mathcal{A}}(\boldsymbol{v}_0) + \boldsymbol{w}_1 \boldsymbol{\Delta}_1(\gamma_{10})\boldsymbol{v}_0 + \boldsymbol{w}_1(\gamma x_{01} + x_{01}\gamma)\boldsymbol{v}_0) + 0 \\ &= h(\boldsymbol{w}_1 \gamma x_{01} \boldsymbol{v}_0 + \boldsymbol{w}_1 x_{01} \gamma \boldsymbol{v}_0) \\ &= \overline{\boldsymbol{w}}_1 \gamma \boldsymbol{v}_0 + \boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0 + \overline{\boldsymbol{w}}_1 \gamma \boldsymbol{v}_0 \\ &= \boldsymbol{w}_1 \gamma_{10} \boldsymbol{v}_0.\end{aligned}$$

Then,

$$\begin{aligned}(\partial^{\mathrm{Cone}(F)} \circ h + h \circ \partial^{\mathrm{Cone}(F)})(\boldsymbol{w}_1 x_{01} \boldsymbol{v}_0) &= \partial^{\mathrm{Cone}(F)}(\overline{\boldsymbol{w}}_1 \boldsymbol{v}_0) + h(\partial_{\mathcal{A}}(\boldsymbol{w}_1) x_{01} \boldsymbol{v}_0 + \boldsymbol{w}_1 x_{01} \partial_{\mathcal{A}}(\boldsymbol{v}_0) + \boldsymbol{w}_1 \boldsymbol{v}_0) \\ &= \widehat{\mathfrak{m}}_1(\overline{\boldsymbol{w}}_1)\boldsymbol{v}_0 + \overline{\boldsymbol{w}}_1 \partial_{\mathcal{A}}(\boldsymbol{v}_0) + \overline{\partial_{\mathcal{A}}(\boldsymbol{w}_1)}\boldsymbol{v}_0 + \overline{\boldsymbol{w}}_1 \partial_{\mathcal{A}}(\boldsymbol{v}_0) + x_{01} \boldsymbol{w}_1 \boldsymbol{v}_0 \\ &= \widehat{\mathfrak{m}}_1(\overline{\boldsymbol{w}}_1)\boldsymbol{v}_0 + \overline{\partial_{\mathcal{A}}(\boldsymbol{w}_1)}\boldsymbol{v}_0 + x_{01}\boldsymbol{w}_1 \boldsymbol{v}_0.\end{aligned}$$

Observe that $\overline{\partial_{\mathcal{A}}(\boldsymbol{w}_1)} = \boldsymbol{\Delta}_1(\overline{\boldsymbol{w}}_1)$ which is one component of $\widehat{\mathfrak{m}}_1(\overline{\boldsymbol{w}}_1)$. Assuming $\boldsymbol{w}_1 = w^1 \dots w^k$ we thus get

$$\begin{aligned}(\partial^{\mathrm{Cone}(F)} \circ h + h \circ \partial^{\mathrm{Cone}(F)})(\boldsymbol{w}_1 x_{01} \boldsymbol{v}_0) &= \sum_{j=1}^{k} (w^1 \dots w^{j-1}(w^j x_{01} + x_{01} w^j) w^{j+1} \dots w^k)\boldsymbol{v}_0 + x_{01}\boldsymbol{w}_1 \boldsymbol{v}_0 \\ &= \boldsymbol{w}_1 x_{01} \boldsymbol{v}_0.\end{aligned}$$

Finally, for a word $a^1 \dots a^k$ in $\mathcal{A}$, we compute

$$\begin{aligned}(\partial^{\mathrm{Cone}(F)} \circ h + h \circ \partial^{\mathrm{Cone}(F)})(a^1 \dots a^k) &= \partial^{\mathrm{Cone}(F)}(x_{01} a^1 \dots a^k) + h \circ \partial_{\mathcal{A}}(a^1 \dots a^k) \\ &= x_{01} \partial_{\mathcal{A}}(a^1 \dots a^k) + F(x_{01} a^1 \dots a^k) + x_{01} \partial_{\mathcal{A}}(a^1 \dots a^k) \\ &= a^1 \dots a^k.\end{aligned}$$ □

Observe that Proposition 6.7 implies that $\widehat{C}_+(\Lambda_0, \Lambda_1)[-n-1]$ is a semifree resolution of the diagonal DG bimodule $\mathcal{A}$. This means in particular that there is a quasi-isomorphism of DG bimodules $\mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}) \simeq \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0, \Lambda_1)[-n-1], \mathcal{A} \otimes \mathcal{A})$. This semifree resolution is also of finite rank so $\mathcal{A}$ is homologically smooth.

Remark 6.8 Such a finite-rank semifree resolution is explicitly given in [27, Section 3.6] for a DG category associated to a quiver (with some specific properties).

We consider now a bimodule map $G\colon \check{C}_-(\Lambda_0,\Lambda_1)\to \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1],\mathcal{A}\otimes\mathcal{A})$ defined on generators by $G(\gamma_{01})=\phi_\gamma$, where $\phi_\gamma\colon \widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]\to\mathcal{A}\otimes\mathcal{A}$ is the bimodule map defined on generators by

$$\phi_\gamma(\beta_{10})=\begin{cases}1\otimes 1 & \text{if } \beta=\gamma,\\ 0 & \text{otherwise,}\end{cases}\qquad \phi_\gamma(x_{01})=0,$$

and similarly $G(y_{01})=\phi_y$ which is the bimodule map defined by $\phi_y(\beta_{10})=0$ and $\phi_y(x_{01})=1\otimes 1$.

Proposition 6.9 *The map G is a quasi-isomorphism of DG bimodules.*

Proof The differential on $\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1],\mathcal{A}\otimes\mathcal{A})$ is given by $D(\phi)=\phi\circ\widehat{\mathfrak{m}}_1+\partial_{\mathcal{A}\otimes\mathcal{A}}\circ\phi$. We want to prove that

$$G\circ\check{\mathfrak{m}}_1+D\circ G=0 \tag{7}$$

and in order to do so we check that (7) is satisfied when applied to each generator of $\check{C}_-(\Lambda_0,\Lambda_1)$. Let first γ_{01} be a generator of $\check{C}_-(\Lambda_0,\Lambda_1)$ which is a long chord. We want to prove

$$G\circ\check{\mathfrak{m}}_1(\gamma_{01})+G(\gamma_{01})\circ\widehat{\mathfrak{m}}_1+\partial_{\mathcal{A}\otimes\mathcal{A}}\circ G(\gamma_{01})=0. \tag{8}$$

The left-hand side being a map $\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]\to\mathcal{A}\otimes\mathcal{A}$, we prove that it vanishes for each generator of $\widehat{C}_+(\Lambda_0,\Lambda_1)$. We divide it into three cases:

(a) the generator γ_{10} which corresponds to γ_{01} in the 2-copy link,

(b) a long chord generator β_{10} which is not γ_{10}, and finally

(c) the Morse generator x_{01}.

(a) We have

$$\partial_{\mathcal{A}\otimes\mathcal{A}}\circ G(\gamma_{01})(\gamma_{10})=\partial_{\mathcal{A}\otimes\mathcal{A}}\circ\phi_\gamma(\gamma_{10})=\partial_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1)=0.$$

Moreover $G(\gamma_{01})\circ\widehat{\mathfrak{m}}_1(\gamma_{10})=\phi_\gamma\circ\widehat{\mathfrak{m}}_1(\gamma_{10})=0$ because γ_{10} cannot be the output of $\widehat{\mathfrak{m}}_1(\gamma_{10})$ for action reasons, and so we get 0 by definition of ϕ_γ. Similarly, $G\circ\check{\mathfrak{m}}_1(\gamma_{01})(\gamma_{10})=0$ because γ_{01} is not an output of $\check{\mathfrak{m}}_1(\gamma_{01})$ for action reasons.

(b) In this case observe first that $\partial_{\mathcal{A}\otimes\mathcal{A}}\circ G(\gamma_{01})(\beta_{10})=\partial_{\mathcal{A}\otimes\mathcal{A}}\circ\phi_\gamma(\beta_{10})=0$. Then, the (rigid) pseudoholomorphic discs contributing to $\phi_\gamma\circ\widehat{\mathfrak{m}}_1(\beta_{10})$ are discs with a positive asymptotic at β_{10}, a negative asymptotic at γ_{10} and potentially negative asymptotics to pure Reeb chords; see Figure 3. Consider such a disc which contributes $v_1\dots v_r\gamma_{10}w_1\dots w_s$ to $\widehat{\mathfrak{m}}_1(\beta_{10})$. Then, note that

$$\phi_\gamma\in\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1],\mathcal{A}\otimes\mathcal{A}),$$

where $\mathcal{A}\otimes\mathcal{A}$ is endowed with the *inner* bimodule structure, which means that

$$\phi_\gamma(v_1\dots v_r\gamma_{10}w_1\dots w_s)=v_1\dots v_r\cdot\phi_\gamma(\gamma_{10})\cdot w_1\dots w_s=v_1\dots v_r\cdot 1\otimes 1\cdot w_1\dots w_s=w_1\dots w_s\otimes v_1\dots v_r.$$

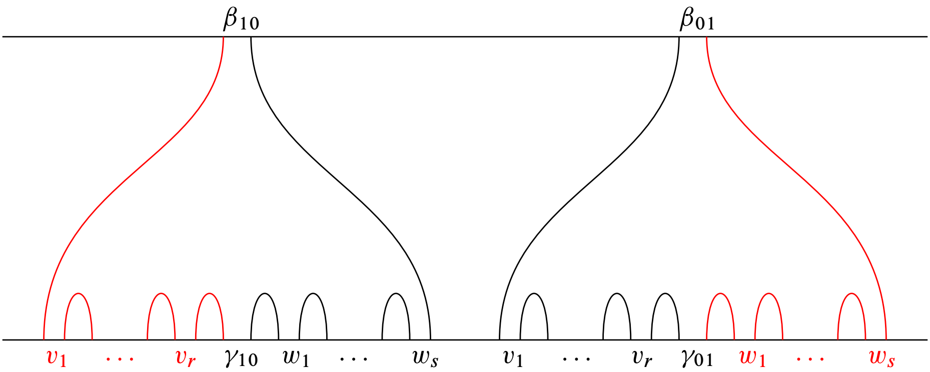

Figure 3: Left: pseudoholomorphic disc contributing to $\phi_\gamma \circ \widehat{\mathfrak{m}}_1(\beta_{10})$; right: pseudoholomorphic discs contributing to $G \circ \check{\mathfrak{m}}_1(\gamma_{01})(\beta_{10})$.

Now let us look at the last term in (8) applied to β_{10}, which is $G \circ \check{\mathfrak{m}}_1(\gamma_{01})(\beta_{10})$. The pseudoholomorphic discs contributing $w_1 \dots w_s \otimes v_1 \dots v_r$ to this term are discs schematized on the right in Figure 3. Indeed, such a disc contributes $w_1 \dots w_s \beta_{01} v_1 \dots v_r$ to $\check{\mathfrak{m}}_1(\gamma_{01})$ and then

$$\begin{aligned} G(w_1 \dots w_s \beta_{01} v_1 \dots v_r)(\beta_{10}) &= (w_1 \dots w_s \cdot G(\beta_{01}) \cdot v_1 \dots v_r)(\beta_{10}) \\ &= (w_1 \dots w_s \cdot \phi_\beta \cdot v_1 \dots v_r)(\beta_{10}) \\ &= w_1 \dots w_s \otimes v_1 \dots v_r, \end{aligned}$$

where for the last equality we use the outer bimodule structure (and the definition of ϕ_β). Finally, observe that the discs schematized in Figure 3 correspond to lifted discs in the terminology of [18] and that there is a bijective correspondence between discs on the left and right of the figure, they both correspond to a rigid disc with boundary on Λ with a pure positive asymptotic at β and pure negative asymptotics at $v_1, \dots, v_r, \gamma, w_1, \dots, w_s$ in this order; see, eg [1, Theorem 3.2]. Thus we get (8) as for any disc contributing to $\phi_\gamma \circ \widehat{\mathfrak{m}}_1(\beta_{10})$ corresponds (bijectively) a disc contributing the same thing to $G \circ \check{\mathfrak{m}}_1(\gamma_{01})(\beta_{10})$.

(c) In this case we have directly that

$$G \circ \check{\mathfrak{m}}_1(\gamma_{01})(x_{01}) + G(\gamma_{01}) \circ \widehat{\mathfrak{m}}_1(x_{01}) + \partial_{\mathcal{A}\otimes\mathcal{A}} \circ G(\gamma_{01})(x_{01}) = 0$$

because y_{01} is never an output of $\check{\mathfrak{m}}_1(\gamma_{01})$ for action reasons (and so $G \circ \check{\mathfrak{m}}_1(\gamma_{01})(x_{01}) = 0$), $\widehat{\mathfrak{m}}_1(x_{01}) = 0$, and $G(\gamma_{01})(x_{01}) = \phi_\gamma(x_{01}) = 0$.

Finally we check that (7) is satisfied when applied to the Morse generator $y_{01} \in \check{C}_-(\Lambda_0, \Lambda_1)$. Recall that $\check{\mathfrak{m}}_1(y_{01}) = \sum_{\gamma \in \mathcal{R}(\Lambda)} \gamma\gamma_{01} + \gamma_{01}\gamma$, so we have

$$G \circ \check{\mathfrak{m}}_1(y_{01})(\gamma_{10}) = \gamma \otimes 1 + 1 \otimes \gamma, \quad G \circ \check{\mathfrak{m}}_1(y_{01})(x_{01}) = 0.$$

On the other side, for a chord γ_{10}, we have

$$\begin{aligned} D \circ G(y_{01})(\gamma_{10}) &= G(y_{01}) \circ \widehat{\mathfrak{m}}_1(\gamma_{10}) + \partial_{\mathcal{A}\otimes\mathcal{A}} \circ G(y_{01})(\gamma_{10}) \\ &= G(y_{01}) \circ \widehat{\mathfrak{m}}_1(\gamma_{10}) + 0 \\ &= G(y_{01})(\boldsymbol{\Delta}_1(\gamma_{10}) + \gamma x_{01} + x_{01}\gamma) \\ &= G(y_{01})(\gamma x_{01} + x_{01}\gamma) = \gamma \otimes 1 + 1 \otimes \gamma. \end{aligned}$$

And finally,

$$\begin{aligned} D \circ G(y_{01})(x_{01}) &= G(y_{01}) \circ \widehat{\mathfrak{m}}_1(\gamma_{10}) + \partial_{\mathcal{A}\otimes\mathcal{A}} \circ G(y_{01})(x_{01}) \\ &= 0 + \partial_{\mathcal{A}\otimes\mathcal{A}}(1 \otimes 1) = 0. \end{aligned}$$

Observe that G admits an inverse $G^{-1}\colon \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A}) \to \check{C}_-(\Lambda_0,\Lambda_1)$ which maps a bimodule morphism $\phi \in \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})$ given by

$$\phi(\gamma_{10}) = \sum_{j=1}^{k_\gamma} \boldsymbol{w}_\gamma^j \otimes \boldsymbol{v}_\gamma^j, \quad \phi(x_{01}) = \sum_{j=1}^{k_x} \boldsymbol{w}_x^j \otimes \boldsymbol{v}_x^j$$

to $G^{-1}(\phi) = \sum_{\gamma\in\mathcal{R}(\Lambda)} \sum_{j=1}^{k_\gamma} \boldsymbol{w}_\gamma^j \gamma_{01} \boldsymbol{v}_\gamma^j + \sum_{j=1}^{k_x} \boldsymbol{w}_x^j y_{01} \boldsymbol{v}_x^j$. □

Proof of Theorem 6.1 From the proof of Proposition 6.7 we obtained that $\mathcal{A}$ is homologically smooth with $\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]$ a finite-dimensional semifree resolution. Consider the shifted Calabi–Yau map

$$\mathcal{CY} := \mathrm{CY}[-n-1]\colon \widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1] \to \check{C}_-(\Lambda_0,\Lambda_1)[-n-1].$$

If Λ_0 is horizontally displaceable then CY is a quasi-isomorphism. Moreover, $\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1]$ is quasi-isomorphic to $\mathcal{A}$ by the map F and the target $\check{C}_-(\Lambda_0,\Lambda_1)[-n-1]$ is quasi-isomorphic to $\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})[-n-1]$ by the map $G[-n-1]$. Remember also that we have $\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})[-n-1] \cong \mathrm{RHom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A}, \mathcal{A}\otimes\mathcal{A})[-n-1]$, so we get a quasi-isomorphism $\mathcal{A} \to \mathcal{A}^![-n-1]$. Finally, we need to check that $\mathcal{CY} = \mathcal{CY}^![-n-1]$. Note that

$$\mathcal{CY}^!\colon \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\check{C}_-(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A}) \to \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})$$

is by definition given by $\mathcal{CY}^!(\phi) = \phi \circ \mathcal{CY}$. Observe that the target of $\mathcal{CY}^!$ is quasi-isomorphic to $\check{C}_-(\Lambda_0,\Lambda_1)$ via the map G^{-1}, while its domain is quasi-isomorphic to $\widehat{C}_+(\Lambda_0,\Lambda_1)$ via the DG bimodule map

$$H\colon \widehat{C}_+(\Lambda_0,\Lambda_1) \to \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\check{C}_-(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})$$

defined on generators by $H(\gamma_{10})(\gamma_{01}) = 1\otimes 1$ and $H(\gamma_{10})$ vanishes on other generators of $\check{C}_-(\Lambda_0,\Lambda_1)$; and $H(x_{01})(y_{01}) = 1\otimes 1$ and vanishes otherwise. Indeed, by a similar argument as for Proposition 6.9, we prove that H is a quasi-isomorphism. First, we check that $H\circ\widehat{\mathfrak{m}}_1 + D\circ H = 0$ by proving that this equation holds when applied to any generator of $\widehat{C}_+(\Lambda_0,\Lambda_1)$. Let γ_{10} be a long chord generator, we prove that

$$H \circ \widehat{\mathfrak{m}}_1(\gamma_{10}) + H(\gamma_{10}) \circ \check{\mathfrak{m}}_1 + \partial_{\mathcal{A}\otimes\mathcal{A}} \circ H(\gamma_{10}) = 0. \tag{9}$$

Notice that the left-hand side is a bimodule map in $\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\check{C}_-(\Lambda_0,\Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})$ so we check that it vanishes when applied to any generator of $\check{C}_-(\Lambda_0,\Lambda_1)$.

