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Endomorphisms of Artin groups of type D

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We determine a classification of the endomorphisms of the Artin group $A[D_n]$ of type D_n for $n \geq 6$. In particular we determine its automorphism group and its outer automorphism group. We also determine a classification of the homomorphisms from $A[D_n]$ to the Artin group $A[A_{n-1}]$ of type A_{n-1} and a classification of the homomorphisms from $A[A_{n-1}]$ to $A[D_n]$ for $n \geq 6$. We show that any endomorphism of the quotient $A[D_n]/Z(A[D_n])$ lifts to an endomorphism of $A[D_n]$ for $n \geq 4$. We deduce a classification of the endomorphisms of $A[D_n]/Z(A[D_n])$, we determine the automorphism and outer automorphism groups of $A[D_n]/Z(A[D_n])$, and we show that $A[D_n]/Z(A[D_n])$ is co-Hopfian for $n \geq 6$. The results are algebraic in nature but the proofs are based on topological arguments (curves on surfaces and mapping class groups).

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1 Introduction

Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S, with coefficients in $\mathbb{N} \cup \{\infty\}$, such that $m_{s,s} = 1$ for all $s \in S$ and $m_{s,t} = m_{t,s} \geq 2$ for all $s, t \in S$ with $s \neq t$. Such a matrix is usually represented by a labeled graph Γ, called a *Coxeter graph*, defined as follows. The set of vertices of Γ is S. Two vertices $s, t \in S$ are connected by an edge if $m_{s,t} \geq 3$, and this edge is labeled with $m_{s,t}$ if $m_{s,t} \geq 4$.

If a and b are two letters and m is an integer ≥ 2, then we denote by $\Pi(a, b, m)$ the word $aba \cdots$ of length m. In other words $\Pi(a, b, m) = (ab)^{m/2}$ if m is even and $\Pi(a, b, m) = (ab)^{(m-1)/2} a$ if m is odd. Let Γ be a Coxeter graph and let $M = (m_{s,t})_{s,t \in S}$ be its Coxeter matrix. With Γ we associate a group $A[\Gamma]$, called the *Artin group* of Γ, defined by the following presentation:

$$A[\Gamma] = \langle S \mid \Pi(s, t, m_{s,t}) = \Pi(t, s, m_{s,t}) \text{ for } s, t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle.$$

The *Coxeter group* of Γ, denoted by $W[\Gamma]$, is the quotient of $A[\Gamma]$ by the relations $s^2 = 1$ for $s \in S$.

Despite the popularity of Artin groups, little is known on their automorphisms and even less on their endomorphisms. The most emblematic cases are the braid groups and the right-angled Artin groups. Recall that the *braid group* on $n+1$ strands is the Artin group $A[A_n]$ where A_n is the Coxeter graph depicted in Figure 1, and an Artin group $A[\Gamma]$ is called a *right-angled Artin group* if $m_{s,t} \in \{2, \infty\}$ for all

Figure 1: The Coxeter graph A_n.

$s, t \in S$ with $s \neq t$. The automorphism group of $A[A_n]$ was determined by Dyer and Grossman [26] and the set of its endomorphisms by Castel [12] for $n \geq 5$, by Chen, Kordek and Margalit [17] for $n \geq 3$ and by Orevkov [35] for $n \geq 2$ (see also Bell and Margalit [2] and Kordek and Margalit [31]). On the other hand there are many articles studying automorphism groups of right-angled Artin groups (see Charney and Vogtmann [15; 16], Day [23; 24], Laurence [33] and Bregman, Charney and Vogtmann [8] for example), but almost nothing is known on endomorphisms of these groups.

Apart from these two families little is known on automorphisms of Artin groups. The automorphism groups of two-generator Artin groups were determined by Gilbert, Howie, Metaftsis and Raptis [29], the automorphism groups of the Artin groups of type B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$ were determined by Charney and Crisp [14], the automorphisms groups of some 2-dimensional Artin groups were determined by Crisp [20] and by An and Cho [1], the automorphism groups of large-type free-of-infinity Artin groups were determined by Vaskou [43], and the automorphism group of $A[D_4]$ was determined by Soroko [41]. On the other hand, as far as we know the set of endomorphisms of an Artin group is not determined for any Artin group except for those of type A_n.

Recall that an Artin group $A[\Gamma]$ is of *spherical type* if $W[\Gamma]$ is finite. The study of spherical-type Artin groups began in the early 1970s with works by Brieskorn [9; 10], Brieskorn and Saito [11] and Deligne [25], which marked in a way the beginning of the theory of Artin groups. This family, and that of right-angled Artin groups, are the two most-studied and best-understood families of Artin groups and, obviously, any question on Artin groups first arises for Artin groups of spherical type and for right-angled Artin groups. Here we are interested in Artin groups of spherical type, and more particularly in those of type D_n.

An Artin group $A[\Gamma]$ is called *irreducible* if Γ is connected. If $\Gamma_1, \dots, \Gamma_l$ are the connected components of Γ, then $A[\Gamma] = A[\Gamma_1] \times \cdots \times A[\Gamma_l]$ and $W[\Gamma] = W[\Gamma_1] \times \cdots \times W[\Gamma_l]$. In particular $A[\Gamma]$ is of spherical type if and only if $A[\Gamma_i]$ is of spherical type for all $i \in \{1, \dots, l\}$. So to classify Artin groups of spherical type it suffices to classify those which are irreducible. Finite irreducible Coxeter groups, and hence irreducible Artin groups of spherical type, were classified by Coxeter [18; 19]. There are four infinite families, A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$ and $I_2(m)$ $(m \geq 5)$, and six "sporadic" groups, E_6, E_7, E_8, F_4, H_3 and H_4. As mentioned above, the automorphism group of $A[\Gamma]$ for Γ of type A_n $(n \geq 1)$, B_n $(n \geq 2)$ and $I_2(m)$ $(m \geq 5)$ is known. The next step is therefore to understand the automorphism group of $A[D_n]$ for $n \geq 5$ (the case $\Gamma = D_4$ is known by Soroko [41]). The Coxeter graph D_n is illustrated in Figure 2.

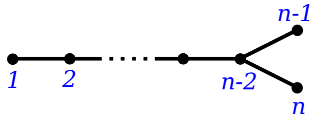

Figure 2: The Coxeter graph D_n.

Here we determine a complete and precise classification of the endomorphisms of $A[D_n]$ for $n \geq 6$ (see Theorem 2.3). In particular we determine the automorphism group and the outer automorphism group of $A[D_n]$ for $n \geq 6$ (see Corollary 2.6). We also determine a complete and precise classification of the homomorphisms from $A[D_n]$ to $A[A_{n-1}]$ (see Theorem 2.1) and a complete and precise classification of the homomorphisms from $A[A_{n-1}]$ to $A[D_n]$ (see Theorem 2.2). Note that all these results were announced but not proved in Castel [13]; actually the proofs turn out to be much more difficult than the first author thought when he announced them. Note also that our techniques cannot be used to treat the cases $n = 4$ and $n = 5$. In particular we do not know how to determine $\mathrm{Aut}(A[D_5])$.

From our main result we deduce a classification of the endomorphisms of the group $A[D_n]/Z(A[D_n])$ for $n \geq 6$, where $Z(A[D_n])$ denotes the center of $A[D_n]$ (see Theorem 2.8). Then we determine the automorphism group and the outer automorphism group of $A[D_n]/Z(A[D_n])$ (see Corollary 2.10), and we show that $A[D_n]/Z(A[D_n])$ is co-Hopfian (see Corollary 2.11). These results follow from Theorem 2.3 and Proposition 2.7, which states that any endomorphism of $A[D_n]/Z(A[D_n])$ lifts to an endomorphism of $A[D_n]$. Such results were previously known for braid groups, that is, Artin groups of type A_n (see Bell and Margalit [2]). Note that the application of our main result to the study of $A[D_n]/Z(A[D_n])$ was not present in an earlier version of the paper. It was suggested to us by the referee, for which we extend our warm thanks.

A *geometric representation* of an Artin group is a homomorphism from the group to a mapping class group (see Section 3 for more details). In order to achieve our goals we make a study of a particular geometric representation of $A[D_n]$ previously introduced by Perron and Vannier [40] with one boundary component replaced by a puncture. This geometric representation will be the key tool for many of our proofs. Overall, although the results of the paper are algebraic in nature, the proofs are mostly based on topological arguments (on curves on surfaces and mapping class groups).

The paper is organized as follows. In Section 2 we give the main definitions and precise statements of the main results. Section 3 is dedicated to the study of some geometric representations of Artin groups of type A_n and type D_n. In Section 4 we determine the homomorphisms from $A[D_n]$ to $A[A_{n-1}]$, in Section 5 we determine the homomorphisms from $A[A_{n-1}]$ to $A[D_n]$, and in Section 6 we determine the endomorphisms of $A[D_n]$. In Section 7 we determine the endomorphisms of $A[D_n]/Z(A[D_n])$.

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2 Definitions and statements

For $n \geq 4$ we denote by $s_1, \dots, s_{n-1}$ the standard generators of $A[A_{n-1}]$ numbered as in Figure 1 and by $t_1, \dots, t_n$ the standard generators of $A[D_n]$ numbered as in Figure 2.

Let Γ be a Coxeter graph. For $X \subset S$ we denote by $A_X = A_X[\Gamma]$ the subgroup of $A = A[\Gamma]$ generated by X, by $W_X = W_X[\Gamma]$ the subgroup of $W = W[\Gamma]$ generated by X, and by Γ_X the full subgraph of Γ spanned by X. We know from van der Lek [34] that A_X is the Artin group of Γ_X and from Bourbaki [7] that W_X is the Coxeter group of Γ_X. A subgroup of the form A_X is called a *standard parabolic subgroup* of A and a subgroup of the form W_X is called a *standard parabolic subgroup* of W.

For $w \in W$ we denote by $\lg(w)$ the word length of w with respect to S. A *reduced expression* for w is an expression $w = s_1 s_2 \cdots s_l$ of minimal length, that is, such that $l = \lg(w)$. Let $\omega\colon A \to W$ be the natural epimorphism which sends s to s for all $s \in S$. This epimorphism has a natural set-section $\tau\colon W \to A$ defined as follows. Let $w \in W$ and let $w = s_1 s_2 \cdots s_l$ be a reduced expression for w. Then $\tau(w) = s_1 s_2 \cdots s_l \in A$. We know from Tits [42] that the definition of $\tau(w)$ does not depend on the choice of its reduced expression.

Assume Γ is of spherical type. Then W has a unique element of maximal length, denoted by w_S, which satisfies $w_S^2 = 1$ and $w_S S w_S = S$. The *Garside element* of A is defined to be $\Delta = \Delta[\Gamma] = \tau(w_S)$. We know that $\Delta S \Delta^{-1} = S$ and, if Γ is connected, then the center $Z(A)$ of A is an infinite cyclic group generated by either Δ or Δ^2 (see Brieskorn and Saito [11]). For $X \subset S$ we denote by w_X the element of maximal length in W_X and by $\Delta_X = \Delta_X[\Gamma] = \tau(w_X)$ the Garside element of A_X.

If $\Gamma = A_{n-1}$, then

$$\Delta = (s_{n-1} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1} s_{n-2}) s_{n-1},$$

$\Delta s_i \Delta^{-1} = s_{n-i}$ for all $1 \le i \le n-1$ and $Z(A)$ is generated by Δ^2. If $\Gamma = D_n$, then

$$\Delta = (t_1 \cdots t_{n-2} t_{n-1} t_n t_{n-2} \cdots t_1)(t_2 \cdots t_{n-2} t_{n-1} t_n t_{n-2} \cdots t_2) \cdots (t_{n-2} t_{n-1} t_n t_{n-2})(t_{n-1} t_n).$$

If n is even, then $\Delta t_i \Delta^{-1} = t_i$ for all $1 \le i \le n$ and $Z(A)$ is generated by Δ. If n is odd, then $\Delta t_i \Delta^{-1} = t_i$ for all $1 \le i \le n-2$, $\Delta t_{n-1} \Delta^{-1} = t_n$, $\Delta t_n \Delta^{-1} = t_{n-1}$ and $Z(A)$ is generated by Δ^2.

If G is a group and $g \in G$, then we denote by $\mathrm{ad}_g\colon G \to G$, $h \mapsto g h g^{-1}$, the conjugation map by g. We say that two homomorphisms $\varphi_1, \varphi_2\colon G \to H$ are *conjugate* if there exists $h \in H$ such that $\varphi_2 = \mathrm{ad}_h \circ \varphi_1$.

A homomorphism $\varphi\colon G \to H$ is called *abelian* if its image is an abelian subgroup of H. A homomorphism $\varphi\colon G \to H$ is called *cyclic* if its image is a cyclic subgroup of H. If $G = A[A_{n-1}]$, then $\varphi\colon A[A_{n-1}] \to H$ is abelian if and only if it is cyclic, if and only if there exists $h \in H$ such that $\varphi(s_i) = h$ for all $1 \le i \le n-1$. Similarly, if $G = A[D_n]$, then $\varphi\colon A[D_n] \to H$ is abelian if and only if it is cyclic, if and only if there exists $h \in H$ such that $\varphi(t_i) = h$ for all $1 \le i \le n$.

Two automorphisms $\zeta, \chi \in \mathrm{Aut}(A[D_n])$ play a central role in our study. These are defined by

$$\zeta(t_i) = t_i \quad \text{for } 1 \le i \le n-2, \qquad \zeta(t_{n-1}) = t_n, \qquad \zeta(t_n) = t_{n-1}, \qquad \chi(t_i) = t_i^{-1} \quad \text{for } 1 \le i \le n.$$

Both are of order 2 and commute, and hence they generate a subgroup of $\mathrm{Aut}(A[D_n])$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If n is odd, then ζ is the conjugation map by $\Delta = \Delta[D_n]$. On the other hand, if n is even, then ζ is not an inner automorphism (see Paris [36]). The automorphism χ is never inner.

Two other homomorphisms play an important role in our study. The first, $\pi\colon A[D_n] \to A[A_{n-1}]$, is defined by

$$\pi(t_i) = s_i \quad \text{for } 1 \le i \le n-2, \qquad \pi(t_{n-1}) = \pi(t_n) = s_{n-1}.$$

The second, $\iota\colon A[A_{n-1}] \to A[D_n]$, is defined by

$$\iota(s_i) = t_i \quad \text{for } 1 \le i \le n-1.$$

Observe that $\pi \circ \iota = \mathrm{id}_{A[A_{n-1}]}$, and hence π is surjective, ι is injective and $A[D_n] \simeq \mathrm{Ker}(\pi) \rtimes A[A_{n-1}]$. We refer to Crisp and Paris [21] for a detailed study on this decomposition of $A[D_n]$ as a semidirect product.

Let $n \ge 4$. For $p \in \mathbb{Z}$ we define a homomorphism $\alpha_p\colon A[D_n] \to A[A_{n-1}]$ by

$$\alpha_p(t_i) = s_i \Delta^{2p} \quad \text{for } 1 \le i \le n-2, \qquad \alpha_p(t_{n-1}) = \alpha_p(t_n) = s_{n-1}\Delta^{2p},$$

where $\Delta = \Delta[A_{n-1}]$ is the Garside element of $A[A_{n-1}]$. Note that $\alpha_0 = \pi$.

Set $Y = \{t_1, \dots, t_{n-1}\}$. For $p, q \in \mathbb{Z}$ we define a homomorphism $\beta_{p,q}\colon A[A_{n-1}] \to A[D_n]$ by

$$\beta_{p,q}(s_i) = t_i \Delta_Y^{2p} \Delta^{\kappa q} \quad \text{for } 1 \le i \le n-1,$$

where $\Delta = \Delta[D_n]$ is the Garside element of $A[D_n]$, $\Delta_Y = \Delta_Y[D_n]$, $\kappa = 2$ if n is odd, and $\kappa = 1$ if n is even. Note that $\beta_{0,0} = \iota$. Note also that, by Paris [36, Theorem 1.1], the centralizer of Y in $A[D_n]$ is the free abelian group of rank 2 generated by Δ_Y^2 and Δ^κ.

For $p \in \mathbb{Z}$ we define the homomorphism $\gamma_p\colon A[D_n] \to A[D_n]$ by

$$\gamma_p(t_i) = t_i \Delta^{\kappa p} \quad \text{for } 1 \le i \le n,$$

where $\Delta = \Delta[D_n]$ is the Garside element of $A[D_n]$, $\kappa = 2$ if n is odd, and $\kappa = 1$ if n is even. Note that $\gamma_0 = \mathrm{id}$.

Concerning $A[A_{n-1}]$, we define an automorphism $\bar\chi\colon A[A_{n-1}] \to A[A_{n-1}]$ by

$$\bar\chi(s_i) = s_i^{-1} \quad \text{for } 1 \le i \le n-1,$$

and for $p \in \mathbb{Z}$ we define an endomorphism $\bar\gamma_p\colon A[A_{n-1}] \to A[A_{n-1}]$ by

$$\bar\gamma_p(s_i) = s_i \Delta^{2p} \quad \text{for } 1 \le i \le n-1,$$

where Δ is the Garside element of $A[A_{n-1}]$.

The main results of this paper are the following.

