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# On the resolution of kinks of curves on punctured surfaces

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Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a surface with marked points $\mathbb{M} \subseteq \partial\Sigma \neq \varnothing$ and punctures $\mathbb{P} \subseteq \Sigma \setminus \partial\Sigma$. We show that for every curve $\gamma$ on $\Sigma \setminus \mathbb{P}$, the curve obtained by resolving the kinks of $\gamma$ in any order is uniquely determined, up to homotopy in $\Sigma \setminus \mathbb{P}$, by the 2-orbifold homotopy class of $\gamma$, in which the punctures are interpreted to be orbifold points of order 2. Our proof relies on an application of the diamond lemma.

57K20; 13F60, 18B40



## 1 Introduction

Surfaces with marked points are classical mathematical objects that, after the appearance of works by Assem, Brüstle, Charbonneau-Jodoin, and Plamondon [6], Fock and Goncharov [8], Fomin, Shapiro, and Thurston [9], and Labardini-Fragoso [16], have suffused both cluster algebras and the representation theory of algebras during the last 15 years, with remarkable connections between geometry and representation theory discovered in works by Haiden, Katzarkov, and Kontsevich [11], Lekili and Polishchuk [18], and Opper, Plamondon, and Schroll [21]. Recent developments by Amiot and Brüstle [2], Amiot and Plamondon [5], and Labardini-Fragoso, Schroll, and Valdivieso [17] have discovered that, somewhat mysteriously, it is sometimes necessary to interpret the punctures not as holes, but as orbifold points of order 2.

When a puncture is regarded as an orbifold point of order 2 there are certain loops, originally of infinite order, that are declared to have order 2, namely each loop closely wrapping around such puncture. The

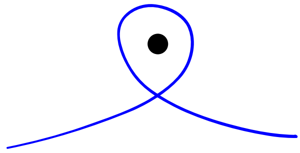

Figure 1

2-*orbifold fundamental groupoid* of a surface $(\Sigma, \mathbb{M}, \mathbb{P})$ with marked points on the boundary $\mathbb{M} \subseteq \partial\Sigma$ and punctures $\mathbb{P} \subseteq \Sigma \setminus \partial\Sigma$ is thus defined as the quotient groupoid of the topological fundamental groupoid of $(\Sigma, \mathbb{M}, \mathbb{P})$ obtained by treating all punctures as orbifold points of order 2.

Intuitively, a *kink* of a curve $\gamma$ on $\Sigma \setminus \mathbb{P}$ is a segment $\kappa$ of $\gamma$ that turns out to be one of the aforementioned loops, and that cannot be dissolved with any homotopy rel $\{0, 1\}$ of curves on $\Sigma \setminus \mathbb{P}$; see Figure 1. However, the notion of a *kink* of an arbitrary morphism in the topological fundamental groupoid of $(\Sigma, \mathbb{M}, \mathbb{P})$, that is, of an arbitrary homotopy class rel $\{0, 1\}$ of curves connecting two given points on $\Sigma \setminus \mathbb{P}$, is not as easy to define in terms of arbitrary representatives of the morphism as one would like, because an arbitrary representative can be a very complicated curve even if the morphism possesses some nice representative — consider, for instance, a curve that has not only self-crossings and self-tangencies, but also different segments that it traverses multiple times, perhaps with immediate backtrackings of some of them.

Here we define the notion of a kink of a curve $\gamma$ in terms of the associated walk on the (leafy) dual graph of an arbitrary triangulation of signature zero. We show that $\gamma$ having kinks is independent of the triangulation of signature zero taken, and that the curve obtained by resolving the kinks of $\gamma$ in any order is uniquely determined, up to homotopy in $\Sigma \setminus \mathbb{P}$, by the 2-orbifold homotopy class of $\gamma$, in which the punctures are interpreted to be orbifold points of order 2. Our proof uses a nontrivial application of the diamond lemma.

Let us describe the contents of the paper in some detail. A *triangulation of signature zero* is an ideal triangulation with the property that every puncture is enclosed by a self-folded triangle. Such a triangulation $\tau$ has its associated *dual graph* $G(\tau)$, whose vertices are the triangles of $\tau$ and the boundary segments of $(\Sigma, \mathbb{M}, \mathbb{P})$, with an edge connecting two triangles each time they share an arc of $\tau$, and with an edge between a boundary segment and the unique triangle containing it. To simplify the treatment of morphisms in the fundamental groupoid of the graph, we introduce the *leafy dual graph* $G^{\leaf}(\tau)$, which is a slight enlargement of $G(\tau)$. An important feature of $G^{\leaf}(\tau)$ is that every nonidentity morphism $f$ in the fundamental groupoid $\pi_1(G^{\leaf}(\tau))$ can be uniquely represented as a backtrack-free walk on $G^{\leaf}(\tau)$; we call such a walk the *standard form* of $f$.

The orientation of $\Sigma$ provides $G(\tau)$ with a natural structure of a *ribbon graph*, which we extend to a ribbon graph structure for $G^{\leaf}(\tau)$. We use this ribbon structure of $G^{\leaf}(\tau)$ to define the notion of a *kink* of any backtrack-free walk on $G^{\leaf}(\tau)$, ie of any nonidentity morphism in $\pi_1(G^{\leaf}(\tau))$ written in standard form. We define the *resolution* of a kink of a backtrack-free walk $w$ on $G^{\leaf}(\tau)$ as the result of applying a purely combinatorial operation that mimics the topological replacement sketched in Figure 2 and produces another backtrack-free walk $w'$ representing the same morphism as $w$ in the 2-orbifold fundamental

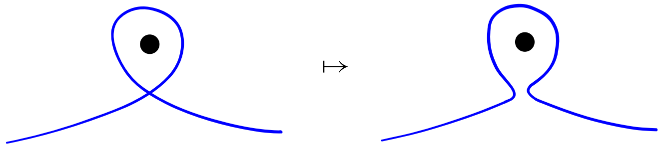

Figure 2

groupoid $\pi_1^{\text{orb}}(G^{⚲}(\tau))$, but whose class in the fundamental groupoid $\pi_1(G^{⚲}(\tau))$ may differ from that of $w$. We prove that the number of kinks decreases every time the resolution of a kink is applied, and that if two distinct kinks of a backtrack-free walk are resolved, producing backtrack-free walks $w'$ and $w''$, then it is always possible to resolve some sequence of kinks of $w'$ and some sequence of kinks of $w''$ so that the resulting backtrack-free walks coincide. This enables us to apply the *diamond lemma* and deduce that, no matter the order in which the kinks of $w$ are resolved, one always arrives at the same kink-free backtrack-free walk without kinks after finitely many steps.

To establish the corresponding result for curves on $\Sigma \setminus \mathbb{P}$ we proceed as follows. The *ribbon surface* $\Sigma(G^{⚲}(\tau))$ can be naturally embedded in $\Sigma$. As usual, there are strong deformation retractions $\Sigma \setminus \mathbb{P} \to \Sigma(G^{⚲}(\tau)) \to G^{⚲}(\tau)$, whose composition we denote by $\rho$. We use these retractions to define the notion of a *kink of a curve $\gamma$ with respect to $\tau$* as a kink of the standard form of $\rho(\gamma)$. We show that for any two triangulations $\tau$ and $\sigma$ of signature zero, $\gamma$ has a kink with respect to $\tau$ if and only if $\gamma$ has a kink with respect to $\sigma$. Furthermore, the constructions and definition make it transparent that $\rho$ induces a commutative diagram of groupoids

$$\begin{array}{ccc}
\pi_1(\Sigma \setminus \mathbb{P}, E) & \xrightarrow[\cong]{\rho_\#} & \pi_1(G^{⚲}(\tau), E) \\
\downarrow \mathfrak{p} & & \downarrow \mathfrak{p} \\
\pi_1^{\text{orb}}(\Sigma \setminus \mathbb{P}, E) & \xrightarrow[\bar{\rho}_\#]{\cong} & \pi_1^{\text{orb}}(G^{⚲}(\tau), E)
\end{array}$$

whose horizontal arrows are isomorphisms, where the vertical arrows are the canonical projections of quotient groupoids. From this, the fact that the resolution of kinks of a backtrack-free walk $w$ on $G^{⚲}(\tau)$ does not affect the 2-orbifold homotopy class of $w$, and the uniqueness result from the last line of the previous paragraph, we deduce the main result of this paper:

**Theorem 1.1** *Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a surface with nonempty boundary, and let $E \subseteq \partial\Sigma$ be a set containing exactly one point from the relative interior of each boundary segment of $(\Sigma, \mathbb{M}, \mathbb{P})$.*

- *Given $u_0, v_0 \in E$, there is exactly one function*

$$\iota\colon \pi_{1,\mathbb{P}}^{\text{orb}}(\Sigma, E)(u_0, v_0) \to \pi_1(\Sigma \setminus \mathbb{P}, E)(u_0, v_0)$$

*such that*

(1) $\mathfrak{p} \circ \iota = \mathbb{1}$,

(2) *for every $f \in \pi_{1,\mathbb{P}}^{\text{orb}}(\Sigma, E)(u_0, v_0)$ there exists a representative curve $\gamma \in \iota(f)$ that has no kinks.*

- *There is exactly one function*
$$\iota\colon \pi_{1,\mathbb{P}}^{\mathrm{orb,free}}(\Sigma, E)/\sim \to \pi_1^{\mathrm{free}}(\Sigma\setminus\mathbb{P}, E)/\sim$$
*such that*
(1) $\mathfrak{p}\circ\iota = \mathbb{1}$,
(2) *for every* $f\in\pi_{1,\mathbb{P}}^{\mathrm{orb,free}}(\Sigma, E)/\sim$ *there exists a representative curve* $\gamma\in\iota(f)$ *that has no kinks*,
*where* $\sim$ *is the equivalence relation that identifies each closed curve with its opposite orientation.*

The paper is organized as follows. In Section 2 we present the formal definition of a *quotient subgroupoid* of a groupoid by what we call a *normal multilocular subgroup*. We also recall the definition of the fundamental groupoid of a graph $G$, and that, provided $G$ is loop-free, every nonidentity morphism $f$ in the fundamental groupoid of $G$ can be uniquely represented as a walk on $G$ without backtrackings; we call this walk the *standard form* of $f$. In Section 3 we recall the definition of the dual graph $G(\tau)$ of a triangulation $\tau$ of signature zero, and introduce the *leafy dual graph* $G^{\mathsf{?}}(\tau)$. In Section 4 we define the combinatorial notions of a *kink* of a morphism belonging to the fundamental groupoid $\pi_1(G^{\mathsf{?}}(\tau))$, and of the *resolution* of a kink, and show that the resolution of kinks satisfies the confluence conditions that allow us to apply the diamond lemma. In Section 5 we introduce the notion of a *kink of a curve* with respect to a triangulation of signature zero, and show that the absence of kinks in a curve is independent of the triangulation with respect to which it is considered. In Section 6 we see that the (2-orbifold) fundamental groupoid of $(\Sigma,\mathbb{M},\mathbb{P})$ is isomorphic to the (2-orbifold) fundamental groupoid of $G^{\mathsf{?}}(\tau)$. In Section 7 we present our main result.

# 2 Background

## 2.1 Groupoids, normal multilocular subgroups, and quotient groupoids

Recall that a *groupoid* is a category in which every morphism is invertible. Thus, for every object $x$ of a groupoid $\Gamma$, the endomorphism set $\Gamma(x,x)$ is a group under composition.

**Definition 2.1** A *multilocular subgroup* of a groupoid $\Gamma$ is a collection $H=(H(x))_{x\in\mathrm{obj}(\Gamma)}$ of subgroups $H(x)\subseteq\Gamma(x,x)$. A *normal multilocular subgroup* is a multilocular subgroup $H=(H(x))_{x\in\mathrm{obj}(\Gamma)}$ with the property that for every two objects $x$ and $y$ of $\Gamma$, every $h\in H(x)$, and every morphism $g\in\Gamma(x,y)$, we have $ghg^{-1}\in H(y)$.

Suppose that $H$ is a normal multilocular subgroup of the groupoid $\Gamma$. For each pair of objects $x$ and $y$ of $\Gamma$, let $\equiv_H\subseteq\Gamma(x,y)\times\Gamma(x,y)$ be the relation defined by the rule
$$f\equiv_H g \iff \text{there exists } h\in H(x) \text{ such that } f=gh.$$
A routine exercise shows that $\equiv_H$ is an equivalence relation. Moreover, if $x$, $y$, and $z$ are objects of $\Gamma$ and $f_1,g_1\in\Gamma(x,y)$ and $f_2,g_2\in\Gamma(y,z)$ are morphisms such that $f_1\equiv_H g_1$ and $f_2\equiv_H g_2$, then, taking $h_1\in H(x)$ and $h_2\in H(y)$ such that $f_1=g_1h_1$ and $f_2=g_2h_2$, we have
$$f_2f_1=g_2h_2g_1h_1=g_2g_1(g_1^{-1}h_2g_1)h_1,$$

with $(g_1^{-1}h_2g_1)h_1 \in H(x)$, which shows that $f_2 f_1 \equiv_H g_2 g_1$ too, just as in the case of groups and normal subgroups.

