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An RBG construction of integral surgery homeomorphisms

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We generalize the RBG construction of Manolescu and Piccirillo to produce pairs of knots with the same n-surgery, and investigate the possibility of constructing exotic definite four-manifolds using n-surgery homeomorphisms.

57K10, 57K18, 57K40

1 Introduction

Manolescu and Piccirillo [12] introduced RBG links, a kind of 3-component framed links in S^3 that produce knot pairs with the same 0-surgery. (Similar constructions appeared earlier in Akbulut's work [2; 3].) RBG links are relevant for an approach to constructing exotic definite 4-manifolds. The strategy is to find a knot pair (K, K'), such that $S^3_0(K) \cong S^3_0(K')$, K is H-slice in some 4-manifold W (ie bounds a null-homologous disk in $W \setminus \operatorname{int} B^4$) and K' is not H-slice in W. Then, one can construct a new 4-manifold W' (an exotic copy of W) by carving out a neighborhood of the slice disk bounded by K, and gluing back the trace of 0-surgery on K' using some 0-surgery homeomorphism.

In [12], they focused on a class of RBG links called special RBG links, and experimented on a 6-parameter family of RBG links. They used Rasmussen's s-invariant to obstruct K' from being H-slice in W, and collected several knots K where the usual invariants obstructing H-sliceness vanish. Later on, however, Nakamura showed that these knots K are not slice. He developed a method in [13] to stably relate the traces of K and K', and obstruct K from being H-slice using $s(K') \neq 0$.

In this paper, for $n \in \mathbb{Z}$, we generalize RBG links to $|n|$-RBG links, which can be used to produce knot pairs (K, J) such that $S^3_l(K) \cong S^3_m(J)$ with $l, m \in \{n, -n\}$.

Definition 1.1 An *$|n|$-RBG link* $L = \{(R, r), (B, b), (G, g)\}$ is a 3-component framed link in S^3, with framings $r \in \mathbb{Q}$ and $b, g \in \mathbb{Z}$, together with homeomorphisms

$$\psi_B\colon S^3_{r,g}(R, G) \to S^3 \quad \text{and} \quad \psi_G\colon S^3_{r,b}(R, B) \to S^3,$$

such that $H_1(S^3_{r,b,g}(R, B, G); \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

Remark 1.2 A certain type of $|n|$-RBG links was defined in the context of Legendrian knots by Casals, Etnyre and Kegel [6, Definition 3.19].

Theorem 1.3 *Any $|n|$-RBG link $L = \{(R,r),(B,b),(G,g)\}$ has an associated knot pair (K_B, K_G) and a homeomorphism $\phi_L\colon S^3_{f_b}(K_B) \to S^3_{f_g}(K_G)$ with $f_b, f_g \in \{n,-n\}$. Conversely, given a homeomorphism $\phi\colon S^3_l(K) \to S^3_m(J)$ with $l,m \in \{n,-n\}$, there exists an $|n|$-RBG link L_ϕ such that the associated knot pair is (K,J) and $\phi_{L_\phi} = \phi$ up to isotopy.*

As in [12], one can attempt to use n-surgery homeomorphisms to construct exotic 4-manifolds. A knot K is said to be *n-slice* in W, if $K \subset \partial(W \setminus \operatorname{int} B^4)$ bounds a properly embedded disk D with self-intersection number $-n$. If there is another knot K' with $S^3_n(K) \cong S^3_n(K')$, by removing a tubular neighborhood of D and gluing back the trace of n-surgery on K', we obtain a new 4-manifold W' such that K' is n-slice in W'. For $W = \#^l\,\overline{\mathbb{CP}}^2$, we have that W' is homeomorphic to W. If K' is not n-slice in W, then W' is not diffeomorphic to W. Moreover, there is an adjunction inequality for the Rasmussen's s-invariant for knots which are n-slice in W; this was conjectured by Manolescu, Marengon, Sarkar and Willis [11] and was proved by Ren [18]. Thus, one can try to use the s-invariant and a pair of knots with the same n-surgery to construct an exotic $\#^l\,\overline{\mathbb{CP}}^2$.

1.1 n-special RBG links

We define n-special RBG links, for which the associated knot pairs are easier to find diagrammatically and the associated knots have the same n-surgery.

Definition 1.4 A link $L = \{(R,r),(B,b),(G,g)\}$ with linking matrix M_L, is called an *n-special RBG link*, if $b = g = 0$, $n = -\det(M_L)$, and there exist link isotopies

$$R \cup B \cong R \cup \mu_R \cong R \cup G,$$

where μ_R is the meridian of R.

As in [12], one can experiment on parametrized families of n-special RBG links to look for knot pairs that share the same n-surgery, such that one of the knots is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$ and the other is not n-slice in any $\#^m\,\overline{\mathbb{CP}}^2$.

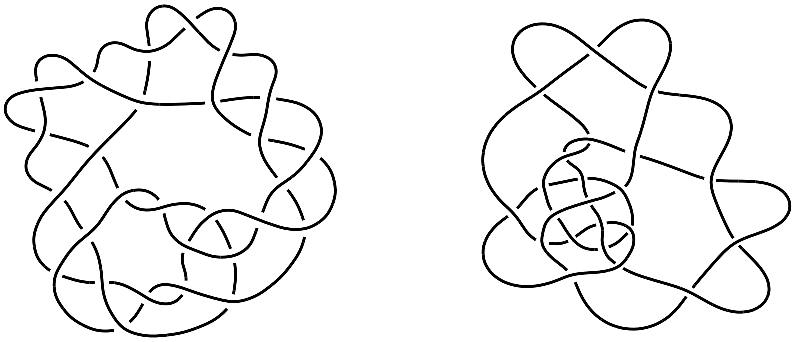
Nakamura [13] showed that for special RBG links such that $R = U$ and K_B is H-slice in W, Rasmussen's s-invariant cannot be used to obstruct K_G from being H-slice. We generalize Nakamura's theorem to n-special RBG links.

Theorem 1.5 *Let $L = \{(R,r),(B,0),(G,0)\}$ be an n-special RBG link with n nonnegative.*

(a) *If R is r-slice in some $\#^m\,\mathbb{CP}^2$ and K_B is n-slice in $\#^l\,\overline{\mathbb{CP}}^2$, then $s(K_G) \le n - \sqrt{n}$.*

(b) *If R is $(r-1)$-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then $s(K_G) \le n + 1 - \sqrt{n+1}$.*

If $n > 0$, the above theorem leaves open the possibility of using the s-invariant to detect exotic pairs of definite 4-manifolds from n-special RBG links where R is the unknot. For example, we have:



$K(-2,1,3)$ $K(-2,1,2)$

Figure 1

Theorem 1.6 *If the knot* $K(-2,1,3)$ *from the left-hand side of Figure 1 is* 3*-slice in some* $\#^m\,\overline{\mathbb{CP}}^2$*, then there exists an exotic* $\#^m\,\overline{\mathbb{CP}}^2$*.*

Theorem 1.5 also gives a new way of obstructing knots from being n-slice, by finding another knot with the same n-surgery and checking its s-invariant. (This generalizes the $n=0$ method, which was used by Piccirillo [17] in her proof that the Conway knot is not slice, and then extended by Nakamura [13].) For example, we prove the following theorem.

Theorem 1.7 *The knots in Figure 1 are not* 2*-slice in any* $\#^m\,\overline{\mathbb{CP}}^2$*.*

1.2 n-peculiar RBG links

We will also define a different class of $|n|$-RBG links called *n-peculiar RBG links* (see Section 5), for which the red components R are rationally framed and the associated knot pairs can be obtained diagrammatically.

Definition 1.8 A link $\{(R,r),(B,b),(G,g)\}$ is called an n-peculiar RBG link, if there exists $t\in\mathbb{Z}$ such that

- $R=U$ and B,G are meridians of R,
- $b=g=1/r+1/t$,
- $n=(g+b-2l)-t(l-b)^2$,

where $l=\operatorname{lk}(B,G)$ under an orientation of L such that $\operatorname{lk}(B,R)=\operatorname{lk}(G,R)=1$.

This gives a new construction of RBG links when $n=0$, for which Nakamura's obstruction in [13] does not immediately apply; so, in principle, they can potentially produce exotic 4-spheres.