(a) For the generator γ_{01} corresponding to γ_{10} we have $H\circ\widehat{\mathfrak{m}}_1(\gamma_{10})(\gamma_{01}) = H(\gamma_{10})\circ\check{\mathfrak{m}}_1(\gamma_{01}) = 0$ because γ_{10} never appears as an output of $\widehat{\mathfrak{m}}_1(\gamma_{10})$, and γ_{01} never appears as an output of $\check{\mathfrak{m}}_1(\gamma_{01})$. Moreover, $\partial_{\mathcal{A}\otimes\mathcal{A}} \circ H(\gamma_{10})(\gamma_{01}) = \partial_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) = 0$ and thus we get (9).

(b) For a generator β_{01} which is a long chord not equal to γ_{01} we have $\partial_{\mathcal{A}\otimes\mathcal{A}} \circ H(\gamma_{10})(\beta_{01}) = 0$. Then we can prove $H \circ \widehat{\mathfrak{m}}_1(\gamma_{10})(\beta_{01}) + H(\gamma_{10}) \circ \check{\mathfrak{m}}_1(\beta_{01}) = 0$ by following the same argument as in the proof of Proposition 6.9 considering pseudoholomorphic discs as in Figure 3 but exchanging β_{10} with γ_{10} and β_{01} with γ_{01}.

(c) Finally for the generator y_{01} we have first $\partial_{\mathcal{A}\otimes\mathcal{A}}(\gamma_{10})(y_{01}) = 0$. Then by definition $\widehat{\mathfrak{m}}_1(\gamma_{10}) = \boldsymbol{\Delta}_1(\gamma_{10}) + \gamma x_{01} + x_{01}\gamma$ so

$$H \circ \widehat{\mathfrak{m}}_1(\gamma_{10})(y_{01}) = H(\boldsymbol{\Delta}_1(\gamma_{10}))(y_{01}) + H(\gamma x_{01} + x_{01}\gamma)(y_{01}) = 0 + \gamma \otimes 1 + 1 \otimes \gamma$$

For the other term, we have $\check{\mathfrak{m}}_1(y_{01}) = \sum_{\gamma\in\mathcal{R}(\Lambda)} \gamma\gamma_{01} + \gamma_{01}\gamma$ and thus $H(\gamma_{10}) \circ \check{\mathfrak{m}}_1(y_{01}) = 1\otimes\gamma + \gamma\otimes 1$, which gives (9).

It remains to prove that $H \circ \widehat{\mathfrak{m}}_1(x_{01}) + D \circ H(x_{01}) = 0$. First, for a long chord β_{01} generator of $\check{C}_-(\Lambda_0, \Lambda_1)$ we have $H \circ \widehat{\mathfrak{m}}_1(x_{01})(\beta_{01}) = \partial_{\mathcal{A}\otimes\mathcal{A}} \circ H(x_{01})(\beta_{01}) = 0$ (recall $\widehat{\mathfrak{m}}_1(x_{01}) = 0$), and $H(x_{01}) \circ \check{\mathfrak{m}}_1(\beta_{01}) = 0$ because y_{01} is never an output of $\check{\mathfrak{m}}_1(\beta_{01})$ for action reasons. Finally we have $H \circ \widehat{\mathfrak{m}}_1(x_{01})(y_{01}) = 0$, $H(x_{01}) \circ \check{\mathfrak{m}}_1(y_{01}) = 0$ as y_{01} is never an output of $\check{\mathfrak{m}}_1(y_{01})$ and $\partial_{\mathcal{A}\otimes\mathcal{A}} \circ H(x_{01})(y_{01}) = \partial_{\mathcal{A}\otimes\mathcal{A}}(1\otimes 1) = 0$.

The map H admits an inverse that is defined similarly as the inverse of G so we skip writing the formula.

Thus we get a DG bimodule map

$$G^{-1} \circ \mathcal{CY}^! \circ H : \widehat{C}_+(\Lambda_0, \Lambda_1) \to \check{C}_-(\Lambda_0, \Lambda_1).$$

We compute this map on generators:

$$G^{-1} \circ \mathcal{CY}^! \circ H(\gamma_{10}) = G^{-1} \circ H(\gamma_{10}) \circ \mathcal{CY}.$$

And $H(\gamma_{10}) \circ \mathcal{CY} \in \mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\widehat{C}_+(\Lambda_0, \Lambda_1)[-n-1], \mathcal{A}\otimes\mathcal{A})$ is given by a count of bananas or strips with a positive asymptotic at γ_{01}, ie

$$\begin{aligned} H(\gamma_{10}) \circ \mathcal{CY}(\beta_{10}) &= H(\gamma_{10})\Bigg(\sum_{\xi_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\cup\{y_{01}\}} \sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(\xi_{01}; \boldsymbol{\delta}_0, \beta_{10}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1\, \xi_{01}\, \boldsymbol{\delta}_0\Bigg) \\ &= \sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(\gamma_{01}; \boldsymbol{\delta}_0, \beta_{10}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1 \otimes \boldsymbol{\delta}_0 \end{aligned}$$

and

$$H(\gamma_{10}) \circ \mathcal{CY}(x_{01}) = \sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(\gamma_{01}; \boldsymbol{\delta}_0, x_{01}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1 \otimes \boldsymbol{\delta}_0 .$$

So we get

$$G^{-1} \circ \mathcal{CY}^! \circ H(\gamma_{10}) = \sum_{\beta_{10},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(\gamma_{01}; \boldsymbol{\delta}_0, \beta_{10}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1\, \beta_{01}\, \boldsymbol{\delta}_0 + \sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(\gamma_{01}; \boldsymbol{\delta}_0, x_{01}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1\, y_{01}\, \boldsymbol{\delta}_0$$

while

$$\mathcal{CY}(\gamma_{10}) = \sum_{\substack{\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1}} \#\mathcal{M}^0(\beta_{01}; \boldsymbol{\delta}_0, \gamma_{10}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1\, \beta_{01}\, \boldsymbol{\delta}_0 + \sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1} \#\mathcal{M}^0(y_{01}; \boldsymbol{\delta}_0, \gamma_{10}, \boldsymbol{\delta}_1)\, \boldsymbol{\delta}_1\, y_{01}\, \boldsymbol{\delta}_0 .$$

Similarly we can compute

$$G^{-1}\circ\mathcal{CY}^{!}\circ H(x_{01})=\sum_{\beta_{10},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0(y_{01};\boldsymbol{\delta}_0,\beta_{10},\boldsymbol{\delta}_1)\,\boldsymbol{\delta}_1\,\beta_{01}\,\boldsymbol{\delta}_0$$

while

$$\mathcal{CY}(x_{01})=\sum\#\mathcal{M}^0(\beta_{01};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1)\,\boldsymbol{\delta}_1\,\beta_{01}\,\boldsymbol{\delta}_0\,.$$

So in order to get $\mathcal{CY}=(G^{-1}\circ\mathcal{CY}^{!}\circ H)[-n-1]:=G^{-1}[-n-1]\circ\mathcal{CY}[-n-1]\circ H[-n-1]$ we need to check that:

(a) For any $\gamma_{10}\in\mathcal{R}(\Lambda_1,\Lambda_0)$, $\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)$, and their corresponding $\gamma_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)$ and $\beta_{10}\in\mathcal{R}(\Lambda_1,\Lambda_0)$, we have

$$\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0(\beta_{01};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)=\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0(\gamma_{01};\boldsymbol{\delta}_0,\beta_{10},\boldsymbol{\delta}_1).$$

(b) For any $\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)$ and its corresponding $\beta_{10}\in\mathcal{R}(\Lambda_1,\Lambda_0)$, we have

$$\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0(\beta_{01};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1)=\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0(y_{01};\boldsymbol{\delta}_0,\beta_{10},\boldsymbol{\delta}_1).$$

The point (a) follows from [18, Theorem 3.6]. Indeed the count of rigid bananas with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1)$ and positive asymptotics at β_{10} and γ_{01} is in bijective correspondence with the count of rigid strips with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1^A)$, where Λ_1^A is a translation of Λ_1 in the positive Reeb direction by $A\gg 0$ such that the only mixed Reeb chords are from Λ_0 to Λ_1^A, with a positive asymptotic at β_{10}^A and a negative asymptotic at γ_{10}^A, where β_{10}^A and γ_{10}^A are the mixed chords of $\Lambda_0\cup\Lambda_1^A$ corresponding to β_{10} and γ_{01}, respectively. By [18, Theorem 3.6], this count of rigid strips with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1^A)$ corresponds to the count of strips with boundary on $\mathbb{R}\times\Lambda$ with a positive asymptotic at β (the pure chord corresponding to β_{10}) and a negative asymptotic at γ. Similarly, the count of rigid bananas with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1)$ and positive asymptotics at β_{01} and γ_{10} is in bijective correspondence with the count of rigid strips with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1^{-A})$ with a positive asymptotic at β_{01}^{-A} and negative asymptotic at γ_{01}^{-A}. This last count is also in bijective correspondence with the count of rigid strips with boundary on $\mathbb{R}\times\Lambda$, positively asymptotic to β and negatively asymptotic to γ.

For (b), by [18, Theorem 3.6] a rigid strip with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1)$, a positive asymptotic at β_{01} and a negative asymptotic at x_{01} corresponds to a generalized disc consisting of a disc with boundary on $\mathbb{R}\times\Lambda$ having a positive asymptotic at β together with a negative gradient flow line of f from the minimum (remember that the maximum Reeb chord $x_{01}\in\mathcal{R}(\Lambda_0,\Lambda_1)$ corresponds to the minimum critical point x of f) to a point on the boundary of the disc. But there is no nonconstant negative gradient flow line flowing from the minimum so the boundary of the disc has to pass by x (more precisely the boundary crosses $\mathbb{R}\times\{x\}$ as we are in the cylindrical setting). On the other side, a rigid banana with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1)$ and positively asymptotic to β_{10} and y_{01} corresponds to a rigid generalized disc consisting of a disc with boundary on $\mathbb{R}\times\Lambda$ positively asymptotic to β together with a negative gradient flow line of f from a point on the boundary of the disc to the maximum critical point y. Again, this flow line must be constant and the boundary of the disc must pass through $\mathbb{R}\times\{y\}$. Now we check that the count of these two types of discs is the same.

If $\dim\Lambda=1$, assume $\mathcal{M}^0(\beta_{01};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1)$ is not empty, ie assume that there is a pseudoholomorphic disc with boundary on $\mathbb{R}\times\Lambda$ passing through $\mathbb{R}\times\{x\}$, positively asymptotic to a chord β and negatively asymptotic to the words $\boldsymbol{\delta}_0,\boldsymbol{\delta}_1$. The boundary of the disc is transverse to $\mathbb{R}\times\{x\}$. In particular, it passes also through all $\mathbb{R}\times\{\mathrm{pt}\}$ for every points pt sufficiently close to x on Λ. If the function f is chosen so that its critical points are sufficiently close to each other, we get the equality in (b).

If $\dim\Lambda\geq 2$, we use the results in [12]. Denote by $\mathcal{M}_\Lambda^{\{*\}}(\beta;\boldsymbol{\delta}_0,\boldsymbol{\delta}_1)$ the moduli space of pseudoholomorphic discs with boundary on $\mathbb{R}\times\Lambda$, positively asymptotic to β, negatively asymptotic to the words $\boldsymbol{\delta}_0$ and $\boldsymbol{\delta}_1$, and having a marked point $*$ on the boundary of the disc in the domain which is situated between the puncture mapped to the last Reeb chord of the word $\boldsymbol{\delta}_0$ and the puncture mapped to the first Reeb chord of the word $\boldsymbol{\delta}_1$. There is still an action of $\mathbb{R}$ on this moduli space. There is a smooth evaluation map

$$ev\colon \mathcal{M}_\Lambda^{\{*\}}(\beta;\boldsymbol{\delta}_0,\boldsymbol{\delta}_1)\to\Lambda$$

given by $ev(u)=u(*)$. The evaluation map takes values in Λ (instead of the general $\mathbb{R}\times\Lambda$) as we are in the cylindrical setting. By a generalization of [30, Chapter 3], every point of Λ is a regular value of the evaluation map, and so are in particular the minimum and maximum Morse critical points x and y. Now, using the transversality results in [12, Section 8], we have that the evaluation map is proper and thus $\#ev^{-1}(x)=\#ev^{-1}(y)$. Finally, the 0-dimensional moduli spaces $\mathcal{M}_\Lambda^0(\beta_{01};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1)$ and $\mathcal{M}_\Lambda^0(y_{01};\boldsymbol{\delta}_0,\beta_{10},\boldsymbol{\delta}_1)$ are, respectively, identified with $ev^{-1}(x)$ and $ev^{-1}(y)$. □

6.5 Relation to Sabloff's duality

Sabloff [33] proved that linearized Legendrian contact homology and cohomology of a Legendrian knot satisfy some duality. It was then generalized to all connected Legendrians in $P\times\mathbb{R}$ in [18]. Given an augmentation $\varepsilon\colon\mathcal{A}\to\mathbb{Z}_2$ of the C-E algebra of a Legendrian sphere of dimension n, its linearized Legendrian contact homology and cohomology fit into a duality exact sequence which gives

$$\begin{aligned}\mathrm{LCH}_n^\varepsilon(\Lambda)&\simeq\mathrm{LCH}_\varepsilon^{-1}(\Lambda)\oplus\langle[\Lambda]\rangle,\\ \mathrm{LCH}_\varepsilon^n(\Lambda)&\simeq\mathrm{LCH}_{-1}^\varepsilon(\Lambda)\oplus\langle[\mathrm{pt}]\rangle,\\ \mathrm{LCH}_k^\varepsilon(\Lambda)&\simeq\mathrm{LCH}_\varepsilon^{n-k-1}(\Lambda)\quad\text{for } k\neq -1,n,\end{aligned}$$

where $[\Lambda]$ corresponds to the fundamental class in $H_n(\Lambda)$, [pt] corresponds to the point class in $H_0(\Lambda)$ and the grading is the usual grading of the C-E algebra $|\gamma|=\mathrm{CZ}(\gamma)-1$.

The Calabi–Yau structure we established on the C-E algebra recovers Sabloff duality in the following way. First, for the C-E algebra $\mathcal{A}$ of Λ, define $\mathcal{A}^e=\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$. Given an augmentation ε of $\mathcal{A}$ as above, we can see our coefficient field $\boldsymbol{k}$, which is equal to $\mathbb{Z}_2$, as a DG left $\mathcal{A}^e$-module $\boldsymbol{k}_\varepsilon$ whose structure map is given by

$$\mathcal{A}^e\otimes\boldsymbol{k}_\varepsilon\to\boldsymbol{k}_\varepsilon,\quad (\boldsymbol{a},\boldsymbol{b})\otimes m\mapsto\varepsilon(\boldsymbol{a})m\varepsilon(\boldsymbol{b}),$$

and differential vanishes. Observe now that the DG $\mathcal{A}$-bimodules $\widehat{C}_+(\Lambda_0,\Lambda_1)$ and $\check{C}_-(\Lambda_0,\Lambda_1)$ can be viewed as right DG $\mathcal{A}^e$-modules with module structure given by $c\cdot(\boldsymbol{a},\boldsymbol{b})=\boldsymbol{b}c\boldsymbol{a}$. By taking the tensor product of $\widehat{C}_+(\Lambda_0,\Lambda_1)$, resp. $\check{C}_-(\Lambda_0,\Lambda_1)$, with $\boldsymbol{k}_\varepsilon$, the Calabi–Yau structure on $\mathcal{A}$ induces a quasi-isomorphism of chain complexes

$$\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon \xrightarrow{\simeq} \check{C}_-(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon. \tag{10}$$

On the left-hand side, the differential which we denote by $\widehat{\mathfrak{m}}_1^\varepsilon$ is given by

$$\widehat{\mathfrak{m}}_1^\varepsilon(\gamma_{10})=\sum_{\beta_{10},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{10};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\cdot\beta_{10}\varepsilon(\boldsymbol{\delta}_0)\varepsilon(\boldsymbol{\delta}_1).$$

There is no term with output x_{01} because the two bananas having positive asymptotics at x_{01} and γ_{10} give the same contribution ($x_{01}\varepsilon(\gamma)+\varepsilon(\gamma)x_{01}=0$). Then $\widehat{\mathfrak{m}}_1^\varepsilon(x_{01})=0$. Observe that $\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon$ is the direct sum of two subcomplexes: one generated by long chords (all mixed chords from Λ_0 to Λ_1), and the other generated by x_{01} having trivial differential. Moreover, the complex generated by long chords and with differential $\widehat{\mathfrak{m}}_1^\varepsilon$ is quasi-isomorphic to the linearized Legendrian contact homology of Λ, $\mathrm{LCH}^\varepsilon_{n-*}(\Lambda)$; see [18] (a cycle consisting only of long chords of $\widehat{C}_+$-grading $*$ in $\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon$ corresponds to a cycle in $\mathrm{LCH}^\varepsilon_{n-*}(\Lambda)$ which is graded with the standard C-E grading). So

$$\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon\simeq\mathrm{LCH}^\varepsilon_{n-*}(\Lambda)\oplus\langle x_{01}\rangle.$$

Remember also that $\mathrm{CZ}(x_{01})=n$ so the standard C-E grading for x_{01} is $n-1$ while its $\widehat{C}$-grading is $n+1$. Let us turn to the right-hand side of (10) now. The differential $\check{\mathfrak{m}}_1^\varepsilon$ is given by

$$\check{\mathfrak{m}}_1^\varepsilon(\gamma_{01})=\sum_{\beta_{01},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1)\cdot\beta_{01}\varepsilon(\boldsymbol{\delta}_0)\varepsilon(\boldsymbol{\delta}_1)$$

and $\check{\mathfrak{m}}_1^\varepsilon(y_{01})=\sum_{\gamma\in\mathcal{R}(\Lambda)}\gamma_{01}\varepsilon(\gamma)+\varepsilon(\gamma)\gamma_{01}=0$. Again we have a direct sum of two subcomplexes: one generated by long Reeb chords from Λ_1 to Λ_0 and quasi-isomorphic to the linearized Legendrian contact cohomology $\mathrm{LCH}_\varepsilon^{*-1}(\Lambda)$ [18], and the other generated by y_{01} and with trivial differential. So

$$\check{C}_-(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\boldsymbol{k}_\varepsilon\simeq\mathrm{LCH}_\varepsilon^{*-1}(\Lambda)\oplus\langle y_{01}\rangle.$$

We get thus a quasi-isomorphism

$$\mathrm{LCH}^\varepsilon_{n-*}(\Lambda)\oplus\langle x_{01}\rangle\xrightarrow{\simeq}\mathrm{LCH}_\varepsilon^{*-1}(\Lambda)\oplus\langle y_{01}\rangle,$$

which gives $\mathrm{LCH}^\varepsilon_n(\Lambda)\simeq\mathrm{LCH}_\varepsilon^{-1}(\Lambda)\oplus\langle y_{01}\rangle$ (remember that $\mathrm{CZ}(y_{01})=0$ so the standard C-E algebra degree of y_{01} is -1). On the other side, cycles in $\mathrm{LCH}_\varepsilon^n(\Lambda)$ come from cycles in $\mathrm{LCH}^\varepsilon_{-1}(\Lambda)$ but also from x_{01} under the Calabi–Yau map CY. Finally when $*\neq 0,n+1$ we have $\mathrm{LCH}^\varepsilon_{n-*}(\Lambda)\simeq\mathrm{LCH}_\varepsilon^{*-1}(\Lambda)$ so we have recovered Sabloff duality.

