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Powell's conjecture on the Goeritz group of S^3 is stably true

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In 1980, J Powell proposed that, for every genus g, five specific elements suffice to generate the Goeritz group $\mathcal{G}_g$ of genus g Heegaard splittings of S^3. Powell's conjecture remains undecided for $g \geq 4$. Let $\mathcal{P}_g \subset \mathcal{G}_g$ denote the subgroup generated by Powell's elements. Here we show that, for each genus g, the natural function $\mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ is trivial.

57K35, 57M50, 57M60

1 Introduction

Following early work of Goeritz [3], the *genus g Goeritz group* $\mathcal{G}_g$ of the 3-sphere can be described as the isotopy classes of orientation-preserving homeomorphisms of the 3-sphere that leave the standard genus g Heegaard surface T_g invariant, with its orientation preserved. Goeritz identified a finite set of generators for $\mathcal{G}_2$. In 1980, J Powell [6] extended Goeritz's set of generators to a set of five elements that he believed would generate the Goeritz group for any fixed higher-genus splitting, but his proof contained a serious gap. In [2], the Powell conjecture (as it is now called) was confirmed for $\mathcal{G}_3$, and, in [7], it is pointed out that one of Powell's proposed generators is redundant, so in fact only four of Powell's proposed generators need to be considered.

Powell's view of the Goeritz group, which we will adopt, is framed somewhat differently. Following Johnson and McCullough [4] (who extend the notion to arbitrary compact orientable manifolds), consider the space of left cosets $\text{Diff}(S^3)/\text{Diff}(S^3, T_g)$, where $\text{Diff}(S^3, T_g)$ consists of those orientation-preserving diffeomorphisms of S^3 that carry T_g to itself. The fundamental group $\widetilde{\mathcal{G}}_g = \pi_1(\text{Diff}(S^3)/\text{Diff}(S^3, T_g))$ of this space projects to the genus g Goeritz group $\mathcal{G}_g$ (with kernel $\pi_1(\text{SO}(3)) = \mathbb{Z}_2$ [6, page 197]) as follows: a nontrivial element is represented by an isotopy of T_g in S^3 that begins with the identity and ends with a diffeomorphism of S^3 that takes T_g to itself; this diffeomorphism of the pair (S^3, T_g) represents an element of the Goeritz group as defined earlier. The advantage of this point of view is that an element of $\widetilde{\mathcal{G}}_g$ can be viewed quite vividly: it is an excursion of T_g in S^3 that begins and ends with the standard picture of $T_g \subset S^3$. Such excursions are how Powell defines his proposed generators [6, Figure 4].

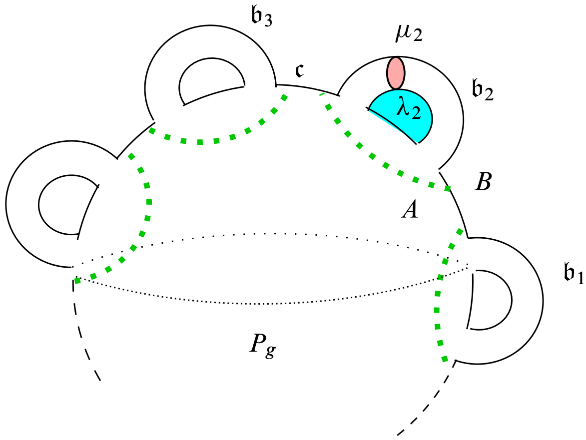

Figure 1: Powell's picture of T_g.

2 Powell's generators reexpressed

Powell's description of $T_g \subset S^3$ for $g \geq 2$ begins with a round 2-sphere in S^3, to which is then connect-summed a standard unknotted torus at each of g points in a circumference $\mathfrak{c}$ of the sphere. These summands $\mathfrak{b}_i$ for $1 \leq i \leq g$ will be called the standard genus one bubbles; their complement is a g-punctured sphere P_g. See Figure 1. For the Heegaard splitting $S^3 = A \cup_{T_g} B$, let $a_i \subset \mathfrak{b}_i \cap T_g$ be a circle (unique up to isotopy in the punctured torus $\mathfrak{b}_i \cap T_g$) that bounds a disk μ_i in the solid torus $A \cap \mathfrak{b}_i$. Similarly, let $b_i \subset \mathfrak{b}_i \cap T_g$ be a circle (unique up to isotopy rel a_i in the punctured torus $\mathfrak{b}_i \cap T_g$) that bounds a disk λ_i in the solid torus $B \cap \mathfrak{b}_i$, chosen so that $|a_i \cap b_i| = 1$. We will call a_i and b_i, respectively, the meridian and longitude of the summand $\mathfrak{b}_i$. Similarly, μ_i and λ_i will be called the meridian and longitudinal disks of $\mathfrak{b}_i$.

In terms of this picture, here are Powell's four proposed generators (the fifth being redundant):

- D_ω is the homeomorphism $(\mathfrak{b}_1, \mathfrak{b}_1 \cap T_g) \to (\mathfrak{b}_1, \mathfrak{b}_1 \cap T_g)$ shown in Figure 2, left. It preserves both μ_1 and λ_1 but reverses the orientation of each. We will call this the *standard flip* on $\mathfrak{b}_1$.
- D_η is the homeomorphism $(S^3, T_g) \to (S^3, T_g)$ shown in Figure 2, center: The punctured sphere P_g is rotated by $2\pi/g$ along the circumference $\mathfrak{c}$ and each standard bubble $\mathfrak{b}_i$ is moved to $\mathfrak{b}_{i+1}$, $i \in \mathbb{Z}/g$, sending each μ_i to μ_{i+1} and λ_i to λ_{i+1}. Fix an orientation for the disks μ_1 and λ_1, and observe that $(D_\eta)^{i-1}$ can be used to fix an orientation on each pair μ_i, λ_i, which is then preserved by D_η.
- Let $v \subset P_g$ be the subarc of $\mathfrak{c}$ connecting $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Let B be the reducing ball for the splitting T_g obtained by attaching a 1-handle regular neighborhood of v to the bubbles $\mathfrak{b}_1$ and $\mathfrak{b}_2$. $D_{\eta_{12}}$ is the homeomorphism $(B, B \cap T_g) \to (B, B \cap T_g)$ shown in Figure 2, right. It exchanges the meridian disks μ_1 and μ_2 and the longitudinal disks λ_1 and λ_2, preserving the orientation of each. We will call this the *standard exchange* of $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

