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Weinstein presentations for high-dimensional antisurgery

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We give an algorithm for describing the Weinstein presentation of Weinstein subdomains obtained by carving out regular Lagrangians. Our work generalizes previous work in dimension 3 and requires a novel Legendrian isotopy move (the "boat move") that changes the local index of Reeb chords in a front projection. As applications, we describe presentations for certain exotic Weinstein subdomains and give explicit descriptions of P-loose Legendrians.

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1 Introduction

Weinstein domains [26] are exact symplectic manifolds equipped with symplectic handlebody decompositions, analogous to CW complexes in topology. These domains are relatively easy to construct by consecutively attaching handles along isotropic spheres in contact manifolds. Weinstein presentations or diagrams keep track of these isotropic spheres and their interactions with each other, and make computation of invariants, like the wrapped Fukaya category, tractable via surgery formulas and gluing formulas; see Bourgeois, Ekholm, and Eliashberg [4] and Ganatra, Pardon, and Shende [14].

A wealth of symplectically exotic Weinstein domains can be constructed as *subdomains* of more standard Weinstein domains, obtained by *carving out* Lagrangian disks. For example, Sylvan and the second author [20] showed that if $n \geq 5$, the standard cotangent bundle T^*S^n has infinitely many Weinstein subdomains that are diffeomorphic to T^*S^n but pairwise nonsymplectomorphic. They also constructed P-loose Legendrians as subdomains of the sector T^*D^n and showed that these P-loose Legendrians are smoothly isotopic but not Legendrian isotopic. The contact analog of carving out Lagrangian disks — *contact antisurgery* — is important for the construction of contact structures; for example, any contact structure on S^{2n-1} is obtained by doing a single contact surgery and antisurgery on the standard contact structure $(S^{2n-1}, \xi_{\mathrm{std}})$; see Lazarev [18]. Weinstein subdomains are also attractive from the point of view of categorical invariants; their wrapped Fukaya categories are localizations of the Fukaya category of the ambient domain by the localization formula in [14]. Finally, any Weinstein domain deformation retracts to its singular Lagrangian skeleton; therefore the question of studying Weinstein subdomains of a fixed domain X is precisely the question of finding singular Lagrangian skeleta in X.

Carving out Lagrangian disks and forming Weinstein subdomains appear naturally in two other geometric problems. First, Weinstein subdomains arise naturally when relating complements of toric divisors and their (partial) smoothings. That is, $X \setminus D$ is a Weinstein subdomain of $X \setminus \widetilde{D}$ for a Weinstein domain X, divisor $D \subset X$, and smoothing $\widetilde{D}$ of D. In 4 dimensions, the Weinstein presentations of such manifolds have been related in this context by Acu, Capovilla-Searle, Gadbled, Marinković, Starkston, and the fourth author [2]. They define a necessary condition on a Delzant polytope of the toric manifold, which ensures that the complement of a corresponding partial smoothing of the toric divisor supports a Weinstein structure. They give an algorithm to construct an explicit Weinstein presentation for the complement of such a partially smoothed toric divisor.

Second, carving out Lagrangian disks plays a role in constructing convex hypersurfaces in contact manifolds. See Honda and Huang [17] and Sackel [24] for background. Briefly, a Weinstein convex hypersurface $H^{2n} \subset Y^{2n+1}$ is the union of two Weinstein hypersurfaces H_1^{2n} and H_2^{2n} glued along a contactomorphism of the contact boundaries ∂H_1^{2n} and ∂H_2^{2n}. One can modify Weinstein hypersurfaces via contact handle attachment. Index $k \le n$ handles correspond to usual Weinstein handle attachment to both H_1 and H_2 along some framed isotropic sphere in $\partial H_1 \cong \partial H_2$. However, index-$(n+1)$ contact handles, which play a crucial role in *bypasses*, are attached along a sphere S^n in H^{2n} that is the union of a Lagrangian disk $D_1 \subset H_1^{2n}$ and a Lagrangian disk $D_2 \subset H_2^{2n}$ that have the same Legendrian boundary in $\partial H_1 \cong \partial H_2$. After the index-$(n+1)$ handle is attached, the resulting Weinstein hypersurface is the union of the *subdomains* $H_1 \backslash D_1$ and $H_2 \backslash D_2$, the result of carving out D_1 and D_2 from H_1 and H_2, respectively. Hence, the problem of explicitly describing convex hypersurfaces as unions of Weinstein hypersurfaces is closely related to the problem of explicitly describing Weinstein subdomains.

On the other hand, Weinstein presentations have not been described yet for the constructions in [20], nor does there exist a general procedure for describing explicit Weinstein presentations for general Weinstein subdomains. For instance, the construction of P-flexible Weinstein manifolds was relatively inexplicit due to the fact that it was not clear how the carving out/antisurgery modified the front of the original Legendrian. In particular, the front projection of these P-loose Legendrians was not known.

Our goal here is to remedy this situation. We focus on the problem of constructing explicit Weinstein presentations of exotic Weinstein subdomains constructed by carving out Lagrangian disks. We want the presentation of such a subdomain to be in terms of a Weinstein presentation of the original Weinstein domain, which is compatible with the Lagrangian disk. In the process, we introduce a new Legendrian isotopy move, called the boat move.

As a concrete application, we give an explicit front projection for the P-loose Legendrians constructed indirectly in [20]. For any collection of integers P, a Legendrian Λ is said to be *P-loose* if it is isotopic to $\Lambda \# \Lambda_P$, the connected sum of Λ and Λ_P, where Λ_P is a P-loose Legendrian unknot, defined in $\mathbb{R}^{2n+1}$ for $n \ge 4$. This operation of taking connect sum with Λ_P can be used to make P-loose Legendrian representatives of any smooth n-dimensional knot type. A Weinstein manifold constructed via handle attachments along P-loose Legendrians is called *P-flexible*. In [20], it was shown that P-loose

Legendrians have properties that generalize those of loose Legendrians, which were introduced by Murphy in [21]. If $0 \in P$, then Λ_P is a loose Legendrian unknot. In general, Λ_P is not necessarily loose but has Legendrian dga (with loop space coefficients) equal to $\mathbb{Z}[1/P]$; see Section 4. Furthermore, for any Legendrian $\Lambda \subset Y$, the Chekanov–Eliashberg dga satisfies

$$\mathrm{CE}(\Lambda \# \Lambda_P) \cong \mathrm{CE}(\Lambda)[P^{-1}],$$

and hence vanishes with $\mathbb{Z}/P\mathbb{Z}$ coefficients.

Main results

For any Weinstein subdomain X_0 of a domain X, the complement $X \setminus X_0$ admits the structure of a Weinstein cobordism C. This cobordism has a subcritical part C_{sub} that does not change invariants like the Fukaya category, and some critical handles H_i^n for $i = 1, \ldots, l$, with Lagrangian cocore disks $L_i^n \subset X$. Hence X_0 can be described, up to subcritical cobordism, as $X \setminus \left(\bigcup_{i=1}^l L_i\right)$.

Conversely, given any Lagrangian disk $L \subset X$, $X \setminus L$ is an exact subdomain of X. The contact boundary $\partial(X \setminus L)$ of $X \setminus L$ is obtained from the contact boundary ∂X of X by doing *antisurgery*, or $(+1)$-contact surgery, along the Legendrian sphere $\partial L \subset \partial X$. To ensure that $X \setminus L$ is Weinstein, we assume that L is *regular* (see Eliashberg, Ganatra, and Lazarev [13]), which implies that our starting Weinstein presentation for X is compatible with L, as we describe later.

Our main goal is to give explicit constructions (Weinstein presentations and handlebody decompositions) for Weinstein manifolds obtained via antisurgery. To do this we introduce a new family of n-dimensional Legendrian moves in the front projection. We construct Legendrian isotopies called D^k-*suspensions* that consist of a Legendrian isotopy ψ of an $(n-k)$-dimensional slice suspended over a k-dimensional disk.

Proposition 1.1 *Given a Legendrian isotopy $\psi\colon D^{n-k} \times [0,1]_t \to \mathbb{R}^{2(n-k)+1}$ which is the identity near ∂D^{n-k} and t-independent near $\partial[0,1]$, its D^k-suspension $\Sigma_{D^k}\{\psi\}$ is a Legendrian in $\mathbb{R}^{2n+1}$, and is Legendrian isotopic to $D^k \times D^{n-k}$ relative to the boundary.*

We call the D^k-suspension of an $(n-k)$-dimensional Reidemeister 1 move an (n,k)-*boat move*; see for instance Figure 9. Using these boat moves, we can construct our desired presentations for Weinstein domains obtained via antisurgery.

In dimension 3, there are several existing results, for example in work of Ding and Geiges [10], explaining how to do antisurgery along Legendrian circles that admit Lagrangian disk fillings. Our main contribution is that we are able to replicate such explicit constructions in higher dimensions by first using (n,k)-boat moves. We do this assuming we know a Weinstein presentation for the original manifold.

Next we state our main result. Let $L \subset X$ be a regular Lagrangian disk in a Weinstein domain X^{2n}. By [13], any regular Lagrangian disk $L \subset X$ in a Weinstein domain X can be presented as

$$D^n \subset T^*D^n \cup \left(\bigcup_i H_i\right),$$

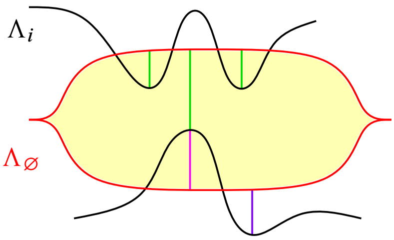

Figure 1: The green Reeb chords are bounded by the flying saucer and go from the black Λ_i to the red $\Lambda_\varnothing$; these have critical points on Λ_i with local indices 1, 0, and 1 for the height difference Morse function from Λ_i to $\Lambda_\varnothing$. We will apply $(1,0)$, $(1,1)$, and $(1,0)$ boat moves at these critical points, respectively, which correspond to doing a Reidemeister 1 move at the two index-1 critical points and nothing at the middle index-0 critical point. The purple Reeb chord is not bounded by the flying saucer. The pink Reeb chord is bounded by the flying saucer but does not go from Λ_i to $\Lambda_\varnothing$.

where H_i are Weinstein handles attached to T^*D^n in the complement of $\partial D^n \subset T^*D^n$. For simplicity, we assume here that all of these handles have index n, although this assumption can be removed. Let

$$\Lambda_\varnothing \cup \Big(\bigcup_i \Lambda_i\Big) \subset \mathbb{R}^{2n-1}$$

be the Legendrian link formed by the following Legendrians. First, $\Lambda_\varnothing = \partial D^n$ is the standard Legendrian unknot with front projection in $\mathbb{R}^n$ given by the "flying saucer" (with a S^{n-2}-family of cusps and no other singularities). Second, Λ_i are the attaching Legendrians of H_i. Let C_i denote the set of Reeb chords from Λ_i to $\Lambda_\varnothing$ with front projection contained in the subset of $\mathbb{R}^n$ bounded by the flying saucer front projection $\pi(\Lambda_\varnothing) \subset \mathbb{R}^n$, and let $C = \bigcup_i C_i$; see Figure 1. We assume all such chords are nondegenerate, and furthermore correspond to critical points of a local Morse function whose indices we call the local index of the Reeb chord; in particular, C is finite.

Theorem 1.2 *There is a Weinstein presentation for the Weinstein subdomain $X \setminus L \subset X$ with the following properties*:

- *The Weinstein presentation of $X \setminus L$ has one more $(n-1)$-handle than the Weinstein presentation for X.*
- *The n-handles for $X \setminus L$ are in one-to-one correspondence with the n-handles of X.*
- *The attaching sphere Λ_i' of the n-handle H_i' of $X \setminus L$ is obtained from the attaching sphere Λ_i of the corresponding handle H_i of X in the following way*: *for each Reeb chord γ in C_i with local index k, we apply an $(n, n-k)$-boat move to Λ_i and do a cusp connected sum with a Legendrian that goes through the new $(n-1)$-handle one time.*

Remark 1.3 The assumptions for Theorem 1.2 can be weakened. For example, as explained in Lemma 4.1, any Legendrian link in $\mathbb{R}^{2n+1}$ can be perturbed by a C^∞-small isotopy so that all Reeb chords in C are nondegenerate and correspond to critical points of a Morse function; however, perturbing a degenerate Reeb chord may result in a larger (finite) number of nondegenerate Reeb chords.

