Atiyah–Segal completion for the Hermitian K-theory of symplectic groups

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We study equivariant Hermitian K-theory for representations of symplectic groups, especially SL_2. The results are used to establish an Atiyah–Segal completion theorem for Hermitian K-theory and symplectic groups.

1 Introduction

In [38], a completion theorem for Hermitian K-theory of schemes with trivial torus action is established. Let X be a regular Noetherian separated scheme over $\mathrm{Spec}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ with a trivial action of a split torus $T \cong \mathbb{G}_m^t$. We let $\mathrm{IO} = \ker(\mathrm{GW}_0^{T,[0]}(X) \to \mathrm{GW}_0^{[0]}(X))$ be the Hermitian version of the augmentation ideal. Define

$$\mathrm{GW}^{T,[n]}(X)_{\mathrm{IO}}^{\wedge}$$

as the derived completion of the $\mathrm{GW}^{T,[0]}(X)$-module $\mathrm{GW}^{T,[n]}(X)$ with respect to IO; see [39] for details. Also define

$$\mathrm{GW}^{[n]}(\mathrm{B}T_X) = \lim_n \mathrm{GW}^{[n]}((\mathbb{P}_X^n)^{\times t}).$$

In [37, Proposition 8.2.2], it is shown that this definition is in fact an equivalence of GW-spectra of motivic spaces. In [38, Theorem 3.2.3], the following completion theorem for Hermitian K-theory is established.

Theorem 1.0.1 *For all $i, n \in \mathbb{Z}$, the natural map*

$$\pi_i(\mathrm{GW}^{T,[n]}(X)_{\mathrm{IO}}^{\wedge}) \xrightarrow{\cong} \mathrm{GW}_i^{[n]}(\mathrm{B}T_X)$$

is an isomorphism.

In the case of algebraic K-theory, the corresponding result for split tori T, due to Totaro [47] for K_0 and Knizel–Neshitov [26] for higher K-groups, was the first important step for establishing more general versions of an Atiyah–Segal style completion theorem for linear algebraic groups, due to Krishna [28], Tabuada–van den Bergh [45] and Carlsson–Joshua [12], and their proofs all rely on a reduction to the case of a split torus. The proof for algebraic K-theory and tori T is easier than Rohrbach's theorem above on Hermitian K-theory because the classifying space BT is a product of copies of $\mathbb{P}^\infty$, and algebraic K-theory is orientable in the sense of Levine–Morel and Panin. As of yet, there is no completion theorem for Hermitian K-theory in the case of general linear groups GL_n, although [40] contains a partial result in this direction by computing the Hermitian K-theory of even-dimensional Grassmannians.

We restrict our attention to the Hermitian K-theory of symmetric forms and symplectic forms in degree zero, which is the $\mathbb{Z}/2\mathbb{Z}$-graded Grothendieck–Witt group $GW^\pm$. The results can likely be extended to higher Hermitian K-theory over general base schemes using the machinery of derived completion as in [38], in line with the recent work of Tabuada–van den Bergh [45] and Carlsson–Joshua [12]. An extension of the completion theorem for Hermitian K-theory to schemes with nontrivial actions seems more difficult, as all the known proofs in algebraic K-theory rely on the equivariant localization theorem, whose analogue in Hermitian K-theory has not yet been established, and on a suitable geometric equivariant decomposition theorem. We refer to the last subsection for further details.

In a series of fundamental articles, Panin and Walter establish the theory of symplectic oriented cohomology theories on smooth algebraic varieties. In this setting, the one-dimensional torus GL_1 is replaced by $Sp_2 = SL_2$. They also show that Hermitian K-theory is a symplectic orientable theory, which in particular implies the following computation of BSp_{2n} over a base field k (see [32; 34, Theorem 9.1]):

$$BO^{*,*}(BSp_{2n}) \cong BO^{*,*}(\operatorname{Spec} k)\llbracket b_1, \ldots, b_n \rrbracket,$$

where the b_i are the *Borel classes* of [34, Section 8] and the right-hand side is the ring of graded power series over $BO^{*,*}(\operatorname{Spec} k)$; compare, eg, [27, Section 6.3]. From this point of view, one might argue that the conjectural Atiyah–Segal completion result for Sp_{2n} and Hermitian K-theory is the correct analogue of the completion theorem for GL_n and algebraic K-theory, and should provide computations involving free polynomial rings that are easier than Rohrbach's theorem above. We will show that this is indeed the case and prove the following theorem as Corollary 3.3.13 below.

Theorem 1.0.2 *Let k be a field of characteristic not two. There is a canonical map of $GW^\pm(k)$-algebras*

$$GW^\pm(\mathbf{Rep}(Sp_{2r})) \to GW^\pm(BSp_{2r}),$$

which exhibits $GW^\pm(BSp_{2r})$ as the completion of $GW^\pm(\mathbf{Rep}(Sp_{2r}))$ with respect to $IO_{Sp_{2r}}$.

The strategy for GW-theory consists in some sense in systematically replacing GL_1 by Sp_2, and more generally GL_n by Sp_{2n} in all steps of the proof for complex or algebraic K-theory.

This approach has already been used in computations for Chow–Witt groups, see, eg, [19], and is motivated in part by the fact that real realization of Sp_{2n} has $U(n)$ as a maximal compact subgroup.

The techniques used in this proof, which build on and slightly generalize those of Morel–Voevodsky and Panin–Walter, can be used to construct geometric classifying spaces for other groups G and other Grothendieck topologies τ. A general result about geometric descriptions of classifying spaces $\mathrm{B}_\tau G$ in motivic homotopy theory is given in Theorem 3.2.17.

Readers familiar with the proof of [4] might consider still another strategy for proving general Atiyah–Segal completion theorems for real and Hermitian K-theory, respectively: rather than Sp_{2n} and its subgroup $\mathrm{Sp}_2^{\times n}$, one should work with GL_n and its maximal split torus, but both equipped with a suitable involution that up to homotopy becomes complex conjugation under complex realization. There are indeed nice models for these in the algebro-geometric setting, which are explained in the introduction of Section 2. In particular, the results in Section 2.3 apply to these algebraic groups with involution. However, we are not yet able to perform all necessary equivariant GW-computations for these algebraic models of GL_n and split tori with involution, and the authors hope to return to this topic in the future.