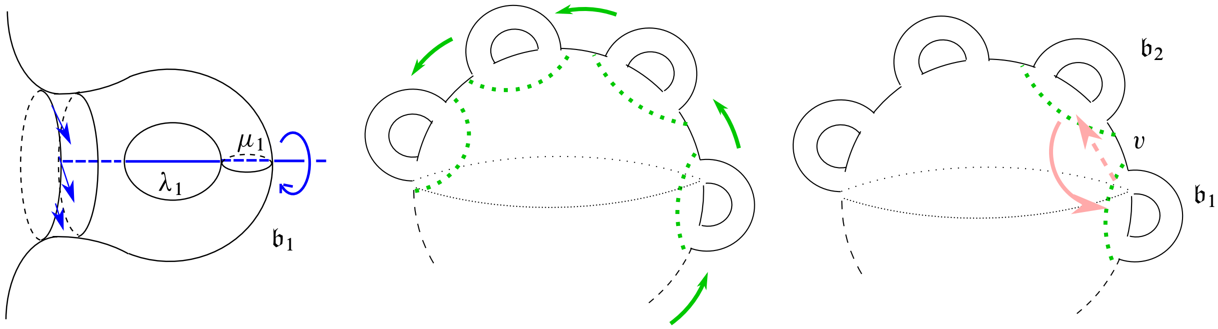

Figure 2: Powell's D_ω, the standard flip on $\mathfrak{b}_1$ (left), D_η (center), and $D_{\eta_{12}}$, the standard exchange of $\mathfrak{b}_1$ and $\mathfrak{b}_2$ (right).

- Let $v \subset T_g$ be an arc connecting $\partial\mu_1$ to $\partial\ell_2$, chosen so that inside the bubbles the interior of v is disjoint from the circles a_1, a_2, b_1, b_2, and outside the bubbles v is a subarc of $\mathfrak{c}$. The complex $\mu_1 \cup \ell_2 \cup v$ is called the *standard eyeglass* in T_g; see Figure 3. Powell's generator D_θ, illustrated in several panels of [6, Figure 4], makes use of this structure; it can be diagrammed more simply as in Figure 3 and is called the *standard eyeglass twist*. This move will be generalized later; see Figure 6.

As noted earlier, in the exact sequence

$$1 \to \pi_1(\mathrm{SO}(3)) \to \widetilde{\mathcal{G}}_g = \pi_1(\mathrm{Diff}(S^3)/\mathrm{Diff}(S^3, T_g) \to \mathcal{G}_g \to 1,$$

Powell's proposed generators actually lie in $\widetilde{\mathcal{G}}_g$, so it makes sense to note the easy:

Proposition 2.1 *Powell's proposed generators generate $\mathcal{G}_g$ if and only if they generate $\widetilde{\mathcal{G}}_g$.*

Proof One direction is obvious; the other follows immediately from the observation that the 2π-rotation $(D_\eta)^g$ is a Powell move that also represents the nontrivial element of $\pi_1(\mathrm{SO}(3))$. □

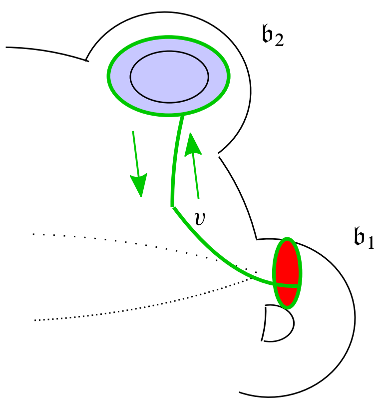

Figure 3: Powell's D_θ, the standard eyeglass twist.

Definition 2.2 The Powell group $\mathcal{P}_g$ is the subgroup of the Goeritz group $\mathcal{G}_g$ that is generated by Powell's four proposed generators D_θ, D_ω, $D_{\eta_{12}}$, D_η. Any element of $\mathcal{P}_g$ (that is, any composition of these generators) will be called a *Powell move*.

Here is another easy line of argument: Both of Powell's moves D_η and $D_{\eta_{12}}$ can be thought of as induced by a homeomorphism on the g-punctured sphere P_g, and such a homeomorphism also defines an element of the g-stranded braid group $\mathcal{B}_g$ of the sphere. Furthermore, $(D_\eta)^{i-1}$ will conjugate the standard exchange $D_{\eta_{12}}$ to a similar exchange (which we will denote by ϕ_i and also call standard) between $\mathfrak{b}_i$ and $\mathfrak{b}_{i+1}$, so each standard exchange ϕ_i is a Powell move. Moreover, viewed as an element of the braid group $\mathcal{B}_g$, each exchange ϕ_i for $1 \le i \le g-1$ just described is exactly one of the $g-1$ canonical half-twist generators σ_i of $\mathcal{B}_g$ (see [1, Chapter 9]). Similarly, $(D_\eta)^{i-1}$ will conjugate D_ω to a flip in the bubble $\mathfrak{b}_i$, which we will call the standard flip in $\mathfrak{b}_i$ and denote by $\omega_i \in \mathcal{P}_g$.