**Definition 2.2** Suppose $\Gamma$ is a groupoid and $H$ is a normal multilocular subgroup of $\Gamma$. The *quotient groupoid* $\Gamma/H$ is the category having the same objects as $\Gamma$, with

$$\begin{aligned} (\Gamma/H)(x,y) &:= \Gamma(x,y)/{\equiv_H} && \text{for } x,y \in \mathrm{obj}(\Gamma), \\ [f_2][f_1] &:= [f_2 f_1] && \text{for } x,y,z \in \mathrm{obj}(\Gamma),\ f_1 \in \Gamma(x,y), \text{ and } f_2 \in \Gamma(y,z). \end{aligned}$$

It is obvious that $\Gamma/H$ is indeed a groupoid, and that the canonical projection $\mathfrak{p}\colon \Gamma \to \Gamma/H$ is a full covariant functor, essentially surjective. Furthermore, the first isomorphism theorem is satisfied:

**Theorem 2.3** *If $\Gamma_1$ and $\Gamma_2$ are groupoids and $F\colon \Gamma_1 \to \Gamma_2$ is a full covariant functor, then, setting $H := \big(\ker\big(\Gamma_1(x,x) \xrightarrow{F} \Gamma_2(F(x),(F(x)))\big)\big)_{x \in \mathrm{obj}\,\Gamma_1}$, there is a unique covariant functor $\overline{F}\colon \Gamma_1/H \to \Gamma_2$ such that the diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{F} & \Gamma_2 \\ \downarrow & \nearrow_{\overline{F}} & \\ \Gamma_1/H & & \end{array}$$

*commutes. The functor $\overline{F}$ is fully faithful, and if $F$ is bijective on objects (resp. essentially surjective), then $\overline{F}$ is an isomorphism of categories (resp. an equivalence of categories).*

It is straightforward to verify that if $\Gamma$ is a groupoid and $\mathcal{F} = \{H_\alpha\}_{\alpha \in I}$ is a nonempty class of normal multilocular subgroups of $\Gamma$, then $\big(\bigcap_{\alpha \in I} H_\alpha(x)\big)_{x \in \mathrm{obj}(\Gamma)}$ is again a normal multilocular subgroup of $\Gamma$. Hence any collection $S = (S(x))_{x \in \mathrm{obj}(\Gamma)}$ of subsets $S(x) \subseteq \Gamma(x,x)$ generates a normal multilocular subgroup $H$ of $\Gamma$. Such $H$ has the property that for each $x \in \mathrm{obj}(\Gamma)$, the normal subgroup of $\Gamma(x,x)$ generated by $S(x)$ is contained in $H(x)$, but the containment may be proper if $H(x)$ is not a characteristic subgroup of $\Gamma(x,x)$. In any case, it is readily seen that $H(x)$ coincides with the subgroup of $\Gamma(x,x)$ generated by $\bigcup_{y \in \mathrm{obj}(\Gamma)} \{g^{-1}hg \mid h \in S(y),\ g \in \Gamma(x,y)\}$.

## 2.2 The fundamental groupoid of a graph

Let $G$ be a finite graph. As is common practice, whenever we want to think of $G$ as a topological space, we shall identify $G$ with a 1-dimensional CW-complex having the vertices of $G$ as 0-cells, and the edges of $G$ as 1-cells.

Define a category $\pi_1(G)$ as follows. Its objects are the vertices of $G$. Given two objects $u$ and $v$ of $\pi_1(G)$, we set

$$\pi_1(G)(u,v) := P(u,v)/{\simeq},$$

where $P(u,v)$ is the set of all continuous curves from $u$ to $v$ in $G$ seen as a topological space, and $\simeq\subseteq P(u,v)\times P(u,v)$ is the equivalence relation of homotopy relative to extreme points. Composition in $\pi_1(G)$ is induced by concatenation of curves. It is well known, and easy to see, that $\pi_1(G)$ is a groupoid, the *fundamental groupoid of* $G$.

**Definition 2.4** Let $G$ be a loop-free graph.

(1) A *backtrack-free walk* on $G$ is a finite sequence

$$f=(u_0,e_1,u_1,e_2,\dots,e_{n-1},u_{n-1},e_n,u_n),$$

where $u_0,u_1,\dots,u_n$ are vertices of $G$, and $e_1,\dots,e_n$ are edges of $G$, such that

- for $l=1,\dots,n$, the edge $e_l$ connects the vertices $u_l$ and $u_{l-1}$,
- for $l=1,\dots,n$, the edges $e_l$ and $e_{l-1}$ are not equal.

(2) A *closed backtrack-free walk* is a backtrack-free walk $f=(u_0,e_1,\dots,e_n,u_n)$ such that $u_0=u_n$ and $e_1\neq e_n$.

(3) We say that two closed backtrack-free walks

$$f=(u_0,e_1,u_1,\dots,u_{n-1},e_n,u_n) \quad \text{and} \quad g=(v_0,d_1,v_1,\dots,v_{m-1},d_m,v_m)$$

are *rotationally equivalent*, and write $f\sim_{\mathrm{rot}} g$, if $n=m$ and for some index $k\in\{0,\dots,n-1\}$ we have

$$g=(u_k,e_{k+1},u_{k+1},\dots,u_{n-1},e_n,u_0,e_1,u_1,\dots,u_{k-1},e_k,u_k).$$

**Remark 2.5** Let $G$ be a loop-free graph. Rotational equivalence is an equivalence relation on the set of closed backtrack-free walks. The equivalence class of such an $f$ will be denoted by $[F]_{\mathrm{rot}}$.

Let $G$ be a loop-free graph. A backtrack-free walk $(u_0,e_1,u_1,e_2,\dots,e_{n-1},u_{n-1},e_n,u_n)$ determines a morphism from $u_0$ to $u_n$ in the fundamental groupoid $\pi_1(G)$ by concatenating the curves obtained by parametrizing each edge $e_l$ as a continuous curve from $u_{l-1}$ to $u_l$. Similarly, each rotational equivalence class $C$ determines an element of the so-called *free-homotopy fundamental group* $\pi_1^{\mathrm{free}}(G)$ by taking, for any representative $f\in C$ (thus $C=[f]_{\mathrm{rot}}$), the free-homotopy class of the closed curve $\mathbb{S}^1\to G$ defined by concatenating the curves obtained by parametrizing each edge $e_l$ as a continuous curve from $u_{l-1}$ to $u_l$.

The next result is a consequence of eg [19, Chapter 4].

**Theorem 2.6** *Suppose that $G$ is a loop-free graph.*

1. *For every pair of vertices $u_0$ and $v_0$ of $G$, every morphism $f:u_0\to v_0$ in the fundamental groupoid $\pi_1(G)$ can be represented uniquely as a backtrack-free walk $(u_0,e_1,u_1,e_2,\dots,e_{n-1},u_{n-1},e_n,v_0)$.*
2. *Every free-homotopy class belonging to $\pi_1^{\mathrm{free}}(G)$ can be represented uniquely as a rotational equivalence class of closed backtrack-free walks on $G$.*

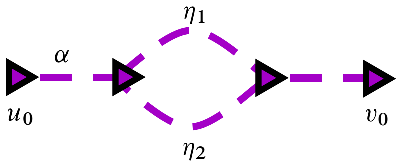


Figure 3

**Definition 2.7** Let $G$ be a loop-free graph, and let $f$ be a morphism in $\pi_1(G)$ (resp. a free-homotopy class belonging to $\pi_1^{\text{free}}(G)$). The unique backtrack-free walk (resp. the unique rotational equivalence class of closed backtrack-free walks) representing $f$ in Theorem 2.6 will be called the *standard form* of $f$ and written as $\phi(f)$.

Thus, if $\phi(f) = (u_0, e_1, u_1, e_2, \dots, e_{n-1}, u_{n-1}, e_n, v_0)$, then in the fundamental groupoid $\pi_1(G)$ we have the equality

$$f = (u_0, e_1, u_1) * (u_1, e_2, u_2) * \cdots * (u_{n-2}, e_{n-1}, u_{n-1}) * (u_{n-1}, e_n, v_0),$$

where $*$ is the operation induced by concatenation of paths (written from left to right, that is, opposite to the usual way of composing morphisms in a category, ie $\circ = *^{\text{op}}$ for $\pi_1(G)$). In particular, $\mathbb{1}_{u_0} = \phi(\mathbb{1}_{u_0}) = (u_0)$.

Notice that in the situation of Theorem 2.6, if $(u_0, e_1, \dots, e_n, v_0)$ is the standard form of the nonidentity morphism $f: u_0 \to v_0$, then $(v_0, e_n, \dots, e_1, u_0)$ is the standard form of the inverse $f^{-1}: v_0 \to u_0$.

By abuse of notation, we denote any given backtrack-free walk $(u_0, e_1, u_1, e_2, \dots, e_{n-1}, u_{n-1}, e_n, u_n)$ or rotation class of backtrack-free walks $[(u_0, e_1, u_1, e_2, \dots, e_{n-1}, u_{n-1}, e_n, u_n)]_{\text{rot}}$ simply by

$$(u_0, e_1, e_2, \dots, e_{n-1}, e_n, u_n).$$

**Example 2.8** The graph $G$ depicted in Figure 3 does not have loops, so Theorem 2.6 can be applied to it. The endomorphism group $\pi_1(G)(u_0, u_0)$ is infinite cyclic, generated by either $f = (u_0, \alpha, \eta_1, \eta_2, \alpha, u_0)$ or $f^{-1} = (u_0, \alpha, \eta_2, \eta_1, \alpha, u_0)$. For $n > 0$, the standard forms of $f^n$ and $f^{-n}$ are, respectively,

$$(u_0, \alpha, \underbrace{\eta_1, \eta_2, \eta_1, \eta_2, \dots, \eta_1, \eta_2}_{\text{length}=2n}, \alpha, u_0) \quad \text{and} \quad (u_0, \alpha, \underbrace{\eta_2, \eta_1, \eta_2, \eta_1, \dots, \eta_2, \eta_1}_{\text{length}=2n}, \alpha, u_0).$$

## 3 The (leafy) dual graph of a triangulation of signature zero

The following notion is in sync with [9, Definition 9.1 and Section 9.2].

**Definition 3.1** Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a (possibly punctured) surface with nonempty boundary. An ideal triangulation $\tau$ of $(\Sigma, \mathbb{M}, \mathbb{P})$ is said to have *signature zero* if every puncture is enclosed by a self-folded triangle of $\tau$.

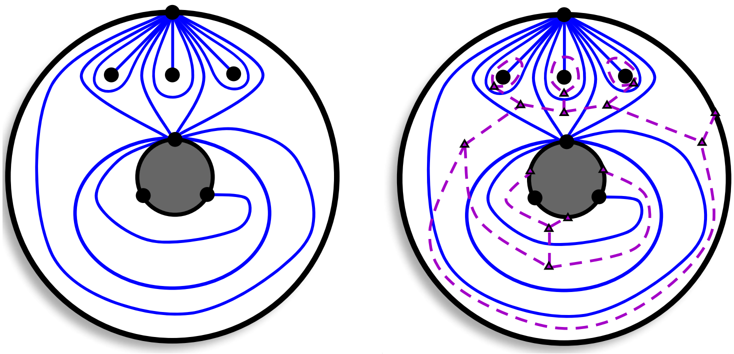

Figure 4

For each ideal triangulation $\tau$ of signature zero, the *dual graph* $G(\tau)$ is defined as follows. The vertices of $G(\tau)$ are the triangles of $\tau$ and the boundary segments of $(\Sigma, \mathbb{M}, \mathbb{P})$. For each arc $k$ of $\tau$ we put an edge connecting the triangles that share $k$, and for each boundary segment $s$ of $(\Sigma, \mathbb{M}, \mathbb{P})$ we put an edge connecting $s$ to the unique triangle of $\tau$ that contains it. Notice that

- the self-folded triangles of $\tau$ are precisely the vertices incident to loops of the graph $G(\tau)$,
- every triangle of $\tau$ has valency 3 as a vertex of $G(\tau)$, whereas each boundary segment has valency 1.

We turn $G(\tau)$ into a ribbon graph (or fat graph) in a natural way by letting the cyclic order on the edges incident to each vertex $v$ of $G(\tau)$ be the clockwise order around $v$, according to the orientation of $\Sigma$.

**Example 3.2** In Figure 4 we can see a triangulation $\tau$ of an annulus with one marked point on one boundary component, three on the other, and three punctures. We can also see the dual graph $G(\tau)$ drawn on the surface.

**Definition 3.3** Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a (possibly punctured) surface with nonempty boundary, and $\tau$ a signature-zero ideal triangulation of $(\Sigma, \mathbb{M}, \mathbb{P})$. The *leafy dual graph* of $\tau$ is the graph $G^{⫯}(\tau)$ obtained from $G(\tau)$ after applying the following combinatorial procedure. For each self-folded triangle $v$ of $\tau$,

(1) split its corresponding loop $\eta_v$ into two distinct edges $\eta_{v,1}$ and $\eta_{v,2}$, each connecting $v$ to a newly introduced vertex $w_v$,

(2) introduce a leaf $\ell_v$ incident to $w_v$; call $z_v$ the vertex of $\ell_v$ distinct from $w_v$.

We extend the ribbon graph structure of $G(\tau)$ to a ribbon graph structure of $G^{⫯}(\tau)$ as indicated in Figure 5.

**Remark 3.4** We stress the fact that all the vertices of $G(\tau)$ are vertices of $G^{⫯}(\tau)$ as well, and that every boundary segment of $(\Sigma, \mathbb{M}, \mathbb{P})$ is a vertex of $\tau$.