This article, particularly Section 2, contains results about the *Real* (sometimes *quadratic* or *Hermitian*) representation ring of a large class of reductive groups with involution, such as Sp_{2n} and GL_n with involution. This builds on previous work of Calmès–Hornbostel [11] and Zibrowius [49] and is of independent interest. As an example of how some of the techniques of this paper extend to such Real groups, we have included in Section 3.4 a short construction of a classifying space for the multiplicative group $\mathbb{G}_m$ with a nontrivial involution.

2 The Grothendieck–Witt rings of some Real split reductive groups

Recall that the proof of Atiyah–Segal completion for topological KO-theory in [4] relies on the splitting of [3, Proposition (5.2)] for topological KR-theory for Real groups, ie for groups with an involution. We now discuss the algebraic analogue of such groups.

A *Real* group is a an algebraic group G with an involution $\iota\colon G \to G$. This involution is assumed to be a group homomorphism. We denote by **Rep**(G) the abelian category of finite-dimensional representations

of G, and equip it with a duality $\vee_\iota$ as follows. Recall first that, irrespective of any involution on G, associating with a representation E of G the dual representation $E^\vee := \operatorname{Hom}_G(E, k)$ defines a duality $\vee$ on $\mathbf{Rep}(G)$. More precisely, we obtain a category with duality $(\mathbf{Rep}(G), \vee, \eta)$, where η denotes the canonical double-dual identification. Given the involution ι on G, we define the associated duality $\mathbf{Rep}(G)$ as the composition $\vee_\iota := \vee \circ \iota^*$, ie $E^{\vee_\iota} := (\iota^* E)^\vee$. We thus obtain a category with duality $\mathbf{Rep}(G, \iota) := (\mathbf{Rep}(G), \vee_\iota, \eta_\iota)$.

In this section, we compute the Grothendieck–Witt ring

$$\mathrm{GW}^\pm(\mathbf{Rep}(G, \iota)) := \mathrm{GW}^+(\mathbf{Rep}(G, \iota)) \oplus \mathrm{GW}^-(\mathbf{Rep}(G, \iota))$$

in cases when G is split reductive, the involution ι restricts to a maximal torus T, and all irreducible representations are self-dual with respect to $\vee_\iota$.

Example 2.0.1 Consider (T, inv), where T is a split torus of rank r and inv is the involution given by $z \mapsto z^{-1}$. All representations of T are symmetric with respect to $\vee_{\mathrm{inv}}$.

Example 2.0.2 Consider (GL_n, ι) with $\iota(A) := (A^{-1})^T$, an involution that restricts to the involution inv on the standard maximal torus of GL_n. All representations of GL_n are symmetric with respect to $\vee_\iota$.

Example 2.0.3 Consider $(\mathrm{Sp}_{2n}, \mathrm{id})$. All representations of Sp_{2n} are self-dual with respect to $\vee_{\mathrm{id}}$. Some are symmetric, some are antisymmetric (see Example 2.3.15 for more details).

2.1 Reductive groups — the setup

Let k be a field. Let (G, B, T) be a triple consisting of a connected split reductive group G over k with $T \subset B \subset G$, where B is a Borel subgroup and T is a maximal torus of rank t. We write $X^* := \operatorname{Hom}(T, \mathbb{G}_m)$ and $X_* := \operatorname{Hom}(\mathbb{G}_m, T)$ for the character lattice and the cocharacter lattice, respectively, both isomorphic to $\mathbb{Z}^t$, and $\langle -, - \rangle \colon X^* \times X_* \to \mathbb{Z}$ for the canonical pairing between them. Let $\Phi = \Phi(G, T)$ denote the associated set of roots, $\Phi^+ = \Phi(B, T)$ the set of positive roots associated with the Borel subgroup B as in [13, Proposition 1.4.4], and $\Delta \subset \Phi^+$ the set of simple positive roots. We write α^{co} for the coroot associated with a root α.[1] Let $X^+ \subset X^*$ denote the cone of dominant characters determined by Φ^+, also known as the (closed, integral) fundamental Weyl chamber [13, Equation (1.5.3)]. Explicitly, the relation between Φ^+ and X^+ can be described as

$$(2.1.1) \qquad X^+ = \{ x \in X^* \mid \langle x, \alpha^{\mathrm{co}} \rangle \geq 0 \text{ for all } \alpha \in \Delta \},$$

$$(2.1.2) \qquad \Phi^+ = \{ \alpha \in \Phi \mid \langle x, \alpha^{\mathrm{co}} \rangle \geq 0 \text{ for all } x \in X^+ \}.$$

The dominant characters parametrize the irreducible representations of G [43, lemme 5]: for each $x \in X^+$, there is an irreducible G-representation E_x of highest weight x, unique up to isomorphism. We fix one

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[1] We deviate from the universally agreed notation $\alpha^\vee$ for α^{co} to avoid any confusion with the duality $\vee$ on $\mathbf{Rep}(G)$.

here	[13]	[43]	
X^*	$X(T)$	M	character lattice
X^+	C	P	dominant cone/ [fundamental] Weyl chamber
Φ	Φ	R	set of roots
Φ^+	Φ^+	R^+	set of positive roots
Δ	Δ	—	set of simple roots
X_*	$X^\vee(T)$	—	cocharacter lattice

Table 1: Translation between notation used here and notation in some of our references.

such representation for each $x \in X^+$. When G is simply connected, $X^+ \cong \mathbb{N}_0^n$, with basis given by fundamental weights $\omega_1, \ldots, \omega_n$. Following [43, section 3.6], we define a partial order on X^* as follows:

Definition 2.1.1 For $x, y \in X^*$, we write $x \leq y$ if and only if $y - x$ can be written as a $\mathbb{Z}$-linear combination of elements of Φ^+ with nonnegative coefficients.

Other definitions abound in the literature. The ordering defined here is slightly different from both orderings defined in [9, Chapter VI, Definition 2.2].

Finally, let $W = W(G, T) = W(\Phi)$ be the associated Weyl group. We refer to any translate $wX^+ \subset X^*$ as a Weyl chamber. The Weyl group acts simply transitively on the set of Weyl chambers. For the convenience of the reader, a partial translation of the notation used here and the notation used in some of the references is provided by Table 1.

2.2 The representation ring

Lemma 2.2.1 *For dominant weights $x, y \in X^+$,*

$$E_x \cdot E_y = E_{x+y} + \sum_z E_z$$

in $\mathrm{K}_0(\mathbf{Rep}(G))$, where the sum is over a finite number of $z \in X^+$ such that $z < x + y$.

