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An overtwisted convex hypersurface in higher dimensions

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We show that the germ of the contact structure surrounding a certain kind of convex hypersurface is overtwisted. We then find such hypersurfaces close to any plastikstufe with toric core, thereby proving that the existence of a plastikstufe with toric core implies overtwistedness. All proofs in this article are explicit, and we hope that the methods used here might hint at a deeper understanding of the size of neighborhoods in contact manifolds.

In the appendix we reprove in a concise way that the Legendrian unknot is loose if the ambient manifold contains a large enough neighborhood of a 2-dimensional overtwisted disk. Additionally we prove the folklore result that the singular distribution induced on a hypersurface Σ of a contact manifold (M, ξ) determines the germ of the contact structure around Σ.

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1 Introduction

The fundamental distinction between tight and overtwisted contact structures was discovered first by Eliashberg in dimension 3 [5], and then generalized to arbitrary dimensions by Borman, Eliashberg and Murphy [2]. The main feature is that an overtwisted contact structure is flexible. However, the original high-dimensional definition makes it practically unverifiable if a given contact structure is overtwisted. Thanks to Casals, Murphy and Presas [3] we know that most previously existing conjectural definitions for overtwistedness are actually equivalent to the one given in [2].

Still many open questions persist. One of them has been settled by Huang [8]: The second author of this article proposed a definition of overtwistedness called plastikstufe [13]. The results in [12] combined with those of [3] showed that certain very specific plastikstufes imply overtwistedness. Huang explained in [8] that this result extends to *any* plastikstufe. However, we are unable to verify several claims made in his proof.

In this article, we reprove Huang's result for the more restrictive case of plastikstufes with toric core (generalizing results by Adachi [1]).

Our strategy is based on the following observation about a certain type of convex hypersurface. Consider for any $C > 0$ the manifold

$$\Sigma_C = \mathbb{D}^2_{\leq\pi} \times (-C, C)^{2n}$$

carrying a singular distribution $\mathcal{D}_C$ given as the kernel of the 1-form $\beta = r \sin r \, d\vartheta - \sum_{j=1}^n t_j \, ds_j$, where (r, ϑ) are polar coordinates on the disk, and (s_j, t_j) are the natural coordinates on the cube $(-C, C)^n \times (-C, C)^n$.

Theorem A *There exists a constant $C_{\mathrm{OT}} > 0$ such that the following holds for every contact manifold (M, ξ) of dimension at least 5. If (M, ξ) admits an embedding of a hypersurface $(\Sigma_C, \mathcal{D}_C)$ with $C > C_{\mathrm{OT}}$ such that ξ induces the singular distribution $\mathcal{D}_C$ on Σ_C, then (M, ξ) is overtwisted.*

Consequently, we call any embedded hypersurface $(\Sigma_C, \mathcal{D}_C)$ with $C > C_{\mathrm{OT}}$ an *overtwisted convex disk*.

Remark (a) This definition is closely related to the characterization of overtwisted contact structures in terms of "large neighborhoods" given in [3; 15]. Our result states that instead of considering large embedded balls, it is already sufficient to find a large hypersurface.

(b) In the model used in Section 4, one sees directly that there is an obvious contact vector field Z ($= \partial_z$ in fact) which is transverse to the overtwisted convex disk. It is tempting to try to characterize the constant C_{OT} more explicitly by trying to recognize some sort of higher-dimensional Giroux criterion for convex hypersurfaces. Unfortunately, the dividing set for Z is noncompact, and we have not succeeded in finding a more suitable contact vector field.

(c) Even though we are unable to give a specific value for the size parameter that appears in the definition of overtwistedness in [2] (and the equivalent formulation via large neighborhoods in [3]), it follows from our argument that the size parameter can be chosen uniformly for all dimensions.

Proof of Theorem A We show in Sections 3 and 4 that every neighborhood of $\mathbb{D}^2_{\leq\pi} \times \bigl(-C, C\bigr)^{2n}$ contains the embedding of a certain type of open subset $B(h) \times \bigl(-\frac{5}{6}C, \frac{5}{6}C\bigr)^{2n}$ of arbitrary height $h > 0$. See Corollary 4.3 for the details. As explained first in [12, page 1813], the Legendrian unknot is loose if C is chosen sufficiently large. We give in Appendix A a streamlined proof of this statement.

It was proved in [3] that any contact manifold in which the unknot is loose is overtwisted. □

Question *Is it possible to explicitly show that in any neighborhood of a hypersurface Σ_C with $C > C_{\mathrm{OT}}$ one can embed a hypersurface $\Sigma_{C'}$ with $C' > 2C$ as in the analogous claim for loose charts* [11, Proposition 4.4]?

Choose a $\rho > 0$ and define V_ρ as in Appendix A. It is likely that the contact germs around the overtwisted convex disk $(\Sigma_C, \mathcal{D}_C)$ correspond to thick neighborhoods of the form $\bigl(\mathbb{R}^3 \times V_\rho, \ker(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}})\bigr)$. It was shown in [3] that these neighborhoods are for sufficiently large $\rho > 0$ overtwisted. However while it is obvious that the contact germs considered here embed into such thick neighborhoods, it is not obvious how to directly prove the converse. By taking instead the detour over the loose unknots, we are able to split our argument cleanly into precise steps.

Even if it might be unclear at the moment if Theorem A is more than a curious observation, it allows us to prove in an extremely elementary way that every contact manifold containing a plastikstufe with *toric* core is overtwisted, as claimed in [8].

In Appendix A, we show that the Legendrian unknot is loose in a large neighborhood of an overtwisted 2-disk; in Appendix B we prove the folklore result that a hypersurface in a contact manifold determines together with the induced singular distribution the germ of the contact structure.

Acknowledgments

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2 A plastikstufe with toric core implies overtwistedness

Denote the cylindrical coordinates on $\mathbb{R}^3$ by (r, ϑ, z). The 1-form

$$\alpha_{\mathrm{OT}} = \cos(r)\,dz + r\sin(r)\,d\vartheta$$

is then a well-defined contact form. The disk $\mathbb{D}_{\mathrm{OT}} := \{(r, \vartheta, z) \mid r \le \pi,\ z = 0\}$ is overtwisted, and we will call it the *standard* overtwisted disk.

We choose a small cylindrical box of height h around $\mathbb{D}_{\mathrm{OT}}$ of the form

$$B(h) := \mathbb{D}^2_{<\pi+\delta} \times (-h, h). \tag{2-1}$$

Let (M, ξ) be a $(2n+3)$-dimensional contact manifold that contains a plastikstufe of the form $\mathbb{D}_{\mathrm{OT}} \times \mathbb{T}^n$. We will show that we can find an arbitrarily "large" hypersurface $\mathbb{D}^2_{\le\pi} \times [-C, C]^{2n}$ in any neighborhood of the plastikstufe by successively unwinding each of the $\mathbb{S}^1$-factors of the torus. This way we obtain the following corollary.

Corollary 2.1 *Every contact manifold that contains a plastikstufe $\mathbb{D}_{\mathrm{OT}} \times \mathbb{T}^n$ with toric core also admits an embedding of an overtwisted convex disk.*

Proof There is a neighborhood of the plastikstufe that is contactomorphic to an open neighborhood of $\mathbb{D}_{\mathrm{OT}} \times \mathbb{T}^n$ in

$$\left(\mathbb{R}^3 \times T^*\mathbb{T}^n, \alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\right);$$

compare [14, Theorem I.1.3.]. Choosing $\varepsilon > 0$ and $\delta > 0$ small enough, we can assume that this neighborhood is contactomorphic to a product of the form $B(\varepsilon) \times \mathbb{D}_{<\delta}\left(T^*\mathbb{T}^n\right)$, where $B(\varepsilon) \subset \mathbb{R}^3$ is a cylindrical box as defined in (2-1) and $\mathbb{D}_{<\delta}\left(T^*\mathbb{T}^n\right)$ is the disk bundle of radius δ in $T^*\mathbb{T}^n$.

