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Linear bounds of the crosscap number of knots

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Kalfagianni and Lee found two-sided bounds for the crosscap number of an alternating link in terms of certain coefficients of the Jones polynomial. We show here that we can find similar two-sided bounds for the crosscap number of Conway sums of strongly alternating tangles. Then we find families of links for which these coefficients of the Jones polynomial and the crosscap number grow independently. These families will enable us to show that neither linear bound generalizes for all links.

57K10, 57K14, 57K16

1 Introduction

In [17] Kalfagianni and Lee showed that the crosscap number of an alternating link admits two-sided linear bounds in terms of certain coefficients of the Jones polynomial of the link. The purpose of this paper is twofold; we first generalize the result of [17] for links that are the Conway sum of strongly alternating tangles. Second, we construct families of knots obstructing the generalization of the result of [17] to arbitrary knots.

For a link L let

$$V(L)=\alpha_L t^n+\beta_L t^{n-1}+\cdots+\beta'_L t^{m+1}+\alpha'_L t^m,$$

where α_L and α'_L are nonzero, be the Jones polynomial, and let $T_L=|\beta_L|+|\beta'_L|$.

Definition 1.1 For a nonorientable connected surface S, bounded by a link L, the *crosscap number* is defined to be

$$C(S)=2-\chi(S)-k_L,$$

where k_L is the number of components of L. Then the *crosscap number* for a link L, denoted $C(L)$ will be the minimum crosscap number of all nonorientable surfaces bounded by the link.

Quantum knot invariants, such as the crosscap number, have historically been used to better understand classical knot invariants and the geometry of knots and links. In the eighties Kauffman [18] and Murasugi [25] showed that the span of the Jones polynomial realizes the crossing number for alternating links. More recent results from Futer, Kalfagianni, and Purcell showed how the coefficients of the colored Jones polynomial store information about the geometry of incompressible surfaces in the link complement and their strong relations to geometric structures and in particular hyperbolic geometry [11; 12; 13]. For example, specific coefficients of the colored Jones polynomial can coarsely define the volume of large families of hyperbolic links [8; 10]. Further, Dasbach and Lin [7] did this for all hyperbolic alternating links. The volume conjecture [23] predicts that the asymptotics of the colored Jones polynomial can be used to calculate the volume of all hyperbolic links.

More recently, for alternating links Kalfagianni and Lee showed that there exist linear bounds for the crosscap number with respect to certain coefficients of the Jones polynomial (see Theorem 1.1 in [17]). Before their work, the best known lower bound for all alternating links was $C(K) \geq 1$. On the other side, Clark [4] observed that for any knot the crosscap number is bounded by

$$C(K) \leq 2g(K) + 1,$$

where $g(K)$ is the orientable genus of the knot. Kalfagianni and Lee's results gave an exact calculation of the crosscap number for 283 prime alternating knots on Knotinfo and improvements of 1472 prime alternating knots. Kindred [19] has also made progress in calculating the crosscap number, calculating it for alternating knots with less than 13 crossings. We hope our results in this paper will pave the way for similar calculations for nonalternating knots which are less understood.

Another source of motivation comes from the work that has been done concerning the orientable genus of links and the Alexander polynomial and Heegard Floer homology of a link. Crowell [5] and Murasugi [24] have independently shown that the orientable genus of an alternating link is half the degree span of the Alexander polynomial of the knot. The Heegard Floer homology is now known to detect the genus of links [22]. There is also an algorithm using normal surface theory to calculate the orientable knot genus [2; 15]. The hope with the crosscap number is to find parallel results to Heegard Floer and orientable genus using quantum invariants such as the Jones polynomial and Khovanov homology.

In this paper we will start by finding two-sided linear bounds for $C(L)$, where L is a Conway sum of tangles in terms of T_L. We also define many of the necessary terms for Theorem 1.2 in that section, which we state here.

Theorem 1.2 *Let T_1 and T_2 be nonsplittable, twist reduced, strongly alternating tangles whose Conway sum is a link L. Let $C(L)$ be the crosscap number of L and k_L be the number of components of L. Then*

$$\left\lceil \tfrac{1}{6} T_L \right\rceil - k_L \leq C(L) \leq 2T_L + k_L + 8.$$

A key ingredient in the proof of Theorem 1.2 is Theorem 1.3 which we state below. Theorem 1.3 gives us bounds for $C(L)$ in terms of the crosscap numbers of the closures of the tangles which sum to L.

Theorem 1.3 *Let T_1 and T_2 be nonsplittable, twist reduced, strongly alternating tangles, and let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} and K_{iD} be the links formed by the numerator and denominator closures of T_i, respectively, $i \in \{1,2\}$. We have*

$$m - 2 \leq C(L) \leq m + 2,$$

where $m = \min\{C(K_{1N}) + C(K_{2N}), C(K_{1D}) + C(K_{2D})\}$.

Having Theorem 1.3 at hand we use a result of [17] and the additivity of twist numbers for strongly alternating tangles to find bounds for $C(L)$ in terms of the twist number of L. Then using these bounds and a generalization of Theorem 1.6 from [9] by Futer, Kalfagianni, and Purcell gives us Theorem 1.2.

In Section 5 we show that for arbitrary knots the crosscap number and T_L are independent. Specifically we show:

Theorem 1.4 (a) *There exists a family of links for which $T_L \leq 2$, but $C(L)$ is arbitrarily large.*

(b) *There exists a family of links for which $C(L) \leq 3$, but T_L is arbitrarily large.*

To show part (a) of Theorem 1.4, we use work by Teragaito [27] to find a family of torus knots $T(p,q)$ where $C(T(p,q))$ grows with q and q can be made arbitrarily large. On the other hand we will show that $T_{T(p,q)} \leq 2$.

For part (b) we will introduce a family of Whitehead doubles for which the crosscap number is always bounded by 3 but $|\beta'_W|$ can be made arbitrarily large. Clark [4] showed that for all links $C(L) \leq 2g(L)+1$ which shows that for all Whitehead doubles, $C(W) \leq 3$. On the other hand, using work by Stoimenow [26], we are able to compute $|\beta'_W|$ for B-adequate links. Then we find a family of B-adequate Whitehead doubles for which $|\beta'_W|$ can be made arbitrarily large.

All the links constructed in Theorem 1.4 are nonhyperbolic. This leaves the question of whether Theorem 1.2 may be generalized to all hyperbolic links. See Section 6 for more details.

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2 Crosscap bounds on connection of two strongly alternating tangles

In this section we will work to prove Theorem 1.3.

2.1 Preliminaries and the upper bound of Theorem 1.3

We start with a couple definitions.

Definition 2.1 A *tangle* is a graph in the plane contained within a box which intersects the box at the four corners with one-valent vertices, with all other vertices, contained inside the box, four-valent, and given over/under crossing data. We label the four one-valent vertices NW, NE, SE, SW, positioned according to Figure 1.

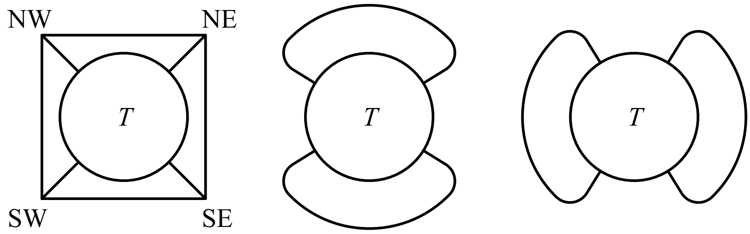

Figure 1: Left: a tangle inside a box with directional strands labeled. Center: numerator closure. Right: denominator closure.

Definition 2.2 The *closure* of a tangle is the link which results when we connect the NW and NE points along the box and SW and SE points along the box as seen in the center panel of Figure 1, this is called the *numerator closure*. If we close as in the right-hand panel of Figure 1 we call it the *denominator closure*. A tangle is *strongly alternating* if both closures are prime, alternating, and contain no nugatory crossings.

Definition 2.3 A *Conway sphere* is a 2-sphere which intersects a knot or link transversely in four points. A *Conway sum* is a sum of tangles as shown in Figure 2. For our purposes, a Conway sphere Σ will be positioned such that it intersects a Conway sum at the four one-valent vertices for one of the tangles in the sum. If we let S be a spanning surface for our Conway sum, then $S \cap \Sigma$ will contain two arcs and a possibly empty collection of simple closed curves as in Figure 3.

We are now ready to begin proving Theorem 1.3. We separate it into the upper and lower bounds, beginning with the upper bound:

Lemma 2.4 *Let T_1 and T_2 be a pair of nonsplittable, strongly alternating tangles. Let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} and K_{iD} be the numerator and denominator closures, respectively, of T_1 and T_2. If $C(L)$ is the crosscap number of L, then*

$$C(L) \leq \min\{C(K_{1N}) + C(K_{2N}) + 2, C(K_{1D}) + C(K_{2D}) + 2\}.$$

Proof Start with a pair of strongly alternating tangles, T_1 and T_2. Let K_{1N} and K_{2N} be the links acquired by the numerator closures of the tangles. Let S_1 and S_2 be nonorientable spanning surfaces

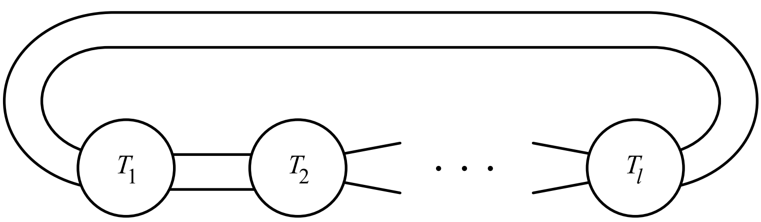

Figure 2: An example of a Conway sum of l tangles.