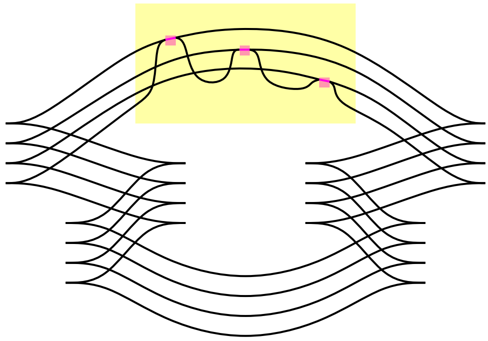

Figure 2: A low-dimensional slice of a P-loose Legendrian which cuts through the three boat move cusp connect sums, represented by the square regions. Away from a neighborhood of this slice, the black Legendrians may be linked in the shaded region but are parallel outside of it.

Also, the assumption that X takes the form $T^*D^n \cup \left(\bigcup_i H_i^n\right)$ can be generalized to allow subcritical handles. Generically, there will be no Reeb chords between $\Lambda_\varnothing$ and the attaching spheres of these subcritical handles, but the attaching spheres of the critical handles may interact with these subcritical handles. Hence, only a portion of these attaching spheres map to the contact boundary $\partial_\infty T^*D^n$, resulting in a Legendrian subset $\Lambda_i \subset \partial_\infty T^*D^n$ (which are not necessarily spheres but manifolds with boundary). In that case, a generalization of Theorem 1.2 holds by considering Reeb chords between $\Lambda_\varnothing$ and the subset Λ_i.

Additionally, as a main application of Theorem 1.2, we construct explicitly the P-loose Legendrians in the Weinstein presentations of P-flexible Weinstein domains.

Corollary 1.4 *For integers $p \geq 0$ and $n \geq 2$, the P-loose Legendrian unknot, $\Lambda_P \subset \mathbb{R}^{2n+1}$ for $P = \{p\}$, is a Legendrian submanifold consisting of four loose Legendrian unknots which are completely parallel away from a bounded region disjoint from the loose charts. Within this bounded region they may be linked* (*in a way that depends on p*) *and are connected via three boat moves and cusp connected sum gluings* (see Figure 2); *the front projection in this bounded region has no singularities except for the singularities in the local boat moves.*

Structure

In Section 2, we give some background on antisurgery and P-loose Legendrians. In Section 3, we discuss the boat move and prove Proposition 1.1. In Section 4, we present our construction for producing Weinstein presentations for subdomains and prove Theorem 1.2 and Corollary 1.4. We then use our construction to give explicit examples of Weinstein antisurgery manifolds and conclude with some open questions.

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2 Background

2.1 Contact surgery and Weinstein manifolds

Let M be an n-dimensional manifold. In the topological setting, we perform k-surgery on M, for $k \leq n$, by first removing a tubular neighborhood $N(S^k) \cong S^k \times D^{n-k}$ of a k-sphere, and then gluing back a copy of $D^{k+1} \times S^{n-k-1}$ along boundary $S^{n-k} \times S^{n-k-1}$. A choice of framing, that is, the particular identification $\varphi\colon N(S^k) \cong S^k \times D^{n-k}$, specifies our gluing, which then determines our surgered manifold up to diffeomorphism.

When the k-sphere is a Legendrian sphere, Λ, inside M, a contact manifold, there is a canonical framing, providing an identification $\Lambda \cong S^k \subset \mathbb{R}^{k+1}$. To see this, we recall some facts and notation from standard contact geometry. The contact hyperplane bundle supports a symplectic form given by $d\alpha|_\xi$. We may additionally endow this bundle with an almost-complex structure J that is compatible with $d\alpha$. Then, along λ, the tangent bundle TM can be decomposed as $TM|_\Lambda = T\Lambda \oplus \langle R_\alpha \rangle \oplus J(T\Lambda)$, and so $N(\Lambda) \cong \langle R_\alpha \rangle \oplus J(T\Lambda)$ for a contact form α and corresponding Reeb vector field R_α. The bundle $\langle R_\alpha \rangle \oplus J(T\Lambda)$ can be identified with the stabilized tangent bundle of Λ, which carries a canonical trivialization after an identification of Λ with $S^k \subset \mathbb{R}^{k+1}$ — this trivialization is the canonical framing. At this point, we drop the dimension k of our sphere from the notation, and refer to such a surgery as $(\pm m/k)$-surgery, where $\pm m/k$ is a fraction relating our choice of framing to the canonical framing. For example, attaching a critical Weinstein handle affects the contact boundary by a (-1)-surgery.

2.2 Legendrian moves

Legendrian moves refer to the replacement of a Legendrian by something Legendrian isotopic that differs from it only within a Darboux neighborhood. These are often depicted via front diagrams.

Legendrian Reidemeister moves are analogous to knot diagram moves which preserve the topological knot type. They usually refer to replacements in the front diagrams of Legendrian knots in a contact 3-manifold depicted in Figure 3. These fully characterize Legendrian isotopies for 1-dimensional Legendrians in contact 3-manifolds.

Theorem 2.1 [25] *Two front diagrams represent Legendrian isotopic Legendrian knots if and only if they are related by regular homotopy and a sequence of moves shown in Figure 3.*

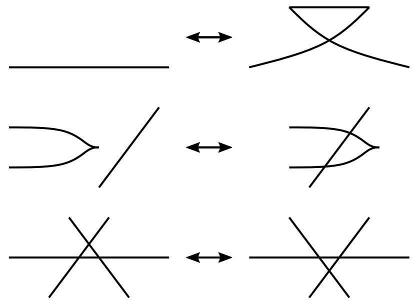

Figure 3: The Legendrian Reidemeister moves.

We will also be using higher-dimensional first Reidemeister moves. For any Legendrian submanifold $\Lambda \subset (Y, \xi)$, the Legendrian submanifold Λ' obtained by replacing a graphical portion of the front of Λ (with respect to a particular Darboux chart) by the rightmost front depicted in Figure 4 is Legendrian isotopic to Λ. To see this, note that the first arrow is the same as one of the higher-dimensional first Reidemeister moves described in [5]. The subsequent two arrows of Figure 4 are front diagrams of Legendrian isotopies where the domed part of the front is pushed inwards via an isotopy of the $(x_1, \dots, x_n, z)$-plane that does not have vertical tangencies at all times. We refer to this replacement within a Darboux chart as a *k-dimensional first Reidemeister move*.[1] Let

$$R1_k : [-1, 1] \times D^k \to \mathbb{R}^{2k+1}$$

denote the associated Legendrian isotopy. Note that the fronts depicted in Figure 4 have S^{k-2}-symmetry about the z-axis passing through the point of singularity.

Remark 2.2 The fronts depicted in Figure 4, and therefore fronts obtained whenever we apply the n-dimensional R1, are not generic [8] as there is a singularity at the cone point. This does not matter for our purposes, but is important to keep in mind especially if computing Legendrian contact homology.

Remark 2.3 Higher-dimensional versions of the second and third Legendrian Reidemeister moves also exist; see for instance their depictions in [6, Section 2.4]. In higher dimensions, these three moves in the front do not fully characterize all Legendrian isotopies. Generally, we can isotope Legendrian fronts past each other by the Reeb flow, as long as there are no Reeb chords between them.

In addition to Reidemeister moves, we will use Legendrian handleslides and the addition or removal of a canceling pair of Weinstein handles in our diagrammatic calculus. These are moves that allow us to pass between different Weinstein handle diagrams of equivalent Weinstein domains. In other words, before and after these moves, the Legendrians depicted are isotopic in the surgered contact boundaries (but maybe not isotopic in the original contact boundaries).

[1] Replacing the figures in Figure 4 by their reflections about the $(x_1, \dots, x_n)$-plane gives analogous Legendrian moves. We omit these in the discussion for simplicity.

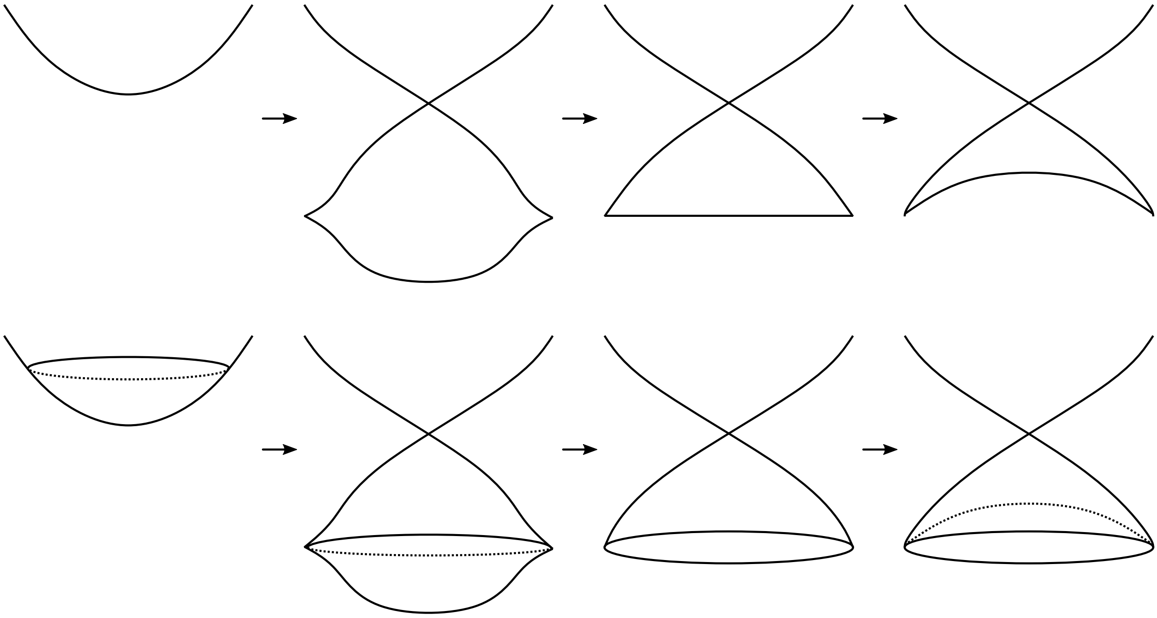

Figure 4: A Legendrian Reidemeister 1 and isotopy for a knot (top) and a surface (bottom).

Following Casals and Murphy [6], we depict the effect of Legendrian handleslides in Figure 5. A handleslide over a $(+1)$-surgery curve produces a cone singularity, while sliding over a (-1)-surgery produces a circle of cusp singularities. Meanwhile, for a Weinstein manifold of dimension $2n$, the addition or removal of a canceling pair refers to adding in an n-handle and an $(n-1)$-handle such that the attaching sphere of the n-handle and the belt sphere of the $(n-1)$-handle intersect exactly once.

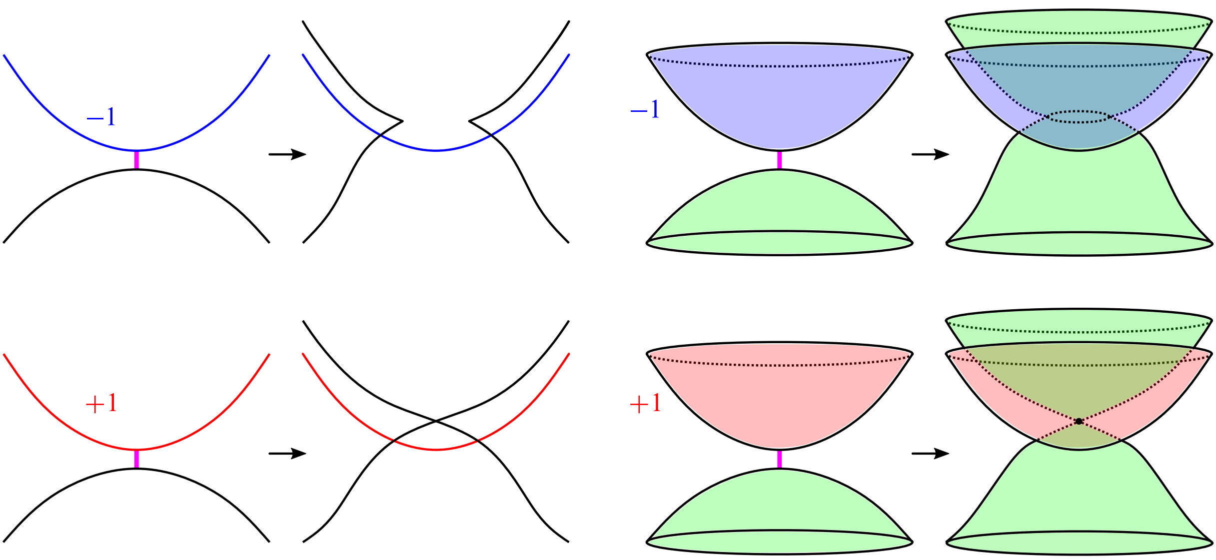

Figure 5: Handleslides over (-1) and $(+1)$ Legendrians (in blue and red, respectively) in 3 and 5 dimensions.