Theorem 2.1 *Let $n \ge 5$. Let $\varphi\colon A[D_n] \to A[A_{n-1}]$ be a homomorphism. Then up to conjugation we have one of the following two possibilities:*

(1) *φ is cyclic.*

(2) *There exist $\psi \in \langle \chi \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \alpha_p \circ \psi$.*

Theorem 2.2 *Let $n \geq 6$. Let $\varphi\colon A[A_{n-1}] \to A[D_n]$ be a homomorphism. Then up to conjugation we have one of the following two possibilities*:

(1) *φ is cyclic.*

(2) *There exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q}$.*

Theorem 2.3 *Let $n \geq 6$. Let $\varphi\colon A[D_n] \to A[D_n]$ be a homomorphism. Then up to conjugation we have one of the following three possibilities*:

(1) *φ is cyclic.*

(2) *There exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q} \circ \pi$.*

(3) *There exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \psi \circ \gamma_p$.*

From Theorem 2.3 we deduce a classification of the injective endomorphisms and of the automorphisms of $A[D_n]$ as follows.

Corollary 2.4 *Let $n \geq 6$. Let $\varphi\colon A[D_n] \to A[D_n]$ be an endomorphism. Then φ is injective if and only if there exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that φ is conjugate to $\psi \circ \gamma_p$.*

Proof Let $\varphi\colon A[D_n] \to A[D_n]$ be an endomorphism. By Theorem 2.3 we have one of the following three possibilities, up to conjugation:

(1) φ is cyclic.

(2) There exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q} \circ \pi$.

(3) There exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \psi \circ \gamma_p$.

If φ is cyclic, then $\varphi(t_{n-1}) = \varphi(t_n)$, and hence φ is not injective. If there exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q} \circ \pi$, then, again, $\varphi(t_{n-1}) = \varphi(t_n)$, and hence φ is not injective. So, if φ is injective, then there exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that φ is conjugate to $\psi \circ \gamma_p$.

It remains to show that, if $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$, then $\psi \circ \gamma_p$ is injective. Since the elements of $\langle \zeta, \chi \rangle$ are automorphisms, it suffices to show that γ_p is injective. We denote by $z\colon A[D_n] \to \mathbb{Z}$ the homomorphism which sends t_i to 1 for all $1 \leq i \leq n$. It is easily seen that $\gamma_p(u) = u\Delta^{\kappa p z(u)}$ for all $u \in A[D_n]$. Let $u \in \operatorname{Ker}(\gamma_p)$. Then $1 = \gamma_p(u) = u\Delta^{\kappa p z(u)}$, and hence $u = \Delta^q$ where $q = -\kappa p z(u)$. We have $z(\Delta) = n(n-1)$, and hence $z(u) = qn(n-1)$, thus

$$1 = \gamma_p(u) = \Delta^q \Delta^{\kappa p q n(n-1)} = \Delta^{q(1+\kappa p n(n-1))}.$$

Since $1 + \kappa p n(n-1) \neq 0$, this equality implies that $q = 0$, and hence $u = 1$. So γ_p is injective. □

Corollary 2.5 *Let $n \geq 6$. Let $\varphi\colon A[D_n] \to A[D_n]$ be an endomorphism. Then φ is an automorphism if and only if it is conjugate to an element of $\langle \zeta, \chi \rangle$.*

Proof Clearly, if φ is conjugate to an element of $\langle \zeta, \chi \rangle$, then φ is an automorphism. Conversely, suppose that φ is an automorphism. We know from Corollary 2.4 that there exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that φ is conjugate to $\psi \circ \gamma_p$. Thus, up to conjugation and up to composing on the left by ψ^{-1}, we can assume that $\varphi = \gamma_p$. It remains to show that $p = 0$.

Again let $z\colon A[D_n] \to \mathbb{Z}$ be the homomorphism which sends t_i to 1 for all $1 \le i \le n$. Recall that $\gamma_p(u) = u\Delta^{\kappa p z(u)}$ for all $u \in A[D_n]$. For $u \in A[D_n]$, we have

$$(z \circ \gamma_p)(u) = (1 + n(n-1)\kappa p) z(u) \in (1 + n(n-1)\kappa p)\mathbb{Z}.$$

Since γ_p is an automorphism, $z \circ \gamma_p$ is surjective, and hence $\mathbb{Z} = \mathrm{Im}(z \circ \gamma_p) \subset (1 + n(n-1)\kappa p)\mathbb{Z}$. It follows that $(1 + n(n-1)\kappa p) \in \{\pm 1\}$, and hence $p = 0$. □

By combining Corollary 2.5 with Crisp and Paris [21, Theorem 4.9] we immediately obtain the following.

Corollary 2.6 *Let $n \ge 6$.*

(1) *If n is even, then*

$$\mathrm{Aut}(A[D_n]) = \mathrm{Inn}(A[D_n]) \rtimes \langle \zeta, \chi \rangle \simeq (A[D_n]/Z(A[D_n])) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}),$$

and $\mathrm{Out}(A[D_n]) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $Z(A[D_n])$ denotes the center of $A[D_n]$.

(2) *If n is odd, then*

$$\mathrm{Aut}(A[D_n]) = \mathrm{Inn}(A[D_n]) \rtimes \langle \chi \rangle \simeq (A[D_n]/Z(A[D_n])) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

and $\mathrm{Out}(A[D_n]) \simeq \mathbb{Z}/2\mathbb{Z}$.

We denote by $Z(A[D_n])$ the center of $A[D_n]$, we set $A_Z[D_n] = A[D_n]/Z(A[D_n])$ and we denote by $\xi\colon A[D_n] \to A_Z[D_n]$ the canonical projection. For each $1 \le i \le n$, we set $t_{Z,i} = \xi(t_i)$. Note that an endomorphism $\varphi\colon A[D_n] \to A[D_n]$ induces an endomorphism $\varphi_Z\colon A_Z[D_n] \to A_Z[D_n]$ if and only if $\varphi(Z(A[D_n])) \subset Z(A[D_n])$. We say that an endomorphism $\psi\colon A_Z[D_n] \to A_Z[D_n]$ *lifts* if there exists an endomorphism $\varphi\colon A[D_n] \to A[D_n]$ such that $\varphi_Z = \psi$. Then we call φ a *lift* of ψ. In Section 7 we prove the following.

Proposition 2.7 *Let $n \ge 4$. Then every endomorphism of $A_Z[D_n]$ lifts.*

From this proposition combined with Theorem 2.3 we will deduce the following.

Theorem 2.8 *Let $n \ge 6$. Let $\varphi_Z\colon A_Z[D_n] \to A_Z[D_n]$ be an endomorphism. Then we have one of the following two possibilities, up to conjugation:*

(1) *φ_Z is cyclic.*

(2) *$\varphi_Z \in \langle \zeta_Z, \chi_Z \rangle$.*

In addition to Theorem 2.8 we have the following.

Proposition 2.9 *Let $n \geq 4$. There are only finitely many conjugacy classes of cyclic endomorphisms of $A_Z[D_n]$.*

Proof Let $\varphi_Z\colon A_Z[D_n] \to A_Z[D_n]$ be a cyclic endomorphism. There exists $g_Z \in A_Z[D_n]$ such that $\varphi_Z(t_{Z,i}) = g_Z$ for all $1 \leq i \leq n$. We denote by Δ the Garside element of $A[D_n]$, and we set $\kappa = 2$ if n is odd and $\kappa = 1$ if n is even. We have $1 = (\varphi_Z \circ \xi)(\Delta^\kappa) = g_Z^{\kappa n(n-1)}$, and hence g_Z is of finite order. By Bestvina [3, Theorem 4.5] there are finitely many conjugacy classes of finite subgroups in $A_Z[D_n]$. Since $\langle g_Z \rangle$ is a finite subgroup of $A_Z[D_n]$, it follows that there are finitely many choices for g_Z, up to conjugation. □

In Lemma 7.1 we will show that if n is even then $\langle \zeta_Z, \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$, and if n is odd then $\langle \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$. Furthermore, it is well known and can be easily proved (arguing as in the proof of Cumplido and Paris [22, Proposition 3.1(4)], for example) that the center of $A[\Gamma]/Z(A[\Gamma])$ is trivial for any $A[\Gamma]$ of spherical type. These two remarks combined with Theorem 2.8 imply the following.

Corollary 2.10 *Let $n \geq 6$.*

(1) *If n is even, then*

$$\mathrm{Aut}(A_Z[D_n]) = \mathrm{Inn}(A_Z[D_n]) \rtimes \langle \zeta_Z, \chi_Z \rangle \simeq A_Z[D_n] \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Aut}(A[D_n]),$$

and $\mathrm{Out}(A_Z[D_n]) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Out}(A[D_n])$.

(2) *If n is odd, then*

$$\mathrm{Aut}(A_Z[D_n]) = \mathrm{Inn}(A_Z[D_n]) \rtimes \langle \chi_Z \rangle \simeq A_Z[D_n] \rtimes (\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Aut}(A[D_n]),$$

and $\mathrm{Out}(A_Z[D_n]) \simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Out}(A[D_n])$.

A group G is said to be *co-Hopfian* if every injective endomorphism of G is an isomorphism. Another direct consequence of Theorem 2.8 is the following.

Corollary 2.11 *Let $n \geq 6$. Then $A_Z[D_n]$ is co-Hopfian.*

In addition to the case D_n for $n \geq 6$ shown in Corollary 2.11, the Coxeter graphs Γ for which we know that $A[\Gamma]/Z(A[\Gamma])$ is co-Hopfian are the Coxeter graphs A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$ for $n \geq 2$ (see Bell and Margalit [2]). Note that, for $\widetilde{A}_n$ and $\widetilde{C}_n$, the center $Z(A[\Gamma])$ is trivial, and hence the above remark means that the Artin group itself is co-Hopfian.

3 Geometric representations

Let Σ be an oriented compact surface possibly with boundary, and let $\mathcal{P}$ be a finite set of punctures in the interior of Σ. We denote by $\mathrm{Homeo}^+(\Sigma, \mathcal{P})$ the group of homeomorphisms of Σ that preserve the orientation, that are the identity on a neighborhood of the boundary of Σ and that setwise leave invariant $\mathcal{P}$.

The *mapping class group* of the pair $(\Sigma, \mathcal{P})$, denoted by $\mathcal{M}(\Sigma, \mathcal{P})$, is the group of isotopy classes of elements of $\mathrm{Homeo}^+(\Sigma, \mathcal{P})$. If $\mathcal{P} = \varnothing$, then we write $\mathcal{M}(\Sigma, \varnothing) = \mathcal{M}(\Sigma)$, and if $\mathcal{P} = \{x\}$ is a singleton, then we write $\mathcal{M}(\Sigma, \mathcal{P}) = \mathcal{M}(\Sigma, x)$. We only give definitions and results on mapping class groups that we need for our proofs and we refer to Farb and Margalit [28] for a complete account on the subject.

Recall that a *geometric representation* of an Artin group A is a homomorphism from A to a mapping class group. Their study is the main ingredient of our proofs. Important tools for constructing and understanding them are Dehn twists and essential reduction systems. So, we start by recalling their definitions and their main properties.

A *circle* of $(\Sigma, \mathcal{P})$ is the (nonoriented) image of an embedding $a\colon \mathbb{S}^1 \hookrightarrow \Sigma \setminus (\partial\Sigma \cup \mathcal{P})$. It is called *generic* if it does not bound any disk containing 0 or 1 puncture and if it is not parallel to any boundary component. The isotopy class of a circle a is denoted by $[a]$. We denote by $\mathcal{C}(\Sigma, \mathcal{P})$ the set of isotopy classes of generic circles of $(\Sigma, \mathcal{P})$. The *intersection number* of two classes $[a], [b] \in \mathcal{C}(\Sigma, \mathcal{P})$ is $i([a], [b]) = \min\{|a' \cap b'| \mid a' \in [a] \text{ and } b' \in [b]\}$. The set $\mathcal{C}(\Sigma, \mathcal{P})$ is endowed with a simplicial complex structure, where a finite set $\mathcal{A}$ is a simplex if $i([a], [b]) = 0$ for all $[a], [b] \in \mathcal{A}$. This complex is called the *curve complex* of $(\Sigma, \mathcal{P})$.

By a *Dehn twist* we mean a right Dehn twist and the (right) Dehn twist along a circle a of $(\Sigma, \mathcal{P})$ will be denoted by T_a. The following is an important tool for constructing and understanding geometric representations of Artin groups. Its proof can be found in Farb and Margalit [28, Section 3.5].

Proposition 3.1 *Let Σ be a compact oriented surface and let $\mathcal{P}$ be a finite collection of punctures in the interior of Σ. Let a and b be two generic circles of $(\Sigma, \mathcal{P})$.*

1. *We have $T_a T_b = T_b T_a$ if and only if $i([a], [b]) = 0$.*
2. *We have $T_a T_b T_a = T_b T_a T_b$ if and only if $i([a], [b]) = 1$.*

Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$. A simplex $\mathcal{A}$ of $\mathcal{C}(\Sigma, \mathcal{P})$ is called a *reduction system* for f if $f(\mathcal{A}) = \mathcal{A}$. In that case any element of $\mathcal{A}$ is called a *reduction class* for f. A reduction class $[a]$ is an *essential reduction class* if, for all $[b] \in \mathcal{C}(\Sigma, \mathcal{P})$ such that $i([a], [b]) \neq 0$ and for all $m \in \mathbb{Z} \setminus \{0\}$, we have $f^m([b]) \neq [b]$. In particular, if $[a]$ is an essential reduction class and $[b]$ is any reduction class, then $i([a], [b]) = 0$. We denote by $\mathcal{S}(f)$ the set of essential reduction classes for f. The following gathers some key results on $\mathcal{S}(f)$ that will be useful later.

Theorem 3.2 (Birman, Lubotzky and McCarthy [6]) *Let Σ be a compact oriented surface and let $\mathcal{P}$ be a finite set of punctures in the interior of Σ. Let $f \in \mathcal{M}(\Sigma, \mathcal{P})$.*

1. *If $\mathcal{S}(f) \neq \varnothing$, then $\mathcal{S}(f)$ is a reduction system for f. In particular, if $\mathcal{S}(f) \neq \varnothing$, then $\mathcal{S}(f)$ is a simplex of $\mathcal{C}(\Sigma, \mathcal{P})$.*
2. *We have $\mathcal{S}(f^n) = \mathcal{S}(f)$ for all $n \in \mathbb{Z} \setminus \{0\}$.*
3. *We have $\mathcal{S}(g f g^{-1}) = g(\mathcal{S}(f))$ for all $g \in \mathcal{M}(\Sigma, \mathcal{P})$.*

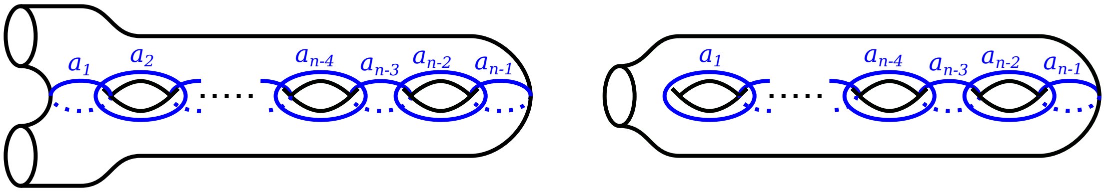

Figure 3: The geometric representation of $A[A_{n-1}]$ for n even (left) and n odd (right).

The following is well known and is a direct consequence of Birman, Lubotzky and McCarthy [6] (see also Castel [12, Corollaire 3.45]). It will be often used in our proofs.

Proposition 3.3 *Let Σ be an oriented compact surface of genus ≥ 2 and let $\mathcal{P}$ be a finite set of punctures in the interior of Σ. Let $f_0 \in Z(\mathcal{M}(\Sigma,\mathcal{P}))$ be a central element of $\mathcal{M}(\Sigma,\mathcal{P})$, let $\mathcal{A} = \{[a_1],\dots,[a_p]\}$ be a simplex of $\mathcal{C}(\Sigma,\mathcal{P})$ and let $k_1,\dots,k_p$ be nonzero integers. Let $g = T_{a_1}^{k_1} T_{a_2}^{k_2} \cdots T_{a_p}^{k_p} f_0$. Then $\mathcal{S}(g) = \mathcal{A}$.*

Let $n \geq 4$. If n is even, then Σ_n denotes the surface of genus $\frac{1}{2}(n-2)$ with two boundary components, and if n is odd, then Σ_n denotes the surface of genus $\frac{1}{2}(n-1)$ with one boundary component. Consider the circles $a_1,\dots,a_{n-1}$ drawn in Figure 3. Then by Proposition 3.1 we have a geometric representation $\rho_A\colon A[A_{n-1}] \to \mathcal{M}(\Sigma_n)$ which sends s_i to T_{a_i} for all $1 \leq i \leq n-1$. The following is well known; it is a direct consequence of Birman and Hilden [5], and its proof is explicitly given in Perron and Vannier [40].

Theorem 3.4 (Birman and Hilden [5]) *Let $n \geq 4$. Then $\rho_A\colon A[A_{n-1}] \to \mathcal{M}(\Sigma_n)$ is injective.*

The following is proved in Castel [12] for $n \geq 6$ using the geometric representation ρ_A defined above. It is proved in Chen, Kordek and Margalit [17] for $n \geq 5$ with a different method.