We can now apply to this neighborhood Lemma 2.3 in dimension 5, or Lemma 2.4 in the general case to find a hypersurface of the form $\mathbb{D}^2_{\leq\pi} \times (-C, C)^n \times (-a, a)^n$ for $a = \delta/\left(2\sqrt{n}\right)$, and where $C > 0$ can be chosen to be arbitrarily large. The singular distribution induced by the contact structure agrees with the kernel of $r \sin r \, d\vartheta - \sum_{j=1}^n t_j \, ds_j$, where (r, ϑ) are polar coordinates on the disk, and (s_j, t_j) are the natural coordinates on the rectangle $(-C, C) \times (-a, a)$.

If $C > 0$ is chosen larger than $2C_{\mathrm{OT}}^2/a$, then it suffices to apply to each coordinate pair (s_j, t_j) the diffeomorphism $(s_j, t_j) \mapsto \left(\mu^{-1} s_j, \mu t_j\right)$ with $\mu = 2C_{\mathrm{OT}}/a$ to obtain the desired overtwisted convex disk $(\Sigma_{\widetilde{C}}, \mathcal{D}_{\widetilde{C}})$ for some appropriate $\widetilde{C} > C_{\mathrm{OT}}$. □

Remark 2.2 The initial version of a plastikstufe $\mathbb{D}^2 \times S$ introduced in [13] differed slightly from the form $\mathbb{D}_{\mathrm{OT}} \times S$ used here, because the boundary of $\mathbb{D}_{\mathrm{OT}} \times S$ is composed of singular leaves of the foliation. The presence of either type of a plastikstufe in a contact manifold implies by [14, Theorem I.1.3.] the other one: The plastikstufe determines the germ of the contact structure which will be contactomorphic to an open subset of $\left(\mathbb{R}^3 \times T^*S, \alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\right)$. This allows us to modify the plastikstufe along its boundary by a C^0-small deformation to move from one model to the other one.

We will now show how to "unwrap" the toric plastikstufe. Consider for simplicity first a contact manifold of dimension 5 so that $\mathbb{T}^n = \mathbb{S}^1$.

Lemma 2.3 *Choose any $\varepsilon > 0$ and $\delta > 0$, and let $B(\varepsilon) \times \mathbb{D}_{<\delta}\left(T^*\mathbb{S}^1\right)$ be a neighborhood of a plastikstufe $\mathbb{D}_{\mathrm{OT}} \times \mathbb{S}^1$ in $\left(\mathbb{R}^3 \times T^*\mathbb{S}^1, \alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\right)$.*

For any arbitrarily large $C > 0$ it is possible to embed the hypersurface

$$S_C := \mathbb{D}^2_{\leq\pi} \times \left(-C, C\right) \times \left(-\tfrac{\delta}{2}, \tfrac{\delta}{2}\right)$$

into $B(\varepsilon) \times \mathbb{D}_{<\delta}\left(T^*\mathbb{S}^1\right)$ such that the contact structure induces the singular distribution

$$\mathcal{D} := \ker\left(r \sin r \, d\vartheta - t \, ds\right)$$

on S_C. Here (r, ϑ) are polar coordinates on the disk, and (s, t) are the natural coordinates on the rectangle $\left(-C, C\right) \times \left(-\tfrac{\delta}{2}, \tfrac{\delta}{2}\right)$.

Proof Define for any $\hbar > 0$ an embedding

$$\Psi_\hbar \colon \mathbb{D}^2_{\le\pi} \times \mathbb{R}^2 \to \mathbb{R}^3 \times T^*\mathbb{S}^1, \quad (r, \vartheta; s, t) \mapsto (r, \vartheta, \hbar s; q = e^{is}, p = t + \hbar \cos r),$$

"reeling up" the hypersurface along the $\mathbb{S}^1$-core of the plastikstufe. One easily verifies that

$$\Psi_\hbar^*(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}) = r \sin r \, d\vartheta - t \, ds$$

so that the singular distribution induced by $\ker(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}})$ is indeed equal to $\mathcal{D}$.

Choose $\hbar = \varepsilon/C$ and suppose that $\hbar < \frac{\delta}{2}$. We see that $\Psi_\hbar(S_C)$ lies in the given neighborhood $B(\varepsilon) \times \mathbb{D}_{<\delta}(T^*\mathbb{S}^1)$. □

For general dimensions, the embedding is only slightly more complicated. Denote the standard coordinates on $\mathbb{T}^n$ by $\boldsymbol{q} = (q_1, \dots, q_n)$ and those on $T^*\mathbb{T}^n$ by $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \dots, q_n; p_1, \dots, p_n)$.

Lemma 2.4 *Any neighborhood $B(\varepsilon) \times \mathbb{D}_{<\delta}(T^*\mathbb{T}^n)$ of a plastikstufe $\mathbb{D}_{\mathrm{OT}} \times \mathbb{T}^n$ with $\varepsilon > 0$ and $\delta > 0$ arbitrarily small contains an embedded hypersurface of the form*

$$S_C := \mathbb{D}^2_{\le\pi} \times \left\{ (s_1, \dots, s_n; t_1, \dots, t_n) \in \mathbb{R}^{2n} \;\middle|\; |s_j| < C, \ |t_j| < \frac{\delta}{2\sqrt{n}} \right\}$$

with $C > 0$ arbitrarily large such that the contact structure induces the singular distribution

$$\mathcal{D} = \ker\Bigl(r \sin r \, d\vartheta - \sum_{j=1}^{n} t_j \, ds_j \Bigr)$$

on S_C.

Proof Choose constants $\hbar_1, \dots, \hbar_n > 0$ that are linearly independent over $\mathbb{Q}$, and define a map $\Psi \colon \mathbb{D}^2_{\le\pi} \times \mathbb{R}^{2n} \to \mathbb{R}^3 \times T^*\mathbb{T}^n$ by

$$(r, \vartheta; s_1, \dots, s_n; t_1, \dots, t_n) \mapsto \Bigl(r, \vartheta, z = \sum_{j=1}^{n} \hbar_j s_j; q_1 = e^{is_1}, \dots, q_n = e^{is_n}; p_1 = t_1 + \hbar_1 \cos r, \dots, p_n = t_n + \hbar_n \cos r \Bigr).$$

It is easy to verify that $\Psi^*(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}) = r \sin r \, d\vartheta - (t_1 \, ds_1 + \cdots + t_n \, ds_n)$ induces the distribution $\mathcal{D}$ on S_C. It is also immediately clear that Ψ is an immersion.

To see that Ψ is injective, use first that the images of two points $(r, \vartheta; s_1, \dots, s_n; t_1, \dots, t_n)$ and $(r', \vartheta'; s'_1, \dots, s'_n; t'_1, \dots, t'_n)$ by Ψ can only agree if $r = r'$, $\vartheta = \vartheta'$, and $t_j = t'_j$ for all $j = 1, \dots, n$, and if $s_j - s'_j$ is for every $j = 1, \dots, n$ an integer multiple of 2π. The equation $\hbar_1 s_1 + \cdots + \hbar_n s_n = \hbar_1 s'_1 + \cdots + \hbar_n s'_n$ implies that $\hbar_1(s_1 - s'_1) + \cdots + \hbar_n(s_n - s'_n) = 0$, but by our assumption that the $\hbar_j$ are linearly independent over $\mathbb{Q}$ it follows that all coefficients $s_1 - s'_1$ need to vanish so that Ψ is injective.