Proof By [43, théorème 4], there is an injective ring morphism

$$\mathrm{ch}_G : \mathrm{K}_0(\mathbf{Rep}(G)) \to \mathbb{Z}[X^*]$$

with image $\mathbb{Z}[X^*]^W$. By [43, lemme 5], $\mathrm{ch}_G(E_x) = e^x + \sum_i e^{x_i}$ for a finite number of $x_i \in X^*$ such that $x_i < x$, and similarly for $\mathrm{ch}_G(E_y)$ and $\mathrm{ch}_G(E_{x+y})$. It follows that

$$\mathrm{ch}_G(E_x)\,\mathrm{ch}_G(E_y) = e^{x+y} + (\text{terms smaller than } x + y),$$

where by "terms smaller than $x + y$" we mean a $\mathbb{Z}$-linear combination of terms e^z with $z \in X^*$ such that $z < x + y$. On the other hand, $\mathrm{ch}_G(E_x)\,\mathrm{ch}_G(E_y) = \mathrm{ch}_G(E_x \otimes E_y)$, and $E_x \otimes E_y$ can be decomposed into

a sum of irreducible representations $E_1, \ldots, E_N$. Writing $\mathrm{ch}_G(E_k) = e^{w_k} + (\text{terms smaller than} w_k)$, we find that

$$\mathrm{ch}_G(E_x)\,\mathrm{ch}_G(E_y) = \sum_{k=1}^{N} (e^{w_k} + \text{terms smaller than } w_k).$$

It follows by comparison that $w_k \leq x + y$ for each k, and that $w_k = x + y$ for exactly one k. $\qquad\square$

Lemma 2.2.2 *There is no $z \in X^+$ such that $z < 0$.*

Proof Let $\Delta \subset \Phi^+$ denote the set of simple roots. If $z < 0$ then $z = \sum_{\alpha \in \Delta} a_\alpha \alpha$ with $a_\alpha \leq 0$ for all $\alpha \in \Delta$ and $a_\alpha < 0$ for at least one α. In particular, z lies in the span of Φ, so we can pass from the root system $\Phi \subset X^* \otimes_{\mathbb{Z}} \mathbb{R}$ to the *reduced* root system $\Phi \subset \Phi \otimes_{\mathbb{Z}} \mathbb{R}$, to which the results of [7, chapitre VI, section 1, n°10] apply. Let σ be the half-sum of all coroots α^{co} corresponding to roots $\alpha \in \Phi^+$. Then by [7, chapitre VI, section 1, n°10, corollaire], $\langle \alpha, \sigma \rangle = \sum_{\alpha \in \Delta} a_\alpha$. This number is smaller than zero by assumption, so $z \notin X^+$. $\qquad\square$

Proposition 2.2.3 *Suppose X^+ can be split into a direct sum $X^+ \cong \mathbb{N}_0^n \oplus \mathbb{Z}^m$. Pick elements $\omega_1, \ldots, \omega_n \in X^+$ and $\zeta_1, \ldots, \zeta_m \in X^+$ corresponding to an $\mathbb{N}_0$-basis of $\mathbb{N}_0^n$ and a $\mathbb{Z}$-basis of $\mathbb{Z}^m$, respectively. There is a unique ring homomorphism*

$$\mathbb{Z}[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \to \mathrm{K}_0(\mathbf{Rep}(G))$$

taking each w_i to E_{ω_i} and each z_j to E_{z_j}, and this ring homomorphism is an isomorphism.

Example 2.2.4 For simply connected G, $X^+ \cong \mathbb{N}_0^n$ with basis $\omega_1, \ldots, \omega_n$ the fundamental weights. The proposition shows that $\mathrm{K}_0(\mathbf{Rep}(G))$ is a polynomial ring over $\mathbb{Z}$ on generators $E_{\omega_1}, \ldots, E_{\omega_n}$.

Example 2.2.5 For $G = T$ a split torus of rank m, $X^+ = X^* \cong \mathbb{Z}^m$. Pick a $\mathbb{Z}$-basis $\zeta_1, \ldots, \zeta_m$ of X^*. The proposition shows that $\mathrm{K}_0(\mathbf{Rep}(T))$ is a ring of Laurent polynomials with the one-dimensional representations E_{ζ_i} as generators.

Proof of Proposition 2.2.3 It suffices to show that there is a well-defined ring map

$$f : \frac{\mathbb{Z}[w_1, \ldots, w_n, z_1, \ldots, z_m, z_1', \ldots, z_m']}{(z_j z_j' - 1 \mid j = 1, \ldots, m)} \to \mathrm{K}_0(\mathbf{Rep}(G))$$

that sends w_i to E_{ω_i}, z_j to E_{ζ_j} and $z_j' \to E_{-\zeta_j}$, and that this map is an isomorphism.

To see that f is well defined, note that by Lemma 2.2.1

$$f(z_j) f(z_j') = E_{\zeta_j} \cdot E_{-\zeta_j} = E_0 + \sum_z E_z,$$

where the sum is over certain $z \in X^+$ such that $z < 0$. By Lemma 2.2.2, this is the empty sum. So $f(z_j) f(z_j') = E_0 = 1$, as required.

To see that f is an isomorphism, we associate the Laurent monomial $M^x := w_1^{a_1} \cdots w_n^{a_n} \cdot z_1^{b_1} \cdots z_m^{b_m}$ with the element $x = \sum_i a_i \omega_i + \sum_j b_j \zeta_j \in X^+$. This defines a bijection between a $\mathbb{Z}$-basis of the Laurent ring and X^+. Under f, the basis element M^x maps to $\prod_i (E_{\omega_i})^{a_i} \prod_j (E_{\zeta_j})^{b_j}$. By Lemma 2.2.1, we can rewrite this element as

$$(2.2.1) \qquad\qquad f(M^x) = E_x + \sum_z E_z,$$

for certain $z \in X^+$ with $z < x$. Using [7, chapitre VI, section 3, n°4, lemme 4] as in the proof of [43, lemme 6], we deduce that the elements $f(M^x)$ for $x \in X^+$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[X^*]^W$. $\qquad\square$

Remark 2.2.6 (alternative generators) More generally, under the assumptions of Proposition 2.2.3, we can choose generators for $K_0(\mathbf{Rep}(G))$ as follows. Take ω_i and $\zeta_j \in X^+$ as before. For $i \in \{1, \ldots, n\}$, pick classes $e_{\omega_i} \in K_0(\mathbf{Rep}(G))$ such that

$$e_{\omega_i} = E_{\omega_i} + (\text{smaller terms}),$$

in $K_0(\mathbf{Rep}(G))$, where (smaller terms) refers to a $\mathbb{Z}$-linear combination of irreducible representations E_x indexed by finitely many $x \in X^+$ with $x < \omega_i$. Then again we have a ring isomorphism

$$\mathbb{Z}[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \xrightarrow{\cong} K_0(\mathbf{Rep}(G))$$

taking each w_i to e_{ω_i} and each z_j to E_{ζ_j}. Indeed, this follows with the same proof, as the key identity (2.2.1) still holds for these more general generators.