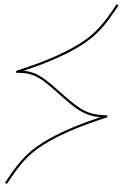

Figure 6: A 1-dimensional stabilized Legendrian arc.

2.3 Loose and P-loose Legendrians, flexible and P-flexible domains

In [21], Murphy introduced a class of Legendrians called loose Legendrians. These Legendrians are characterized by an explicit local model.

Definition 2.4 A loose unknot $\Lambda_l \subset (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ for $n \geq 2$ is a Legendrian that is formally isotopic to the Legendrian unknot, and where there is an $\mathbb{R}^3$ slice that intersects Λ_l transversely and the front projection of $\Lambda_l \cap \mathbb{R}^3$ is the 1-dimensional stabilized arc (see Figure 6). A Legendrian $\Lambda \subset (Y, \xi)$ is loose if it is Legendrian isotopic to $\Lambda \# \Lambda_l \subset Y \# \mathbb{R}^{2n+1} \cong Y$.

Remark 2.5 Since we prefer to work with closed Legendrians, the above definition involves loose Legendrian unknots (which are spheres) instead of the original definition, which involves loose Legendrian *charts*, which are Legendrian disks; these definitions are equivalent.

We emphasize that there is not a canonical model for the loose Legendrian unknot (except in dimension 1, which must be excluded for reasons explained below). For example, we can take any codimension-zero subdomain U near the cusp of the Legendrian unknot $\Lambda_\varnothing$ and push through U past the cusp to create Λ_U (see Figure 7). Then Λ_U is always loose and if U has Euler characteristic zero, then Λ_U is Legendrian isotopic to the loose Legendrian unknot. Alternatively, one can take any *closed* codimension-1 submanifold V of $\Lambda_\varnothing$ and add a 1-dimensional zigzag spun by V to create Λ_V. Again if V has Euler characteristic zero, Λ_V is a Legendrian unknot. Another way to see the nonexistence of a canonical model is to observe that there is no canonical way to extend the 1-dimensional stabilized arc to a higher-dimensional Legendrian disk so that the extension is standard near the boundary.

This indicates a proliferation of loose Legendrians unknots, making it impossible to speak of *the* loose Legendrian unknot. However, the main result proven in [21] about loose Legendrians is that they satisfy an h-principle, which essentially says that, if loose Legendrians Λ_1 and Λ_2 in (Y, ξ) are smoothly isotopic (and satisfy additional tangential data), then they are Legendrian isotopic. This means that the

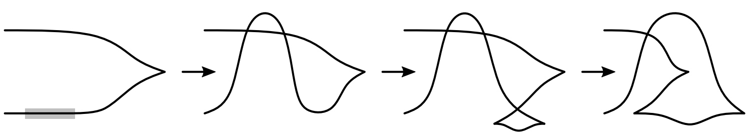

Figure 7: Near the cusp of the Legendrian, we push through the highlighted subdomain, which perturbs the knot into becoming loose.

symplectic geometry of loose Legendrians reduces just to classical differential topology. In particular, all loose Legendrian unknots are Legendrian isotopic. For $n = 1$, the local model still makes sense but the h-principle does not hold, and therefore we speak of loose Legendrians only for $n \geq 2$.

Using loose Legendrians, Cieliebak and Eliashberg defined the class of flexible Weinstein domains [7]:

Definition 2.6 A Weinstein domain is *flexible* if the attaching spheres for its half-dimensional handles are loose Legendrians.

More generally, we can consider flexible Weinstein sectors. A sector is equivalently a Weinstein domain with the extra data of a Weinstein hypersurface in its contact boundary. A flexible sector is one for which the ambient domain is flexible and the Weinstein hypersurface is loose (in the sense that all core disks of its critical handles are loose Legendrians). We point out that, in work of Murphy and Siegel [22], these domains are called *explicitly* flexible and Weinstein domains are called flexible if they admit a Weinstein homotopy to an explicitly flexible one; this notion of flexibility is tautologically preserved under Weinstein homotopy.

2.4 Antisurgery construction of P-loose Legendrian unknots

In this section we review the construction of P-loose Legendrian unknots via antisurgery in [20]. There, the authors considered Lagrangians disks $D_P \subset T^*D^n$ (described below) with Legendrian boundary $\partial D_P \subset \partial_\infty T^*D^n$ disjoint from the boundary of the zero-section ∂D^n. Then they carved out these disks to obtain subdomains $T^*D^n \setminus D_P$ of T^*D^n.

To construct the P-loose Legendrian unknot Λ_P, we view $T^*D^n \setminus D_P$ as a Weinstein sector, or equivalently as a Weinstein domain $B^{2n} \setminus D_P$ with a Legendrian stop in its contact boundary. This can be done as follows. As a Weinstein domain, $B^{2n} \setminus D_P$ is just $B^{2n} \cup H^{n-1}$. Since ∂D^n and ∂D_P are disjoint, ∂D^n can be viewed as a Legendrian in the carved-out domain $B^{2n} \cup H^{n-1} = B^{2n} \setminus D_P$. Next, we recover B^{2n} by attaching a flexible handle along any Legendrian that is loose in the complement of $\partial D^n \subset \partial(B^{2n} \cup H^{n-1})$. The image of ∂D^n in this new B^{2n} is denoted by Λ_P, the P-loose unknot.

To complete the construction of Λ_P, we need to describe the construction of the Lagrangian disks $D_P \subset T^*D^n$. These disks were originally introduced by Abouzaid and Seidel in [1, Section 3b]. Let $U \subset S^{n-1}$ be a compact codimension-zero submanifold with smooth boundary. Let

$$f : S^{n-1} \to \mathbb{R}$$

be a C^1-small function with zero as a regular value, so that f is strictly negative in the interior of U, zero on ∂U, and strictly positive on $S^{n-1} \setminus \overline{U}$. Next, we consider S^{n-1} as the $\frac{1}{2}$-radius sphere $S^{n-1}_{1/2} \subset D^n$ and extend f to a smooth Morse function (again called)

$$f : D^n \to \mathbb{R},$$

so that f is C^0-small in the $\frac{1}{2}$-radius disk and satisfies

$$f(tq) = |t|^2 f(q) \quad \text{for } q \in S^{n-1}_{1/2} \text{ and } 1 \le t \le 2.$$

Let $\Gamma(df)$ be the graph of df in T^*D^n and let

$$D_U = \Gamma(df) \cap B^{2n}.$$

Since f is homogeneous for $|q| \ge \frac{1}{2}$ and 0 is a regular value of g, D_U has Legendrian boundary (with respect to the standard radial Liouville vector field on B^{2n}) that is disjoint from ∂D^n. Furthermore, there is a Lagrangian isotopy $\Gamma(d(sf))_{s\in[0,1]}$ from the zero-section $D^n \subset T^*D^n$ to D_U (that intersects the stop ∂D^n precisely when $s = 0$).

Now let U be a P-Moore space: a CW complex whose reduced singular cohomology is isomorphic to $\mathbb{Z}/P\mathbb{Z}$ in some degree. For example, one can take the mapping cone of the degree-P map

$$\varphi_P : S^k \to S^k.$$

Then the resulting Lagrangian disk D_U is called D_P, and is the Lagrangian disk D_P used earlier to construct Λ_P.

Remark 2.7 The above construction of $D_U \subset T^*D^n$ requires a smooth embedding $U \subset S^{n-1}$. If U is a P-Moore space, this is possible only if $n \ge 5$, and hence there exist P-loose Legendrians of dimension at least 4.

The sector $T^*D^n \setminus D_U$ is obtained by doing antisurgery on the boundary of D_U, while keeping track of the original stop ∂D^n of T^*D^n. In particular, the Legendrian boundaries

$$\Lambda_U := \partial D_U \quad \text{and} \quad \Lambda^{n-1} := \partial D^n$$

are linked in a way we now describe. Note that we can view $\Lambda^{n-1} \subset \partial T^*D^n$ as the Legendrian unknot in $\mathbb{R}^{2n-1}$.

Proposition 2.8 *Λ_U is contained in a small neighborhood of ∂D^n, it is given by the 1-jet of $f|_{\partial D^n}$, and its front projection in $\partial D^n \times \mathbb{R}$ is given by the graph of $f|_{\partial D^n}$. In particular, Λ_U can be isotoped so that its front is obtained from ∂D^n by a negative Reeb pushoff on U', a smaller open set $U' \subset U$, and a positive Reeb pushoff in $\partial D^n \setminus U''$ for some larger open set $U'' \supset U$. See Figure 8.*

Proof As stated in the construction of $D_U = \Gamma(df)$ above, to ensure that D_U has Legendrian boundary, we consider T^*D^n as the Liouville domain

$$\left(B^{2n}, \frac{1}{2}\left(\sum_{i=1}^{n} x_i\, dy_i - y_i\, dx_i\right)\right)$$

equipped with a stop ∂D^n, which we identify with the Legendrian unknot. The Liouville vector field associated to the Liouville form $\frac{1}{2}\left(\sum_{i=1}^n x_i\, dy_i - y_i\, dx_i\right)$ is the radial vector field $\frac{1}{2}\sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i})$.

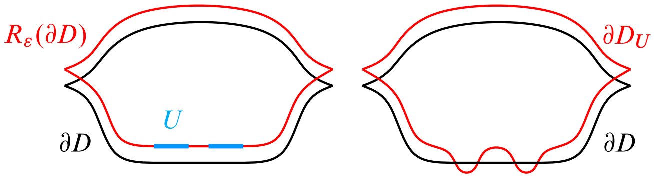

Figure 8: Left: Two parallel Legendrian unknots. The blue region in the top Legendrian is the smooth subdomain U. Right: The Legendrian link $\partial D_U \cup \partial D$, with ∂D_U in red and ∂D in black. This figure is obtained by pushing through U in the top Legendrian down past the bottom Legendrian.

So the fact that f has the form $t^2 f(\theta)$ near $\partial D^n = S^{n-1}$, with coordinate θ, implies that this Liouville vector field is tangent to $\Gamma(df)$ and hence $\Gamma(df)$ has Legendrian boundary. Near ∂D^n,

$$\left(B^{2n}, \frac{1}{2}\left(\sum_{i=1}^{n} x_i\, dy_i - y_i\, dx_i\right)\right)$$

equals

$$(T^*S^{n-1}, \lambda_{T^*S^{n-1}}) \times \big(T^*[1-\varepsilon, 1], \tfrac{1}{2}(t\, dp - p\, dt)\big) = \big(T^*S^{n-1} \times T^*[1-\varepsilon, 1], \lambda_{T^*S^{n-1}} + \tfrac{1}{2}(t\, dp - p\, dt)\big).$$

To see this, note that both of these Liouville structures have the radial vector field as their Liouville vector field, which determines the Liouville form. The (convex) contact boundary of this domain is $T^*S^{n-1} \times T^*_1[1-\varepsilon, 1]$, the restriction to the cotangent fiber at 1. The induced contact form is

$$\lambda_{T^*S^{n-1}} + \tfrac{1}{2}\, dp,$$

given by restricting the Liouville form $\lambda_{T^*S^{n-1}} + \frac{1}{2}(t\, dp - p\, dt)$ to the cotangent fiber at 1. Furthermore, this contact boundary is contactomorphic to

$$(J^1(S^{n-1}) = T^*S^{n-1} \times \mathbb{R}_z, \lambda_{T^*S^{n-1}} + dz)$$

via the map $z = \frac{1}{2} p$.