Theorem 3.5 (Castel [12], Chen, Kordek and Margalit [17] and Orevkov [35]) *Let $n \geq 5$. Let $\varphi\colon A[A_{n-1}] \to A[A_{n-1}]$ be a homomorphism. Then up to conjugation we have one of the following two possibilities*:

(1) *φ is cyclic.*

(2) *There exist $\psi \in \langle \bar{\chi} \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \psi \circ \bar{\gamma}_p$.*

Let $n \geq 6$. Pick a puncture x in the interior of Σ_n and consider the circles $d_1,\dots,d_n$ drawn in Figure 4. Then by Proposition 3.1 we have a geometric representation $\rho_D\colon A[D_n] \to \mathcal{M}(\Sigma_n, x)$ which sends t_i to T_{d_i} for all $1 \leq i \leq n$. On the other hand, the embedding of $\mathrm{Homeo}^+(\Sigma_n, x)$ into $\mathrm{Homeo}^+(\Sigma_n)$ induces a surjective homomorphism $\theta\colon \mathcal{M}(\Sigma_n, x) \to \mathcal{M}(\Sigma_n)$ whose kernel is naturally isomorphic to $\pi_1(\Sigma_n, x)$ (see Birman [4]). It is easily seen that

$$\theta(T_{d_i}) = T_{a_i} \quad \text{for } 1 \leq i \leq n-2, \qquad \theta(T_{d_{n-1}}) = \theta(T_{d_n}) = T_{a_{n-1}},$$

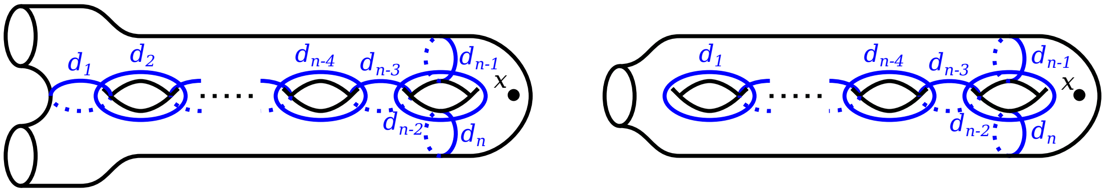

Figure 4: The geometric representation of $A[D_n]$ for n even (left) and n odd (right).

and hence we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Ker}(\pi) & \longrightarrow & A[D_n] & \xrightarrow{\pi} & A[A_{n-1}] & \longrightarrow & 1 \\ & & \downarrow \bar\rho & & \downarrow \rho_D & & \downarrow \rho_A & & \\ 1 & \longrightarrow & \mathrm{Ker}(\theta) & \longrightarrow & \mathcal{M}(\Sigma_n, x) & \xrightarrow{\theta} & \mathcal{M}(\Sigma_n) & \longrightarrow & 1 \end{array} \tag{3-1}$$

where we denote by $\bar\rho\colon \mathrm{Ker}(\pi) \to \mathrm{Ker}(\theta)$ the restriction of ρ_D to $\mathrm{Ker}(\pi)$.

The proof of the following can be found in Perron and Vannier [40, Theorem 1] with few modifications. As this result is central in our paper, for the sake of completeness we give a proof. Note that our proof is a little shorter than that of Perron and Vannier [40] because it uses results from Crisp and Paris [21] which were not known and it does not need to deal with some Dehn twist along a boundary component, but our arguments are essentially the same.

Theorem 3.6 (Perron and Vannier [40]) *Let $n \geq 4$.*

1. *The homomorphism $\bar\rho\colon \mathrm{Ker}(\pi) \to \mathrm{Ker}(\theta)$ is an isomorphism.*
2. *The geometric representation $\rho_D\colon A[D_n] \to \mathcal{M}(\Sigma_n, x)$ is injective.*

Proof Part (2) is a consequence of (1) because of the following. Suppose $\bar\rho$ is an isomorphism. Then, since ρ_A is injective, ρ_D is injective by the five lemma applied to (3-1).

Now, we prove (1). We know from Crisp and Paris [21, Proposition 2.3] that $\mathrm{Ker}(\pi)$ is a free group of rank $n-1$. We also know from Birman [4] that $\mathrm{Ker}(\theta) = \pi_1(\Sigma_n, x)$, which is also a free group of rank $n-1$. Recall that a group G is Hopfian if every surjective endomorphism $G \to G$ is an isomorphism. It is well known that free groups of finite rank are Hopfian (see de la Harpe [30, Chapter III, Section 19]), and hence in order to show that $\bar\rho$ is an isomorphism it suffices to show that $\bar\rho$ is surjective.

Set $f_{n-1} = T_{d_{n-1}}^{-1} T_{d_n}$. Note that $t_{n-1}^{-1} t_n \in \mathrm{Ker}(\pi)$ and $f_{n-1} = \bar\rho(t_{n-1}^{-1} t_n)$. In particular $f_{n-1} \in \mathrm{Im}(\bar\rho) \subset \mathrm{Ker}(\theta) = \pi_1(\Sigma_n, x)$. This element, seen as an element of $\pi_1(\Sigma_n, x)$, is represented by the loop drawn in Figure 5. For $2 \leq i \leq n-1$ we define $f_{n-i} \in \pi_1(\Sigma_n, x) \subset \mathcal{M}(\Sigma_n, x)$ by induction on i by setting $f_{n-i} = T_{d_{n-i}} f_{n-i+1} T_{d_{n-i}}^{-1} f_{n-i+1}^{-1}$. The element f_{n-i}, viewed as an element of $\pi_1(\Sigma_n, x)$, is represented by the loop drawn in the left-hand side of Figure 6 if $i = 2j$ is even, and by the loop drawn in the right-hand side of Figure 6 if $i = 2j+1$ is odd, where we compose paths from right to left. Observe that $f_1, \dots, f_{n-1}$

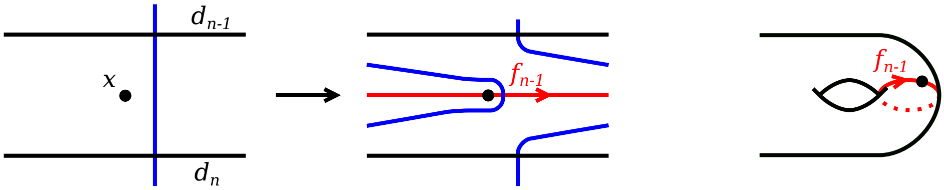

Figure 5: The loop $f_{n-1} \in \pi_1(\Sigma_n, x)$.

generate $\pi_1(\Sigma_n, x)$. So, in order to show that $\bar\rho$ is surjective, it suffices to show that $f_{n-i} \in \mathrm{Im}(\bar\rho)$ for all $i \in \{1, \dots, n-1\}$. We argue by induction on i. We already know that $f_{n-1} = \bar\rho(t_{n-1}^{-1} t_n) \in \mathrm{Im}(\bar\rho)$. Suppose $i \geq 2$ and $f_{n-i+1} \in \mathrm{Im}(\bar\rho)$. Let $u \in \mathrm{Ker}(\pi)$ such that $f_{n-i+1} = \bar\rho(u)$. Since $\mathrm{Ker}(\pi)$ is a normal subgroup of $A[D_n]$, we have $t_{n-i} u t_{n-i}^{-1} \in \mathrm{Ker}(\pi)$; hence $t_{n-i} u t_{n-i}^{-1} u^{-1} \in \mathrm{Ker}(\pi)$, and therefore

$$f_{n-i} = T_{d_{n-i}} f_{n-i+1} T_{d_{n-i}}^{-1} f_{n-i+1}^{-1} = \bar\rho(t_{n-i} u t_{n-i}^{-1} u^{-1}) \in \mathrm{Im}(\bar\rho). \qquad \square$$

Our last preliminary on geometric representations is a result implicitly proved in Castel [13, Section 3.2], and it is in this theorem that we need the assumption $n \geq 6$.

Theorem 3.7 (Castel [13]) *Let $n \geq 6$. Let $\varphi\colon A[A_{n-1}] \to \mathcal{M}(\Sigma_n, x)$ be a noncyclic homomorphism. Then there exist generic circles $c_1, \dots, c_{n-1}$ in $\Sigma_n \setminus \{x\}$, $\varepsilon \in \{\pm 1\}$ and $g \in \mathcal{M}(\Sigma_n, x)$ such that*

(a) *$|c_i \cap c_j| = 1$ if $|i-j| = 1$ and $|c_i \cap c_j| = 0$ if $|i-j| \geq 2$, for all $1 \leq i, j \leq n-1$,*

(b) *g commutes with T_{c_i} for all $1 \leq i \leq n-1$,*

(c) *$\varphi(s_i) = T_{c_i}^{\varepsilon} g$ for all $1 \leq i \leq n-1$.*

Proof Assume n is even. Let ∂_1 and ∂_2 be the two boundary components of Σ_n. We denote by $\hat\Sigma_n$ the closed surface obtained from Σ_n by gluing a disk D_1 along ∂_1 and a disk D_2 along ∂_2. Moreover, we choose a point $\hat x_1$ in the interior of D_1 and a point $\hat x_2$ in the interior of D_2, and we set $\hat{\mathcal{P}} = \{x, \hat x_1, \hat x_2\}$. Assume n is odd. Let ∂ be the boundary component of Σ_n. We denote by $\hat\Sigma_n$ the closed surface obtained from Σ_n by gluing a disk D along ∂. Moreover, we choose a point $\hat x$ in the interior of D and we set $\hat{\mathcal{P}} = \{x, \hat x\}$. For each n we denote by $\mathcal{PM}(\hat\Sigma_n, \hat{\mathcal{P}})$ the subgroup of $\mathcal{M}(\hat\Sigma_n, \hat{\mathcal{P}})$ formed by the isotopy classes of elements in $\mathrm{Homeo}^+(\hat\Sigma_n, \hat{\mathcal{P}})$ which pointwise fix $\hat{\mathcal{P}}$. The embedding of Σ_n into $\hat\Sigma_n$ induces a surjective homomorphism $\varpi\colon \mathcal{M}(\Sigma_n, x) \to \mathcal{PM}(\hat\Sigma_n, \hat{\mathcal{P}})$. If n is even, then the kernel of ϖ is the free abelian group of rank 2 generated by T_{∂_1} and T_{∂_2}, and if n is odd, then the kernel of ϖ is the cyclic group generated by T_∂. In both cases $\mathrm{Ker}(\varpi)$ is contained in the center of $\mathcal{M}(\Sigma_n, x)$.

Figure 6: The loop $f_{n-i} \in \pi_1(\Sigma_n, x)$.

Let $\varphi\colon A[A_{n-1}] \to \mathcal{M}(\Sigma_n, x)$ be a noncyclic homomorphism. Assume that $\varpi \circ \varphi$ is cyclic. Then there exists $\hat{g} \in \mathcal{PM}(\hat{\Sigma}_n, \hat{\mathcal{P}})$ such that $(\varpi \circ \varphi)(s_i) = \hat{g}$ for all $1 \le i \le n-1$. Let $g \in \mathcal{M}(\Sigma_n, x)$ be such that $\varpi(g) = \hat{g}$. For each $1 \le i \le n-1$ there exists $h_i \in \operatorname{Ker}(\varpi) \subset Z(\mathcal{M}(\Sigma_n, x))$ such that $\varphi(s_i) = g h_i$. Let $1 \le i \le n-2$. Then

$$g^3 h_i^2 h_{i+1} = \varphi(s_i s_{i+1} s_i) = \varphi(s_{i+1} s_i s_{i+1}) = g^3 h_i h_{i+1}^2.$$

Hence $h_i = h_{i+1}$. This shows that $\varphi(s_i) = g h_1$ for all $1 \le i \le n-1$, and hence that φ is cyclic, which is a contradiction. So $\varpi \circ \varphi$ is not cyclic.

To differentiate Dehn twists in $\mathcal{M}(\Sigma_n, x)$ from those in $\mathcal{PM}(\hat{\Sigma}_n, \hat{\mathcal{P}})$, for a circle c in $\hat{\Sigma}_n \setminus \hat{\mathcal{P}}$ we denote by $\hat{T}_c$ the Dehn twist in $\mathcal{PM}(\hat{\Sigma}_n, \hat{\mathcal{P}})$ along c. By Castel [13, Theorem 1] there exist generic circles $c_1, \dots, c_{n-1}$ in $\hat{\Sigma}_n \setminus \hat{\mathcal{P}}$, $\varepsilon \in \{\pm 1\}$ and $\hat{g} \in \mathcal{PM}(\hat{\Sigma}_n, \hat{\mathcal{P}})$ such that

(1) $|c_i \cap c_j| = 1$ if $|i-j| = 1$ and $|c_i \cap c_j| = 0$ if $|i-j| \ge 2$, for all $1 \le i, j \le n-1$,

(2) $\hat{g}$ commutes with $\hat{T}_{c_i}$ for all $1 \le i \le n-1$,

(3) $(\varpi \circ \varphi)(s_i) = \hat{T}_{c_i}^{\varepsilon} \hat{g}$ for all $1 \le i \le n-1$.

Clearly, we can choose each c_i siting in the interior of Σ_n. Let $g \in \mathcal{M}(\Sigma_n, x)$ be such that $\varpi(g) = \hat{g}$. It is easily shown with Castel [13, Lemma 3.2.1] that g and T_{c_i} commute for all $1 \le i \le n-1$. Furthermore, for each $1 \le i \le n-1$, there exists $h_i \in \operatorname{Ker}(\varpi) \subset Z(\mathcal{M}(\Sigma_n, x))$ such that $\varphi(s_i) = T_{c_i}^{\varepsilon} g h_i$. Let $1 \le i \le n-2$. Then

$$\begin{aligned} T_{c_i}^{\varepsilon} T_{c_{i+1}}^{\varepsilon} T_{c_i}^{\varepsilon} g^3 h_i^2 h_{i+1} = \varphi(s_i s_{i+1} s_i) = \varphi(s_{i+1} s_i s_{i+1}) &= T_{c_{i+1}}^{\varepsilon} T_{c_i}^{\varepsilon} T_{c_{i+1}}^{\varepsilon} g^3 h_i h_{i+1}^2 \\ &= T_{c_i}^{\varepsilon} T_{c_{i+1}}^{\varepsilon} T_{c_i}^{\varepsilon} g^3 h_i h_{i+1}^2, \end{aligned}$$

and hence $h_{i+1} = h_i$. So there exists $h \in \operatorname{Ker}(\varpi)$ such that $\varphi(s_i) = T_{c_i}^{\varepsilon} g h$ and gh commutes with T_{c_i} for all $1 \le i \le n-1$. □

4 Homomorphisms from $A[D_n]$ to $A[A_{n-1}]$

Proof of Theorem 2.1 Let $n \ge 5$. Let $\varphi\colon A[D_n] \to A[A_{n-1}]$ be a homomorphism. By precomposing φ with $\iota\colon A[A_{n-1}] \to A[D_n]$, we obtain a homomorphism $\varphi \circ \iota\colon A[A_{n-1}] \to A[D_n] \to A[A_{n-1}]$, and hence, by Theorem 3.5, one of the following two possibilities holds:

- $\varphi \circ \iota$ is cyclic.
- There exist $\psi \in \langle \bar{\chi} \rangle$ and $p \in \mathbb{Z}$ such that $\varphi \circ \iota$ is conjugate to $\psi \circ \bar{\gamma}_p$.

Suppose $\varphi \circ \iota$ is cyclic. Then there exists $u \in A[A_{n-1}]$ such that $(\varphi \circ \iota)(s_i) = \varphi(t_i) = u$ for all $1 \le i \le n-1$. Moreover,

$$\varphi(t_n) = \varphi(t_{n-2} t_n) \varphi(t_{n-2}) \varphi(t_n^{-1} t_{n-2}^{-1}) = \varphi(t_{n-2} t_n) \varphi(t_1) \varphi(t_n^{-1} t_{n-2}^{-1}) = \varphi(t_1) = u,$$

and hence φ is cyclic.

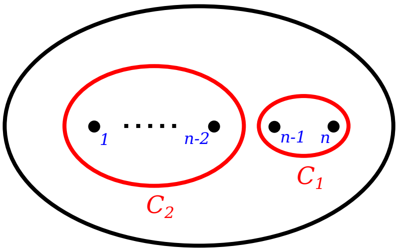

Figure 7: Circles in the punctured disk.

So, up to conjugating and replacing φ by $\varphi \circ \chi$ if necessary, we can assume that there exists $p \in \mathbb{Z}$ such that $\varphi \circ \iota = \bar{\gamma}_p$. This means that $\varphi(t_i) = (\varphi \circ \iota)(s_i) = s_i \Delta^{2p}$ for all $1 \le i \le n-1$, where Δ is the Garside element of $A[A_{n-1}]$. Now we turn to showing that $\varphi = \alpha_p$.

Set $Y = \{s_1, \dots, s_{n-3}\}$. By Paris [37, Theorem 5.1] the centralizer of the group $\langle s_1, \dots, s_{n-3}, s_{n-1} \rangle$ in $A[A_{n-1}]$ is generated by Δ^2, Δ_Y^2 and s_{n-1}, where $\Delta_Y = \Delta_Y[A_{n-1}]$. These three elements pairwise commute and generate a copy of $\mathbb{Z}^3$. Set $u = \varphi(t_n)$. Since u commutes with $\varphi(t_i) = s_i \Delta^{2p}$ for all $i \in \{1, \dots, n-3, n-1\}$ and Δ^2 is central in $A[A_{n-1}]$, u belongs to the centralizer of $\langle s_1, \dots, s_{n-3}, s_{n-1} \rangle$, and hence there exist $k_1, k_2, k_3 \in \mathbb{Z}$ such that $u = s_{n-1}^{k_1} \Delta_Y^{2k_2} \Delta^{2k_3}$.