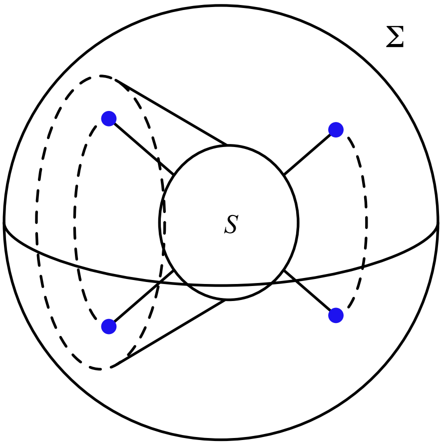

Figure 3: Here we see a tangle contained within a Conway sphere. The dots represent the intersection of T_i with Σ. Then the dotted lines are the intersections of S with Σ.

which realize the crosscap numbers of K_{1N} and K_{2N}, respectively. We find a spanning surface S of L by attaching S_1 and S_2 with a pair of bands bounded by the strands along which the Conway sum was taken. As we do not cut S_1 and S_2, S will also be nonorientable.

Now we study the relationship between $C(S)$ and the sum of $C(S_1)$ and $C(S_2)$. We remind the reader that $C(S) = 2 - \chi(S) - k$. The difference between S and the disjoint union of S_1 and S_2 is the two connecting bands used to construct S. So, $\chi(S) = \chi(S_1) + \chi(S_2) - 2$.

Next we compare the number of link components in L with the total in K_{1N} and K_{2N}. The gluing of the East strands of K_{1N} to the West strands of K_{2N} will reduce the number of components by 1 as we are connecting two disjoint links. The other attachment can increase or decrease the number of components by 1, or keep it the same. Then $k_L = k_{K_{1N}} + k_{K_{2N}} - \epsilon$ where $\epsilon = 0, 1$, or 2.

Now we substitute for $\chi(S)$ and k_L to find

$$C(S) = 2 - \chi(S_1) - \chi(S_2) + 2 - k_{K_{1N}} - k_{K_{2N}} + \epsilon = C(K_{1N}) + C(K_{2N}) + \epsilon.$$

Hence

$$C(L) \leq C(S) = C(K_{1N}) + C(K_{2N}) + 2,$$

as $\epsilon = 2$, will give the weakest upper bound. By the same argument with the denominator closures of T_1 and T_2, we find that

$$C(L) \leq C(K_{1D}) + C(K_{2D}) + 2. \qquad \square$$

Here we remark that Lemma 2.4 will hold even if we take two general tangles. Notice in the proof that we do not use the fact that T_1 or T_2 are nonsplittable or strongly alternating. As this will not hold true for the other statements we included these hypotheses for uniformity.

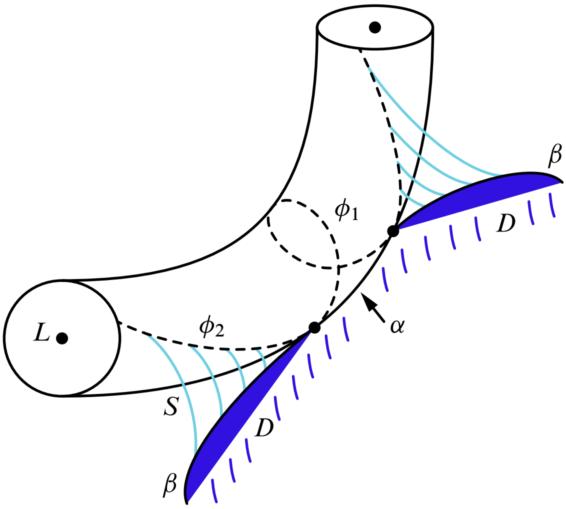

Figure 4: This figure shows a case where L is meridianally boundary compressible.

2.2 Technical lemmas

Before we show the lower bound, we will need some more background, as well as some technical results. Lemma 2.5 below was discussed in the proof of Lemma 2.4.

Lemma 2.5 *Let L be the Conway sum of the tangles T_1 and T_2, and let K_1 and K_2 be closures of T_1 and T_2. If k_L is the number of link components for L, and k_1 and k_2 the number of link components for K_1 and K_2, respectively, then $k_L = k_1 + k_2 - \epsilon$ for $\epsilon = 0, 1, 2$.*

Definition 2.6 Let L be a link in S^3 and let $N(L)$ be a neighborhood of L. A spanning surface S of L in S^3 is defined to be *meridianally boundary compressible* if there exists a disk D embedded in $S^3 \setminus N(L)$ such that $\partial D = \alpha \cup \beta$ where $\alpha = D \cap \partial N(L)$ and $\beta = D \cap S$. Notice both α and β are arcs, β does not cut off a disk of S, $\partial D \cap \partial S$ cuts ∂S into two arcs ϕ_1, ϕ_2 and $\alpha \cup \phi_i$ is a meridian of the link for one of $i = 1, 2$ as shown in Figure 4. A spanning surface is said to be *meridianally boundary incompressible* if no such disk exists.

Definition 2.7 Given an alternating projection of a link L on S^2, we modify it so that in a neighborhood of each crossing, we have a ball whose equator lies on S^2 such that the over strand runs over the ball and the under goes underneath; see Figure 5. We call every such ball a *Menasco ball*, and we call such an embedding of L relative to S^2 a *Menasco projection* P with n crossings.

Definition 2.8 We say that a surface S intersects a Menasco ball B_i in a *crossing band* if $S \cap B_i$ consists of a disc bounded by the over and under strands on ∂B_i along with opposite arcs along the equator of B_i. We refer the reader to Figures 28–30 in [1].

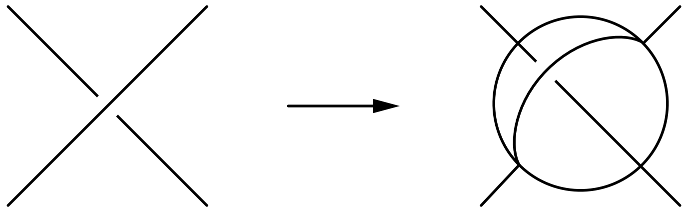

Figure 5: Here we put a crossing into Menasco form, with the over strand running across the top of the ball and the lower strand running along the bottom.

Let $F = S^2 \setminus \bigcup_i B_i$, where the B_i are the Menasco balls for L. Given an incompressible (not necessarily meridianally) surface S spanning L, we can isotope S so that:

(i) $S \cap F$ is a collection of simple closed curves and arcs with endpoints on L or the equator of a Menasco ball.

(ii) S is disjoint wherever possible from the interior of the B_i, including along $N(L)$. The only exception will be at crossing bands.

We say such a surface isotoped in this way is in *Menasco form*.

We will also need Lemma 5.1 from [1] by Adams and Kindred which we state here as Lemma 2.9. Lemma 2.9 is required to prove Lemma 2.10, which is proven as Corollary 5.2 in [1]. Lemma 2.10 is essential to our proof of the lower bound, as it guarantees the connectedness of any spanning surface of the numerator or denominator closures of tangles that we consider.

Lemma 2.9 *An incompressible and meridianally boundary incompressible surface S spanning an alternating link L can be isotoped relative to a given nontrivial Menasco projection P to obtain a crossing band.*

Lemma 2.10 *Any spanning surface for a nonsplittable, alternating link is connected.*

Proof This proof uses induction on the number of crossings a nonsplittable, alternating link contains. As the unknot contains only one link component, any spanning surface for the unknot must be connected. Now consider a nonsplittable alternating link L which has n crossings, and let S be a spanning surface for L. Then S is either incompressible and meridianally boundary incompressible or a finite sequence of compressions take S to an incompressible and meridianally boundary incompressible surface S'. Choose a reduced alternating diagram of L and put S' into Menasco form relative to L. By Lemma 2.9, we can isotope S' such that S' contains a crossing band in at least one of the Menasco balls, M. Further, when we cut open the link along M, we find a spanning surface S'' for a nonsplittable alternating link with fewer crossings L'. Part of the equator of M replaces the crossing strands and guarantees that S'' is a spanning surface of L'. Then, by induction, as S'' is a spanning surface for a link of $n-1$ crossings, it is connected. Regluing in the crossing band does not disconnect our surface, showing that S' and S are connected. □

Lemma 2.11 *Let Σ be a Conway sphere which intersects a Conway sum L of two strongly alternating tangles T_1 and T_2 in S^3 such that Σ separates T_1 and T_2. If we let S be a spanning surface of L and $S \cap \Sigma$ contains a simple closed curve γ such that γ does not separate the two arcs in $S \cap \Sigma$ on Σ, then there exists an isotopy on S which will eliminate γ.*

Proof Assume there is only one closed curve γ contained in $S \cap \Sigma$. First we consider the case where cutting S along γ and gluing in disks along the resulting boundary components results in two disconnected closed components. This implies that S has a disconnected closed component, contradicting that S is a crosscap realizing surface.

Next assume that cutting along γ and gluing in disks does not result in a closed surface component. Then cutting S along Σ and gluing disks along the copies of γ will result in spanning surfaces for a closure of T_1 and a closure of T_2, both of which are connected by Lemma 2.10. Reversing this procedure everywhere but γ results in a new connected surface S' which also spans L. But as S' has two additional disks, $\chi(S') = \chi(S) + 2$, showing that $C(S') < C(S)$ contradicting that S is a crosscap realizing spanning surface.

The only remaining possibility is if cutting S along γ and gluing disks to the two resulting boundaries results in a single closed surface component, U. Then U will separate S^3 into two disjoint spaces. As γ does not separate the two arcs on Σ, one side of U must not contain any part of S. But then we can move U to the opposite side of Σ and reglue it to S along γ to find an isotopy of S for which γ is no longer in $\Sigma \cap S$.

In the case that we have multiple such closed curves along Σ we do the same as above starting with the innermost closed curve. The innermost closed curve in this case is the one that bounds an empty disk on Σ. □

By Lemma 2.11, we can choose S such that the only simple closed curves in $\Sigma \cap S$ are those which bound two disks each containing an arc. Next we show that S can be chosen so that $S \cap \Sigma$ contains at most one such simple closed curve.