2.3 The ±-symmetric representation ring

From now on, we always assume $\mathrm{char}(k) \neq 2$.

We now assume that G is equipped with an involution ι (possibly trivial) *that restricts to the chosen maximal torus T*, and study the associated duality $\vee_\iota$ on $\mathbf{Rep}(G)$ introduced at the beginning of the section.

The following lemmas describe the highest weight of the $\vee_\iota$-dual of an irreducible representation. As we have assumed that ι restricts to T, we have induced involutions ι^* and ι_* on the character lattice X^* and the cocharacter lattice X_*, respectively, compatible with the canonical pairing $\langle -, - \rangle$. For example, the involutions inv^* and inv_* induced by the involution from Example 2.0.1 are given by $-\mathrm{id}$. Interpreting characters as one-dimensional representations, we see that the dualities $\vee$ and $\vee_\iota$ also define involutions on X^*. Explicitly, $\vee = -\mathrm{id}$ on X^*, hence $\vee_\iota = -\iota^*$ on X^*.

Lemma 2.3.1 *There is a unique involution $w_\iota \in W$ whose action on Weyl chambers agrees with this action of $\vee_\iota = -\iota^*$ on the Weyl chambers. In particular, $-\omega_\iota \iota^*$ defines an involution of the fundamental Weyl chamber X^+.*

Example 2.3.2 If $\iota|_T = \mathrm{id}$, then $\vee_\iota = -\mathrm{id}$ on X^*, and $w_\iota = w_0$, the longest element of the Weyl group. If $\iota|_T = \mathrm{inv}$ is the inversion from Example 2.0.1, then $\vee_\iota = \mathrm{id}$ on X^*, and $w_\iota = 1$.

Proof The case when $\iota = \mathrm{id}$ is standard. For the general case, note that both involutions ι^* and $\vee$ on X^* send roots to roots. (For ι^*, this follows from ι is a group homomorphism compatible with the inclusion $T \subset G$; see [13, above Theorem 1.3.15].) The claim then follows from the fact that the Weyl group acts simply transitively on the set of Weyl chambers. $\qquad\square$

Lemma 2.3.3 *Given an irreducible G-representation E_x with highest weight x, the dual representation $(\iota^* E_x)^\vee$ has highest weight $-w_\iota \iota^* x$, where w_ι is as in the previous lemma.*

Proof Let $\Omega_E \subset X^*$ be the set of weights of $E := E_x$. As $\vee_\iota = -\iota^*$ on X^*, we have $\Omega_{(\iota^* E)^\vee} = -\iota^* \Omega_E$. Now consider an arbitrary weight $z \in \Omega_{(\iota^* E)^\vee}$. As $\Omega_{(\iota^* E)^\vee}$ is invariant under the action of the Weyl group, we also have $\omega_\iota z \in \Omega_{(\iota^* E)^\vee}$. So $\omega_\iota z \in (-\iota^* \Omega_E)$. As x is the highest weight of E, this means that $\omega_\iota z = -\iota^*(x - \sum_{b \in \Delta} m_b b)$ for certain $m_b \in \mathbb{Z}_{\geq 0}$. So $z = -\omega_\iota \iota^* x + \sum_{b \in w_\iota \iota_*(\Delta)} m'_b b$ for certain $m'_b \in \mathbb{Z}_{\geq 0}$. As noted in Lemma 2.3.1, $\omega_\iota \iota^*(X^+) = -X^+$. In view of (2.1.2), this implies that $w_\iota \iota_*(\Delta) \subseteq -\Phi^+$. So $z = -\omega_\iota \iota^* x - \sum_{b \in \Phi^+} m''_b b$, for certain $m''_b \in \mathbb{Z}_{\geq 0}$. It follows that $-w_\iota \iota^* x$ is the highest weight of $(\iota^* E)^\vee$. $\qquad\square$

Definition 2.3.4 A dominant character $x \in X^+$ is $(\vee_\iota\text{-})$*self-dual* if $x = -w_\iota \iota^* x$, where w_ι is as in Lemma 2.3.1.

Definition 2.3.5 We call a representation E of G *self-dual* if there exists an isomorphism $\phi \colon E \to E^{\vee_\iota}$. We call a self-dual representation E *symmetric* or *antisymmetric* if ϕ can be chosen to be symmetric or antisymmetric, respectively.

By Lemma 2.3.3, the irreducible representation with highest weight x is $\vee_\iota$-self-dual if and only if x is $\vee_\iota$-self-dual. In general, a representation may be both symmetric and antisymmetric. However, in our setting a self-dual *irreducible* representation is always either symmetric or antisymmetric, but never both:

Lemma 2.3.6 *Let E be an irreducible representation of a split reductive group G. If there exists an isomorphism $\phi \colon E \to (\iota^* E)^\vee$, then ϕ is either symmetric or antisymmetric, and any isomorphism $E \to (\iota^* E)^\vee$ is a multiple of ϕ by an (invertible) scalar.*

Proof This is essentially [11, Corollary 2.3], which relies on [11, Lemma 1.21]. Note that [11, Section 1] is phrased in the generality of abelian categories with duality and hence applies to the category of G-representations regardless of our choice of duality on this category. $\qquad\square$

In light of this lemma, we make the following definition.

Definition 2.3.7 The *sign $s(x)$* of a dominant character $x \in X^+$ is an element of $\{-1, 0, 1\}$, which is 1 if E_x is symmetric, -1 if E_x is antisymmetric, and 0 if E_x is not $\vee_\iota$-self-dual.