Every edge of the leafy dual graph $G^{⫯}(\tau)$ is incident to a vertex of valency 3. Given a valency-3 vertex $u$ of $G^{⫯}(\tau)$ and an edge $e$ of $G^{⫯}(\tau)$ containing $u$, we use the ribbon graph structure of $G^{⫯}(\tau)$ to define

$$\begin{aligned} e^{+,u} &:= \text{the edge incident to } u \text{ which is preceded by } e \text{ around } u, \\ e^{-,u} &:= \text{the edge incident to } u \text{ which is followed by } e \text{ around } u. \end{aligned} \tag{3-1}$$

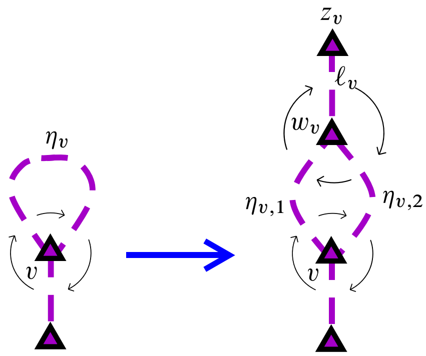


Figure 5

**Example 3.5** Consider the triangulation $\tau$ from Example 3.2. On the left-hand side of Figure 6 we can see the ribbon graph $G(\tau)$. On the right-hand side of the figure we can see the ribbon graph $G^{⫯}(\tau)$.

Although the next result is quite standard, we briefly sketch its proof, essentially following [1, Section 2.3; 3, Section 2.2; 4, Section 3.2]. Along the way, we emphasize that the standard construction of the ribbon surface $\Sigma(G^{⫯}(\tau))$ associated to $G^{⫯}(\tau)$ allows us to see it as the result of gluing very rigid hexagonal ribbons. This will be very useful later on, to fix a very specific class of curves for which the notion of a "kink" is easy to define, and with the property that every morphism in the fundamental groupoid $\pi_1(\Sigma(G^{⫯}(\tau)))$ is represented by at least one curve in the class.

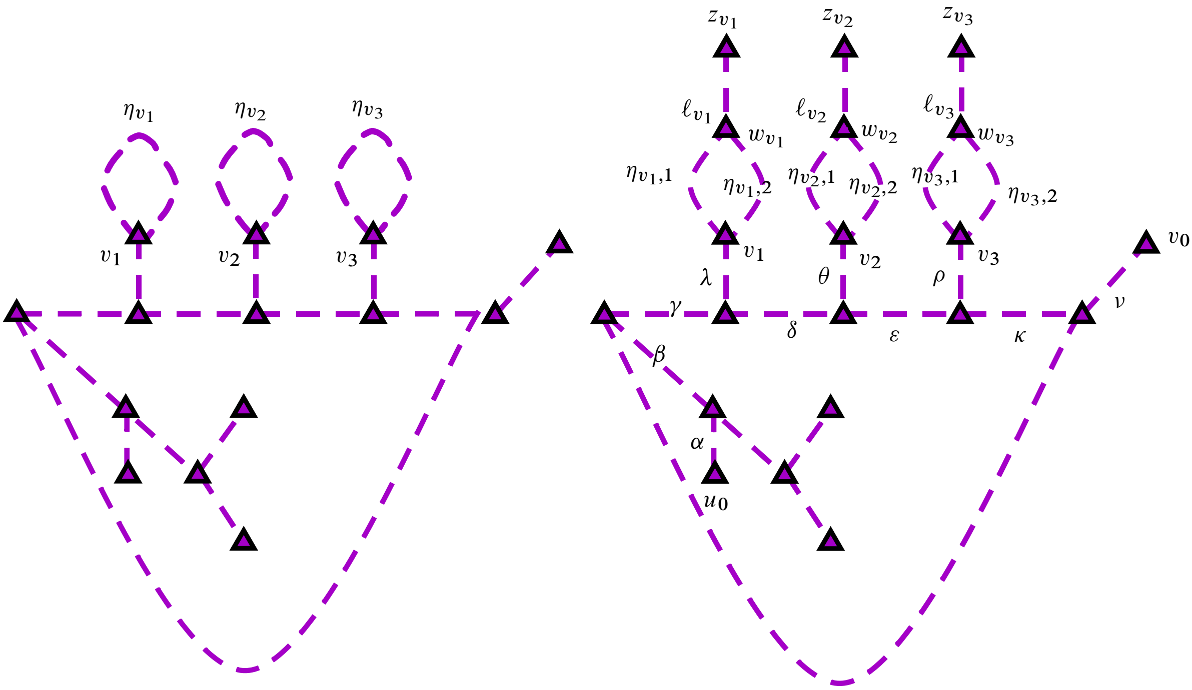


Figure 6

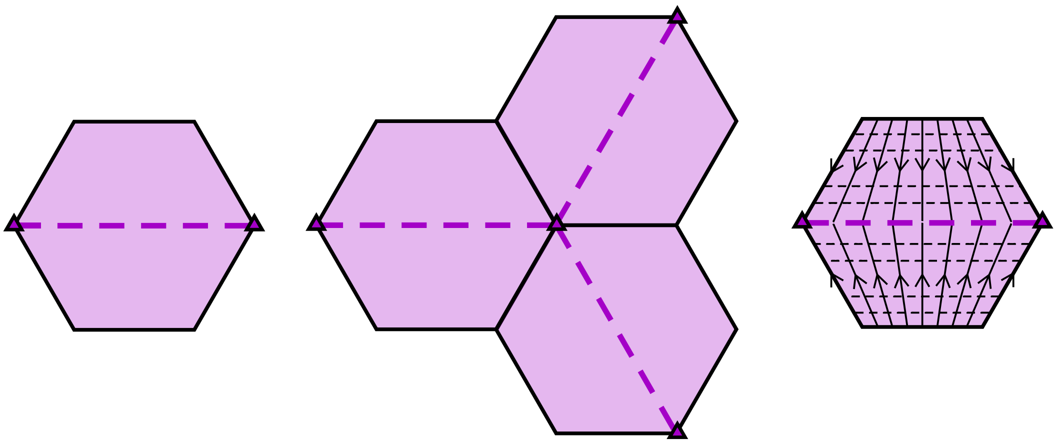

Figure 7

**Theorem 3.6** *The leafy dual graph $G^{♀}(\tau)$ is a strong deformation retract of $\Sigma \setminus \mathbb{P}$.*

**Proof** For each edge $e$ of $G^{♀}(\tau)$, let $R_e$ be an open regular oriented Euclidean hexagon inscribed in the unit circle in the complex plane, and take an embedding of $e$ into $R_e$ as a Euclidean straight line segment joining a pair of radially opposite vertices. We fix such an $R_e$ and such an embedding of $e$ into $R_e$ once and for all; see Figure 7, left.

For each valency-3 vertex $u$ of $G^{♀}(\tau)$, take an edge $e$ of $G^{♀}(\tau)$ containing $u$, and glue the hexagons $R_e$, $R_{e^{+,u}}$, and $R_{e^{-,u}}$ as indicated in Figure 7, center; see also (3-1). Denote by $\Sigma(G^{♀}(\tau))$ the result of performing this gluing over all valency-3 vertices of $G^{♀}(\tau)$. We shall refer to $\Sigma(G^{♀}(\tau))$ as the *ribbon surface* of $G^{♀}(\tau)$. We can take a natural open piecewise-differentiable continuous injective function

$$\iota_\tau \colon \Sigma(G^{♀}(\tau)) \to \Sigma \setminus (\mathbb{P} \cup \partial\Sigma) \tag{3-2}$$

that embeds $\Sigma(G^{♀}(\tau))$ as an open subsurface of $\Sigma \setminus \mathbb{P}$. Similarly to [1, Section 2.3; 3, Section 2.2; 4, Section 3.2], it is not hard to see that this embedding admits a strong deformation retraction $\Sigma \setminus \mathbb{P} \to \Sigma(G^{♀}(\tau))$.

For each edge $e$ of $G^{♀}(\tau)$, let $\varrho_e \colon R_e \to e$ be the piecewise-linear strong deformation retraction sketched in Figure 7, right. Define $\varrho \colon \Sigma(G^{♀}(\tau)) \to G^{♀}(\tau)$ by setting $\varrho|_{R_e} := \varrho_e$ for every edge $e$. Then $\varrho$ is a strong deformation retraction. Composing $\varrho$ with a strong deformation retraction $\Sigma \setminus \mathbb{P} \to \Sigma(G^{♀}(\tau))$ from the previous paragraph, we obtain a strong deformation retraction $\rho \colon \Sigma \setminus \mathbb{P} \to G^{♀}(\tau)$. $\square$

**Example 3.7** For the ribbon graph $G^{♀}(\tau)$ from Figure 6, the ribbon surface $\Sigma(G^{♀}(\tau))$ can be visualized in Figure 8. Strictly speaking, the picture is incorrect, since the hexagonal ribbons $R_e$ are not drawn as regular hexagons congruent to each other. Drawing them as such would interfere with an intuitive visualization of $\Sigma(G^{♀}(\tau))$.

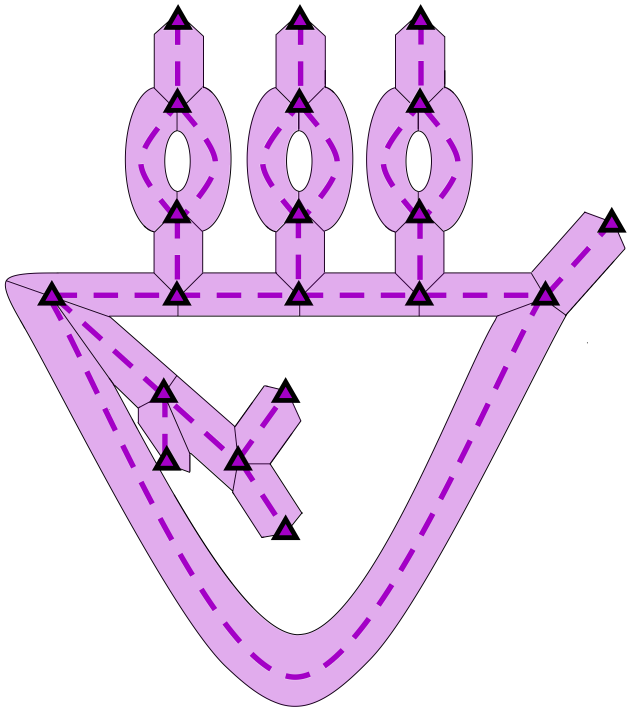

Figure 8

## 4 Kinks of walks on graphs

From this point on we will work only with triangulations of signature zero. By construction, no edge of $G^{\dagger}(\tau)$ is a loop. Hence Theorem 2.6 can be applied. Let $f$ be either

(4-1) a nonidentity morphism in $\pi_1(G^{\dagger}(\tau))$ between vertices of $G(\tau)$ distinct from all the self-folded triangles of $\tau$, or

(4-2) a noncontractible free-homotopy class belonging to $\pi_1^{\mathrm{free}}(G^{\dagger}(\tau))$.

Define a finite sequence $\boldsymbol{s}(f)=(\varepsilon_1,\varepsilon_2,\dots)$ of signs $\varepsilon_j\in\{+,-\}$ as follows. Let $\phi(f)=(u_0,e_1,\dots,e_n,v_0)$ be the standard form of $f$. If $n=1$, we set $\boldsymbol{s}(f)$ to be the empty sequence. Otherwise, for $j=1,\dots,n-1$, set $\varepsilon_j$ to be the sign with the property that

$$e_{j+1}=e_j^{\varepsilon_j,u_j}.$$

If $f$ belongs to $\pi_1^{\mathrm{free}}(G^{\dagger}(\tau))$, we further set $\varepsilon_n$ to be the sign with the property that

$$e_1=e_n^{\varepsilon_n,u_n}.$$

We define

$$\boldsymbol{s}(f):=\begin{cases}(\varepsilon_1,\dots,\varepsilon_{n-1}) & \text{if } f \text{ is a morphism in } \pi_1(G),\\ (\varepsilon_1,\dots,\varepsilon_{n-1},\varepsilon_n) & \text{if } f \text{ belongs to } \pi_1^{\mathrm{free}}(G^{\dagger}(\tau)).\end{cases}$$

Notice that if $f\in\pi_1^{\mathrm{free}}(G^{\dagger}(\tau))$, then $\boldsymbol{s}(f)$ is defined up to a cyclic rotation of its entries.

**Example 4.1** Consider the vertices $u_0$ and $v_0$ of the graph $G^{\dagger}(\tau)$ shown in Figure 6. For the following morphisms $f: u_0 \to v_0$ (written in standard form) in $\pi_1(G^{\dagger}(\tau))$, we can see the sign sequence $s(f)$ on the right (the commas have been omitted for space reasons):

$$\begin{aligned} f_1 &= (u_0\alpha\beta\gamma\delta\varepsilon\kappa\nu v_0), & s(f_1) &= (+----+),\\ f_2 &= (u_0\alpha\beta\gamma\lambda\eta_{v_1,1}\eta_{v_1,2}\lambda\delta\varepsilon\kappa\nu v_0), & s(f_2) &= (+-++-++--+),\\ f_3 &= (u_0\alpha\beta\gamma\lambda\eta_{v_1,1}\eta_{v_1,2}\eta_{v_1,1}\eta_{v_1,2}\lambda\delta\varepsilon\kappa\nu v_0), & s(f_3) &= (+-++---++--+),\\ f_4 &= (u_0\alpha\beta\gamma\lambda\eta_{v_1,2}\eta_{v_1,1}\lambda\delta\varepsilon\kappa\nu v_0), & s(f_4) &= (+-+-+-+--+). \end{aligned}$$

**Definition 4.2** Let $\tau$ be a triangulation of signature zero, and $G^{\dagger}(\tau)$ its leafy dual graph.

(1) If $f$ is a morphism as in (4-1), then a *kink* of $f$ is a segment $(e_j, e_{j+1}, \dots, e_m)$ of the standard form $\phi(f) = (u_0, e_1, \dots, e_n, u_n)$, with the property that $r := \frac{1}{2}(m-j-3)$ is a positive integer and there exists a self-folded triangle $v$ of $\tau$, and distinct indices $i, k \in \{1, 2\}$, such that either

(a) $e_{j+r+1} = \eta_{v,i}$ and $e_{j+r+2} = \eta_{v,k}$,

(b) for $t = 1, \dots, r$, we have $e_{j+t} = e_{m-t}$,

(c) $e_{j+r} = e_{j+r+3} \notin \{\eta_{v,k}, \eta_{v,i}\}$, and

(d) $\varepsilon_j = \varepsilon_{j+r+1} = \varepsilon_{m-1}$,

or

(i) $e_{j+1} = e_{j+3} = \cdots = e_{j+2r+1} = \eta_{v,i}$ and $e_{j+2} = e_{j+4} = \cdots = e_{j+2r+2} = \eta_{v,k}$, and

(ii) $e_j = e_m \notin \{\eta_{v,i}, \eta_{v,k}\}$.