Lemma 2.12 *There exists a spanning surface S for L, where L is the Conway sum of two strongly alternating tangles, such that $C(S) = C(L)$ and $\Sigma \cap S$ contains at most one closed curve.*

Proof By Lemma 2.11 we can assume that if $\Sigma \cap S$ contains closed curves $\gamma_1, \ldots, \gamma_n$, they each split Σ such that the two arcs lie on opposite disks. Assume we have $n > 1$ such closed curves in $\Sigma \cap S$, and let γ_1 and γ_2 be such that γ_1 bounds a disk on Σ such that no other γ_i are in the disk and γ_2 bounds a disk where the only closed curve in it is γ_1.

We now find a spanning surface S' such that $C(S') = C(L)$ and $S' \cap \Sigma$ contains $n-2$ closed curves. We start with S and cut along γ_1 and γ_2 and then glue in annuli whose boundaries are a copy of γ_1 and a copy of γ_2. Then as the Euler characteristic of an annulus is 0, this cutting and gluing operation will result in $\chi(S) = \chi(S')$.

It remains to show that S' will be a connected surface. As in the proof of Lemma 2.11 we cut S along Σ and glue in disks along each γ_i except for γ_1 and γ_2 which we glue a pair of annuli. This results in spanning surfaces S_1 and S_2 for closures of T_1 and T_2 which by Lemma 2.10 are connected. If we reverse this procedure everywhere except γ_1 and γ_2 the result will be S' and as we only remove disks before regluing, S' will be connected. Hence, we have found a spanning surface S' for L such that $C(S') = C(L)$ and $S' \cap \Sigma$ contains two fewer closed curves. Hence, repeating for all such pairs of closed curves in $S \cap \Sigma$ we will find the claim. □

Lemma 2.13 *We can choose a surface S which spans a link L such that $C(S) = C(L)$ and cutting along Σ will not give us a closed surface component.*

Proof This was shown in the proof of Lemma 2.11. □

2.3 Lower bound of Theorem 1.3

In this subsection we will prove the lower bound of Theorem 1.3, which will be restated as Lemma 2.14. We start by discussing what happens when we cut our link L along Σ. Assume that S is a nonorientable spanning surface for L with $C(S) = C(L)$. The two arcs on Σ will define how we close T_1 and T_2 after cutting. We let K_1 and K_2 be these closures. To see that K_1 and K_2 are the numerator or denominator closures consider a crossing which as a vertex in the tangle graph is adjacent to a one-valent vertex. If the exterior regions for a tangle are the faces bounded by the box in the graph then one of the two exterior regions adjacent to the crossing must be included in S. This means the boundary of this region will result in the numerator or denominator closures.

We are now ready to prove the lower bound of Theorem 1.3.

Lemma 2.14 *Let T_1 and T_2 be nonsplittable, strongly alternating tangles and L the link resulting from the Conway sum of T_1 and T_2. Also, let S be a spanning surface of L such that $C(L) = C(S)$. Then*

$$C(K_1) + C(K_2) - 2 \leq C(L).$$

Proof Let S be a nonorientable spanning surface for L such that $C(L) = C(S)$. By Lemma 2.12 S can be chosen such that the intersection of S with Σ contains at most one closed curve γ and that both disks γ bounds contain arcs. We cut S along Σ and if γ exists we glue a disk to each copy to get spanning surfaces S_1 and S_2 for K_1 and K_2, respectively. By Lemma 2.13 we know that S_1 and S_2 will not have closed components and by Lemma 2.10 S_1 and S_2 must be connected as K_1 and K_2 are alternating.

If k_1 and k_2 are the number of link components for K_1 and K_2, respectively, then by Lemma 2.5 $k_1 + k_2 - \epsilon = k_L$ where $\epsilon = 0, 1, 2$.

Next we consider how the Euler characteristics of S and the sum of the Euler characteristics of S_1 and S_2 will be related. We know that $\Sigma \cap S$ contains two arcs and at most one closed curve by Lemma 2.12. Then cutting the two arcs along Σ will increase the Euler characteristic by 2. Assume there are n closed curves. Gluing disks along the two copies after cutting will further increase the Euler characteristic by $2n$. The final consideration we have to make is whether S_1 and S_2 are nonorientable. Let $t = 0, 1, 2$ be the number of S_i which are orientable. We will have to add t half twist bands to make sure all the S_i are nonorientable decreasing the Euler characteristic by t. Now we see that $\chi(S) = \chi(S_1) + \chi(S_2) - 2 - 2n + t$. Then

$$C(S_1) + C(S_2) = 4 - \chi(S) - 2 - 2n + t - k_L - \epsilon = C(L) - 2n + t - \epsilon.$$

Simplifying we find

$$C(K_1) + C(K_2) + 2n - t + \epsilon \leq C(L).$$

This will be the weakest when $n = 0$, $t = 2$, and $\epsilon = 0$. Hence we find $C(K_1) + C(K_2) - 2 \leq C(L)$. □

We restate Theorem 1.3:

Theorem 1.3 *Let T_1 and T_2 be nonsplittable, twist reduced, strongly alternating tangles. Let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} be the link formed by the numerator closure of T_i, $i \in \{1, 2\}$, similarly K_{iD} will be the link formed by the denominator closure. If we let $m = \min\{C(K_{1N}) + C(K_{2N}), C(K_{1D}) + C(K_{2D})\}$, then*

$$C(K_1) + C(K_2) - 2 \leq C(L) \leq m + 2.$$

Theorem 1.3 follows directly from Lemmas 2.4 and 2.14. A result similar to Theorem 1.3 exists for the cross cap number of connected sums. In particular, Clark [4] showed with a strategy similar to our own, that if K_1 and K_2 are knots, then

$$C(K_1) + C(K_2) - 1 \leq C(K_1 \# K_2) \leq C(K_1) + C(K_2).$$

3 Crosscap number, twist number, and the Jones polynomial

3.1 Twist number bounds

Now we have a relationship between the crosscap numbers of the Conway sum of two tangles and the closures of the tangles which compose it. Unfortunately, the bounds depend upon the tangles and which closures we take. But we can use Theorem 1.3 to find bounds for $C(L)$ entirely dependent upon L. Before proceeding with the statements, we will need a definition.

Definition 3.1 The *twist number* of a link diagram or a tangle diagram is the number of twist regions a link diagram contains, where a *twist region* is a maximal collection of bigon regions contained end to end. We call a link diagram *twist reduced* if any simple closed curve which meets the link diagram transversely at four points, with two points adjacent to one crossing and the other two another crossing, bounds a possibly empty collection of bigons arranged end to end between the two crossings.

We take a brief pause to mention that we can take the Conway sum of more than two tangles. In particular, for tangles $T_1, T_2, \dots, T_n$, we can glue the eastern strands of T_i to the western strands of T_{i+1}. Then we glue the eastern strands of T_n to the western strands of T_1. See Figure 2 for an example.

Lemma 3.2 *Let $T_1, T_2, \dots, T_n$ be strongly alternating tangle diagrams whose Conway sum is a link diagram $D(L)$. Then $\operatorname{tw}(D(L)) = \sum_{i=1}^n \operatorname{tw}(T_i)$, where $\operatorname{tw}(D(L))$ is the twist number for $D(L)$ and $\operatorname{tw}(T_i)$ the twist number for the tangle diagram T_i.*

Proof First notice that taking the sum of tangles will not result in new twist regions. This is because, when taking a Conway sum, crossings that shared a twist region will still share a twist region and we introduce no new crossings. Therefore, $\operatorname{tw}(D(L)) \le \sum_{i=1}^n \operatorname{tw}(T_i)$.

Now assume that $\operatorname{tw}(D(L)) < \sum_{i=1}^n \operatorname{tw}(T_i)$. Then for some i, a twist region in T_i and a twist region in T_{i+1} become one region in L. This implies there exists a simple closed curve γ which transversely intersects $D(L)$ twice in T_i and twice in T_{i+1}. If we think back to T_i lying in a unit square, then γ must intersect the north and south edges or the east and west edges of the square. In the first case, this shows that the denominator closure is not prime, and the second, the numerator closure is not prime. But as T_i is strongly alternating, this would be a contradiction, therefore $\operatorname{tw}(D(L)) = \sum_{i=1}^n \operatorname{tw}(T_i)$. □

Next we consider the relationship between the twist number of a tangle diagram and the twist numbers of diagrams of its closures.

Lemma 3.3 *Let T be a strongly alternating tangle diagram, and let $D(K)$ be the link diagram which comes from the numerator or denominator closure. Then*

$$\operatorname{tw}(T) - 2 \le \operatorname{tw}(D(K)) \le \operatorname{tw}(T).$$

Proof The upper bound is true as we are not adding crossings when closing a tangle, and hence cannot create new twist regions.

The lower bound stems from the fact that when we choose a closure for T we create two new potential bigons. If either region is a bigon, then it joins two twist regions. If both regions are bigons, the twist number is reduced by 2. In Figure 6 we see an example of a tangle where the lower bound is sharp for both closures. □

We will also need Theorem 3.8 from [17] which we state here. This theorem allows us to relate the cross cap numbers of the closures of strongly alternating tangles to twist numbers of their diagrams.

Theorem 3.4 *Let $L \subset S^3$ be a link of k_L components with a prime, twist reduced, alternating diagram $D(L)$. Suppose that $D(L)$ has $\operatorname{tw}(D(L)) \ge 2$ twist regions. Let $C(L)$ denote the crosscap number of L. We have*

$$\left\lceil \tfrac{1}{3} \operatorname{tw}(D(L)) \right\rceil + 2 - k_L \le C(L) \le \operatorname{tw}(D(L)) + 2 - k_L.$$

Furthermore, both bounds are sharp.