7 Products and higher-order structure maps

In this section we define chain complexes $\check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ and $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ over $\mathbb{Z}_2$, obtained from $\check{C}_-(\Lambda_0,\Lambda_1)$ and $\widehat{C}_+(\Lambda_0,\Lambda_1)$ by the bimodule tensor product with the diagonal bimodule $\mathcal{A}$. We show that these complexes admit product structures and that the Calabi–Yau morphism (2) induced on them preserves the products in homology.

Notation 7.1 In the following we will consider Reeb chords with boundary on a 3-copy (and even more) of a Legendrian. Previously we wrote γ_{01} for a Reeb chord from Λ_1 to Λ_0 and γ_{10} for the corresponding Reeb chords from Λ_0 to Λ_1. Unless specified, this "correspondence" doesn't apply anymore in this section, that is, we will denote Reeb chords of the 3-copy by γ_{ij} with $1\le i\neq j\le 2$, but γ_{ij} is not necessarily the chord from Λ_j to Λ_i which corresponds to γ_{ji}, unless specified. But we will still use x_{ij} and y_{ij} to, respectively, denote maximum and minimum Morse Reeb chords between different 2-copies.

7.1 Chain complexes for Hochschild homology and cohomology

We start by describing the chain complex $(\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1),\widehat{\mathfrak{m}}_1)$, where by abuse of notation we also denote by $\widehat{\mathfrak{m}}_1$ the differential but this should not create any confusion. The vector space $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ is infinite-dimensional, generated over $\mathbb{Z}_2$ by elements of the form $\gamma_{10}\boldsymbol{a}$ and $x_{01}\boldsymbol{a}$ where $\boldsymbol{a}=a_1\dots a_k$ denotes a word of Reeb chords of Λ. The differential is given by

$$\widehat{\mathfrak{m}}_1(\gamma_{10}\boldsymbol{a})=\sum_{\beta_{10},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{10};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1)\cdot\beta_{10}\,\boldsymbol{\delta}_0\,\boldsymbol{a}\,\boldsymbol{\delta}_1+x_{01}\gamma\boldsymbol{a}+x_{01}\boldsymbol{a}\gamma+\sum_{j=1}^{k}\gamma_{10}a_1\dots a_{j-1}\partial_{\mathcal{A}}(a_j)a_{j+1}\dots a_k,$$

where γ is the pure Reeb chord corresponding to γ_{10}, and

$$\widehat{\mathfrak{m}}_1(x_{01}\boldsymbol{a})=\sum_{j=1}^{k}x_{01}a_1\dots a_{j-1}\partial_{\mathcal{A}}(a_j)a_{j+1}\dots a_k.$$

See Figure 4.

This complex computes the Hochschild homology of $\mathcal{A}$, by definition. Indeed, $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ is equal to the tensor product $\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\mathcal{A}$ where $\widehat{C}_+(\Lambda_0,\Lambda_1)$ is viewed as a DG right $\mathcal{A}^e$-module as in Section 6.5, and $\mathcal{A}$ is viewed as a DG left $\mathcal{A}^e$-module with $(a,b)\cdot a_1\dots a_k=aa_1\dots a_kb$. Remember moreover that $\widehat{C}_+(\Lambda_0,\Lambda_1)$ is a semifree resolution of $\mathcal{A}$, so the homology of $\widehat{C}_+(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\mathcal{A}$ is isomorphic to the Hochschild homology of $\mathcal{A}$. Similarly, the complex $(\check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1),\check{\mathfrak{m}}_1)$ is generated over $\mathbb{Z}_2$ by elements $\gamma_{01}\boldsymbol{a}$ and $y_{01}\boldsymbol{a}$, and

$$\check{\mathfrak{m}}_1(\gamma_{01}a_1\dots a_k)=\sum_{\beta_{01},\boldsymbol{\delta}_0,\boldsymbol{\delta}_1}\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1)\cdot\beta_{01}\,\boldsymbol{\delta}_0\,a_1\dots a_k\,\boldsymbol{\delta}_1+\sum_{j=1}^{k}\gamma_{01}a_1\dots a_{j-1}\partial_{\mathcal{A}}(a_j)a_{j+1}\dots a_k$$

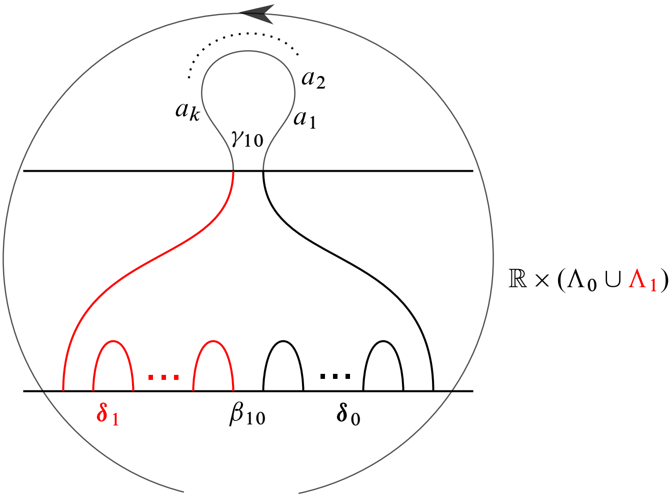

Figure 4: Pseudoholomorphic disc contributing to $\widehat{\mathfrak{m}}(\gamma_{10}a_1\dots a_k)$. Observe that the "bubble" at the top is not a pseudoholomorphic disc but a way to write the cyclic word $\gamma_{10}a_1\dots a_k$. The contribution $\beta_{10}\boldsymbol{\delta}_0 a_1\dots a_k\boldsymbol{\delta}_1$ of the disc is given by the output mixed chord β_{10} followed by a word of Reeb chords as they appear along the boundary of the disc when following it counterclockwise from β_{10}.

and

$$\check{\mathfrak{m}}_1(y_{01}\boldsymbol{a}) = \sum_{\gamma\in\mathcal{R}(\Lambda)} \gamma_{01}\gamma\boldsymbol{a} + \gamma_{01}\boldsymbol{a}\gamma + \sum_{j=1}^{k} y_{01}a_1\dots a_{j-1}\partial_{\mathcal{A}}(a_j)a_{j+1}\dots a_k,$$

where γ_{10} is the mixed chord corresponding to γ. Again, we can check that

$$\check{C}^{\mathrm{cyc}}_{-}(\Lambda_0,\Lambda_1)\simeq \check{C}_{-}(\Lambda_0,\Lambda_1)\otimes_{\mathcal{A}^e}\mathcal{A}\simeq \mathrm{RHom}_{\mathcal{A}^e}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\simeq \mathrm{RHom}_{\mathcal{A}^e}(\mathcal{A},\mathcal{A})$$

and thus the complex $(\check{C}^{\mathrm{cyc}}_{-}(\Lambda_0,\Lambda_1),\check{\mathfrak{m}}_1)$ computes the Hochschild cohomology of $\mathcal{A}$. By Theorem 6.1 we thus get an isomorphism between Hochschild homology and cohomology of the C-E algebra of a horizontally displaceable Legendrian sphere in $\mathbb{R}\times P$.

7.2 Product structures

In [29] the author defined a product structure and more generally A_∞-structure maps on the complex $\mathrm{RFC}(\Lambda_0,\Lambda_1)$ in the case when Λ_0,Λ_1 admit augmentations of their C-E algebras. When Λ_1 is a small negative push-off of Λ_0, this product extends naturally to the chain complex $\mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ (cone of the banana map induced on the cyclic model $\widehat{C}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ and with values in $\check{C}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$). This product is defined by counting the same type of pseudoholomorphic discs as the one counted to get a product on $\mathrm{RFC}(\Lambda_0,\Lambda_1)$ but keeping the negative pure Reeb chords asymptotics as coefficients instead of turning them into elements of $\mathbb{Z}_2$ with augmentations. It doesn't seem possible however to define a product directly on the $\mathcal{A}$-bimodule $\mathrm{RFC}(\Lambda_0,\Lambda_1)$, because there is no good way to deal with the

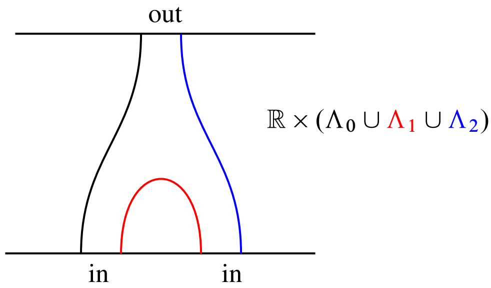

Figure 5: Pseudoholomorphic building with boundary on $\mathbb{R}\times(\Lambda_0\cup\Lambda_1\cup\Lambda_2)$ contributing to the product on $\check{C}_{-}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$.

coefficients (see [6] for constructions of A_∞-structures with coefficients in a noncommutative algebra). The pseudoholomorphic discs we count have boundary on the cylinder over a 3-copy $\Lambda_0\cup\Lambda_1\cup\Lambda_2$ of a Legendrian Λ. This 3-copy is given by $\Lambda_0=\Lambda$ and then Λ_1 and Λ_2 are small perturbed negative push-offs of Λ (Λ_2 is a slightly more negative push-off than Λ_1) such that $\Lambda_0\cup\Lambda_1$, $\Lambda_0\cup\Lambda_2$ and $\Lambda_1\cup\Lambda_2$ are 2-copies as described in Section 6.1. Observe that $\check{C}_{-}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ is a subcomplex of $\mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$, and that the restriction of the product in $\mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ to $\check{C}_{-}^{\mathrm{cyc}}$ takes values in $\check{C}_{-}^{\mathrm{cyc}}$; see [29]. So we get a well-defined product on $\check{C}_{-}^{\mathrm{cyc}}$. More precisely, given $\Lambda_0\cup\Lambda_1\cup\Lambda_2$ a 3-copy of Λ there is a degree-0 map $\check{\mathfrak{m}}_2\colon \check{C}_{-}^{\mathrm{cyc}}(\Lambda_1,\Lambda_2)\otimes\check{C}_{-}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)\to\check{C}_{-}^{\mathrm{cyc}}(\Lambda_0,\Lambda_2)$ defined by

$$\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\gamma_{01}\boldsymbol{a_0})=\sum_{\substack{\gamma_{02}\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}}\#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot\gamma_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2,$$

where γ_{ij} can also be the minimum Morse Reeb chord. And this map $\check{\mathfrak{m}}_2$ satisfies the Leibniz rule $\check{\mathfrak{m}}_1\circ\check{\mathfrak{m}}_2+\check{\mathfrak{m}}_2(\mathrm{id}\otimes\check{\mathfrak{m}}_1)+\check{\mathfrak{m}}_2(\check{\mathfrak{m}}_1\otimes\mathrm{id})=0$. This product is the standard "two negative inputs one positive output" product; see Figure 5. As for the differential, the word of pure Reeb chords in the output element is obtained by following the boundary of the pseudoholomorphic disc counterclockwise from the mixed output Reeb chord. Because of the coefficients in the C-E algebra, verifying the Leibniz rule for the product on $\check{C}_{-}^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ involves slightly more terms than for the Leibniz rule for the product on the $\mathbb{Z}_2$ vector space $C(\Lambda_0,\Lambda_1)$ as done in [29], so we detail it now. We want to prove

$$\check{\mathfrak{m}}_1\circ\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\gamma_{01}\boldsymbol{a}_0)+\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\check{\mathfrak{m}}_1(\gamma_{01}\boldsymbol{a}_0))+\check{\mathfrak{m}}_2(\check{\mathfrak{m}}_1(\gamma_{12}\boldsymbol{a}_1),\gamma_{01}\boldsymbol{a}_0)=0.$$

The left-hand side can be rewritten

$$\begin{aligned}\sum_{\substack{\gamma_{02}\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}}&\#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot\check{\mathfrak{m}}_1(\gamma_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2)\\ &\qquad+\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\check{\mathfrak{m}}_1(\gamma_{01})\boldsymbol{a}_0+\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0))+\check{\mathfrak{m}}_2(\check{\mathfrak{m}}_1(\gamma_{12})\boldsymbol{a}_1+\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0),\gamma_{01}\boldsymbol{a}_0)\\ =\sum&\#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot(\check{\mathfrak{m}}_1(\gamma_{02})\,\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2+\gamma_{02}\partial_{\mathcal{A}}(\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2))\\ &\qquad+\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\check{\mathfrak{m}}_1(\gamma_{01})\boldsymbol{a}_0)+\check{\mathfrak{m}}_2(\check{\mathfrak{m}}_1(\gamma_{12})\boldsymbol{a}_1,\gamma_{01}\boldsymbol{a}_0)\\ &\qquad+\check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0))+\check{\mathfrak{m}}_2(\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0),\gamma_{01}\boldsymbol{a}_0).\end{aligned}$$

We separate the terms having the boundary of an $\boldsymbol{a}_i$ from the other terms to get

$$\begin{aligned}= \sum \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot\big(\check{\mathfrak{m}}_1(\gamma_{02})\,\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2 + \gamma_{02}\partial_{\mathcal{A}}(\boldsymbol{\delta}_0)\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2 \\ + \boldsymbol{\delta}_0\,\boldsymbol{a}_0\partial_{\mathcal{A}}(\boldsymbol{\delta}_1)\boldsymbol{a}_1\,\boldsymbol{\delta}_2 + \boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\partial_{\mathcal{A}}(\boldsymbol{\delta}_2)\big) \\ + \check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\check{\mathfrak{m}}_1(\gamma_{01})\boldsymbol{a}_0) + \check{\mathfrak{m}}_2(\check{\mathfrak{m}}_1(\gamma_{12})\boldsymbol{a}_1,\gamma_{01}\boldsymbol{a}_0) \\ + \sum \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot(\gamma_{02}\,\boldsymbol{\delta}_0\,\partial_{\mathcal{A}}(\boldsymbol{a}_0)\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2 + \boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\partial_{\mathcal{A}}(\boldsymbol{a}_1)\,\boldsymbol{\delta}_2) \\ + \check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0)) + \check{\mathfrak{m}}_2(\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0),\gamma_{01}\boldsymbol{a}_0)\end{aligned}$$

The two first lines vanish as the algebraic contributions of pseudoholomorphic buildings in the boundary of moduli spaces of type $\mathcal{M}^1_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)$. The two last lines vanish also, because by definition of the product we have

$$\sum_{\substack{\gamma_{02}\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}} \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot\gamma_{02}\,\boldsymbol{\delta}_0\,\partial_{\mathcal{A}}(\boldsymbol{a}_0)\,\boldsymbol{\delta}_1\,\boldsymbol{a}_1\,\boldsymbol{\delta}_2 = \check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1,\gamma_{01}\partial_{\mathcal{A}}(\boldsymbol{a}_0))$$

and

$$\sum_{\substack{\gamma_{02}\\ \boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}} \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{02};\boldsymbol{\delta}_0,\gamma_{01},\boldsymbol{\delta}_1,\gamma_{12},\boldsymbol{\delta}_2)\cdot\gamma_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{a}_0\,\boldsymbol{\delta}_1\,\partial_{\mathcal{A}}(\boldsymbol{a}_1)\,\boldsymbol{\delta}_2 = \check{\mathfrak{m}}_2(\gamma_{12}\partial_{\mathcal{A}}(\boldsymbol{a}_1),\gamma_{01}\boldsymbol{a}_0),$$

and so the Leibniz rule is satisfied. Moreover, it follows from [18, Theorem 5.5] that the product $\check{\mathfrak{m}}_2$ is unital with the unit given by the minimum Morse Reeb chord, ie we have

$$\begin{aligned}\check{\mathfrak{m}}_1(y_{01}) = \check{\mathfrak{m}}_1(y_{12}) = 0,\\ \check{\mathfrak{m}}_2(\gamma_{12}\boldsymbol{a}_1, y_{01}) = \gamma_{02}\boldsymbol{a}_1,\\ \check{\mathfrak{m}}_2(y_{12},\gamma_{01}\boldsymbol{a}_0) = \gamma_{02}\boldsymbol{a}_0\end{aligned}$$

for all $\gamma_{01}\in\mathcal{R}(\Lambda_0,\Lambda_1)$, $\gamma_{12}\in\mathcal{R}(\Lambda_1,\Lambda_2)$ and $\gamma_{02}\in\mathcal{R}(\Lambda_0,\Lambda_2)$ corresponding to the *same* chord (either a long chord or a Morse chord).