Near its boundary, our Lagrangian is given by

$$d(t^2 f(\theta)) = t^2 d_\theta f + 2t f(\theta)\, dt,$$

and hence its Legendrian restriction to the contact boundary $T^*S^{n-1} \times T^*_1[1-\varepsilon, 1]$ is $(d_\theta f, 2f(\theta))$. Under the contactomorphism $z = \frac{1}{2} p$, this Legendrian maps to $(d_\theta f, f(\theta))$, the 1-jet of the function $f(\theta)$. Furthermore, we can isotopy f through functions vanishing precisely on ∂U so that f is equal to -1 on U' for a smaller open set $U' \subset U$ and equal to $+1$ on $S^{n-1} \setminus U''$ for a larger open neighborhood $U'' \supset U$, giving us the claimed result. ◻

One of our goals is to is to do antisurgery on ∂D_U and find a presentation for ∂D^n in the resulting contact manifold.

3 Constructions of higher-dimensional Legendrian isotopy moves

In this section we describe constructions of some higher-dimensional Legendrian isotopies. These isotopies are compactly supported, so we may view these as higher-dimensional Legendrian moves. In the subsequent section we will use these Legendrian moves in the construction of handle diagrams for Weinstein manifolds obtained via antisurgery.

3.1 Suspensions of Legendrian isotopies

Consider a Darboux chart with coordinates $x_1,\dots,x_n,y_1,\dots,y_n,z$ and contact form $\alpha = dz - \sum_{i=1}^n y_i\,dx_i$. Let us use $\mathbb{R}^{2k}$ to refer to the span of $x_1,\dots,x_k,y_1,\dots,y_k$ and $\mathbb{R}^{2(n-k)+1}$ to refer to the span of $x_{k+1},\dots,x_n,y_{k+1},\dots,y_n,z$. Then, with the contact form

$$\alpha_{n-k} := dz - \sum_{i=k+1}^{n} y_i\,dx_i,$$

$(\mathbb{R}^{2(n-k)+1}, \alpha_{n-k})$ is a contact manifold. With this notation in mind, we may view

$$\mathbb{R}^{2n+1} = \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)+1} \cong T^*\mathbb{R}^n \times \mathbb{R}^{2(n-k)+1}.$$

Consider the Legendrian (with boundary) in this Darboux chart given by

$$D^k \times \{0\} \times D^{n-k} \times \{0\} \subset \mathbb{R}^k_{x_1,\dots,x_k} \times \mathbb{R}^k_{y_1,\dots,y_k} \times \mathbb{R}^{n-k}_{x_{k+1},\dots,x_n} \times \mathbb{R}^{n-k+1}_{y_{k+1},\dots,y_n,z}.$$

For ease of notation we write $D^k \times D^{n-k} := D^k \times \{0\} \times D^{n-k} \times \{0\}$. We view $D^k \times D^{n-k}$ as a D^k-parameter family of Legendrian D^{n-k}'s. Let $s \in D^k$ and $\theta \in D^{n-k}$ denote arbitrary elements.

Consider a Legendrian isotopy

$$\psi\colon D^{n-k} \times [0,1]_t \to \mathbb{R}^{2(n-k)+1}.$$

We use the same notation $\psi\colon \mathbb{R}^{2(n-k)+1} \times [0,1] \to \mathbb{R}^{2(n-k)+1}$ to denote a contact isotopy that extends this Legendrian isotopy. Assume that ψ is the identity near ∂D^{n-k} and is t-independent near $\partial[0,1]$, ie

$$\frac{\partial \psi}{\partial t}(\theta,0) = \frac{\partial \psi}{\partial t}(\theta,1) = 0 \quad \text{for all } \theta \in D^{n-k}.$$

Fix a smooth "bump function" $\beta_k\colon D^k \to [0,1]$ on the parameter space such that

- β_k has a unique critical point which is a maximum at 0 with $\beta_k(0) = 1$,
- β_k has radial symmetry, that is, $\beta_k(t) = \beta_k(t')$ whenever $|t| = |t'|$, and
- $\beta_k|_{\partial D^k} \equiv 0$.

Definition 3.1 The *D^k-suspension of a Legendrian isotopy*, $\Sigma_{D^k}\{\psi\}$, is the unique Legendrian lift of

$$\{(s, \psi(\theta, \beta_k(s))) \mid s \in D^k, \theta \in D^{n-k}\}$$

under the projection

$$\Pi\colon T^*\mathbb{R}^k \times \mathbb{R}^{2(n-k)+1} \to \mathbb{R}^k \times \mathbb{R}^{2(n-k)+1},$$
$$(x_1,\dots,x_k,y_1,\dots,y_k,x_{k+1},\dots,y_n,z) \mapsto (x_1,\dots,x_k,x_{k+1},\dots,y_n,z).$$

The Legendrian condition implies that the momentum coordinates $y_1, \dots, y_k$ of $T^*\mathbb{R}^k$, and thus the Legendrian submanifold itself, can be uniquely recovered from its projection $\Pi(\Sigma_{D^k}\{\psi\})$. The boundary of $D^k \times D^{n-k}$ is equal to $\partial D^k \times D^{n-k} \cup D^k \times \partial D^{n-k}$. By our assumptions on ψ and β_k, and uniqueness of the Legendrian lift,

$$\partial \Sigma_{D^k}\{\psi\} = \partial(D^k \times D^{n-k}).$$

We now prove Proposition 1.1 from the introduction, which states that $\Sigma_{D^k}\{\psi\}$ is Legendrian isotopic to $D^k \times D^{n-k}$ relative to boundary.

Proof of Proposition 1.1 We can define the isotopy by setting, for every $\tau \in [0,1]$, $\varphi_\tau(D^k \times D^{n-k})$ to be the unique Legendrian lift of

$$\{(s, \psi(\theta, \tau\beta_k(s))) \mid s \in D^k, \theta \in D^{n-k}\}$$

under the canonical projection Π. Therefore φ_τ is a Legendrian embedding and φ is a Legendrian isotopy from $D^n \times D^{n-k}$ to $\Sigma_{D^k}(\psi)$.

As $\partial \Sigma_{D^k}\{\psi\} = \partial(D^k \times D^{n-k})$, for every $\tau \in [0,1]$, φ_τ is the identity on the boundary $\partial(D^k \times D^{n-k})$. □

Remark 3.2 The word "suspension" in the context of Legendrians has been used to denote suspensions when the parameter space is S^k. In [9] the suspension $\Sigma_{S^k}\{\Lambda_\theta\}$ is defined similarly to Definition 3.1 for S^k-parametrized families of (closed) Legendrian embeddings. If the S^k-family is taken to be a constant family, then one recovers the S^k-spun of Legendrians, which first appeared in [11] for $k = 1$ and then was extended to general k in [15]. This construction was also studied by Sabloff and Sullivan, who called it "front spinning" in [23].

In this paper, our construction is "local". We obtain a Legendrian with boundary within a Darboux chart. This is in contrast to the previous constructions where the obtained Legendrian is closed, that is, has no boundary.

Another small point of difference from earlier constructions is that instead of beginning with a D^k-parametrized family of Legendrian embeddings, we first create such a family from a 1-parameter Legendrian isotopy.

We may generalize the suspension construction to the case when the initial Legendrian has a graphical front for a nonzero function. That is, instead of $D^k \times D^{n-k}$, consider Legendrian N contained in a Darboux chart so that the front of N in $D^n \times D^{n-k} \times \mathbb{R}$ is given by

$$\pi(N) = \Gamma(f_1 + f_2) = \{(x_1, \dots, x_k, x_{k+1}, \dots, x_n, f_1(x_1, \dots, x_k) + f_2(x_{k+1}, \dots, x_n))\}$$

for two smooth functions

$$f_1 \colon D^k \to \mathbb{R} \quad \text{and} \quad f_2 \colon D^{n-k} \to \mathbb{R}.$$

Let Λ_{f_2} be the unique Legendrian lift in $\mathbb{R}^{2(n-k)+1}$ of the front $\Gamma(f_2)$. Again consider a Legendrian isotopy

$$\psi \colon \Lambda_{f_2} \times [0,1] \to \mathbb{R}^{2(n-k)+1}$$

which is the identity near the boundary of Λ_{f_2} and satisfies

$$\frac{\partial\psi}{\partial t}(\theta,0)=\frac{\partial\psi}{\partial t}(\theta,1)=0 \quad \text{for all } \theta\in\Lambda_{f_2}.$$

Definition 3.3 The $\Gamma(f_1)$*-suspension of a Legendrian isotopy*, $\Sigma_{\Gamma(f_1)}\{\psi\}$, is the unique Legendrian lift of

$$\Pi(\Sigma_{\Gamma(f)}\{\psi\})=\{(s,(0,0,f_1(s))+\psi_{\beta_k(s)}(\theta))\in D^k\times\mathbb{R}^{2(n-k)+1}\mid s\in D^k,\theta\in\Lambda_{f_2}\}.$$

Here $s\in D^k$ is in the first k position coordinates, $(0,0,f_1(s))\in\mathbb{R}^{n-k}\times\mathbb{R}^{n-k}\times\mathbb{R}$, and $\psi_{\beta_k(s)}(\theta)\in\mathbb{R}^{2(n-k)+1}$. Analogous to Proposition 1.1, we have the following proposition. We do not include a detailed proof as it would merely be a slight tweak of the proof of Proposition 1.1.

Proposition 3.4 *The* $\Gamma(f)$*-suspension*, $\Sigma_{\Gamma(f)}\{\psi\}$, *is Legendrian isotopic to* N *relative to boundary.*

3.2 The boat move

In this section we introduce a new Legendrian move that can be viewed as a generalization of the Reidemeister 1 move. Introducing a boat is a way of swapping nonmaximum critical points in a graphical front with maxima and some cusp singularities.

Consider a Legendrian $\Lambda\subset(Y,\xi)$ with front equal to the graph of a Morse function, ie the front in some Darboux chart is $\Gamma(f)$ for $f\colon D^n\to\mathbb{R}$ a Morse function. Suppose there exists a minimum. Then the n-dimensional Reidemeister 1 move swaps in a maximum for a minimum, as seen in Figure 4. We want to convert other index critical points to maxima as well.

Definition 3.5 Suppose the front of an open subset $\Lambda_0\subset\Lambda$ locally is given as the graph of a Morse function $f\colon D^n\to\mathbb{R}$ with an index-k critical point at $\mathbf{0}$. By the Morse lemma, there exist local coordinates $x_1,\dots,x_n$ such that

$$f(x_1,\dots,x_n)=-x_1^2-\dots-x_k^2+x_{k+1}^2+\dots+x_n^2.$$

Locally we can extend $(x_1,\dots,x_n)$ to Darboux coordinates $(x_1,\dots,x_n,y_1,\dots,y_n,z)$, as any diffeomorphism of $\mathbb{R}^n$ can be extended to a contactomorphism of the 1-jet space, $J^1(\mathbb{R}^n)$. If we set

$$f_1\colon\mathbb{R}^k\to\mathbb{R},\qquad f_1(x_1,\dots,x_k)=-x_1^2-\dots-x_k^2,$$
$$f_2\colon\mathbb{R}^{n-k}\to\mathbb{R},\quad f_2(x_{k+1},\dots,x_n)=x_{k+1}^2+\dots+x_n^2,$$

we are in the setup of Definition 3.3. We define the (n,k)*-boat move* to be the replacement of the graphical open subset Λ_0 by the $\Gamma(f_1)$-suspension of $R1_{n-k}$, $\Sigma_{\Gamma(f_1)}(R1_{n-k})$. Let $B_{n,k}$ denote the resulting Legendrian (locally). We refer to $B_{n,k}$ as the (n,k)*-boat*.

Remark 3.6 If $k=n$, the boat move does not change anything. If $k=0$, the boat move is equal to the n-dimensional first Reidemeister move.