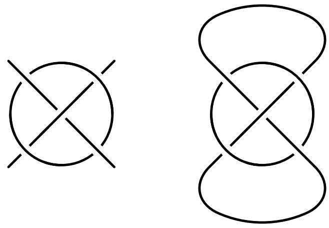

Figure 6: Left: a strongly alternating tangle with 5 twist regions. Right: the numerator closure with 3 twist regions. The denominator also results in 3 twist regions, showing the sharpness of the lower bound.

Now that we have that the twist number is additive for strongly alternating tangles by Lemma 3.2 and can relate $C(K_i)$ and $\operatorname{tw}(T_i)$ by Lemma 3.3 and Theorem 3.4, we are ready to state Theorem 3.5

Theorem 3.5 *Let T_1 and T_2 be diagrams of nonsplittable, strongly alternating, twist reduced tangles whose Conway sum is a link diagram $D(L)$. Let $C(L)$ be the crosscap number of L, $\operatorname{tw}(D(L))$ be the twist number of $D(L)$ and k_L be the number of link components in L. Then*

$$\left\lceil \tfrac{1}{3}\operatorname{tw}(D(L))\right\rceil - k_L \le C(L) \le \operatorname{tw}(D(L)) + 4 - k_L.$$

Proof We start with a lower bound in the proof of Lemma 2.14 with ambiguity on ϵ where $k_1+k_2-\epsilon=k_L$. Then $C(K_1)+C(K_2)-2+\epsilon \le C(L)$. Let $D(K_1)$ and $D(K_2)$ be the diagrams of K_1 and K_2 that arise from cutting L as in Theorem 1.3. Notice that $\operatorname{tw}(D(K_i)) \ge 2$ for $i=1,2$ as T_i is strongly alternating, and tangle diagrams with twist number 1 will have a nonprime closure. Then by Theorem 3.4 we find for $i \in \{1,2\}$ that $\left\lceil \frac{1}{3}\operatorname{tw}(D(K_i))\right\rceil + 2 - k_i \le C(K_i)$ where K_i has k_i link components. So

$$\left\lceil \tfrac{1}{3}\operatorname{tw}(D(K_1))\right\rceil + \left\lceil \tfrac{1}{3}\operatorname{tw}(D(K_2))\right\rceil + 2 + \epsilon - k_1 - k_2 \le C(L).$$

By Lemma 3.3 $\operatorname{tw}(T_i) - 2 \le \operatorname{tw}(D(K_i))$ and from Lemma 3.2 we know $\operatorname{tw}(T_1)+\operatorname{tw}(T_2)=\operatorname{tw}(D(L))$. Combining the two lemmas shows

$$\left\lceil \tfrac{1}{3}\operatorname{tw}(D(L))\right\rceil - 2 \le \left\lceil \tfrac{1}{3}(\operatorname{tw}(T_1)-2)\right\rceil + \left\lceil \tfrac{1}{3}(\operatorname{tw}(T_2)-2)\right\rceil \tag{1}$$

$$\le \left\lceil \tfrac{1}{3}\operatorname{tw}(D(K_1))\right\rceil + \left\lceil \tfrac{1}{3}\operatorname{tw}(D(K_2))\right\rceil. \tag{2}$$

Finally substituting in $k_1+k_2=k_l+\epsilon$, we find

$$\left\lceil \tfrac{1}{3}\operatorname{tw}(D(L))\right\rceil - k_L \le C(L).$$

Now we consider the upper bound. Similar to the lower bound we start with a step from Lemma 2.4, $C(L) \le \min\{C(K_{1N})+C(K_{2N})+\epsilon, C(K_{1D})+C(K_{2D})+\epsilon\}$. By Lemma 3.3 we see that $\operatorname{tw}(D(K_{iN})) \le \operatorname{tw}(T_i)$ and $\operatorname{tw}(D(K_{iD})) \le \operatorname{tw}(T_i)$ and then by Theorem 3.4,

$$C(L) \le \operatorname{tw}(T_1)+\operatorname{tw}(T_2)+4+\epsilon-k_1-k_2.$$

Then substituting for $k_1+k_2=k_L+\epsilon$ and considering Lemma 3.2,

$$C(L) \le \operatorname{tw}(D(L)) + 4 - k_L. \qquad \square$$

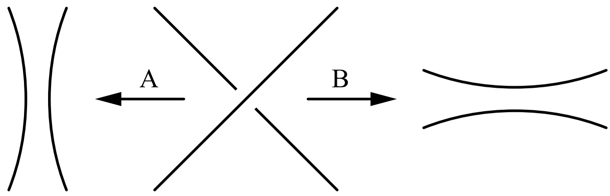

Figure 7: Given a crossing, we can resolve it to either the A- or B-resolution.

3.2 Jones polynomial bounds

From here we work to find bounds in terms of T_L, but first we have to generalize Theorem 1.6 in [9] which will allow us to relate the twist number of the diagram of a link L to T_L. Theorem 1.6 from [9] only considers knots but we want a similar result for links. We start with some necessary definitions and then a generalization of Lemma 5.4 from [9] which is a necessary piece of our generalization of Theorem 1.6.

Definition 3.6 If we let $D(L)$ be the diagram of a link L we define the *A-resolution* and *B-resolution* as shown in Figure 7. Then a *Kauffman state* is a choice of resolutions for each crossing in a link diagram.

Definition 3.7 Next we construct a *all A* (*resp all B*) *state graph* $\mathbb{G}_A(D(L))$ (resp $\mathbb{G}_B(D(L))$) by adding edges where we performed resolutions and then contracting the simple closed curves to vertices. Let e_A (resp e_B) be the number of edges in $\mathbb{G}_A(D(L))$ (resp $\mathbb{G}_B(D(L))$). Further, if we identify all parallel edges (edges which share two vertices) we find the *reduced state graph* $\mathbb{G}'_A(D(L))$ (resp $\mathbb{G}'_B(D(L))$). Let e'_A (resp e'_B) be the number of edges in $\mathbb{G}'_A(D(L))$ (resp $\mathbb{G}'_B(D(L))$).

Now we are ready to define what it means for a link to be adequate.

Definition 3.8 We call a link diagram *A-adequate* (resp *B-adequate*) if the A-state (resp B-state) graph of the diagram has no one-edge loops. A link diagram is called *adequate* if it is both A-adequate and B-adequate, and a link is adequate if it has a diagram which is adequate.

Here we recall some terminology from [9].

Definition 3.9 Let $D(L)$ be the link diagram obtained by taking the Conway sum of strongly alternating tangles $T_1, \dots, T_n$. Let $\ell_{\mathrm{in}}(D(L))$ denote the loss of edges in $\mathbb{G}_A(D(L))$ and $\mathbb{G}_B(D(L))$ as we pass from $e_A + e_B$ to $e'_A + e'_B$ which come from equivalent crossings in the same tangle T_i. Then let $\ell_{\mathrm{ext}}(D(L))$ be the number of edges we lose from identification when we take the Conway sum. It follows that $\ell_{\mathrm{in}}(D(L)) + \ell_{\mathrm{ext}}(D(L)) = e_A + e_B - e'_A - e'_B$.

For an alternating tangle diagram T, the vertices of $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ are in one-to-one correspondence with the regions of T. For the state graph of a tangle, if we consider the tangle lying within a disk, we have four exterior regions bounded by the disk. This means our state graphs have *interior vertices* those whose region lie entirely within the interior of the disk and two *exterior vertices* with corresponding region, with sides on the boundary of the disk.

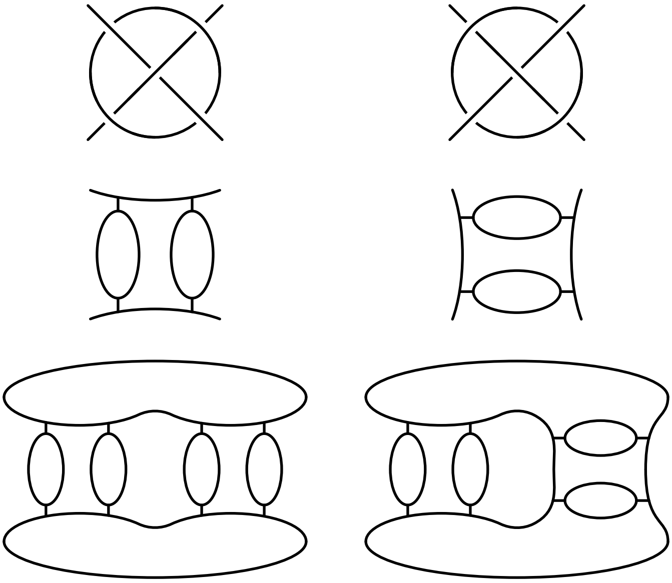

Figure 8: Top: two strongly alternating tangles. Middle: the all A-resolution diagrams for the tangles. Bottom left: the all A-state diagram for the Conway sum of two of the left tangles which has all admissible bridges. Bottom right: the all A-state diagram for the Conway sum of one of both tangles; here the bridges are all inadmissible.

For a tangle T_i, a *bridge* of $\mathbb{G}_A(T_i)$ or $\mathbb{G}_B(T_i)$ is a subgraph consisting of an interior vertex v, and edges e', e'' which connect v to the exterior vertices v' and v''. We call the bridge *inadmissible* if the vertices become identified in $\mathbb{G}_A(D(L))$ or $\mathbb{G}_B(D(L))$; see Figure 8.

Now we find an upper bound for $\ell_{\mathrm{ext}}(D(L))$ in terms of the twist number. This work will largely follow the proof of Lemma 5.4 in [9].