Organization of the paper In Section 2, we generalize the RBG construction of zero surgeries to integral surgeries. We discuss the construction of a potential exotic pair by cutting and pasting of n-traces. In Section 3, we introduce n-special RBG links, for which the associated knot pairs can be obtained diagrammatically. In Section 4, we generalize Nakamura's sliceness obstruction to an n-sliceness obstruction using n-special RBG links, and we give examples where the n-sliceness of knots in $\#^m \overline{\mathbb{CP}}^2$ is obstructed by this new method. In Section 5, we discuss n-peculiar RBG links.

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2 $\langle n \rangle$-surgery homeomorphisms

Let n be an integer and $|n|$ be its absolute value. We use $\langle n \rangle$ to denote the set $\{n, -n\}$.

Definition 1.1 An $|n|$*-RBG link* $L = \{(R,r),(B,b),(G,g)\}$ is a 3-component framed link in S^3, with framings $r \in \mathbb{Q}$ and $b, g \in \mathbb{Z}$, together with homeomorphisms

$$\psi_B : S^3_{r,g}(R,G) \to S^3 \quad \text{and} \quad \psi_G : S^3_{r,b}(R,B) \to S^3,$$

such that $H_1(S^3_{r,b,g}(R,B,G);\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

Remark 2.1 In [12], RBG links are defined to be rationally framed for 0-surgeries. In the case of $\pm n$-surgeries with $n \neq 0$, we restrict the framings b, g of B, G to be integers, so that we can pin down the surgery coefficient to $\pm n$ from the homological condition: $H_1(S^3_{r,b,g}(R,B,G);\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

Definition 2.2 Given a pair of framed knots $\{(K, f_K), (J, f_J)\}$, a homeomorphism $\phi : S^3_{f_K}(K) \to S^3_{f_J}(J)$ is called an $\langle n \rangle$*-surgery homeomorphism*, if $f_K, f_J \in \{n, -n\}$. If $f_K = f_J = n$, then we call ϕ an n*-surgery homeomorphism*.

We generalize Theorem 1.2 of [12] to $|n|$-RBG links as in the following theorem, which is a rephrasing of Theorem 1.3 from the introduction.

Theorem 2.3 *Any $|n|$-RBG link $L = \{(R,r),(B,b),(G,g)\}$ has an associated knot pair (K_B, K_G) and an $\langle n \rangle$-surgery homeomorphism $\phi_L : S^3_{f_b}(K_B) \to S^3_{f_g}(K_G)$. Conversely, given an $\langle n \rangle$-surgery homeomorphism $\phi : S^3_l(K) \to S^3_m(J)$, there exists an $|n|$-RBG link L_ϕ such that the associated knot pair is (K, J) and $\phi_{L_\phi} = \phi$ up to isotopy.*

Proof Given an $|n|$-RBG link $L = \{(R,r),(B,b),(G,g)\}$, we associate to it two framed knots $(K_B, f_b), (K_G, f_g)$ and an $\langle n \rangle$-surgery homeomorphism ϕ_L as follows: First, let (K_B, f_b) be $\psi_B(B,b)$,

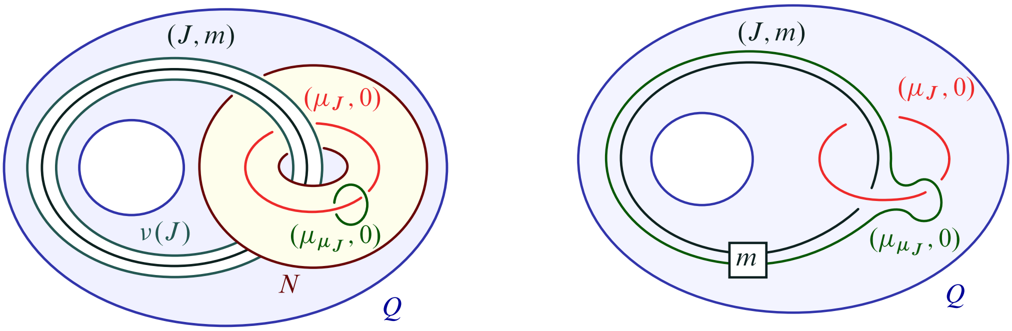


Figure 2

and let (K_G, f_g) be $\psi_G(G, g)$. Since a homeomorphism maps an integer framing to an integer framing, f_b, f_g are integers. Then, extend ψ_B (resp. ψ_G) to $\widetilde{\psi}_B\colon S^3_{r,b,g}(R, B, G) \to S^3_{f_b}(K_B)$ (resp. $\widetilde{\psi}_G\colon S^3_{r,b,g}(R, B, G) \to S^3_{f_g}(K_G)$) by gluing back tubular neighborhoods of B and K_B (resp. G and K_G) according to the framings. Since $H_1(S^3_{r,b,g}(R, B, G); \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, we have $f_b, f_g \in \{n, -n\}$. Finally, define ϕ_L as

$$\phi_L := \widetilde{\psi}_G \circ \widetilde{\psi}_B^{-1}\colon S^3_{f_b}(K_B) \to S^3_{f_g}(K_G).$$

Conversely, given an $\langle n\rangle$-surgery homeomorphism $\phi\colon S^3_l(K) \to S^3_m(J)$, define an $|n|$-RBG link L_ϕ as follows.

Fix setwise representatives of $S^3_l(K), S^3_m(J)$ by specifying the knots $K, J \subset S^3$ and the surgery tubular neighborhoods $\nu(K), \nu(J)$. Pick a meridian $\mu_J \subset S^3 \setminus \nu(J)$. Up to isotopy, we can assume that ϕ^{-1} maps μ_J into $S^3 \setminus \nu(K)$. Choose a tubular neighborhood Q of J which contains μ_J, and pick a tubular neighborhood N of μ_J such that $N \subset Q \setminus \nu(J)$ and $\phi^{-1}(N) \subset S^3 \setminus \nu(K)$. Pick a meridian μ_{μ_J} of μ_J in N. See Figure 2, left. Let L_ϕ be $\{\phi^{-1}(\mu_J, 0), (K, l), (\phi^{-1}(\mu_{\mu_J}), 0)\}$.

Let $\widetilde{N}$ be the manifold obtained by surgery on N along $\{(\mu_J, 0), (\mu_{\mu_J}, 0)\}$, and let $\widetilde{Q}$ be the manifold obtained by surgery on Q along $\{(\mu_J, 0), (J, m), (\mu_{\mu_J}, 0)\}$. Extend $\phi\colon S^3_l(K) \to S^3_m(J)$ to $\widetilde{\phi}: S^3_{l,r,0}(K, R, G) \to S^3_{m,0,0}(J, \mu_J, \mu_{\mu_J})$. Let $\widetilde{\psi}_B$ be the slam-dunk map (as in Figure 5.30 in [8]) in $\widetilde{\phi}^{-1}(\widetilde{N})$ and the identity map on $S^3_l(K) \setminus \phi^{-1}(N)$.

Let η be the composition map $\phi \circ \widetilde{\psi}_B \circ (\widetilde{\phi})^{-1}$, which is identity outside of $\widetilde{N}$. Slide μ_{μ_J} over J and cancel the pair (J, μ_J) in $\widetilde{Q}$, as in Figure 2, right. Together, they induce a homeomorphism ψ which is identity on $S^3 \setminus Q$. Let $\widetilde{\psi}_G$ be $\psi \circ \widetilde{\phi}$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc} & & S^3_{r,l,0}(R, B, G) & \xrightarrow{\widetilde{\psi}_B} & S^3_l(K) \\ & \swarrow^{\widetilde{\psi}_G} & \downarrow{\scriptstyle\widetilde{\phi}} & & \downarrow{\scriptstyle\phi} \\ S^3_m(J) & \xleftarrow[\psi]{} & S^3_{0,m,0}(\mu_J, J, \mu_{\mu_J}) & \xrightarrow{\eta} & S^3_m(J) \end{array}$$