Proposition 3.7 *The* (n,k)*-boat* $B_{n,k}$ *is Legendrian isotopic to* Λ_0 *relative to boundary.*

Proof As the boat move is a special case of the suspension defined in Definition 3.3, $B_{n,k}$ is Legendrian isotopic to Λ_0 relative to boundary, directly from Proposition 1.1. □

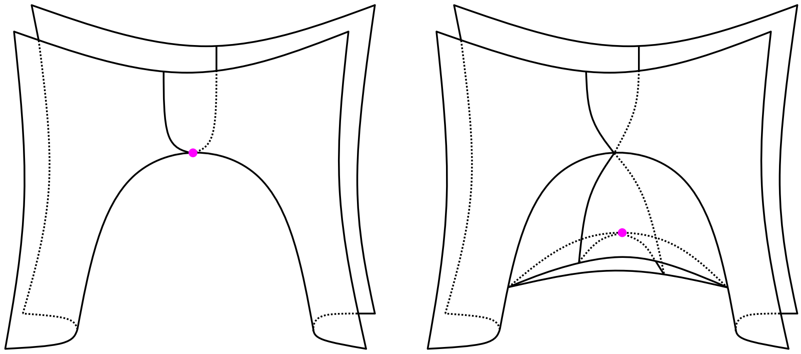

Figure 9: A (2,1)-boat move which changes a saddle point into a maximum.

Remark 3.8 The boat move gets its name because the $(2,1)$-boat looks like a overturned boat or canoe, as seen in Figure 9. The $(2,1)$-boat is very similar to the unigerm $A_3^{e,\pm}$, birth of cuspidal lips; see [3] or [16]. The main difference is the parametrization of the Legendrian front before the move.

Our goal for introducing the boat was that we wanted all the Reeb chords to correspond to maxima of the front. To do so, we will use the following proposition:

Proposition 3.9 *Decompose the front of $B_{n,k}$ into a disjoint union of a finite number of graphical components away from nonsmooth points. All but one of these graphical components are graphs of functions with no critical points. Further, the unique component with critical points has a unique critical point that is a maximum. Moreover, the tangent planes at the cusps are not parallel to the $(x_1,\ldots,x_n)$-plane.*

Proof The front of $B_{n,k}$ has a critical point only at points where both the front of $R1_{n-k}$ and f_1 have critical points. This happens at only one point, namely $x_1=\cdots=x_n=0$. By construction of f_1, $x_1=\cdots=x_n=0$ is assumed to be a maximum in the first k coordinates. The Reidemeister move then converts the critical point to a maximum in the last $n-k$ coordinates, thus proving the proposition.

We get that the cusps are not parallel to the $(x_1,\ldots,x_n)$-plane as the $(n-k)$-dimensional Reidemeister front does not have cusps parallel to the $(x_{k+1},\ldots,x_n)$-plane. □

Remark 3.10 Consider a Reeb chord with one end on the pink dot in the left figure of Figure 9. This Reeb chord survives the boat move. One may be confused about how the grading of this Reeb chord as an element of the Legendrian dga changes, since the local Morse index is changing. However, note that the relationship between the grading and local Morse index is not the typical one [12], as the cusps we use do not rotate the Lagrangian planes the way typical cusps in the literature do. As such, one can check that the associated Maslov indices of loops of tangent vectors from the top to the bottom of the Reeb chord actually stay the same under our boat moves, preserving the grading.

4 Weinstein presentations of subdomains

In this section we prove Theorem 1.2 and Corollary 1.4. We then apply this theorem to construct several explicit examples of handle decompositions for Weinstein manifolds obtained via antisurgery. We conclude with some open questions.

4.1 Proof of Theorem 1.2

Recall that Theorem 1.2 required that certain Reeb chords be nondegenerate and correspond to critical points of Morse functions. The following result shows that locally, Reeb chords can always be assumed to correspond to critical points of a Morse function. Given a parametrized Reeb chord $c\colon [0,T] \to Y$, we will let $\partial_- c$ and $\partial_+ c$ denote the negative and positive endpoints $c(0)$ and $c(T)$, respectively, of c.

Lemma 4.1 *Let $\Lambda_2 \subset J^1(\Lambda_1)$ be a compact Legendrian submanifold of the 1-jet space of a smooth manifold Λ_1, equipped with the standard contact structure. Then we may perturb Λ_2 by a C^∞-small isotopy so that there are finitely many Reeb chords c_i from the zero-section Λ_1 of $J^1(\Lambda_1)$ to Λ_2 and so that, in a neighborhood of the positive endpoint $\partial_+ c_i \in \Lambda_2$, the Legendrian Λ_2 looks like (x, df, f) for f a Morse function on a neighborhood (in Λ_1) of the negative endpoint $\partial_- c_i \in \Lambda_1$.*

Proof Let $\pi\colon J^1(\Lambda_1) \cong T^*(\Lambda_1) \times \mathbb{R} \to T^*(\Lambda_1)$ be the Lagrangian projection onto the cotangent bundle. We will use the notation Λ_1 and Λ_2 for both the original Legendrians as well as their images under the projection. By the Thom transversality theorem, we can perturb Λ_2 by a C^∞-small Hamiltonian isotopy in $T^*\Lambda_1$ so that Λ_1 and Λ_2 intersect transversely (this then corresponds to a C^∞-small Legendrian isotopy in the lift). Let $q \in \Lambda_1$ be one of these intersection points. Since $T\Lambda_2$ intersects $T\Lambda_1$ transversely at q, we have that $T\Lambda_2$ is graphical over the cotangent fiber $T(T_q^*\Lambda_1)$, and hence given by dQ, where $Q\colon T_q^*\Lambda_1 \to \mathbb{R}$ is a quadratic form. Now we may apply a C^∞-small isotopy in the space of quadratic forms from Q to a nondegenerate quadratic form $Q'\colon T_q^*\Lambda_1 \to \mathbb{R}$. This isotopy induces an isotopy of Lagrangian planes from dQ to dQ' in $T^*(T_q^*\Lambda_1)$. The Lagrangian plane dQ' is transverse to $T_q\Lambda_1$ since it is still graphical over $T_q^*\Lambda_1$ and now it is transverse to $T_q^*\Lambda_1$ since the quadratic form is nondegenerate. This (linear) isotopy of Lagrangian planes can be lifted to C^∞-small linear Hamiltonian isotopy. By taking a bump function, we get a compactly supported C^∞-small Hamiltonian function that takes Λ_2 to a Lagrangian Λ_2' that intersects Λ_1 at transverse intersection points $q \in \Lambda_1$ where Λ_2' also intersects the fiber $T_q^*\Lambda_1$ transversely. Since the isotopy is C^∞-small, no new intersection points are introduced, which means no new Reeb chords are introduced. Finally, we observe, because of these transversality conditions, $\pi(\Lambda_2')$ locally looks like df, where f is a Morse function locally defined near q. □

Remark 4.2 In Lemma 4.3, the Reeb chords we investigate may be interpreted as lying between Λ and Λ_- or between Λ and Λ_+. This is relevant when we refer to the Reeb chords as being represented as a maximum or a minimum of a Morse function in a suitable neighborhood; what is maximum with regard to a function on Λ_- will be a minimum with regard to a function on Λ_+. Here it should be understood

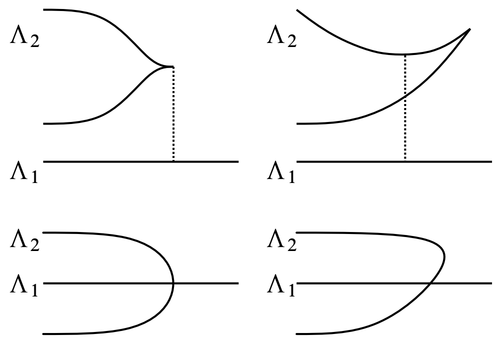

Figure 10: Front and Lagrangian projections of two Legendrians Λ_1 (zero-section) and Λ_2. Top left: there is a Reeb chord (depicted by the dotted line) between the Legendrians Λ_2 and Λ_1 with endpoint on the cusp of Λ_2. Bottom left: the corresponding Lagrangian projection. Top right: after Legendrian isotopy of Λ_2, the Reeb chord between Λ_1 and Λ_2 has endpoint on a smooth branch of Λ_2 which is locally described by the graph of a Morse function on Λ_1 (with minimum corresponding to the Reeb chord endpoint). Bottom right: the corresponding Lagrangian projection.

that we can only handleslide Λ past Λ_- at Reeb chords represented by maxima with respect to a locally defined Morse function on Λ_+ (resp. minima with respect to a locally defined Morse function on Λ_-). The following lemma uses suitable boat moves to turn all critical points into such maxima (resp. minima).

Lemma 4.3 *Suppose Λ is a Legendrian sphere in a contact manifold (M_0, ξ_0), and that (-1)-surgery on Λ produces the contact manifold (M', ξ'). Suppose Λ_+ and Λ_- are a pair of n-dimensional Legendrian spheres which are completely parallel (that is, Λ_- is the ε Reeb pushoff of Λ_+ for $\varepsilon > 0$) in (M_0, ξ_0) but not necessarily parallel in (M', ξ'), ie they may be distinctly linked with Λ.*

Let (M, ξ) be the $(2n+1)$-dimensional contact manifold obtained from (M_0, ξ_0) by (-1)-surgery along Λ and Λ_-, and $(+1)$-surgery along Λ_+. Then there exists a Legendrian submanifold $\Lambda_f \subset (M_0, \xi_0)$ such that (M, ξ) can be obtained by only a (-1)-surgery along the components of Λ_f.

Proof To obtain Λ_f, our goal will be to cancel Λ_+ with Λ_- in a surgery diagram of (M, ξ). In order to do so, we need Λ_+ and Λ_- to be completely parallel, ie they must be identically linked with Λ in (M_0, ξ_0). We will parallelize Λ_+ and Λ_- by performing a sequence of Legendrian isotopies and handleslides that preserve the resulting surgered contact manifold (M, ξ); see the example of Figure 11.

Consider a neighborhood $U \cong J^1(\Lambda_+)$ of Λ_+ in M_0 that contains Λ_- as the graph of a constant positive function ε and the subset $\Lambda \cap U$, which we will also call Λ. By Lemma 4.1, we may perturb Λ by a Legendrian isotopy so that we have finitely many Reeb chords, and in a neighborhood of each such Reeb chord the height difference is a Morse function. In general, there may be many other large-period Reeb chords in M_0 from Λ to Λ_- that leave U, but these will not affect our construction, which is local to Λ_-.

If we attempt to isotope Λ_- towards Λ_+ by pushing Λ_- in the z-coordinate in the front, we are obstructed whenever f has a Morse critical point on a part of Λ between Λ_- and Λ_+. These critical points

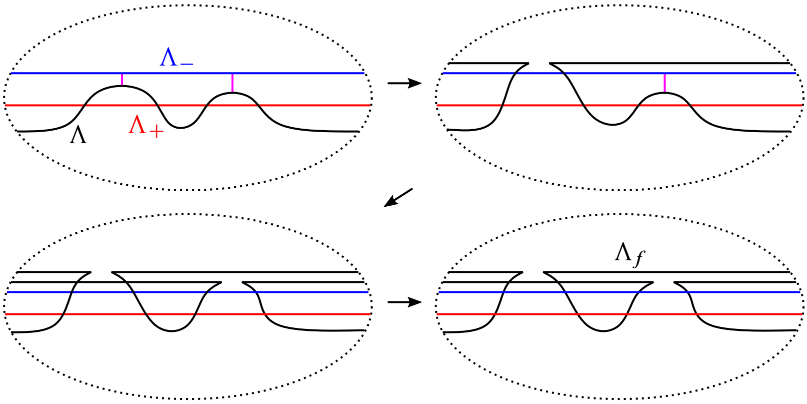

Figure 11: Assuming they are parallel outside of the dotted circle, the red Legendrian Λ_+ and blue Legendrian Λ_-, where we perform a $(+1)$ and (-1) Legendrian surgery, respectively, are made parallel by handlesliding the black Legendrian Λ over Λ_- at its maximum. The Λ_+ and Λ_- are then canceled, leaving only the new black Legendrian Λ_f.

correspond to nondegenerate Reeb chords between Λ_- and Λ and must be removed. We do so one at a time: we always work with the smallest remaining critical value, that is, the shortest remaining Reeb chord that obstructs our isotopy.