Lemma 3.10 *Let T_1 and T_2 be strongly alternating tangles whose Conway sum is a link diagram $D(L)$. Let k_L be number of link components in L. Then*

$$\ell_{\mathrm{ext}}(D(L)) \leq \tfrac{1}{2}\,\mathrm{tw}(D(L)) + k_L + 4.$$

Proof For $T \in \{T_1, T_2\}$ let $b_A(T)$, $b_B(T)$ be the number of bridges in $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$, respectively. Then the contribution of T to ℓ_{ext} will be at most $b_A(T) + b_B(T)$. Any other edge identification from moving to the reduced graph will still be counted by ℓ_{int}.

If b is a bridge there are two possibilities:

(i) The edges e', e'' do not come from the resolutions of a single twist region.

(ii) The edges e', e'' come from the resolutions of a single twist region.

For type-(ii) bridges the two crossings which result in the edges are the only two in their respective twist region. Otherwise the two edges will not be adjacent to the exterior vertices or this would no longer constitute a twist region.

For type-(i) bridges when we pass from $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ to $\mathbb{G}'_A(T)$ and $\mathbb{G}'_B(T)$ the contributions to ℓ_{ext} is half the number of twist regions involved in such bridges. Unlike in [9] we can have more than one type-(ii) bridge, as each additional type-(ii) bridge creates a new link component.

Case 1 Suppose that $b_A(T) \geq 3$ or $b_B(T) \geq 3$. Without loss of generality let $b_A(T) \geq 3$. Then $b_B(T) = 0$. If $b_B(T)$ were not zero then the B state bridge would cross the A state bridges, implying two internal vertices which is not a bridge.

There can be any number of type-(ii) bridges, but each bridge beyond the first will add a new link component. If we have only type-(ii) bridges $b_A(T) \leq k_T$ where k_T is the number of tangle components. On the other hand if we only have type-(i) bridges $b_A(T) \leq \frac{1}{2}\operatorname{tw}(T)$. Then for any mix of bridges we find that

$$b_A(T) + b_B(T) \leq \tfrac{1}{2}\operatorname{tw}(T) + k_T.$$

Case 2 In this case we will consider $b_A(T) = b_B(T) = 2$. Then k must be at least 2, as the bridges in $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ will create a square resulting in a tangle second component. Also, as $\mathbb{G}_A(T)$ has two bridges there are at least two twist regions in T. Then

$$b_A(T) + b_B(T) \leq \tfrac{1}{2}\operatorname{tw}(T) + k_T + 1.$$

Case 3 Either $b_A(T) \leq 2$ and $b_B(T) \leq 1$ or $b_A(T) \leq 1$ and $b_B(T) \leq 2$. Without loss of generality consider the first possibility. Then $b_A(T) + b_B(T)$ is at most three. If it's less than three we see that $k_T + 1 \geq 2$ so we need only consider when they sum to three. But as with the previous case we will have at least two twist regions as $\mathbb{G}_A(T)$ has two bridges. Thus,

$$b_A(T) + b_B(T) \leq \tfrac{1}{2}\operatorname{tw}(T) + k_T + 1.$$

Then by Lemma 3.2 we know that the twist number is additive over Conway sums. By Lemma 2.5 $k_1 + k_2 \leq k_L + 2$. Then we find the bound

$$\ell_{\text{ext}} \leq \sum_{i=1}^{2} b_A(T_i) + b_B(T_i) \leq \tfrac{1}{2}\operatorname{tw}(D(L)) + k_L + 4. \qquad \square$$

Now we have the tools necessary to prove the main lemma needed to find bounds for $C(L)$ in terms of T_L.

Lemma 3.11 *Let $T_1, \dots, T_n$ be strongly alternating tangles whose Conway sum is a link diagram $D(L)$ for a link L. Then letting β_L and β'_L be the second and second-to-last coefficients of the Jones polynomial of L, $T_L = |\beta_L| + |\beta'_L|$, and k_L the number of link components, we have*

$$\tfrac{1}{2}\operatorname{tw}(D(L)) - k_L - 2 \leq T_L \leq 2\operatorname{tw}(D(L)).$$

Proof We start by noting that we can mutate the link L in such a way that it either is alternating or the sum of T and T' where T is a positive strongly alternating tangle and T' is a negative strongly alternating tangle, without changing its Jones polynomial [20]. The positive and negative refer to whether the northwest strand originates from an overcrossing or an undercrossing. In the former case we have a stronger result by Dasbach and Lin [7] that $T_L = \mathrm{tw}(L)$.

We will assume that L is not alternating. Then work by Lickorish and Thistlethwaite [21] shows that $D(L)$ is adequate. Further, by Propositions 1 and 5 of [21] we have $v_A + v_B = c$ where v_A is the number of vertices in $\mathbb{G}_A(D(L))$, v_B the same in $\mathbb{G}_B(D(L))$ and c the number of crossings in $D(L)$. Every edge we lose when passing from $\mathbb{G}_A(D(L))$ and $\mathbb{G}_B(D(L))$ to $\mathbb{G}'_A(D(L))$ and $\mathbb{G}'_B(D(L))$ comes from either multiple edges in a twist region or an inadmissible bridge. By Lemma 5.2 in [9] we see that the number of edges lost due to twist regions is $c - \mathrm{tw}(D(L))$ which is equivalent to ℓ_{in}. This is because the edges which share a twist region are equivalent to those who become identified in the reduction before taking the Conway sum. On the other hand type-(i) inadmissible bridges for strongly alternating tangles will be precisely those which we lose from identification when taking the Conway sum, $\ell_{\mathrm{ext}}(D(L))$. Then the first part of the move from (4) to (5) below will come from the equality $e'_A + e'_B - e_A - e_B = -(c - \mathrm{tw}(D(L)) + \ell_{\mathrm{ext}})$. Work by Stoimenow shows that for an adequate link diagram (see [7] for a proof)

$$\begin{aligned} T_L &= e'_A + e'_B - v_A - v_B + 2 && (3)\\ &= (e'_A + e'_B - e_A - e_B) + e_A + (e_B - v_A - v_B) + 2 && (4)\\ &= -(c - \mathrm{tw}(D(L)) + \ell_{\mathrm{ext}}) + c + (c - v_A - v_B) + 2 && (5)\\ &\geq \mathrm{tw}(D(L)) - \ell_{\mathrm{ext}} + 2 && (6)\\ &\geq \mathrm{tw}(D(L)) - \tfrac{1}{2}\mathrm{tw}(D(L)) - k_L - 4 + 2 = \tfrac{1}{2}\mathrm{tw}(D(L)) - k_L - 2. && (7) \end{aligned}$$

The upper bound on T_L was shown by Futer, Kalfagianni and Purcell in [8]. □

Theorem 1.2 *Let T_1 and T_2 be nonsplittable, twist reduced, strongly alternating tangles whose Conway sum is a link L. If $C(L)$ is the crosscap number of L, $T_L = |\beta_L| + |\beta'_L|$ and k_L is the number of link components in L we find that*

$$\left\lceil \tfrac{1}{6} T_L \right\rceil - k_L \leq C(L) \leq 2T_L + k_L + 8.$$

Proof This follows immediately from Theorem 3.5 and Lemma 3.11. □

Corollary 3.12 *Let T_1 and T_2 be twist reduced, nonsplittable, strongly alternating tangles whose Conway sum is a link L. Assume that $\mathrm{tw}(T_i) = \mathrm{tw}(K_{iN}) = \mathrm{tw}(K_{iD})$. If $C(L)$ is the crosscap number of L, $T_L = |\beta_L| + |\beta'_L|$ and k_L is the number of link components in L, we have*

$$\left\lceil \tfrac{1}{6} T_L \right\rceil + 2 - k_L \leq C(L) \leq 2T_L + k_L + 8.$$

4 Generalizing to larger Conway sums of tangles

Our goal in this section is to generalize Theorem 1.2 to Conway sums of more than two tangles. A *Conway sum* of more than 2 tangles is a closure where we connect diagrams of the tangles $T_1, T_2, \dots, T_l$ linearly west to east shown in Figure 2. As with the case of the sum of two tangles, if we let L be our Conway sum and S a spanning surface, cutting S along a Conway sphere intersecting T_i will result in a spanning surface for either K_{iN} or K_{iD}, dictating the closure for the tangle. When we cut L, we position l Conway spheres such that Σ_i intersects L at the directional strands of T_i. We note that $S^3 \setminus \bigcup_i \Sigma_i$ will not be a sphere, but we are concerned with the surfaces within the interior of each Σ_i. $S \setminus \cup_i \Sigma_i$ will be a collection of bands and tubes which we consider in the Euler characteristic change. This section will have similar results to the previous sections but with a factor for the number of tangles. We start with the following lemma which is a generalization of Lemma 2.5.

Lemma 4.1 *Let L be the Conway sum of the tangles $T_1, T_2, \dots, T_l$, and $K_1, K_2, \dots, K_l$ are closures of the tangles. Then if k_L is the number of link components for L, and $k_1, k_2, \dots, k_l$ the number of link components for each link, respectively, then $k_L = \sum_{i=1}^{l} k_i - l + \epsilon$ for $\epsilon = 0, 1, 2$.*

Theorem 4.2 *Let $T_1, T_2, \dots, T_l$ be nonsplittable, strongly alternating tangles, and let L be the Conway sum that results from the l tangles. If K_i is the closure of T_i resulting from cutting the crosscap realizing spanning surface for L for all $i \in \{1, 2, \dots, l\}$ and K_{iN} is the numerator closure of T_i and K_{iD} the denominator closure, then we have*

$$\sum_{i=1}^{l} C(K_i) - l \le C(L) \le \min\left\{ \sum_{i=1}^{l} C(K_{iN}) + l, \sum_{i=1}^{l} C(K_{iD}) + 2 \right\}.$$

Proof This proof will largely follow the work we did in Lemmas 2.14 and 2.4. We will start by considering the upper bound.