For an abelian category $\mathcal{A}$ with duality, ie with a fixed involution $\vee$ and double-dual identification ω, we write $\mathrm{GW}^+(\mathcal{A}, \vee, \omega)$ and $\mathrm{GW}^-(\mathcal{A}, \vee, \omega)$ for the Grothendieck–Witt groups of symmetric and antisymmetric forms over $\mathcal{A}$, respectively. Taking $\mathcal{A}$ to be the category of finite-dimensional k-vector spaces with its usual duality, this construction yields the usual groups $\mathrm{GW}^+(k)$ and $\mathrm{GW}^-(k)$ of symmetric and antisymmetric bilinear forms over k. The tensor product furnishes us with a $\mathbb{Z}/2$-graded ring structure on the direct sum $\mathrm{GW}^\pm(k) := \mathrm{GW}^+(k) \oplus \mathrm{GW}^-(k)$. As it is convenient to think of the grading group $\mathbb{Z}/2$ multiplicatively as $\{\pm 1\}$, we will refer to this grading as a $\pm$-grading in all that follows. Note that $\mathrm{GW}^-(k)$ does not depend on the field k. Explicitly, $\mathrm{GW}^-(k) = \mathbb{Z} \cdot H^-$ as a group, and

$$(2.3.1) \qquad \mathrm{GW}^\pm(k) = \frac{\mathrm{GW}^+(k)[H^-]}{((H^-)^2 - 2H^+)}$$

as a ring, where $H^\pm \in \mathrm{GW}^\pm(k)$ are the respective hyperbolic planes.

More generally, for the category with duality $(\mathbf{Rep}(G), \iota)$ introduced at the beginning of this section, we obtain a $\pm$-graded algebra $\mathrm{GW}^\pm(\mathbf{Rep}\, G, \iota)$ over the $\pm$-graded ring $\mathrm{GW}^\pm(k)$. The following theorem is an analogue of [11, Theorem 2.10] for Grothendieck–Witt theory, under the assumption that all characters are self-dual so as to eliminate the existence of hyperbolic elements.

Proposition 2.3.8 *Assume that all $x \in X^+$ are $\vee_\iota$-self-dual. Choose an isomorphism $\phi_x \colon E_x \to (\iota^* E_x)^\vee$ for each $x \in X^+$. Then the classes $[E_x, \phi_x]$ form a basis of the $\mathrm{GW}^\pm(k)$-module $\mathrm{GW}^\pm(\mathbf{Rep}(G), \iota)$.*

Proof The proof is the same as for [11, Theorem 2.10], noting that the obstruction from [11, Remark 2.11] has been removed with the assumption that all dominant characters are self-dual. The main ingredients are [11, Corollary 1.14], which provides additive decompositions of both $\mathrm{GW}^+(\mathbf{Rep}(G))$ and $\mathrm{GW}^-(\mathbf{Rep}(G))$, and [11, Corollary 1.38], which identifies the summands of $\mathrm{GW}^+(\mathbf{Rep}(G)$ corresponding to symmetric characters and the summands of $\mathrm{GW}^-(\mathbf{Rep}(G))$ corresponding to antisymmetric characters. For a full proof, [11, Corollary 1.38] and the preceding [11, Proposition 1.37] need to be mildly generalized to include all signs. (For example, strictly speaking the identification of $\mathrm{GW}(\mathcal{A}_i)$ with $\mathrm{GW}_-(\mathcal{A}_\mathbb{1})$ as a $\mathrm{GW}(\mathcal{A}_\mathbb{1})$-module is missing from [11, Corollary 1.38 (i)], as the tensor unit $\mathbb{1}$ is assumed to be symmetric, not just δ-symmetric, throughout [11, Section 1.4].) $\qquad\square$

Remark 2.3.9 Even in the presence of non-self-dual characters, it is easy to describe the $\mathrm{GW}^\pm(k)$-module structure of $\mathrm{GW}^\pm(\mathbf{Rep}(G), \iota)$. Let $H^+(E)$ and $H^-(E)$ denote the symmetric and antisymmetric hyperbolic space associated with a representation E, respectively. The results quoted from [11] in the proof above show that every pair of non-self-dual dominant characters $(z, -w_\iota z)$ contributes a copy of $\mathbb{Z}$ generated by $H^+(E_z)$ to $\mathrm{GW}^+(\mathbf{Rep}(G), \iota))$ and a copy of $\mathbb{Z}$ generated by $H^-(E_z)$ to $\mathrm{GW}^-(\mathbf{Rep}(G), \iota))$. However, we want to concentrate on the case when $\mathrm{GW}^\pm(\mathbf{Rep}(G), \iota))$ is a *free* $\mathrm{GW}^\pm(k)$-module here.

Lemma 2.3.10 *Consider the duality $\vee_\iota$ on $\mathbf{Rep}\, G$. Suppose (V, ϕ) is an ϵ-symmetric representation of G such that $V = E_x + (\text{smaller terms})$ in $K_0(\mathbf{Rep}(G))$, for some $x \in X^+$ and $\epsilon \in \{\pm\}$. Then there exists an ϵ-symmetric isomorphism ϕ_x on E_x such that*

$$(V, \phi) = (E_x, \phi_x) + (\text{smaller terms})$$

in $\mathrm{GW}^\epsilon(\mathbf{Rep}(G, \iota))$, where (smaller terms) refers to a $\mathrm{GW}(k)$-linear combination of terms T_z indexed by finitely many $z \in X^+$ with $z < x$ such that, for each z, T_z is either of the form (E_z, ϕ_z) (in case z is ϵ-symmetric) or of the form $H^\epsilon(E_z)$ (in case z is not).

Proof Using the decomposition of $\mathrm{GW}^\epsilon(\mathbf{Rep}(G, \iota))$ of [11, Corollaries 1.14 and 1.38] as in the proof of Proposition 2.3.8 above, we can write (V, ϕ) as a $\mathrm{GW}(k)$-linear combination of elements (E_z, ϕ_z) with $z \in X^+$ such that ϕ_z is ϵ-symmetric, and of elements $H^\epsilon(E_v)$ with $v \in X^+$ such that E_v is not ϵ-self-dual:

$$(V, \phi) = \sum_{\substack{z:s(z)=\epsilon}} \alpha_z \cdot (E_z, \phi_z) + \sum_{\substack{v:s(v)=-\epsilon \\ \text{or } s(v)=0}} a_v \cdot H^\epsilon(E_v).$$