(2) If $f$ is a free-homotopy class as in (4-2), then a *kink* of $f$ is a kink of any representative of the rotational equivalence class of the standard form $\phi(f)$.

We say that a kink has *multiplicity* 1 or $r+1$ according to whether it satisfies the first or second set of conditions above. Furthermore, the *core* of $\kappa$ is its segment $(e_{j+r+1}, e_{j+r+2})$ if the multiplicity of $\kappa$ is 1, and its segment $(e_{j+1}, e_{j+2}, \dots, e_{j+2r+1}, e_{j+2r+2})$ if the multiplicity of $\kappa$ is greater than 1.

**Example 4.3** With reference to Example 4.1, the morphisms $f_1$ and $f_2$ do not have kinks whatsoever. The morphisms $f_3$ and $f_4$ do have kinks (again, for space reasons we omit the commas):

$$\begin{aligned} f_3 &= (u_0\alpha\beta\gamma\underbrace{\lambda\eta_{v_1,1}\eta_{v_1,2}\eta_{v_1,1}\eta_{v_1,2}\lambda}_{\text{mult}=2,\,j=4,\,m=9,\,r=1}\delta\varepsilon\kappa\nu v_0), & s(f_3) &= (+-++---++--+),\\ f_4 &= (u_0\alpha\beta\underbrace{\gamma\lambda\eta_{v_1,2}\eta_{v_1,1}\lambda\delta}_{\text{mult}=1,\,j=3,\,m=8,\,r=1}\varepsilon\kappa\nu v_0), & s(f_4) &= (+-\boxed{+}-\boxed{+}-\boxed{+}--+). \end{aligned}$$

For the morphism $f_4$, the signs $\varepsilon_j = \varepsilon_{j+r+1} = \varepsilon_{m-1}$ from Definition 4.2(d) appear enclosed in squares.

**Definition 4.4** Let $\tau$ be a triangulation of signature zero, and $G^{\text{♀}}(\tau)$ its leafy dual graph.

(1) Suppose that $f$ is a morphism as in (4-1) and that $\kappa = (e_j, e_{j+1}, \dots, e_m)$ is a kink of $f$, and set $r := \frac{1}{2}(m - j + 3)$ as in Definition 4.2. A *partial resolution* of $\kappa$ in $f$ is the sequence $\tilde{\rho}_\kappa(f)$ obtained after applying one of the following two combinatorial operations to $f$:

- If $\kappa$ has multiplicity 1, switch the $(j+r+1)^{\text{th}}$ and $(j+r+2)^{\text{th}}$ entries of $f$.
- If $\kappa$ has multiplicity $r+1 > 1$, choose an index $s \in \{j+1, \dots, j+2r+1\}$ and switch the $s^{\text{th}}$ and $(s+1)^{\text{th}}$ entries of $f$.

(2) Suppose that $f$ is a free-homotopy class as in (4-2), and that $\kappa = (e_j, e_{j+1}, \dots, e_m)$ is a kink of $f$. Suppose further that $\phi(f) = \{f_1, \dots, f_n\}$ (see Definitions 2.7 and 4.2, as well as Remark 2.5), and that $\kappa$ appears as a kink of $f_j \in \phi(f)$. A *partial resolution* of $\kappa$ in $f$ is the sequence $\tilde{\rho}_\kappa(f_j)$ obtained by applying one of the two combinatorial operations just described to $f_j$.

Suppose that $f$ is as in (4-1) or (4-2), and $\kappa$ is a kink of $f$. If $f$ is a morphism and $\kappa$ has multiplicity 1, then the sequence $\tilde{\rho}_\kappa(f)$ is well defined, ie uniquely determined by $f$ and $\kappa$, whereas if $k$ has multiplicity $r+1 > 1$, then there is ambiguity in the definition of $\tilde{\rho}_\kappa(f)$ as a sequence. Furthermore, if $f$ is a free-homotopy class, then $\kappa$ does not appear as a kink of all the representatives $f_1, \dots, f_n$, as rotation of walks eventually breaks the appearance of $\kappa$ as a subsequence. However, we do have the following result, whose proof is immediate:

**Proposition 4.5** *Let $\tau$ be a triangulation of signature zero and $G^{\text{♀}}(\tau)$ its leafy dual graph, and let $f$ be as in (4-1) or (4-2).*

(1) *If $f$ is a morphism, then for any kink $\kappa$ of $f$, the sequence $\phi(\tilde{\rho}_\kappa(f))$ is uniquely determined by $f$ and $\kappa$. Thus, for any kink $\kappa$ of multiplicity greater than 1, the sequence $\phi(\tilde{\rho}_\kappa(f))$ is independent of the index $s$ chosen in Definition 4.4.*

(2) *If $f$ is a free-homotopy class, with standard form $\phi(f) = \{f_1, \dots, f_n\}$, and $\kappa$ appears as a kink of $f_i$ and $f_j$, with such appearances being brought to each other by the corresponding rotations that bring $f_i$ and $f_j$ to each other, then $[\phi(\tilde{\rho}_\kappa(f_i))]_{\text{rot}} = [\phi(\tilde{\rho}_\kappa(f_j))]_{\text{rot}}$.*

*Furthermore, the number of kinks of $\phi(\tilde{\rho}_\kappa(f))$ or $[\phi(\tilde{\rho}_\kappa(f_j))]_{\text{rot}}$, counted with multiplicity, is strictly less than the number of kinks of $f$, counted with multiplicity.*

**Definition 4.6** In the situation of Proposition 4.5,

$$\rho_\kappa(f) := \begin{cases} \phi(\tilde{\rho}_\kappa(f)) & \text{if } f \text{ is a morphism,} \\ [\phi(\tilde{\rho}_\kappa(f_j))]_{\text{rot}} & \text{if } f \text{ is a free-homotopy class,} \end{cases}$$

is the *resolution* of $\kappa$ in $f$.

**Lemma 4.7** *Let $\tau$ be a triangulation of signature zero, and $G^{\text{♀}}(\tau)$ its leafy dual graph, and let $f$ be as in (4-1) or (4-2).*

- *Resolving a kink of multiplicity 1 makes it disappear but does not affect the length of $f$.*

- *Resolving a kink of multiplicity* 2 *makes it disappear and makes the length of* $f$ *drop by at least* 4.
- *Resolving a kink* $\kappa$ *of* $f$ *of multiplicity* 3 *makes the multiplicity of* $\kappa$ *drop by* 2 *or* 3 *and the length of* $f$ *drop by* 4.
- *Resolving a kink* $\kappa$ *of* $f$ *of multiplicity greater than* 3 *makes the multiplicity of* $\kappa$ *drop by* 2 *and the length of* $f$ *drop by* 4.

The proof of Lemma 4.7 is left to the reader.

**Lemma 4.8** *Let* $u_0, v_0, w_0$ *be vertices of* $G(\tau)$ *distinct from all the self-folded triangles of* $\tau$. *Suppose* $f=(u_0,e_1,e_2,\dots,e_n,v_0)$ *and* $g=(v_0,e_n,\dots,e_2,d_1,w_0)$ *are nonidentity morphisms in* $\pi_1(G^{\dagger})$, *both written in standard form, with* $e_1\neq d_1$, *and that* $\kappa=(e_j,\dots,e_m)$ *is a kink of* $(e_2,\dots,e_n)$.

(1) *If the multiplicity of* $\kappa$ *is* 1, *then for* $r:=\frac{1}{2}(m-j-3)$ *we have*

$$\phi(\rho_\kappa(f)*g)=(u_0,e_1,\dots,e_{j+r-1},\underbrace{e_{j+r},e_{j+r+2},e_{j+r+1},e_{j+r+2},e_{j+r+1},e_{j+r}}_{\text{entries } j+r \text{ to } j+r+5},e_{j+r-1},\dots,d_1,w_0).$$

(2) *If the multiplicity of* $\kappa$ *is greater than* 1, *then*

$$\phi(\rho_\kappa(f)*g)=(u_0,e_1,e_2,\dots,e_{j-1},\underbrace{e_j,e_{j+1},e_{j+2},e_{j+1},e_{j+2},e_j}_{\text{entries } j \text{ to } j+5},e_{j-1},\dots,e_2,d_1,w_0).$$

*In any case,* $\phi(\rho_\kappa(f)*g)$ *has a kink* $\kappa'$ *of multiplicity* 2, *such that*

$$\phi(f*g)=(u_0,e_1,d_1,w_0)=\rho_{\kappa'}(\phi(\rho_\kappa(f)*g)).$$

**Proof** In the case where the multiplicity of $\kappa$ is 1, the computation of $\phi(\rho_\kappa(f)*g)$ can be seen in Figure 9, whereas in Figure 10 we have included the computation of $\phi(\rho_\kappa(f)*g)$ in the case where the multiplicity of $\kappa$ is exactly 3. The computation of $\phi(\rho_\kappa(f)*g)$ in the case where the multiplicity of $\kappa$ is an arbitrary integer greater than 1 is completely analogous.

With $\phi(\rho_\kappa(f)*g)$ explicitly computed, the assertions of the lemma can be directly verified. □

**Proposition 4.9** *Let* $\tau$ *be a triangulation of signature zero and* $G^{\dagger}(\tau)$ *its leafy dual graph, and let* $\varphi$ *be as in (4-1) or (4-2). If* $\kappa_0$ *and* $\kappa_0'$ *are distinct kinks of* $\varphi$, *then there exist finite sequences of kinks* $(\kappa_1,\dots,\kappa_l)$ *and* $(\kappa_1',\dots,\kappa_{l'}')$ *such that*

(1) *for each* $t=1,\dots,l$, $\kappa_t$ *is a kink of* $\rho_{\kappa_{t_1}}\rho_{\kappa_{t-2}}\cdots\rho_{\kappa_0}(\varphi)$,

(2) *for each* $t=1,\dots,l'$, $\kappa_t'$ *is a kink of* $\rho_{\kappa_{t_1}'}\rho_{\kappa_{t-2}'}\cdots\rho_{\kappa_0'}(\varphi)$,

(3) $\rho_{\kappa_l}\cdots\rho_{\kappa_1}\rho_{\kappa_0}(\varphi)=\rho_{\kappa_{l'}'}\cdots\rho_{\kappa_1'}\rho_{\kappa_0'}(\varphi)$ *if* $\varphi$ *is a morphism,*

(3′) $\rho_{\kappa_l}\cdots\rho_{\kappa_1}\rho_{\kappa_0}(\varphi)$ *is equal to* $\rho_{\kappa_{l'}'}\cdots\rho_{\kappa_1'}\rho_{\kappa_0'}(\varphi)$ *or its opposite orientation if* $\varphi$ *is a free-homotopy class as in (4-2).*

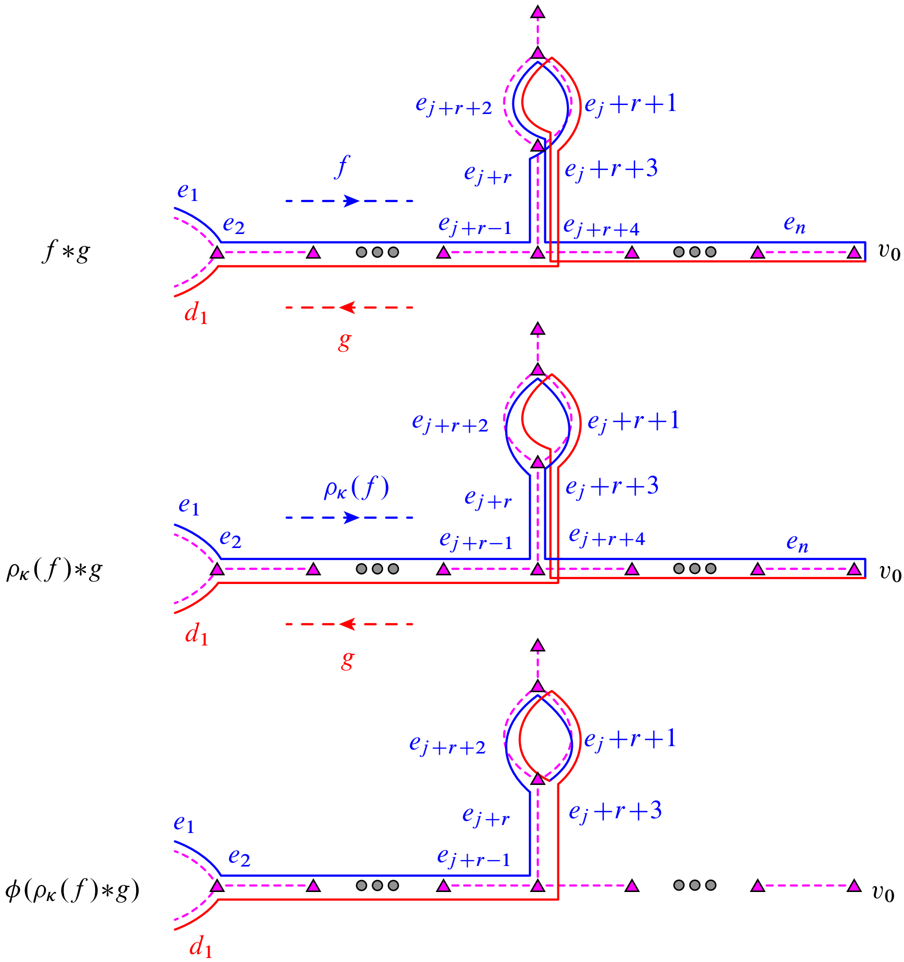


Figure 9

**Proof** Throughout the proof we will assume, without loss of generality, that $\varphi$ is already written in standard form.