First we consider the case where we have K_{iN} for all $i \in \{1, 2, \dots, l\}$, and spanning surfaces S_i for each K_{iN} such that $C(S_i) = C(K_{iN})$. Unlike in Lemma 2.4 the NW and NE strands connect to different tangles and we will find the same for the SW and SE strands. Then the spanning surface resulting from the Conway sum will have northern and southern disks attached to each of the S_i by a band as seen in Figure 9. Let this resulting surface be S.

By this construction $\chi(S) = \sum_{i=1}^{l} \chi(S_i) - 2l + 2$ where the $2l$ comes from the bands connecting each surface to the disks, and the 2 from the disks themselves. Then by Lemma 4.1 we see that $k_L = \sum_{i=1}^{l} k_{iN} - l + \epsilon$. Then

$$C(S) = 2 - \left(\sum_{i=1}^{l} \chi(S_i) - 2l + 2 \right) - \left(\sum_{i=1}^{l} k_{iN} - l + \epsilon \right) = \sum_{i=1}^{l} C(K_{iN}) + l - \epsilon.$$

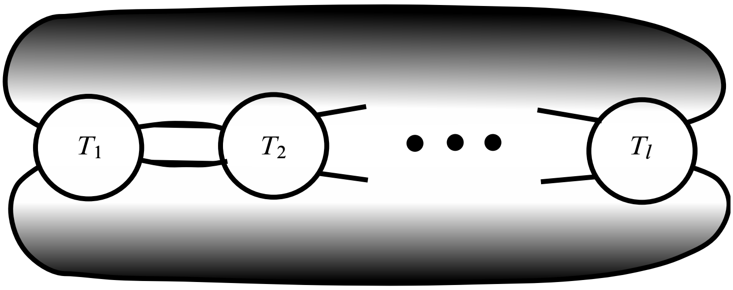

Figure 9: We see here that a surface would have to fill the shaded areas to connect the numerator closures of the tangles since the western and eastern boundaries of the S_i connect to separate tangles.

Then we see the weakest upper bound is when $\epsilon = 0$, so

$$C(L) \le C(S) = \sum_{i=1}^{l} C(K_{iN}) + l.$$

Meanwhile the all denominator closure case will be similar to when $l = 2$. In particular $\chi(S) = \sum_{i=1}^{l} \chi(S_i) - l$ as we add bands to connect each of the S_i to their neighboring surfaces. By Lemma 4.1 and similar computations to the numerator closure case we find $C(L) \le \sum_{i=1}^{l} C(K_{iD}) + 2$. We have a 2 instead of an l as we added half the number of bands in constructing S. Then we take the minimum of the denominator and numerator bounds to find an upper bound for $C(L)$.

Now we consider the lower bound. Let S be a spanning surface for L such that $C(S) = C(L)$. Similar to Lemma 2.14 we will be considering the surfaces $S_1, S_2, \dots, S_l$ that result from cutting along the Conway spheres Σ_i. By a similar argument to the one for Lemma 2.12, S can be chosen so that $\Sigma_i \cap S$ contains at most one closed curve for all i. Notice that if any of the steps in Lemma 2.12 were to disconnect the surface outside Σ_i, then for some other Σ_j, $i \ne j$, T_j would span a disconnected surface which is a contradiction to Lemma 2.10. Then when we cut along the Conway spheres we see that we are at most cutting along two arcs and a closed curve.

We know from Lemma 4.1 that $k_L = \sum_{i=1}^{l} k_i - l + \epsilon$ for $\epsilon = 0, 1, 2$. If we have closed curves along a Σ_i, when we cut we will have to add disks to both resulting boundary components which increases the Euler characteristic by 2. For any surface resulting from cutting that is orientable we will have to add in a half twist band to make it nonorientable. Each such half twist band reduces the Euler characteristic by 1. Then $\sum_{i=1}^{l} \chi(S_i) = \chi(S) + t + c - b$ where $t = l$ or $t = 2l - 2$ depending on if the K_i are the denominator or numerator closures, c the number of closed curves on the Σ_i which can be as large as l and b the number of twist bands added to make the S_i nonorientable which also has maximum l. The value of t arises from the bands which sit in $S^3 \setminus \bigcup_i \Sigma_i$.

Now we see that

$$\sum_{i=1}^{l} C(S_i) = 2l - \sum_{i=1}^{l} \chi(S_i) - \sum_{i=1}^{l} k_i = 2l - \chi(S) - t - c + b - k_L - l + \epsilon.$$

The weakest upper bound for $\sum_{i=1}^{l} C(S_i)$ is when t and c are minimal and b and ϵ are maximal. So this will be when we do not have any closed curves and each of the resulting S_i need half twist bands to make them nonorientable. So

$$\sum_{i=1}^{l} C(S_i) = 2l - \chi(S) - l + l - k_L - l + 2 = C(L) + l.$$

Then moving the l to the other side we see that $\sum_{i=1}^{l} C(K_i) - l \leq C(L)$, showing the claim. □

As in Section 3 we will now use Theorem 3.4 to find bounds for $C(L)$ in terms of $\mathrm{tw}(D(L))$ where $\mathrm{tw}(D(L))$ is the twist number for a diagram of L.

Theorem 4.3 *Let $T_1, T_2, \ldots, T_l$ be nonsplittable, twist reduced, strongly alternating tangles and let $D(L)$ be the link diagram for the link L which results from taking the Conway sum of the tangles. Let $\mathrm{tw}(D(L))$ denote the twist number of $D(L)$ and $C(L)$ the crosscap number for L. Then*

$$\left\lceil \tfrac{1}{3}\,\mathrm{tw}(D(L)) \right\rceil + 2 - k_L \leq C(L) \leq \mathrm{tw}(D(L)) + l + 2 - k_L.$$

Proof We start with the bounds from Theorem 4.2. From here we use Theorem 3.4 and Lemma 3.3 to get bounds on $C(L)$ with respect to the twist numbers of diagrams of $T_1, \ldots, T_l$. Combining these two statements we find

$$\sum_{i=1}^{l} \left\lceil \tfrac{1}{3}(\mathrm{tw}(T_i) - 2) \right\rceil + l - \sum_{i=1}^{l} k_i \leq C(L) \leq \sum_{i=1}^{l} \mathrm{tw}(T_i) + 2l + 2 - \sum_{i=1}^{l} k_i,$$

where k_i is the number of link components for each K_i.

We know that the twist number of strongly alternating tangles is additive over a Conway sum by Lemma 3.2. So the only detail left to consider is the relationship between k_L and $\sum_{i=1}^{l} k_i$. By Lemma 4.1 and a similar argument to that in Theorem 3.5 we find the claim:

$$\left\lceil \tfrac{1}{3}\big(\mathrm{tw}(D(L)) - 2l\big) \right\rceil + 2 - k_L \leq c(L) \leq \mathrm{tw}(D(L)) + l + 2 - k_L. \qquad \square$$

The final piece of our puzzle is to find bounds in terms of T_L. To do this we use Lemma 3.11 and Theorem 4.3 and the result follows.

Theorem 4.4 *Let $T_1, T_2, \ldots, T_l$ be nonsplittable, twist reduced, strongly alternating tangles and let L be the link which results from taking the Conway sum. Then let $C(L)$ be the crosscap number and k_L the number of link components in L. Then*

$$\left\lceil \tfrac{1}{6}(T_L - 2l) \right\rceil + 2 - k_L \leq C(L) \leq 2T_L + l + 6 + k_L.$$

With an additional constraint on our tangles we find the following corollary:

Corollary 4.5 *Let $T_1, T_2, \dots, T_l$ be nonsplittable, twist reduced, strongly alternating tangles such that $\operatorname{tw}(T_i) = \operatorname{tw}(D(K_{iN})) = \operatorname{tw}(D(K_{iD}))$ for all $i \in \{1, \dots, l\}$. Let L be the link which results from taking the Conway sum, $C(L)$ the crosscap number, and k_L the number of link components in L. Then*

$$\left\lceil \tfrac{1}{6} T_L \right\rceil + 2 - k_L \le C(L) \le 2T_L + l + 6 + k_L.$$

5 Families where T_L and the crosscap number are independent

We begin by recalling the following theorem from [17] which gives linear bounds for the crosscap number of an alternating link in terms of T_L, where $T_L = |\beta_L| + |\beta'_L|$ and β_L and β'_L are the second and second-to-last coefficients of the Jones polynomial of L, respectively.

Theorem 5.1 *Let L be a nonsplit, prime alternating link with k components and with crosscap number $C(L)$. Suppose that K is not a $(2, p)$ torus link. We have*

$$\left\lceil \tfrac{1}{3} T_L \right\rceil + 2 - k \le C(L) \le T_L + 2 - k,$$

where T_L is as above. Furthermore, both bounds are sharp.

In the previous sections we showed Theorem 5.1 generalizes to Conway sums of strongly alternating tangles. In this section we will show that Theorem 5.1 does not generalize to arbitrary knots.

Theorem 1.4 (a) *There exists a family of links for which $T_L \le 2$, but $C(L)$ is arbitrarily large.*

(b) *There exists a family of links for which $C(L) \le 3$, but T_L is arbitrarily large.*

5.1 Part (a) of Theorem 1.4

In this section we will consider the family of torus knots $T(p,q)$, where $q = j$ and $p = 2 + 2jk$ for odd $j > 1$ and all natural numbers k. This family will allow us to prove part (a) of Theorem 1.4. We start with the following definition from Teragaito [27].

Definition 5.2 We define the value $N(p,q)$ from [27] for fractions $\frac{p}{q}$, where p and q are coprime, to begin write $\frac{p}{q}$ as a continued fraction

$$\frac{p}{q} = [a_0, a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where the a_i are integers, $a_0 \ge 0$, $a_i > 0$ for $1 \le i \le n$, and $a_n > 1$. A continued fraction of this form is unique (see [14]). Now we recursively define b_i as

$$b_0 = a_0, \qquad b_i = \begin{cases} a_i & \text{if } b_{i-1} \ne a_{i-1} \text{ or if } \sum_{j=0}^{i-1} b_j \text{ is odd,} \\ 0 & \text{if } b_{i-1} = a_{i-1} \text{ and } \sum_{j=0}^{i-1} b_j \text{ is even.} \end{cases}$$

Then, $N(p,q) = \frac{1}{2} \sum_{i=1}^{n} b_i$. We say a torus knot K is even if the product of p and q is even and we say K is odd otherwise. Using these definitions we can state Theorem 1.1 from [27].