We still need to verify that the image of Ψ lies in the neighborhood $B(\varepsilon) \times \mathbb{D}_{<\delta}(T^*\mathbb{T}^n)$. To respect the z-height, it suffices to choose $\hbar_1 + \cdots + \hbar_n < \varepsilon/C$, so that the z-coordinate of Ψ is bounded by ε. For the radius of the fibers in $\mathbb{D}_{<\delta}(T^*\mathbb{T}^n)$ choose $\hbar_j < \delta/(2\sqrt{n})$ so that Ψ also stays inside the δ-disk bundle of $T^*\mathbb{T}^n$. □

3 The standard overtwisted contact structure on $\mathbb{R}^3$

For a cylindrical box of height h around the standard overtwisted disk $\mathbb{D}_{\mathrm{OT}}$ in $(\mathbb{R}^3, \alpha_{\mathrm{OT}})$ of the form

$$B(h) := \mathbb{D}^2_{<\pi+\delta} \times (-h, h),$$

it is well known that the choice of h is not relevant for the contactomorphism type. Below we will give a contact vector field that can be used to prove this fact by hand, but instead one can also easily convince oneself that all $B(h)$ are *overtwisted at infinity* which uniquely characterizes by [6] a contact structure on $\mathbb{R}^3$.

The main technical problem that we will deal with in this article is to show that the choice of the h-parameter also remains largely irrelevant for the contactomorphism type when we take the product with a Liouville domain.

We will now discuss a contact vector field X whose flow compresses any large box $B(h)$ into an arbitrarily small neighborhood of the standard overtwisted disk. Ideally we would like X to be a strict contact vector field or at least to have a constant scaling factor c such that $\mathcal{L}_X \alpha_{\mathrm{OT}} = c \cdot \alpha_{\mathrm{OT}}$. Unfortunately such a vector field cannot exist: firstly, X should be contracting and thus it needs to reduce the total volume. This implies that c would have to be strictly negative on a predominant part of its domain. On the other hand, c cannot be *everywhere* strictly negative as this would allow us to squeeze with the strategy of Section 4 a high-dimensional overtwisted chart into an arbitrarily "thin" set, thus contradicting the existence of tight contact manifolds.

The following vector field arose in discussions with Patrick Massot around 2010:

$$X := -z\,\partial_z - \frac{r\cos(r)\sin(r)}{r + \cos(r)\sin(r)}\,\partial_r \tag{3-1}$$

is well defined and induces a contact flow on $(\mathbb{R}^3, \alpha_{\mathrm{OT}})$.

Even though this vector field might at first appear overly complicated, note that all coordinates in X are uncoupled. This allows us to see that its time T flow preserves the ϑ-coordinate, and it contracts the z-coordinate by the factor e^{-T}. The radial coordinate is fixed on every cylinder of radius $r \in \frac{\pi}{2}\mathbb{N}$. The cylinders of radius $r \in \frac{\pi}{2} + \pi\mathbb{N}$ are repelling; the cylinders of radius $r \in \pi\mathbb{N}$ are attracting, in the sense that all points between these cylinders are pushed away from the repelling cylinder towards the attracting one; see Figure 1.

The height of the box $B(h)$ is squeezed by the flow of X by an exponential factor, while the radial direction also shrinks, without becoming ever smaller than π though. This way an arbitrarily tall box $B(h)$ can be squeezed by a contactomorphism into an arbitrarily small neighborhood of the standard overtwisted disk. In fact, one can convince oneself that the image of a box $B(h)$ for a certain choice of $\delta > 0$ will be of the form $\mathbb{D}^2_{<\pi+\delta'} \times (-h', h')$ for some smaller $h' > 0$ and $\delta' > 0$.

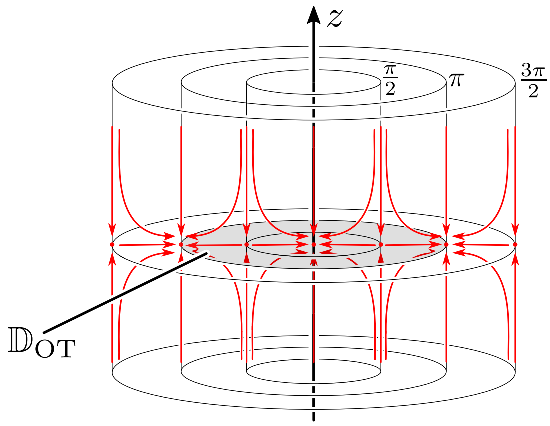

Figure 1: The flow of X preserves the cylinders of radius $r \in \frac{\pi}{2} + \pi\mathbb{N}$. The boundary of the standard overtwisted disk sits on an attracting cylinder.

Finally note that $\mathcal{L}_X \alpha_{\mathrm{OT}} = g \cdot \alpha_{\mathrm{OT}}$, where

$$g(r, \vartheta, z) = -\frac{\cos r\,(r\cos r + \sin r)}{r + \cos r \sin r}. \tag{3-2}$$

As we claimed above, g takes both positive and negative values. More precisely:

Lemma 3.1 *The function $g\colon B(h) \to \mathbb{R}$ is everywhere negative on $B(h)$ except for the domain lying between $r_m = \pi/2$ and $r_M \approx 2.03$ such that $r_M = -\tan r_M$. See also the graph in Figure 2.*

Proof The function g only depends on the radial coordinate r. Its denominator is everywhere positive while the numerator changes once its sign at $r = \pi/2$, where $\cos r$ vanishes, and then again at $r \approx 2.03$, where $r\cos r + \sin r$ vanishes. □

We can also read off from the graph, Figure 2, that g is everywhere smaller than 0.1; ie, even though g becomes positive it only becomes very slightly so.

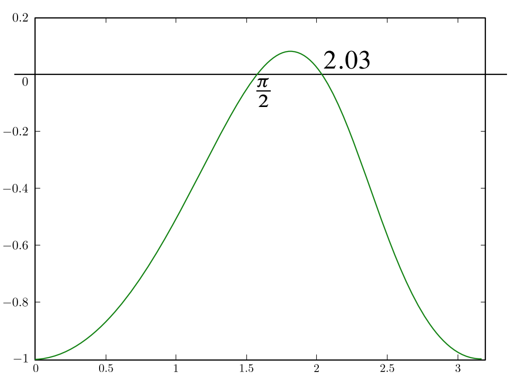

Figure 2: The graph of $g = -\dfrac{\cos r\,(r\cos r + \sin r)}{r + \cos r \sin r}$.

4 Contactomorphism on product structure

Let (M, ξ) be a contact manifold with contact form α. Assume that X is a contact vector field such that $\mathcal{L}_X \alpha = g \cdot \alpha$ for some function $g \colon M \to \mathbb{R}$.

Choose an exact symplectic manifold $(W, d\lambda)$ that has a Liouville vector field Y, then we easily check that the contact form of $\bigl(M \times W, \alpha + \lambda\bigr)$ is preserved by the vector field

$$\widehat{X} = X + g \cdot Y.$$

Note that even if Y points outwards and is expanding on $(W, d\lambda)$, the behavior of $\widehat{X}$ on the product manifold $M \times W$ is controlled in W-direction by the sign of the function g that might take positive or negative values.

We consider now the main example we will be interested in: Let $B(h) \subset \bigl(\mathbb{R}^3, \alpha_{\mathrm{OT}}\bigr)$ be a box of height h as defined in (2-1), and let $\bigl(L, \langle \cdot\,, \cdot \rangle\bigr)$ be a Riemannian manifold that does not need to be closed or geodesically complete. We denote the disk bundle of radius c in $\bigl(T^*L, d\lambda_{\mathrm{can}}\bigr)$ by

$$\mathbb{D}_{<c}\bigl(T^*L\bigr) := \bigl\{\nu \in T^*L \mid \|\nu\| < c\bigr\}.$$

Proposition 4.1 *There exists a positive constant $\mu_0 < \frac{7}{6}$ such that every contact domain*

$$\Bigl(B(h) \times \mathbb{D}_{<c}\bigl(T^*L\bigr), \ker\bigl(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\bigr)\Bigr)$$

with $c > 0$ can be embedded by a contactomorphism into

$$\Bigl(B(h') \times \mathbb{D}_{<\mu_0 \cdot c}\bigl(T^*L\bigr), \ker\bigl(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\bigr)\Bigr)$$

independently of the choices of $h, h' > 0$.