Notation 7.2 In the following, we will write only γ_{01} instead of the more general $\gamma_{01}\boldsymbol{a}$ for an element in $\check{C}^{\mathrm{cyc}}_{-}(\Lambda_0,\Lambda_1)$. This is just in order to reduce a bit the length of formulas. We will also do the same for elements in $\widehat{C}^{\mathrm{cyc}}_{+}(\Lambda_0,\Lambda_1)$. In particular we'll define a product $\widehat{\mathfrak{m}}_2$ only for pairs of inputs $(\gamma_{21},\gamma_{10})$, where the inputs can also be maximum Morse Reeb chords, but keeping in mind that the rule to define more generally $\widehat{\mathfrak{m}}_2(\gamma_{21}\boldsymbol{a}_1,\gamma_{10}\boldsymbol{a}_0)$ is the same as for the product $\check{\mathfrak{m}}_2$. Namely, the output will contain the words $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ in a larger word of pure Reeb chords obtained by following the boundary of pseudoholomorphic discs involved in the definition of $\widehat{\mathfrak{m}}_2(\gamma_{21},\gamma_{10})$ counterclockwise.

So let us now construct this product on $\widehat{C}^{\mathrm{cyc}}_{+}(\Lambda_0,\Lambda_1)$. From a 3-copy of Λ we define a map

$$\widehat{\mathfrak{m}}_2\colon \widehat{C}^{\mathrm{cyc}}_{+}(\Lambda_1,\Lambda_2)\otimes\widehat{C}^{\mathrm{cyc}}_{+}(\Lambda_0,\Lambda_1)\to\widehat{C}^{\mathrm{cyc}}_{+}(\Lambda_0,\Lambda_2)$$

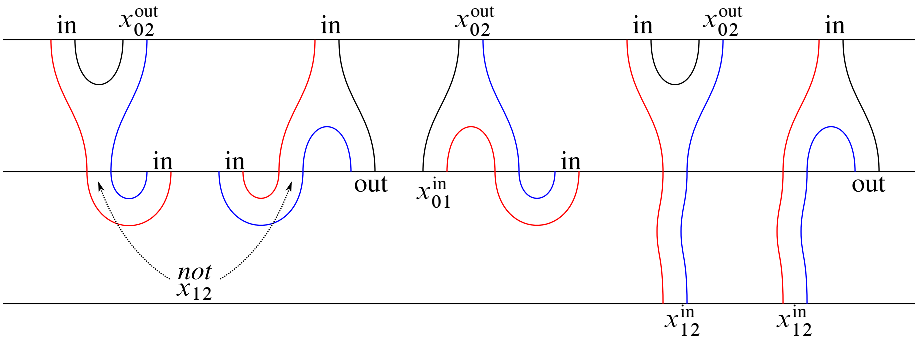

Figure 6: Pseudoholomorphic buildings contributing to the product $\widehat{\mathfrak{m}}_2$ on $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$.

by counting 2-level pseudoholomorphic buildings as shown on Figure 6. More precisely, for generators γ_{10}, x_{01} of $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$ and generators γ_{21}, x_{12} of $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_1,\Lambda_2)$ we have

$$\widehat{\mathfrak{m}}_2(\gamma_{21},\gamma_{10}) = \sum_{\beta_{12}\in\mathcal{R}^l(\Lambda_1,\Lambda_2)\cup\{y_{12}\}} \sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_1'\\ \boldsymbol{\delta}_2,\boldsymbol{\delta}_2'}} \#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\cdot\#\mathcal{M}^0_{\Lambda_{12}}(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_1'\,\boldsymbol{\delta}_2\,\boldsymbol{\delta}_2'$$
$$+\sum_{\gamma_{20}}\sum_{\beta_{12}\in\mathcal{R}^l(\Lambda_1,\Lambda_2)\cup\{y_{12}\}} \sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_1'\\ \boldsymbol{\delta}_2,\boldsymbol{\delta}_2'}} \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\#\mathcal{M}^0_{\Lambda_{12}}(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2) \cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_1'\,\boldsymbol{\delta}_2\,\boldsymbol{\delta}_2',$$

$$\widehat{\mathfrak{m}}_2(\gamma_{21},x_{01}) = \sum_{\beta_{12}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_1'\\ \boldsymbol{\delta}_2,\boldsymbol{\delta}_2'}} \#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\#\mathcal{M}^0_{\Lambda_{12}}(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_1'\,\boldsymbol{\delta}_2\,\boldsymbol{\delta}_2',$$

$$\widehat{\mathfrak{m}}_2(x_{12},\gamma_{10}) = \sum_{\beta_{12}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_1'\\ \boldsymbol{\delta}_2,\boldsymbol{\delta}_2'}} \#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\#\mathcal{M}^0_{\Lambda_{12}}(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_1'\,\boldsymbol{\delta}_2\,\boldsymbol{\delta}_2'$$
$$+\sum_{\gamma_{20},\beta_{12}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_1'\\ \boldsymbol{\delta}_2,\boldsymbol{\delta}_2'}} \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\#\mathcal{M}^0_{\Lambda_{12}}(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2)\cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_1'\,\boldsymbol{\delta}_2\,\boldsymbol{\delta}_2',$$

$$\widehat{\mathfrak{m}}_2(x_{12},x_{01}) = 0.$$

Observe that in the definition of $\widehat{\mathfrak{m}}_2(\gamma_{21},x_{01})$ and $\widehat{\mathfrak{m}}_2(x_{12},\gamma_{10})$, the "connecting" chord β_{12} will automatically be in $\mathcal{R}^l(\Lambda_1,\Lambda_2)\cup\{y_{12}\}$. In the first case, it has to be a Morse chord for action reasons, and can not be x_{12} for degree reasons. In the second case it can not be x_{12} also for degree reasons.

By [29, Proposition 2] we can check that this map $\widehat{\mathfrak{m}}_2$ is of degree 0. Then we have:

Proposition 7.3 *The map $\widehat{\mathfrak{m}}_2$ satisfies $\widehat{\mathfrak{m}}_1\circ\widehat{\mathfrak{m}}_2+\widehat{\mathfrak{m}}_2(\mathrm{id}\otimes\widehat{\mathfrak{m}}_1)+\widehat{\mathfrak{m}}_2(\widehat{\mathfrak{m}}_1\otimes\mathrm{id})=0$, ie $\widehat{\mathfrak{m}}_2$ descends to a well-defined map on homology.*

Proof We prove the proposition for each type of pair of inputs. For a pair of inputs $(\gamma_{21},\gamma_{10})$, we obtain the Leibniz rule by considering the algebraic contributions of pseudoholomorphic buildings in the

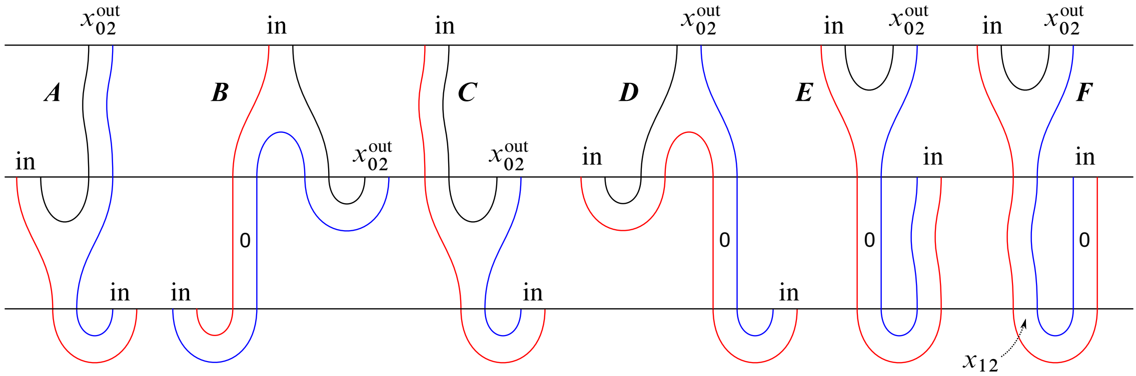

Figure 7: Pseudoholomorphic buildings in the boundary of (11) and (12).

boundary of the compactification of the following products of moduli spaces:

$$\mathcal{M}^1(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^0(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2), \tag{11}$$

$$\mathcal{M}^0(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^1(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2), \tag{12}$$

$$\mathcal{M}^1(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^0(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2), \tag{13}$$

$$\mathcal{M}^0(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^1(\beta_{12};\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2). \tag{14}$$

See Figure 7 for buildings in the boundary of the compactification of (11) and (12), and Figure 8 for those in the boundary of the compactification of (13) and (14). In each case (and in all the rest of the proof), we omit to draw the buildings which appear twice, ie both in the boundary of (11) and (12) or both in the boundary of (13) and (14), because they cancel each other over $\mathbb{Z}_2$.

According to Lemma 6.3(c), the building $\boldsymbol{A}$ either never appears or appears twice, so we can ignore it.

Consider now the building $\boldsymbol{B}$. Removing the banana containing the output x_{02} puncture gives a building contributing to the component of $\widehat{\mathfrak{m}}_2(\gamma_{21},\gamma_{10})$ taking values in the generators of $\widehat{C}_+^{\text{cyc}}(\Lambda_0,\Lambda_2)$ of the form $\beta_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2$. Adding the banana with positive asymptotic at x_{02} and using Lemma 6.3(b), we

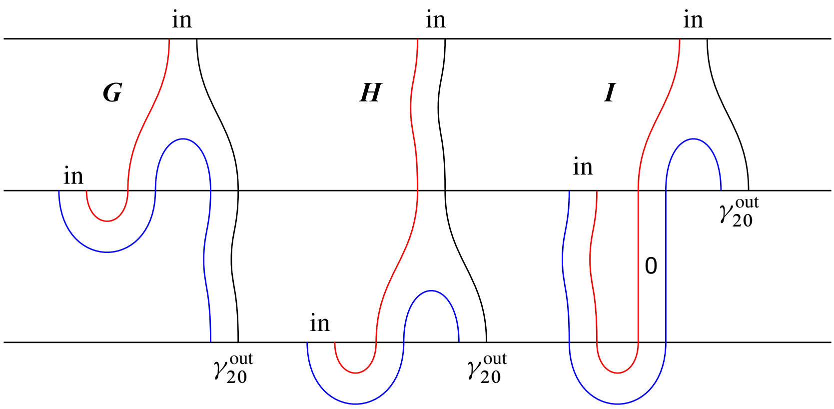

Figure 8: Pseudoholomorphic buildings in the boundary of (13) and (14).

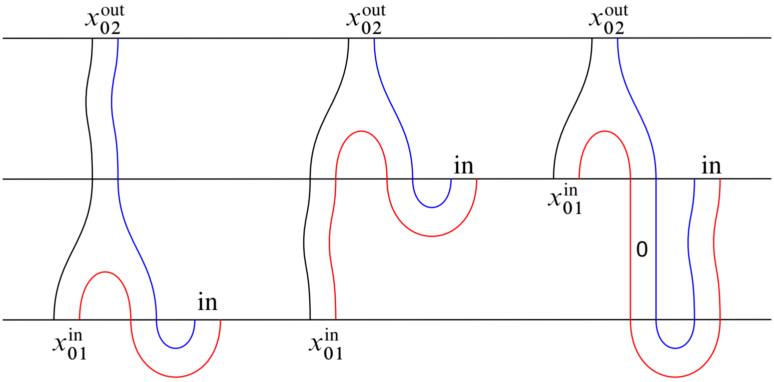

Figure 9: Pseudoholomorphic buildings in the boundary of (15) and (16).

get that the building contributes $x_{02}\beta\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2 + x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2\,\beta$, where β is the pure Reeb chord of Λ corresponding to β_{01}. Thus, this type $\boldsymbol{B}$ of buildings, together with $\boldsymbol{G}$, contribute to $\widehat{\mathfrak{m}}_1 \circ \widehat{\mathfrak{m}}_2(\gamma_{21}, \gamma_{10})$.

The buildings of type $\boldsymbol{C}$ and $\boldsymbol{D}$, together with $\boldsymbol{H}$ contribute to $\widehat{\mathfrak{m}}_2(\gamma_{21}, \widehat{\mathfrak{m}}_1(\gamma_{10}))$. Indeed the only slightly subtle thing here is about building $\boldsymbol{D}$. By assumption, the mixed connecting chord from Λ_2 to Λ_1 is not the maximum Morse Reeb chord x_{12}. But then, if such a building exists this connecting chord must be the minimum Morse chord y_{12} for action reasons, because the component of the building with three mixed asymptotics has a positive asymptotic at the Morse chord x_{02}. For degree reasons, the connecting mixed chord from Λ_1 to Λ_0 must then be x_{01} and the banana having it as a positive asymptotic contributes to the component of $\widehat{\mathfrak{m}}_1(\gamma_{01})$ taking values in $\langle x_{01}\rangle^{\mathrm{cyc}}_{\mathcal{A}\text{-}\mathcal{A}}$ (this component actually vanishes for elements $\gamma_{10}\boldsymbol{a}$ where $\boldsymbol{a} = 1$, but it doesn't in the general case).

Finally, let us consider the building $\boldsymbol{F}$. Usually, these types of buildings cancel by pairs but because of the assumption about the connecting chord, the buildings of type $\boldsymbol{F}$ arise by degenerating the banana but will never arise by degenerating the top level with three mixed asymptotic. Observe then that the buildings $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{I}$ contribute to $\widehat{\mathfrak{m}}_2(\widehat{\mathfrak{m}}_1(\gamma_{21}), \gamma_{10})$, and we have thus proved the Leibniz rule for the pair of inputs $(\gamma_{21}, \gamma_{10})$.

For a pair of inputs (γ_{21}, x_{01}), the Leibniz rule restricts to $\widehat{\mathfrak{m}}_2(\widehat{\mathfrak{m}}_1(\gamma_{21}), x_{01}) = 0$ because $\widehat{\mathfrak{m}}_1$ vanishes on the maximum Reeb chord. Let us consider the pseudoholomorphic buildings in the boundary of the compactification of the following products of moduli spaces:

$$\mathcal{M}^1(x_{02}; \boldsymbol{\delta}_0, x_{01}, \boldsymbol{\delta}_1, \beta_{12}, \boldsymbol{\delta}_2') \times \mathcal{M}^0(\beta_{12}; \boldsymbol{\delta}_1', \gamma_{21}, \boldsymbol{\delta}_2), \tag{15}$$

$$\mathcal{M}^0(x_{02}; \boldsymbol{\delta}_0, x_{01}, \boldsymbol{\delta}_1, \beta_{12}, \boldsymbol{\delta}_2') \times \mathcal{M}^1(\beta_{12}; \boldsymbol{\delta}_1', \gamma_{21}, \boldsymbol{\delta}_2). \tag{16}$$

See Figure 9 for a schematic picture of these buildings. As before, the first building can be ignored. The second building never appears either, for action reasons. The third building finally contributes to $\widehat{\mathfrak{m}}_2(\widehat{\mathfrak{m}}_1(\gamma_{21}), x_{01})$. For the pair of inputs (x_{12}, γ_{10}), the Leibniz rule restricts to

$$\widehat{\mathfrak{m}}_1 \circ \widehat{\mathfrak{m}}_2(x_{12}, \gamma_{10}) + \widehat{\mathfrak{m}}_2(x_{12}, \widehat{\mathfrak{m}}_1(\gamma_{10})) = 0.$$

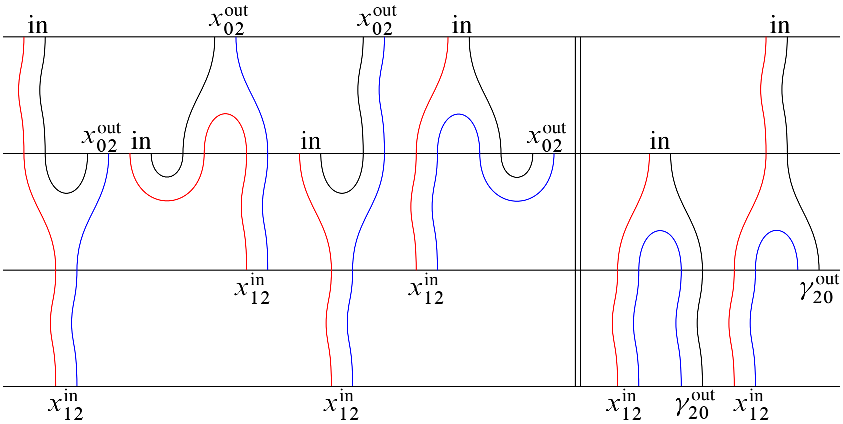

Figure 10: Left: pseudoholomorphic buildings in the boundary of (17) and (18); right: pseudo-holomorphic buildings in the boundary of (19) and (20).