Theorem 5.3 *Let K be the nontrivial torus knot of type (p,q), where $p,q>0$ and let F be a nonorientable spanning surface of K with $C(F)=C(K)$.*

(i) *If K is even, then $C(K)=N(p,q)$ and the boundary slope of F is pq.*

(ii) *If K is odd, then $C(K)=N(pq-1,p^2)$ (resp $N(pq+1,p^2)$) and the boundary slope of F is $pq-1$ (resp $pq+1$) if $xq\equiv -1 \pmod p$ has an even (resp odd) solution x satisfying $0<x<p$.*

We take advantage of (i) from Theorem 5.3 to both construct our family of torus links and prove Proposition 5.5 below. We also need the following lemma, which gives an explicit formula for the Jones polynomial of a torus knot originally given in Proposition 11.9 of [16], which allows us to calculate T_L for torus knots.

Lemma 5.4 *The Jones polynomial for a torus knot $T(p,q)$ is given by*

$$V(T(p,q))=t^{(p-1)(q-1)/2}\frac{1-t^{p+1}-t^{q+1}+t^{p+q}}{1-t^2}.$$

Proposition 5.5 *Let $L=T(p,q)$ be the family of torus knots where $q>1$ is odd and $p=2+2qk$ for $k\in\mathbb{N}$. Then $T_L\leq 2$ but $C(T(p,q))$ can be made arbitrarily large.*

Proof Let $q>1$ be odd, and $p=2+2qk$ where k is a natural number. To show that for all such torus knots, $L=T(p,q)$, $C(L)$ does not have a universal upper bound with respect to T_L, we will show that as k goes to ∞, $C(L)$ also goes to ∞, but $T_L\leq 2$. We start by computing the crosscap number of $T(p,q)$ using Theorem 5.3.

First we notice that

$$\frac{p}{q}=\frac{2+2qk}{q}=2k+\frac{1}{1+\frac{1}{2}}.$$

Then $A=[2k,1,2]$. Then by Definition 5.2 $B=[2k,0,2]$. Finally, as pq is even,

$$C(L)=N(p,q)=\tfrac{1}{2}(2k+0+2)=k+1.$$

Then as $k\to\infty$, $C(L)$ also goes to ∞.

Next by Lemma 5.4 we know that

$$\begin{aligned}V(L)&=t^{(p-1)(q-1)/2}\frac{1-t^{p+1}-t^{q+1}+t^{p+q}}{1-t^2}\\&=t^{((2+2qk)q-(2+2qk)-q+1)/2}\frac{1-t^{2+2qk+1}-t^{q+1}+t^{2+2qk+q}}{1-t^2}\\&=t^{((2+2qk)q-(2+2qk)-q+1)/2}(-t^{2qk+q}-t^{2qk+q-2}-\cdots-t^{2+2qk+1}+t^{q-1}+\cdots+t^2+1).\end{aligned}$$

The last step arises from taking the polynomial division. Therefore, given our choices of p and q we see that $T_L\leq 2$. □

Then Proposition 5.5 shows part (a) of Theorem 1.4.

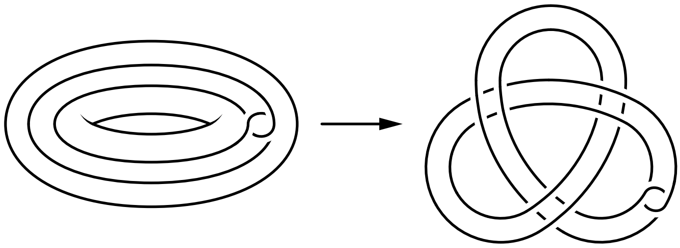

Figure 10: Here we see the unknot with a clasp contained inside the torus, then we see the resulting *positive* Whitehead double with the blackboard framing when map the torus to the trefoil.

5.2 Part (b) of Theorem 1.4

In this section we will work to prove part (b) of Theorem 1.4 and the following theorem:

Theorem 5.6 *There does not exist a universal linear lower bound on $C(L)$ for all links L in terms of T_L.*

To prove this we will introduce a family of links for which $C(L)$ is uniformly bounded but T_L can be made arbitrarily large. These links will be constructed by using the Whitehead double defined here:

Definition 5.7 The *Whitehead double* of a knot L is the satellite of the unknot clasped inside of the torus. We call it a *positive* Whitehead double if the clasp is as in Figure 10 and a *negative* Whitehead double if not.

The particular family is defined in this next theorem:

Theorem 5.8 *Let $K_1, K_2, \dots, K_n$ be alternating knots such that $\beta'_{K_i} \neq 0$. Then we let K be the connect sum of $K_1, K_2, \dots, K_n$ such that K is alternating and $W_-(K)$ be the negative Whitehead double of K using the blackboard framing. Then $C(L) \leq 3$ and $|\beta'_L| \geq n$.*

Lemma 5.9 *If a link is B-adequate then the negative Whitehead double of the link using the blackboard framing is also B-adequate.*

A similar statement was proven in [3] as Proposition 7.1. They show it for the untwisted negative Whitehead double of a knot with nonnegative writhe. The writhe of the knot introduces extra twists into the diagram of the untwisted Whitehead double, which can interfere with adequacy around the clasp.

Proof We start by showing that the blackboard two-cabling will be B-adequate. This is shown by Lickorish in [20] for n-cablings. Let D be a B-adequate diagram for our link and D^2 the two-cabling. Notice in D^2 there will be four copies of each crossing in D. Then when we have the all B-resolution state we will end up with four parallel strands instead of two as we did in D. If we were to have a one-edge loop, then two of the strands are part of the same state circle. But these state circles are copies of the state circles for D so this would contradict that D is adequate.

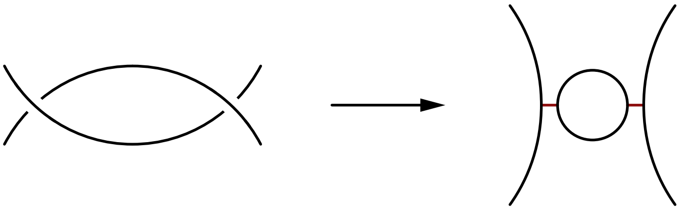

Figure 11: The result of the B-resolution on the clasp of a negative Whitehead double.

Now we want to look at the negative Whitehead double of D using the blackboard framing. If we let the Whitehead double be $W_-(D)$ we will see that $G'(W_-(D))$ will be the same as $G'(D^2)$ but with an additional vertex and 2 new edges as we see in Figure 11. As resolving the clasp does not create a one-edge loop we see that $W_-(D)$ is B-adequate. $\square$

Here we remind the reader that $G'_B(D(L))$ for a link diagram $D(L)$ is the reduced all B-state graph. We continue with the following lemma:

Lemma 5.10 *If $W_-(D(L))$ is the negative Whitehead double of a B-adequate link diagram $D(L)$ using the blackboard framing, and $b(G)$ the first Betti number for a graph, then $b\big(G'_B\big(W_-(D(L))\big)\big) = b\big(G'_B(D(L))\big) + 1$.*

Proof Dasbach and Lin [6] showed in Lemma 2.5 that if D^2 is the two-cabling of a B-adequate link diagram then $b(\mathbb{G}'_B(D^2)) = b\big(\mathbb{G}'_B(D(L))\big)$. For a graph G, $b(G) = e - v + 1$, where e is the number of edges and v the number of vertices. In the reduced graph when we take the two-cabling every parallel copy of a state circle will also produce a new edge. Hence, the change in v and e will be the same between $\mathbb{G}'_B(D^2)$ and $\mathbb{G}'_B(D(L))$. Then when we move to $W_-(D(L))$ the clasp will add 2 edges and 1 vertex as we see in Figure 11. Then we see that $b\big(\mathbb{G}'_B\big(W_-(D(L))\big)\big) = b(\mathbb{G}'_B(D^2)) + 1 = b\big(\mathbb{G}'_B(D(L))\big) + 1$. $\square$

The two previous lemmas allow us to see that the blackboard framing of the negative Whitehead double of an alternating link will be B-adequate. Also, we have a formula for the Betti number of the Whitehead double in relation to the first Betti number of the original link. The only remaining piece of the puzzle is to get from the first Betti number of the reduced B-state graph to the second-to-last coefficient of the Jones polynomial. This comes from the following result proven by Stoimenow in Proposition 3.1 of [26].

Lemma 5.11 *If $D(L)$ is a B-adequate, connected diagram for a link, then in the representation of the Jones polynomial, $V(D(L))$, we have $\alpha'_{D(L)} = \pm 1$, $\alpha'_{D(L)}\beta'_{D(L)} \le 0$, and*

$$|\beta'_{D(L)}| = e' - v' + 1 = b\big(G'_B(D(L))\big),$$

where $\mathbb{G}'_B$ is the reduced all B-state graph and e' and v' are the number of edges and vertices of the graph $G'_B(D(L))$, respectively.

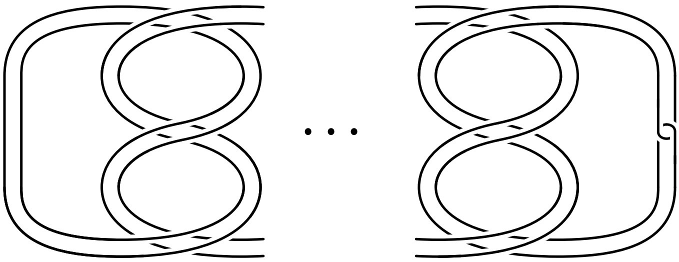

Figure 12: The negative Whitehead double of the connect sum of m trefoil knots.