The coefficients a_v may be taken in $\mathbb{Z}$, as $a \cdot H^\epsilon(b) = H^{\delta\epsilon}(F(a)b)$ for any $a \in \mathrm{GW}^\delta(\mathbf{Rep}(G, \iota))$ and any $b \in K_0(\mathbf{Rep}(G))$. By [11, Remark 1.15], we even know that the coefficients α_z of (E_z, ϕ_z) can be chosen to be actual symmetric forms over k of positive rank, and that the coefficients $a_v \in \mathbb{Z}$ of $H^\epsilon(E_v)$ are positive. Applying the forgetful map, we thus find that

$$V = \sum_z (\mathrm{rank}\, \alpha_z) E_z + \sum_v 2 a_v E_v$$

in $K_0(\mathbf{Rep}(G))$. As the irreducible representations E_x with x ranging over X^+ form a $\mathbb{Z}$-basis of $K_0(\mathbf{Rep}(G))$, we deduce by comparing this decomposition with the given decomposition of (V, ϕ) that one of the z's in the first sum must be equal to x, with α_x of rank one, and that for all other z's and all v's we have $z < x$ and $v < x$, respectively. □

The following is a careful restatement of [11, Lemma 2.14]:

Lemma 2.3.11 *For any two self-dual $x, y \in X^+$, the sum $x + y \in X^+$ is also self-dual. Moreover, given ϵ_x- and ϵ_y-symmetric isomorphisms $\phi_x: E_x \to (\iota^* E_x)^\vee$ and $\phi_y: E_y \to (\iota^* E_y)^\vee$, respectively, there exists an $\epsilon_x \epsilon_y$-symmetric isomorphism $\phi: E_{x+y} \to (\iota^* E_{x+y})^\vee$ such that*

$$(E_x, \phi_x) \cdot (E_y, \phi_y) = (E_{x+y}, \phi) + (\text{smaller terms}),$$

in $\mathrm{GW}^{\epsilon_x \epsilon_y}(\mathbf{Rep}(G, \iota))$, where (smaller terms) is to be read as in Lemma 2.3.10.

Proof The claim that $x + y$ is self-dual is immediate from the definitions. Also, $(E_x, \phi_x) \otimes (E_y, \phi_y)$ is an $\epsilon_x \epsilon_y$-symmetric representation and hence defines an element of $\mathrm{GW}^{\epsilon_x \epsilon_y}(\mathbf{Rep}(G, \iota))$. From Lemma 2.2.1, we know that $E_x \cdot E_y = E_{x+y} + (\text{smaller terms})$, so we can apply Lemma 2.3.10. □

The following result should be compared to [11, Theorem 2.16]. It is an analogue, with more restrictive hypotheses, of the description of the usual representation ring provided in Proposition 2.2.3.

Proposition 2.3.12 *Suppose that all $x \in X^+$ are $\vee_\iota$-self-dual. Assume in addition that X^+ can be split into a direct sum $X^+ \cong \mathbb{N}_0^n \oplus \mathbb{Z}^m$. Pick elements $\omega_1, \ldots, \omega_n \in X^+$ and $\zeta_1, \ldots, \zeta_m \in X^+$ corresponding to an $\mathbb{N}_0$-basis of $\mathbb{N}_0^n$ and a $\mathbb{Z}$-basis of $\mathbb{Z}^m$, respectively. Pick (anti)symmetric isomorphisms ϕ_{ω_i} and ϕ_{ζ_j} for each of the representations E_{ω_i} and E_{ζ_j}, respectively. Consider the ring $GW^{\pm}(k)[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$ as a graded $GW^{\pm}(k)$-algebra with generators w_i and z_j in degrees $s(w_i)$ and $s(z_j)$, respectively. Then there is a unique graded $GW^{\pm}(k)$-algebra homomorphism*

$$f : GW^{\pm}(k)[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \to GW^{\pm}(\mathbf{Rep}(G, \iota))$$

taking each w_i to $(E_{\omega_i}, \phi_{\omega_i})$ and each z_j to $(E_{\zeta_j}, \phi_{\zeta_j})$, and this homomorphism is an isomorphism.

For the one-dimensional representations E_{ζ_j} we have $E_{\zeta_j}^\vee \cong E_{-\zeta_j}$ with respect to the usual duality $\vee = \vee_{\mathrm{id}}$. So, for $m > 0$, the assumption that each E_ζ be self-dual with respect to $\vee_\iota$ cannot hold with respect to the usual duality $\vee$.

Example 2.3.13 Take $(G, \iota) = (T, \mathrm{inv})$, a split torus of rank m equipped with the inversion from Example 2.0.1. In this case, $X^+ = X^* \cong \mathbb{Z}^m$, and all characters are self-dual symmetric. The proposition shows that $GW^{\pm}(\mathbf{Rep}(T), \mathrm{inv})$ is a ring of Laurent polynomials over $GW^{\pm}(k)$ with one-dimensional symmetric representations as generators.

Example 2.3.14 Take $(G, \iota) = (GL_n, \iota)$, the general linear group equipped with the involution from Example 2.0.2. The fundamental representations of GL_n are given by the exterior powers of the standard n-dimensional representation V [24, Part II, Section 2.15]. More precisely, in the notation from above, $E_{\omega_i} = \Lambda^i(V)$ for $i = 1, \ldots, n-1$ and $E_{\zeta_1} = \Lambda^n(V)$. As noted in Example 2.0.2, we can choose a nondegenerate $\vee_\iota$-symmetric form ϕ on V, so that we obtain elements $\lambda_i := (\Lambda^i(V), \Lambda^i(\phi))$ in $GW^+(\mathbf{Rep}(GL_n, \iota))$. The above proposition then shows that

$$GW^{\pm}(\mathbf{Rep}(GL_n, \iota)) \cong GW^{\pm}(k)[\lambda_1, \ldots, \lambda_{n-1}, \lambda_n, \lambda_n^{-1}].$$

Example 2.3.15 Take $(G, \iota) = (Sp_{2n}, \mathrm{id})$, as in Example 2.0.3. The irreducible representation E_{ω_i} is symmetric for even i and antisymmetric for odd i [8, Chapter VIII, Table 1], so we can choose a nondegenerate $(-1)^i$-symmetric form ϕ_i on each E_{ω_i}. The proposition then shows that $GW^{\pm}(\mathbf{Rep}(Sp_{2n}), \mathrm{id})$ is a polynomial algebra over $GW^{\pm}(k)$ on n generators $(E_{\omega_1}, \phi_{\omega_1}), \ldots, (E_{\omega_n}, \phi_n)$, with (E_{ω_i}, ϕ_i) of degree $(-1)^i$. (We will see in Lemma 2.4.2, using the generalization of Proposition 2.3.12 discussed in Remark 2.3.16, that alternative polynomial generators are again given by the exterior powers of the standard representation.)