Proof For $h' \ge h$ the claim is obvious; for $h' < h$ the strategy is to use the flow of the vector field $\widehat{X} := X + g \cdot Y$ with X and g introduced in the previous section, and Y the Liouville vector field on T^*L that is defined by $\iota_Y d\lambda_{\mathrm{can}} = \lambda_{\mathrm{can}}$.

By writing Y in a coordinate chart, one easily convinces oneself that the time t flow of Y simply consists of multiplying the fiber of T^*L by e^t. In particular, even if L is open or has boundary there is no danger that the flow of $\widehat{X}$ escapes transversely through the fibers of T^*L. Now let us study the behavior of the flow $\Phi_T^{\widehat{X}}$ in more details.

Recall that the coordinates are uncoupled by the flow of X given in (3-1). We can thus write

$$\Phi_T^X\bigl(r, \vartheta, z\bigr) = \bigl(F(r, T), \vartheta, e^{-T} z\bigr),$$

where $F(r, T)$ is the solution of the ODE

$$y'(t) = -\frac{y(t) \cos y(t) \sin y(t)}{y(t) + \cos y(t) \sin y(t)} \quad \text{and} \quad y(0) = r.$$

The flow $\Phi_T^{\widehat{X}}$ is therefore of the form

$$\Phi_T^{\widehat{X}}\big(r,\vartheta,z;\boldsymbol{q},\boldsymbol{p}\big) = \Big(\Phi_T^{X}\big(r,\vartheta,z\big);\boldsymbol{q}, e^{G(r,T)}\cdot\boldsymbol{p}\Big).$$

That is, the flow on the $\mathbb{R}^3$-factor simply reduces to the corresponding flow of X and can be evaluated independently of the T^*L-part; the flow on the cotangent bundle factor is obtained by multiplying the fiber direction by a positive function e^G that can be computed via

$$G(r,t) = \int_0^t g\big(\Phi_s^X(r,\vartheta,z)\big)\,ds = \int_0^t g\big(F(r,s)\big)\,ds. \tag{4-1}$$

If T is chosen to be $T=\ln\frac{h}{h'}$, it follows that $\Phi_T^{\widehat{X}}$ squeezes the first factor of $B(h)\times\mathbb{D}_{<c}\big(T^*L\big)$ into $B(h')$. By Lemma 4.2 below, $G(r,t)<\ln\frac{7}{6}$ for any point in $B(h)$ and any $t\geq 0$. This implies as desired that the initial domain is squeezed into $B(h')\times\mathbb{D}_{<(7/6)c}\big(T^*L\big)$. □

Lemma 4.2 *The function $G(r,t)$ given in* (4-1) *is bounded from above by* $\ln\frac{7}{6}$ *for all $r\in[0,\pi+\delta)$ and all $t\in[0,\infty)$.*

The sharp upper bound in the lemma is $\ln\frac{2r_M\sin r_M}{\pi}$ with r_M specified in Lemma 3.1.

Proof Denote the r-coefficient of the vector field X by

$$f(r) = -\frac{r\cos(r)\sin(r)}{r+\cos(r)\sin(r)}.$$

Then $F(r,t)$ is the flow of the field $X_r(r):=f(r)\,\partial_r$ on $[0,\pi+\delta)$; that is, F is the solution of the ordinary differential equation $\partial_t F(r,t) = f\big(F(r,t)\big)$ with initial condition $F(r,0)=r$.

The only critical points of X_r are the points $r\in\frac{\pi}{2}\mathbb{N}$; see also Figure 1. Furthermore, recall that $r=\frac{\pi}{2}$ and $r=\frac{3\pi}{2}$ are repelling, and that $r=\pi$ is an attracting critical point.

According to Lemma 3.1, the function g is everywhere on $[0,\pi+\delta)$ negative except for the interval $[r_m,r_M]$ with $r_m=\frac{\pi}{2}$, and $r_M\approx 2.03$ given by $r_M=-\tan r_M$.

Since all trajectories of X_r starting in $\big[0,\frac{\pi}{2}\big]$ are trapped inside this interval, the function $G(r,t)$ will be negative for all $r\in\big[0,\frac{\pi}{2}\big]$ and all $t\geq 0$. Similarly, the points in $[\pi,\pi+\delta)$ are pulled by the flow towards $r=\pi$ without ever crossing this point. Thus $G(r,t)$ will also be negative for all $r\in[\pi,\pi+\delta)$ and all $t\geq 0$.

It only remains to understand the behavior of $G(r,t)$ for $r\in\big(\frac{\pi}{2},\pi\big)$. Since f is strictly positive on this interval, it follows that, for every initial value $r\in\big(\frac{\pi}{2},\pi\big)$,

$$F(r,\cdot)\colon\mathbb{R}\to\big(\tfrac{\pi}{2},\pi\big)$$

is an orientation preserving diffeomorphism, and in particular there is a unique time $T_r\in\mathbb{R}$ such that $F(r,T_r)=r_M$.

For every fixed $r \in \left(\frac{\pi}{2}, r_M\right]$ and all positive $t \geq 0$, the upper bound of $G(r,t)$ in (4-1) is then given by

$$G(r, T_r) = \int_0^{T_r} g\big(F(r,s)\big)\, ds,$$

because g is strictly positive up to $t = T_r$ so that $G(r,\cdot)$ increases up to that moment; for all later times $t > T_r$, the trajectory $F(r,t)$ lies in the zone $[r_M, \pi)$ where g is negative so that $G(r,t) \leq G(r,T_r)$ for all $t \geq T_r$.

To compute $G(r,T_r)$ use that $F(r,\cdot)\colon \mathbb{R} \to \left(\frac{\pi}{2}, \pi\right)$ is for every choice of $r \in \left(\frac{\pi}{2}, \pi\right)$ a diffeomorphism, so that we can substitute $u = F(r,s)$ in the integral using that $\frac{du}{ds} = f\big(F(r,s)\big) = f(u)$, and obtain

$$\begin{aligned} G(r, T_r) &= \int_0^{T_r} g\big(F(r,s)\big)\, ds = \int_{F(r,0)}^{F(r,T_r)} \frac{g(u)}{f(u)}\, du = \int_r^{r_M} \frac{u\cos u + \sin u}{u \sin u}\, du \\ &= \ln\big(u \sin u\big)\Big|_r^{r_M} = \ln \frac{r_M \sin r_M}{r \sin r}. \end{aligned}$$

The denominator $r \sin r$ is increasing on $\left[\frac{\pi}{2}, r_M\right]$ so that its smallest value on this interval is attained at $r = \frac{\pi}{2}$. We obtain the estimate

$$G(r,t) \leq \ln \frac{r_M \sin r_M}{r \sin r} < \ln \frac{r_M \sin r_M}{\frac{\pi}{2} \sin \frac{\pi}{2}} = \ln \frac{2 r_M \sin r_M}{\pi} < \ln \tfrac{7}{6}. \qquad \square$$

In particular, we obtain the following result.

Corollary 4.3 *Let L be a manifold that does not need to be closed, and let*

$$\Sigma := \mathbb{D}^2_{\leq \pi} \times \mathbb{D}_{<c_0}\big(T^*L\big)$$

be a hypersurface in a contact manifold (M,ξ) such that the singular distribution induced by ξ on the hypersurface agrees with the kernel of the 1-form $\beta = r \sin r\, d\vartheta + \lambda_{\mathrm{can}}$, where (r,ϑ) denotes the polar coordinates on $\mathbb{D}^2_{\leq\pi}$.