Consider the critical point, say q, with least critical value. If q has Morse index 0, then in the front projection we see a local maximum with respect to Λ_+ (see Remark 4.2). We remove it by performing a handleslide of Λ over Λ_- along the Reeb chord c between Λ_- and Λ corresponding to q. We then isotope Λ_- towards Λ_+. See Figure 11. Note that after the handleslide, there is exactly one fewer Reeb chord from Λ to Λ_- contained in U. First, the handleslide along c removes c. Furthermore, the rest of the Reeb chords remain and no other Reeb chords from Λ to Λ_- are produced in U. To see this, we note that the handleslide modifies Λ by first pushing the maximum through Λ_- and doing a cusp connected sum with a positive Reeb pushoff $R_\varepsilon(\Lambda_-)$ of Λ_-. The first step removes the Reeb chord c but does not create new Reeb chords since the pushing through the maximum can be obtained by increasing the function, without creating new critical points (this is not true if we push through an index $k > 0$ critical point as we discuss in the next paragraph). The second step of cusp connect summing with $R_\varepsilon(\Lambda_-)$ creates a family of Reeb chords from Λ_- to Λ but no Reeb chords from Λ to Λ_- since there are no Reeb chords from the positive pushoff $R_\varepsilon(\Lambda_-)$ to Λ_- contained in U. As before, there may be large-period Reeb chords from the handleslid Λ to Λ_- (for example from $R_\varepsilon(\Lambda_-)$ to Λ_-) that leave the neighborhood U. However, these large Reeb chords do not affect our construction, whose goal is to remove local Reeb chords that obstruct the isotopy between Λ_- and Λ_+ in U.

If q has Morse index $n-k$, for $0 \le k < n$, we first perform a (n,k)-boat move to Λ, as defined in Definition 3.5. (If the critical point has Morse index n, then it is a minimum in the front projection, so we perform an $(n,0)$-boat move, which is just an n-dimensional first Reidemeister move). By Proposition 3.9,

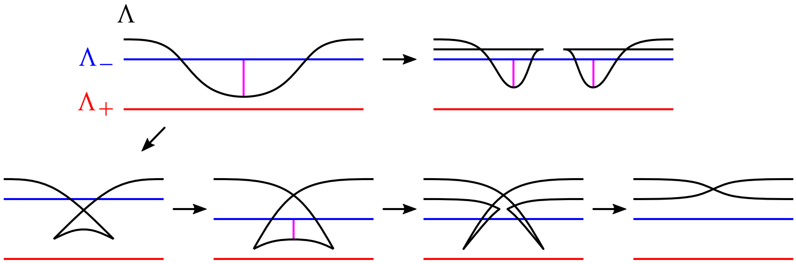

Figure 12: Top row: doing a -1 handleslide of Λ over Λ_- at an index-1 critical point (of the height difference function) creates two more Reeb chords. Bottom row: first doing a boat move at the index-1 critical point (a first Reidemeister move) and then a -1 handleslide over Λ_- removes all Reeb chords. We observe that the resulting Legendrian is a crossing connect sum of Λ and Λ_- which appears when doing a $+1$ handleslide; see Figure 5.

the (n,k)-boat move is a Legendrian isotopy that converts the index-$(n-k)$ critical point q to a maximum, say q', in the front projection. We now have a Reeb chord, say $\gamma_{q'}$, between Λ and Λ_- corresponding to q'. Proposition 3.9 implies that within a small neighborhood no other Reeb chords are created in this process. To see this note that such Reeb chords only exist when Λ and Λ_- are parallel in the front projection. Within the Darboux neighborhood where we perform the boat move, the front of Λ_- is given by the $(x_1,\dots,x_n)$-plane. As Proposition 3.9 says no other smooth critical points are created and the tangent spaces at the newly created cusp points are not parallel to the $(x_1,\dots,x_n)$-plane, the boat move does not create any additional Reeb chords within this neighborhood. Outside the neighborhood, we do not make any changes to both Λ and Λ_-. Perhaps we make changes to some Reeb chords but these are not within our neighborhood.

We isotope Λ_- towards q' until $\gamma_{q'}$ is uninterrupted. Next, we handleslide Λ over Λ_- along the Reeb chord $\gamma_{q'}$ at q'. We can then isotope Λ_- further towards Λ_+. This process is fully illustrated in Figures 12 and 13.

We repeat this process until no obstructing nondegenerate Reeb chords remain and Λ_+ and Λ_- are in canceling position. Once canceled, we are left with a single Legendrian Λ_f in the surgery diagram of (M,ξ). In summary, Λ_f corresponds to Λ in the following way: for each Reeb chord γ from Λ to Λ_- with local index $n-k$, we applied an (n,k)-boat move to Λ and did a cusp connected sum with Λ_-. □

Remark 4.4 If we handleslide over a Reeb chord corresponding to an index-k critical point without first doing a boat move, then new Reeb chords are created; see the top row of Figure 12. By first doing the boat move, which does not create any new Reeb chords but only changes the local index of the existing Reeb chord, we ensure that handlesliding removes that Reeb chord without creating any new chords.

Remark 4.5 Although Lemma 4.3 is stated in the language of contact manifolds and contact (± 1)-surgeries, the result also holds with Weinstein homotopies. We recall that (-1)-surgeries correspond to Weinstein handle attachment while $(+1)$-surgeries correspond to handle removal, and in general do not

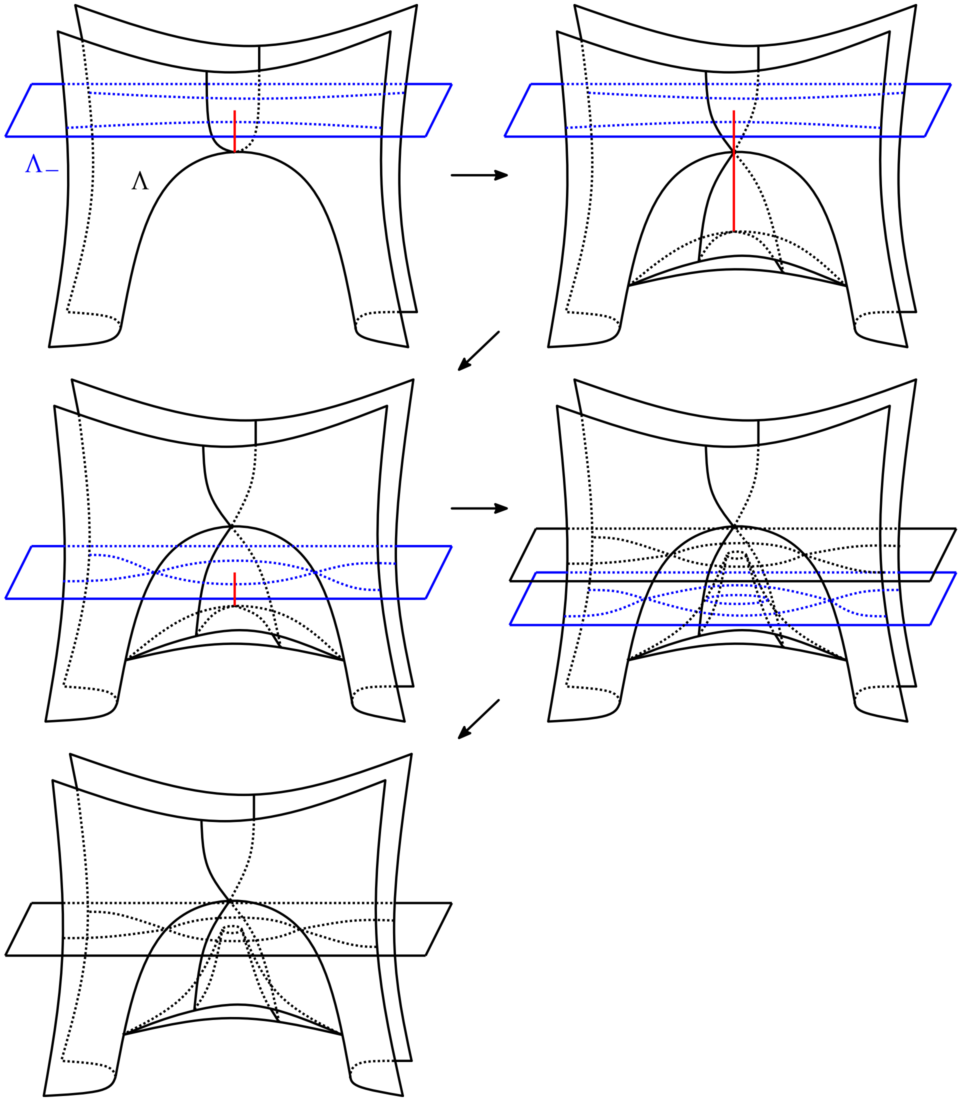

Figure 13: The process of performing a boat move locally for an index-1 critical point on a surface Legendrian to enable a handleslide: (1) The black Legendrian Λ has an index-1 critical point at which there is a red Reeb chord to the blue Legendrian Λ_-. (2) Perform a boat move at the critical point; the Reeb chord now meets the Λ at a maximum in the front diagram. (3) Isotope Λ_- downwards; the Reeb chord is now uninterrupted in the front diagram. (4) Perform a handleslide along the Reeb chord, producing a small circle of cusps in the middle of the boat. (5) Isotope the Λ_- past the boat.

produce fillable contact manifolds. However, in our case, all surgery moves are handleslides are over the (-1)-Legendrian, and hence we can view the $(+1)$-Legendrian as a placeholder for the boundary of the Lagrangian disk, which will be removed at the last step. Hence all our contact moves are really Weinstein homotopies.

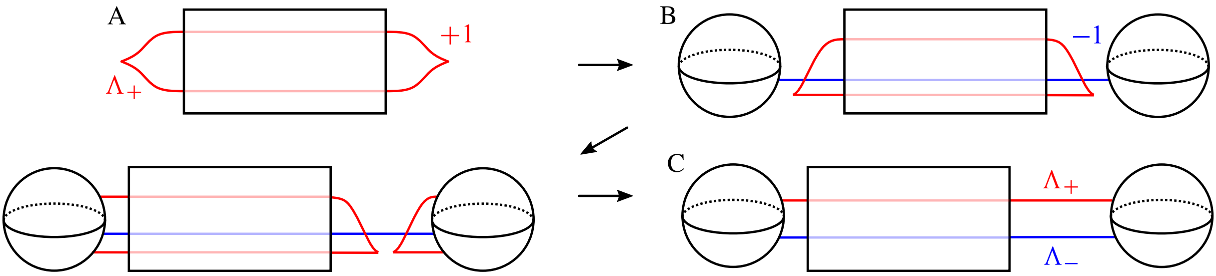

Figure 14: The diagrams A, B, and C. In the top left, we see a red Legendrian unknot in the boundary of a Weinstein domain. This unknot may be linked with attaching handles of the domain away from its cusps, represented by the black rectangle. This diagram is related to the figure on the bottom right by the addition of a canceling pair and a single handleslide.

Remark 4.6 In the next two proofs, we abuse terminology and refer interchangeably to Weinstein diagrams and Weinstein domains, so we sometimes say "attach a handle to the Weinstein diagram". We will also refer to the carving out of a Lagrangian disk as "antisurgery" along its boundary Legendrian. We hope this abuse of terminology will not confuse the reader but, in fact, make the proof easier to read.

Proof of Theorem 1.2 Recall that $L \subset X$ is of the form $X = T^*D^n \cup H_i^n$ and $L = D^n \subset T^*D^n$. Then the Weinstein sector $X \setminus L$ is obtained by antisurgery along the Legendrian knot ∂L which corresponds to the unknot $\partial D^n \subset T^*D^n$. From now onward, we will denote the knot ∂L by Λ_+, and depict it in red in all figures. Let $\Lambda = \bigcup_i \Lambda_i$ denote the link consisting of all the attaching spheres of the H_i^n.

We may isotope Λ_+ so that the front projection of Λ is disjoint from the cusps of Λ_+, giving us a surgery diagram, say diagram A, for $\partial(X \setminus L)$, as depicted in the top left of Figure 14. We introduce a canceling pair to diagram A — an $(n-1)$-handle and a critical handle Λ_- such that Λ_- is a parallel pushoff of the bottom arc of Λ_+ — to obtain a new surgery diagram, say diagram B, again for $\partial(X \setminus L)$, as depicted in top right of Figure 14. We observe that diagram B is in turn equivalent to a "diagram C" of the form of the bottom right of Figure 14, where Λ_+ and Λ_- both traverse across the $n-1$ handle exactly once each.