It is well known that $A[A_{n-1}]$ is naturally isomorphic to the mapping class group $\mathcal{M}(\mathbb{D}, \mathcal{P})$, where $\mathbb{D}$ denotes the disk and $\mathcal{P} = \{x_1, \dots, x_n\}$ is a set of n punctures in the interior of $\mathbb{D}$. In this identification s_{n-1}^2 corresponds to the Dehn twist along the circle c_1 depicted in Figure 7, Δ_Y^2 corresponds to the Dehn twist along the circle c_2 depicted in the same figure and Δ^2 corresponds to the Dehn twist along a circle parallel to $\partial\mathbb{D}$. By Proposition 3.3 we have $\mathcal{S}(u^2) \subseteq \{c_1, c_2\}$, where $c_1 \in \mathcal{S}(u^2)$ if and only if $k_1 \neq 0$ and $c_2 \in \mathcal{S}(u^2)$ if and only if $k_2 \neq 0$. We know that $\varphi(t_1^2) = s_1^2 \Delta^{4p}$, and hence $\mathcal{S}(\varphi(t_1^2))$ is formed by a single circle containing two marked points in its interior. Since t_1^2 and t_n^2 are conjugate $\varphi(t_1^2)$ and $\varphi(t_n^2) = u^2$ are conjugate, and hence, by Theorem 3.2, $\mathcal{S}(u^2)$ is also formed by a single circle containing two marked points in its interior. It follows that $\mathcal{S}(u^2) = \{c_1\}$, and hence $k_1 \neq 0$ and $k_2 = 0$. It remains to show that $k_1 = 1$ and $k_3 = p$.

From the equality $t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n$ it follows that $s_{n-2} s_{n-1}^{k_1} s_{n-2} \Delta^{4p+2k_3} = s_{n-1}^{k_1} s_{n-2} s_{n-1}^{k_1} \Delta^{2p+4k_3}$, and hence

$$(s_{n-2} s_{n-1}^{k_1} s_{n-2})(s_{n-1}^{k_1} s_{n-2} s_{n-1}^{k_1})^{-1} = \Delta^{2k_3 - 2p}.$$

We know from Paris [38, Corollary 2.6] that $A_{\{s_{n-2}, s_{n-1}\}}[A_{n-1}] \cap \langle \Delta \rangle = \{1\}$, and hence

$$(s_{n-2} s_{n-1}^{k_1} s_{n-2})(s_{n-1}^{k_1} s_{n-2} s_{n-1}^{k_1})^{-1} = \Delta^{2k_3 - 2p} = 1.$$

Let $z \colon A[A_{n-1}] \to \mathbb{Z}$ be the homomorphism which sends s_i to 1 for all $1 \le i \le n-1$. We have

$$0 = z(1) = z((s_{n-2} s_{n-1}^{k_1} s_{n-2})(s_{n-1}^{k_1} s_{n-2} s_{n-1}^{k_1})^{-1}) = 1 - k_1,$$

and hence $k_1 = 1$. Moreover, $\Delta^{2k_3 - 2p} = 1$ and Δ is of infinite order; thus $k_3 = p$. □

5 Homomorphisms from $A[A_{n-1}]$ to $A[D_n]$

The formula in the following lemma is a crucial point in various proofs, including those of Lemma 5.4 and Theorem 2.8.

Lemma 5.1 *Let $n \geq 1$. Then*

$$\Delta[A_n]^2 = (s_1 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_1)(s_2 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_2) \cdots (s_{n-1} s_n^2 s_{n-1}) s_n^2.$$

Proof We argue by induction on n. The case $n = 1$ is trivial, and hence we can assume that $n \geq 2$ and that the induction hypothesis holds. Recall that

$$\Delta[A_n] = (s_1 \cdots s_n)\Delta[A_{n-1}] = \Delta[A_{n-1}](s_n \cdots s_1).$$

Moreover, it is easily checked that $s_i(s_n \cdots s_1) = (s_n \cdots s_1)s_{i+1}$ for all $1 \leq i \leq n-1$. By the induction hypothesis,

$$\Delta[A_{n-1}]^2 = (s_1 \cdots s_{n-2} s_{n-1}^2 s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1}^2 s_{n-2}) s_{n-1}^2.$$

Hence

$$\begin{aligned}
\Delta[A_n]^2 &= (s_1 \cdots s_n)\Delta[A_{n-1}]^2(s_n \cdots s_1)\\
&= (s_1 \cdots s_n)\big((s_1 \cdots s_{n-2} s_{n-1}^2 s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1}^2 s_{n-2}) s_{n-1}^2\big)(s_n \cdots s_1)\\
&= (s_1 \cdots s_n)(s_n \cdots s_1)\big((s_2 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_2) \cdots (s_{n-1} s_n^2 s_{n-1}) s_n^2\big)\\
&= (s_1 \cdots s_{n-1} s_n^2 s_{n-1} \cdots s_1) \cdots (s_{n-1} s_n^2 s_{n-1}) s_n^2.
\end{aligned}$$

□

Now, Lemmas 5.2–5.8 are preliminaries to the proof of Theorem 2.2.

Lemma 5.2 *Let $n \geq 6$. Let $\varphi\colon A[A_{n-1}] \to A[D_n]$ be a homomorphism. If $\pi \circ \varphi\colon A[A_{n-1}] \to A[A_{n-1}]$ is cyclic, then φ is cyclic.*

Proof Assume $\pi \circ \varphi$ is cyclic. Then there exists $u \in A[A_{n-1}]$ such that $(\pi \circ \varphi)(s_i) = u$ for all $1 \leq i \leq n-1$. For $3 \leq i \leq n-1$ we set $v_i = \varphi(s_i s_1^{-1})$. We have $\pi(v_i) = uu^{-1} = 1$, and hence $v_i \in \mathrm{Ker}(\pi)$. We have

$$(s_3 s_1^{-1})(s_4 s_1^{-1})(s_3 s_1^{-1}) = s_3 s_4 s_3 s_1^{-3} = s_4 s_3 s_4 s_1^{-3} = (s_4 s_1^{-1})(s_3 s_1^{-1})(s_4 s_1^{-1}),$$

and hence $v_3 v_4 v_3 = v_4 v_3 v_4$. Since $\mathrm{Ker}(\pi)$ is a free group (see Crisp and Paris [21, Proposition 2.3]) and two elements in a free group either freely generate a free group or commute, the existence of such equality implies that $v_3 v_4 = v_4 v_3$. It follows that $v_3 v_4 v_3 = v_3 v_4^2$; hence $v_3 = v_4$, and therefore

$$\varphi(s_3)\varphi(s_1)^{-1} = v_3 = v_4 = \varphi(s_4)\varphi(s_1)^{-1}.$$

So $\varphi(s_3) = \varphi(s_4)$. We conclude by Castel [13, Lemma 3.1.1] that φ is cyclic. □

Let $n \geq 6$. If n is odd then Σ_n has one boundary component, which we denote by ∂, and we denote by T_∂ the Dehn twist along ∂. If n is even then Σ_n has two boundary components, which we denote by ∂_1 and ∂_2, and we denote by T_{∂_1} and T_{∂_2} the Dehn twists along ∂_1 and ∂_2, respectively. It is known that the center of $\mathcal{M}(\Sigma_n)$, denoted by $Z(\mathcal{M}(\Sigma_n))$, is the cyclic group generated by T_∂ if n is odd, and it is a free abelian group of rank 2 generated by T_{∂_1} and T_{∂_2} if n is even (see Paris and Rolfsen [39, Theorem 5.6], for example).

Lemma 5.3 *Let $n \geq 2$. Let $f \in \mathcal{M}(\Sigma_n)$ such that $f T_{a_i}^2 = T_{a_i}^2 f$ for all $1 \leq i \leq n-1$. Then $f^2 \in Z(\mathcal{M}(\Sigma_n))$.*

Proof Assume n is odd. The case where n is even can be proved in the same way. Let $f \in \mathcal{M}(\Sigma_n)$ such that $f T_{a_i}^2 = T_{a_i}^2 f$ for all $1 \leq i \leq n-1$. Since $f T_{a_i}^2 f^{-1} = T_{a_i}^2$ we have $f([a_i]) = [a_i]$ (see Farb and Margalit [28, Section 3.3]). The mapping class f may reverse the orientation of each a_i up to isotopy, but f^2 preserves the orientation of all a_i up to isotopy, and hence f^2 can be represented by an element of $\mathrm{Homeo}^+(\Sigma_n)$ which is the identity on a (closed) regular neighborhood Σ' of $\bigcup_{i=1}^{n-1} a_i$. We observe that Σ' is a surface of genus $\frac{1}{2}(n-1)$ with one boundary component, ∂', and that $\partial \cup \partial'$ bounds a cylinder C. This implies that $f^2 \in \mathcal{M}(C) \subset \mathcal{M}(\Sigma_n)$. Since $\mathcal{M}(C) = \langle T_\partial \rangle = Z(\mathcal{M}(\Sigma_n))$, we conclude that $f^2 \in Z(\mathcal{M}(\Sigma_n))$. □

Lemma 5.4 *Let $n \geq 3$. We set $m = n-1$ if n is odd and $m = n-2$ if n is even. Let $1 \leq k \leq m$. Let c be a generic circle of $\Sigma_n \setminus \{x\}$ such that $c \cap d_i = \varnothing$ for $1 \leq i \leq k-2$, $|c \cap d_{k-1}| = 1$ if $k \geq 2$, $c \cap d_k = \varnothing$ and c is isotopic to d_k in Σ_n. Then there exists $g \in \mathrm{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \leq i \leq k-1$ and $g([c]) = [d_k]$.*

Proof We identify D_3 with A_3 in this proof to treat the cases $k = 2$ and $k = 1$. We first assume that k is even. If c is isotopic in $\Sigma_n \setminus \{x\}$ to d_k, then it suffices to take $g = \mathrm{id}$. So we can assume that c and d_k are not isotopic in $\Sigma_n \setminus \{x\}$. Since c and d_k are isotopic in Σ_n, by Epstein [27, Lemma 2.4] there exists a cylinder C in Σ_n whose boundary components are d_k and c. Since c and d_k are not isotopic in $\Sigma_n \setminus \{x\}$, this cylinder must contain the puncture x.

Let Σ' be a regular neighborhood of $\left(\bigcup_{i=1}^{k-1} d_i\right) \cup C$. The surface Σ' contains the cylinder C with boundaries c and d_k, having the puncture x in it, and d_{k-1} intersects c and d_k once. Hence an arc of the curve d_{k-1} connects a point on c with a point on d_k within the cylinder C, and it may wind around the cylinder in different ways (see Figure 8). However, by applying suitable Dehn twists about c and d_k, one can unwind this arc to the simplest case, shown in Figure 9. Hence, up to homeomorphism of the surface Σ_n, we may assume that the circles $d_1, \dots, d_k, c$ are arranged as in Figure 9.

By Proposition 3.1 there are homomorphisms $\psi_1 \colon A[D_{k+1}] \to \mathcal{M}(\Sigma_n, x)$ and $\psi_2 \colon A[A_k] \to \mathcal{M}(\Sigma_n, x)$ defined by

$$\psi_1(t_i) = T_{d_i} \quad \text{for } 1 \leq i \leq k, \qquad \psi_1(t_{k+1}) = T_c,$$
$$\psi_2(s_i) = T_{d_i} \quad \text{for } 1 \leq i \leq k-1, \qquad \psi_2(s_k) = T_c.$$

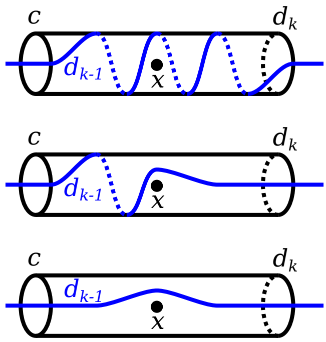

Figure 8: The intersection of C with d_{k-1}.

We denote by $\Delta_{D,k}$ the Garside element of $A[D_{k+1}]$ and by $\Delta_{A,k}$ the Garside element of $A[A_k]$, and we set $g=\psi_1(\Delta_{D,k})\psi_2(\Delta_{A,k}^{-2})$. We have $\Delta_{D,k}t_i\Delta_{D,k}^{-1}=t_i$ for all $1\le i\le k-1$, $\Delta_{D,k}t_{k+1}\Delta_{D,k}^{-1}=t_k$ and $\Delta_{A,k}^2 s_i\Delta_{A,k}^{-2}=s_i$ for all $1\le i\le k$. Hence $gT_{d_i}g^{-1}=T_{g(d_i)}=T_{d_i}$ for all $1\le i\le k-1$ and $gT_cg^{-1}=T_{g(c)}=T_{d_k}$. It follows that $g([d_i])=[d_i]$ for all $1\le i\le k-1$ and $g([c])=[d_k]$ (see Farb and Margalit [28, Fact 3.6]).

Since c and d_k are isotopic in Σ_n, the corresponding Dehn twists T_c and T_{d_k} are equal in $\mathcal{M}(\Sigma_n)$, and hence for T_c and T_{d_k}, viewed on the surface $\Sigma_n\setminus\{x\}$, we have $\theta(T_c)=\theta(T_{d_k})$. Moreover,

$$\Delta_{D,k}=(t_1\cdots t_{k-1}t_kt_{k+1}t_{k-1}\cdots t_1)\cdots(t_{k-1}t_kt_{k+1}t_{k-1})(t_kt_{k+1}),$$
$$\Delta_{A,k}^2=(s_1\cdots s_{k-1}s_k^2s_{k-1}\cdots s_1)\cdots(s_{k-1}s_k^2s_{k-1})s_k^2,$$

(see Lemma 5.1 for the second equality); hence $\theta(\psi_1(\Delta_{D,k}))=\theta(\psi_2(\Delta_{A,k}^2))$, and therefore $\theta(g)=1$. So $g\in\mathrm{Ker}(\theta)$.

Now assume k is odd. If c is isotopic in $\Sigma_n\setminus\{x\}$ to d_k, then we can take $g=\mathrm{id}$. So we can assume that c and d_k are not isotopic in $\Sigma_n\setminus\{x\}$. Since c and d_k are isotopic in Σ_n, there exists a cylinder C in Σ_n whose boundary components are d_k and c. Since c and d_k are not isotopic in $\Sigma_n\setminus\{x\}$, this cylinder must contain the puncture x. Let Σ' be a closed regular neighborhood of $\left(\bigcup_{i=1}^{k-1}d_i\right)\cup C$. Then Σ' is a surface of genus $\frac{1}{2}(k-1)$ with two boundary components and the circles $d_1,\dots,d_{k-1},d_k,c$ are arranged as shown in Figure 10. Since $k\le m$ and k is odd, $\frac{1}{2}(k-1)$ is strictly less than the genus of Σ_n; hence we can choose a subsurface Σ'' of Σ_n of genus $\frac{1}{2}(k+1)$, with one boundary component, and containing Σ'. We can also choose a generic circle e in $\Sigma''\setminus\{x\}$ such that $|e\cap d_1|=1$, $|e\cap c|=1$ if $k=1$, $e\cap d_i=\varnothing$ for all $2\le i\le k$ and $e\cap c=\varnothing$ if $k\ge 2$ (see Figure 10). By Proposition 3.1 there are homomorphisms

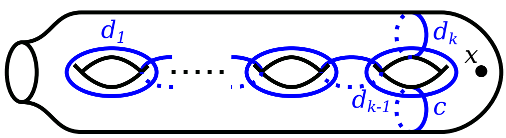

Figure 9: The regular neighborhood of $\left(\bigcup_{i=1}^{k-1}d_i\right)\cup C$ when k is even.

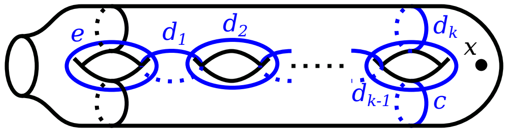

Figure 10: The regular neighborhood of $\left(\bigcup_{i=1}^{k-1} d_i\right) \cup C$ when k is odd

$\psi_1 \colon A[D_{k+2}] \to \mathcal{M}(\Sigma_n, x)$ and $\psi_2 \colon A[A_{k+1}] \to \mathcal{M}(\Sigma_n, x)$ defined by

$$\psi_1(t_1) = T_e, \qquad \psi_1(t_i) = T_{d_{i-1}} \quad \text{for } 2 \le i \le k+1, \qquad \psi_1(t_{k+2}) = T_c,$$
$$\psi_2(s_1) = T_e, \qquad \psi_2(s_i) = T_{d_{i-1}} \quad \text{for } 2 \le i \le k, \qquad \psi_2(s_{k+1}) = T_c.$$

We denote by $\Delta_{D,k+1}$ the Garside element of $A[D_{k+2}]$ and by $\Delta_{A,k+1}$ the Garside element of $A[A_{k+1}]$, and we set $g = \psi_1(\Delta_{D,k+1})\psi_2(\Delta_{A,k+1}^{-2})$. Then, as in the case where k is even, we have $g([d_i]) = [d_i]$ for all $1 \le i \le k-1$, $g([c]) = [d_k]$ and $g \in \mathrm{Ker}(\theta)$. □

The following lemma is the extension of Lemma 5.4 to the case $c \cap d_k \neq \varnothing$.