Proposition 2.3 *The Powell group $\mathcal{P}_g$ is also generated by these $g+1$ elements of $\mathcal{P}_g$:*

$$D_\theta, \quad D_\omega, \quad \{\phi_i \mid 1 \le i \le g-1\}.$$

Proof In the braid group $\mathcal{B}_g$, the composition $\sigma_1\sigma_2\cdots\sigma_{g-1}$ visibly has the effect of moving g points evenly distributed in the circumference $\mathfrak{c} \subset S^2$ by a rotation of $2\pi/g$. (See also [1, Chapter 9].) It follows that, for some even k (easily but unimportantly found to be $k=2$; see below), $D_\eta = (D_\omega)^k \phi_1\phi_2\cdots\phi_{g-1}$ in $\mathcal{P}_g$. Thus D_η lies in the subgroup of $\mathcal{P}_g$ generated by D_θ, D_ω and $\{\phi_i \mid 1 \le i \le g-1\}$, as does $D_{\eta_{12}} = \phi_1$. Thus each of the generators D_θ, D_ω, $D_{\eta_{12}}$, D_η lies in the subgroup of $\mathcal{P}_g$ generated by D_θ, D_ω and $\{\phi_i \mid 1 \le i \le g-1\}$, as required. □

Figure 4 demonstrates the equality $D_\eta = (D_\omega)^2\phi_1\phi_2\cdots\phi_{g-1}$ (composed right to left) in the case $g=4$. The red arrows in each $\mathfrak{b}_i$ are meant to show orientation within the bubble: the arrows indicate the normal direction to a_i. In the left panel, $\phi_1\phi_2\phi_3$ rotates the red arrow in $\mathfrak{b}_4$ through an angle $-\frac{3\pi}{2}$ to the red arrow in $\mathfrak{b}_1$; in the right panel, D_η rotates it $\frac{\pi}{2}$.

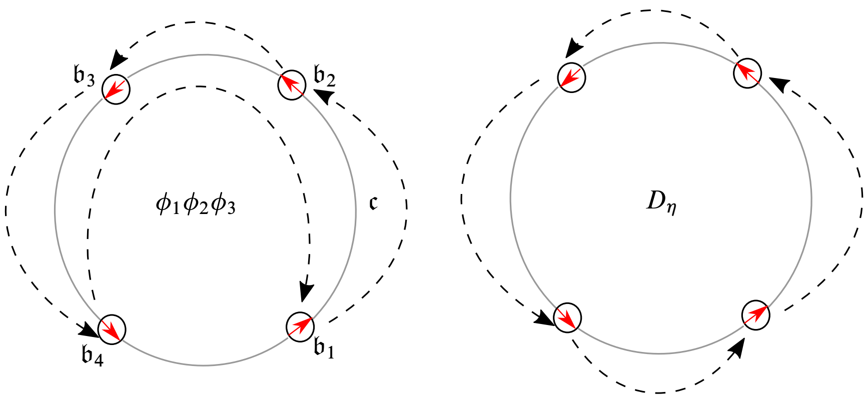

Figure 4: $D_\eta = (D_\omega)^2\phi_1\phi_2\phi_3$.

3 What the Powell group can do

We have seen above that the Powell group $\mathcal{P}_g$ contains any automorphism of T_g obtained by operating on the planar surface P_g by an element of the braid group $\mathcal{B}_g$. In [2, Sections 1–4], broader types of automorphisms are also shown to lie in $\mathcal{P}_g$. For example, [2, Lemma 1.4] says this:

Lemma 3.1 *Any braid move of a collection of standard bubbles over their complementary surface is a Powell move.* □

Figure 5 shows an example in the case of two standard bubbles.

In the case of a single standard bubble $\mathfrak{b}_i$, a braid move is just an isotopy of $\mathfrak{b}_i$ through some path in $T_g - \mathfrak{b}_i$ that returns $\mathfrak{b}_i$ to itself. This will be called a *standard bubble move* of $\mathfrak{b}_i$ and is a Powell move. More generally, if $\mathfrak{b}$ is an arbitrary bubble, such an isotopy of $\mathfrak{b}_i$ through some path in $T_g - \mathfrak{b}_i$ is called a (*generic*) *bubble move*. A generic bubble move is not obviously Powell, but see Corollary 3.5 below.

Here is one consequence: Let $q\colon (S^3, T_{g+1}) \to (S^3, T_g)$ be the quotient map that shrinks the standard bubble $\mathfrak{b}_{g+1} \subset S^3$ to a point $\star \in T_g$. Suppose $\tau\colon (S^3, T_g) \to (S^3, T_g)$ represents an element in $\mathcal{G}_g$ and let α be an embedded path in T_g from $\star$ to $\tau^{-1}(\star)$. Let $\alpha^{\star}_{\tau^{-1}(\star)}\colon (S^3, T_g) \to (S^3, T_g)$ be a homeomorphism whose support lies in a regular neighborhood of α and pushes $\star$ to $\tau^{-1}(\star)$. (See [5, Homogeneity lemma].) Then $\tau\alpha^{\star}_{\tau^{-1}(\star)}(\star) = \star$, so τ induces $\tau_\alpha\colon (S^3, T_{g+1}) \to (S^3, T_{g+1})$ by replacing a neighborhood of $\star$ with the standard bubble $\mathfrak{b}_{g+1}$. (The construction can be extended to the case in which the path α is not necessarily embedded, but is in general position, by breaking α up into a finite sequence of embedded paths.)

Proposition 3.2 *For any genus $g \geq 2$, the construction $\tau \to \tau_\alpha$ described above determines a function*

$$\iota^+\colon \mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$$

with the property that $\iota^+(\mathcal{P}_g)$ is trivial.