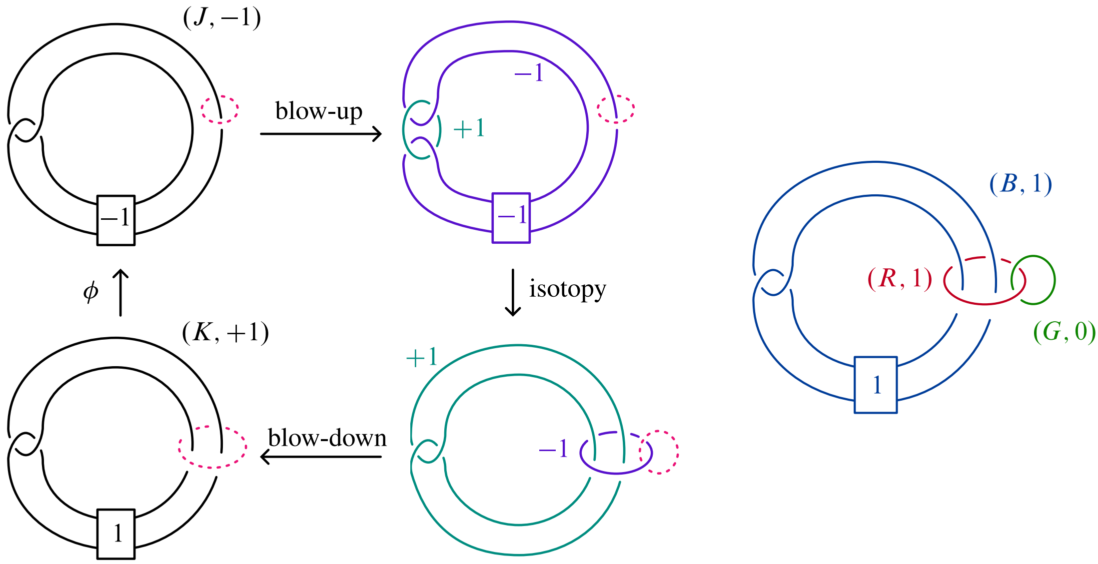


Figure 3

Since $\psi \circ \eta^{-1}$ is identity outside of Q and $\mathrm{MCG}(S^1 \times D^2, S^1 \times S^1)$ is trivial, $\psi \circ \eta^{-1}$ is isotopic to the identity. Therefore, $\widetilde{\psi}_G \circ \widetilde{\psi}_B^{-1}$ is isotopic to ϕ. Undo the surgery on B (resp. G), we obtain ψ_B (resp. ψ_G) from $\widetilde{\psi}_B$ (resp. $\widetilde{\psi}_G$). □

Example 2.4 Consider a homeomorphism ϕ between the 1-surgery on the figure-eight knot K and the (-1)-surgery on the right-handed trefoil J in Figure 3, left, which is an analogue of Figure 23 in [4].

We can construct the corresponding $|1|$-RBG link (Figure 3, right) by chasing the image of $(\mu_J, 0)$ under the map ϕ^{-1}.

For the rest of the paper, we will mostly be concerned with n-surgery homeomorphisms (rather than those that change an n-surgery to a $(-n)$-surgery).

2.1 *n*-slice knots from *n*-surgery homeomorphisms

Let X be a smooth, closed, oriented 4-manifold, and let $X^\circ = X \setminus \mathrm{int}(B^4)$. Let K be a knot in $\partial X^\circ \cong S^3$. Suppose K bounds a properly embedded disk D in X°. There exists a tubular neighborhood $\nu(D) \cong D^2 \times D^2$, where D is identified with $D^2 \times \{0\}$. Pick a point $p \in \partial D^2$ and denote $S^1 \times \{p\}$ by K_D. Following Section 2.2 of [13], we make the following definition.

Definition 2.5 The knot $K \subset \partial X^\circ$ is *n-slice*, if $\mathrm{lk}(K, K_D) = n$ in ∂X°.

Denote $X^\circ \setminus \nu(D)$ by $E(D)$ and denote the trace of the n-surgery along K by $X_n(K)$. By the trace embedding lemma [9, Lemma 3.3], if K is n-slice in X, then $-X_n(K)$ is smoothly embedded in X. In particular, we have that $[D]\cdot[D] = -n$.

Now, given an n-surgery homeomorphism $\phi\colon S^3_n(K) \to S^3_n(J)$, we define

$$X_{(D,\phi)} = -X_n(J) \cup_\phi E(D).$$

For $n=0$, if the disk D is not null-homologous, then it is possible that $X_{(D,\phi)}$ is not homeomorphic to X. Note that if X is definite, such as $\#^m\,\mathbb{CP}^2$, every 0-slice disk is null-homologous.

Example 2.6 [10, Example 5.3] Let X be $S^2\times S^2$. Since $S^2\times D^2 \cong X_0(U)$, $S^2\times D^2$ is the exterior of some disk in X°. Let $\phi\colon S^3_0(U)\to S^3_0(U)$ be a homeomorphism that maps the 0-framed meridian of U to a 1-framed meridian of U. Then $X_{(D,\phi)} \cong \mathbb{CP}^2\,\#\,\overline{\mathbb{CP}}^2$.

However, for $X = \#^m\,\overline{\mathbb{CP}}^2$, we have that $X_{(D,\phi)}$ is homeomorphic to X.

Proposition 2.7 *Let $\phi\colon S^3_n(K)\to S^3_n(J)$ be an n-surgery homeomorphism and D be an n-slice disk bounded by K. If X is simply connected and negative definite with $n\neq 0$, then $X_{(D,\phi)}$ is homeomorphic to X.*

Proof Denote the intersection pairing of a 4-manifold M on $H_2(M;\mathbb{Q})$ by Q_M. Since $n\neq 0$, we have $H_2(S^3_n(K);\mathbb{Q}) = H_1(S^3_n(K);\mathbb{Q}) = 0$. Thus, by the Mayer–Vietoris sequence for X,

$$H_2(X;\mathbb{Q}) \cong H_2(E(D);\mathbb{Q}) \oplus H_2(-X_n(K);\mathbb{Q}).$$

Similarly, for $X_{(D,\phi)}$, we have $H_2(X_{(D,\phi)};\mathbb{Q}) \cong H_2(E(D);\mathbb{Q}) \oplus H_2(-X_n(J);\mathbb{Q})$. Thus, Q_X is isomorphic to $Q_{E(D)} \oplus Q_{-X_n(K)}$ over $\mathbb{Q}$, and $Q_{X_{(D,\phi)}}$ is isomorphic to $Q_{E(D)} \oplus Q_{-X_n(J)}$ over $\mathbb{Q}$. Since $Q_{X_n(K)} \cong Q_{X_n(J)} \cong (n)$ and X is negative definite, we have that $X_{(D,\phi)}$ is also negative definite. By Donaldson's theorem, the intersection forms of X and $X_{(D,\phi)}$ are diagonalizable over $\mathbb{Z}$. As in the proof of Lemma 3.3 in [12] for the case $n=0$, $X_{(D,\phi)}$ is simply connected. Hence, by Freedman's theorem $X_{(D,\phi)}$ is homeomorphic to X. □

Proposition 2.8 *The knot J is n-slice in $X_{(D,\phi)}$.*

Proof The knot trace is $X_{-n}(-J) = B^4 \cup_{(-J,-n)} \{\text{2-handle}\}$. Remove the B^4 from $X_{(D,\phi)}$, and the core of the 2-handle gives an n-slice disk of J. □

2.2 Extendability over *n*-traces

For $(X, X_{(D,\phi)})$ to be a potential exotic pair, we need that the n-surgery homeomorphism

$$\phi\colon S^3_n(K)\to S^3_n(J)$$

does not extend smoothly to an n-trace diffeomorphism $\Phi\colon X_n(K)\to X_n(J)$. In some cases, one can see that ϕ actually extends smoothly over n-traces.

Example 2.9 Given an $|n|$-RBG link L such that $(R,r)=(U,0)$, B and G are meridians of R and ψ_B (resp. ψ_G) is induced by sliding B (resp. G) over R and a slam-dunk. Replacing (R,r) by a dotted circle and doing the same diagram calculus, we obtain a diffeomorphism from $X_n(K_B)$ to $X_n(K_G)$, extending ϕ_L. Note that L is an n-special RBG link with $(R,r)=(U,0)$ (see Definition 1.4).

Generalizing Definition 3.12 in [12], we say that an n-surgery homeomorphism $\phi\colon S^3_n(K)\to S^3_n(J)$ has *property* U, if there exists a choice of surgery diagrams of $S_n(K)$ and $S_n(J)$, such that ϕ sends a 0-framed meridian of K to a 0-framed curve γ which appears unknotted in the diagram of $S_n(J)$.

Theorem 2.10 *If ϕ has property U, then there exists a diffeomorphism $\Phi\colon X_n(K)\to X_n(J)$ with $\Phi|_\partial=\phi$.*

Proof This is a generalization of Theorem 3.13 in [12]. □

For example, all links in Example 2.9 have property U.