Suppose first that $\varphi$ is a morphism as in (4-1). We distinguish five cases:

(i) when each of $\kappa_0$ and $\kappa_0'$ has multiplicity greater than 2,

(ii) when one of $\kappa_0$ and $\kappa_0'$ has multiplicity greater than 2 and the other one has multiplicity equal to 2,

(iii) when each of $\kappa_0$ and $\kappa_0'$ has multiplicity equal to 2,

(iv) when one of $\kappa_0$ and $\kappa_0'$ has multiplicity equal to 2 and the other one has multiplicity equal to 1,

(v) when each of $\kappa_0$ and $\kappa_0'$ has multiplicity equal to 1.

**Case 1** Using Lemma 4.7, it is easy to verify that if each of $\kappa_0$ and $\kappa_0'$ has multiplicity greater than 2, then $\kappa_0$ is a kink of $\rho_{\kappa_0'}(\varphi)$, $\kappa_0'$ is a kink of $\rho_{\kappa_0}(\varphi)$, and $\rho_{\kappa_0}(\rho_{\kappa_0'}(\varphi)) = \rho_{\kappa_0'}(\rho_{\kappa_0}(\varphi))$, so the proposition follows in this case.

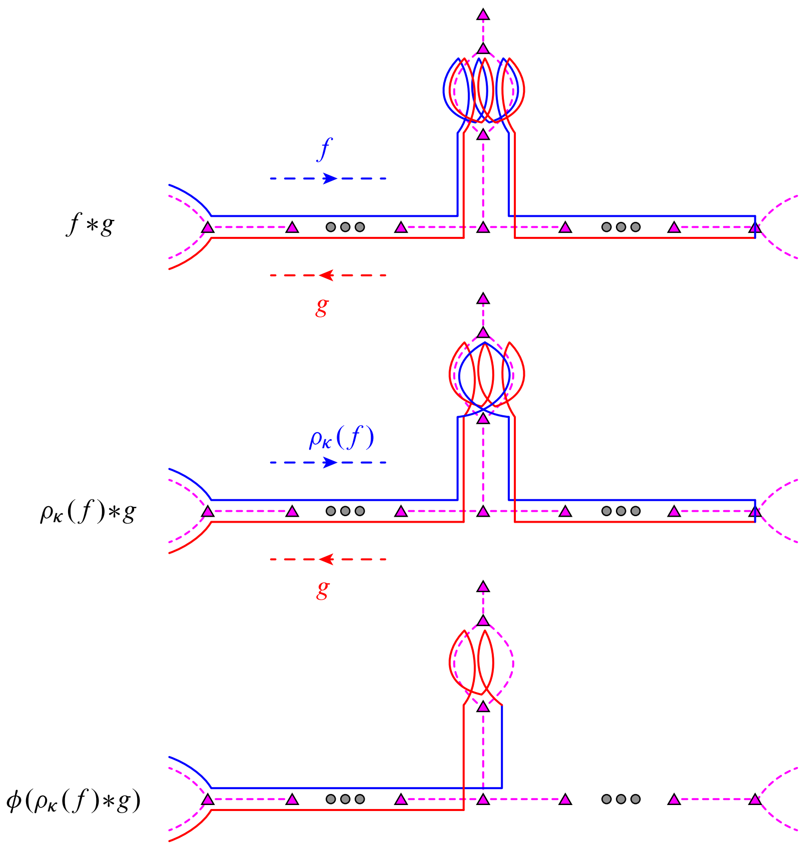


Figure 10

**Case 2** Suppose that the multiplicity of $\kappa_0$ is 2 and the multiplicity of $\kappa'_0$ is greater than 2. Then $\kappa_0$ is a kink of $\rho_{\kappa'_0}(\varphi)$ by Lemma 4.7, and $\varphi$ can be written as a concatenation

$$\varphi = \alpha * f * \kappa_0 * g * \beta,$$

where $\alpha$, $f$, $g$, and $\beta$ are morphisms in $G^{\wr}(\tau)$ between vertices of $G(\tau)$ distinct from all the self-folded triangles of $\tau$, with $f = (u_0, e_1, e_2, \dots, e_n, v_0)$ and $g = (v_0, e_n, \dots, e_2, d_1, w_0)$, and either

$$\begin{cases} e_1 = d_1 \text{ and } \alpha = \mathbb{1}_{u_0} = \beta, & \text{or} \\ e_1 \neq d_1. \end{cases}$$

If $e_1 \neq d_1$, then either $\kappa'_0$ is disjoint from $f$ and from $g$, or it overlaps one of them. If $\kappa'_0$ is entirely contained in $f$, then we can apply Lemma 4.8 to obtain a kink $\kappa'_1$ of $\rho_{\kappa_0}\rho_{\kappa'_0}(\varphi) = \phi(\alpha * \rho_{\kappa'_0}(f) * g * \beta)$ such that

$$\rho_{\kappa_0}(\varphi) = \phi(\alpha * f * g * \beta) = \rho_{\kappa'_1}\rho_{\kappa_0}\rho_{\kappa'_0}(\varphi),$$

which is the assertion of the proposition. Essentially the same argument shows that the proposition holds also when $\kappa'_0$ is entirely contained in $g$, or is disjoint from $f$ and from $g$, or overlaps one of them without being entirely contained in either.

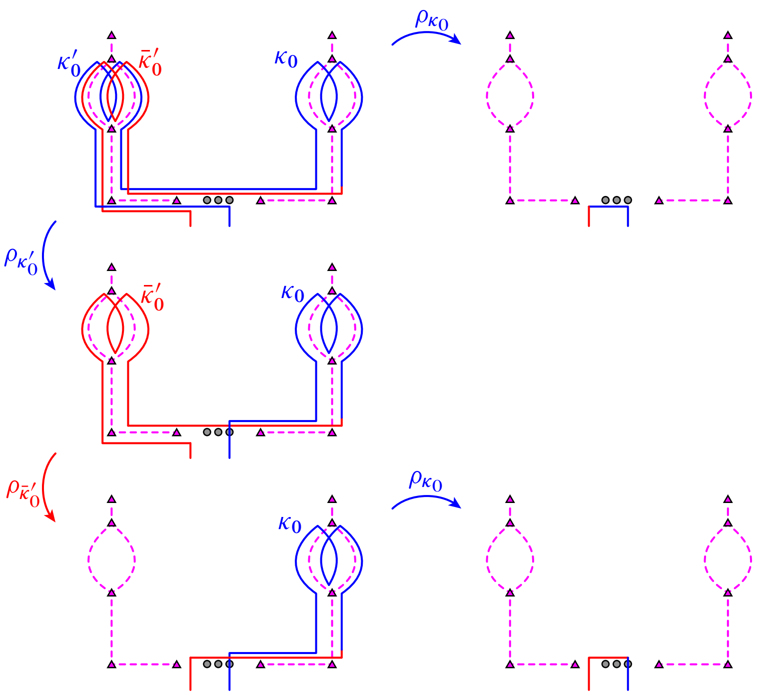


Figure 11

The same argument works also if $e_1 = d_1$ and $\alpha = \mathbb{1}_{u_0} = \beta$; notice that in this case $\rho_{\kappa_0}(\varphi) = \mathbb{1}_{u_0}$.

**Case 3** Suppose that each of $\kappa_0$ and $\kappa'_0$ has multiplicity equal to 2. Then $\varphi$ can be written as concatenations

$$\alpha * f * \kappa_0 * g * \beta = \varphi = \alpha' * f' * \kappa'_0 * g' * \beta',$$

where $\alpha$, $\alpha'$, $f$, $f'$, $g$, $g'$, $\beta$, and $\beta'$ are morphisms in $G^{\dagger}(\tau)$ between vertices of $G(\tau)$ that are distinct from all the self-folded triangles of $\tau$, with $f = (u_0, e_1, e_2, \dots, e_n, v_0)$, $g = (v_0, e_n, \dots, e_2, d_1, w_0)$, $f' = (u'_0, e'_1, e'_2, \dots, e'_n, v'_0)$, $g' = (v'_0, e'_n, \dots, e'_2, d'_1, w'_0)$, and either

$$\begin{cases} e_1 = d_1 \text{ and } \alpha = \mathbb{1}_{u_0} = \beta, & \text{or} \\ e_1 \neq d_1. \end{cases}$$

If $e_1 \neq d_1$, then $\kappa'_0$ either is disjoint from $f$ and from $g$, or it overlaps one of them. We analyze the situation where $\kappa'_0$ is entirely contained in $f$, and leave in the hands of the reader the analysis of the remaining possibilities.

So, suppose $\kappa'_0$ is entirely contained in $f$. Then $\kappa_0$ either is disjoint from $f'$ and from $g'$, or it overlaps one of them. If $\kappa_0$ is disjoint from both $f'$ and $g'$, then Figure 11 shows us a kink $\bar{\kappa}'_0$ of $\rho_{\kappa'_0}(\varphi)$ such that $\kappa_0$ is a kink of $\rho_{\bar{\kappa}'_0}\rho_{\kappa'_0}(\varphi)$ and

$$\rho_{\kappa_0}(\varphi) = \rho_{\kappa_0}\rho_{\bar{\kappa}'_0}\rho_{\kappa'_0}(\varphi).$$

If $\kappa_0$ is entirely contained in $f$, then Figure 12 shows us kinks $\bar{\kappa}_0$ and $\bar{\kappa}'_0$ such that

$$\rho_{\bar{\kappa}'_0}\rho_{\kappa'_0}(\varphi) = \rho_{\bar{\kappa}_0}\rho_{\kappa_0}(\varphi).$$

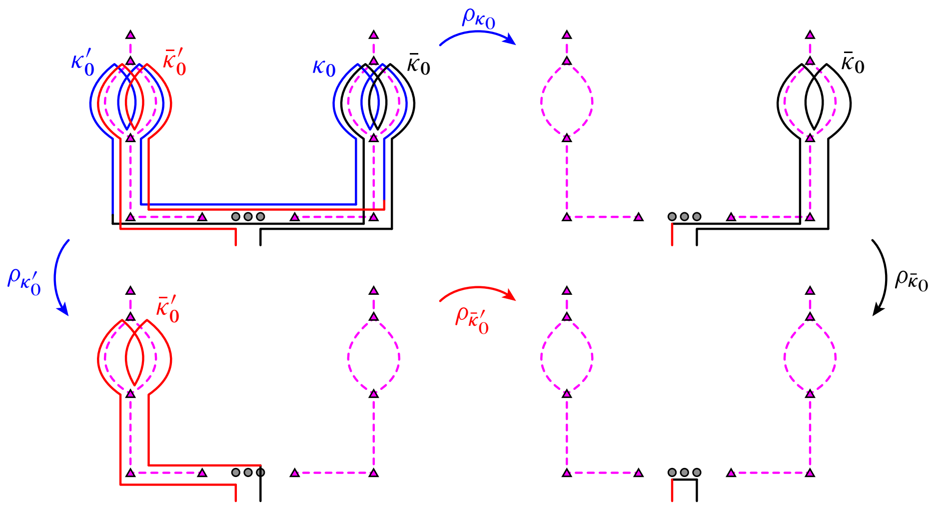


Figure 12

We leave the analysis of the cases where $\kappa_0$ is entirely contained in $g'$, or it overlaps $f'$ or $g'$ without being entirely contained in either, to the reader.

**Case 4** If one of $\kappa_0$ and $\kappa_0'$ has multiplicity equal to 2 and the other one has multiplicity equal to 1, one can resort to Lemma 4.8 as we did in Case 2 to obtain the assertion of the proposition.

**Case 5** If each of $\kappa_0$ and $\kappa_0'$ has multiplicity equal to 1, then $\rho_{\kappa_0'}\rho_{\kappa_0}(\varphi) = \rho_{\kappa_0}\rho_{\kappa_0'}(\varphi)$.

Suppose now that $\varphi$ is a free-homotopy class as in (4-2). If both $\kappa_0$ and $\kappa_0'$ appear as kinks of at least one of the closed backtrack-free walks $f$ representing $\varphi$, considering these walks as morphisms in $\pi_1(G^{\text{♀}}(\tau))$, and if the standard forms of the morphisms $\rho_{\kappa_0}(f)$ and $\rho_{\kappa_0'}(f)$ are closed backtrack-free walks (see Definition 2.4), then the result follows from the proof we have just given above. Otherwise, $\varphi$ is represented by one of the closed backtrack-free walks depicted in Figures 13 and 14 (in Figure 13, $\kappa_0$ and $\kappa_0'$ are images of antipodal segments of $\mathbb{S}^1$, whereas in Figure 14 they are not; see Remark 4.10 below).