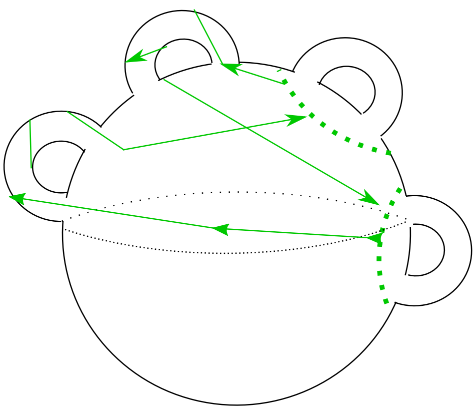

Figure 5

Proof The construction above assigns the homeomorphism $\tau_\alpha\colon (S^3, T_{g+1}) \to (S^3, T_{g+1})$ to τ; we are interested in the coset that τ_α represents in $\mathcal{G}_{g+1}/\mathcal{P}_{g+1}$. The first observation is that the coset does not depend on the path α. Indeed, suppose β is a different path in T_g from $\star$ to $\tau^{-1}(\star)$. Then τ_α and τ_β differ by a standard bubble move on $\mathfrak{b}_{g+1}$ along the path $\bar{\alpha}\beta$, and by Lemma 3.1 such a bubble move is in $\mathcal{P}_{g+1}$. Thus τ_α and τ_β represent the same coset in $\mathcal{G}_{g+1}/\mathcal{P}_{g+1}$.

Claim *This assigned coset in $\mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ is independent of the choice of τ as a representative of a given element of $\mathcal{G}_g$.*

Proof Suppose that $\tau_1\colon (S^3, T_g) \to (S^3, T_g)$ represents the same element in $\mathcal{G}_g$ as $\tau = \tau_0$. This means that there is an isotopy $\tau_t, 0 \le 1$ from τ_0 to τ_1. Such an isotopy defines a path β in T_g from $\tau_0^{-1}(\star)$ to $\tau_1^{-1}(\star)$ given by $\beta(t) = \tau_t^{-1}(\star)$. Then the concatenation $\alpha_1 = \alpha * \beta$ is a path from $\star$ to $\tau_1^{-1}(\star)$ and so determines a coset in $\mathcal{G}_{g+1}/\mathcal{P}_{g+1}$, as described just before the proposition. That is, the coset is represented by $(\tau_1)_{\alpha_1}$, the homeomorphism obtained from $\tau_1(\alpha_1)^*_{\tau_1^{-1}(\star)}$ by replacing a neighborhood of $\star$ with the standard bubble $\mathfrak{b}_{g+1}$.

There is now a straightforward way to construct an isotopy from $(\tau_0)_{\alpha_0}$ to $(\tau_1)_{\alpha_1}$: For each $0 \le t \le 1$, let $\beta_t\colon I \to T_g$ be the path given by $\beta_t(s) = \beta(st)$. Then β_t is a path from $\beta(0) = \tau_0^{-1}(\star)$ to $\beta(t) = \tau_t^{-1}(\star)$, so, for each $0 \le t \le 1$, $\alpha_t = \alpha * \beta_t$ is a path from $\star$ to $\tau_t^{-1}(\star)$. Using this path, construct a homeomorphism $(\tau_t)_{\alpha_t}\colon (S^3, T_{g+1}) \to (S^3, T_{g+1})$ in the same way we constructed $(\tau_0)_{\alpha_0}$ and $(\tau_1)_{\alpha_1}$. Then, letting t now run from 0 to 1, $(\tau_t)_{\alpha_t}\colon (S^3, T_{g+1}) \to (S^3, T_{g+1})$ defines an isotopy from $(\tau_0)_{\alpha_0}$ to $(\tau_1)_{\alpha_1}$, as required. Thus $(\tau_0)_{\alpha_0}$ and $(\tau_1)_{\alpha_1}$ represent the same element of $\mathcal{G}_{g+1}$. This proves the claim. $\lhd$

Since the assigned coset in $\mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ is independent of the choice of α and the choice of representative of a class in $\mathcal{G}_g$, the construction defines a function $\iota^+\colon \mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ without ambiguity.

To show that $\iota^+(\mathcal{P}_g)$ lies in $\mathcal{P}_{g+1}$, consider the generators of $\mathcal{P}_g$ described in Proposition 2.3. Each involves at most two standard bubbles, so the same will be true once ι^+ carries them to $\mathcal{G}_{g+1}$. Following Lemma 3.1, arbitrary exchanges of standard bubbles are Powell moves in $\mathcal{G}_{g+1}$, so, up to Powell moves, we can view the generators of $\iota^+(\mathcal{P}_g)$ as each involving only the standard bubbles $\mathfrak{b}_1$ and $\mathfrak{b}_2$, so each is a standard generator of $\mathcal{P}_{g+1}$. $\square$

Another useful lemma requires the following generalization of the standard eyeglass move D_θ, originally [2, Definition 2.1]:

Definition 3.3 For $S^3 = A \cup_T B$ a Heegaard splitting of S^3, an *eyeglass* $\eta \subset S^3$ is the union of two properly embedded disjoint disks $\ell_a \subset A$ and $\ell_b \subset B$ (the *lenses* of η), with an embedded arc $v \subset T - \partial(\ell_a \cup \ell_b)$ (the *bridge* of η) connecting their boundaries. The 1-complex $\eta \cap T$ (the bridge, together with the boundary of the lenses) is called the *frame* of η.

The eyeglass η defines a natural automorphism $(S^3, T) \to (S^3, T)$, with support in a ball neighborhood of η, as illustrated in Figure 6. This automorphism is called an *eyeglass twist* based on η.

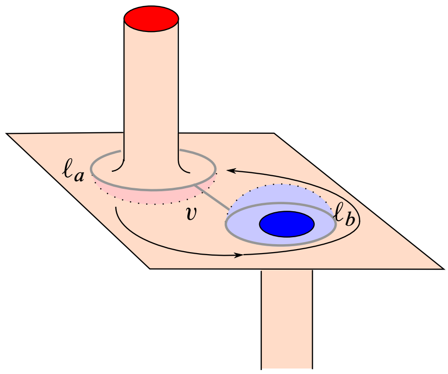

Figure 6: An eyeglass twist.