If the mapping class group of the n-surgeries is trivial, then if there exists some $\phi\colon S^3_n(K)\to S^3_n(J)$ which does not extend over the trace, then the traces $X_n(K)$ and $X_n(J)$ are not diffeomorphic. In general, it is hard to obstruct extensibility smoothly, but we have obstructions for extending homeomorphically over the traces.

Proposition 2.11 *Let $\phi\colon S^3_n(K)\to S^3_n(J)$ be an n-surgery homeomorphism, with $n\neq 0$, which induces $f_*\colon H_1(S^3_n(K))\to H_1(S^3_n(J))$. Then, f extends to a trace homeomorphism $\Phi\colon X_n(K)\to X_n(J)$ if and only if $f_*([\mu_K])=\pm[\mu_J]$.*

Proof If $f_*([\mu_K])=\pm[\mu_J]$, then we can lift f_* to an isometry $\Lambda\colon H_2(X_n(K))\to H_2(X_n(J))$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(X_n(K)) & \longrightarrow & H_2(X_n(K),S^3_n(K)) & \longrightarrow & H_1(S^3_n(K)) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\Lambda} & & {\scriptstyle\Lambda^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \\
0 & \longrightarrow & H_2(X_n(J)) & \longrightarrow & H_2(X_n(K),S^3_n(K)) & \longrightarrow & H_1(S^3_n(K)) & \longrightarrow & 0
\end{array}$$

Since $n\neq 0$, the geometric obstruction $\theta(f,\Lambda)$ vanishes for any morphism (f,Λ). Moreover, the Kirby–Siebenmann invariants $\Delta(X_n(K))\equiv\Delta(X_n(J))\equiv 0 \pmod 2$. Then the result follows from [5, Corollary 0.8(i)]. □

Remark 2.12 There exist homeomorphisms of n-surgeries that do not map meridian to meridian. For instance, consider an n-special RBG link with $l=2,\ r=3$ (see Section 3). One can chase the meridian within the link diagram and construct a homeomorphism f which maps meridian $[\mu_{K_G}]$ to $3[\mu_{K_B}]$. By Proposition 2.11, we have that f is not extensible over the n-trace.

Proposition 2.13 *Let n be an integer such that $\{l \mid l^2 = 1\} = \{\pm 1\} \subset \mathbb{Z}/n\mathbb{Z}$. Every n-surgery homeomorphism extends over traces.*

Proof Consider the linking form Q of M. Since n-traces have intersection form (n), we have $Q([\mu_K],[\mu_K]) = Q([\mu_J],[\mu_J]) = 1/n \pmod{\mathbb{Z}}$. Let $f_*([\mu_K]) = l[\mu_J]$. Since Q is invariant under f_*, we have that $l^2 = 1 \pmod{n}$. The result follows from Proposition 2.11. □

For example, when $n = 1, 2, 4, p^k$ or $2p^k$ with p an odd prime, every n-surgery homeomorphism extends over the traces homeomorphically. For other n, it is possible that $\phi\colon S^3_n(K) \to S^3_n(J)$ does not map meridian to meridian homologically; in such a case, if we also have that the mapping class group of $S^3_n(K)$ is trivial, then we conclude that their n-traces $X_n(K)$ and $X_n(J)$ are not homeomorphic.

3 n-special RBG links

Definition 1.4 A link $L = R \cup B \cup G$, with framings $r, b, g \in \mathbb{Z}$, respectively, and a linking matrix M_L, is called an *n-special RBG link*, if

- $b = g = 0$,
- there exist link isotopies $R \cup B \cong R \cup \mu_R \cong R \cup G$,
- $n = -\det(M_L)$.

We obtain a knot diagram of (K_G, f_g) as follows. Choose a diagram of an n-special RBG link L. Isotope L such that $B = \mu_R$. Slide G over R such that G does not intersect some meridian disk Δ_B of R bounded by B. Cancel the pair (B, R). Denote the image of (G, g) by (G', g'). Similarly, we can obtain (K_B, f_b).

Remark 3.1 Since there is only one orientation preserving homeomorphism of S^3 up to isotopy, in the standard (empty) diagram of S^3, the framed knot (K_G, f_g) is isotopic to (G', g') and (K_B, f_b) is isotopic to (B', b').

Notice that isotopies and slides do not change the determinant of the linking matrix, and a slam-dunk changes the sign of the linking matrix. Therefore, the homeomorphism ϕ_L induced by L is an n-surgery homeomorphism.

Let ψ_B (ψ_G) be the homeomorphism induced by a sequence of Kirby moves. Note that an n-special RBG link, together with ψ_B and ψ_G, is an $|n|$-RBG link.

Orient the link L such that $\mathrm{lk}(R, G) = \mathrm{lk}(R, B) = 1$ and denote $\mathrm{lk}(B, G)$ by l. Then

$$n = -\det M_L = l(rl - 2).$$

We can get arbitrary n by setting $l = \pm 1$ and changing r.

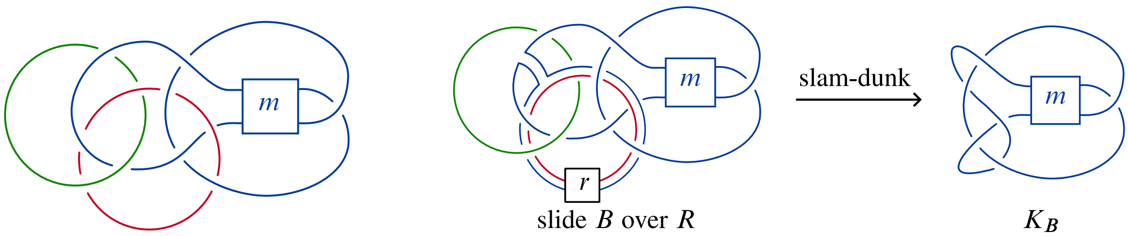


Figure 4: Left: a one parameter family of n-special RBG links. Right: obtaining the diagram of K_B with one slide.

Example 3.2 Consider a family of special RBG links with $l = -1$ in Figure 4, left.

The knot diagram for K_B can be computed by sliding B over R and canceling the pair (G, R) (see Figure 4, right). Similarly, to get K_G, we isotope the link diagram such that B becomes a circle, and let Δ_B be the inner domain bounded by B. Slide G over R along Δ_B, and cancel the pair (B, R) (see Figure 5, left).

If $m = 1$, K_B is the figure-eight knot, and K_G is also the figure-eight knot for any r. For $r = -1$, $m = 0$, we have $S^3_1(K_B) \cong S^3_1(K_G)$. Using SnapPy [7], we identify K_B as 6_2, which is the mirror of 6_2 in the Rolfsen knot table, and K_G as $K13n3596$. This gives an example of small knots which have the same 1-surgery.

Remark 3.3 By Theorem 3.7 in [1], for a knot K with an annulus presentation, one can construct another knot K' via the $(*n)$ operation, such that $S^3_n(K) \cong S^3_n(K')$.

In Example 3.2 with $m = 0$, the knot K_B has an annulus presentation as shown in Figure 5, right. The knot K' obtained by applying $(*(-1))$ operation on the mirror of K_B, is the mirror of the knot K_G in Example 3.2.

There are, however, some cases where an n-special RBG link produces identical knots.

Proposition 3.4 *Let $L = \{(R, r), (B, 0), (G, 0)\}$ be an n-special RBG link. If $(R, r) = (U, 0)$ and R bounds a disk Δ_R such that $|\Delta_R \cap B| = |\Delta_R \cap G| = 1$, then $K_B = K_G$.*

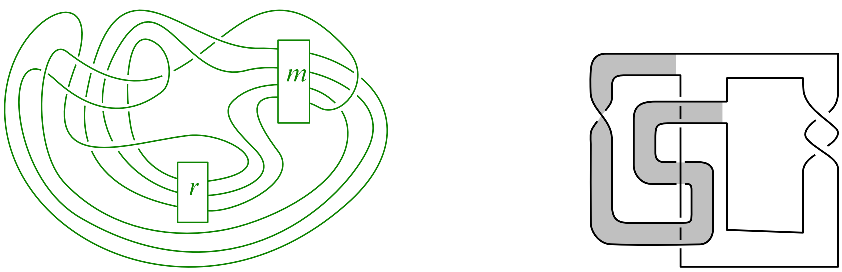


Figure 5: Left: the diagram of K_G. Right: an annulus presentation of K_B in Figure 4, right, with $m = 0$.