We consider the boundary of the compactification of

$$\mathcal{M}^1(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^0(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2), \tag{17}$$

$$\mathcal{M}^0(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^1(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2) \tag{18}$$

and of

$$\mathcal{M}^1(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^0(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2), \tag{19}$$

$$\mathcal{M}^0(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\beta_{12},\boldsymbol{\delta}_2')\times\mathcal{M}^1(\beta_{12};\boldsymbol{\delta}_1',x_{12},\boldsymbol{\delta}_2), \tag{20}$$

whose different components are schematized in Figure 10. By arguments similar as before, the algebraic contributions of the second and third buildings vanish. The first and sixth contribute to $\widehat{\mathfrak{m}}_2(x_{12},\widehat{\mathfrak{m}}_1(\gamma_{10}))$, and the fourth and fifth to $\widehat{\mathfrak{m}}_1\circ\widehat{\mathfrak{m}}_2(x_{12},\gamma_{10})$. Finally, for a pair (x_{12},x_{01}) all the terms of the Leibniz rule vanish by definition. □

Remark 7.4 The product on $\widehat{C}_+^{\text{cyc}}(\Lambda_0,\Lambda_1)$ is defined by a count of 2-level pseudoholomorphic buildings, which arise in the boundary of the compactification of 1-dimensional moduli spaces. In particular, there is no canonical choice for the buildings we choose to count to define the product, but there is a choice up to homotopy. In particular, we could choose the product to be the map

$$\widehat{\boldsymbol{d}}_2\colon \widehat{C}_+^{\text{cyc}}(\Lambda_1,\Lambda_2)\otimes\widehat{C}_+^{\text{cyc}}(\Lambda_0,\Lambda_1)\to\widehat{C}_+^{\text{cyc}}(\Lambda_0,\Lambda_2)$$

defined by

$$\begin{aligned}\widehat{\boldsymbol{d}}_2(\gamma_{21},\gamma_{10}) &= \\ &\sum_{\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\cup\{y_{01}\}}\ \sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_0',\\ \boldsymbol{\delta}_1,\boldsymbol{\delta}_1',\boldsymbol{\delta}_2}} \#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,\beta_{01},\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2)\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0',\gamma_{10},\boldsymbol{\delta}_1)\cdot x_{02}\boldsymbol{\delta}_0\boldsymbol{\delta}_0'\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\boldsymbol{\delta}_2 \\ &+\sum_{\gamma_{20}}\ \sum_{\beta_{01}\in\mathcal{R}^l(\Lambda_0,\Lambda_1)\cup\{y_{01}\}}\ \sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}_0',\\ \boldsymbol{\delta}_1,\boldsymbol{\delta}_1',\boldsymbol{\delta}_2}} \#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{20};\boldsymbol{\delta}_0,\beta_{01},\boldsymbol{\delta}_1',\gamma_{21},\boldsymbol{\delta}_2)\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}_0',\gamma_{10},\boldsymbol{\delta}_1) \\ &\hspace{20em}\cdot\gamma_{20}\boldsymbol{\delta}_0\boldsymbol{\delta}_0'\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\boldsymbol{\delta}_2,\end{aligned}$$

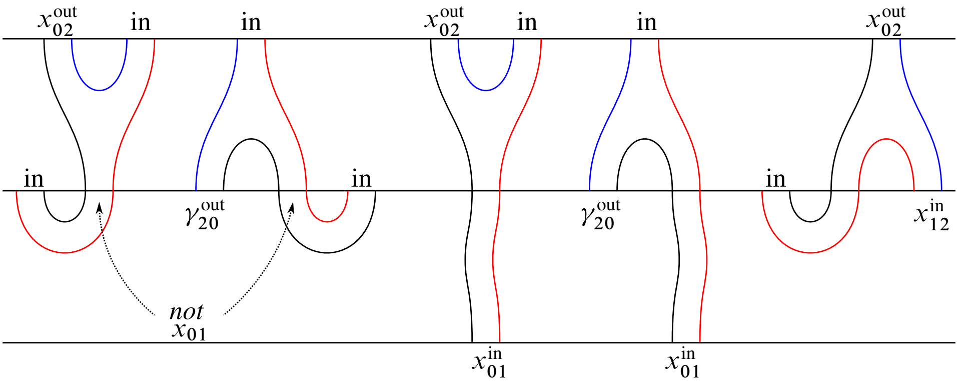

Figure 11: Pseudoholomorphic buildings contributing to $\widehat{\boldsymbol{d}}_2(x_{12},\gamma_{01})$.

$$\begin{aligned}\widehat{\boldsymbol{d}}_2(\gamma_{21},x_{01})=&\sum_{\beta_{01}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}'_0,\\ \boldsymbol{\delta}_1,\boldsymbol{\delta}'_1,\boldsymbol{\delta}_2}}\#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,\beta_{01},\boldsymbol{\delta}'_1,\gamma_{21},\boldsymbol{\delta}_2)\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}'_0,x_{01},\boldsymbol{\delta}_1)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}'_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}'_1\,\boldsymbol{\delta}_2\\ &+\sum_{\gamma_{20},\beta_{01}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}'_0,\\ \boldsymbol{\delta}_1,\boldsymbol{\delta}'_1,\boldsymbol{\delta}_2}}\#\mathcal{M}^0_{\Lambda_{012}}(\gamma_{20};\boldsymbol{\delta}_0,\beta_{01},\boldsymbol{\delta}'_1,\gamma_{21},\boldsymbol{\delta}_2)\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}'_0,x_{01},\boldsymbol{\delta}_1)\cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}'_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}'_1\,\boldsymbol{\delta}_2,\end{aligned}$$

$$\widehat{\boldsymbol{d}}_2(x_{12},\gamma_{10})=\sum_{\beta_{01}}\sum_{\substack{\boldsymbol{\delta}_0,\boldsymbol{\delta}'_0\\ \boldsymbol{\delta}_1,\boldsymbol{\delta}'_1,\boldsymbol{\delta}_2}}\#\mathcal{M}^0_{\Lambda_{012}}(x_{02};\boldsymbol{\delta}_0,\beta_{01},\boldsymbol{\delta}'_1,x_{12},\boldsymbol{\delta}_2)\#\mathcal{M}^0_{\Lambda_{01}}(\beta_{01};\boldsymbol{\delta}'_0,\gamma_{10},\boldsymbol{\delta}_1)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}'_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}'_1\,\boldsymbol{\delta}_2,$$

$$\widehat{\boldsymbol{d}}_2(x_{12},x_{01})=0.$$

See Figure 11. We show that the maps $\widehat{\boldsymbol{d}}_2$ and $\widehat{\mathfrak{m}}_2$ are homotopic. Let us define a (degree -1) map $\boldsymbol{h}\colon \widehat{C}^{\text{cyc}}_+(\Lambda_1,\Lambda_2)\otimes\widehat{C}^{\text{cyc}}_+(\Lambda_0,\Lambda_1)\to\widehat{C}^{\text{cyc}}_+(\Lambda_0,\Lambda_2)$ by

$$\begin{aligned}\boldsymbol{h}(\gamma_{21},\gamma_{10})=&\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}\#\mathcal{M}^0(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2\\ &+\sum_{\gamma_{20}}\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2}\#\mathcal{M}^0(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2)\cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2,\\ \boldsymbol{h}(\gamma_{21},x_{01})=&\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}\#\mathcal{M}^0(x_{02};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2\\ &+\sum_{\gamma_{20}}\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2}\#\mathcal{M}^0(\gamma_{20};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2)\cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2,\\ \boldsymbol{h}(x_{12},\gamma_{10})=&\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1,\boldsymbol{\delta}_2}\#\mathcal{M}^0(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,x_{12},\boldsymbol{\delta}_2)\cdot x_{02}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2\\ &+\sum_{\gamma_{20}}\sum_{\boldsymbol{\delta}_0,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2}\#\mathcal{M}^0(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,x_{12},\boldsymbol{\delta}_2)\cdot\gamma_{20}\,\boldsymbol{\delta}_0\,\boldsymbol{\delta}_1\,\boldsymbol{\delta}_2,\\ \boldsymbol{h}(x_{12},x_{01})=&\;0.\end{aligned}$$

See Figure 12.

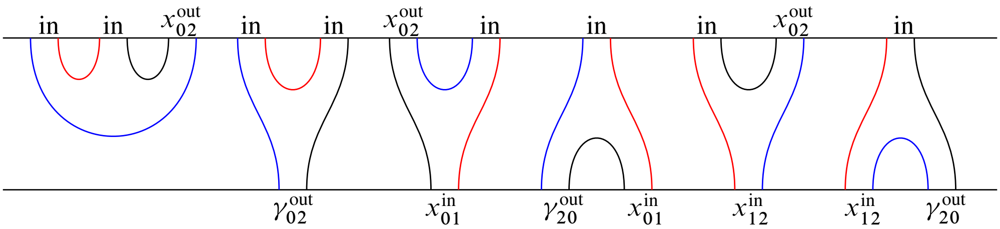

Figure 12: Pseudoholomorphic discs contributing to the map $\boldsymbol{h}$.

Now we check that

$$\widehat{\mathfrak{m}}_2 + \widehat{d}_2 = \boldsymbol{h}(\mathrm{id} \otimes \widehat{\mathfrak{m}}_1) + \boldsymbol{h}(\widehat{\mathfrak{m}}_1 \otimes \mathrm{id}) + \widehat{\mathfrak{m}}_1 \circ \boldsymbol{h} \tag{21}$$

by studying the boundary of the compactification of 1-dimensional moduli spaces of discs as the ones shown in Figure 12. Namely, in order to prove (21) for the pair of generators $(\gamma_{21}, \gamma_{10})$, we study the boundary of the compactification of

$$\mathcal{M}^1(x_{02}; \boldsymbol{\delta}_0, \gamma_{10}, \boldsymbol{\delta}_1, \gamma_{21}, \boldsymbol{\delta}_2) \tag{22}$$

and

$$\mathcal{M}^1(\gamma_{20}; \boldsymbol{\delta}_0, \gamma_{10}, \boldsymbol{\delta}_1, \gamma_{21}, \boldsymbol{\delta}_2). \tag{23}$$

The corresponding pseudoholomorphic buildings are schematized in Figures 13 and 14.

Observe that buildings of type $\boldsymbol{A}$, type $\boldsymbol{G}$ and those of type $\boldsymbol{B}$ and $\boldsymbol{H}$ where the connecting chord is x_{01} (for both types $\boldsymbol{B}$ and $\boldsymbol{H}$) are the ones contributing exactly to $\boldsymbol{h}(\gamma_{21}, \widehat{\mathfrak{m}}_1(\gamma_{10}))$. Similarly, the buildings of types $\boldsymbol{C}$ and $\boldsymbol{I}$ together with the buildings of types $\boldsymbol{D}$ and $\boldsymbol{J}$ where the connecting chord is x_{12} contribute to $\boldsymbol{h}(\widehat{\mathfrak{m}}_1(\gamma_{21}), \gamma_{10})$. The contribution of buildings of type $\boldsymbol{E}$, if such buildings exist, vanishes according to Lemma 6.3(c). The buildings of types $\boldsymbol{F}$ and $\boldsymbol{K}$ contribute to $\widehat{\mathfrak{m}}_1 \circ \boldsymbol{h}(\gamma_{21}, \gamma_{10})$. We are thus counting buildings which algebraically contribute to the right-hand side of (21) for the pair of generators $(\gamma_{21}, \gamma_{10})$. Finally, observe that the buildings of types $\boldsymbol{B}$ and $\boldsymbol{H}$, when the connecting chord is not the Morse chord x_{01}, are exactly the ones contributing to $\widehat{\boldsymbol{d}}_2(\gamma_{21}, \gamma_{10})$, and that those of types $\boldsymbol{D}$

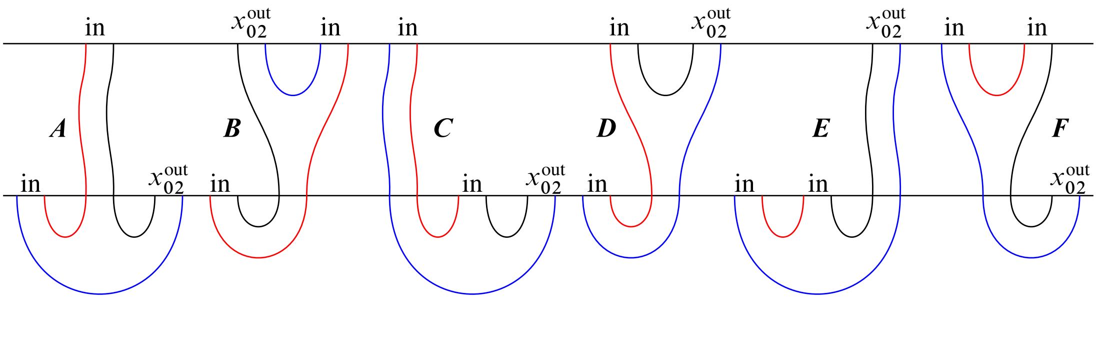

Figure 13: Pseudoholomorphic buildings in the boundary of (22).

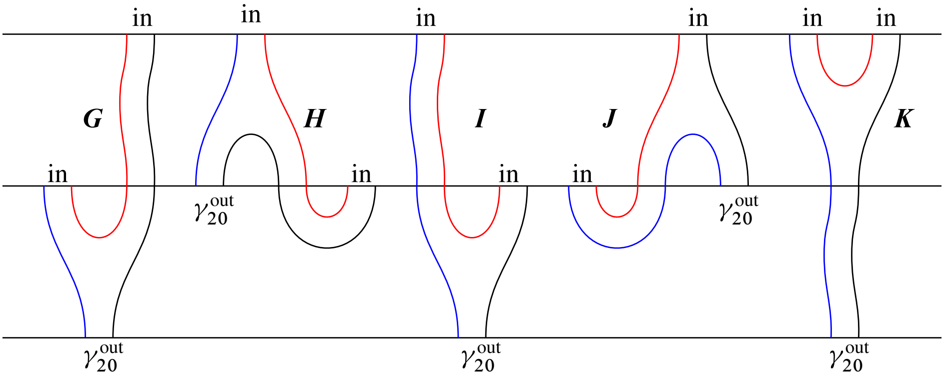

Figure 14: Pseudoholomorphic buildings in the boundary of (23).

and $\boldsymbol{J}$ with the connecting chord not being x_{21} contribute to $\widehat{\mathfrak{m}}_2(\gamma_{21},\gamma_{10})$. These give the left-hand side of (21) and thus we have proved this equation for the pair $(\gamma_{21},\gamma_{10})$.

In order to prove this equation for a pair (γ_{21},x_{01}), we consider the pseudoholomorphic buildings in the boundary of the compactification of

$$\mathcal{M}^1(x_{02};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2), \tag{24}$$

$$\mathcal{M}^1(\gamma_{20};\boldsymbol{\delta}_0,x_{01},\boldsymbol{\delta}_1,\gamma_{21},\boldsymbol{\delta}_2), \tag{25}$$

which we schematized in Figure 15.

The buildings of types $\boldsymbol{A}'$ and $\boldsymbol{F}'$ contribute to $\widehat{\boldsymbol{d}}_2(\gamma_{21},x_{01})$ and the buildings of type $\boldsymbol{C}'$ contribute to $\widehat{\mathfrak{m}}_2(\gamma_{21},x_{01})$ (observe that the connecting chord in this case cannot be x_{12}, for example, for degree reasons). This gives the left-hand side of (21). The other buildings contribute algebraically to the right-hand side. Indeed, buildings of types $\boldsymbol{B}'$and $\boldsymbol{G}'$ contribute to $\boldsymbol{h}(\widehat{\mathfrak{m}}_1(\gamma_{21}),x_{01})$, the contribution of the buildings of type $\boldsymbol{D}'$ vanishes according to Lemma 6.3(c), and buildings of types $\boldsymbol{E}'$ and $\boldsymbol{H}'$ contribute to $\widehat{\mathfrak{m}}_1\circ\boldsymbol{h}(\gamma_{21},x_{01})$. The remaining term in the relation, $\boldsymbol{h}(\gamma_{21},\widehat{\mathfrak{m}}(x_{01}))$, vanishes because $\widehat{\mathfrak{m}}_1(x_{01})=0$.

We continue by proving (21) for a pair (x_{12},γ_{10}). We study the boundary of the compactification of

$$\mathcal{M}^1(x_{02};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,x_{12},\boldsymbol{\delta}_2), \tag{26}$$

$$\mathcal{M}^1(\gamma_{20};\boldsymbol{\delta}_0,\gamma_{10},\boldsymbol{\delta}_1,x_{12},\boldsymbol{\delta}_2) \tag{27}$$

for which the pseudoholomorphic buildings are schematized in Figure 16.

Observe first that the contribution of the buildings of type $\boldsymbol{B}''$ vanishes as before. Then, buildings of type $\boldsymbol{C}''$, $\boldsymbol{D}''$ and $\boldsymbol{G}''$ contribute to $(\widehat{\mathfrak{m}}_2+\widehat{d}_2)(x_{12},\gamma_{10})$. The buildings of type $\boldsymbol{A}''$ and $\boldsymbol{F}''$ are exactly the ones contributing to $\boldsymbol{h}(x_{12},\widehat{\mathfrak{m}}_1(\gamma_{10}))$ (indeed there is also a component of $\widehat{\mathfrak{m}}_1(\gamma_{10})$ with output the Morse chord x_{01} but then $h(x_{12},x_{01})=0$ so these do not contribute). The buildings of type $\boldsymbol{E}''$ and $\boldsymbol{H}''$ are the ones contributing to $\widehat{\mathfrak{m}}_1\circ h(x_{12},\gamma_{10})$, and we get thus (21) because $\boldsymbol{h}(\widehat{\mathfrak{m}}(x_{12}),\gamma_{10})=0$.

Finally, (21) is trivially satisfied for the pair of inputs (x_{12},x_{01}) because both sides vanish.

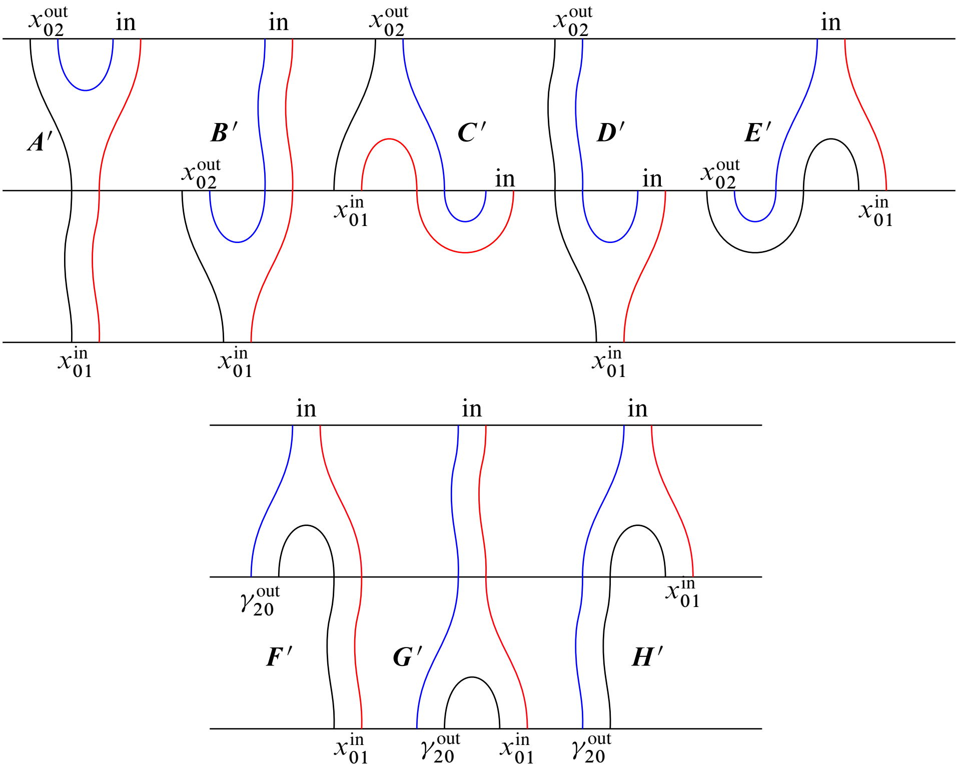

Figure 15: Pseudoholomorphic buildings in the boundary of (24) (top) and (25) (bottom).