Since the support of an eyeglass twist lies entirely in a ball, it is isotopic in S^3 to the identity. Thus any eyeglass twist of the standard splitting $A \cup_{T_g} B$ lies in $\mathcal{G}_g$.

Remark Figure 6 shows the lens disk ℓ_a twisting once around the lens disk ℓ_b, but this is an artifact of the figure; it could equally well have been drawn symmetrically, with ℓ_b twisting once around the lens disk ℓ_a.

Suppose $c \subset T_g$ separates T_g into two components: T_A containing the punctured tori in $g_1 < g$ standard bubbles, and T_B containing the punctured tori in the other $g_2 = g - g_1$ standard bubbles.

Lemma 3.4 [2, Lemma 3.4] *Suppose η is an eyeglass in T_g whose lenses consist of a disk $a \subset A$ with $\partial a \subset T_A$ and a disk $b \subset B$ with $\partial b \subset T_B$. Suppose further that the bridge v intersects c exactly once. Then an eyeglass twist along η is a Powell move.*

Corollary 3.5 *Suppose $\mathfrak{b}$ is a generic bubble disjoint from the standard meridian μ_i and standard longitude λ_i of a standard bubble $\mathfrak{b}_i$, and γ is an embedded arc in $T_g - \mathfrak{b}$ from $\partial\mathfrak{b}$ to a_i (resp. b_i) whose interior is disjoint from both a_i and b_i. Then a bubble move of $\mathfrak{b}$ around the path $\overline{\gamma} a_i \gamma$ (resp. $\overline{\gamma} b_i \gamma$) is a Powell move.*

Proof Let $c \subset T_g$ be the boundary of a regular neighborhood of $a_i \cup b_i$, separating the standard bubble $\mathfrak{b}_i$ from the rest of T. Then the arc γ intersects c in one point and the bubble move around $\overline{\gamma} a_i \gamma$, say, can be described as an eyeglass move along an eyeglass with one lens $\mu_i \subset A$, the other lens the disk $\partial\mathfrak{b} \cap B$, and bridge the arc γ. The result then follows from Lemma 3.4. □

4 Stabilization and topological conjugacy

Definition 4.1 Orientation-preserving homeomorphisms $h_1, h_2 \colon (S^3, T_g) \to (S^3, T_g)$ are *topologically conjugate* if there is an orientation-preserving homeomorphism $g \colon (S^3, T_g) \to (S^3, T_g) \in \mathcal{G}_g$ such that $g^{-1} h_1 g = h_2$. A homeomorphism topologically conjugate to a standard flip is called a *generic flip*; a homeomorphism topologically conjugate to a standard exchange is called a *generic exchange*.

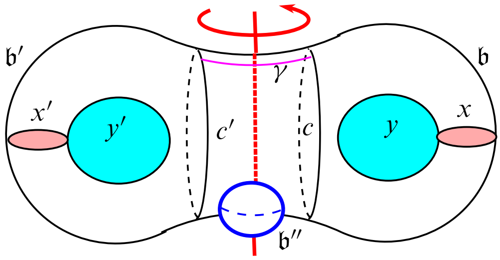

Figure 7: A bubble exchange, via a Goeritz move in $\mathcal{G}_2$.

For example, suppose $\mathfrak{b}$ is *any* genus one bubble for T_g, so $T \cap \mathfrak{b}$ is a once-punctured unknotted torus whose meridian bounds a properly embedded disk $x \in A$ and longitude bounds a properly embedded disk $y \in B$. Let $h\colon (S^3, T_g) \to (S^3, T_g) \in \mathcal{G}_g$ be the homeomorphism which is the identity outside $\mathfrak{b}$ but looks like Figure 2, left, inside $\mathfrak{b}$, with μ_1 replaced by x and λ_1 by y. Then it follows from [10] that h is a generic flip, and from the definition that any generic flip can be described in this way.

Similarly, suppose $\mathfrak{b}$ and $\mathfrak{b}'$ are any two disjoint genus one bubbles for T_g, and $\gamma \subset T$ is a path in their complement between the circles $c = \partial\mathfrak{b} \cap T$ and $c' = \partial\mathfrak{b}' \cap T$. A generic exchange is as shown in Figure 2, right, with $\mathfrak{b}_1$, $\mathfrak{b}_2$, v replaced respectively with $\mathfrak{b}$, $\mathfrak{b}'$, γ. The generic exchange can also be depicted as a generalization of one of Goeritz's original generators, as in Figure 7. In this depiction, the visible genus two splitting is rotated π around the vertical red axis, as Goeritz did. But in our case the blue ball at the base is a genus $g-2$ bubble $\mathfrak{b}''$ and the complement of $\mathfrak{b}''$, the genus two splitting we see, is the tube sum of bubbles $\mathfrak{b}$ and $\mathfrak{b}'$ along the arc γ.

It follows immediately from Lemma 3.1 that, if both $\mathfrak{b}$ and $\mathfrak{b}'$ are standard, then the exchange is a Powell move, even though γ can be arbitrary. More surprising and useful is the following extension:

Proposition 4.2 *If one of the bubbles in a generic exchange is standard, then the exchange is a Powell move.*

We defer the proof in order to describe how the proposition fits in to the main result of this paper.

Proposition 4.3 *For $\mathfrak{b}_{g+1}$ the standard bubble in (S^3, T_{g+1}), suppose the following two assumptions are true:*

1. *Any generic bubble exchange between $\mathfrak{b}_{g+1}$ and a disjoint genus one bubble in (S^3, T_{g+1}) is a Powell move. In other words, Proposition 4.2 is true.*
2. *Any eyeglass twist in (S^3, T_{g+1}) in which the eyeglass frame is disjoint from $\mathfrak{b}_{g+1}$ is a Powell move.*

Then the function $\iota^+\colon \mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ defined in Proposition 3.2 is trivial.