Let $c > 0$ be such that $\mu_0 c < c_0$ with the constant $\mu_0 < \frac{7}{6}$ in Proposition 4.1. Then we can embed the contact domain

$$\Big(B(h) \times \mathbb{D}_{<c}\big(T^*L\big), \ker\big(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\big)\Big)$$

of “width” $c > 0$ and of any chosen “height” $h > 0$ into an arbitrarily small neighborhood of the hypersurface $\mathbb{D}^2_{\leq\pi} \times \mathbb{D}_{<c_0}\big(T^*L\big)$.

Proof The induced singular distribution of a hypersurface determines by Proposition B.1 the germ of the contact structure on a neighborhood of the hypersurface. We can thus assume that Σ has a neighborhood U that is contactomorphic to $\big(B(\varepsilon) \times \mathbb{D}_{<c_0}\big(T^*L\big), \ker\big(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}}\big)\big)$ for small $\varepsilon > 0$. By Proposition 4.1 we can thus embed $B(h) \times \mathbb{D}_{<c}\big(T^*L\big)$ into U. $\square$

Appendix A The Legendrian unknot is loose in a sufficiently large overtwisted chart

In this appendix, we provide an elementary proof that Legendrian unknots are loose in ambient manifolds containing a large neighborhood of an overtwisted chart. This was first hinted at in [12]. A proof was given by Huang in [8] but he used piecewise smooth Legendrians so that a lot of the potential clarity was lost.

The key idea here is that for Legendrians that are in product form, any isotopy on the first factor can be trivially extended to the second one to obtain a global isotopy. If the Legendrian is only locally in product form, this construction only provides a local isotopy. To globalize it, we want to smoothen it out so that local isotopy glues to the identity outside the considered neighborhood. This interpolation requires a sufficient amount of space.

Except for a certain 3-dimensional result that is accepted as a black-box, the proof then boils down to a careful inspection of the original definition of looseness given by Murphy [11].

Let $\big(\mathbb{R}^{2n+1}, \xi_0 = \ker\big(dz - \sum_{j=1}^{n} y_j\, dx_j\big)\big)$ with coordinates $(\boldsymbol{x}, \boldsymbol{y}, z) = (x_1, \dots, x_n, y_1, \dots, y_n, z)$ be the standard contact space. The *Legendrian unknot* Λ_0 in $\mathbb{R}^{2n+1}$ can be given by the embedding

$$\mathbb{S}^n \hookrightarrow (\mathbb{R}^{2n+1}, \xi_0), (\boldsymbol{x}, s) \mapsto \big(\boldsymbol{x}, -s\boldsymbol{x}, \tfrac{1}{3}s^3\big). \tag{A-1}$$

By extension, any Legendrian Λ in a contact manifold is called a *Legendrian unknot* if there exists a Darboux chart containing Λ in its interior such that Λ agrees in the chart with Λ_0.

Let (M, ξ) be a contact manifold. We want to study Legendrians in M that look locally like product submanifolds in the following sense: Suppose that there is an open subset $U \subset M$ that is diffeomorphic to $U_M \times U_W$ where U_M is a manifold that carries a contact form α, and U_W is an open Liouville domain with Liouville form λ such that $\xi|_U = \ker(\alpha + \lambda)$. Then assume that a Legendrian Λ satisfies

$$\Lambda \cap \big(U_M \times U_W\big) = L \times N,$$

where L is a Legendrian in $\big(U_M, \alpha\big)$ and N is an exact Lagrangian in $\big(U_W, \lambda\big)$ with $\lambda|_{TN} = 0$. We do not assume in general that L or N are closed.

The key notion we want to study in this appendix is due to Murphy [11].

Definition Let Λ be a Legendrian in the manifold (M, ξ) that is locally in product form $L \times Z_\rho$ in a chart $\big(C \times V_\rho, \ker(\alpha_0 + \lambda_{\mathrm{can}})\big)$, where

- $C = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in (-1, 1)\}$ is a cube with side lengths 2 and standard contact form $\alpha_0 = dz - y\, dx$,
- $V_\rho = \{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2n-2} \mid \|\boldsymbol{q}\| < \rho,\ \|\boldsymbol{p}\| < \rho\}$ with Liouville form $\lambda_{\mathrm{can}} = -\sum_j p_j\, dq_j$,
- L is a properly embedded Legendrian arc whose front is a zig-zag and which is equal to the set $\{y = z = 0\}$ near the boundary,
- $Z_\rho = \big\{(\boldsymbol{q}, \boldsymbol{p}) \in V_\rho \mid \boldsymbol{p} = \mathbb{0}\big\}$.

We say that Λ is *loose* if $\rho > 1$.

Remark A.1 If we replace the cube C in the definition above with any cube of side lengths smaller than 2, then it can be seen with the argument in [11, Proposition 4.4] that the corresponding Legendrian is still loose.

The result we want to show in this appendix is the following proposition.

Proposition A.2 *There exists a $\rho_0 > 0$ (that is independent of the dimension of V_ρ) such that the Legendrian unknot Λ_0 is loose in every contact manifold*

$$\big(B(1) \times V_\rho, \ker(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}})\big)$$

for which $\rho > \rho_0$. Here, $\alpha_{\mathrm{OT}} = \cos(r)\,dz + r\sin(r)\,d\vartheta$ denotes the standard overtwisted contact form on $B(1)$; see also Section 3.

Proof By [4] or [7], see also the details in [12], the Legendrian unknot is in any overtwisted 3-manifold the stabilization of another Legendrian knot L_1. More precisely, let $\big(B(1), \alpha_{\mathrm{OT}}\big)$ be the cylindrical box surrounding an overtwisted disk described in Section 3. We can assume that there is a Darboux chart U_1 centered around a point of L_1 such that

- the restriction of α_{OT} agrees in the coordinates of the Darboux chart with the standard form $dz - y\,dx$,
- U_1 is a cube of size $\varepsilon_1 < 1$,
- $L_1 \cap U_1$ is the Legendrian arc $\{y = z = 0\}$.

We stabilize L_1 inside U_1 by adding a zig-zag. The resulting knot is then a Legendrian unknot L_0.

In particular, there exists a Darboux chart U_0 in $B(1)$ such that L_0 lies in standard position (A-1) inside U_0. The restriction $\alpha_{\mathrm{OT}}|_{U_0}$ with respect to the coordinates of U_0 will be of the form $e^{f(x,y,z)}\big(dz - y\,dx\big)$ for some smooth function f that probably cannot be chosen to be equal to 0. Nonetheless, we can assume that there are constants $c_0 > 0$, and $C_0 > 1$ such that $c_0 \le e^f \le C_0$.

Using the Darboux chart U_0 in $B(1)$, we can find in the product contact manifold

$$\big(B(1) \times V_\rho, \ker(\alpha_{\mathrm{OT}} + \lambda_{\mathrm{can}})\big)$$

a higher-dimensional Darboux chart $U_0 \times V_{\rho'}$ for $\rho' = \rho/C_0$ embedded by

$$U_0 \times V_{\rho'} \to U_0 \times V_\rho, (x, y, z; \boldsymbol{q}, \boldsymbol{p}) \mapsto \big(x, y, z; \boldsymbol{q}, e^{f(x,y,z)}\boldsymbol{p}\big).$$

If $\rho' > 1$, we can embed the Legendrian unknot Λ_0 into this Darboux chart using the standard map (A-1)

$$\mathbb{S}^{n+1} \hookrightarrow \big(U_0 \times V_{\rho'}, \xi_0\big), (x_0, \boldsymbol{x}, s) \mapsto \big(x_0, -sx_0, \tfrac{1}{3}s^3; \boldsymbol{x}, -s\boldsymbol{x}\big).$$

The intersection of Λ_0 with the slice $B(1) \times \{\mathbb{0}\}$ is precisely the unknot L_0 in U_0 that we had singled out in $B(1)$.