Lemma 5.5 *Let $n \ge 3$. Set $m = n-1$ if n is odd and $m = n-2$ if n is even. Let $1 \le k \le m$. Let c be a generic circle of $\Sigma_n \setminus \{x\}$ such that $c \cap d_i = \varnothing$ for $1 \le i \le k-2$, $|c \cap d_{k-1}| = 1$ if $k \ge 2$, and c is isotopic to d_k in Σ_n. Then there exists $g \in \mathrm{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le k-1$ and $g([c]) = [d_k]$.*

Proof We argue by induction on $i([c],[d_k])$, which is computed on the surface $\Sigma_n \setminus \{x\}$ and not on Σ_n. The case $i([c],[d_k]) = 0$ is proved in Lemma 5.4, and hence we can assume that $i([c],[d_k]) \ge 1$ and that the induction hypothesis holds. Note that now c and d_k cannot be isotopic in $\Sigma_n \setminus \{x\}$ since $i([c],[d_k]) \neq 0$. We can assume without loss of generality that $i([c],[d_k]) = |c \cap d_k|$. Since c and d_k are isotopic in Σ_n, there exists a bigon D in Σ_n cobounded by an arc of d_k and an arc of c as shown in Figure 11. We can choose this bigon to be minimal in the sense that its interior intersects neither c nor d_k. The bigon D cannot intersect d_i for $1 \le i \le k-2$ and one can easily modify c so that D does not intersect d_{k-1} either. Since c and d_k are not isotopic in $\Sigma_n \setminus \{x\}$, D necessarily contains the puncture x in its interior. We choose a circle c' parallel to c except in the bigon D, where it follows the arc of d_k which borders D as illustrated in Figure 11. By construction $c' \cap d_i = \varnothing$ for $1 \le i \le k-2$, $|c' \cap d_{k-1}| = 1$ if $k \ge 2$, and c' is isotopic to d_k in Σ_n. Moreover $i([c'],[d_k]) \le |c' \cap d_k| < |c \cap d_k| = i([c],[d_k])$. By the induction hypothesis there exists $g_1 \in \mathrm{Ker}(\theta)$ such that $g_1([d_i]) = [d_i]$ for all $1 \le i \le k-1$ and $g_1([c']) = [d_k]$.

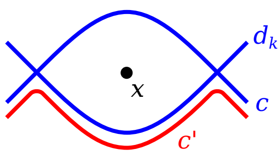

Figure 11: The bigon cobounded by c and d_k.

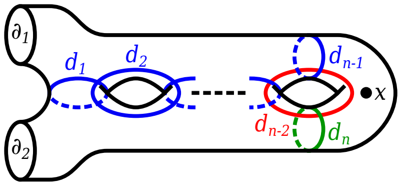

Figure 12: The circles $d_1, \dots, d_n$.

By Farb and Margalit [28, Lemma 2.9], we can choose $G_1 \in \mathrm{Homeo}^+(\Sigma_n, x)$ which represents g_1 such that $G_1(d_i) = d_i$ for all $1 \le i \le k-1$ and $G_1(c') = d_k$. We set $c'' = G_1(c)$. Then $c'' \cap d_i = \varnothing$ for $1 \le i \le k-2$, $|c'' \cap d_{k-1}| = 1$ if $k \ge 2$, $c'' \cap d_k = \varnothing$ and c'' is isotopic to d_k in Σ_n. By Lemma 5.4 there exists $g_2 \in \mathrm{Ker}(\theta)$ such that $g_2([d_i]) = [d_i]$ for all $1 \le i \le k-1$ and $g_2([c'']) = [d_k]$. We set $g = g_2 \circ g_1$. Then $g \in \mathrm{Ker}(\theta)$, $g([d_i]) = [d_i]$ for all $1 \le i \le k-1$ and $g([c]) = [d_k]$. □

Lemma 5.6 *Let $n \ge 4$ be even. Let c be a generic circle of $\Sigma_n \setminus \{x\}$ such that $c \cap d_i = \varnothing$ for all $1 \le i \le n-3$, $|c \cap d_{n-2}| = 1$, $c \cap d_{n-1} = \varnothing$ and c is isotopic to d_{n-1} in Σ_n. Then we have one of the following two possibilities*:

(1) *c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.*

(2) *There exists $g \in \mathrm{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$ and $g([c]) = [d_n]$.*

Proof The surface Σ_n is a surface of genus $\frac{1}{2}(n-2)$ with two boundary components ∂_1 and ∂_2. We assume that the circles $d_1, \dots, d_{n-1}, d_n$ are arranged as in Figure 12. Let Ω be the surface obtained by cutting Σ_n along $\bigcup_{i=1}^{n-1} d_i$. Then Ω has two connected components Ω_1 and Ω_2. Each of these components is a cylinder that we represent by a square with a hole in the middle, as shown in Figure 13. Two opposite sides of each square represent arcs of d_{n-2}, one side represents an arc of d_{n-1} and the last side represents a union of arcs of $d_1, \dots, d_{n-3}$. The boundary of the hole represents ∂_1 for Ω_1 and ∂_2 for Ω_2. The puncture x sits inside Ω_2. The trace of the circle c in Ω is a simple arc ℓ, either in Ω_1 or in Ω_2.

Suppose ℓ is in Ω_1. Let q be the intersection point of c with d_{n-2}. Then q is represented in Ω_1 by two points q_1 and q_2 on two opposite sides of Ω_1, as shown in Figure 13, and ℓ is a simple arc connecting q_1 with q_2. Up to isotopy pointwise fixing the boundary of Ω_1, there exist exactly two simple arcs in Ω_1

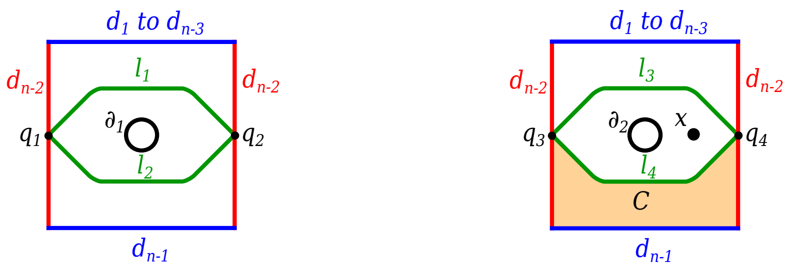

Figure 13: The surface Ω with components Ω_1 (left) and Ω_2 (right).

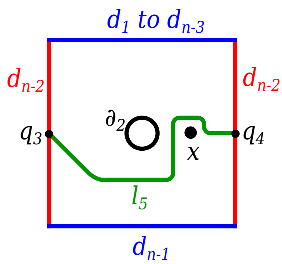

Figure 14: The arc ℓ_5.

connecting q_1 to q_2 that are represented by the arcs ℓ_1 and ℓ_2 depicted in Figure 13. The arc ℓ cannot be isotopic to ℓ_1, otherwise c would not be isotopic to d_{n-1} in Σ_n. So ℓ is isotopic to ℓ_2 in Ω_1, which implies that c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.

Now suppose ℓ is in Ω_2. Let q be the intersection point of c with d_{n-2}. Then q is represented in Ω_2 by two points q_3 and q_4 on two opposite sides of Ω_2, as shown in Figure 13, and ℓ is a simple arc connecting q_3 with q_4. Up to isotopy (in Ω_2 and not in $\Omega_2 \setminus \{x\}$) pointwise fixing the boundary of Ω_2, there exist exactly two simple arcs in Ω_2 connecting q_3 to q_4 that are represented by the arcs ℓ_3 and ℓ_4 depicted in Figure 13. The arc ℓ cannot be isotopic to ℓ_3 in Ω_2, otherwise c would not be isotopic to d_{n-1} in Σ_n. So ℓ is isotopic to ℓ_4 in Ω_2. Let $\{F_t\colon \Omega_2 \to \Omega_2\}_{t\in[0,1]}$ be an isotopy such that $F_0 = \mathrm{id}$, $F_1(\ell) = \ell_4$ and F_t is the identity on the boundary of Ω_2 for all $t \in [0,1]$. The arc ℓ_4 divides Ω_2 into two parts: the lower one, which does not contain the hole bordered by ∂_2 and the puncture x, and the upper one, which contains the hole bordered by ∂_2 and the puncture x, as shown in Figure 13.

Suppose $F_1(x)$ is in the upper part. Let C be the domain of Ω_2 bounded by ℓ_4, two arcs of d_{n-2} and an arc of d_{n-1}, as shown in Figure 13. Let $C' = F_1^{-1}(C)$. Then C' is a domain of Ω_2 bounded by ℓ, two arcs of d_{n-2} and an arc of d_{n-1}, and C' does not contain the puncture x. The existence of such a domain implies that c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.

Now suppose $F_1(x)$ is in the lower part. We can assume without loss of generality that the trace of d_n on Ω_2 is the simple arc ℓ_5 drawn in Figure 14. We can choose an isotopy $\{F'_t\colon \Omega_2 \to \Omega_2\}_{t\in[0,1]}$ such that $F'_0 = \mathrm{id}$, $F'_1(\ell_4) = \ell_5$, F'_t is the identity on the boundary of Ω_2 for all $t \in [0,1]$, and $F'_1(F_1(x)) = x$. Let $\widetilde{F}\colon \Sigma_n \to \Sigma_n$ be the homeomorphism which is $F'_1 \circ F_1$ on Ω_2 and is the identity outside Ω_2, and let $g \in \mathcal{M}(\Sigma_n, x)$ be the mapping class represented by $\widetilde{F}$. Then $g \in \mathrm{Ker}(\theta)$, $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$, and $g([c]) = [d_n]$. □

Remark The element g at the end of the proof of Lemma 5.6 is not necessarily trivial. For example, ℓ can be as shown in Figure 15 up to isotopy and, in this case, g must be nontrivial. In fact, g can be any element of the fundamental group $\pi_1(\Omega_2, x)$, which is an infinite cyclic group, seen as a subgroup of $\mathcal{M}(\Sigma_n, x)$.

The following lemma is the extension of Lemma 5.6 to the case $c \cap d_k \neq \varnothing$.

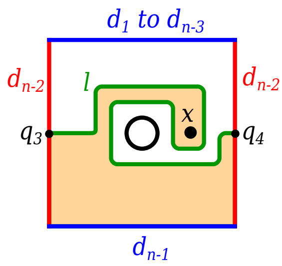

Figure 15: An arc ℓ nonisotopic to ℓ_5.

Lemma 5.7 *Let $n \geq 4$ be even. Let c be a generic circle of $\Sigma_n \setminus \{x\}$ such that $c \cap d_i = \varnothing$ for all $1 \leq i \leq n-3$, $|c \cap d_{n-2}| = 1$ and c is isotopic to d_{n-1} in Σ_n. Then there exists $g \in \mathrm{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \leq i \leq n-2$, and either $g([c]) = [d_{n-1}]$ or $g([c]) = [d_n]$.*

Proof In this proof the intersection number of two circles is computed on the surface $\Sigma_n \setminus \{x\}$ and not on Σ_n. We can assume that $|c \cap d_{n-1}| = i([c],[d_{n-1}])$ and $|c \cap d_n| = i([c],[d_n])$. We argue by induction on $|c \cap d_{n-1}| + |c \cap d_n| = i([c],[d_{n-1}]) + i([c],[d_n])$. The case $|c \cap d_{n-1}| = 0$ follows directly from Lemma 5.6, and the case $|c \cap d_n| = 0$ is proved in the same way by replacing d_{n-1} with d_n. So we can assume that $i([c],[d_{n-1}]) = |c \cap d_{n-1}| \geq 1$, $i([c],[d_n]) = |c \cap d_n| \geq 1$ and that the induction hypothesis holds. Note that now c and d_{n-1} cannot be isotopic in $\Sigma_n \setminus \{x\}$. Since c and d_{n-1} are isotopic in Σ_n, there exists a bigon D in Σ_n cobounded by an arc of d_{n-1} and an arc of c (see Figure 16). Since c and d_{n-1} are not isotopic in $\Sigma_n \setminus \{x\}$, this bigon necessarily contains the puncture x. We can choose D to be minimal in the sense that its interior does not intersect c and d_{n-1}. Moreover, up to exchanging the roles of d_{n-1} and d_n if necessary, we can also assume that d_n does not intersect the interior of D. Clearly D does not intersect d_i for any $1 \leq i \leq n-3$ and, up to replacing c with an isotopic circle, we can assume that D does not intersect d_{n-2} either. Let c' be a circle parallel to c except in the bigon D, where it follows the arc of d_{n-1}, which borders D as illustrated in Figure 16. We have $c' \cap d_i = \varnothing$ for all $1 \leq i \leq n-3$, $|c' \cap d_{n-2}| = 1$ and c' is isotopic to d_{n-1} in Σ_n. We also have $i([c'],[d_{n-1}]) < i([c],[d_{n-1}])$ and $i([c'],[d_n]) \leq i([c],[d_n])$; hence by the induction hypothesis there exists $g_1 \in \mathrm{Ker}(\theta)$ such that $g_1([d_i]) = [d_i]$ for all $1 \leq i \leq n-2$, and either $g_1([c']) = [d_{n-1}]$ or $g_1([c']) = [d_n]$. Without loss of generality we can assume that $g_1([c']) = [d_{n-1}]$. We choose $G_1 \in \mathrm{Homeo}^+(\Sigma_n, x)$ which represents g_1 such that $G_1(d_i) = d_i$ for all $1 \leq i \leq n-2$ and $G_1(c') = d_{n-1}$. We set $c'' = G_1(c)$. Then $c'' \cap d_i = \varnothing$ for all $1 \leq i \leq n-3$, $|c'' \cap d_{n-2}| = 1$, $c'' \cap d_{n-1} = \varnothing$ and c'' is isotopic to d_{n-1} in Σ_n. By Lemma 5.6 there exists $g_2 \in \mathrm{Ker}(\theta)$ such that $g_2([d_i]) = [d_i]$ for all $1 \leq i \leq n-2$, and either $g_2([c'']) = [d_{n-1}]$ or $g_2([c'']) = [d_n]$. We set $g = g_2 \circ g_1$. Then $g \in \mathrm{Ker}(\theta)$, $g([d_i]) = [d_i]$ for all $1 \leq i \leq n-2$, and either $g([c]) = [d_{n-1}]$ or $g([c]) = [d_n]$. □

Lemma 5.8 *Let $n \geq 6$. Let $c_1, \dots, c_{n-1}$ be generic circles in $\Sigma_n \setminus \{x\}$ such that*

(a) *$|c_i \cap c_j| = 1$ if $|i-j| = 1$ and $|c_i \cap c_j| = 0$ if $|i-j| \geq 2$, for all $1 \leq i, j \leq n-1$,*

(b) *c_i is isotopic to d_i in Σ_n for all $1 \leq i \leq n-1$.*

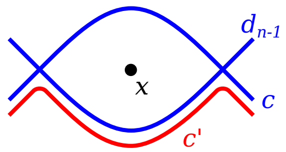

Figure 16: The bigon cobounded by c and d_{n-1}.

Then:

(1) *If n is odd, then there exists $g \in \mathrm{Ker}(\theta)$ such that $g([c_i]) = [d_i]$ for all $1 \le i \le n-1$.*

(2) *If n is even, then there exists $g \in \mathrm{Ker}(\theta)$ such that $g([c_i]) = [d_i]$ for all $1 \le i \le n-2$, and either $g([c_{n-1}]) = [d_{n-1}]$ or $g([c_{n-1}]) = [d_n]$.*

Proof For $1 \le k \le n-2$ we construct by induction on k an element $g_k \in \mathrm{Ker}(\theta)$ such that $g_k([c_i]) = [d_i]$ for all $1 \le i \le k$. Assume $k = 1$. Then, by Lemma 5.5 applied to $k = 1$, there exists $g_1 \in \mathrm{Ker}(\theta)$ such that $g_1([c_1]) = [d_1]$. Suppose $2 \le k \le n-1$ and g_{k-1} is constructed. We choose $G_{k-1} \in \mathrm{Homeo}^+(\Sigma_n, x)$ which represents g_{k-1} such that $G_{k-1}(c_i) = d_i$ for all $1 \le i \le k-1$, and we set $c'_k = G_{k-1}(c_k)$. Note that, since $g_{k-1} \in \mathrm{Ker}(\theta)$, the circle c'_k is isotopic to c_k in Σ_n. Then, by Lemma 5.5, there exists $h_k \in \mathrm{Ker}(\theta)$ such that $h_k([d_i]) = [d_i]$ for all $1 \le i \le k-1$ and $h_k([c'_k]) = [d_k]$. We set $g_k = h_k \circ g_{k-1}$. Then $g_k([c_i]) = [d_i]$ for all $1 \le i \le k$. Note that when n is odd we can extend the induction to $k = n-1$ and conclude the proof here by setting $g = g_{n-1}$. The case where n is even requires an extra argument.

Assume n is even. We choose $G_{n-2} \in \mathrm{Homeo}^+(\Sigma_n, x)$ which represents g_{n-2} and such that $G_{n-2}(c_i) = d_i$ for all $1 \le i \le n-2$, and we set $c'_{n-1} = G_{n-2}(c_{n-1})$. Again, since $g_{n-2} \in \mathrm{Ker}(\theta)$, the circle c'_{n-1} is isotopic to c_{n-1} in Σ_n. By Lemma 5.7 there exists $h_{n-1} \in \mathrm{Ker}(\theta)$ such that $h_{n-1}([d_i]) = [d_i]$ for all $1 \le i \le n-2$, and either $h_{n-1}([c'_{n-1}]) = [d_{n-1}]$ or $h_{n-1}([c'_{n-1}]) = [d_n]$. We set $g = h_{n-1} \circ g_{n-2}$. Then $g([c_i]) = [d_i]$ for all $1 \le i \le n-2$, and either $g([c_{n-1}]) = [d_{n-1}]$ or $g([c_{n-1}]) = [d_n]$. □

Proof of Theorem 2.2 Let $n \ge 6$ and let $\varphi\colon A[A_{n-1}] \to A[D_n]$ be a homomorphism. Composing φ with π, we get a homomorphism $\pi \circ \varphi\colon A[A_{n-1}] \to A[D_n] \to A[A_{n-1}]$. We know by Theorem 3.5 that we have one of the following possibilities:

- $\pi \circ \varphi$ is cyclic.
- There exist $\psi \in \langle \bar{\chi} \rangle$ and $p \in \mathbb{Z}$ such that $\pi \circ \varphi$ is conjugate to $\psi \circ \bar{\gamma}_p$.