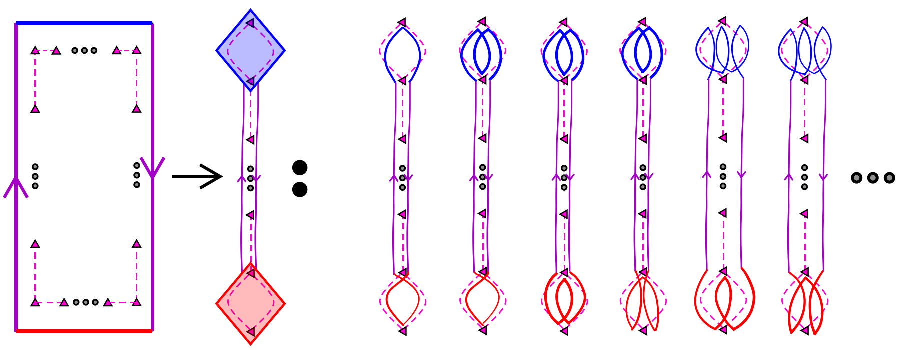

Figure 13

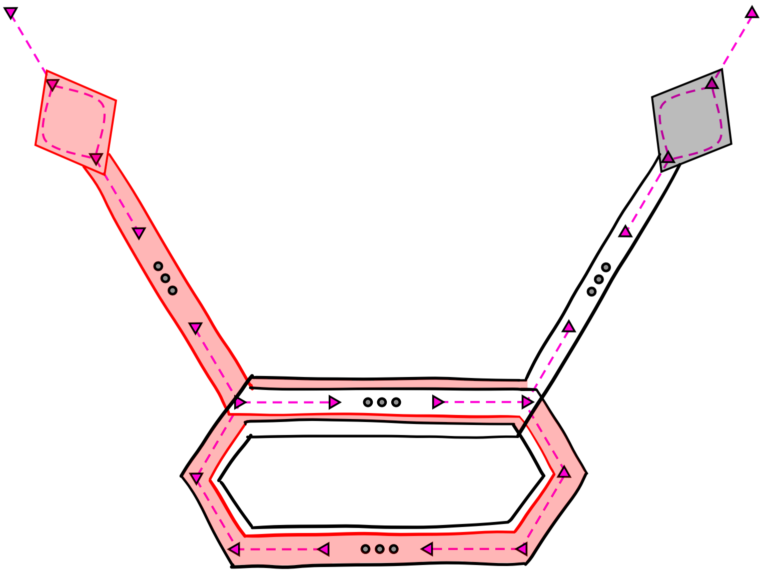

Figure 14

In the situation of Figure 13, the assertion of the proposition follows easily (it is here that one may have to take opposite orientations of the closed curves). The situation of Figure 14 is dealt with in Figure 15, where the assertion of the proposition is proved in the cases where the multiplicities of $\kappa_0$ and $\kappa_0'$ are less than 3. The proposition is very easy to verify in the cases where the multiplicities are at least 3 (see Lemma 4.7). □

**Remark 4.10** Consider a rectangle $Q$, made into a graph in such a way that the number of edges constituting the left vertical side of $Q$ is equal to the number of edges constituting the right vertical side of $Q$, and each of the horizontal sides of $Q$ is conformed by an even number of edges (the number of edges in the top side of $Q$ is not necessarily equal to the number of edges in the bottom side). What we mean by "$\kappa_0$ and $\kappa_0'$ are images of antipodal segments of $\mathbb{S}^1$" in the final part of the proof of Proposition 4.9 is that the backtrack-free walk representing $\varphi$ is given by a graph homomorphism $Q \to G^{\text{♀}}(\tau)$ as depicted in Figure 13, ie sending the horizontal sides of $Q$ to the respective subsequences $(e_{j+1}, \dots, e_{j+2r+2}) = (\eta_{v,i}, \eta_{v,k}, \dots, \eta_{v_i}, \eta_{v,k})$ of $\kappa_0$ and $\kappa_0'$, and the vertical sides of $Q$ to opposite orientations of the same walk on $G^{\text{♀}}(\tau)$. Depending on the parities of $\kappa_0$ and $\kappa_0'$, this sometimes has the effect that resolving $\kappa_0$ first and resolving $\kappa_0'$ first result in opposite orientations of the same closed walk on $G^{\text{♀}}(\tau)$.

**Theorem 4.11** *Let $\tau$ be a triangulation of signature zero, $G^{\text{♀}}(\tau)$ be its leafy dual graph, and $f$ be as in (4-1) or (4-2). Suppose that $(\kappa_1, \dots, \kappa_l)$ and $(\kappa_1', \dots, \kappa_{l'}')$ are finite sequences of kinks such that*

(1) *for each $t = 1, \dots, l$, $\kappa_t$ is a kink of $\rho_{\kappa_{t_1}} \rho_{\kappa_{t-2}} \cdots \rho_{\kappa_0}(f)$,*

(2) *for each $t = 1, \dots, l'$, $\kappa_t'$ is a kink of $\rho_{\kappa_{t_1}'} \rho_{\kappa_{t-2}'} \cdots \rho_{\kappa_0'}(f)$,*

(3) *neither $\rho_{\kappa_l} \cdots \rho_{\kappa_1} \rho_{\kappa_0}(f)$ nor $\rho_{\kappa_{l'}'} \cdots \rho_{\kappa_1'} \rho_{\kappa_0'}(f)$ has kinks.*

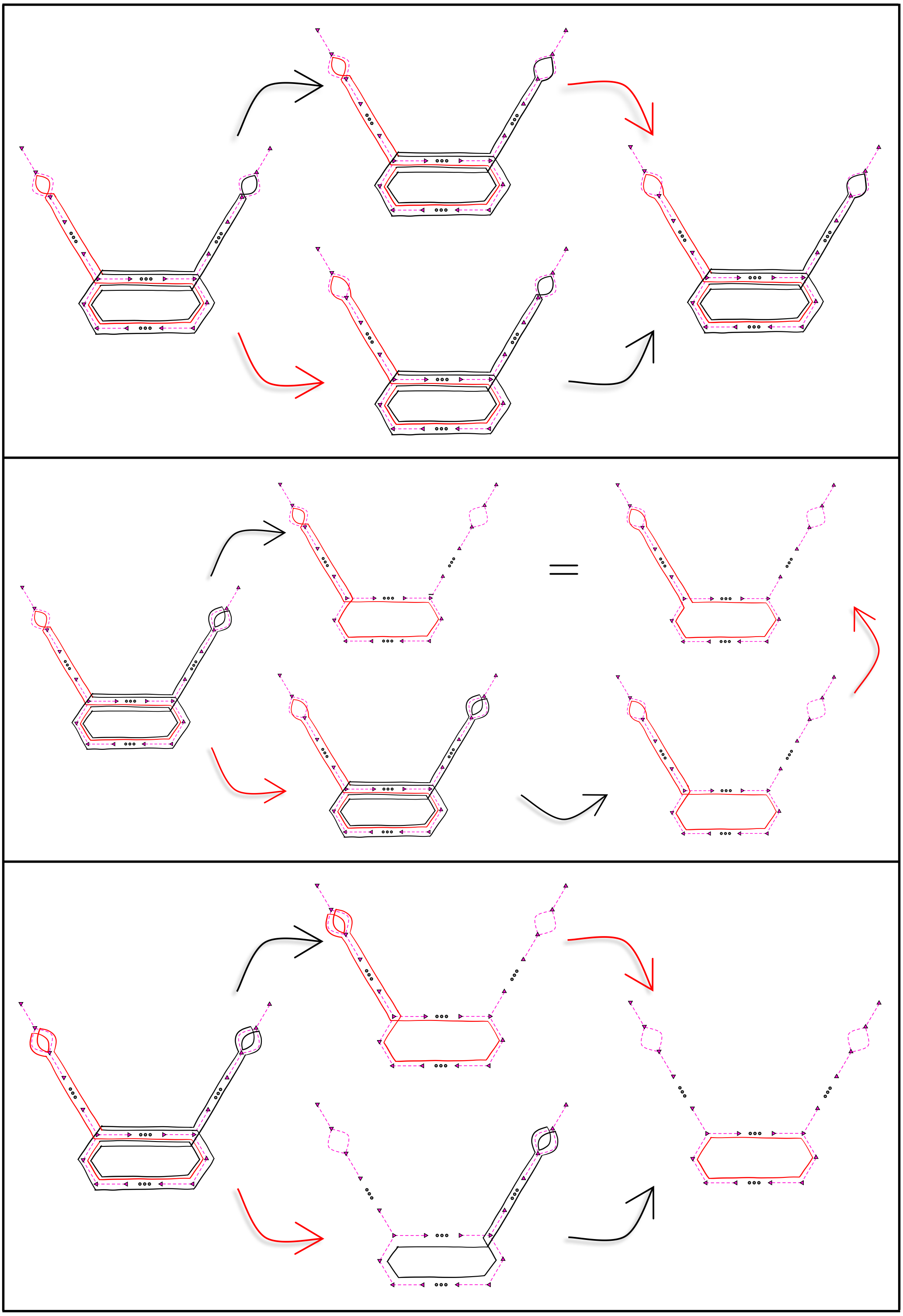

Figure 15

*Then*

- $\rho_{\kappa_l} \cdots \rho_{\kappa_1} \rho_{\kappa_0}(f) = \rho_{\kappa'_{l'}} \cdots \rho_{\kappa'_1} \rho_{\kappa'_0}(f)$ *if* $\varphi$ *is a morphism,*
- $\rho_{\kappa_l} \cdots \rho_{\kappa_1} \rho_{\kappa_0}(\varphi)$ *is equal to* $\rho_{\kappa'_{l'}} \cdots \rho_{\kappa'_1} \rho_{\kappa'_0}(\varphi)$ *or its opposite orientation if* $\varphi$ *is a free-homotopy class as in (4-2).*

**Proof** By Propositions 4.5 and 4.9, resolutions of kinks satisfy the hypotheses of the diamond lemma; see [14, Lemma 2.4], [20, Theorem 3] or [23, Theorem 1.7.10]. The result follows. □

## 5 Kinks of curves via kinks of walks on graphs

**Definition 5.1** A continuous function $\gamma\colon [0,1] \to \Sigma(G^{\dagger}(\tau))$ connecting valency-1 vertices of $G(\tau)$ (resp. continuous closed curve $\gamma\colon \mathbb{S}^1 \to \Sigma(G^{\dagger}(\tau))$) is called a *curve adapted to the ribbon structure of* $\Sigma(G^{\dagger}(\tau))$ if there is a finite partition $0 = t_0 < t_1 < \cdots < t_l = 1$ of the interval $[0,1]$ such that

(1) for $i = 1, \dots, l$, there is an edge $e_i$ of $G^{\dagger}(\tau)$ such that the image of the open interval $(t_{i-1}, t_i)$ (resp. the circle segment $e^{2\pi i (t_{i-1}, t_i)}$) under $\gamma$ is entirely contained in the interior of $R_{e_i}$,

(2) for $i = 2, \dots, l$, the edges $e_{i-1}$ and $e_i$ are distinct,

(3) for $i = 1, \dots, l$, the image $\gamma([t_{i-1}, t_i]) \subseteq R_{e_i}$ (resp. $\gamma(e^{2\pi i [t_{i-1}, t_i]}) \subseteq R_{e_i}$) has exactly one element in common with each fiber of the map $\varrho_{e_i}\colon R_{e_i} \to e_i$ introduced in the proof of Theorem 3.6,

(4) for $i, j = 1, \dots, l$, if $i \neq j$, then $\gamma([t_{i-1}, t_i]) \cap \gamma([t_{j-1}, t_j])$ (resp. $\gamma(e^{2\pi i [t_{i-1}, t_i]}) \cap \gamma(e^{2\pi i [t_{j-1}, t_j]})$) is a finite set.

Notice that if $\gamma\colon [0,1] \to \Sigma(G^{\dagger}(\tau))$ is a curve adapted to the ribbon structure of $\Sigma(G^{\dagger}(\tau))$, then the sequence $(\gamma(0), e_1, e_2, \dots, e_l, \gamma(1))$ is a morphism written in standard form in the fundamental groupoid $\pi_1(G^{\dagger}(\tau))$.

The next result can be seen to be a consequence of Theorems 2.6 and 3.6. Alternatively, it follows from [13].