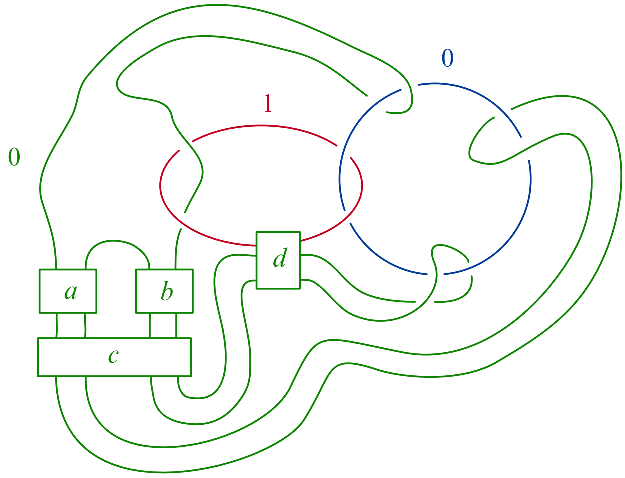


Figure 6

Proof Slide B over G such that the resulting blue component B' does not intersect with Δ_R. Since R has framing 0, we can cancel the pair (R, G). Since these induce a homeomorphism from $S^3_{0,0}(R, G)$ to S^3, the knot K_B is isotopic to B'. Similarly, the knot K_G is obtained by sliding G over B using the same band as above and therefore has the same diagram as K_B. □

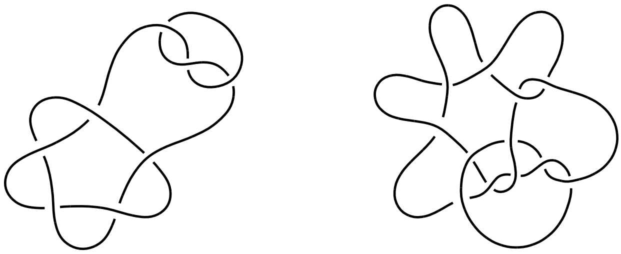
Proposition 3.5 *Let L be an n-special RBG link. If B bounds a properly embedded disk Δ_B such that $|\Delta_B \cap R| = 1$, $|\Delta_B \cap G| < 2$, and if G bounds a properly embedded disk Δ_G such that $|\Delta_G \cap R| = 1$, $|\Delta_G \cap B| \leq 2$, then $K_B = K_G$. (All intersections are required to be transverse.)*

Proof This is a generalization of Proposition 4.11 in [12], which was for $n = 0$. The proof in [12] is independent of the framings of the RBG link. □

Remark 3.6 From Example 3.2, we see that there exists a special RBG link with disks Δ_G, Δ_B, such that $|\Delta_G \cap R| = 1$, $|\Delta_B \cap R| = 1$, and $|\Delta_G \cap B| = 1$, $|\Delta_B \cap G| = 3$, but the associated knots K_B, K_G are not isotopic.

Example 3.7 Consider a family of special RBG links with four twisting boxes as in Figure 6. Since the linking number l between B and G is -1, if $r = 1$, then $n = 3$. Therefore, we obtain a family of 3-special RBG links parametrized by the numbers of twists (a, b, c, d). For each choice of (a, b, c, d), we denote the green knot associated to the link by $K_G(a, b, c, d)$, and the corresponding blue knot by $K_B(a, b, c, d)$.

For example, let $(a, b, c, d) = (-2, 1, -1, -1)$. The knot $K_G(-2, 2, -1, -1)$ is the nonhyperbolic knot $T(-2, 3) \# T(2, 5)$, and the knot $K_B(-2, 2, -1, -1)$ is recognized as $K12n121$ by SnapPy [7] (see Figure 7). Since $K_G(-2, 2, -1, -1)$ and $K_B(-2, 2, -1, -1)$ are generated by a 3-special RBG link, they have the same 3-surgery. Thus, 3 is a noncharacterizing slope for $T(-2, 3) \# T(2, 5)$, recovering an example in [20].



$K_G(-2,2,-1,-1)$ $K_B(-2,2,-1,-1)$

Figure 7

This family also produces knot pairs with small crossing numbers (summing up to 20). When $(a,b,c,d) = (1,2,-1,-1)$, we have that $S^3_3(6_3) \cong S^3_3(K14n15962)$. For $(a,b,c,d) = (0,1,-1,-1)$, we obtain that $S^3_3(6_2) \cong S^3_3(K14n10164)$, which can also be obtained from Example 3.2 with $r = 1$, $m = 0$. Finally, when $(a,b,c,d) = (-1,2,-1,-1)$, the link generates the knot pair $(K_G, K_B) = (10_{125}, 10_{132})$, whose 3-surgeries are isometric hyperbolic 3-manifolds. All of the knot pairs in this example have the property that $s(K_B) = s(K_G)$.

4 Potential exotica

In this section, we follow the recipe given in Section 2.1 and look for exotica using n-special links and the Rasmussen's s-invariant. More specifically, we can use n-special RBG links to find knot pairs (K, J) that share the same n-surgery. If K is n-slice in $\#^m \overline{\mathbb{CP}}^2$ and J is not n-slice in $\#^m \overline{\mathbb{CP}}^2$ for some m, then we can build an exotic negative-definite 4-manifold (see Proposition 2.7). We first discuss the use of the s-invariant in obstructing a knot from being n-slice and then present an example which generates potential exotica.

4.1 *n*-special RBG link and the *s*-invariant

We generalize the results of Nakamura [13] to n-special RBG links.

Proposition 4.1 (analogue of Lemma 3.1 in [13]) *Let $L = \{(R,r),(B,0),(G,0)\}$ be an n-special RBG link. If R is r-slice in a smooth oriented closed 4-manifold W, then*

$$X_n(K_B) \# W \cong X_n(K_G) \# W.$$

Proof Let Z be the 4-manifold obtained by attaching two 2-handles to $W^\circ \setminus \nu(D)$ along $\{(B,0),(G,0)\}$, where D is an r-slice disk of R. Since $(B,0)$ is isotopic to $(\mu_R, 0)$, the 2-handle attached along $(B,0)$ fills $\nu(D)$. Slide G over R such that G does not intersect some meridian disk Δ_B bounded by B. This changes $(G,0)$ to (K_G,n) and induces $Z \cong W^\circ \cup_{(K_G,n)} 2h \cong X_n(K_G) \# W$. Similarly, $Z \cong X_n(K_B) \# W$, and therefore $X_n(K_B) \# W \cong X_n(K_G) \# W$. □

Corollary 4.2 (analogue of Corollary 3.2 in [13]) *Let $L=\{(R,r),(B,0),(G,0)\}$ be an n-special RBG link, such that R is r-slice in W. If K_B is n-slice in X, then K_G is n-slice in $X\#-W$.*

Proof If K_B is n-slice in X, then $-X_n(K_B)$ smoothly embeds in X, which implies that $-X_n(K_B)\#-W$ smoothly embeds in $X\#-W$. By Proposition 4.1, $-X_n(K_G)\#-W$ also smoothly embeds in $X\#-W$. Therefore, K_G is n-slice in $X\#-W$. □

Proposition 4.3 (analogue of Lemma 3.11 in [13]) *Let $L=\{(R,r),(B,0),(G,0)\}$ be an n-special RBG link. If R is $(r-1)$-slice in some closed 4-manifold W, then K_B, K_G are $(n+1)$-slice in $W\#\overline{\mathbb{CP}}^2$.*

Proof We use $X_{f_K,f_J}(K,J)$ to denote the 4-manifold obtained by attaching two 2-handles along the framed link $\{(K,f_K),(J,f_J)\}$ to a 4-ball.

Consider the slides of B over R which are applied to L in obtaining the knot K_B. These induce a diffeomorphism between $X_{r,n+1}(R,K_B)$ and $X_{r,1}(R,B)$. Next, Slide R over B to separate B and R, and obtain a diffeomorphism $X_{r,1}(R,B)\cong X_{r-1}(R)\#\mathbb{CP}^2$. Since R is $(r-1)$-slice in W, the trace $-X_{r-1}(R)$ smoothly embeds in W, which implies that $-X_{r-1}(R)\#\overline{\mathbb{CP}}^2$ smoothly embeds in $W\#\overline{\mathbb{CP}}^2$. Since $-X_{n+1}(K_B)\subset -X_{r,n+1}(R,K_B)$, the trace $-X_{n+1}(K_B)$ smoothly embeds in $W\#\overline{\mathbb{CP}}^2$, ie K_B is $(n+1)$-slice in $W\#\overline{\mathbb{CP}}^2$. □

From now on, we assume that n is a nonnegative integer.