Proof We begin with two claims, assuming the two assumptions are true:

Claim 1 *Suppose $\mathfrak{b}$ is a bubble in T_{g+1} that is disjoint from $\mathfrak{b}_{g+1}$. Then the generic flip in $\mathfrak{b}$ is a Powell move.*

Proof Recall from the discussion before Proposition 2.3 that the standard flip ω_{g+1} of $\mathfrak{b}_{g+1}$ is a Powell move. Suppose $\mathfrak{b}$ is a generic bubble disjoint from $\mathfrak{b}_{g+1}$. Choose an embedded path β from $\mathfrak{b}_{g+1}$ to $\mathfrak{b}$ and let ρ be the bubble exchange of $\mathfrak{b}_{g+1}$ and $\mathfrak{b}$ along β, a Powell move by assumption (1). Then $\rho^{-1}\omega_{g+1}\rho$ is the generic flip of $\mathfrak{b}$ and is the composition of Powell moves. This proves the claim. □

Claim 2 *Suppose $\mathfrak{b}$ and $\mathfrak{b}'$ are disjoint bubbles in T_{g+1} that are also disjoint from $\mathfrak{b}_{g+1}$. Suppose $\alpha \subset T_{g+1}$ is an arc between them that is also disjoint from $\mathfrak{b}_{g+1}$. Then the generic exchange of $\mathfrak{b}$ with $\mathfrak{b}'$ along α is a Powell move.*

Proof Consider the 3-punctured genus $g-2$ surface $T_- = T_{g+1} - \mathfrak{b} \cup \mathfrak{b}' \cup \mathfrak{b}_{g+1}$. Let $\beta \subset T_-$ (resp. $\beta' \subset T_-$) be embedded arcs from $\mathfrak{b}$ (resp. $\mathfrak{b}'$) to $\mathfrak{b}_{g+1}$, chosen so that α, β, β' are all disjoint and lie in a disk in T_-.

By assumption (1), the bubble exchanges ρ, ρ' along, respectively, β, β' are Powell moves. Thus the composition $\rho^{-1}\rho'\rho$ is a Powell move. But $\rho^{-1}\rho'\rho\colon T_- \to T_-$ is easily seen to be isotopic to the exchange of $\mathfrak{b}$ and $\mathfrak{b}'$ along α, proving the claim. (The composition $\rho'\rho\rho'^{-1}$ is also isotopic to this exchange. Indeed the consequent isotopy between $\rho^{-1}\rho'\rho$ and $\rho'\rho\rho'^{-1}$ acting on a 3-punctured disk in T_- is the source of the standard braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.) □

Following the claims, we exploit the central result of [9]: the Goeritz group $\mathcal{G}_g$ is generated by eyeglass twists and topological conjugates of generators of $\mathcal{P}_g$. We consider the generators of $\mathcal{P}_g$ from Proposition 2.3. These consist of Powell's D_θ, which is itself an eyeglass twist; Powell's D_ω, which is a standard flip; and the collection $\{\phi_i\}$ of standard exchanges. Any topological conjugate of an eyeglass twist is an eyeglass twist, any topological conjugate of a standard flip is a generic flip, and any topological conjugate of a standard exchange ϕ_i is a generic exchange. Assumption (2) above says that ι^+ takes any eyeglass twist in $\mathcal{G}_g$, (its frame disjoint from the point $\star$ by general position) to $\mathcal{P}_{g+1} \subset \mathcal{G}_{g+1}$. Similarly, Claim 1, using assumption (1), says that ι^+ takes any generic flip in $\mathcal{G}_g$ to $\mathcal{P}_{g+1}$, and Claim 2, also using assumption (1), says that ι^+ takes any generic exchange in $\mathcal{G}_g$ to $\mathcal{P}_{g+1}$. Since each generator of $\mathcal{G}_g$ is taken to $\mathcal{P}_{g+1}$, $\iota^+\colon \mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ is trivial, as required. □

We now show that in fact both assumptions in Proposition 4.3 are true. The proofs of both are highly visual. We first prove assumption (2) of Proposition 4.3.

Proposition 4.4 *Suppose $\tau \in \mathcal{G}_{g+1}$ is an eyeglass twist whose eyeglass η is disjoint from the standard bubble $\mathfrak{b}_{g+1}$. Then τ is a Powell move.*

Proof Let $\{\ell_a, \ell_b, v\}$ be the frame of η. Let u be an arc such that

- the ends of u lie on a_{g+1} near the point $a_{g+1} \cap b_{g+1}$ and on ℓ_a near the point $\ell_a \cap v$,
- u crosses the circle $c = \partial\mathfrak{b}_{g+1}$ once, and
- u is otherwise disjoint from $a_{g+1} \cup b_{g+1}$ and η.

See Figure 8.

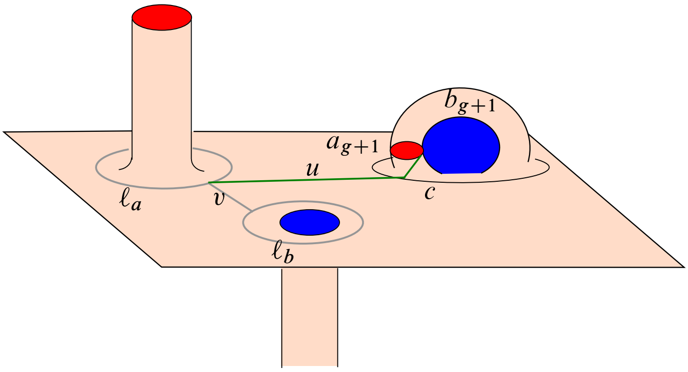

Figure 8

Let η' be the eyeglass given by $\{\mu_{g+1}, \ell_b, u \cup v\}$. We know from Lemma 3.4 that an eyeglass twist τ' along η' is a Powell move. Let ℓ'_a be the band sum of ℓ_a and μ_{g+1} along u and observe, by watching the motion of ℓ_b, that the composition $\tau\tau'$ is an eyeglass twist τ_+ whose eyeglass is η_+ given by $\{\ell'_a, \ell_b, v\}$.