Remark 7.5 The pseudoholomorphic buildings contributing to the component of the product $\widehat{\mathfrak{m}}_2$ with long chords as inputs and output can also be used to build a product structure on the linearized Legendrian contact *homology* complex, which will be unital when the Legendrian submanifold is horizontally displaceable.

It is well known that there is a (nonunital) product structure (even an A_∞-structure) on the linearized Legendrian contact cohomology, defined first in [11] and generalized to the bilinearized case in [1]. This product can be computed directly from the C-E algebra, and equivalently by a count of pseudoholomorphic discs with boundary on a 3-copy $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ of Λ negatively asymptotic to a chord from Λ_0 to Λ_1 and to a chord from Λ_1 to Λ_2 (the inputs), and positively asymptotic to a chord from Λ_0 to Λ_2 (the output), and potentially having pure negative asymptotics which are augmented. In this case, the mixed chords considered are never Morse chords.

In [32], for knots in $\mathbb{R}^3$ the authors consider the complex generated by mixed chords from Λ_1 to Λ_0 in a 2-copy of Λ, which they denote by $\mathrm{Hom}_+(\varepsilon_0, \varepsilon_1)$. They define then a product (as well as an A_∞-structure) on this complex by a count of similar curves as above (two negative mixed inputs and one positive mixed output) but the main difference is that in this case the Morse chords can be inputs and outputs. The homology of the complex $\mathrm{Hom}_+(\varepsilon_0, \varepsilon_1)$ is isomorphic to the Legendrian contact homology of Λ. This is implied by the acyclicity of the complex of the 2-copy which holds for knots

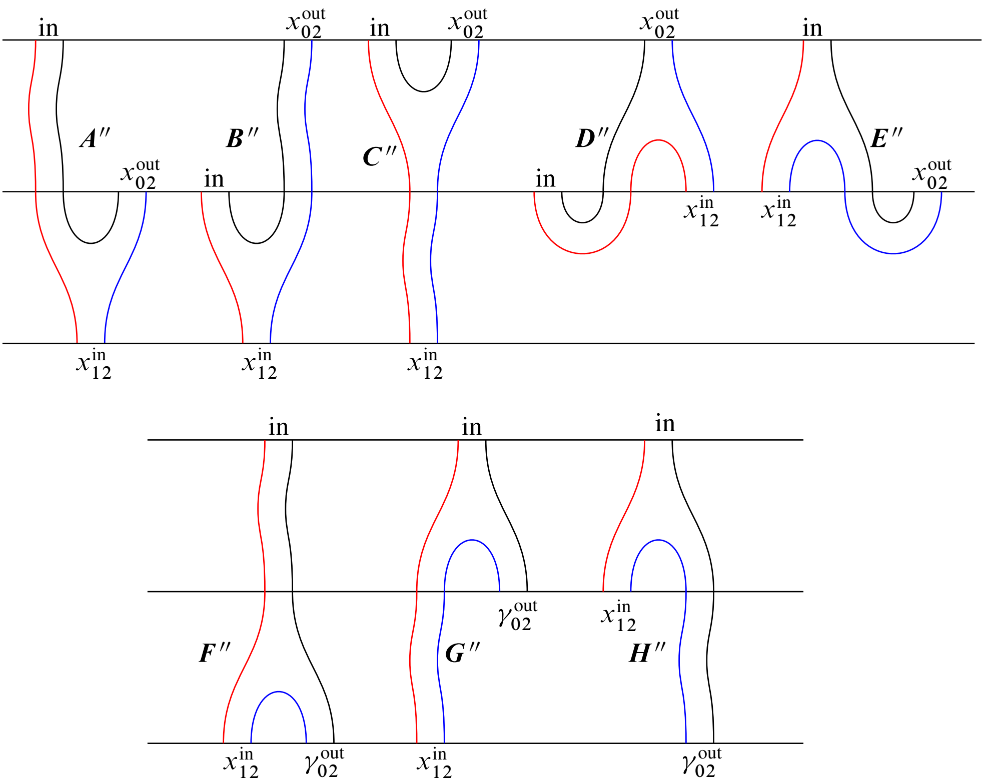

Figure 16: Pseudoholomorphic buildings in the boundary of (26) (top) and (27) (bottom).

in $\mathbb{R}^3$; see [32, Proposition 5.4]. In particular, the product on Hom_+ doesn't give canonically a product on Legendrian contact homology.

By [4, Proposition 2.7], the (bi)linearization of the DG bimodule $(C_+(\Lambda_0,\Lambda_1),\boldsymbol{\Delta}_1)$ by augmentations ε_0 and ε_1 is canonically isomorphic to the bilinearized Legendrian contact homology complex $\mathrm{LCC}_*^{\varepsilon_0,\varepsilon_1}(\Lambda)$. We claim that the only-long-chords-asymptotics component of the product $\widehat{\mathfrak{m}}_2$, when linearized by augmentations, gives a product on the Legendrian contact homology complex, the proof of this being schematized in Figure 8. In the presence of a filling of Λ and under the hypothesis of horizontal displaceability, it is possible to prove that this new product on Legendrian contact homology is isomorphic to the product on the Hom_+ complex, because both are isomorphic to the product on the singular cohomology of the filling (through the Ekholm–Seidel isomorphism [13; 16]). This will be investigated more precisely in a forthcoming paper with Georgios Dimitroglou Rizell, where we describe a *relative Calabi–Yau structure* carried by the C-E algebra. This relative structure allows in particular to show the isomorphism between the products on LCH_* and Hom_+ without the presence of a filling.

7.3 Products under the Calabi–Yau morphism

Let us consider the Calabi–Yau map CY induced on the $\mathbb{Z}_2$-modules, denoted by

$$\mathrm{CY}_1\colon \widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)\to \check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1).$$

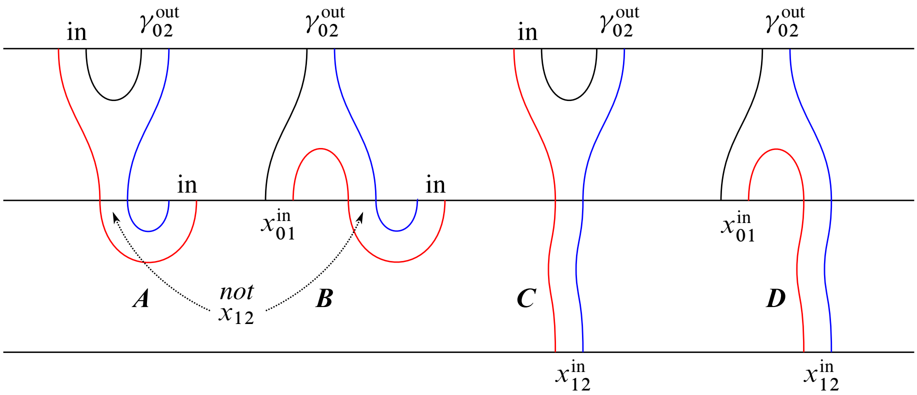

Figure 17: Pseudoholomorphic buildings contributing to the map CY_2.

Theorem 7.6 *The map* CY_1 *preserves the product structures in homology*, *ie*

$$\mathrm{CY}_1 \circ \widehat{\mathfrak{m}}_2 + \check{\mathfrak{m}}_2(\mathrm{CY}_1, \mathrm{CY}_1) = 0$$

is satisfied in homology.

Proof In this proof we will use the notation $\widehat{\mathfrak{m}}_2^+$ and $\widehat{\mathfrak{m}}_2^x$ to denote the components of $\widehat{\mathfrak{m}}_2$ with values in $C_+^{\text{cyc}}(\Lambda_0, \Lambda_1)$ and $\langle x_{01}\rangle_{\mathcal{A}\text{-}\mathcal{A}}^{\text{cyc}}$, respectively.

Given a 3-copy $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ of Λ, we define a (degree -1) map

$$\mathrm{CY}_2 \colon \widehat{C}_+^{\text{cyc}}(\Lambda_1, \Lambda_2) \otimes \widehat{C}_+^{\text{cyc}}(\Lambda_0, \Lambda_1) \to \check{C}_-^{\text{cyc}}(\Lambda_0, \Lambda_2)$$

by a count of pseudoholomorphic buildings as shown in Figure 17. Similarly as for the product $\widehat{\mathfrak{m}}_2$ we require that the connecting chord from Λ_2 to Λ_1 in the buildings is not the maximum Morse Reeb chord x_{12}. Observe however that this is automatically satisfied for action reasons for the buildings of types $\boldsymbol{C}$ and $\boldsymbol{D}$.

By considering buildings in the boundary of the compactification of one-dimensional products of moduli spaces of types $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ in Figure 17, one proves

$$\check{\mathfrak{m}}_1 \circ \mathrm{CY}_2 + \check{\mathfrak{m}}_2(\mathrm{CY}_1 \otimes \mathrm{CY}_1) + \mathrm{CY}_1 \circ \widehat{\mathfrak{m}}_2 + \mathrm{CY}_2(\mathrm{id} \otimes \widehat{\mathfrak{m}}_1) + \mathrm{CY}_2(\widehat{\mathfrak{m}}_1 \otimes \mathrm{id}) = 0, \tag{28}$$

which shows that CY_1 preserves the products in homology. In Figure 18 we depicted the different types of buildings in the boundary of the compactification of 1-dimensional products of moduli spaces of type $\boldsymbol{A}$. The buildings of type $\boldsymbol{A1}$ together with those of type $\boldsymbol{A3}$, where the connecting chord from Λ_1 to Λ_0 is x_{01}, contribute to $\mathrm{CY}_2(\gamma_{21}, \widehat{\mathfrak{m}}_1(\gamma_{10}))$. Those of type $\boldsymbol{A2}$ contribute to $\mathrm{CY}_1 \circ \widehat{\mathfrak{m}}_2^x(\gamma_{21}, \gamma_{10})$ when the connecting chord from Λ_2 to Λ_0 is x_{02}, and to $\check{\mathfrak{m}}_1 \circ \mathrm{CY}_2(\gamma_{21}, \gamma_{10})$ otherwise. The buildings of type $\boldsymbol{A3}$ when the connecting chord from Λ_1 to Λ_0 is not x_{01} contribute to $\check{\mathfrak{m}}_2(\mathrm{CY}_1(\gamma_{21}), \mathrm{CY}_1(\gamma_{10}))$. Those of type $\boldsymbol{A4}$ contribute to $\mathrm{CY}_1 \circ \widehat{\mathfrak{m}}_2^+(\gamma_{21}, \gamma_{10})$ and finally those of types $\boldsymbol{A5}$ and $\boldsymbol{A6}$ contribute to $\mathrm{CY}_2(\widehat{\mathfrak{m}}_1(\gamma_{21}), \gamma_{10})$. This gives (28) for the pair of inputs $(\gamma_{21}, \gamma_{10})$. In Figure 19 we consider the broken discs in the boundary of 1-dimensional products of moduli spaces of type $\boldsymbol{B}$. The buildings $\boldsymbol{B1}$ contribute to $\mathrm{CY}_1 \circ \widehat{\mathfrak{m}}_2^x(\gamma_{21}, x_{01})$

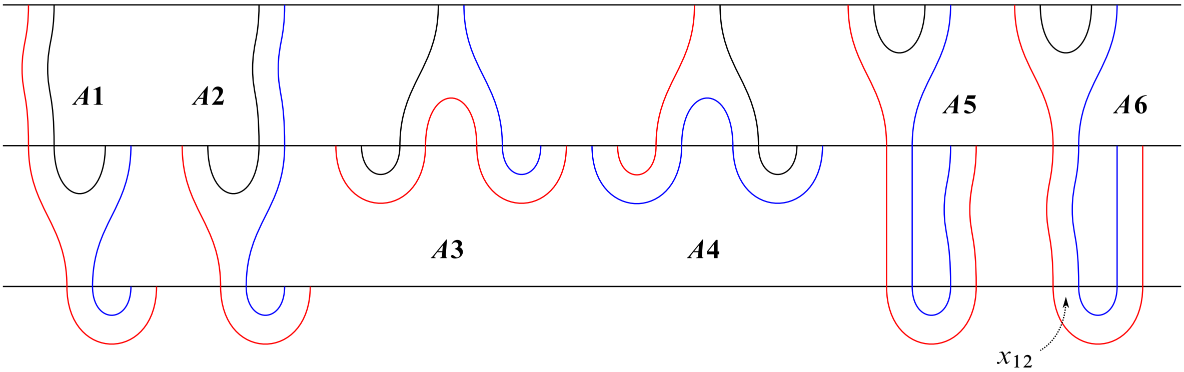

Figure 18: Pseudoholomorphic buildings in the boundary of 1-dimensional products of moduli spaces of type $\boldsymbol{A}$.

when the connecting chord from Λ_2 to Λ_0 is the maximum Morse chord x_{02}, and to $\check{\mathfrak{m}}_1 \circ \mathrm{CY}_2(\gamma_{21}, x_{01})$ otherwise. Those of type $\boldsymbol{B2}$ contribute to $\check{\mathfrak{m}}_2(\mathrm{CY}_1(\gamma_{21}), \mathrm{CY}_1(x_{01}))$ (none of the connecting chord is a maximum).The buildings of types $\boldsymbol{B3}$ and $\boldsymbol{B4}$ finally contribute to $\mathrm{CY}_2(\widehat{\mathfrak{m}}_1(\gamma_{21}), x_{01})$. The sum of these contributions gives (28) for the pair of inputs (γ_{21}, x_{01}) (observe that some terms in the relation vanish by definition). Similarly this relation can be checked for pairs of inputs (x_{12}, γ_{10}) and (x_{12}, x_{01}) by considering broken discs in the boundary of 1-dimensional products of moduli spaces of type $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively; see Figure 20. □

From Theorem 7.6, we deduce that the product $\widehat{\mathfrak{m}}_2$ has a unit in homology represented by any cycle in $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0, \Lambda_1)$ which is sent to the minimum Morse Reeb chord y_{01} by the map CY_1.

7.4 A_∞-structure

We can go further and define for each $d \geq 3$ maps $\check{\mathfrak{m}}_d$ and $\widehat{\mathfrak{m}}_d$ of degree $2-d$ by counting pseudoholomorphic discs with boundary on a $(d+1)$-copy of Λ (defined in an analogous way as the 2- and 3-copies).

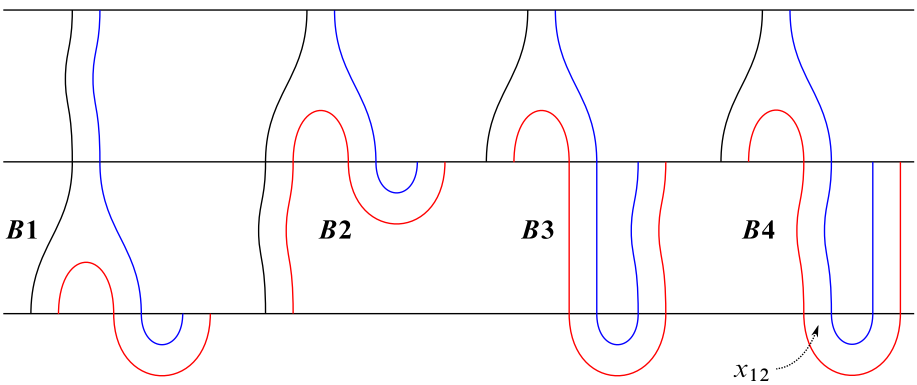

Figure 19: Pseudoholomorphic buildings in the boundary of 1-dimensional products of moduli spaces of type $\boldsymbol{B}$.

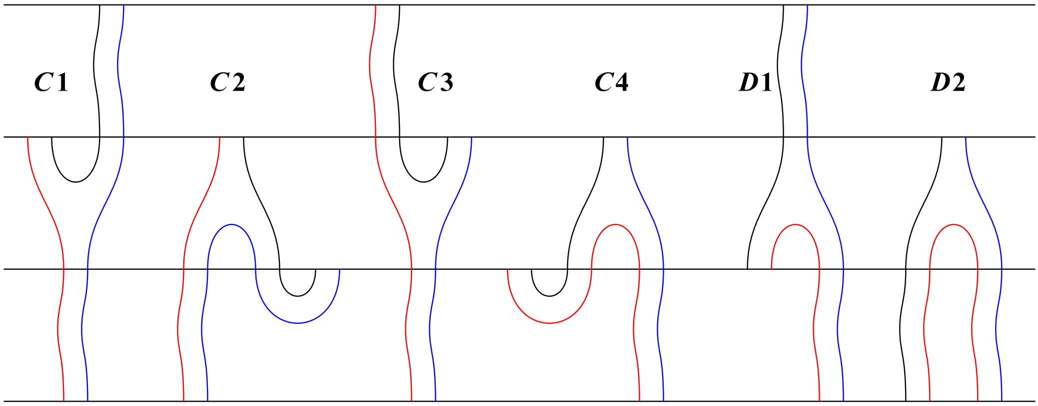

Figure 20: Pseudoholomorphic buildings in the boundary of 1-dimensional products of moduli spaces of type $\boldsymbol{C}$ and $\boldsymbol{D}$.

Given a $(d+1)$-copy $\Lambda_0 \cup \cdots \cup \Lambda_d$ of Λ, the maps

$$\check{\mathfrak{m}}_d : \check{C}^{\mathrm{cyc}}_{-}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \check{C}^{\mathrm{cyc}}_{-}(\Lambda_0, \Lambda_1) \to \check{C}^{\mathrm{cyc}}_{-}(\Lambda_0, \Lambda_d)$$

are given by a count of pseudoholomorphic discs having mixed negative asymptotics corresponding to inputs and one positive asymptotic which is the output. The map $\check{\mathfrak{m}}_d$ has degree $2-d$. The fact that these maps satisfy the A_∞-equations goes back to [1; 11]. In our case we have to take extra care of the coefficients in the C-E algebra but it works exactly the same as in the case $d=2$ we treated in Section 7.2.