We note that, in diagram C, Λ_+ and Λ_- are parallel except in how they are linked with Λ. So we can apply Lemma 4.3 to obtain a surgery presentation that contains only (-1)-surgery along Legendrian submanifolds, say diagram D. Then this diagram D corresponds to a Weinstein handle diagram where each (-1)-surgery corresponds to a critical handle attachment.

Note that the Weinstein diagram in diagram D has exactly one more index-$(n-1)$ handle than the Weinstein presentation for X. Further, if we denote the attaching spheres of the index-n handles H_i in diagram D by Λ_i', respectively, the construction in Lemma 4.3 implies that the Λ_i' are exactly as described in the theorem statement. □

As a corollary, we explain how to describe the P-loose Legendrian unknot in $\mathbb{R}^{2n-1} \subset S^{2n-1} = \partial B^{2n}$. To do so, we first describe a front diagram of the knot in $S^{n-1} \times \mathbb{R}^n \subset \partial(B^{2n} \cup H^{n-1})$ and then attach a flexible handle to make the ambient space ∂B^{2n}.

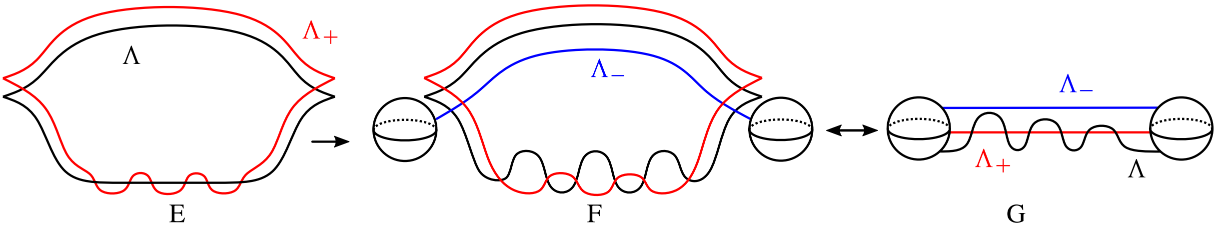

Figure 15: A canceling pair of handles, represented by a pair of spheres and a blue Legendrian, is added just below the cusps of the linked red and black Legendrians. This diagram is equivalent to the final one via two handleslides.

Proof of Corollary 1.4 Recall from Section 2.4 and [20] that the P-loose Legendrian unknot,

$$\Lambda_P \subset S^{2n-1} = \partial B^{2n}$$

is obtained as follows. First, to construct a P-Moore space, consider the CW complex $S^1 \cup_P D^2$, the result of attaching D^2 to S^1 along the degree-p map $\partial D^2 = S^1 \to S^1$. If $n \geq 5$, then this CW complex embeds into S^{n-1}, as observed in [1], and we let U be a neighborhood of this CW complex. Next, we observe that U has a Morse function (with gradient outward-pointing near the boundary of U) that has three critical points, one of index 0 and 1 for the S^1, and one of index 2 for the D^2.

Next, we carve out a disk $D_U \subset T^*D^n$ from T^*D^n to obtain $T^*D^n \setminus D_U$, which as an unstopped Weinstein domain and is equivalent to $B^{2n} \cup H^{n-1}$. Then Λ_P is

$$\partial D \subset \partial(B^{2n} \cup H^{n-1} \cup H_{\text{flex}}),$$

where H_{flex} is a flexible Weinstein handle attached along a Legendrian which is loose in the complement of $\partial D \subset B^{2n} \cup H^{n-1}$. Additionally, H_{flex} is in canceling position with H^{n-1}. We will explicitly construct the Weinstein handle decomposition of this $B^{2n} \cup H^{n-1}$ to obtain an explicit front diagram for Λ_P.

We begin by using the construction from Proposition 2.8. In the boundary $S^{2n-1} = \partial B^{2n}$, we consider a pair of canceling contact surgeries along parallel $(n-1)$-dimensional Legendrian unknots. We label the Legendrian unknot corresponding to the (-1)-surgery Λ and the Legendrian unknot corresponding to the $(+1)$-surgery Λ_+. Here Λ_+ and Λ represent Legendrian boundaries ∂D_U and ∂D, respectively. Note that there exists a P-Moore space $U \subset \Lambda_+$ by our assumption. We perturb Λ_+ past the Λ by "pushing" the subdomain U so that Λ_+ and Λ are no longer parallel. Let us refer to this surgery diagram as diagram E.

To diagram E, we insert a canceling pair of handles consisting of an $(n-1)$-handle H^{n-1} and an n-handle H^n, just below the cusps of Λ_+ and Λ; see Figure 15. Note that as a contact surgery curve on the boundary, the n-handle attachment corresponds to a (-1)-surgery along a Legendrian knot. Let us denote the Legendrian knot corresponding to this (-1)-surgery by Λ_-, and refer to this Weinstein diagram as diagram F. We see from Figure 15 that diagram F is equivalent to diagram G after some handleslides and Legendrian isotopies.

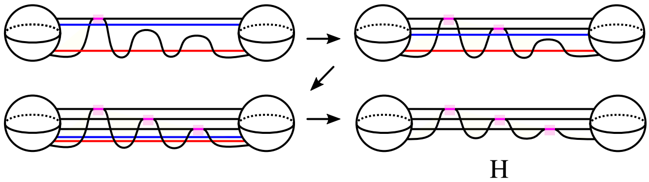

Figure 16: Applying Lemma 4.3 to diagram G involves three instances of a boat move isotopy followed by a handleslide. Then Λ_+ and Λ_- are completely parallel; thus the respective surgeries on these knots cancel.

We are now set up exactly as in the proof of Theorem 1.2, namely diagram G is of the form of diagram C in the proof of Theorem 1.2. So we similarly apply Lemma 4.3 to obtain the required Weinstein diagram, say diagram H. Since the P-Moore space has three critical points, Λ in diagram G also has three critical points, of index 0, 1, and 2 when considering the Morse function f of Lemma 4.3. Thus in applying the lemma, we perform three instances of a boat move and a cusp connect sum to Λ in diagram G to obtain diagram H; see Figure 16.

Finally, to obtain Λ_P in B^{2n}, that is, to make the ambient manifold B^{2n}, we attach a flexible handle, H_{flex}, to cancel out H^{n-1} in diagram H. This amounts to attaching a loose Legendrian Λ_{flex} that winds around the $(n-1)$-handle once, and is in the complement of Λ in diagram H. This gives us diagram I, which is depicted in the top left of Figure 17, with Λ_{flex} denoted in pink.

Next, in diagram I, we slide Λ repeatedly over H_{flex} until Λ no longer passes through the H^{n-1}. We then cancel H^{n-1} with H_{flex}. We are left with a surgery diagram, say diagram J, that consists of a single Legendrian Λ_P in B^{2n}. This process is illustrated in Figure 17. We see that Λ_P consists of four loose Legendrian unknots which are completely parallel, away from a bounded region where they are linked (in a way that depends on p) and are connected via three boat moves and cusp connected sum gluings.

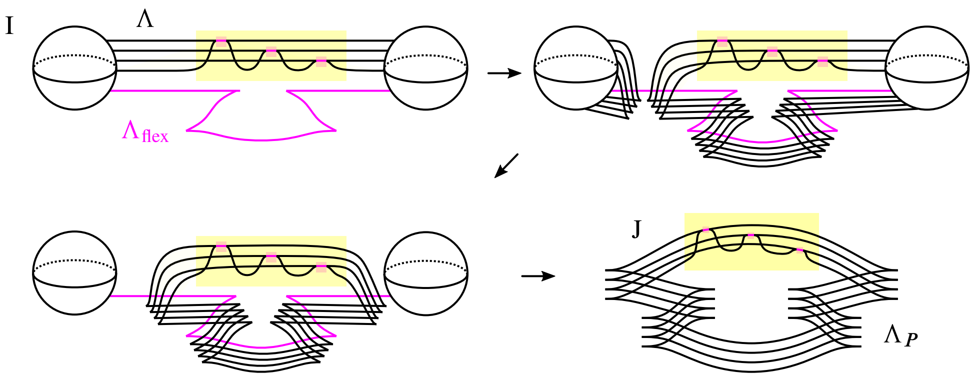

Figure 17: The final step in the construction: In the complement of the black Legendrian Λ which traverses the $(n-1)$-handle some number of times, we attach a flexible handle along the pink loose Legendrian Λ_{flex}. We then repeatedly slide Λ over Λ_{flex} to detach it from the $(n-1)$-handle. Finally, we cancel Λ_{flex} with the $(n-1)$-handle.

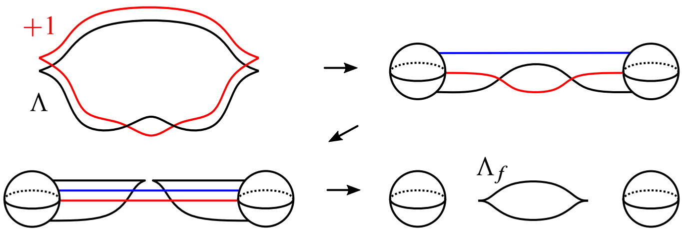

Figure 18: Our construction applied to the case where $U = D^k$. We first add a canceling pair and slide the red and black Legendrians over the blue Legendrian. Then at the maximum on the black Legendrian, we slide over the blue Legendrian. Passing the cusps over the $n-1$ handle and canceling the red with the blue, we obtain a max tb unknotted sphere in the complement of the $n-1$ handle.

As a Weinstein diagram, diagram J depicts the Weinstein sector (B^{2n}, Λ_P), where Λ_P is a P-loose Legendrian unknot. □

4.2 Explicit examples

We now construct several exotic Weinstein manifolds as applications of the construction from Theorem 1.2 and Corollary 1.4. In all these examples, we consider antisurgery on the Lagrangian disk obtained by perturbing the boundary $S^{n-1} = \partial D^n$ of the zero-section $D^n \subset T^*D^n$ in a neighborhood $U \subset S^{n-1}$, as in the construction of P-loose Legendrians (see Section 2.4).

Example 4.7 Suppose $U = D^{n-1} \subset S^{n-1}$ is a disk. Then we may choose the Morse function $g\colon U \to \mathbb{R}$ to have a single critical point of index 0. So we obtain Λ_U with a single maximum. Applying the construction from Theorem 1.2, we obtain a Legendrian Λ_f by a single handleslide (see Figure 18). The resulting Legendrian, Λ_f, is a standard Legendrian unknot in the complement of the $(n-1)$-handle, H^{n-1}.

Here Λ_f is not loose, and is the Legendrian unknot in the subcritical domain $B^{2n} \cup H^{n-1}$. Indeed, we have constructed the Weinstein diagram of $T^*D^n \cup H^{n-1}$, ie T^*D^n with a subcritical handle attached along a subcritical isotropic sphere in a Darboux chart. Λ_+ is the Legendrian unknot and bounds the Lagrangian unknot; carving out the Lagrangian unknot is equivalent to attaching an index-$(n-1)$ handle.

Example 4.8 Let $U \subset S^{n-1}$ be the disconnected union of a codimension-zero submanifold U' and a disk D^{n-1}. In this case, after cancellation, we expect the remaining Legendrian to be loose. This is because $D^n_{U' \coprod D^{n-1}}$ is Lagrangian isotopic to $D^n_{U'} \natural T^*_0 D^n$, and by [19], $T^*D^n \setminus (D_{U'} \natural T^*_0 D^n)$ is obtained from the subcritical sector $T^*D^n \setminus (D_{U'} \coprod T^*_0 D^n)$ by attaching a flexible handle. We can also see this from our explicit construction as follows. Since U is disconnected, we may isotope Λ_U so that the Reeb chords coming from $g|_{U'}$, the U'-perturbation, are shorter than the Reeb chord coming from the maximum of $g|_{D^{n-1}}$, the D^{n-1}-perturbation (see Figure 19).

Next, we apply the construction of Theorem 1.2 to obtain the Legendrian attaching sphere Λ_f as follows. A combination of boat moves and handleslides will first remove all the critical points coming from U'.