Proof of Proposition 2.3.12   We have seen in the proof of Proposition 2.2.3 that $E_{\zeta_j} \cdot E_{-\zeta_j} = 1$ in $K_0(\mathbf{Rep}(G))$. In particular, each E_{ζ_j} is one-dimensional. We can therefore pick symmetric isomorphisms $\phi_{-\zeta_j}$ on each $E^\vee_{-\zeta_j}$ such that

$$(E_{\zeta_j}, \phi_{\zeta_j}) \cdot (E_{-\zeta_j}, \phi_{-\zeta_j}) = 1$$

in $GW(\mathbf{Rep}(G, \iota))$. It follows that we have a well-defined $GW^\pm(k)$-algebra homomorphism

$$f : GW^\pm(k)[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \to GW^\pm(\mathbf{Rep}(G, \iota))$$

sending w_i to $(E_{\omega_i}, \phi_{\omega_i})$, z_j to (E_{z_j}, ϕ_{z_j}), and z'_j to (E_{-z_j}, ϕ_{-z_j}).

To see that this map is an isomorphism, we argue exactly as in the proof of Proposition 2.2.3 and associate the Laurent monomial $M^x := w_1^{a_1} \cdots w_n^{a_n} \cdot z_1^{b_1} \cdots z_m^{b_m}$ with the element $x = \sum_i a_i \omega_i + \sum_j b_j \zeta_j \in X^+$. Under the map f above, the basis element M^x maps to

$$\prod_{i=1}^n (E_{\omega_i}, \phi_{\omega_i})^{a_i} \cdot \prod_{j=1}^m (E_{\zeta_j}, \phi_{\zeta_j})^{b_j},$$

and it suffices to show that these classes form a basis of $GW^\pm(\mathbf{Rep}(G))$ considered as $GW^\pm(k)$-module. By Lemma 2.3.11, we can rewrite $f(M^x)$ as

(2.3.2) $$f(M^x) = (E_x, \phi_x) + \sum_z a_z (E_z, \phi_z),$$

where ϕ_x is an appropriately chosen (anti)symmetric isomorphism on E_x, the sum is over certain $z \in X^+$ such that $z < x$, and $a_z \in GW^\pm(k)$. (Remember all $z \in X^+$ are self-dual by assumption.) Proposition 2.3.8 tells us that, if we equip each irreducible representation E_x with the isomorphism ϕ_x that appears here, the elements (E_x, ϕ_x) form a $GW^\pm(k)$-module basis for $GW^\pm(\mathbf{Rep}(G))$. The claim then again follows from [7, chapitre VI, section 3, n°4, lemme 4].   □

Just as in the case of the usual representation ring (Remark 2.2.6), we can and sometimes will pick more general generators than the ones specified in Proposition 2.3.12.

Remark 2.3.16   (alternative generators)   Under the assumptions of Proposition 2.3.12, we can more generally choose generators for $GW(\mathbf{Rep}(G, \iota))$ as follows. As before, pick (anti)symmetric isomorphisms ϕ_{ω_i} and ϕ_{ζ_j} for each of the representations E_{ω_i} and E_{ζ_j}, respectively. Pick homogeneous classes $e_{\omega_i} \in GW^\pm(\mathbf{Rep}(G, \iota))$ for $i \in \{1, \ldots, n\}$ such that

$$e_{\omega_i} = (E_{\omega_i}, \phi_{\omega_i}) + \text{(smaller terms)},$$

where (smaller terms) is to be read as in Lemma 2.3.10. Then again we have an isomorphism of graded $GW^\pm(k)$-algebras

$$f : GW^\pm(k)[w_1, \ldots, w_n, z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \to GW^\pm(\mathbf{Rep}(G, \iota))$$

taking each w_i to e_{ω_i} and each z_j to $(E_{\zeta_j}, \phi_{\zeta_j})$. Indeed, this follows with the same proof, as the key identity (2.3.2) still holds for these more general generators.

2.4 Some details on symplectic groups

Now let $G = \mathrm{Sp}_{2n}$ be the symplectic group, ie the group of automorphisms of the antisymmetric form

$$(2.4.1) \qquad J := \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix},$$

and T the maximal torus, of rank n, consisting of matrices

$$(2.4.2) \qquad \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}$$

with $D \in \mathrm{GL}_n$ is diagonal; see, eg, [13, Exercise 2.4.6]. The Weyl group of G is the wreath product $S_2 \wr S_n$, that is, the semidirect product $S_2^n \rtimes S_n$ with the permutation action of S_n on S_2^n. The character lattice X^* of G is a free $\mathbb{Z}$-module of rank n, with dominant characters $X^+ \cong \mathbb{N}_0^n$.

If $n = 1$, then $G = \mathrm{SL}_2 = \mathrm{Sp}_2$ and $X^+ \cong \mathbb{N}_0$. The standard 2-dimensional representation V of G preserves the standard nondegenerate antisymmetric bilinear form ϕ, yielding an element $[V, \phi] \in \mathrm{GW}^-(\mathbf{Rep}(G))$. Note that V is a simple representation of highest weight 1, which is fundamental. By Proposition 2.3.12, $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2))$ therefore is a polynomial algebra over $\mathrm{GW}^\pm(k)$ on one generator (V, ϕ). It will be convenient for us to use the rank-zero element $b := (V, \phi) - H^-$ as a generator instead, so that we have

$$(2.4.3) \qquad \mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2)) \cong \mathrm{GW}^\pm(k)[b].$$

We think of b as a "first Borel class", and note that b generates the graded augmentation ideal of $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2))$ (see Definition 3.3.1).

For $n > 1$, it will be convenient for us to replace the standard polynomial generators E_{ω_i} of $\mathrm{K}_0(\mathbf{Rep}(\mathrm{Sp}_{2n}))$ and the corresponding generators of $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_{2n}))$ from Example 2.3.15 by alternative polynomial generators given by the exterior powers $\Lambda^i(V)$ and $\Lambda^i(V, \phi)$, respectively, where V is the $2n$-dimensional standard representation.