By slightly deforming Λ_0 close to $L_0 \times \{\mathbb{0}\}$ we can assume that Λ_0 is locally in product form $L_0 \times Z_\varepsilon$ for some small constant $\varepsilon > 0$. This can be easily seen using the front projection (even though "seeing" the front projection requires from dimension 7 on some experience). More explicitly, let $g\colon [0,1] \to [0,1]$ be a monotonous smooth function such that g is equal to 0 in a neighborhood of 0 and 1 in a neighborhood of 1. We can then consider the deformed sphere $\mathbb{S}_g^{n+1} \subset \mathbb{R}^{n+2}$ given by the equation

$$x_0^2 + s^2 + g\big(\boldsymbol{x}^2\big) \cdot \boldsymbol{x}^2 = 1.$$

Interpolating linearly between the equation of the round sphere and the deformed one, one sees that both are isotopic to each other.

We define a Legendrian embedding of $\mathbb{S}_g^{n+1}$ by

$$\mathbb{S}_g^{n+1} \hookrightarrow \big(U_0 \times V_{\rho'}, \xi_0\big), (x_0, \boldsymbol{x}, s) \mapsto \Big(x_0, -sx_0, \tfrac{1}{3}s^3;; \boldsymbol{x}, -s\boldsymbol{x}\big(g(\boldsymbol{x}^2) + \boldsymbol{x}^2 g'(\boldsymbol{x}^2)\big)\Big).$$

We denote this deformed sphere by Λ_0'. It is Legendrian isotopic to the initial Legendrian unknot, and it is composed of a cylindrical part $L_0 \times Z_\delta$ for small values of $\boldsymbol{x}$ in $U_0 \times V_\delta$.

Recall that the Darboux chart $U_1 \times V_{\rho'}$ had been embedded into the ball $B(1) \times V_\rho$ via the map $(x, y, z; \boldsymbol{q}, \boldsymbol{p}) \mapsto \big(x, y, z; \boldsymbol{q}, e^{f(x,y,z)}\boldsymbol{p}\big)$. By looking at the preimage of this embedding, it follows that Λ_0' also has a cylindrical segment in $B(1) \times V_\rho$. More explicitly, we have shown that after an isotopy the Legendrian unknot Λ_0 in $B(1) \times V_\rho$ has a cylindrical part of the form $L_0 \times Z_{\delta'}$ in the open ball $B(1) \times V_{\delta'}$ with $\delta' = \delta / c_0$.

We stretch out Λ_0' in the V_ρ-direction of $B(1) \times V_\rho$ using an isotopy $(x, y, z; \boldsymbol{q}, \boldsymbol{p}) \mapsto \big(x, y, z; e^t \boldsymbol{q}, e^{-t} \boldsymbol{p}\big)$ where the maximal size of $t \geq 0$ depends on the width of V_ρ. If there is enough space, then we can suppose that the cylindrical part found above expands to be of the form $L_0 \times Z_\eta$ in the open ball $B(1) \times V_\eta$ with $\eta > 1$.

If we now consider the Darboux chart $U_1 \subset B(1)$, we see that the intersection of the deformed Legendrian sphere with $U_1 \times V_\eta$ is $L \times Z_\eta$, where L is a Legendrian arc in U_1 whose front is a zig-zag, just as in the definition of looseness. If $\eta > 1$ and if U_1 is a cube of size smaller than 1 (see Remark A.1), then the deformed unknot and thus also Λ_0 are loose. □

Appendix B Contact germ along a hypersurface

A folklore result states that a hypersurface in a contact manifold determines the germ of the contact structure surrounding it. Not having found a proof for dimension > 3 in the literature we have decided to add it here.

Proposition B.1 *Let (M_0, ξ_0) and (M_1, ξ_1) be two $(2n+1)$-dimensional contact manifolds, and let Σ be a (not necessarily closed) $2n$-dimensional manifold. Assume that there are two embeddings*

$$\iota_0 \colon \Sigma \hookrightarrow M_0 \quad \text{and} \quad \iota_1 \colon \Sigma \hookrightarrow M_1$$

such that the singular distributions $\mathcal{D}_0 = (D\iota_0)^{-1}(\xi_0)$ and $\mathcal{D}_1 = (D\iota_1)^{-1}(\xi_1)$ agree.

Then there exist a neighborhood $U_0 \subset M_0$ of $\iota_0(\Sigma)$, a neighborhood $U_1 \subset M_1$ of $\iota_1(\Sigma)$, and a contactomorphism

$$\Phi\colon (U_0, \xi_0) \to (U_1, \xi_1)$$

such that $\Phi \circ \iota_0 = \iota_1$.

Remark B.2 To be able to apply Proposition B.1 to a hypersurface Σ with nonempty boundary, one needs to attach a small collar along $\partial\Sigma$, and extend the embeddings ι_0 and ι_1 in such a way that the singular distributions $\mathcal{D}_0 = (D\iota_0)^{-1}(\xi_0)$ and $\mathcal{D}_1 = (D\iota_1)^{-1}(\xi_1)$ agree.

We split the proof of Proposition B.1 into several lemmas. The first one is due to Giroux, but we learned about it from [9].

Lemma B.3 *Let Σ be a (not necessarily closed) manifold carrying a (cooriented) singular distributions $\mathcal{D}$ that is given as the kernel of a 1-form β such that $d\beta$ does not vanish at the singular points of $\mathcal{D}$; that is, at the points where $\beta = 0$.*

If β' is any other 1-form such that $\mathcal{D} = \ker \beta'$ inducing the same coorientation, and such that $d\beta'$ does not vanish either at the singular points of $\mathcal{D}$, then there exists a smooth positive function $f : \Sigma \to \,]0, \infty[$ such that

$$\beta = f \cdot \beta'.$$

Proof Denote the set of all regular points of the distributions $\mathcal{D}$ by $U_{\mathrm{reg}} = \{p \in \Sigma \mid \mathcal{D}_p \neq T_p\Sigma\}$. On U_{reg}, we can simply define f to be the quotient $\beta(X)/\beta'(X)$, where X is any vector field on U_{reg} that is transverse to $\mathcal{D}$. We are thus left with studying the singular points $p \notin U_{\mathrm{reg}}$ of $\mathcal{D}$, where β and β' both vanish, and proving that f extends to a nonvanishing smooth function.

Use a coordinate chart for Σ centered at $p \in \Sigma \setminus U_{\mathrm{reg}}$ with coordinates $\boldsymbol{x} = (x_1, \dots, x_n)$. We can then write

$$\beta = g_1\, dx_1 + \dots + g_n\, dx_n \quad \text{and} \quad \beta' = g_1'\, dx_1 + \dots + g_n'\, dx_n$$

with functions $g_1, \dots, g_n$ and $g_1', \dots, g_n'$ such that all g_j and all g_j' vanish at the origin. In fact for each j, the two functions g_j and g_j' vanish precisely on the same subset. By our assumption $d\beta' \neq 0$ at p, so that we can assume after possibly permuting the coordinates that $(\partial g_1'/\partial x_2)(\mathbb{0}) \neq 0$.

We will now show that f extends in the chart smoothly to a neighborhood of the origin such that $g_1(\boldsymbol{x}) = f(\boldsymbol{x}) \cdot g_1'(\boldsymbol{x})$. Note that $\{\boldsymbol{x} \mid g_1'(\boldsymbol{x}) \neq 0\}$ is a subset of U_{reg} so that f is a well-defined function on this subset. The condition $(\partial g_1'/\partial x_2)(\mathbb{0}) \neq 0$ allows us to apply the implicit function theorem to find a new set of coordinates $\boldsymbol{y} = (y_1, \dots, y_n)$ for which g_1' simplifies to $g_1'(\boldsymbol{y}) = y_2$. In this new chart, we obtain that f is defined in particular for all points $\{y_2 \neq 0\} \subset U_{\mathrm{reg}}$.