Theorem 4.4 *Let $L=\{(R,r),(B,0),(G,0)\}$ be an n-special RBG link.*

(a) *If R is r-slice in some $\#^m\mathbb{CP}^2$ and K_B is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$, then K_G is n-slice in $\#^{l+m}\,\overline{\mathbb{CP}}^2$.*

(b) *If R is $(r-1)$-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then K_B, K_G are $(n+1)$-slice in $\#^{m+1}\,\overline{\mathbb{CP}}^2$.*

Proof (a) If R is r-slice in some $\#^m\mathbb{CP}^2$, and K_B is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$, then by Corollary 4.2, K_G is n-slice in $\#^{l+m}\,\overline{\mathbb{CP}}^2$.

(b) If R is $(r-1)$-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then by Proposition 4.3, K_B, K_G are $(n+1)$-slice in $\#^{m+1}\,\overline{\mathbb{CP}}^2$. □

For obstructing n-sliceness in $\#^l\,\overline{\mathbb{CP}}^2$, we will use the following adjunction inequality for the s-invariant in [18].

Theorem 4.5 [18, Corollary 1.4] *Let $W=\#^l\,\overline{\mathbb{CP}}^2$. Let $K\subset\partial W^\circ=S^3$ be a knot, and $\Sigma\subset W^\circ$ a properly, smoothly embedded oriented surface with no closed components, such that $\partial\Sigma=K$. Then*

$$s(K)\le 2g(\Sigma)-|[\Sigma]|-[\Sigma]\cdot[\Sigma].$$

Proposition 4.6 *If K is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$, then $s(K)\le n-\sqrt{n}$. If J is $(-n)$-slice in some $\#^l\,\mathbb{CP}^2$, then $s(J)\ge -n+\sqrt{n}$.*

Proof Let $\{e_1,\dots,e_l\}$ be an orthonormal basis of $H_2(\#^l\,\overline{\mathbb{CP}}^2)$, and $[D]=a_1e_1+a_2e_2+\cdots+a_le_l$ in $H_2(\#^l\,\overline{\mathbb{CP}}^2)$. Since K is n-slice, $-[D]\cdot[D]=a_1^2+\cdots+a_l^2=n$. Since

$$|[D]|^2=(|a_1|+\cdots+|a_l|)^2\ge a_1^2+\cdots+a_l^2=n,$$

we have $s(K)\le n-\sqrt{n}$ by Theorem 4.5. For the second part, consider the mirror of J, which is n-slice in $\#^l\,\overline{\mathbb{CP}}^2$, and obtain that $s(J)\ge -n+\sqrt{n}$. □

Note that a similar inequality holds for the τ-invariant (see [13, Corollary 2.13]).

Theorem 1.5 *Let $L=\{(R,r),(B,0),(G,0)\}$ be an n-special RBG link.*

(a) *If R is r-slice in some $\#^m\,\mathbb{CP}^2$ and K_B is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$, then $s(K_G)\le n-\sqrt{n}$.*

(b) *If R is $(r-1)$-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then $s(K_G)\le n+1-\sqrt{n+1}$.*

Proof If R is r-slice in some $\#^m\,\mathbb{CP}^2$ and K_B is n-slice in some $\#^l\,\overline{\mathbb{CP}}^2$, then by Corollary 4.2, K_G is n-slice in $\#^{l+m}\,\overline{\mathbb{CP}}^2$, then by Proposition 4.6, $s(K_G)\le n-\sqrt{n}$.

If R is $(r-1)$-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then by Proposition 4.3, K_G is $(n+1)$-slice in $\#^{m+1}\,\overline{\mathbb{CP}}^2$. Then, by Proposition 4.6, $s(K_G)\le n+1-\sqrt{n+1}$. □

Given an n-special RBG link L, one can modify the link to L' by decreasing the framing r of R. Notice that the diagrams of K_G can be obtained by adding full twists on parallel strands of the diagram of K'_G of the modified link. We will use the following lemma, which generalize Proposition 7.6(8) in [11], to get more precise bounds of $s(K_G)$.

Lemma 4.7 *Let K be a knot. Let K' be obtained by adding one positive twist to parallel strands of K. If the algebraic intersection between the parallel stands and the twisting disk Δ_{T} is l, then we have $s(K')\le s(K)-|l|+l^2$.*

Proof Let T be the boundary of Δ_{T}. We build $W=\overline{\mathbb{CP}}^2\setminus(\mathrm{int}\,B^4\cup\mathrm{int}\,B^4)$ by attaching a (-1)-framed 2-handle along T to $S^3\times\{1\}\subset S^3\times I$. Consider the annulus $A=K\times I\subset S^3\times I$. Choose a point $p\in K$ which is not on the twisting disk Δ_{T}. Pick a tubular neighborhood N of $\{p\}\times I$ in W, such that $N\cap(\Delta_{\mathrm{T}}\times\{1\})=\varnothing$. Let (D,X) be $(A\setminus N,W\setminus N)\cong(D^2,\overline{\mathbb{CP}}^2\setminus\mathrm{int}\,B^4)$. The boundary of D is $-K\,\#\,K'$. Since K intersect Δ algebraically l-times, D represents the class $l[\overline{\mathbb{CP}}^1]$ in $H^2(\overline{\mathbb{CP}}^2)$. Applying Theorem 4.5 to (D,X), we have $s(-K\,\#\,K')\le g(D)-|[D]|-[D]\cdot[D]=-|l|+l^2$. Thus, $s(K')\le s(K)-|l|+l^2$. □

Proposition 4.8 *Let $L = \{(R, r), (B, 0), (G, 0)\}$ be an n-special RBG link, such that $R = U$, $r \geq 0$. Suppose there exists a disk Δ_G, such that $\partial\Delta_G = G$, and B, R intersect Δ_G geometrically once. Suppose also that there exists a disk Δ_R, such that $\partial\Delta_R = R$, and B, G intersect Δ_R geometrically once. If K_B is n-slice in some $\#^l\, \overline{\mathbb{CP}}^2$, then $s(K_G) \leq n - \sqrt{n}$.*

Proof Since there exists a disk Δ_G, such that $\partial\Delta_G = G$, and B, R intersect Δ_G geometrically once, the knot K_B can be obtained by sliding B over R once, and therefore the diagram of K_B does not depend on r. Let $L^0 = \{(U, 0), (B, 0), (G, 0)\}$. Proposition 3.4 implies that $K_B^0 = K_G^0$.

Observe that K_G can be obtained from K_G^0 by adding an r-positive twist box on parallel strands. Since $|\Delta_G \cap B| = 1$ implies $|\operatorname{lk}(B, G)| = 1$, the parallel strands link the twist box algebraically once. By Lemma 4.7, $s(K_G) \leq s(K_G^0) \leq n - \sqrt{n}$. □

Proposition 4.9 *Let $L[r] = \{(R, r), (B, 0), (G, 0)\}$ be an n-special RBG link such that $R = U$, $r > 1$. Let l be the linking number between B and G. If $l^2 \leq n$, then $s(K_G) \leq n - \sqrt{n}$.*

Proof Consider the $(n-l^2)$-special RBG link $L[r-1] = \{(R, r-1), (B, 0), (G, 0)\}$. Since $r > 1$, U is $(r-2)$-slice in some $\#^m\, \overline{\mathbb{CP}}^2$. By Theorem 1.5, $s(K_G[r-1]) \leq n - l^2 + 1 - \sqrt{n - l^2 + 1}$. The knot K_G of $L[r]$ can be obtained by adding one positive full twists along parallel strands from $K_G[r-1]$. Since $\operatorname{lk}(B, G) = l$, the algebraic intersection number between the parallel strands and the twist box is l. By Lemma 4.7, we have that $s(K_G) \leq s(K_G[r-1]) - |l| + l^2$. Thus, $s(K_G) \leq n - |l| + 1 - \sqrt{n - l^2 + 1}$. If $l^2 \leq n$, then $n - |l| + 1 - \sqrt{n - l^2 + 1} \leq n - \sqrt{n}$. The result follows. □

4.2 Experiments

Consider an n-special RBG link with $R = U$. If $r \leq 0$, then R is r-slice in $\#^{|r|}\, \mathbb{CP}^2$. By Theorem 1.5, the n-sliceness of K_B implies that $s(K_G) \leq n - \sqrt{n}$. Therefore, we cannot use Proposition 4.6 to obstruct the n-sliceness of K_G. If $r > 0$, then R is $(r-1)$-slice in $\#^{r-1}\, \overline{\mathbb{CP}}^2$. By Theorem 1.5, we only have $s(K_G) \leq n + 1 - \sqrt{n+1}$. Now, pick an integer n such that there exists an even integer $2q$ satisfying $n - \sqrt{n} < 2q \leq n + 1 - \sqrt{n+1}$ (eg $n = 3, 6, 8, 11, 13, 15$ etc). If one can find an n-special RBG link with $s(K_G) = 2q$, then since $s(K_G) > n - \sqrt{n}$, we have that K_G is not n-slice by Proposition 4.6. Hence, Theorem 1.5 leaves open the possibility to use the s-invariant to obstruct K_G from being n-slice.