By Lemma 5.2, if $\pi \circ \varphi$ is cyclic, then φ is cyclic. So we can assume that there exist $\psi \in \langle \bar{\chi} \rangle$ and $p \in \mathbb{Z}$ such that $\pi \circ \varphi$ is conjugate to $\psi \circ \bar{\gamma}_p$. Up to conjugating and composing φ on the left by χ if necessary, we can assume that $\pi \circ \varphi = \bar{\gamma}_p$, that is, $(\pi \circ \varphi)(s_i) = s_i \Delta_A^{2p}$, where Δ_A denotes the Garside element of $A[A_{n-1}]$.

Set $U = \rho_A(\Delta_A^2)$. If n is odd, then $U^2 = T_\partial$, where ∂ is the boundary component of Σ_n, and if n is even, then $U = T_{\partial_1} T_{\partial_2}$, where ∂_1 and ∂_2 are the two boundary components of Σ_n (see Labruère and Paris [32, Proposition 2.12]). In particular $U^2 \in Z(\mathcal{M}(\Sigma_n))$ in both cases.

By Theorem 3.7 there exist generic circles $c_1, \dots, c_{n-1}$ in $\Sigma_n \setminus \{x\}$, $\varepsilon \in \{\pm 1\}$ and $f_0 \in \mathcal{M}(\Sigma_n, x)$ such that

(a) $|c_i \cap c_j| = 1$ if $|i-j| = 1$ and $|c_i \cap c_j| = 0$ if $|i-j| \ge 2$, for all $1 \le i, j \le n-1$,

(b) f_0 commutes with T_{c_i} for all $1 \le i \le n-1$,

(c) $(\rho_D \circ \varphi)(s_i) = T_{c_i}^{\varepsilon} f_0$ for all $1 \le i \le n-1$.

For $1 \le i \le n-1$ we denote by b_i the circle in Σ_n obtained by composing $c_i : \mathbb{S}^1 \to \Sigma_n \setminus \{x\}$ with the embedding $\Sigma_n \setminus \{x\} \hookrightarrow \Sigma_n$. In addition we set $g_0 = \theta(f_0)$. Then $(\theta \circ \rho_D \circ \varphi)(s_i) = T_{b_i}^{\varepsilon} g_0$ for all $1 \le i \le n-1$. Note that, since $\theta \circ \rho_D = \rho_A \circ \pi$ (see (3-1)), we also have $(\theta \circ \rho_D \circ \varphi)(s_i) = (\rho_A \circ \bar{\gamma}_p)(s_i) = \rho_A(s_i \Delta_A^{2p}) = T_{a_i} U^p$ for all $1 \le i \le n-1$, where the a_i are the circles depicted in Figure 3.

Claim *We have $\varepsilon = 1$, $g_0 = U^p$ and b_i is isotopic to a_i in Σ_n for all $1 \le i \le n-1$.*

Proof of the claim Note that $g_0 = \theta(f_0)$ commutes with $T_{b_i} = \theta(T_{c_i})$ and $U = \rho_A(\Delta_A^2)$ commutes with $T_{a_i} = \rho_A(s_i)$; hence $T_{b_i}^{2\varepsilon} g_0^2 = (T_{b_i}^{\varepsilon} g_0)^2 = (T_{a_i} U^p)^2 = T_{a_i}^2 U^{2p}$. Since g_0^2 commutes with $T_{b_i}^{2\varepsilon} g_0^2 = T_{a_i}^2 U^{2p}$ and $U^2 \in Z(\mathcal{M}(\Sigma_n))$, g_0^2 commutes with $T_{a_i}^2$ for all $1 \le i \le n-1$. By Lemma 5.3 it follows that $g_0^4 \in Z(\mathcal{M}(\Sigma_n))$. By Proposition 3.3 applied to $\mathcal{M}(\Sigma_n)$ we deduce that $\mathcal{S}(T_{a_i}^4 U^{4p}) = \mathcal{S}(T_{b_i}^{4\varepsilon} g_0^4) = \{[a_i]\} = \{[b_i]\}$, and hence $[a_i] = [b_i]$ for all $1 \le i \le n-1$. Then $T_{a_i}^{4-4\varepsilon} = U^{-4p} g_0^4$; hence, by Proposition 3.3, $4 - 4\varepsilon = 0$, and therefore $\varepsilon = 1$. Finally, from the equality $T_{a_i} U^p = T_{a_i} g_0$ it follows that $g_0 = U^p$. □

From the claim it follows that c_i is isotopic to d_i in Σ_n. Hence, by Lemma 5.8, there exists $g \in \mathrm{Ker}(\theta)$ such that $g([c_i]) = [d_i]$ for all $1 \le i \le n-2$, $g([c_{n-1}]) = [d_{n-1}]$ if n is odd, and either $g([c_{n-1}]) = [d_{n-1}]$ or $g([c_{n-1}]) = [d_n]$ if n is even. These equalities imply that $g T_{c_i} g^{-1} = T_{d_i}$ for $1 \le i \le n-2$, $g T_{c_{n-1}} g^{-1} = T_{d_{n-1}}$ if n is odd, and either $g T_{c_{n-1}} g^{-1} = T_{d_{n-1}}$ or $g T_{c_{n-1}} g^{-1} = T_{d_n}$ if n is even. By Theorem 3.6(1) there exists $v \in \mathrm{Ker}(\pi)$ such that $\rho_D(v) = g$. So, up to composing φ on the left by ad_v first, and composing on the left by ζ if necessary after, we can assume that $(\rho_D \circ \varphi)(s_i) = T_{d_i} f_0$ for all $1 \le i \le n-1$, where f_0 commutes with T_{d_i} for all $1 \le i \le n-1$. Since $T_{d_1} = \rho_D(t_1) \in \mathrm{Im}(\rho_D)$, we have $f_0 \in \mathrm{Im}(\rho_D)$, and hence there exists $u_0 \in A[D_n]$ such that $\rho_D(u_0) = f_0$. Since ρ_D is injective (see Theorem 3.6), we deduce that $\varphi(s_i) = t_i u_0$ for all $1 \le i \le n-1$ and u_0 commutes with t_i for all $1 \le i \le n-1$. We set $Y = \{t_1, \dots, t_{n-1}\}$, $\Delta_Y = \Delta_Y[D_n]$, $\Delta_D = \Delta[D_n]$, $\kappa = 2$ if n is odd, and $\kappa = 1$ if n is even. By Paris [36, Theorem 1.1] the centralizer of Y in $A[D_n]$ is generated by Δ_Y^2 and Δ_D^{κ}, and hence there exist $q, r \in \mathbb{Z}$ such that $u_0 = \Delta_Y^{2q} \Delta_D^{\kappa r}$. We conclude that $\varphi = \beta_{q,r}$. □

6 Endomorphisms of $A[D_n]$

The following lemma is a counterpart of Lemma 5.8 for the case of odd n, and it is a preliminary to the proof of Theorem 2.3.

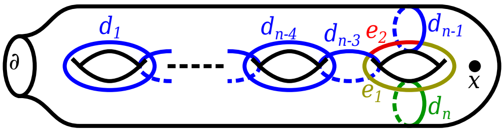

Figure 17: The circles $d_1, \dots, d_n$.

Lemma 6.1 *Let $n \ge 5$ be odd. Let c be a generic circle of $\Sigma_n \setminus \{x\}$ such that $c \cap d_i = \varnothing$ for $1 \le i \le n-3$, $|c \cap d_{n-2}| = 1$, $c \cap d_{n-1} = \varnothing$ and c is isotopic to d_{n-1} in Σ_n. Then we have one of the following three possibilities*:

(1) *c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.*

(2) *There exists $g \in \operatorname{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$ and $g([c]) = [d_n]$.*

(3) *There exists $g \in \operatorname{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le n-2$, $g([d_{n-1}]) = [d_n]$ and $g([c]) = [d_{n-1}]$.*

Proof The surface Σ_n is of genus $\frac{1}{2}(n-1)$ with one boundary component, ∂. We assume that the circles $d_1, \dots, d_{n-1}, d_n$ are arranged as shown in Figure 17. The circles d_{n-3} and d_{n-1} divide d_{n-2} into two arcs, e_1 and e_2, where the arc e_1 intersects d_n and the arc e_2 does not intersect d_n (see Figure 17). Let Ω be the surface obtained by cutting Σ_n along $\bigcup_{i=1}^{n-1} d_i$. Then Ω is a cylinder represented by an octagon with a hole in the middle (see Figure 18). Two opposite sides of this octagon represent arcs of d_{n-1} and two opposite sides represent arcs of $d_1, \dots, d_{n-3}$, as shown in the figure. Two other sides represent arcs of e_1 and the last two sides represent arcs of e_2, arranged as shown in Figure 18. The boundary of the hole represents ∂.

The circle c intersects d_{n-2} in a point q, and q is either on the arc e_1 or on the arc e_2. Suppose first that q is on the arc e_1. Then q is represented on Ω by two points q_1 and q_2 lying on two different sides of Ω that represent e_1, and the trace of c in Ω is a simple arc ℓ connecting q_1 to q_2. Up to isotopy (in Ω and not in $\Omega \setminus \{x\}$) pointwise fixing the boundary of Ω, there are exactly two simple arcs in Ω connecting q_1 to q_2, represented by the arcs ℓ_1 and ℓ_2 depicted in Figure 18. The arc ℓ cannot be isotopic to ℓ_2, otherwise c would not be isotopic to d_{n-1} in Σ_n. So ℓ is isotopic to ℓ_1 in Ω. Let $\{F_t : \Omega \to \Omega\}_{t \in [0,1]}$ be an isotopy such that $F_0 = \mathrm{id}$, $F_1(\ell) = \ell_1$ and F_t is the identity on the boundary of Ω for all $t \in [0,1]$. The arc ℓ_1

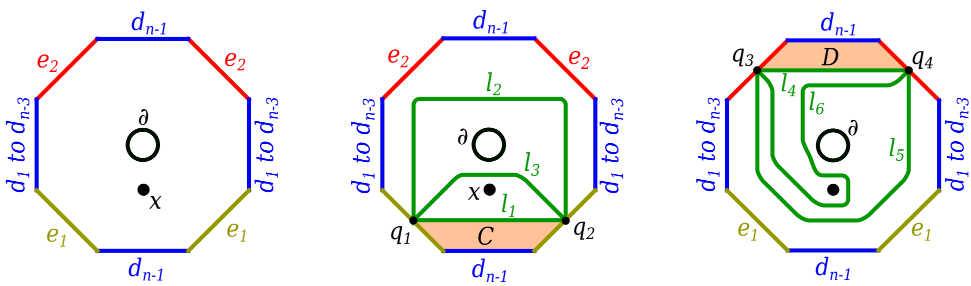

Figure 18: The surface Ω.

divides Ω into two parts: the lower one, which does not contain the hole bounded by ∂ and the puncture x, and the upper one, which contains the hole bounded by ∂ and the puncture x, as shown in Figure 18.

Suppose $F_1(x)$ is in the upper part. Let C be the domain of Ω bounded by ℓ_1, two arcs of e_1 and an arc of d_{n-1}, as shown in Figure 18. Let $C' = F_1^{-1}(C)$. Then C' is a domain of Ω bounded by ℓ, two arcs of e_1 and an arc of d_{n-1} which does not contain the puncture x. The existence of such a domain implies that c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.

Suppose $F_1(x)$ is in the lower part. We can suppose that the trace of d_n on Ω is the arc ℓ_3 depicted in Figure 18. We can choose an isotopy $\{F'_t : \Omega \to \Omega\}_{t \in [0,1]}$ such that $F'_0 = \mathrm{id}$, $F'_1(\ell_1) = \ell_3$, F'_t is the identity on the boundary of Ω for all $t \in [0,1]$, and $F'_1(F_1(x)) = x$. Let $\widetilde{F} : \Sigma_n \to \Sigma_n$ be the homeomorphism which is $F'_1 \circ F_1$ on Ω and is the identity outside Ω, and let $g \in \mathcal{M}(\Sigma_n, x)$ be the mapping class represented by $\widetilde{F}$. Then $g \in \mathrm{Ker}(\theta)$, $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$, and $g([c]) = [d_n]$.

Suppose now that q is on the arc e_2. Then q is represented on Ω by two points q_3 and q_4 lying on two different sides of Ω which represent e_2, and the trace of c in Ω is a simple arc ℓ connecting q_3 to q_4. Up to isotopy (in Ω and not in $\Omega \setminus \{x\}$) pointwise fixing the boundary of Ω, there are exactly two simple arcs in Ω connecting q_3 to q_4 represented by the arcs ℓ_4 and ℓ_5 depicted in Figure 18. The arc ℓ cannot be isotopic to ℓ_5, otherwise c would not be isotopic to d_{n-1} in Σ_n. So ℓ is isotopic to ℓ_4 in Ω. Let $\{F_t : \Omega \to \Omega\}_{t \in [0,1]}$ be an isotopy such that $F_0 = \mathrm{id}$, $F_1(\ell) = \ell_4$ and F_t is the identity on the boundary of Ω for all $t \in [0,1]$. The arc ℓ_4 divides Ω into two parts: the upper one, which does not contain the hole bounded by ∂ and the puncture x, and the lower one, which contains the hole bounded by ∂ and the puncture x, as shown in Figure 18.

Suppose $F_1(x)$ is in the lower part. Let D be the domain of Ω bounded by ℓ_4, two arcs of e_2 and an arc of d_{n-1} as shown in Figure 18. Let $D' = F_1^{-1}(D)$. Then D' is a domain of Ω bounded by ℓ, two arcs of e_2 and an arc of d_{n-1} which does not contain the puncture x. The existence of such a domain implies that c is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.

Suppose $F_1(x)$ is in the upper part. Let c' be the circle drawn in Figure 19. We can assume that the trace of c' on Ω is the arc ℓ_6 drawn in Figure 18. We can choose an isotopy $\{F'_t : \Omega \to \Omega\}_{t \in [0,1]}$ such that $F'_0 = \mathrm{id}$, $F'_1(\ell_4) = \ell_6$, F'_t is the identity on the boundary of Ω for all $t \in [0,1]$, and $F'_1(F_1(x)) = x$. Let $\widetilde{F} : \Sigma_n \to \Sigma_n$ be the homeomorphism which is $F'_1 \circ F_1$ on Ω and is the identity outside Ω, and let $g_1 \in \mathcal{M}(\Sigma_n, x)$ be the mapping class represented by $\widetilde{F}$. Then $g_1 \in \mathrm{Ker}(\theta)$, $g_1([d_i]) = [d_i]$ for all $1 \le i \le n-1$, and $g_1([c]) = [c']$.

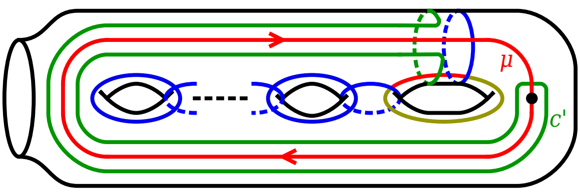

Figure 19: The circle c' and the loop μ.

Let $g_2 \in \pi_1(\Sigma_n, x) = \mathrm{Ker}(\theta)$ be the element represented by the loop μ drawn in Figure 19. Let us mention here that g_2 is not the Dehn twist T_μ along μ, but rather the image of the point-pushing map applied to μ, which is equal to $T_{\mu_1} T_{\mu_2}^{-1}$ for μ_1 and μ_2 the two boundary curves of a small regular neighborhood of μ, as explained in Farb and Margalit [28, Section 4.2.2]. We have $g_2([d_i]) = [d_i]$ for all $1 \le i \le n-2$, $g_2([d_{n-1}]) = [d_n]$ and $g_2([c']) = [d_{n-1}]$. Set $g = g_2 \circ g_1$. Then $g \in \mathrm{Ker}(\theta)$, $g([d_i]) = [d_i]$ for all $1 \le i \le n-2$, $g([d_{n-1}]) = [d_n]$ and $g([c]) = [d_{n-1}]$. □

Proof of Theorem 2.3 Let $n \ge 6$. Let $\varphi\colon A[D_n] \to A[D_n]$ be an endomorphism. Consider the composition homomorphism $\varphi \circ \iota\colon A[A_{n-1}] \to A[D_n] \to A[D_n]$. We know from Theorem 2.2 that we have one of the following two possibilities up to conjugation:

(1) $\varphi \circ \iota$ is cyclic.

(2) There exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi \circ \iota = \psi \circ \beta_{p,q}$.

Suppose $\varphi \circ \iota$ is cyclic. Then there exists $u \in A[D_n]$ such that $\varphi(t_i) = (\varphi \circ \iota)(s_i) = u$ for all $1 \le i \le n-1$. We also have

$$\begin{aligned}\varphi(t_n) = \varphi(t_{n-2} t_n t_{n-2} t_n^{-1} t_{n-2}^{-1}) = \varphi(t_{n-2} t_n) \varphi(t_{n-2}) \varphi(t_n^{-1} t_{n-2}^{-1}) = \varphi(t_{n-2} t_n) \varphi(t_1) \varphi(t_n^{-1} t_{n-2}^{-1}) = \varphi(t_1) \\ = u,\end{aligned}$$

and hence φ is cyclic.