Now let's define the maps $\widehat{\mathfrak{m}}_d$. First, we extend the maps $\boldsymbol{b}_1$ and $\boldsymbol{\Delta}_1$ to higher-order maps

$$\boldsymbol{b}_d, \boldsymbol{\Delta}_d : \mathrm{RFC}^{\mathrm{cyc}}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0, \Lambda_1) \to \mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0, \Lambda_d)$$

for $d \geq 1$ as follows. These maps have degree $2-d$ and were considered in [29, Section 8]; we recall the definitions. For a d-tuple of elements $(c_{d-1}\boldsymbol{a}_{d-1}, \ldots, c_0\boldsymbol{a}_0) \in \mathrm{RFC}^{\mathrm{cyc}}(\Lambda_{d-1}, \Lambda_d) \otimes \cdots \otimes \mathrm{RFC}^{\mathrm{cyc}}(\Lambda_0, \Lambda_1)$ where c_j are mixed chords and $\boldsymbol{a}_j$ words of pure Reeb chords, set

$$\boldsymbol{b}_d(c_{d-1}\boldsymbol{a}_{d-1}, \ldots, c_0\boldsymbol{a}_0) = \sum_{\substack{\gamma_{0d} \\ \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_d}} \#\mathcal{M}^0_{\Lambda_{0\ldots d}}(\gamma_{0d}; \boldsymbol{\delta}_0, c_0, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_{d-1}, c_{d-1}, \boldsymbol{\delta}_d) \cdot \gamma_{0d}\, \boldsymbol{\delta}_0\, \boldsymbol{a}_0\, \boldsymbol{\delta}_1 \ldots \boldsymbol{a}_{d-1}\, \boldsymbol{\delta}_d,$$

$$\boldsymbol{\Delta}_d(c_{d-1}\boldsymbol{a}_{d-1}, \ldots, c_0\boldsymbol{a}_0) = \sum_{\substack{\gamma_{d0} \\ \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_d}} \#\mathcal{M}^0_{\Lambda_{0\ldots d}}(\gamma_{d0}; \boldsymbol{\delta}_0, c_0, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_{d-1}, c_{d-1}, \boldsymbol{\delta}_d) \cdot \gamma_{d0}\, \boldsymbol{\delta}_0\, \boldsymbol{a}_0\, \boldsymbol{\delta}_1 \ldots \boldsymbol{a}_{d-1}\, \boldsymbol{\delta}_d.$$

Observe that for the map $\boldsymbol{b}_d$, the mixed chord in the output is a positive asymptotic of the pseudoholomorphic discs considered to define it, while for $\boldsymbol{\Delta}_d$ it is a negative asymptotic. Thus, for energy reasons, $\boldsymbol{\Delta}_d(c_{d-1}\boldsymbol{a}_{d-1}, \ldots, c_0\boldsymbol{a}_0)$ is automatically 0 if for all $0 \leq j \leq d-1$ we have $c_j \in C(\Lambda_j, \Lambda_{j+1})$.

Notation 7.7 We will denote by $\boldsymbol{b}^x_d$ the component of $\boldsymbol{b}_d$ which takes values in $\langle x_{0d} \rangle^{\mathrm{cyc}}_{\mathcal{A}\text{-}\mathcal{A}}$, and $\boldsymbol{b}^\vee_d$ for the component which takes values in $\check{C}^{\mathrm{cyc}}_{-}(\Lambda_0, \Lambda_d)$.

We make the following observations:

- As ungraded maps, the map CY_1 is equal to the restriction to $\widehat{C}_+^{\mathrm{cyc}}$ of the map $\boldsymbol{b}_1^\vee$, namely it is defined by a count of bananas with two positive asymptotics.
- The maps $\check{\mathfrak{m}}_d$ are equal to the restriction of $\boldsymbol{b}_d^\vee$ to $\check{C}_-^{\mathrm{cyc}}(\Lambda_{d-1},\Lambda_d)\otimes\cdots\otimes\check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$.

With this notation we can also rewrite the differential $\widehat{\mathfrak{m}}_1$ and the product $\widehat{\mathfrak{m}}_2$ as follows (we now drop again the full notation for elements, removing the words of pure Reeb chords):

$$\begin{aligned}\widehat{\mathfrak{m}}_1(\gamma_{10}) &= \boldsymbol{\Delta}_1(\gamma_{10})+\boldsymbol{b}_1^x(\gamma_{10}),\\ \widehat{\mathfrak{m}}_2(\gamma_{21},\gamma_{10}) &= \boldsymbol{b}_2^x(\mathrm{CY}_1(\gamma_{21}),\gamma_{10})+\boldsymbol{\Delta}_2(\mathrm{CY}_1(\gamma_{21}),\gamma_{10}),\\ \widehat{\mathfrak{m}}_2(\gamma_{21},x_{01}) &= \boldsymbol{b}_2^x(\mathrm{CY}_1(\gamma_{21}),x_{01}),\\ \widehat{\mathfrak{m}}_2(x_{12},\gamma_{10}) &= \boldsymbol{b}_2^x(\mathrm{CY}_1(x_{12}),\gamma_{10})+\boldsymbol{\Delta}_2(\mathrm{CY}_1(x_{12}),\gamma_{10}).\end{aligned}$$

Note that $\boldsymbol{\Delta}_2(\mathrm{CY}_1(\gamma_{21}),x_{01})=0$ for action reasons, as well as

$$\boldsymbol{b}_2^x(x_{12},\mathrm{CY}_1(x_{01}))=\boldsymbol{\Delta}_2(\mathrm{CY}_1(x_{12}),x_{01})=0.$$

So we can write a compact formula for the product

$$\widehat{\mathfrak{m}}_2=(\boldsymbol{b}_2^x+\boldsymbol{\Delta}_2)(\mathrm{CY}_1\otimes\mathrm{id}).$$

Finally, observe that the map CY_2 can be rewritten $\mathrm{CY}_2=\boldsymbol{b}_2^\vee(\mathrm{CY}_1\otimes\mathrm{id})$.

Remark 7.8 Very rigorously, the map $\widehat{\mathfrak{m}}_1$ has domain and target the $\mathbb{Z}_2$-module $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)$, so it is actually equal to the sum of the *shifted by* 1 restrictions of the maps $\boldsymbol{\Delta}_1$ and $\boldsymbol{b}_1^x$ (these restrictions have domain $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)[-1]$). So for the formulas for $\widehat{\mathfrak{m}}_1,\widehat{\mathfrak{m}}_2,\mathrm{CY}_1,\mathrm{CY}_2$ as well as for the maps we define below, the reader should consider these equalities as equalities of ungraded maps.

We extend now these formulas, ie we define maps

$$\begin{aligned}\widehat{\mathfrak{m}}_d &: \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{d-1},\Lambda_d)\otimes\cdots\otimes\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)\to\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_d),\\ \mathrm{CY}_d &: \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{d-1},\Lambda_d)\otimes\cdots\otimes\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)\to\check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_d)\end{aligned}$$

for a $(d+1)$-copy of Λ by

(29)
$$\widehat{\mathfrak{m}}_d=\sum_{j=2}^{d}\sum_{\substack{1\le i_2,\dots,i_j\le d-1\\ i_2+\cdots+i_j=d-1}}(\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}),$$

(30)
$$\mathrm{CY}_d=\sum_{j=2}^{d}\sum_{\substack{1\le i_2,\dots,i_j\le d-1\\ i_2+\cdots+i_j=d-1}}\boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}).$$

Observe that $\widehat{\mathfrak{m}}_d$ is of degree $2-d$ while CY_d is of degree $1-d$.

Theorem 7.9 *Let $\Lambda_0 \cup \cdots \cup \Lambda_d$ be a $(d+1)$-copy of Λ_0. Then for any $1 \le k \le d$ and any $(k+1)$-tuple of integers $0 \le s_0 < \cdots < s_k \le d$ we have*

$$\sum_{m=1}^{k} \sum_{n=0}^{k-m} \widehat{\mathfrak{m}}_{k-m+1}(\mathrm{id}^{\otimes k-m-n} \otimes \widehat{\mathfrak{m}}_m \otimes \mathrm{id}^{\otimes n}) = 0, \tag{31}$$

$$\sum_{r=1}^{k} \sum_{\substack{t_1,\dots,t_r \\ t_1+\cdots+t_r=k}} \check{\mathfrak{m}}_r(\mathrm{CY}_{t_r} \otimes \cdots \otimes \mathrm{CY}_{t_1}) + \sum_{m=1}^{k} \sum_{n=0}^{k-m} \mathrm{CY}_{k-m+1}(\mathrm{id}^{\otimes k-m-n} \otimes \widehat{\mathfrak{m}}_m \otimes \mathrm{id}^{\otimes n}) = 0, \tag{32}$$

where

- $\widehat{\mathfrak{m}}_m$ *has domain* $\widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{n+m-1}}, \Lambda_{s_{n+m}}) \otimes \cdots \otimes \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_n}, \Lambda_{s_{n+1}})$,
- $\widehat{\mathfrak{m}}_{k-m+1}$ *and* CY_{k-m+1} *have domain*
  $$\widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{k-1}}, \Lambda_{s_k}) \otimes \cdots \otimes \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{n+m}}, \Lambda_{s_{n+m+1}}) \otimes \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{n-1}}, \Lambda_{s_n}) \otimes \cdots \otimes \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_1}, \Lambda_{s_0}),$$
- *if we define* $\tau_j = \sum_{i=1}^{j} t_i$, *then* CY_{t_j} *has domain*
  $$\widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{\tau_j-1}}, \Lambda_{s_{\tau_j}}) \otimes \cdots \otimes \widehat{C}_+^{\mathrm{cyc}}(\Lambda_{s_{\tau_{j-1}}}, \Lambda_{s_{\tau_{j-1}+1}}),$$

and $\check{\mathfrak{m}}_r$ *has domain*

$$\check{C}_-^{\mathrm{cyc}}(\Lambda_{s_{\tau_{r-1}}}, \Lambda_{s_{\tau_r}}) \otimes \cdots \otimes \check{C}_-^{\mathrm{cyc}}(\Lambda_{s_{\tau_1}}, \Lambda_{s_{\tau_2}}) \otimes \check{C}_-^{\mathrm{cyc}}(\Lambda_{s_{\tau_1}}, \Lambda_{s_0}).$$

To simplify notation, in the following we will assume that the $(k+1)$-tuple of integers $(s_0, \dots, s_k)$ is $(0, \dots, k)$. In order to prove Theorem 7.9 we will use results proved in [29] about the maps $\boldsymbol{b}_d$ and $\boldsymbol{\Delta}_d$ that we recall now.

Lemma 7.10 [29, Lemmas 3 and 5] *Let $\Lambda_0 \cup \cdots \cup \Lambda_d$ be a $(d+1)$-copy of Λ_0. Then for any $1 \le k \le d$ we have*

(a) $$\sum_{m=1}^{k} \sum_{n=0}^{k-m} \boldsymbol{b}_{k-m+1}(\mathrm{id}^{\otimes d-m-n} \otimes (\boldsymbol{b}_m + \boldsymbol{\Delta}_m) \otimes \mathrm{id}^{\otimes n}) = 0,$$

(b) $$\sum_{m=1}^{k} \sum_{n=0}^{k-m} \boldsymbol{\Delta}_{k-m+1}(\mathrm{id}^{\otimes d-m-n} \otimes (\boldsymbol{b}_m + \boldsymbol{\Delta}_m) \otimes \mathrm{id}^{\otimes n}) = 0.$$

Proof of Theorem 7.9 In order to avoid any useless complicated notation and use Lemma 7.10 as it is, we prove the theorem ignoring the grading of maps. This means that we will write $\boldsymbol{b}_1^{\vee}$ (restricted to the appropriate module) for CY_1 and $\boldsymbol{\Delta}_1 + \boldsymbol{b}_1^{x}$ for $\widehat{\mathfrak{m}}_1$. We start by proving (32) which is a proof by induction. For $k = 1, 2$ we have already shown the relation in Lemma 6.5 and the proof of Theorem 7.6, respectively. Now let us prove (32) for $k \ge 3$ inputs, assuming that the relation holds for any number of inputs less or equal to $k-1$. We denote by LHS(32) the left-hand side of (32). Using the fact that $\check{\mathfrak{m}}_r = \boldsymbol{b}_r^\vee$ and the formula for the Calabi–Yau map, we have

$$\mathrm{LHS}(32) = \sum_{j=1}^{k} \sum_{\substack{t_1,\dots,t_j\geq 1\\ t_1+\dots+t_j=k}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{t_j}\otimes\dots\otimes\mathrm{CY}_{t_1}) + \boldsymbol{b}_1^\vee\circ\widehat{\mathfrak{m}}_k \\ + \sum_{m=1}^{k-1}\sum_{n=0}^{k-m}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})(\mathrm{id}^{\otimes k-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n}).$$

We separate the case $n=0$ from the others in the second line and get

$$\sum_{j=1}^{k} \sum_{\substack{t_1,\dots,t_j\geq 1\\ t_1+\dots+t_j=k}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{t_j}\otimes\dots\otimes\mathrm{CY}_{t_1}) + \boldsymbol{b}_1^\vee\circ\widehat{\mathfrak{m}}_k + \sum_{m=1}^{k-1}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\widehat{\mathfrak{m}}_m) \\ + \sum_{m=1}^{k-1}\sum_{n=1}^{k-m}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})(\mathrm{id}^{\otimes k-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n}).$$

We now separate the cases $t_1=1$ and $m=1$ from the others in the first line, and apply a change of variables in the second line (note that for any $2\leq s\leq j$, when $1+\sum_{v=2}^{s-1} i_v \leq n \leq 1+\sum_{v=2}^{s} i_v$, the "inner" $\widehat{\mathfrak{m}}_m$ will be an argument of CY_{i_s}; thus instead of summing over $1\leq n\leq k-m$ we can sum over the variables s, n with $2\leq s\leq j$ and $0\leq n\leq i_s-1$) to obtain

$$\sum_{j=2}^{k} \sum_{\substack{t_2,\dots,t_j\geq 1\\ t_2+\dots+t_j=k-1}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{t_j}\otimes\dots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{CY}_1) + \boldsymbol{b}_1^\vee\circ\widehat{\mathfrak{m}}_k + \sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\widehat{\mathfrak{m}}_1) \\ + \sum_{j=1}^{k} \sum_{\substack{t_1,\dots,t_j\geq 1\\ t_1\geq 2\\ t_1+\dots+t_j=k}} \boldsymbol{b}_j^\vee(\mathrm{CY}_{t_j}\otimes\dots\otimes\mathrm{CY}_{t_1}) + \sum_{m=2}^{k-1}\sum_{r=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_r\geq 1\\ i_2+\dots+i_r=k-m}} \boldsymbol{b}_r^\vee(\mathrm{CY}_{i_r}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\widehat{\mathfrak{m}}_m) \\ + \sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{s=2}^{j}\sum_{m=1}^{i_s}\sum_{n=0}^{i_s-m} \boldsymbol{b}_j^\vee(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_s-m+1}(\mathrm{id}^{\otimes i_s-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n})\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}).$$

On the first line the summations are on the same sets of parameters, so we combine them together using that

$$\mathrm{CY}_1+\widehat{\mathfrak{m}}_1 = \boldsymbol{b}_1^\vee + \boldsymbol{\Delta}_1 + \boldsymbol{b}_1^x = \boldsymbol{b}_1 + \boldsymbol{\Delta}.$$

Then we rewrite CY_{t_1} and $\widehat{\mathfrak{m}}_m$ on the second line, as well as $\widehat{\mathfrak{m}}_k$ on the first line, using (29) and (30). Finally, we use (32) in the third line, which is assumed to hold by induction hypothesis. After all these changes we get

$$
\begin{aligned}
&\sum_{j=2}^{k} \sum_{\substack{t_2,\dots,t_j\geq 1\\ t_2+\dots+t_j=k-1}} \boldsymbol{b}_j^{\vee}(\mathrm{CY}_{t_j}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes(\boldsymbol{b}_1+\boldsymbol{\Delta}_1))\\
&+\sum_{r=2}^{k} \sum_{\substack{t_2,\dots,t_r\geq 1\\ t_2+\dots+t_r=k-1}} \boldsymbol{b}_1^{\vee}\circ(\boldsymbol{b}_r^{x}+\boldsymbol{\Delta}_r)(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{id})\\
&+\sum_{j=1}^{k} \sum_{\substack{t_1,\dots,t_j\geq 1\\ t_1\geq 2\\ t_1+\dots+t_j=k}} \sum_{r=2}^{t_1} \sum_{\substack{i_2,\dots,i_r\geq 1\\ i_2+\dots+i_r=t_1-1}} \boldsymbol{b}_j^{\vee}(\mathrm{CY}_{t_j}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes\boldsymbol{b}_r^{\vee}(\mathrm{CY}_{i_r}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}))\\
&+\sum_{m=2}^{k-1} \sum_{j=2}^{k-m+1} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}} \sum_{r=2}^{m} \sum_{\substack{t_2,\dots,t_r\geq 1\\ t_2+\dots+t_r=m-1}} \boldsymbol{b}_j^{\vee}(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes(\boldsymbol{b}_r^{x}+\boldsymbol{\Delta}_r)(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{id}))\\
&+\sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{s=2}^{j}\sum_{r=1}^{i_s} \sum_{\substack{t_1,\dots,t_r\geq 1\\ t_1+\dots+t_r=i_s}} \boldsymbol{b}_j^{\vee}(\mathrm{CY}_{i_j}\otimes\cdots\otimes\boldsymbol{b}_r^{\vee}(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_1})\otimes\mathrm{CY}_{i_{s-1}}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}).
\end{aligned}
$$