For example, we can consider the case where $n = 3$. If there is a 3-special RBG link with $R = U$ and $r > 0$, such that K_G is 3-slice in some $\#^m\, \overline{\mathbb{CP}}^2$ and $s(K_B) = 2$, then by Proposition 4.6, K_B is not 3-slice in any $\#^m\, \overline{\mathbb{CP}}^2$. Thus, such a link would produce an exotic $\#^m\, \overline{\mathbb{CP}}^2$. Note that $3 = n = l(rl-2)$, so $(r, l) \in \{(5, 1), (1, -1), (1, 3)\}$. If $l = 1$, then $r = 5 > 1$, and by Proposition 4.9, we have that $s(K_G) \leq n - \sqrt{n}$, which cannot obstruct the 3-sliceness of K_G. Thus, all potentially useful 3-special RBG links are those with $r = 1$ and $l \in \{-1, 3\}$.

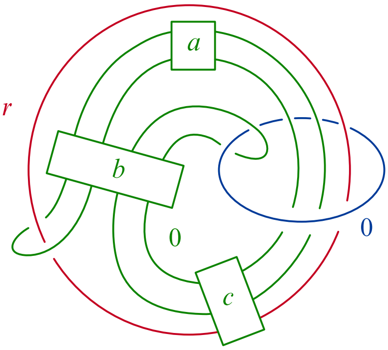


Figure 8: The diagram of $(r+2)$-special RBG links $L(a,b,c;r)$.

Here is a potential situation that could lead to an exotic $\#^m\,\overline{\mathbb{CP}}^2$ using the s-invariant. Consider the family $L(a,b,c;r)$ of special RBG links in Figure 8. Since $l=-1$, the link $L(a,b,c;r)$ is an $(r+2)$-special RBG link. Figure 9 gives the diagrams of $K_B(a,b,c;r)$ and $K_G(a,b,c;r)$. Note that two diagrams in Figure 9 are isotopic for all choices of a,b and c. Thus, we denote $K_B(a,b,c;r)$ by $K(a,b,c)$, and $K_G(a,b,c;r)$ by $K(a,b,r-c)$.

Example 4.10 The link $L(-2,1,-2;1)$ (see Figure 8) is a 3-special RBG link. Using KnotJob [19], one can compute that $s(K_G)=0$ and $s(K_B)=2$. The knot $K_B(-2,1,-2;1)=K(-2,1,-2)$ is recognized by SnapPy [7] as $K9\_533$, and $K_G(-2,1,-2;1)=K(-2,1,3)$ has a diagram with 28 crossings as in Figure 1, left, by KnotJob [19].

From Example 4.10, we obtain the following theorem.

Theorem 1.6 *If the knot $K(-2,1,3)$ from the left-hand side of Figure 1 is 3-slice in some $\#^m\,\overline{\mathbb{CP}}^2$, then there exists an exotic $\#^m\,\overline{\mathbb{CP}}^2$.*

Therefore, we are interested in whether $K(-2,1,3)$ is 3-slice in some $\#^m\,\overline{\mathbb{CP}}^2$. Notice that if we change the colored crossings in Figure 9 by adding 2-handles, we get ribbon knots for any choice of $(a,b,c;r)$.

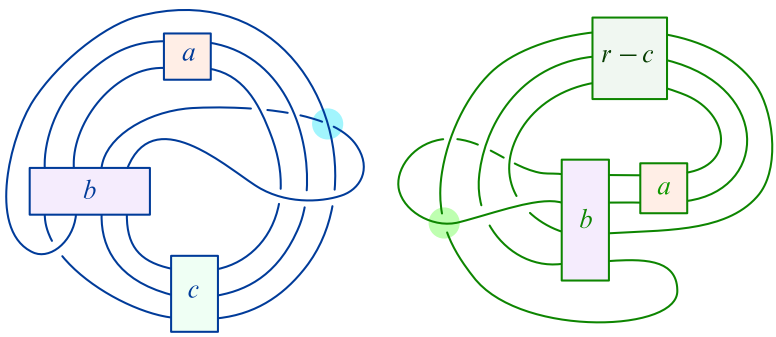


Figure 9: The diagrams of $K_B(a,b,c;r)$ and $K_G(a,b,c;r)$.

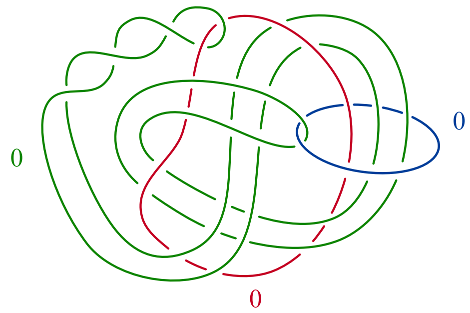


Figure 10: A diagram of the 2-special RBG link $L(-2,1,2;0)$ from the family in Example 4.10.

Therefore, both K_B and K_G are H-slice in $\mathbb{CP}^2$ and 4-slice in $\overline{\mathbb{CP}}^2$. Using SnapPy [7], we compute that $\tau(K(-2,1,3)) = 0$, which does not obstruct 3-sliceness. However, by considering $L(-2,1,-3;0)$, we can obstruct 2-sliceness.

Proposition 4.11 *The knot $K(-2,1,3)$ in Figure 1, left, is not 2-slice in any $\#^m\,\overline{\mathbb{CP}}^2$.*

Proof The knot $K(-2,1,3)$ has the same diagram as $K_G(-2,1,-3;0)$. Consider the 2-special RBG link in the 4-parameter family in Figure 8 with $(a,b,c;r) = (-2,1,-3;0)$. Using KnotJob, we compute that $s(K_B(-2,1,-3;0)) = 2$. However, by Theorem 1.5(a), if $K_G(-2,1,-3;0)$ is 2-slice, then $s(K_B(-2,1,-3;0)) \le 2-\sqrt{2} < 2$. Thus, $K(-2,1,3) = K_G(-2,1,-3;0)$ is not 2-slice in any $\#^m\,\overline{\mathbb{CP}}^2$. □

Although we cannot determine whether $K(-2,1,3)$ is 3-slice, Proposition 4.11 gives an example where we can use an n-special RBG link to obstruct the n-sliceness of a knot. Another example of this type is the 2-special RBG link in Figure 10.

Example 4.12 The 2-special RBG link $L(-2,1,2;0)$ in Figure 10 gives another example where $s(K_B) \ne s(K_G)$. Using SnapPy [7], we recognize K_G as $K9\_533$. Figure 1, right, gives a diagram of K_B. Using KnotJob [19], we compute that $s(K_B) = 0$, $s(K_G) = 2$. Also, using SnapPy [7], we have that $\tau(K_B) = \tau(K_G) = 0$.