So we can assume that there exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi \circ \iota$ is conjugate to $\psi \circ \beta_{p,q}$. We set $Y = \{t_1, \dots, t_{n-2}, t_{n-1}\}$, $\Delta_Y = \Delta_Y[D_n]$, $\Delta_D = \Delta[D_n]$, $\kappa = 2$ if n is odd, and $\kappa = 1$ if n is even. Up to conjugating and composing φ on the left by ζ if necessary, we can assume that there exist $\varepsilon \in \{\pm 1\}$ and $p, q \in \mathbb{Z}$ such that $\varphi(t_i) = (\varphi \circ \iota)(s_i) = t_i^\varepsilon \Delta_Y^{2p} \Delta_D^{\kappa q}$ for all $1 \le i \le n-1$. The remainder of the proof is divided into four cases depending on whether p is zero or not and whether n is even or odd.

Case 1 (n is even and $p \ne 0$) Then Σ_n is a surface of genus $\frac{1}{2}(n-2)$ with two boundary components, ∂_1 and ∂_2, and $\kappa = 1$. We have $\rho_D(t_i) = T_{d_i}$ for $1 \le i \le n-1$ and, by Labruère and Paris [32, Proposition 2.12], $\rho_D(\Delta_Y^2) = T_e T_{\partial_1}$ and $\rho_D(\Delta_D) = T_{\partial_1} T_{\partial_2}$, where e is the circle drawn in Figure 20. Set $f_i = (\rho_D \circ \varphi)(t_i)$ for all $1 \le i \le n$. Then, by the above,

$$f_i = T_{d_i}^\varepsilon T_e^p T_{\partial_1}^{p+q} T_{\partial_2}^q \quad \text{for all } 1 \le i \le n-1.$$

In particular, $\mathcal{S}(f_i) = \{[d_i], [e]\}$ for all $1 \le i \le n-1$. Since t_n is conjugate in $A[D_n]$ to t_1, f_n is conjugate to f_1 in $\mathcal{M}(\Sigma_n, x)$; hence f_n is of the form $f_n = T_{d'}^\varepsilon T_{e'}^p T_{\partial_1}^{p+q} T_{\partial_2}^q$, where d' is a nonseparating circle

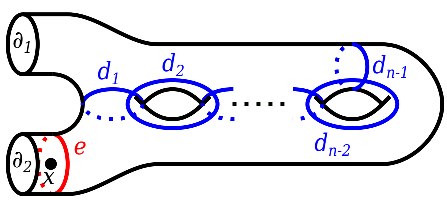

Figure 20: Circles in Σ_n when n is even and $p \ne 0$.

and e' is a circle that separates Σ_n into two components, one being a cylinder containing x and the other being a surface of genus $\frac{1}{2}(n-2)$ with two boundary components, ∂_1 and e', which does not contain x. Moreover, by Theorem 2.1, $(\pi \circ \varphi)(t_{n-1}) = (\pi \circ \varphi)(t_n)$, and hence

$$T_{d_{n-1}}^{\varepsilon} T_e^p T_{\partial_1}^{p+q} T_{\partial_2}^q = \theta(f_{n-1}) = \theta(f_n) = T_{d'}^{\varepsilon} T_{e'}^p T_{\partial_1}^{p+q} T_{\partial_2}^q$$

on Σ_n, that is, $T_{d_{n-1}}^{\varepsilon} T_e^p = T_{d'}^{\varepsilon} T_{e'}^p$ as multitwists on Σ_n. Now we can invoke Farb and Margalit [28, Lemma 3.14] to conclude that each curve of the set $\{d_{n-1}, e\}$ is isotopic to a curve from the set $\{d', e'\}$ in Σ_n. To decide which curve of one set is isotopic to which curve in the other set we observe that removing a puncture does not change the property of a curve being nonseparating, but can make a separating curve peripheral. Since both d_{n-1} and d' are nonseparating, whereas e and e' are both separating or peripheral in Σ_n, we conclude that d_{n-1} is isotopic to d' in Σ_n (and also that e is isotopic to e' in Σ_n).

We have $f_1 f_n = f_n f_1$, and hence by Theorem 3.2(3) we have $f_n(\mathcal{S}(f_1)) = \mathcal{S}(f_1)$. Thus $[e]$ is a reduction class for f_n, and therefore $i([e],[e']) = 0$, because $[e']$ is an essential reduction class for f_n. We can choose representatives e and e' such that $e \cap e' = \varnothing$ either by eliminating bigons, or by choosing geodesic representatives. Let $C, C' \subset \Sigma_n$ be cylinders containing x and having boundaries $\partial_2 \cup e$ and $\partial_2 \cup e'$, respectively. Then either $C \subset C'$ if $e \subset C'$, or $C' \subset C$ if $e' \subset C$, with $x \in C \cap C'$. Say $C \subset C'$. Being a separating circle on Σ_n, e separates C' into two subsurfaces, one containing ∂_2 and x, and the other containing e'. Being a subsurface with two boundary components lying inside a cylinder, the latter must be a cylinder itself. This cylinder establishes an isotopy between e and e' in $\Sigma_n \setminus \{x\}$, and hence $[e] = [e']$. So we can assume that $e = e'$.

Choose representatives d_{n-1} and d' in minimal position in $\Sigma_n \setminus \{x\}$. Denote by C_0 and Σ' the two components into which the curve e separates Σ_n, with C_0 being a cylinder containing x, and Σ' being the rest of the surface Σ_n, containing $d_1, \dots, d_{n-1}$. Suppose $d_{n-1} \cap d' \neq \varnothing$. Then d_{n-1} and d' cobound a bigon. Since d_{n-1} and d' were chosen to be in minimal position in $\Sigma_n \setminus \{x\}$, such a bigon must contain x. This implies that d' has nonempty intersection with the cylinder C_0 which e separates from the rest of the surface Σ_n, and since e and d' are disjoint, d' lies entirely in C_0. This is not possible because any generic circle in C_0 is peripheral in Σ_n and d' is nonseparating in Σ_n. So $d_{n-1} \cap d' = \varnothing$. Then there exists an embedded cylinder C in Σ_n with boundary components d_{n-1} and d'. Since e is disjoint from d' and d_{n-1}, e either lies entirely in C or is disjoint from C. The circle e cannot lie entirely in C because e is peripheral in Σ_n and, since both d_{n-1} and d' are nonseparating in Σ_n, any generic circle lying in C must be nonseparating. So e is disjoint from C, and hence C lies in Σ'. Therefore d_{n-1} is isotopic to d' in $\Sigma_n \setminus \{x\}$. Thus we can also assume $d' = d_{n-1}$.

In conclusion we have $(\rho_D \circ \varphi)(t_{n-1}) = (\rho_D \circ \varphi)(t_n) = T_{d_{n-1}}^{\varepsilon} T_e^p T_{\partial_1}^{p+q} T_{\partial_2}^q$, and hence $\varphi(t_{n-1}) = \varphi(t_n) = t_{n-1}^{\varepsilon} \Delta_Y^{2p} \Delta_D^q$. We conclude that $\varphi = \beta_{p,q} \circ \pi$ if $\varepsilon = 1$ and $\varphi = \chi \circ \beta_{-p,-q} \circ \pi$ if $\varepsilon = -1$.

Case 2 (n is odd and $p \neq 0$) Then Σ_n is a surface of genus $\frac{1}{2}(n-1)$ with one boundary component, ∂, and $\kappa = 2$. We have $\rho_D(t_i) = T_{d_i}$ for $1 \le i \le n-1$ and, by Labruère and Paris [32, Proposition 2.12],

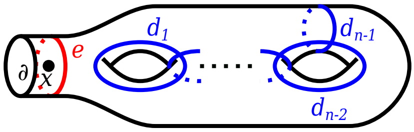

Figure 21: Circles in Σ_n when n is odd and $p \neq 0$.

$\rho_D(\Delta_Y^4) = T_e$ and $\rho_D(\Delta_D^2) = T_\partial$, where e is the circle drawn in Figure 21. Set $f_i = (\rho_D \circ \varphi)(t_i)$ for all $1 \le i \le n$. Then, by the above,

$$f_i^2 = T_{d_i}^{2\varepsilon} T_e^p T_\partial^{2q} \quad \text{for all } 1 \le i \le n-1.$$

In particular, $\mathcal{S}(f_i) = \mathcal{S}(f_i^2) = \{[d_i], [e]\}$ for all $1 \le i \le n-1$. The element t_n is conjugate to t_1 in $A[D_n]$; hence $\varphi(t_n)$ is conjugate to $\varphi(t_1)$ in $A[D_n]$, and therefore there exists $v \in A[D_n]$ such that $\varphi(t_n) = v\varphi(t_1)v^{-1} = (vt_1^\varepsilon v^{-1})(v\Delta_Y^{2p}v^{-1})\Delta_D^{2q}$. The element $\rho_D(vt_1v^{-1})$ is conjugate to $\rho_D(t_1) = T_{d_1}$, and hence $\rho_D(vt_1v^{-1}) = T_{d'}$, where d' is a nonseparating circle. The element $\rho_D(v\Delta_Y^4 v^{-1})$ is conjugate to $\rho_D(\Delta_Y^4) = T_e$, and hence $\rho_D(v\Delta_Y^4 v^{-1}) = T_{e'}$, where e' is a circle that separates Σ_n into two components, one being a cylinder containing x and the other being a surface of genus $\frac{1}{2}(n-1)$ with one boundary component which does not contain x. We also have $f_n^2 = T_{d'}^2 T_{e'}^p T_\partial^{2q}$ and $\mathcal{S}(f_n) = \mathcal{S}(f_n^2) = \{[d'], [e']\}$. By Theorem 2.1 $(\pi \circ \varphi)(t_{n-1}) = (\pi \circ \varphi)(t_n)$, and hence $\theta(f_{n-1}^2) = \theta(f_n^2)$. This implies that d' is isotopic to d_{n-1} in Σ_n.

Since $f_1 f_n = f_n f_1$, by Theorem 3.2(3) we have $f_n^2(\mathcal{S}(f_1)) = \mathcal{S}(f_1)$; hence $[e]$ is a reduction class for f_n^2, and therefore $i([e], [e']) = 0$, because $[e']$ is an essential reduction class for f_n^2. As in Case 1, we can choose representatives e and e' such that $e \cap e' = \varnothing$. Let $C, C' \subset \Sigma_n$ be cylinders containing x and having boundaries $\partial \cup e$ and $\partial \cup e'$, respectively. Then either $C \subset C'$ if $e \subset C'$, or $C' \subset C$ if $e' \subset C$, with $x \in C \cap C'$. Say $C \subset C'$. Being a separating circle on Σ_n, e separates C' into two subsurfaces, one containing ∂ and x, and the other containing e'. Being a subsurface with two boundary components lying inside a cylinder, the latter must be a cylinder itself. This cylinder establishes an isotopy between e and e' in $\Sigma_n \setminus \{x\}$, and hence $[e] = [e']$. So we can assume that $e = e'$, and hence $\rho_D(v\Delta_Y^4 v^{-1}) = T_{e'} = T_e = \rho_D(\Delta_Y^4)$. Since ρ_D is injective, it follows that $v\Delta_Y^4 v^{-1} = \Delta_Y^4$.

Using the same argument as in Case 1, from the fact that d' does not intersect $e' = e$ and that d' is isotopic to d_{n-1} in Σ_n, it follows that d' is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$; hence we can also assume that $d' = d_{n-1}$. Then $\rho_D(vt_1v^{-1}) = T_{d'} = T_{d_{n-1}} = \rho_D(t_{n-1})$, and hence, since ρ_D is injective, $vt_1v^{-1} = t_{n-1}$. At this stage of the proof we have that $\varphi(t_n) = t_{n-1}^\varepsilon (v\Delta_Y^{2p}v^{-1})\Delta_D^{2q}$ and $(v\Delta_Y^{2p}v^{-1})^2 = v\Delta_Y^{4p}v^{-1} = \Delta_Y^{4p}$. It remains to show that $v\Delta_Y^{2p}v^{-1} = \Delta_Y^{2p}$.

By Theorem 2.2 there exist $\psi \in \langle \zeta, \chi \rangle$ and $r, s \in \mathbb{Z}$ such that $\varphi \circ \zeta \circ \iota$ is conjugate to $\psi \circ \beta_{r,s}$. The automorphism ζ is inner since n is odd, and hence we can assume that $\psi \in \langle \chi \rangle$. So there exist $w \in A[D_n]$, $\mu \in \{\pm 1\}$ and $r, s \in \mathbb{Z}$ such that $\varphi(t_i) = wt_i^\mu \Delta_Y^{2r} \Delta_D^{2s} w^{-1}$ for all $1 \le i \le n-2$ and $\varphi(t_n) = wt_{n-1}^\mu \Delta_Y^{2r} \Delta_D^{2s} w^{-1}$. Set $g = \rho_D(w)$. We have $(\rho_D \circ \varphi)(t_i^2) = T_{d_i}^{2\varepsilon} T_e^p T_\partial^{2q} = gT_{d_i}^{2\mu} T_e^r T_\partial^{2s} g^{-1}$ for all $1 \le i \le n-2$ and $(\rho_D \circ \varphi)(t_n^2) = T_{d_{n-1}}^{2\varepsilon} T_e^p T_\partial^{2q} = gT_{d_{n-1}}^{2\mu} T_e^r T_\partial^{2s} g^{-1}$. So $g^{-1}(\mathcal{S}(T_{d_i}^{2\varepsilon} T_e^p T_\partial^{2q})) =$

$\mathcal{S}(T_{d_i}^{2\mu} T_e^r T_\partial^{2s})$, and hence $g^{-1}(\{[d_i],[e]\}) \subset \{[d_i],[e]\}$ for all $1 \le i \le n-1$. This implies $g^{-1}([d_i]) = [d_i]$, and hence g commutes with T_{d_i}; therefore w commutes with t_i for all $1 \le i \le n-1$. Since Δ_Y is in the subgroup of $A[D_n]$ generated by $Y = \{t_1, \dots, t_{n-1}\}$ and Δ_D^2 is central, it follows that $\varphi(t_i) = t_i^\mu \Delta_Y^{2r} \Delta_D^{2s}$ for all $1 \le i \le n-2$ and $\varphi(t_n) = t_{n-1}^\mu \Delta_Y^{2r} \Delta_D^{2s}$. Consider the equality $\varphi(t_1) = t_1^\varepsilon \Delta_Y^{2p} \Delta_D^{2q} = t_1^\mu \Delta_Y^{2r} \Delta_D^{2s}$. Then $t_1^{\varepsilon-\mu} \Delta_Y^{2(p-r)} = \Delta_D^{2(s-q)}$. The right-hand side of this equality lies in the center of $A[D_n]$, the left-hand side lies in $A_Y[D_n]$ and, by Paris [38, Corollary 2.6], the intersection of $A_Y[D_n]$ with the center of $A[D_n]$ is trivial; hence $s = q$ and $t_1^{\varepsilon-\mu} = \Delta_Y^{2(r-p)}$. The element $\Delta_Y^{2(r-p)}$ lies in the center of $A_Y[D_n]$ and $\langle t_1 \rangle$ is a proper parabolic subgroup of $A_Y[D_n]$; hence, again by Paris [38, Corollary 2.6], $t_1^{\varepsilon-\mu} = \Delta_Y^{2(r-p)} = 1$, and therefore $\varepsilon = \mu$ and $r = p$. Here we use that $A[D_n]$ is torsion-free, which follows from Deligne [25], where it is proved that $A[D_n]$ has a finite-dimensional classifying space. So $\varphi(t_n) = t_{n-1}^\varepsilon \Delta_Y^{2p} \Delta_D^{2q}$. We conclude that $\varphi = \beta_{p,q} \circ \pi$ if $\varepsilon = 1$ and $\varphi = \chi \circ \beta_{-p,-q} \circ \pi$ if $\varepsilon = -1$.

Case 3 (n is even and $p = 0$) Then, again, Σ_n is a surface of genus $\frac{1}{2}(n-2)$ with two boundary components, ∂_1 and ∂_2, and $\kappa = 1$. We have $\rho_D(t_i) = T_{d_i}$ for $1 \le i \le n-1$ and, by Labruère and Paris [32, Proposition 2.12], $\rho_D(\Delta_D) = T_{\partial_1} T_{\partial_2}$. Set $f_i = (\rho_D \circ \varphi)(t_i)$ for all $1 \le i \le n$. Then, by the above,

$$f_i = T_{d_i}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q \quad \text{for all } 1 \le i \le n-1.$$

In particular, $\mathcal{S}(f_i) = \{[d_i]\}$ for all $1 \le i \le n-1$. Since t_n is conjugate in $A[D_n]$ to t_1, f_n is of the form $f_n = T_{d'}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q$, where d' is a nonseparating circle.

For $1 \le i \le n-3$ we have $t_i t_n = t_n t_i$; hence $T_{d_i} T_{d'} = T_{d'} T_{d_i}$, and therefore, by Proposition 3.1, $i([d_i],[d']) = 0$. Similarly, $i([d_{n-1}],[d']) = 0$. Since $t_{n-2} t_n t_{n-2} = t_n t_{n-2} t_n$, we have $T_{d_{n-2}} T_{d'} T_{d_{n-2}} = T_{d'} T_{d_{n-2}} T_{d'}$, and hence, by Proposition 3.1, $i([d_{n-2}],[d']) = 1$. So we can assume that $d_i \cap d' = \varnothing$ for $1 \le i \le n-3$, $d_{n-1} \cap d' = \varnothing$ and $|d_{n-2} \cap d'| = 1$. Moreover, by Theorem 2.1, $(\pi \circ \varphi)(t_{n-1}) = (\pi \circ \varphi)(t_n)$; hence $\theta(f_{n-1}) = \theta(f_n)$, and therefore d' is isotopic to d_{n-1} in Σ_n. By Lemma 5.6 it follows that we have one of the following two possibilities:

(1) d' is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$.