Now let η_1 be the eyeglass given by $\{\ell_a, \lambda_{g+1}, u\}$ and let τ_1 be the eyeglass twist along η_1. See Figure 9. Again, from Lemma 3.4, τ_1 is a Powell move. Further observe that $\tau_1^{-1}(\eta_+)$ is an eyeglass η'_+ with lenses μ_{g+1} and ℓ_b and a bridge that intersects c in a single point, so again an eyeglass twist τ'_+ along η'_+ is a Powell move. See Figure 10.

Since $\tau_1^{-1}(\eta_+) = \eta'_+$ it follows that $\tau_+ = \tau_1 \tau'_+ \tau_1^{-1}$. As a composition of Powell moves, τ_+ is a Powell move. But we have earlier shown that $\tau = \tau_+ \tau'^{-1}$, so, as a composition of Powell moves, τ is a Powell move. □

Proposition 4.5 *Suppose $\tau \in \mathcal{G}_{g+1}$ is a bubble exchange between the standard bubble $\mathfrak{b}_{g+1}$ and a generic bubble $\mathfrak{b}$. Then τ is a Powell move.*

Proof The entire proof is contained in Figure 11. In the top panel, the standard bubble $\mathfrak{b}_{g+1}$ is on the right, with the meridian circle a_{g+1} and the longitude circle b_{g+1} both shown in fuchsia. On the left is

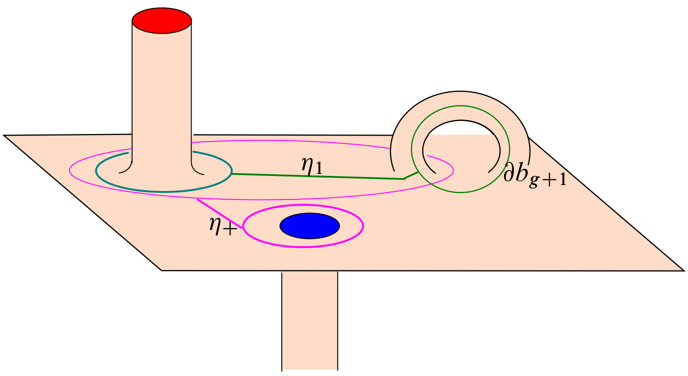

Figure 9

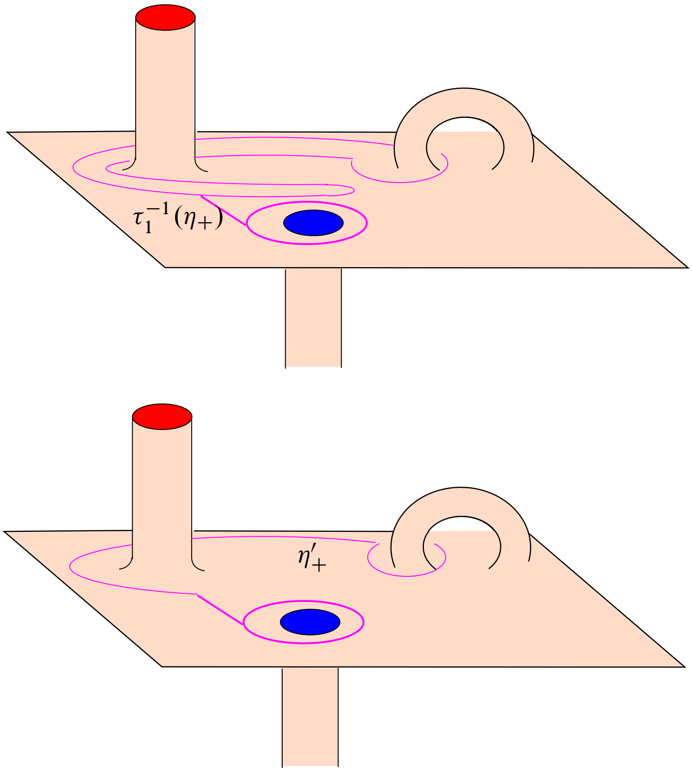

Figure 10

a generic bubble with the boundaries of the meridian $x \subset A$ and longitude $y \subset B$ shown in blue. Also shown there are two eyeglasses η_1 and η_2, in red and green, respectively. Since the bubble on the right is standard, by Lemma 3.4 the eyeglass twists τ_i along η_i are both Powell moves. The direction of the twist along the η_i that we will use is shown by arrows on the longitudinal lenses of η_1 and η_2, parallel respectively to ∂y and b_{g+1}.

The next two panels down show the effect of first τ_1 then τ_2. The following panel shows the effect of a (clockwise π) flip on the standard bubble $\mathfrak{b}_{g+1}$, a Powell move. Then follow two Powell moves: τ_1^{-1} and a final move of the standard bubble around the meridian x of the generic bubble. The final result in the last panel is that the pair μ_{g+1}, λ_{g+1} and the pair x, y have been exchanged.