Using the 2-RBG link in Figure 10, we obtain:

Proposition 4.13 *The knot $K(-2,1,2)$ in Figure 1, right, is not 2-slice in any $\#^m\,\overline{\mathbb{CP}}^2$.*

Proof Since $K(-2,1,2)$ is the blue knot $K_B(-2,1,2;0)$ associated with the 2-special RBG link $L(-2,1,2;0)$ in Example 4.12 (see Figure 10), we apply Theorem 1.5(a) with $n=2$. Since $s(K_G) = 2$ is larger than $n-\sqrt{n} = 2-\sqrt{2}$, the associated knot K_B is not 2-slice in any $\#^m\,\overline{\mathbb{CP}}^2$. Thus, $K(-2,1,2)$ is not 2-slice in any $\#^m\,\overline{\mathbb{CP}}^2$. □

So far, we have proved Theorem 1.7, which is a combination of Propositions 4.11 and 4.13 using Theorem 1.5. We do not know other proofs of Theorem 1.7. In general, there are two other approaches to obstruct knots from being slice n-slice in any $\#^m \overline{\mathbb{CP}}^2$. The first one is to use the τ invariant from [16], where Ozsváth and Szabó proved that the τ invariant of a knot K satisfies the adjunction inequality in a negative definite 4-manifold W with $b_1(W) = 0$; namely

$$2\tau(K) \leq 2g(\Sigma) - |[\Sigma]| - [\Sigma]\cdot[\Sigma],$$

where Σ is a properly, smoothly embedded surface in W without closed components, such that $\partial\Sigma = K$. Similarly to the proof of Proposition 4.6, we have that if K is n-slice in some $\#^m \overline{\mathbb{CP}}^2$, then $2\tau(K) \leq n - \sqrt{n}$. However, we have $s = \tau = 0$ for the knots in Figure 1, so this method does not apply to Theorem 1.7.

The second approach is to use the d-invariant from [15]. Theorem 9.6 in [15] gives a lower bound on the d-invariants of a rational homology sphere that bounds a definite 4-manifold. Moreover, by Proposition 1.6 in [14], we can compute the d-invariant of $S^3_2(K)$ by looking at the full knot Floer complex of K. A calculation of $\widehat{\mathrm{HFK}}$ with SnapPy [7] does not quite determine the full knot complexes of the knots in Figure 1, but suggests that they could be CFK-equivalent to the unknot. Therefore, this method does not apply either.

5 n-peculiar RBG links

In this section, we consider a different construction of $|n|$-RBG links (that are usually not n-special). Let $n \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Consider a two component link $\{(K, n), (\mu_K, r)\}$, where μ_K is a meridian of K. By a Rolfsen twist, there exists a homeomorphism from $S^3_{n,r}(K, \mu_K)$ to $S^3_{n-1/r}(K)$, which restricts to the identity outside a tubular neighborhood of K. If $K = U$ and $n - 1/r = 1/t$ for some $t \in \mathbb{Z}$, then $\{(K, n), (\mu_K, r)\}$ is a surgery diagram for S^3.

Definition 1.8 A link $\{(R, r), (B, b), (G, g)\}$ is called an n-peculiar RBG link, if there exists $t \in \mathbb{Z}$ such that

- $R = U$ and B, G are meridians of R,
- $b = g = 1/r + 1/t$,
- $n = (g + b - 2l) - t(l - b)^2$,

where $l = \mathrm{lk}(B, G)$ under an orientation of L such that $\mathrm{lk}(B, R) = \mathrm{lk}(G, R) = 1$.

We can read out the diagram of K_G from its RBG link diagram in three steps. Let Δ_R be a meridian disk of B such that $\partial\Delta_R = R$. First, isotope G away from Δ_R by sliding G over B. Then, slam dunk R into B, and obtain a link diagram where R is deleted and B has framing $1/t$. Finally, blow down the blue component B by Rolfsen twists. We can get a diagram for K_B in a similar way.

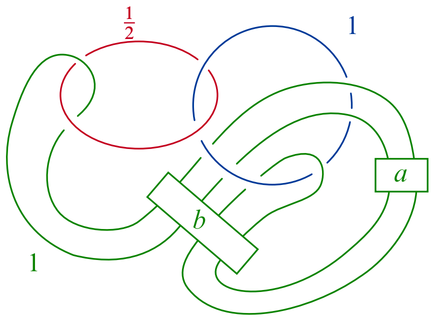


Figure 11: A family of 0-peculiar links.

Lemma 5.1 *An n-peculiar RBG link induces an n-surgery homeomorphism $\phi\colon S^3_n(K_B) \to S^3_n(K_G)$.*

Proof We keep track of the linking matrix M_L of the link L under those three diagram changes. We start with

$$M_L = \begin{bmatrix} r & 1 & 1 \\ 1 & b & l \\ 1 & l & g \end{bmatrix}.$$

Sliding G over B so that $|\Delta_R \cap G| = 0$, the linking matrix becomes

$$\begin{bmatrix} r & 1 & 0 \\ 1 & b & l-b \\ 0 & l-b & g+b-2l \end{bmatrix}.$$

After the slam-dunk of R into B, the linking matrix is

$$\begin{bmatrix} b-1/r & l-b \\ l-b & g+b-2l \end{bmatrix}.$$

Now, blow down the blue component by a Rolfsen twist along B. The framing of K_G is

$$f_g = (g+b-2l) - t(l-b)^2 = n.$$

□

Example 5.2 Consider the family of peculiar RBG links in Figure 11, parametrized by two twisting boxes. Since $l = 1$ and $b = g = 1$, we have $n = 0$. When (a, b) is $(2, -1)$ or $(3, -2)$, the peculiar link $L(a, b)$ generates a knot pair such that $s(K_G) = -2$ and $s(K_B) = 0$. (When $(a, b) = (2, -1)$, the knot K_B is 11_{270}.) However, since the signature of K_B is 2 in each case, the knots K_B are not H-slice in any $\#^m\,\mathbb{CP}^2$, so the knot pairs do not produce any exotic $\#^m\,\mathbb{CP}^2$.

References

[1] **T Abe**, **I D Jong**, **J Luecke**, **J Osoinach**, *Infinitely many knots admitting the same integer surgery and a four-dimensional extension*, Int. Math. Res. Not. 2015 (2015) 11667–11693 MR Zbl

[2] **S Akbulut**, *On 2-dimensional homology classes of 4-manifolds*, Math. Proc. Cambridge Philos. Soc. 82 (1977) 99–106 MR Zbl

[3] **S Akbulut**, *Knots and exotic smooth structures on 4-manifolds*, J. Knot Theory Ramifications 2 (1993) 1–10 MR Zbl

[4] **S Akbulut**, **R Kirby**, *Mazur manifolds*, Michigan Math. J. 26 (1979) 259–284 MR Zbl

[5] **S Boyer**, *Simply-connected 4-manifolds with a given boundary*, Trans. Amer. Math. Soc. 298 (1986) 331–357 MR Zbl

[6] **R Casals**, **J Etnyre**, **M Kegel**, *Stein traces and characterizing slopes*, Math. Ann. 389 (2024) 1053–1098 MR Zbl

[7] **M Culler**, **N M Dunfield**, **M Goerner**, **J R Weeks**, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds* Available at `http://snappy.computop.org`

[8] **R E Gompf**, **A I Stipsicz**, *4-manifolds and Kirby calculus*, Grad. Stud. Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[9] **K Hayden**, **L Piccirillo**, *The trace embedding lemma and spinelessness* (2019) arXiv 1912.13021 To appear in J. Differential Geom.

[10] **O A Ivanov**, **N Y Netsvetaev**, *On the intersection form of the result of gluing manifolds with degenerate intersection forms*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 231 (1995) 169–179 MR Zbl In Russian; translated in J. Math. Sci. 91 (1998) 3440–3447

[11] **C Manolescu**, **M Marengon**, **S Sarkar**, **M Willis**, *A generalization of Rasmussen's invariant, with applications to surfaces in some four-manifolds*, Duke Math. J. 172 (2023) 231–311 MR Zbl

[12] **C Manolescu**, **L Piccirillo**, *From zero surgeries to candidates for exotic definite 4-manifolds*, J. Lond. Math. Soc. 108 (2023) 2001–2036 MR Zbl

[13] **K Nakamura**, *Trace embeddings from zero surgery homeomorphisms*, J. Topol. 16 (2023) 1641–1664 Zbl

[14] **Y Ni**, **Z Wu**, *Cosmetic surgeries on knots in S^3*, J. Reine Angew. Math. 706 (2015) 1–17 MR Zbl

[15] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR Zbl

[16] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR Zbl

[17] **L Piccirillo**, *The Conway knot is not slice*, Ann. of Math. 191 (2020) 581–591 MR Zbl

[18] **Q Ren**, *Lee filtration structure of torus links*, Geom. Topol. 28 (2024) 3935–3960 MR Zbl

[19] **D Schütz**, *KnotJob*, Java software (2015) Available at `https://www.maths.dur.ac.uk/users/dirk.schuetz/knotjob.html`

[20] **K Varvarezos**, *Certain connect sums of torus knots with infinitely many non-characterizing slopes*, preprint (2023) arXiv 2302.05068

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