Consider now the function g_1 represented with respect to the $\boldsymbol{y}$-coordinates. It also vanishes precisely along the hyperplane $\{y_2 = 0\}$ so that there exists a smooth functions $\widetilde{g}_1$ allowing us to write g_1 as

$$g_1(\boldsymbol{y}) = y_2 \widetilde{g}_1(\boldsymbol{y});$$

see, for example, [10, Lemma 2.1]. Using this representation we see that $f(y) = \widetilde{g}_1(y)$ extends to a smooth function on the whole chart so that it obviously satisfies the equation $g_1 = f \cdot g_1'$.

In particular, since U_{reg} is dense in Σ the continuous extension of f is unique and does not depend on our choice of charts. This way, f can be defined smoothly on all of Σ, and it satisfies $\beta = f \cdot \beta'$ on Σ.

It remains to prove that f does not vanish anywhere, but this is clear because if f ever vanished at a singular point p of $\mathcal{D}$, we would find from $\beta = f \cdot \beta'$ that $d\beta = 0$ at p — contrary to our assumption that $d\beta \neq 0$ along the singular set of $\mathcal{D}$. □

Lemma B.4 *Let (M_0, ξ_0) and (M_1, ξ_1) be two $(2n+1)$-dimensional contact manifolds with contact forms α_0 and α_1, respectively, and let Σ be a (not necessarily closed) manifold of dimension $2n$.*

Suppose that there are two embeddings

$$\iota_0 \colon \Sigma \hookrightarrow (M_0, \xi_0) \quad \textit{and} \quad \iota_1 \colon \Sigma \hookrightarrow (M_1, \xi_1)$$

of Σ into M_0 and M_1 such that $\iota_0^*\alpha_0 = \iota_1^*\alpha_1$.

Then, there is a bundle isomorphism

$$\Phi \colon TM_0|_\Sigma \to TM_1|_\Sigma$$

such that:

(i) $\Phi|_{T\Sigma} = \mathrm{id}_{T\Sigma}$ (*we identify here, and in the rest of the proof, $T\Sigma$ with the tangent spaces of $\iota_0(\Sigma)$ and $\iota_1(\Sigma)$*).

(ii) $\alpha_1 \circ \Phi = \alpha_0$ *on* $TM_0|_\Sigma$.

(iii) *The linear interpolation $(1-\tau)\, d\alpha_0 + \tau\, (d\alpha_1 \circ \Phi)$ is for every $\tau \in [0,1]$ a symplectic form on $\xi_0|_\Sigma$.*

Proof Denote the 1-form $\iota_0^*\alpha_0 = \iota_1^*\alpha_1$ by β. To construct the desired bundle isomorphism Φ, we distinguish two types of subsets of Σ: Define

$$U_{\mathrm{reg}} = \{p \in \Sigma \mid \beta_p \neq 0\} \quad \text{and} \quad U_{\mathrm{symp}} = \{p \in \Sigma \mid d\beta_p^n \neq 0\}.$$

Both sets are open and their union covers all of Σ, because $d\beta = \iota_j^* d\alpha_j$ is at every point $p \in \Sigma$ where $T_p\Sigma = \xi_j(p)$ a maximally nondegenerate form on $T_p\Sigma$; that is, $d\beta$ is a symplectic form on $T_p\Sigma$ at every point p where β vanishes.

We construct now separate bundle isomorphisms over U_{reg} and over U_{symp} that we then glue together using a partition of unity.

Over U_{symp}, we can decompose TM_j as $T\Sigma \oplus \mathrm{span}(R_j)$, where R_j is the Reeb vector field of α_j. This allows us to define a first bundle isomorphism

$$\Phi_{\mathrm{symp}} \colon TM_0|_{U_{\mathrm{symp}}} \to TM_1|_{U_{\mathrm{symp}}}$$

by $\Phi_{\mathrm{symp}}(v + cR_0) = v + cR_1$ for every $v \in T\Sigma|_{U_{\mathrm{symp}}}$ and every $c \in \mathbb{R}$. It is easy to verify that α_0 and $\alpha_1 \circ \Phi_{\mathrm{symp}}$ agree on $TM_0|_{U_{\mathrm{symp}}}$. Furthermore, $d\alpha_0$ and $d\alpha_1 \circ \Phi_{\mathrm{symp}}$ also agree, because for any pair of vectors $v, v' \in T\Sigma$, we have $d\alpha_j(v, v') = d\beta(v, v')$ on one hand, and $d\alpha_j(R_j, v) = 0$ on the other, for both $j = 0, 1$.

To construct a bundle isomorphism over U_{reg}, we define the *characteristic foliation* $\mathcal{F}$ on Σ. It is characterized over U_{reg} as the (singular) subdistribution of $\ker\beta = T\Sigma \cap \xi$ on which $d\beta|_{\ker\beta}$ vanishes. A dimension count shows that $\mathcal{F}$ is of dimension 1.

Choose a volume form $d\,\mathrm{vol}_\Sigma$ on Σ, and let X be the vector field determined by the equation

$$\iota_X d\,\mathrm{vol}_\Sigma = \beta \wedge (d\beta)^{n-1}.$$

Since X only vanishes at points where β vanishes, it follows that X is everywhere on U_{reg} nonsingular, and it is easy to convince oneself that the characteristic foliation is generated by X.

Choosing compatible complex structures J_0 on $(\xi_0, d\alpha_0)$ and J_1 on $(\xi_1, d\alpha_1)$, we define two vector fields $Y_0 = J_0 \cdot X$ and $Y_1 = J_1 \cdot X$ along $\iota_0(\Sigma)$ and $\iota_1(\Sigma)$, respectively. These vector fields are everywhere over U_{reg} transverse to Σ and they lie in the kernel of α_j. This way, we can split the tangent bundles as

$$TM_j\big|_{U_{\mathrm{reg}}} = T\Sigma|_{U_{\mathrm{reg}}} \oplus \mathrm{span}(Y_j)\big|_{U_{\mathrm{reg}}},$$

and use these decompositions to define the bundle isomorphism

$$\Phi_{\mathrm{reg}} \colon TM_0|_{U_{\mathrm{reg}}} \to TM_1|_{U_{\mathrm{reg}}}$$

by $\Phi_{\mathrm{reg}}\big(v + cY_0\big) = v + cY_1$ for every $v \in T\Sigma|_{U_{\mathrm{reg}}}$ and every $c \in \mathbb{R}$. Again, we easily check that $\alpha_1 \circ \Phi_{\mathrm{reg}}$ agrees on $TM_0|_{U_{\mathrm{reg}}}$ with α_0 so that $\Phi_{\mathrm{reg}}\big(\xi_0|_{U_{\mathrm{reg}}}\big) = \xi_1|_{U_{\mathrm{reg}}}$.

To understand the interpolation between $d\alpha_0$ and $d\alpha_1 \circ \Phi_{\mathrm{reg}}$, choose at a point $p \in U_{\mathrm{reg}}$ a basis of $\xi_0(p)$ of the form $v_1, \dots, v_{2n-2}, X(p), Y_0(p)$, where the v_j all lie in $\ker\beta$ and are complementary to $X(p)$. Assume they are ordered in such a way that

$$d\alpha_0^{n-1}(v_1, \dots, v_{2n-2}) = d\beta^{n-1}(v_1, \dots, v_{2n-2}) = d\alpha_1^{n-1}(v_1, \dots, v_{2n-2}) > 0.$$

Note that $d\alpha_0(X, \cdot)$ and $d\alpha_1(X, \cdot)$ vanish on all the vectors $v_1, \dots, v_{2n-2}, X$.