We now have the tools to prove Theorem 5.8. But first we show a more specific example of a family which satisfies Theorem 1.4(b).

Proposition 5.12 *Let $W_-(K_m)$ be the negative Whitehead double using the blackboard framing of the connect sum of m trefoils as in Figure 12. Then for all m, $C(W_-(K_m)) \leq 3$ and $T_{W_-(K_m)}$ grows with m. Therefore, $T_{W_-(K_m)}$ can be made arbitrarily large across the family of knots.*

Proof The first part of the lemma is a direct result of [4] by Clark where he shows that $c(K) \leq 2g(K)+1$ where $g(K)$ is the genus of the knot. For any Whitehead double we can find an oriented spanning surface with genus exactly one by taking the annulus with a double twisted band at the clasp. Then $C(W_-(K_m)) \leq 3$ as $g(W_-(K_m)) = 1$.

Now we will compute $\beta'_{W_-(K_m)}$ by finding $\mathbb{G}'_B(W_-(K_m))$. By Lemma 5.11 we only need to find the number of vertices and edges as $W_-(K_m)$ is B-adequate. By Lemma 5.11 and the graph $\mathbb{G}'_B(W_-(K_m))$ shown in Figure 13

$$|\beta'_{W_-(K_m)}| = e\big(\mathbb{G}'_B(W_-(K_m))\big) - v\big(\mathbb{G}'_B(W_-(K_m))\big) + 1 = (5m+3) - (4m+3) + 1 = m. \tag{8}$$

Hence, we have shown that $T_{W_-(K_m)} \geq m$ for all k, proving the claim. □

Here we will introduce a more general family of knots for which Theorem 1.4(b) holds true:

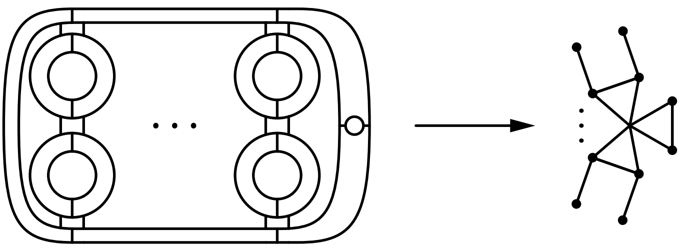

Figure 13: Left: the all B-state circle diagram. Right: the reduced state graph $G'_B(L)$. Notice the disjoint dots are not nodes for the graph but represent that we have k copies of the subgraph on the left.

Theorem 5.8 *Let $K_1, K_2, \dots, K_n$ be alternating knots such that $\beta'_{K_i} \neq 0$. Then let K be the connect sum of $K_1, K_2, \dots, K_n$ such that K is alternating, and let $W_-(K)$ the negative Whitehead double of K using the blackboard framing. Then $C(W_-(K)) \leq 3$ and $|\beta'_{W_-(K)}| \geq n$.*

Proof As in Proposition 5.12 for a Whitehead double such as $W_-(K)$, $C(W_-(K)) \leq 3$. Now we will work to compute $T_{W_-(K)}$. From [20] we know that the Jones polynomial for K will be the product of the Jones polynomials of the K_i. Then as all of the K_i are alternating, $\alpha'_{K_i} = \pm 1$ so $\beta'_K = \sum_{i=1}^n \pm \beta'_{K_i}$. From Lemma 5.11 we know that $\alpha'_{K_i}\beta'_{K_i} \leq 0$ which tells us that the signs of α'_{K_i} and β'_{K_i} do not match. If we let m be the number of the α'_i which are negative, then we see that $\beta'_K = \sum_{i=1}^n (-1)^{m\pm 1}|\beta'_{K_i}|$. Hence, in our sum the signs match so $|\beta'_K| = \sum_{i=1}^n |\beta'_{K_i}|$. By our hypothesis $|\beta'_{K_i}| > 0$ for all i, hence $|\beta'_K| \geq n$.

By Lemma 5.9 we know that $W_-(K)$ will be B-adequate as K is alternating and therefore B-adequate. Then by Lemmas 5.10 and 5.11 we see that $|\beta'_{W_-(K)}| = |\beta'_K| + 1$ and as $|\beta'_K|$ is at least as large as the number of knots in the connect sum so is $|\beta'_{W_-(K)}|$ and further $T_{W_-(K)}$. Then $T_{W_-(K)}$ will grow with n showing that it is unbounded across the family. □

Combining Proposition 5.12 and Theorem 5.8 shows Theorem 1.4(b).

6 Future directions

In Sections 2, 3, and 4, we generalized the work from [17] to bound the crosscap number of sums of strongly alternating tangles. Then in Section 5, introduced infinite families of knots for which their crosscap number and T_L grow independently. The links we considered in Sections 2, 3, and 4 are all hyperbolic, meanwhile those that we constructed in Section 5 are not hyperbolic. This leads to the following question:

Question 6.1 Does Theorem 4.4 generalize for all hyperbolic knots?

A first step for Question 6.1 would be to relax the requirement that the individual tangles be strongly alternating. At the time of writing, this seems reasonable for the first step of our proof, but the uncertainty arises in moving from bounds in terms of the crosscap numbers of individual tangles to the twist number. In particular, alternating is a requirement for our usage of Theorem 3.4. Another potential way to move forward with this question would be to look at adequate links in general, which will be studied in future work.

References

[1] **C Adams**, **T Kindred**, *A classification of spanning surfaces for alternating links*, Algebr. Geom. Topol. 13 (2013) 2967–3007 MR

[2] **I Agol**, **J Hass**, **W Thurston**, *The computational complexity of knot genus and spanning area*, Trans. Amer. Math. Soc. 358 (2006) 3821–3850 MR

[3] **K L Baker**, **K Motegi**, **T Takata**, *The strong slope conjecture for twisted generalized Whitehead doubles*, Quantum Topol. 11 (2020) 545–608 MR

[4] **B E Clark**, *Crosscaps and knots*, Int. J. Math. Math. Sci. 1 (1978) 113–123 MR

[5] **R Crowell**, *Genus of alternating link types*, Ann. of Math. 69 (1959) 258–275 MR

[6] **O T Dasbach**, **X-S Lin**, *On the head and the tail of the colored Jones polynomial*, Compos. Math. 142 (2006) 1332–1342 MR

[7] **O T Dasbach**, **X-S Lin**, *A volumish theorem for the Jones polynomial of alternating knots*, Pacific J. Math. 231 (2007) 279–291 MR

[8] **D Futer**, **E Kalfagianni**, **J S Purcell**, *Dehn filling, volume, and the Jones polynomial*, J. Differential Geom. 78 (2008) 429–464 MR

[9] **D Futer**, **E Kalfagianni**, **J S Purcell**, *Symmetric links and Conway sums: volume and Jones polynomial*, Math. Res. Lett. 16 (2009) 233–253 MR

[10] **D Futer**, **E Kalfagianni**, **J S Purcell**, *Cusp areas of Farey manifolds and applications to knot theory*, Int. Math. Res. Not. 2010 (2010) 4434–4497 MR

[11] **D Futer**, **E Kalfagianni**, **J Purcell**, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Math. 2069, Springer (2013) MR

[12] **D Futer**, **E Kalfagianni**, **J S Purcell**, *Jones polynomials, volume and essential knot surfaces: a survey*, from "Knots in Poland, III: Part I", Banach Center Publ. 100, Polish Acad. Sci. Inst. Math., Warsaw (2014) 51–77 MR

[13] **D Futer**, **E Kalfagianni**, **J S Purcell**, *Quasifuchsian state surfaces*, Trans. Amer. Math. Soc. 366 (2014) 4323–4343 MR

[14] **G H Hardy**, **E M Wright**, *An introduction to the theory of numbers*, 5th edition, Oxford Univ. Press (1979) MR

[15] **J Hass**, **J C Lagarias**, **N Pippenger**, *The computational complexity of knot and link problems*, J. ACM 46 (1999) 185–211 MR

[16] **V F R Jones**, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. 126 (1987) 335–388 MR

[17] **E Kalfagianni**, **C R S Lee**, *Crosscap numbers and the Jones polynomial*, Adv. Math. 286 (2016) 308–337 MR

[18] **L H Kauffman**, *State models and the Jones polynomial*, Topology 26 (1987) 395–407 MR

[19] **T Kindred**, *Crosscap numbers of alternating knots via unknotting splices*, Internat. J. Math. 31 (2020) art. id. 2050057 MR

[20] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Math. 175, Springer (1997) MR

[21] **W B R Lickorish**, **M B Thistlethwaite**, *Some links with nontrivial polynomials and their crossing-numbers*, Comment. Math. Helv. 63 (1988) 527–539 MR

[22] **C Livingston**, **A H Moore**, *KnotInfo: table of knot invariants*, electronic reference (2021) Available at `https://knotinfo.math.indiana.edu`

[23] **H Murakami**, *An introduction to the volume conjecture*, from "Interactions between hyperbolic geometry, quantum topology and number theory", Contemp. Math. 541, Amer. Math. Soc., Providence, RI (2011) 1–40 MR

[24] **K Murasugi**, *On the Alexander polynomial of the alternating knot*, Osaka Math. J. 10 (1958) 181–189 MR Correction in 11 (1959) 95

[25] **K Murasugi**, *Jones polynomials and classical conjectures in knot theory*, Topology 26 (1987) 187–194 MR

[26] **A Stoimenow**, *Coefficients and non-triviality of the Jones polynomial*, J. Reine Angew. Math. 657 (2011) 1–55 MR

[27] **M Teragaito**, *Crosscap numbers of torus knots*, Topology Appl. 138 (2004) 219–238 MR

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