Note that the orientations of x, y and μ_{g+1}, λ_{g+1} are not an issue: if an exchange as above moves an oriented μ_{g+1} to an unwanted orientation of x (or vice versa), then precomposing (or postcomposing) with a standard flip (a Powell move) will fix the problem. □

Corollary 4.6 *The function $\iota^+ : \mathcal{G}_g \to \mathcal{G}_{g+1}/\mathcal{P}_{g+1}$ defined in Proposition 3.2 is trivial.* □

Two final notes The proof of Proposition 4.5 focuses on the standard bubble $\mathfrak{b}_{g+1}$ and the generic bubble $\mathfrak{b}$, both lying in the neighborhood of a disk D that contains the exchange; the exchange map is the

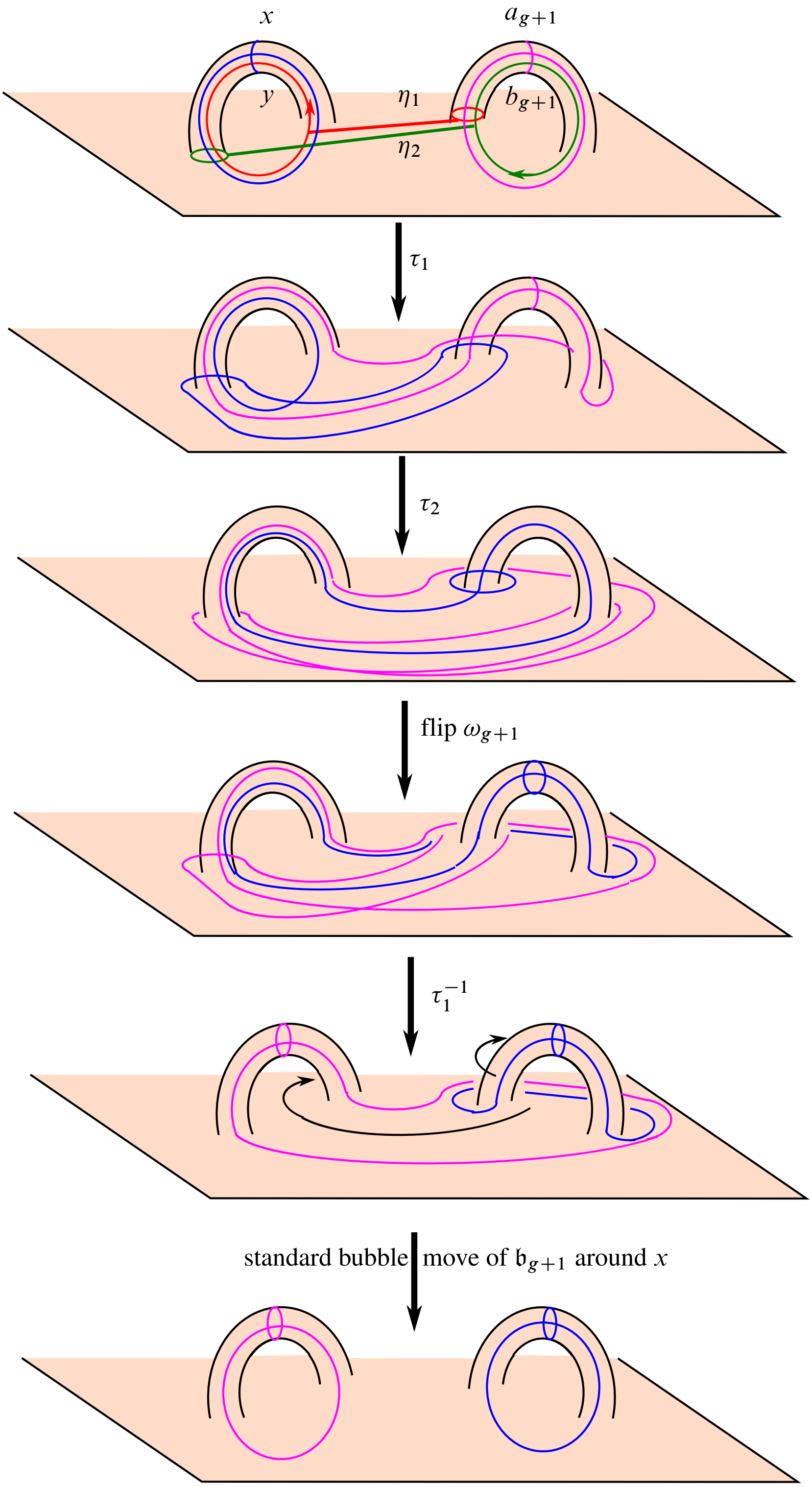

Figure 11: An exchange that is a Powell move.

identity map outside of the neighborhood of D. Left undiscussed is the pair of pants $D-(\mathfrak{b}\cup\mathfrak{b}_{g+1})\subset T_{g+1}$. Observe, though, that any homeomorphism of this pair of pants, fixed on its boundary, is also a Powell move. Here is the argument: Any such homeomorphism can be constructed by Dehn twisting some number of times around each of the boundary components of the pair of pants. A Dehn twist at the boundary of the standard bubble $\mathfrak{b}_{g+1}$ is a Powell move, for it is a composition of two standard flips. Similarly, conjugating this Powell Dehn twist by the exchange just described, also a Powell move, gives a Dehn twist around the bubble $\mathfrak{b}$, so such a Dehn twist is also Powell. Lastly, it is easy to check that the exchange defined above gives a simple (clockwise) half-twist on a collar of ∂D, so a full Dehn twist around that boundary component can be accomplished just by doing the exchange twice.

Because the exchange in Proposition 4.5 moves $\mathfrak{b}_{g+1}$ to another bubble, the exchange itself is not in the image of ι^+ and its construction may seem a bit ad hoc. An online version of this paper [8] contains a lengthy appendix that puts the exchange into a broader context, one involving an order 12 dihedral subgroup of $\mathcal{G}_2$, each of whose elements somewhat naturally defines an element of $\mathcal{G}_{g+1}$ that also lies in $\mathcal{P}_{g+1}$ but not in the image of ι^+. Figure 7 shows the connection between one such element of $\mathcal{G}_2$ and a bubble exchange like that of Proposition 4.5.

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