**Theorem 5.2** *Let* $\tau$ *be a signature-zero triangulation, and let* $\varrho\colon \Sigma(G^{\dagger}(\tau)) \to G^{\dagger}(\tau)$ *be the strong deformation retraction defined in the proof of Theorem 3.6.*

(1) *Every continuous curve* $\delta\colon [0,1] \to \Sigma(G^{\dagger}(\tau))$ *connecting valency-1 vertices of* $G(\tau)$ *is homotopic rel* $\{0,1\}$ *to* $\gamma\colon [0,1] \to \Sigma(G^{\dagger}(\tau))$*, a curve adapted to the ribbon structure of* $\Sigma(G^{\dagger}(\tau))$*, say, with respect to a partition* $0 = t_0 < t_1 < \cdots < t_l = 1$ *of the interval* $[0,1]$*. Furthermore, the sequence* $(\gamma(0), e_1, e_2, \dots, e_l, \gamma(1))$ *from Definition 5.1 is a backtrack-free walk that coincides with the standard form of* $\varrho\delta$*, and is thus uniquely determined by* $\delta$*.*

(2) *Every continuous closed curve* $\delta\colon \mathbb{S}^1 \to \Sigma(G^{\dagger}(\tau))$ *is freely homotopic to* $\gamma\colon \mathbb{S}^1 \to \Sigma(G^{\dagger}(\tau))$*, a closed curve adapted to the ribbon structure of* $\Sigma(G^{\dagger}(\tau))$*, say, with respect to a partition* $0 = t_0 < t_1 < \cdots < t_l = 1$ *of the interval* $[0,1]$*. Furthermore, the sequence* $(\gamma(1), e_1, e_2, \dots, e_l, \gamma(1))$ *from Definition 5.1 is a closed backtrack-free walk whose rotational equivalence class coincides with the standard form of* $\varrho\delta$*, and is thus uniquely determined by* $\delta$*.*

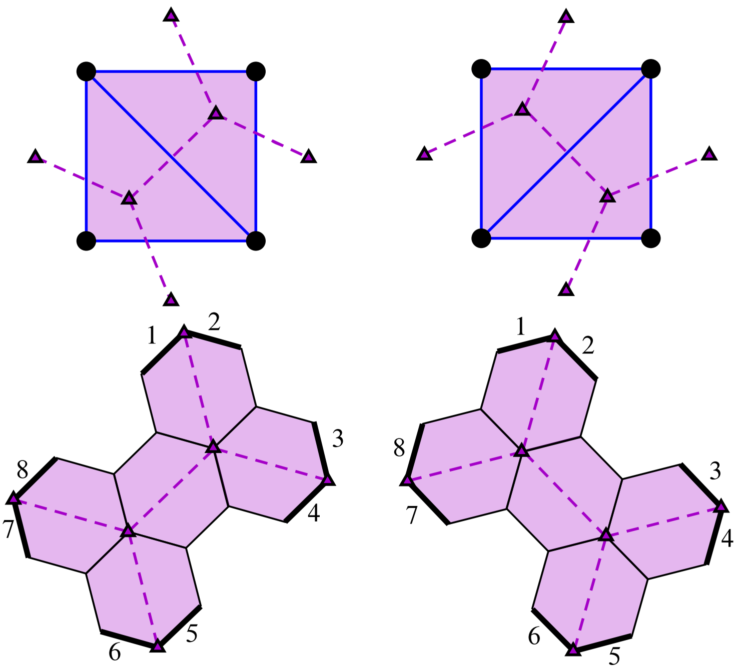


Figure 16

**Definition 5.3** Let $\tau$ be a signature-zero triangulation and $\varrho\colon \Sigma(G^{\natural}(\tau)) \to G^{\natural}(\tau)$ the strong deformation retraction defined in the proof of Theorem 3.6. Let $\gamma\colon [0,1] \to \Sigma(G^{\natural}(\tau))$ (resp. $\gamma\colon \mathbb{S}^1 \to \Sigma(G^{\natural}(\tau))$) be a curve adapted to the ribbon structure of $\Sigma(G^{\natural}(\tau))$ with respect to a partition $0 = t_0 < t_1 < \cdots < t_l = 1$ of the interval $[0,1]$. A *kink of $\gamma$ with respect to $\tau$* is a segment $\gamma|_{[t_j,t_m]}\colon [t_j,t_m] \to \Sigma(G^{\natural}(\tau))$ (resp. $\gamma|_{e^{2\pi i[a,b]}}\colon e^{2\pi i[a,b]} \to \Sigma(G^{\natural}(\tau))$) such that the subsequence $(e_j,\dots,e_m)$ of $\phi(\varrho\delta)$ it determines is a kink of $\phi(\varrho\delta)$.

**Proposition 5.4** *Suppose that $\tau$ and $\sigma$ are triangulations of signature zero. Let $\gamma_1$ and $\gamma_2$ be curves adapted to the ribbon structures of $\Sigma(G^{\natural}(\tau))$ and $\Sigma(G^{\natural}(\sigma))$, respectively. If $\iota_\tau(\gamma_1)$ and $\iota_\sigma(\gamma_2)$ are homotopic rel $\{0,1\}$ (resp. freely homotopic) in $\Sigma$ (see (3-2)), then $\gamma_1$ has at least one kink if and only if $\gamma_2$ has at least one kink.*

**Proof** Any two triangulations of $(\Sigma,\mathbb{M},\mathbb{P})$ of signature zero are related by a finite sequence of $\diamondsuit$-flips; see [9, Definition 9.11]. Hence it suffices to prove the proposition in the case where $\tau$ and $\sigma$ are related by a single $\diamondsuit$-flip. Furthermore, it is enough to show that, *given* $\gamma_1$, *there exists* a $\gamma_2$ with the properties that $\iota_\tau(\gamma_1)$ and $\iota_\sigma(\gamma_2)$ are homotopic rel $\{0,1\}$ (resp. freely homotopic) in $\Sigma$, and $\gamma_1$ has at least one kink if and only if $\gamma_2$ has at least one kink.

There are two types of $\diamondsuit$-flips, namely, the ones sketched in Figures 16 and 17. In each of these figures, we have drawn the subgraphs of $G^{\natural}(\tau)$ and $G^{\natural}(\sigma)$ conformed by the edges incident to the triangles of $\tau$

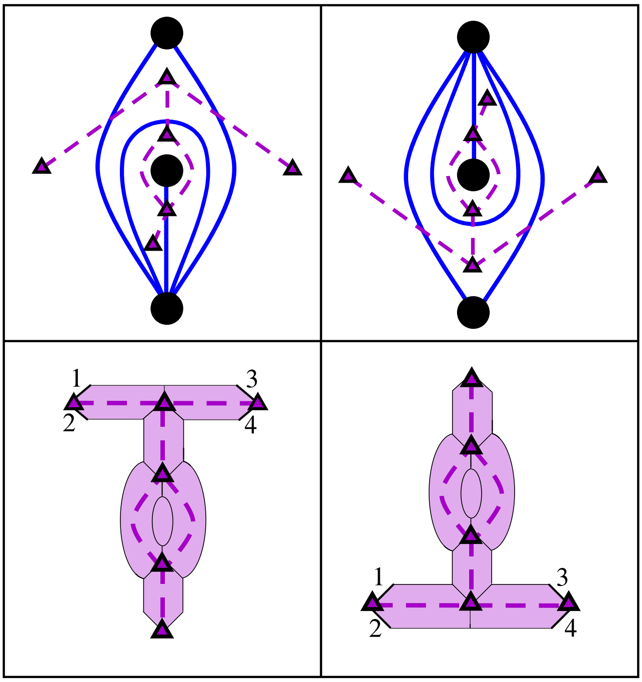


Figure 17

and $\sigma$ that contain the arcs involved in the $\Diamond$-flip. We have also drawn the ribbon subsurfaces of $\Sigma(G^{\dagger}(\tau))$ and $\Sigma(G^{\dagger}(\sigma))$ induced by these subgraphs. We shall denote these ribbon subsurfaces by $S(\tau)$ and $S(\sigma)$.

Outside the portions depicted in Figures 16 and 17, $G^{\dagger}(\tau)$ and $G^{\dagger}(\sigma)$ are identical, and $\Sigma(G^{\dagger}(\tau))$ and $\Sigma(G^{\dagger}(\sigma))$ are identical. Thus, in order to exhibit a $\gamma_2$ if we are given $\gamma_1$, it suffices to replace each segment $s$ of $\gamma_1$ contained in $S(\tau)$ by a segment $s'$ contained in $S(\sigma)$, connecting the endpoints of $s$, and satisfying the conditions in Definition 5.1, in such a way that $\iota_\tau(s)$ and $\iota_\sigma(s')$ are homotopic rel $\{0,1\}$ in $\Sigma \setminus (\mathbb{P} \cup \partial\Sigma)$. This replacement can be carried out combinatorially very easily, and has been sketched in Figure 18.

Letting $\gamma_2$ be the result of performing all of the above replacements, it becomes routine to verify that $\gamma_1$ has a kink with respect to $\tau$ if and only if $\gamma_2$ has a kink with respect to $\sigma$. □

**Remark 5.5** With the current definition of the multiplicity of a kink on a (closed) backtrack-free walk on $G^{\dagger}(\tau)$ (see Definition 4.2), for kinks of positive multiplicity it is not always true that their multiplicity is invariant under $\Diamond$-flips. However, a suitable refinement of the notion of multiplicity can be given that is invariant under $\Diamond$-flips. It would also be interesting to define the notion of a kink with respect to a triangulation not necessarily having signature zero, and proving the corresponding invariance under flips.

**Remark 5.6** By the work of Hansper [12], the two possible notions of an *admissible* string or band on a clannish algebra are equivalent; see also [10]. Now, it is extremely easy to see that if $\tau$ is a triangulation of

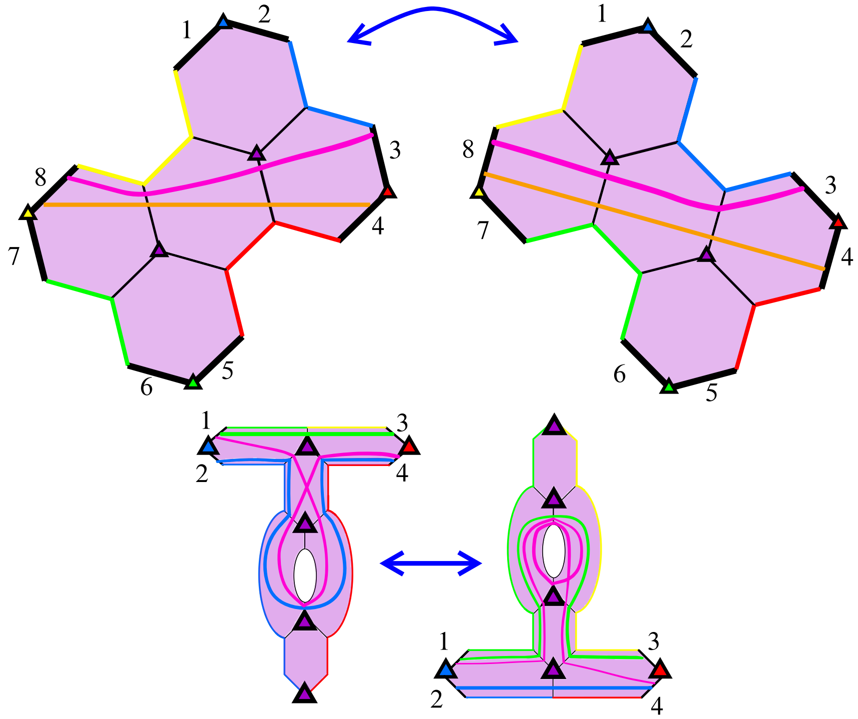


Figure 18

signature zero, then the Jacobian algebra $\Lambda(\tau)$, associated to $\tau$ by the second author in [16], is a clannish algebra. It turns out that the string or band on $\Lambda(\tau)$ determined by a curve on $(\Sigma, \mathbb{M}, \mathbb{P})$ is admissible if and only if the curve itself has no kinks.

## 6 Equivalence of orbifold fundamental groupoids

Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a surface with nonempty boundary. We define a subgroupoid of the fundamental groupoid $\pi_1(\Sigma \setminus \mathbb{P})$ as follows. Form a set $E$ by picking exactly one point from each connected component of $(\partial\Sigma) \setminus \mathbb{M}$. Define $\pi_1(\Sigma \setminus \mathbb{P}, E)$ to be the full subcategory of $\pi_1(\Sigma \setminus \mathbb{P})$ having $E$ as its set of objects.

For each point $u \in E$, each puncture $p \in \mathbb{P}$, and each continuous function $c\colon [0,1] \to \Sigma$ connecting $u$ to $p$ such that $(\partial\Sigma \cup \mathbb{P}) \cap c(0,1) = \varnothing$, take a continuous function $\gamma_{u,p,c}\colon [0,1] \to \Sigma$ with $\gamma_{u,p,c}(0) = u = \gamma_{u,p,c}(1)$ and $(\partial\Sigma \cup \mathbb{P}) \cap c(0,1) = \varnothing$, closely following $c$ but surrounding $p$ clockwise instead of going into it. Thus $\gamma_{u,p,c}$ cuts out a once-punctured monogon based at $u$, the puncture being $p$.

For each $u \in E$ form the set

$$S(u) := \{[\gamma_{u,p,c}]^2 \mid p \in \mathbb{P} \text{ and } c \text{ is as above}\} \subseteq \pi_1(\Sigma \setminus \mathbb{P}, E)(u,u) = \pi_1(\Sigma \setminus \mathbb{P})(u,u).$$

Although $S(u)$ fails to be a subgroup of $\pi_1(\Sigma \setminus \mathbb{P}, E)(u,u)$, the collection $S := (S(u))_{u\in E}$ has the property that for every $u_0, v_0 \in E$, every $h \in S(u_0)$, and every morphism $g \in \pi_1(\Sigma \setminus \mathbb{P}, E)(x,y)$, we have $ghg^{-1} \in S(v_0)$. Thus, if we let $H(u)$ be the subgroup of $\pi_1(\Sigma \setminus \mathbb{P}, E)(u,u)$ generated by $S(u)$, then $H := (H(u))_{u\in E}$ is a normal multilocular subgroup of $\pi_1(\Sigma \setminus \mathbb{P}, E)$.