Define $d\alpha_\tau = (1 - \tau)\, d\alpha_0 + \tau\, (d\alpha_1 \circ \Phi_{\mathrm{reg}})$ for any $\tau \in [0, 1]$. Then we compute for all $\tau \in [0, 1]$ that

$$d\alpha_\tau^n(v_1, \dots, v_{2n-2}, X, Y_0) = n\, d\alpha_\tau(X, Y_0) \cdot d\alpha_\tau^{n-1}(v_1, \dots, v_{2n-2}) > 0,$$

because $d\alpha_\tau(X, Y_0) = (1 - \tau)\, d\alpha_0(X, J_0 X) + \tau\, d\alpha_1(X, J_1 X) > 0$.

We glue now Φ_{reg} and Φ_{symp} to produce a global bundle isomorphism. Choose a smooth function $\rho \colon \Sigma \to [0, 1]$ with support in U_{reg} such that $1 - \rho$ has support in U_{symp} so that ρ and $1 - \rho$ form a partition of unity subordinate to $\big\{U_{\mathrm{reg}}, U_{\mathrm{symp}}\big\}$. Define a bundle homomorphism

$$\Phi \colon TM_0|_\Sigma \to TM_1|_\Sigma$$

by mapping a vector $v \in T_p M_0$ at a point $p \in \Sigma$ to $\Phi(v) = \rho(p) \cdot \Phi_{\mathrm{reg}}(v) + (1-\rho(p)) \cdot \Phi_{\mathrm{symp}}(v)$. It is obvious that Φ is a bundle homomorphism such that $\Phi|_{T\Sigma} = \mathrm{id}_{T\Sigma}$ and such that $\alpha_0 = \alpha_1 \circ \Phi$ on $TM_0|_\Sigma$ proving properties (i) and (ii) in the lemma.

It remains to verify property (iii). Define the interpolation $d\alpha_\tau := (1-\tau)\, d\alpha_0 + \tau\, (d\alpha_1 \circ \Phi)$ for $\tau \in [0,1]$. Since Φ agrees with Φ_{symp} at the points where $\beta = 0$, we obtain that $d\alpha_\tau = d\alpha_0$ is nondegenerate at any such point. We study now the desired property at points at which $\beta \neq 0$ and thus $X \neq 0$.

Since $d\alpha_\tau|_{T\Sigma} = d\beta$ is independent of τ, we see that $d\alpha_\tau(X,\cdot)$ vanishes on every vector that lies in $\ker\beta$. Using the same basis chosen above with $Y_0 = J_0 X$ and $Y_1 = J_1 X$, it follows that the sign of $d\alpha_\tau^n(v_1,\dots,v_{2n-2},X,Y_0) = n\, d\alpha_\tau(X,Y_0) \cdot d\beta^{n-1}(v_1,\dots,v_{2n-2})$ only depends on the sign of the term $d\alpha_\tau(X,Y_0)$.

For this term we obtain $d\alpha_\tau(X,Y_0) = (1-\tau)\, d\alpha_0(X,J_0X) + \tau\, d\alpha_1\bigl(X,\Phi(Y_0)\bigr)$. The first term is clearly positive, and for the second one write $d\alpha_1\bigl(X,\Phi(Y_0)\bigr) = \rho\, d\alpha_1(X,J_1X) + (1-\rho)\, d\alpha_1\bigl(X,\Phi_{\mathrm{symp}}(Y_0)\bigr)$, where the first term is again positive. Recall that $d\alpha_0 = d\alpha_1 \circ \Phi_{\mathrm{symp}}$ so that we can simplify the second term as $d\alpha_1\bigl(X,\Phi_{\mathrm{symp}}(Y_0)\bigr) = d\alpha_1\bigl(\Phi_{\mathrm{symp}}(X),\Phi_{\mathrm{symp}}(Y_0)\bigr) = \bigl(d\alpha_1 \circ \Phi_{\mathrm{symp}}\bigr)(X,Y_0) = d\alpha_0(X,Y_0)$. Thus $d\alpha_\tau(X,Y_0) = (1-\tau)\, d\alpha_0(X,Y_0) + \tau\bigl(\rho\, d\alpha_1(X,Y_1) + (1-\rho)\, d\alpha_0(X,Y_0)\bigr)$ is positive as a convex combination of positive terms, and we have shown property (iii). □

Proof of Proposition B.1 Let α_0 be a contact form for ξ_0, and let α_1 be a contact form for ξ_1. By Lemma B.3 there is a smooth function $f\colon \Sigma \to \mathbb{R}_{>0}$ such that $\iota_0^*\alpha_0 = f \cdot \iota_1^*\alpha_1$. We would like to extend $f \circ \iota_1^{-1}$ to all of M_1 to normalize α_1 globally; in general though, if Σ is not closed, this might be impossible.

Denote the normal bundle of $\iota_1(\Sigma)$ in M_1 by $\nu_1\Sigma \xrightarrow{\pi} \Sigma$, and recall that there is a tubular neighborhood U_1 of $\iota_1(\Sigma)$ that is diffeomorphic to a neighborhood V_1 of the 0-section in $\nu_1\Sigma$ (of course V_1 will generally not have uniform radius in the fiber directions, when Σ is not closed). The function $f \circ \pi$ is a smooth positive function on $\nu_1\Sigma$. We will replace M_1 by the open subset U_1, and use $f \circ \pi$ to rescale α_1 on U_1 so that we can assume that $\iota_0^*\alpha_0 = \iota_1^*\alpha_1$. This allows us to apply Lemma B.4 to obtain a bundle isomorphism Φ between $TM_0|_{\iota_0(\Sigma)}$ and $TM_1|_{\iota_1(\Sigma)}$.

Let U_0 be a tubular neighborhood of $\iota_0(\Sigma)$ in M_0 such that the exponential map $\exp_0$ (with respect to some Riemannian metric) defines a diffeomorphism $\exp_0\colon V_0 \to U_0$, where V_0 is a neighborhood of the 0-section of the normal bundle of Σ in M_0. Similarly, let $\exp_1$ be the exponential map on M_1. By suitably reducing the size of U_0 and U_1, we can assume that

$$\Psi := \exp_1 \circ \Phi \circ \exp_0^{-1}\colon U_0 \to U_1$$

is a diffeomorphism. To simplify our setup, pull-back α_1 to U_0, and work in the fixed ambient manifold U_0. For simplicity we also write α_1 for its pull-back. Then it follows that U_0 contains the submanifold Σ, and carries two contact structures given by contact forms α_0 and α_1 such that α_0 and α_1 agree at all points of Σ, and such that the linear interpolation of $d\alpha_0$ and $d\alpha_1$ is a path of symplectic structures on $\xi_0|_\Sigma = \xi_1|_\Sigma$.

The rest of the proof is an application of the Moser trick: Clearly the interpolation $\alpha_\tau := (1-\tau)\alpha_0 + \tau\alpha_1$ satisfies along Σ for every $\tau \in [0,1]$ the contact condition. There is thus a small neighborhood of Σ in U_0 on which all α_τ are contact forms.

As in the standard proof of Gray stability, we define a vector field X_τ on this neighborhood by the equations

$$\alpha_\tau(X_\tau) = 0 \quad \text{and} \quad d\alpha_\tau(X_\tau, \cdot) = f_\tau \alpha_\tau - \dot{\alpha}_\tau$$

with $f_\tau := \dot{\alpha}_\tau(R_\tau)$, where R_τ is the Reeb field of α_τ. Note that the right-hand side of the second equation vanishes along Σ, thus it follows that $X_\tau(p) = 0$ at every $p \in \Sigma$. By reducing to a smaller neighborhood of Σ in U_0, the flow of X_τ will be defined up to time 1 giving a contact isotopy between ξ_0 and ξ_1.

Composing this isotopy with Ψ, we find the desired contactomorphism. □

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