We can sum the three middle lines together, and change variables on the resulting sum as well as on the last line to get

$$
\begin{aligned}
&\sum_{j=2}^{k} \sum_{\substack{t_2,\dots,t_j\geq 1\\ t_2+\dots+t_j=k-1}} \boldsymbol{b}_j^{\vee}(\mathrm{CY}_{t_j}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes(\boldsymbol{b}_1+\boldsymbol{\Delta}_1))\\
&+\sum_{j=2}^{k} \sum_{\substack{t_2,\dots,t_j\geq 1\\ t_2+\dots+t_j=k-1}} \sum_{r=2}^{j} \boldsymbol{b}_{j-r+1}^{\vee}(\mathrm{CY}_{t_j}\otimes\cdots\otimes\mathrm{CY}_{t_{r+1}}\otimes(\boldsymbol{b}_r+\boldsymbol{\Delta}_r)(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{id}))\\
&+\sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{r=1}^{j-1}\sum_{n=1}^{j-r} \boldsymbol{b}_{j-r+1}^{\vee}(\mathrm{CY}_{i_j}\otimes\cdots\otimes\boldsymbol{b}_r^{\vee}(\mathrm{CY}_{i_{n+r}}\otimes\cdots\otimes\mathrm{CY}_{i_{n+1}})\otimes\mathrm{CY}_{i_n}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}).
\end{aligned}
$$

Finally, note that adding $\boldsymbol{b}_r^{x}+\boldsymbol{\Delta}_r$ to $\boldsymbol{b}_r^{\vee}$ on the last line doesn't change anything because these terms vanish for degree and energy reasons (observe that curves contributing to $\boldsymbol{b}_r^{x}(\mathrm{CY}_{i_{n+r}}\otimes\cdots\otimes\mathrm{CY}_{i_{n+1}})$ would have a unique positive asymptotic at a maximum Morse chord and negative asymptotic which have to be Morse chords for action reasons. These negative asymptotics are in the image of the CY map so can only be minimum Morse chords. For index reasons, such rigid discs do not exist). Then we observe that the first line is the case $r=1, n=0$ of the third line while the second line is the case $r\geq 1, n=0$

of the third line, so we have

$$\mathrm{LHS}(32)=\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}}\sum_{r=1}^{j}\sum_{n=0}^{j-r}\boldsymbol{b}^{\vee}_{j-r+1}(\mathrm{id}^{\otimes j-r-n}\otimes(\boldsymbol{b}_r+\boldsymbol{\Delta}_r)\otimes\mathrm{id}^{\otimes n})(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}),$$

which vanishes by Lemma 7.10(a) and thus we have proven (32). In order to prove (31), we will use (32), which simplifies slightly the notation in the computations below. We have

$$\mathrm{LHS}(31)=\widehat{\mathfrak{m}}_1\circ\widehat{\mathfrak{m}}_k+\widehat{\mathfrak{m}}_k(\mathrm{id}^{\otimes k-1}\otimes\widehat{\mathfrak{m}}_1)+\sum_{m=2}^{k-1}\widehat{\mathfrak{m}}_{k-m+1}(\mathrm{id}^{\otimes k-m}\otimes\widehat{\mathfrak{m}}_m)$$
$$+\sum_{m=1}^{k-1}\sum_{n=1}^{k-m}\widehat{\mathfrak{m}}_{k-m+1}(\mathrm{id}^{\otimes k-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n})$$

and we use (29) to rewrite $\widehat{\mathfrak{m}}_j$ where it appear, except the "inner one" in the last line. We get

$$\begin{aligned}\mathrm{LHS}(31)=&\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}}(\boldsymbol{b}^x_1+\boldsymbol{\Delta}_1)(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})\\
&+\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes(\boldsymbol{b}^x_1+\boldsymbol{\Delta}_1))\\
&+\sum_{m=2}^{k-1}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}}\sum_{r=2}^{m}\sum_{\substack{t_2,\dots,t_r\geq 1\\ t_2+\dots+t_r=m-1}}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)\big(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\\
&\qquad\otimes(\boldsymbol{b}^x_r+\boldsymbol{\Delta}_r)(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{id})\big)\\
&+\sum_{m=1}^{k-1}\sum_{n=1}^{k-m}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})(\mathrm{id}^{\otimes k-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n}).\end{aligned}$$

Observe that the last line can be written

$$\sum_{m=1}^{k-1}\sum_{j=2}^{k-m+1}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}}\sum_{s=2}^{j}\sum_{n=0}^{i_s}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_s}(\mathrm{id}^{\otimes i_s-1-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n})\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})$$
$$=\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}}\sum_{s=2}^{j}\sum_{m=1}^{i_s}\sum_{n=0}^{i_s-m}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)\big(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_s-m+1}(\mathrm{id}^{\otimes i_s-m-n}\otimes\widehat{\mathfrak{m}}_m\otimes\mathrm{id}^{\otimes n})$$
$$\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}\big),$$

where the right-hand side of the equality is obtained after a change of variables. On this last line we can now apply (32) to get

$$\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}}\sum_{s=2}^{j}\sum_{r=1}^{i_s}\sum_{\substack{t_1,\dots,t_r\geq 1\\ t_1+\dots+t_r=i_s}}(\boldsymbol{b}^x_j+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_{s+1}}\otimes\check{\mathfrak{m}}_r(\mathrm{CY}_{t_r}\otimes\cdots\otimes\mathrm{CY}_{t_1})\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}).$$

Using that $\check{\mathfrak{m}}_r = \boldsymbol{b}_r^\vee$, we have

$$\begin{aligned}
\mathrm{LHS}(31) = {} & \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} (\boldsymbol{b}_1^x+\boldsymbol{\Delta}_1)(\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}) \\
& + \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} (\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes(\boldsymbol{b}_1^x+\boldsymbol{\Delta}_1)) \\
& + \sum_{m=2}^{k-1} \sum_{j=2}^{k-m+1} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-m}} \sum_{r=2}^{m} \sum_{\substack{t_2,\dots,t_r\geq 1\\ t_2+\dots+t_r=m-1}} (\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)\big(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2} \\
& \qquad\qquad \otimes(\boldsymbol{b}_r^x+\boldsymbol{\Delta}_r)(\mathrm{CY}_{t_r}\otimes\dots\otimes\mathrm{CY}_{t_2}\otimes\mathrm{id})\big) \\
& + \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{s=2}^{j}\sum_{r=1}^{i_s} \sum_{\substack{t_1,\dots,t_r\geq 1\\ t_1+\dots+t_r=i_s}} (\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)\big(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_{s+1}} \\
& \qquad\qquad \otimes \boldsymbol{b}_r^\vee(\mathrm{CY}_{t_r}\otimes\dots\otimes\mathrm{CY}_{t_1})\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}\big).
\end{aligned}$$

By a change of variables in the third and fourth lines we obtain

$$\begin{aligned}
\mathrm{LHS}(31) = {} & \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} (\boldsymbol{b}_1^x+\boldsymbol{\Delta}_1)(\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}) \\
& + \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} (\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j)(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes(\boldsymbol{b}_1^x+\boldsymbol{\Delta}_1)) \\
& + \sum_{j=2}^{k-1} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{r=2}^{j-1} (\boldsymbol{b}_{j-r+1}^x+\boldsymbol{\Delta}_{j-r+1})(\mathrm{CY}_{i_j}\otimes\dots\otimes\mathrm{CY}_{i_{r+1}}\otimes(\boldsymbol{b}_r^x+\boldsymbol{\Delta}_r)(\mathrm{CY}_{i_r}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id})) \\
& + \sum_{j=2}^{k} \sum_{\substack{i_2,\dots,i_j\geq 1\\ i_2+\dots+i_j=k-1}} \sum_{r=1}^{j-1}\sum_{n=1}^{j-r} (\boldsymbol{b}_{j-r+1}^x+\boldsymbol{\Delta}_{j-r+1})\big(\mathrm{CY}_{i_j}\otimes\dots \\
& \qquad\qquad \otimes \boldsymbol{b}_r^\vee(\mathrm{CY}_{i_{n+r}}\otimes\dots\otimes\mathrm{CY}_{i_{n+1}})\otimes\mathrm{CY}_{i_n}\otimes\dots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}\big).
\end{aligned}$$

Finally, observe that:

- Adding $\boldsymbol{b}_j^\vee$ to $\boldsymbol{b}_j^x+\boldsymbol{\Delta}_j$ on the first line does nothing as it vanishes for energy reasons (when the output is a long chord) or by canceling pairs of discs (when the output of $\boldsymbol{b}_j^\vee$ is y and the dimension of the Legendrian is 1).

- Adding $\boldsymbol{b}_1^\vee$ to $\boldsymbol{b}_1^x+\boldsymbol{\Delta}_1$ on the second line contributes nothing more when the output of $\boldsymbol{b}_1^\vee$ is a long chord (for energy reasons). Let us check that it also vanishes when the output of $\boldsymbol{b}_1^\vee$ is the Morse chord y.

In such a case, energy arguments imply that the terms

$$\mathbf{\Delta}_j(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes \boldsymbol{b}_1^y)$$

vanish because all inputs for $\mathbf{\Delta}_j$ would be negative Reeb chords asymptotics; see the definition of $\mathbf{\Delta}_j$ at the beginning of Section 7.4. Then we consider the terms

$$\boldsymbol{b}_j^x(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes \boldsymbol{b}_1^y).$$

Pseudoholomorphic buildings contributing to such terms should contain a rigid disc with a unique positive asymptotic to x. For energy reasons, the j negative asymptotics must be Morse chords and in particular can only be y's (by definition of the CY maps). For index reasons, such a rigid disc doesn't exist.

- Adding $\boldsymbol{b}_r^\vee$ to $\boldsymbol{b}_r^x+\mathbf{\Delta}_r$ on the third line doesn't contribute either for the same reasons as the previous point.
- Adding $\boldsymbol{b}_r^x+\mathbf{\Delta}_r$ to $\boldsymbol{b}_r^\vee$ on the fourth line does not change anything either: $\mathbf{\Delta}_r(\mathrm{CY}_{i_{n+r}}\otimes\cdots\otimes\mathrm{CY}_{i_{n+1}})$ vanishes for energy reasons and then as before the term $\boldsymbol{b}_r^x(\mathrm{CY}_{i_{n+r}}\otimes\cdots\otimes\mathrm{CY}_{i_{n+1}})$ vanishes also because there is an even number of discs, or no disc at all, with positive asymptotic at x and negative asymptotics only at y's chords.

Moreover, the first line is the case $r=j, n=0$ of the last line, the second line is the case $r=1, n=0$ of the last line, and the third line is the case $2\le r\le j-1, n=0$ of the last line. So we have

$$\mathrm{LHS}(31)=\sum_{j=2}^{k}\sum_{\substack{i_2,\dots,i_j\ge 1\\ i_2+\cdots+i_j=k-1}}\sum_{r=1}^{j}\sum_{n=0}^{j-r}(\boldsymbol{b}_{j-r+1}^x+\mathbf{\Delta}_{j-r+1})\\ (\mathrm{id}^{\otimes j-r-n}\otimes(\boldsymbol{b}_r+\mathbf{\Delta}_r)\otimes\mathrm{id}^{\otimes n})(\mathrm{CY}_{i_j}\otimes\cdots\otimes\mathrm{CY}_{i_2}\otimes\mathrm{id}),$$

which vanishes by Lemma 7.10(a)–(b). □

8 Example: the unknot

The computation done here is a subcase of the computation done in [2, Section 5]. We nevertheless detail it using our notation. Let Λ be the standard $\mathrm{TB}=-1$ unknot and consider a 2-copy and a 3-copy; see Figure 21. We have

$$\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)=\langle a_{10},x_{01}\rangle_{\mathcal{A}\text{-}\mathcal{A}}^{\mathrm{cyc}}\quad\text{and}\quad \check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1)=\langle a_{01},y_{01}\rangle_{\mathcal{A}\text{-}\mathcal{A}}^{\mathrm{cyc}}$$

with $|a_{10}|_{\widehat{C}_+}=0$, $|x_{01}|_{\widehat{C}_+}=2$, $|a_{01}|_{\check{C}_-}=2$ and $|y_{01}|_{\check{C}_-}=0$. Denote by $\boldsymbol{a}^j=a\dots a$ the word consisting of j times the chord a which is the only Reeb chord of Λ. We have, for all $j\ge 0$, $\widehat{\mathfrak{m}}_1(a_{10}\boldsymbol{a}^j)=x_{01}a\boldsymbol{a}^j+x_{01}\boldsymbol{a}^ja=0$, and $\widehat{\mathfrak{m}}_1(x_{01})=0$ as well. On the other side $\check{\mathfrak{m}}_1(a_{01}\boldsymbol{a}^j)=0$, for example, for degree reasons, and $\check{\mathfrak{m}}_1(y_{01}\boldsymbol{a}^j)=a_{01}a\boldsymbol{a}^j+a_{01}\boldsymbol{a}^ja=0$. So both differentials $\widehat{\mathfrak{m}}_1$ and $\check{\mathfrak{m}}_1$ vanish implying that the homologies of $(\widehat{C}_+^{\mathrm{cyc}}(\Lambda_0,\Lambda_1),\widehat{\mathfrak{m}}_1)$ and $(\check{C}_-^{\mathrm{cyc}}(\Lambda_0,\Lambda_1),\check{\mathfrak{m}}_1)$ are infinite-dimensional generated

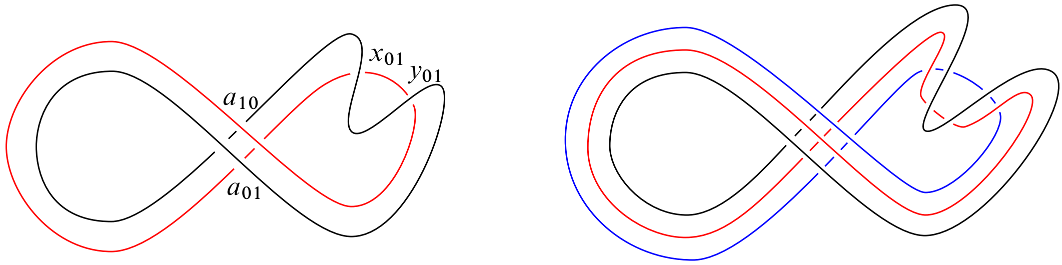

Figure 21: Lagrangian projections of the 2-copy $\Lambda_0 \cup \Lambda_1$ on the left, and of the 3-copy $\Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ on the right.

by all words $a_{10}\boldsymbol{a}^j$, $x_{01}\boldsymbol{a}^j$ and $a_{01}\boldsymbol{a}^j$, $y_{01}\boldsymbol{a}^j$, respectively. In this simple case there is a unique way to define the Calabi–Yau map by degree reasons, but one can also easily see on the figure which are the bananas with a positive asymptotic at a_{10} and the strips with a negative asymptotic at x_{01}. This gives

$$\mathrm{CY}_1(a_{10}\boldsymbol{a}^j) = y_{01}\boldsymbol{a}^j \quad \text{and} \quad \mathrm{CY}_1(x_{01}\boldsymbol{a}^j) = a_{01}\boldsymbol{a}^j.$$

We can then use the Lagrangian projection of the 3-copy to compute the product structures on $\widehat{C}_+^{\mathrm{cyc}}$ and $\check{C}_-^{\mathrm{cyc}}$. The rigid discs asymptotic to generators of $\check{C}_-^{\mathrm{cyc}}$ with one positive asymptotic and two negative asymptotics contribute to the product $\check{\mathfrak{m}}_2$ and one can see that the only ones are those giving

$$\check{\mathfrak{m}}_2(a_{12}\boldsymbol{a}^j, y_{01}\boldsymbol{a}^i) = a_{02}\boldsymbol{a}^{i+j}, \quad \check{\mathfrak{m}}_2(y_{12}\boldsymbol{a}^j, a_{01}\boldsymbol{a}^i) = a_{02}\boldsymbol{a}^{i+j}, \quad \check{\mathfrak{m}}_2(y_{12}\boldsymbol{a}^j, y_{01}\boldsymbol{a}^i) = y_{02}\boldsymbol{a}^{i+j},$$

which expresses the fact that the minimum Morse Reeb chord acts as a unit, ie induces a quasi-isomorphism $\check{C}_-^{\mathrm{cyc}}(\Lambda_1, \Lambda_2) \cong \check{C}_-^{\mathrm{cyc}}(\Lambda_0, \Lambda_2)$. For the product $\widehat{\mathfrak{m}}_2$ we have to find buildings as the one pictured in Figure 6. The only ones are those we depicted on Figure 22. They give the following nontrivial components of the product:

$$\widehat{\mathfrak{m}}_2(a_{21}\boldsymbol{a}^j, a_{10}\boldsymbol{a}^i) = a_{20}\boldsymbol{a}^{i+j}, \quad \widehat{\mathfrak{m}}_2(a_{21}\boldsymbol{a}^j, x_{01}\boldsymbol{a}^i) = x_{02}\boldsymbol{a}^{i+j}, \quad \widehat{\mathfrak{m}}_2(x_{12}\boldsymbol{a}^j, a_{10}\boldsymbol{a}^i) = x_{02}\boldsymbol{a}^{i+j},$$

which translates the fact that the mixed a generator acts as a unit for the product $\widehat{\mathfrak{m}}_2$. The higher-order operations $\check{\mathfrak{m}}_j$ and $\widehat{\mathfrak{m}}_j$ for $j \geq 3$ vanish.

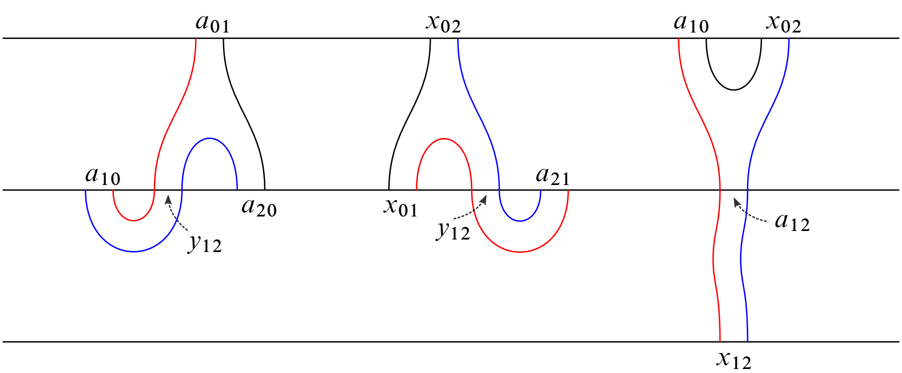

Figure 22: Pseudoholomorphic buildings contributing to the product $\widehat{\mathfrak{m}}_2$ for the unknot.

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Department of Mathematical Sciences, Chalmers University of Technology
Göteborg, Sweden

noelet@hig.se

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