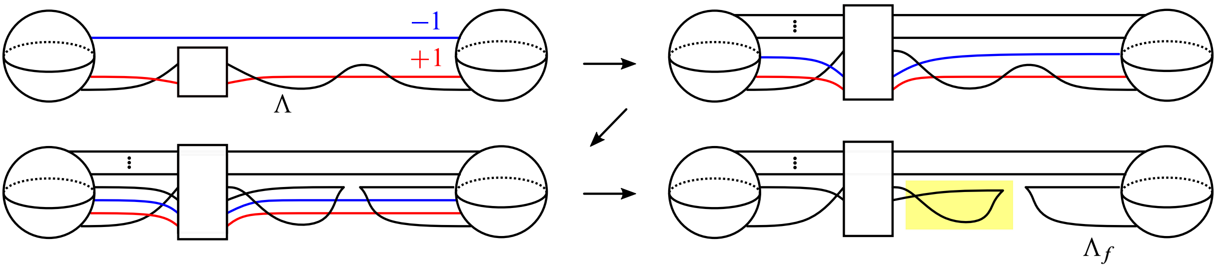

Figure 19: Our construction applied to the case where U is a union of some submanifold (the rectangular region) and a disk (the maximum on the right). We first use boat moves and handleslides to remove all obstructing Reeb chords coming from the boxed region. A single handleslide then puts the red and blue Legendrians in a canceling position. The yellow highlighted region is a loose chart.

Then a single handleslide will remove the critical point coming from D^{n-1} (see Figure 19). We are now in position to cancel Λ_- and Λ_+ as in the proof of Theorem 1.2.

After this cancellation, the resulting Legendrian knot Λ_f has a loose fishtail chart in a transverse slice. By [21], this implies that Λ_f is loose.

Example 4.9 When $0 \in P$, the P-loose Legendrian Λ_P is loose. To see this, first recall that

$$D_U = \Gamma(df) \cap B^{2n} \quad \text{for a function } f\colon D^n \to \mathbb{R}$$

which is an extension of a Morse function $f\colon S^{n-1} \to R$ that is negative on U and positive on the closure of the complement. It is enough to prove that Λ_P is loose for $P = \{0\}$; as discussed in [20, Section 2.2.2], $\Lambda_{P \coprod Q}$ is isotopic to $\Lambda_P \# \Lambda_Q$ and the connected sum of any Legendrian with a loose Legendrian (in a separate Darboux chart) is loose.

Let $U = S^k$, that is, U is a P-Moore space for $P = \{0\}$ since its relative cohomology is $\mathbb{Z} \cong \mathbb{Z}/\{0\}$ in positive degree. Consider U to be embedded in S^{n-1} as the intersection $D^{k+1} \cap S^{n-1}$. Then one can take the function $f\colon D^n \to \mathbb{R}$ to be a perturbation of

$$-(x_1^2 + \cdots + x_{k+1}^2) + x_{k+2}^2 + \cdots + x_n^2.$$

Then $D_U = \Gamma(df)$ is Hamiltonian isotopic to the cotangent fiber, $T_0^* D^n$, in $T^* D^n$. So $T^* D^n \setminus T_0^* D^n$ is just $B^{2n} \cup H^{n-1}$. Further, the stop ∂D_U is a Legendrian that passes through the H^{n-1} exactly one time, ie it is a loose Legendrian.

Next, we will explain how to see this explicitly by following the construction in the proof of Theorem 1.2. We use the notation from the proof of Theorem 1.2 as well. On $U = S^k$, f has two critical points, namely, a maximum and a saddle point. Following the construction, to get an explicit Weinstein diagram we perform a single boat move and two handleslides. After canceling Λ_- and Λ_+, the resulting Legendrian (diagram H) intersects the subcritical $(n-1)$-handle H^{n-1} three times. This is because initially ∂D^n

intersected H^{n-1} once, and each handleslide introduces one new intersection. For $n = 3$, this process is illustrated in Figure 20. In higher dimensions the construction follows analogously.

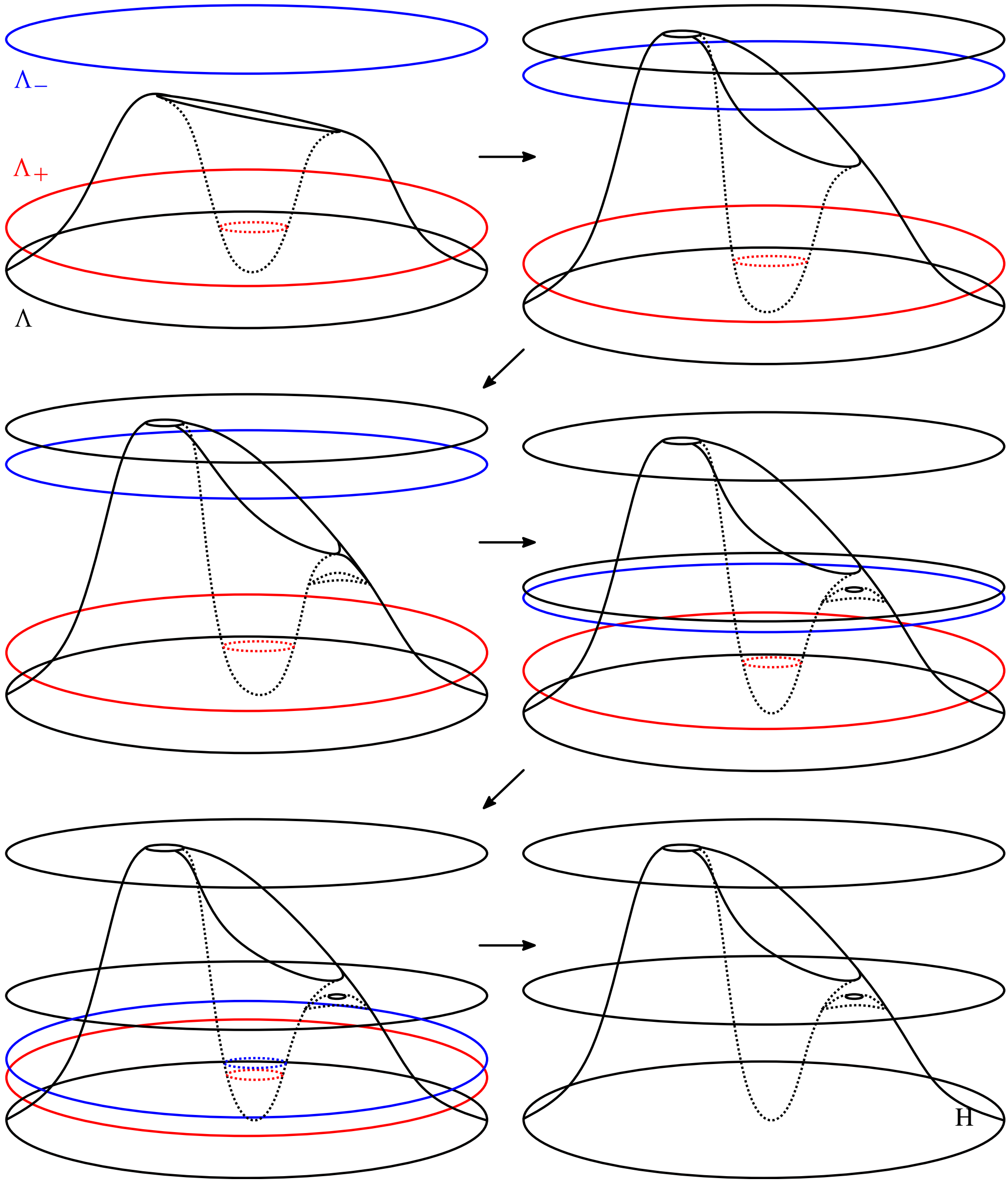

Figure 20: The process to obtain diagram H when $n = 3$. The black Legendrian Λ begins with a maximum and a saddle point between the blue Λ_- and the red Λ_+. Away from the pictured region, Λ_- and Λ_+ are parallel. We perform the following moves: (1) At the maximum, we perform a handleslide. (2) At the saddle point, we perform a boat move. (3) We then isotope Λ_- downwards, and handleslide Λ over Λ_-. (4) We further isotope Λ_- so that it is in canceling position with Λ_+. (5) Finally, we cancel Λ_- and Λ_+, so that only Λ remains.

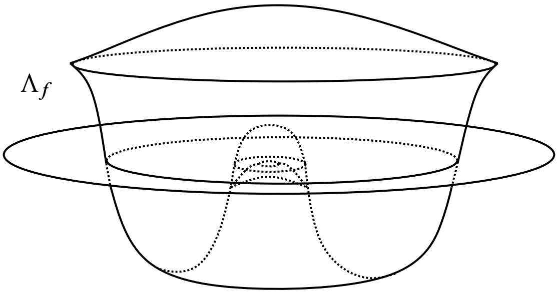

Figure 21: After a Legendrian isotopy which passes the top circle of cusps across the 2-handle, the connected black Legendrian resulting from the recipe in Figure 20 is in canceling position with the subcritical handle whose attaching region is a torus around the middle disk in this dimension.

With diagram H, instead of continuing with the construction in the proof of Theorem 1.2 as is, we first do an additional step. We perform a Legendrian isotopy that passes the circle of cusps over H^{n-1}. The resulting Legendrian, $\Lambda_f = \partial D_U$, passes through H^{n-1} exactly one time.

To conclude, note that Λ_f is now in canceling position with H^{n-1}; see Figure 21. Hence it is loose, and remains loose when the flexible handle H_{flex} is attached alongside it to cancel H^{n-1}. Thus we obtain that Λ_0 is loose.

4.3 Questions

One can construct a loose Legendrian unknot by pushing through any codimension-zero subdomain U (with boundary) past the Legendrian unknot *near a cusp*. As observed in [21], Λ_U is always loose; see Figure 7. If the Euler characteristic of U is 0, then Λ_U is formally Legendrian isotopic to Λ (but not genuinely Legendrian isotopic), and hence called the loose Legendrian unknot. So by the h-principle for loose Legendrians, Λ_U and Λ_V are isotopic if $\chi(U) = \chi(V)$.

Our construction of the P-loose Legendrians also involves pushing through certain codimension-zero subdomains (neighborhoods of P-Moore spaces). However, here the construction is less concrete; one must first push through to create the Lagrangian disk D_U, then carve out D_U, and then attach a flexible handle, ultimately resulting in our recipe above. Hence it is natural to ask whether there is a more direct route towards the construction of these P-loose Legendrians, analogous to the construction of loose Legendrians by Murphy.

Question 4.10 *Can a P-loose Legendrian unknot be constructed more directly by pushing through a P-Moore space past a region of the Legendrian unknot* (*not near a cusp*), *after Legendrian isotopy of the unknot*?

For example, this pushing operation, if it exists, must have the property that if U is disconnected then Λ_U must be loose, as discussed in Figure 21.

Another line of inquiry is to see whether our algorithm can provide an alternative proof of the Ganatra–Pardon–Shende [14] localization formula from the point of view of Legendrian invariants. The localization formula [14] computes the wrapped Fukaya of the subdomain $X \setminus D$ as

$$\mathcal{W}(X \setminus D) \cong \mathcal{W}(X)/D,$$

where $\mathcal{W}(X)/D$ is the algebraic localization of $\mathcal{W}(X)$ by the divisor D. They describe a concrete formula computing the morphism chain complexes, $\mathrm{Hom}_{\mathcal{W}(X)/D}(L, K)$, via a dg bar construction that depends on the morphism $\mathrm{Hom}_{\mathcal{W}(X)}(L, K)$ as well as $\mathrm{Hom}_{\mathcal{W}(X)}(L, D)$ and $\mathrm{Hom}_{\mathcal{W}(X)}(D, K)$. For certain Lagrangians, like the cocores of X, these complexes can all be computed using the Legendrian dga's of the attaching spheres of X and ∂D. Hence the Ganatra–Pardon–Shende surgery formula can be used a priori to describe the Legendrian dga's of the attaching spheres of $X \setminus D$. On the other hand, here we give an explicit geometric Weinstein presentation for $X \setminus D$ and an explicit depiction of its Legendrian attaching spheres.

Question 4.11 *Can one compute the Legendrian dga's of the attaching spheres of $X \setminus D$ produced by Theorem 1.2 directly, giving an alternative direct proof of the Ganatra–Pardon–Shende localization formula?*

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