(2) There exists $g \in \operatorname{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$ and $g([d']) = [d_n]$.

Suppose d' is isotopic to d_{n-1} in $\Sigma_n \setminus \{x\}$. Then $(\rho_D \circ \varphi)(t_n) = T_{d_{n-1}}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q$, and hence, since ρ_D is injective, $\varphi(t_n) = t_{n-1}^\varepsilon \Delta_D^q$. We conclude that $\varphi = \beta_{0,q} \circ \pi$ if $\varepsilon = 1$ and $\varphi = \chi \circ \beta_{0,-q} \circ \pi$ if $\varepsilon = -1$.

Suppose there exists $g \in \operatorname{Ker}(\theta)$ such that $g([d_i]) = [d_i]$ for all $1 \le i \le n-1$ and $g([d']) = [d_n]$. We have

$$(\rho_D \circ \varphi)(t_i) = T_{d_i}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q = g^{-1} T_{d_i}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q g$$

for all $1 \le i \le n-1$ and

$$(\rho_D \circ \varphi)(t_n) = T_{d'}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q = g^{-1} T_{d_n}^\varepsilon T_{\partial_1}^q T_{\partial_2}^q g.$$

By Theorem 3.6 there exists $v \in \operatorname{Ker}(\pi) \subset A[D_n]$ such that $\rho_D(v) = g$. Since ρ_D is injective, it follows that

$$\varphi(t_i) = v^{-1} t_i^\varepsilon \Delta_D^q v \quad \text{for all } 1 \le i \le n.$$

We conclude that $\varphi = \operatorname{ad}_{v^{-1}} \circ \gamma_q$ if $\varepsilon = 1$ and $\varphi = \operatorname{ad}_{v^{-1}} \circ \chi \circ \gamma_{-q}$ if $\varepsilon = -1$.

Case 4 (n is odd and $p=0$) Then, again, Σ_n is a surface of genus $\frac{1}{2}(n-1)$ with one boundary component, ∂, and $\kappa=2$. We have $\rho_D(t_i)=T_{d_i}$ for $1\le i\le n-1$ and, by Labruère and Paris [32, Proposition 2.12], $\rho_D(\Delta_D^2)=T_\partial$. Set $f_i=(\rho_D\circ\varphi)(t_i)$ for all $1\le i\le n$. Then, by the above,

$$f_i=T_{d_i}^{\varepsilon}T_{\partial}^{q}\quad\text{for all } 1\le i\le n-1.$$

In particular, $\mathcal{S}(f_i)=\{[d_i]\}$ for all $1\le i\le n-1$. Since t_n is conjugate in $A[D_n]$ to t_1, f_n is conjugate to f_1 in $\mathcal{M}(\Sigma_n,x)$, and hence f_n is of the form $f_n=T_{d'}^{\varepsilon}T_{\partial}^{q}$ where d' is a nonseparating circle.

For $1\le i\le n-3$ we have $t_it_n=t_nt_i$, and hence $T_{d_i}T_{d'}=T_{d'}T_{d_i}$. Therefore, by Proposition 3.1, $i([d_i],[d'])=0$. Similarly, $i([d_{n-1}],[d'])=0$. Since $t_{n-2}t_nt_{n-2}=t_nt_{n-2}t_n$, we have $T_{d_{n-2}}T_{d'}T_{d_{n-2}}=T_{d'}T_{d_{n-2}}T_{d'}$, and hence, by Proposition 3.1, $i([d_{n-2}],[d'])=1$. So we can assume that $d_i\cap d'=\varnothing$ for $1\le i\le n-3$, $d_{n-1}\cap d'=\varnothing$ and $|d_{n-2}\cap d'|=1$. Moreover, by Theorem 2.1, $(\pi\circ\varphi)(t_{n-1})=(\pi\circ\varphi)(t_n)$; hence $\theta(f_{n-1})=\theta(f_n)$, and therefore d' is isotopic to d_{n-1} in Σ_n. By Lemma 6.1 it follows that we have one of the following three possibilities:

(1) d' is isotopic to d_{n-1} in $\Sigma_n\setminus\{x\}$.

(2) There exists $g\in\mathrm{Ker}(\theta)$ such that $g([d_i])=[d_i]$ for all $1\le i\le n-1$ and $g([d'])=[d_n]$.

(3) There exists $g\in\mathrm{Ker}(\theta)$ such that $g([d_i])=[d_i]$ for all $1\le i\le n-2$, $g([d_{n-1}])=[d_n]$ and $g([d'])=[d_{n-1}]$.

If d' is isotopic to d_{n-1} in $\Sigma_n\setminus\{x\}$, then we prove as in the case where n is even that $\varphi=\beta_{0,q}\circ\pi$ if $\varepsilon=1$ and $\varphi=\chi\circ\beta_{0,-q}\circ\pi$ if $\varepsilon=-1$. Similarly, if there exists $g\in\mathrm{Ker}(\theta)$ such that $g([d_i])=[d_i]$ for all $1\le i\le n-1$ and $g([d'])=[d_n]$, then we prove as in the case where n is even that $\varphi=\mathrm{ad}_{v^{-1}}\circ\gamma_q$ if $\varepsilon=1$ and $\varphi=\mathrm{ad}_{v^{-1}}\circ\chi\circ\gamma_{-q}$ if $\varepsilon=-1$, where v is an element of $\mathrm{Ker}(\pi)\subset A[D_n]$.

Suppose there exists $g\in\mathrm{Ker}(\theta)$ such that $g([d_i])=[d_i]$ for all $1\le i\le n-2$, $g([d_{n-1}])=[d_n]$ and $g([d'])=[d_{n-1}]$. We have

$$(\rho_D\circ\varphi)(t_i)=T_{d_i}^{\varepsilon}T_{\partial}^{q}=g^{-1}T_{d_i}^{\varepsilon}T_{\partial}^{q}g\quad\text{for } 1\le i\le n-2,$$
$$(\rho_D\circ\varphi)(t_{n-1})=T_{d_{n-1}}^{\varepsilon}T_{\partial}^{q}=g^{-1}T_{d_n}^{\varepsilon}T_{\partial}^{q}g,\qquad(\rho_D\circ\varphi)(t_n)=T_{d'}^{\varepsilon}T_{\partial}^{q}=g^{-1}T_{d_{n-1}}^{\varepsilon}T_{\partial}^{q}g.$$

By Theorem 3.6 there exists $v\in\mathrm{Ker}(\pi)\subset A[D_n]$ such that $\rho_D(v)=g$. Since ρ_D is injective, it follows that

$$\varphi(t_i)=v^{-1}t_i^{\varepsilon}\Delta_D^{2q}v\quad\text{for } 1\le i\le n-2,\qquad\varphi(t_{n-1})=v^{-1}t_n^{\varepsilon}\Delta_D^{2q}v,\qquad\varphi(t_n)=v^{-1}t_{n-1}^{\varepsilon}\Delta_D^{2q}v.$$

We conclude that $\varphi=\mathrm{ad}_{v^{-1}}\circ\zeta\circ\gamma_q$ if $\varepsilon=1$ and $\varphi=\mathrm{ad}_{v^{-1}}\circ\zeta\circ\chi\circ\gamma_{-q}$ if $\varepsilon=-1$. □

7 Endomorphisms of $A[D_n]/Z(A[D_n])$

Proof of Proposition 2.7 Let Δ be the Garside element of $A[D_n]$. We set $\kappa=2$ if n is odd and $\kappa=1$ if n is even. Recall that $Z(A[D_n])$ is the cyclic group generated by Δ^κ. Let $\varphi_Z\colon A_Z[D_n]\to A_Z[D_n]$ be an endomorphism. For each $1 \le i \le n-2$ we define $u_i \in A[D_n]$ by induction on i as follows. First choose any $u_1 \in A[D_n]$ such that $\xi(u_1) = \varphi_Z(t_{Z,1})$. Now assume that $2 \le i \le n-2$ and that u_{i-1} is defined. Choose $u_i' \in A[D_n]$ such that $\xi(u_i') = \varphi_Z(t_{Z,i})$. Since $\varphi_Z(t_{Z,i-1}t_{Z,i}t_{Z,i-1}) = \varphi_Z(t_{Z,i}t_{Z,i-1}t_{Z,i})$, there exists $k_i \in \mathbb{Z}$ such that $u_{i-1}u_i'u_{i-1} = u_i'u_{i-1}u_i'\Delta^{\kappa k_i}$. Then set $u_i = u_i'\Delta^{\kappa k_i}$. Note that $\xi(u_i) = \xi(u_i') = \varphi_Z(t_{Z,i})$ and

$$u_{i-1}u_iu_{i-1} = u_{i-1}u_i'u_{i-1}\Delta^{\kappa k_i} = u_i'u_{i-1}u_i'\Delta^{2\kappa k_i} = u_iu_{i-1}u_i.$$

Define in the same way $u_{n-1}, u_n \in A[D_n]$ such that $\xi(u_{n-1}) = \varphi_Z(t_{Z,n-1})$, $\xi(u_n) = \varphi_Z(t_{Z,n})$, $u_{n-2}u_{n-1}u_{n-2} = u_{n-1}u_{n-2}u_{n-1}$ and $u_{n-2}u_nu_{n-2} = u_nu_{n-2}u_n$.

Let $i, j \in \{1, \dots, n\}$ be such that $i \ne j$ and $t_it_j = t_jt_i$. We have $\varphi_Z(t_{Z,i}t_{Z,j}) = \varphi_Z(t_{Z,j}t_{Z,i})$, and hence there exists $l \in \mathbb{Z}$ such that $u_iu_j = u_ju_i\Delta^{\kappa l}$. Recall the homomorphism $z\colon A[D_n] \to \mathbb{Z}$ which sends t_i to 1 for all $1 \le i \le n$. Since $z(\Delta) = n(n-1)$, the previous equality implies that

$$z(u_i) + z(u_j) = z(u_j) + z(u_i) + \kappa l n(n-1).$$

Hence $l = 0$, and therefore $u_iu_j = u_ju_i$.

By the above we have an endomorphism $\varphi\colon A[D_n] \to A[D_n]$ which sends t_i to u_i for all $1 \le i \le n$, and this endomorphism is a lift of φ_Z. □

Proof of Theorem 2.8 Let $n \ge 6$. Let $\varphi_Z\colon A_Z[D_n] \to A_Z[D_n]$ be an endomorphism. We know from Proposition 2.7 that φ_Z admits a lift $\varphi\colon A[D_n] \to A[D_n]$. By Theorem 2.3 we have one of the following three possibilities up to conjugation:

(1) φ is cyclic.

(2) There exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q} \circ \pi$.

(3) There exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \psi \circ \gamma_p$.

Clearly, if φ is cyclic then φ_Z is cyclic.

Now we show that the second case cannot occur. Suppose there exist $\psi \in \langle \zeta, \chi \rangle$ and $p, q \in \mathbb{Z}$ such that $\varphi = \psi \circ \beta_{p,q} \circ \pi$. As ever, we set $\kappa = 2$ if n is odd and $\kappa = 1$ if n is even. Recall that the center of $A[D_n]$ is generated by Δ^κ, where Δ is the Garside element of $A[D_n]$. We need to show that $\varphi(\Delta^\kappa) \notin Z(A[D_n]) = \langle \Delta^\kappa \rangle$, which leads to a contradiction. Since $\psi \in \mathrm{Aut}(A[D_n])$, we have $\psi(Z(A[D_n])) = Z(A[D_n])$, and hence we can assume that $\varphi = \beta_{p,q} \circ \pi$. Let $Y = \{t_1, \dots, t_{n-1}\}$ and let $\Delta_Y = \Delta_Y[D_n]$ be the Garside element of $A_Y[D_n]$. Since

$$\Delta = (t_1 \cdots t_{n-2}t_{n-1}t_nt_{n-2} \cdots t_1) \cdots (t_{n-2}t_{n-1}t_nt_{n-2})(t_{n-1}t_n),$$
$$\Delta[A_{n-1}]^2 = (s_1 \cdots s_{n-2}s_{n-1}^2s_{n-2} \cdots s_1) \cdots (s_{n-2}s_{n-1}^2s_{n-2})s_{n-1}^2,$$

(see Lemma 5.1 for the second equality), we have $\pi(\Delta) = \Delta[A_{n-1}]^2$, and hence

$$\varphi(\Delta^\kappa) = (\beta_{p,q} \circ \pi)(\Delta^\kappa) = \beta_{p,q}(\Delta[A_{n-1}]^{2\kappa}) = \Delta_Y^{2\kappa(1+pn(n-1))}\Delta^{\kappa^2 qn(n-1)}.$$

This element does not belong to $Z(A[D_n]) = \langle \Delta^\kappa \rangle$, because $\kappa(1+pn(n-1)) \ne 0$ and $\langle \Delta_Y^2 \rangle \cap \langle \Delta^\kappa \rangle = \{1\}$.

Suppose we are in the third case. So there exist $\psi \in \langle \zeta, \chi \rangle$ and $p \in \mathbb{Z}$ such that $\varphi = \psi \circ \gamma_p$. We have

$$\gamma_p(\Delta^\kappa) = \Delta^{\kappa(1+\kappa p n(n-1))} \in \langle \Delta^\kappa \rangle,$$

and hence γ_p induces an endomorphism $\gamma_{Z,p}\colon A_Z[D_n] \to A_Z[D_n]$. Moreover, for all $1 \le i \le n$,

$$\gamma_{Z,p}(t_{Z,i}) = \xi(t_i \Delta^{\kappa p}) = \xi(t_i) = t_{Z,i},$$

so $\gamma_{Z,p} = \mathrm{id}$. Clearly ψ is the lift of an element $\psi_Z \in \langle \zeta_Z, \chi_Z \rangle$, and hence $\varphi_Z = \psi_Z \circ \gamma_{Z,p} = \psi_Z$. □

Now, as promised in Section 2, we prove the following.

Lemma 7.1 *Let $n \ge 4$. If n is even, then $\langle \zeta_Z, \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$, and if n is odd, then $\langle \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$.*

Proof We first show that, if $\varphi\colon A[D_n] \to A[D_n]$ is an automorphism such that $\varphi_Z \in \mathrm{Inn}(A_Z[D_n])$, then $\varphi \in \mathrm{Inn}(A[D_n])$. Let $\varphi \in \mathrm{Aut}(A[D_n])$ be such that $\varphi_Z \in \mathrm{Inn}(A_Z[D_n])$. There exists $g_Z \in A_Z[D_n]$ such that $\varphi_Z(t_{Z,i}) = g_Z t_{Z,i} g_Z^{-1}$ for all $1 \le i \le n$. Again, we denote by Δ the Garside element of $A[D_n]$, and we set $\kappa = 2$ if n is odd and $\kappa = 1$ if n is even. Let $g \in A[D_n]$ be such that $\xi(g) = g_Z$. For every $1 \le i \le n$, there exists $k_i \in \mathbb{Z}$ such that $\varphi(t_i) = g t_i g^{-1} \Delta^{\kappa k_i}$. Let $i, j \in \{1, \dots, n\}$ be such that $\{t_i, t_j\}$ is an edge of D_n. From the equality $t_i t_j t_i = t_j t_i t_j$ it follows that

$$g t_i t_j t_i g^{-1} \Delta^{\kappa(2k_i + k_j)} = \varphi(t_i t_j t_i) = \varphi(t_j t_i t_j) = g t_j t_i t_j g^{-1} \Delta^{\kappa(k_i + 2k_j)}.$$

Hence $2k_i + k_j = k_i + 2k_j$, and therefore $k_i = k_j$. Since D_n is a connected graph, it follows that $k_i = k_j$ for all $i, j \in \{1, \dots, n\}$. So there exists $k \in \mathbb{Z}$ such that $\varphi(t_i) = g t_i g^{-1} \Delta^{\kappa k}$ for all $1 \le i \le n$. Recall the homomorphism $z\colon A[D_n] \to \mathbb{Z}$ which sends t_i to 1 for all $1 \le i \le n$. Since φ is an automorphism, we have $\mathrm{Im}(z \circ \varphi) = \mathrm{Im}(z) = \mathbb{Z}$. Furthermore, since $z(\Delta) = n(n-1)$, we have $(z \circ \varphi)(t_i) = 1 + \kappa k n(n-1)$ for all $1 \le i \le n$, and hence $\mathrm{Im}(z \circ \varphi) = (1 + \kappa k n(n-1))\mathbb{Z}$. This implies that $k = 0$, and hence $\varphi = \mathrm{ad}_g \in \mathrm{Inn}(A[D_n])$.

Arguing in a similar way we can see that lifts of ζ_Z and χ_Z in $\mathrm{Aut}(A[D_n])$ are unique. Since we know that $\langle \zeta, \chi \rangle \cap \mathrm{Inn}(A[D_n]) = \{\mathrm{id}\}$ if n is even and $\langle \chi \rangle \cap \mathrm{Inn}(A[D_n]) = \{\mathrm{id}\}$ if n is odd, it follows that $\langle \zeta_Z, \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$ if n is even and $\langle \chi_Z \rangle \cap \mathrm{Inn}(A_Z[D_n]) = \{\mathrm{id}\}$ if n is odd. □

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