**Definition 6.1** [5; 7] The quotient groupoid

$$\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma, E) := \pi_1(\Sigma \setminus \mathbb{P}, E)/H$$

will be called the 2-*orbifold fundamental groupoid* of $(\Sigma, \mathbb{M}, \mathbb{P})$.

Let $\tau$ be a triangulation of $(\Sigma, \mathbb{M}, \mathbb{P})$ having signature zero. Each point $u$ in $E$ belongs to exactly one boundary segment $b_u$ of $\Sigma$. This segment $b_u$ is a vertex of both of the graphs $G(\tau)$ and $G^{⫯}(\tau)$. We will identify each $u$ as the vertex $b_u$ whenever we consider either of these graphs. Under this identification, we define $\pi_1(G^{⫯}(\tau), E)$ to be the full subcategory of $\pi_1(G^{⫯}(\tau))$ having $E$ as set of objects.

For each vertex $u \in E$, each self-folded triangle $v$ of $\tau$, and each morphism $c \in G^{⫯}(\tau)(u,v)$, set $\gamma_{u,v,c} := c * (\eta_{v,1}, \eta_{v_2}) * c^{-1} = c^{-1} \circ (\eta_{v,1}, \eta_{v_2}) \circ c$. For each $u \in E$ form the set

$$R(u) := \{\gamma_{u,v,c}^2 \mid v \text{ is as above and } c \in G^{⫯}(\tau)(u,v)\} \subseteq \pi_1(G^{⫯}(\tau), E)(u,u) = \pi_1(G^{⫯}(\tau))(u,u).$$

Although $R(u)$ fails to be a subgroup of $\pi_1(G^{⫯}(\tau), E)(u,u)$, the collection $R := (R(u))_{u\in E}$ has the property that for every $u_0, v_0 \in E$, every $h \in R(u_0)$, and every morphism $g \in \pi_1(G^{⫯}(\tau), E)(x,y)$, we have $ghg^{-1} \in R(v_0)$. Thus if we let $K(u)$ be the subgroup of $\pi_1(\Sigma \setminus \mathbb{P}, E)(u,u)$ generated by $R(u)$, then $K := (K(u))_{u\in E}$ is a normal multilocular subgroup of $\pi_1(G^{⫯}(\tau), E)$.

**Definition 6.2** The quotient groupoid

$$\pi_{1,\mathbb{P}}^{\mathrm{orb}}(G^{⫯}(\tau), E) := \pi_1(G^{⫯}(\tau), E)/K$$

will be called the 2-*orbifold fundamental groupoid* of the graph $G^{⫯}(\tau)$.

The proof of the next result follows from the definitions.

**Theorem 6.3** *Let $(\Sigma, \mathbb{M}, \mathbb{P})$ be a surface with nonempty boundary, and let $\tau$ be an ideal triangulation of $(\Sigma, \mathbb{M}, \mathbb{P})$ of signature zero. The strong deformation retraction $\rho\colon \Sigma \setminus \mathbb{P} \to G^{⫯}(\tau)$ from Theorem 3.6 induces a commutative diagram of groupoids*

$$\begin{array}{ccc}
\pi_1(\Sigma \setminus \mathbb{P}, E) & \xrightarrow[\cong]{\rho_\#} & \pi_1(G^{⫯}(\tau), E) \\
\Big\downarrow{\scriptstyle \mathfrak{p}} & & \Big\downarrow{\scriptstyle \mathfrak{p}} \\
\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma, E) & \xrightarrow[\bar{\rho}_\#]{\cong} & \pi_{1,\mathbb{P}}^{\mathrm{orb}}(G^{⫯}(\tau), E)
\end{array}$$

*whose horizontal arrows are isomorphisms.*

# 7 Main result: uniqueness of kink-free representative curves

Let $(\Sigma,\mathbb{M},\mathbb{P})$ and $E$ be as in Section 6. For each $u\in E$, define $L(u)$ to be the image, under the projection $\pi_1(\Sigma\setminus\mathbb{P})(u,u)\to\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u,u)$, of the set of homotopy classes of the curves $\gamma_{u,p,c}$ described in the second paragraph of Section 6. Since $\pi_1(\Sigma\setminus\mathbb{P})(u,u)$ is isomorphic to a torsion-free Fuchsian group having a finite Dirichlet polygon of finite hyperbolic area, we can deduce from eg [15, Theorem 3.5.2 and Corollary 4.2.6; 22, Proposition VI.1.4] that $L(u)=\{f\in\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u,u)\mid f\neq\mathbb{1}\text{ and has finite order}\}$, and thus $L(u)=\{f\in\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u,u)\mid \mathbb{1}\neq f=f^{-1}\}$.

**Theorem 7.1** *Let $(\Sigma,\mathbb{M},\mathbb{P})$ be a surface with nonempty boundary.*

- *Given $u_0,v_0\in E$, there exists exactly one function*

$$\iota\colon\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u_0,v_0)\setminus L(u_0)\to\pi_1(\Sigma\setminus\mathbb{P},E)(u_0,v_0)$$

  *such that*

  (1) $\mathfrak{p}\circ\iota=\mathbb{1}$,

  (2) *for every $f\in\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u_0,v_0)\setminus L(u_0)$ there exists a representative curve $\gamma\in\iota(f)$ that has no kinks.*

- *There is exactly one function*

$$\iota\colon\pi_{1,\mathbb{P}}^{\mathrm{orb,free}}(\Sigma,E)/{\sim}\to\pi_1^{\mathrm{free}}(\Sigma\setminus\mathbb{P},E)/{\sim}$$

  *such that*

  (1) $\mathfrak{p}\circ\iota=\mathbb{1}$,

  (2) *for every $f\in\pi_{1,\mathbb{P}}^{\mathrm{orb,free}}(\Sigma,E)/{\sim}$ there exists a representative curve $\gamma\in\iota(f)$ that has no kinks,*

  *where $\sim$ is the equivalence relation that identifies each closed curve with its opposite orientation.*

**Proof** We prove the first statement; the proof of the second statement is similar. By Proposition 5.4 and Theorem 6.3, it is enough to take a triangulation $\tau$ of signature zero, and prove the statements for $\pi_{1,\mathbb{P}}^{\mathrm{orb}}(G^{\gimel}(\tau),E)$ and $\pi_1(G^{\gimel}(\tau),E)$ instead of $\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)$ and $\pi_1(\Sigma\setminus\mathbb{P},E)$.

Take $f\in\pi_1(G^{\gimel}(\tau),E)(u_0,v_0)\setminus\bar\rho_\#(L(u_0))$, where $\bar\rho_\#$ is the morphism of groupoids in Theorem 6.3. Assume $f$ is written in standard form. It is clear that applying resolutions of kinks to $f$ does not affect the class of $f$ in $\pi_{1,\mathbb{P}}^{\mathrm{orb}}(G^{\gimel}(\tau),E)(u_0,v_0)$. Furthermore, by Theorem 4.11, the element $\iota(f)\in\pi_1(G^{\gimel}(\tau),E)(u_0,v_0)$ obtained after fully resolving all kinks from $f$ is independent of the order in which the kinks are resolved. This produces one function

$$\iota\colon\pi_{1,\mathbb{P}}^{\mathrm{orb}}(\Sigma,E)(u_0,v_0)\setminus L(u_0)\to\pi_1(\Sigma\setminus\mathbb{P},E)(u_0,v_0)$$

with the two stated properties.

Uniqueness follows also from Theorem 4.11 after realizing that, by Definition 6.2, the paragraph preceding it, and Lemma 4.8, if $g\in\pi_1(G^{\gimel}(\tau),E)(u_0,v_0)$ is a morphism without kinks and such that $\mathfrak{p}(g)=\mathfrak{p}(f)$, then it is possible to obtain $g$ from $f$ by applying a finite sequence of resolutions of kinks. □

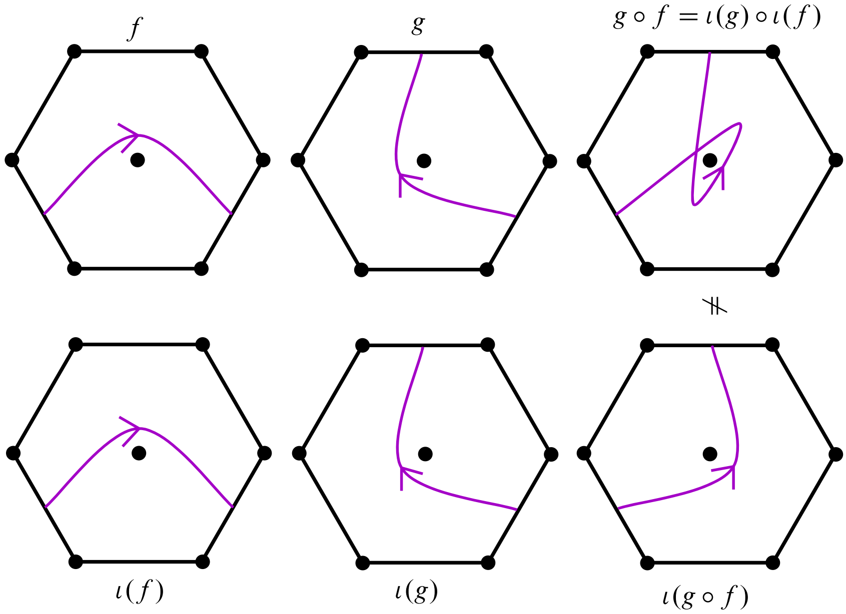


Figure 19

**Remark 7.2** The functions $\iota$ in the first assertion of Theorem 7.1 may fail to constitute a morphism of groupoids; see Figure 19.

## References


[1] **C Amiot**, *The derived category of surface algebras: the case of the torus with one boundary component*, Algebr. Represent. Theory 19 (2016) 1059–1080 MR Zbl

[2] **C Amiot**, **T Brüstle**, *Derived equivalences between skew-gentle algebras using orbifolds*, Doc. Math. 27 (2022) 933–982 MR Zbl

[3] **C Amiot**, **Y Grimeland**, *Derived invariants for surface algebras*, J. Pure Appl. Algebra 220 (2016) 3133–3155 MR Zbl

[4] **C Amiot**, **D Labardini-Fragoso**, **P-G Plamondon**, *Derived invariants for surface cut algebras of global dimension 2, II: The punctured case*, Comm. Algebra 49 (2021) 114–150 MR Zbl

[5] **C Amiot**, **P-G Plamondon**, *The cluster category of a surface with punctures via group actions*, Adv. Math. 389 (2021) art. id. 107884 MR Zbl

[6] **I Assem**, **T Brüstle**, **G Charbonneau-Jodoin**, **P-G Plamondon**, *Gentle algebras arising from surface triangulations*, Algebra Number Theory 4 (2010) 201–229 MR Zbl

[7] **M Chas**, **S Gadgil**, *The extended Goldman bracket determines intersection numbers for surfaces and orbifolds*, Algebr. Geom. Topol. 16 (2016) 2813–2838 MR Zbl

[8] **V V Fock**, **A B Goncharov**, *Dual Teichmüller and lamination spaces*, from "Handbook of Teichmüller theory, I", IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc., Zürich (2007) 647–684 MR Zbl

[9] **S Fomin**, **M Shapiro**, **D Thurston**, *Cluster algebras and triangulated surfaces, I: Cluster complexes*, Acta Math. 201 (2008) 83–146 MR Zbl

[10] **C Geiss**, *On homomorphisms and generically $\tau$-reduced components for skewed-gentle algebras*, preprint (2023) arXiv 2307.10306

[11] **F Haiden**, **L Katzarkov**, **M Kontsevich**, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes Études Sci. 126 (2017) 247–318 MR Zbl

[12] **U Hansper**, *Classification of the indecomposable finite dimensional modules of clannish algebras*, PhD thesis, Universität Bielefeld (2022) Available at https://doi.org/10.4119/unibi/2962952

[13] **J Hass**, **P Scott**, *Intersections of curves on surfaces*, Israel J. Math. 51 (1985) 90–120 MR Zbl

[14] **G Huet**, *Confluent reductions: abstract properties and applications to term rewriting systems*, J. Assoc. Comput. Mach. 27 (1980) 797–821 MR Zbl

[15] **S Katok**, *Fuchsian groups*, Univ. Chicago Press (1992) MR Zbl

[16] **D Labardini-Fragoso**, *Quivers with potentials associated to triangulated surfaces*, Proc. Lond. Math. Soc. 98 (2009) 797–839 MR Zbl

[17] **D Labardini-Fragoso**, **S Schroll**, **Y Valdivieso**, *Derived categories of skew-gentle algebras and orbifolds*, Glasg. Math. J. 64 (2022) 649–674 MR Zbl

[18] **Y Lekili**, **A Polishchuk**, *Derived equivalences of gentle algebras via Fukaya categories*, Math. Ann. 376 (2020) 187–225 MR Zbl

[19] **J P May**, *A concise course in algebraic topology*, Univ. Chicago Press (1999) MR Zbl

[20] **M H A Newman**, *On theories with a combinatorial definition of "equivalence"*, Ann. of Math. 43 (1942) 223–243 MR Zbl

[21] **S Opper**, **P-G Plamondon**, **S Schroll**, *A geometric model for the derived category of gentle algebras*, preprint (2018) arXiv 1801.09659

[22] **H P de Saint-Gervais**, *Uniformization of Riemann surfaces: revisiting a hundred-year-old theorem*, Eur. Math. Soc., Zürich (2016) MR Zbl

[23] **M V Sapir**, *Combinatorial algebra: syntax and semantics*, Springer (2014) MR Zbl

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