Lemma 2.4.1 *In* $\mathrm{K}_0(\mathbf{Rep}(\mathrm{Sp}_{2n}))$, $\Lambda^i V = E_{\omega_i} + \sum_{x : x < \omega_i} n_x E_x$ *for all* $i \in \{1, \ldots, n\}$, *where the sum runs over a finite number of dominant weights x such that $x < \omega_i$ in the partial order of Definition 2.1.1, and $n_x \in \mathbb{N}$. In particular,* $\mathrm{K}_0(\mathbf{Rep}(\mathrm{Sp}_{2n})) \cong \mathbb{Z}[V, \Lambda^2 V, \ldots, \Lambda^n V]$.

Proof We first note that the fundamental weights ω_i of Sp_{2n} satisfy

$$(2.4.4) \qquad \omega_{i+2} > \omega_i$$

for all i, ie for all $i \in \{1, \ldots, n-2\}$. Indeed, we see from [7, planche III] that

$$\omega_{i+2} - \omega_i = \alpha_{i+1} + \left(\sum_{j=i+2}^{r-1} \alpha_j \right) + \alpha_r,$$

where $\alpha_1, \ldots, \alpha_n$ are the simple roots, so $\omega_{i+2} - \omega_i > 0$.

In characteristic zero, $\Lambda^i V$ decomposes as

$$(2.4.5) \qquad \Lambda^i V \cong E_{\omega_i} \oplus E_{\omega_{i-2}} \oplus \cdots \oplus \begin{cases} E_{\omega_1} & \text{if } i \text{ is odd,} \\ k & \text{if } i \text{ is even,} \end{cases}$$

for $i = 1, \ldots, n$ *as a representation*, not just as an element of $K_0(\mathbf{Rep}(Sp_{2n}))$. (See [8, Chapter VIII, Section 13.3 (IV)] for a proof of the corresponding statement for the Lie algebra of Sp_{2n}, and recall that in characteristic zero we can pass from decompositions of a representation of a split semisimple simply connected algebraic group to decompositions of the associated representation of its Lie algebra and vice versa using [22, Section 3.2].) The claim of the lemma is therefore immediate from (2.4.4) in characteristic zero.

Suppose now that k is a field of positive characteristic. Note that Sp_{2n} lifts to a split reductive group scheme $Sp_{2n,\mathbb{Z}}$ over $\mathbb{Z}$ [10, théorème 7.2.0.46], hence the results of [43, section 3.7] apply. Let us temporarily write $Sp_{2n,\mathbb{Q}}$ and $Sp_{2n,k}$ for the symplectic groups over $\mathbb{Q}$ and k, which can be obtained from $Sp_{2n,\mathbb{Z}}$ via base change, and let us denote by $E_{\omega_i,\mathbb{Q}}$ and $E_{\omega_i,k}$ the respective irreducible representations of highest weight ω_i. Consider the category $\mathbf{Rep}(Sp_{2n,\mathbb{Z}})$ of $Sp_{2n,\mathbb{Z}}$-representations that are finitely generated and free over $\mathbb{Z}$, and the associated K-ring $K_0(\mathbf{Rep}\, Sp_{2n,\mathbb{Z}})$, denoted by $R_{\mathbb{Z}}(Sp_{2n})$ in [43, section 3.3].

The base-change homomorphisms

$$i_{\mathbb{Q}} \colon K_0(Sp_{2n,\mathbb{Z}}) \to K_0(Sp_{2n,\mathbb{Q}}) \quad \text{and} \quad i_k \colon K_0(Sp_{2n,\mathbb{Z}}) \to K_0(Sp_{2n,k})$$

are homomorphisms of λ-rings, essentially because exterior powers are compatible with base change [16, Proposition A2.2(b)]. By [43, théorème 5], $i_{\mathbb{Q}}$ is an isomorphism. Better still, we see from [43, lemme 2(a)] that for each representation $E_{\mathbb{Q}}$ of $Sp_{2n,\mathbb{Q}}$ there exists a representation $E_{\mathbb{Z}}$, not just a virtual representation, such that $i_{\mathbb{Q}}(E_{\mathbb{Z}}) = E_{\mathbb{Q}}$. In particular, for each dominant weight ω, we have a representation $E_{\omega,\mathbb{Z}}$ of $Sp_{2n,\mathbb{Z}}$ with $i_{\mathbb{Q}}(E_{\omega,\mathbb{Z}}) = E_{\omega,\mathbb{Q}}$. Note, however, that there is no reason to assume that $i_k(E_{\omega,\mathbb{Z}}) = E_{\omega,k}$ in general. Rather, arguing as in [43, section 3.8, remarque 3], we find that

$$(2.4.6) \qquad i_k(E_{\omega,\mathbb{Z}}) = E_{\omega,k} + \sum_{x < \omega} n_x E_{x,k},$$

where the sum is over all dominant x with $x < \omega$, and only finitely many of the coefficients $n_x \in \mathbb{N}_0$ are nonzero. Indeed, let us write ch_k for the character homomorphism $ch_{Sp_{2n,k}}$ (see Lemma 2.2.1). As we have a maximal torus of Sp_{2n} defined over $\mathbb{Z}$, and as restriction to this maximal torus is compatible with base change, we find that $ch_{Sp_{2n,k}}(i_k(E_{\omega,\mathbb{Z}})) = ch_{Sp_{2n,\mathbb{Q}}}(E_{\omega,\mathbb{Q}})$. Thus, [43, lemme 5(a)], applied both over $\mathbb{Q}$ and over k, yields the following two identities, for certain coefficients $a_x, b_x \in \mathbb{N}_0$:

$$ch_k(i_k(E_{\omega,\mathbb{Z}})) = e^\omega + \sum_{x < \omega} a_x e^x \quad \text{and} \quad ch_k(E_{\omega,k}) = e^\omega + \sum_{x < \omega} b_x e^x.$$

Equation (2.4.6) follows from a comparison of these two identities and the injectivity of ch_k.

On the other hand, the standard representation of Sp_{2n} is already defined over $\mathbb{Z}$, so we have a representation $V_{\mathbb{Z}}$ of $Sp_{2n,\mathbb{Z}}$ with $i_k(V_{\mathbb{Z}}) = V_k$ for arbitrary fields k. Equation (2.4.5), appropriately decorated

with subscripts $(-)_{\mathbb{Z}}$, is therefore equally valid in $K_0(Sp_{2n,\mathbb{Z}})$. The claimed decomposition of $\Lambda^i(V)$ in $K_0(\mathbf{Rep}(Sp_{2n}))$ now follows by combining (2.4.4), this integral version of (2.4.5), and (2.4.6).

The final claim is immediate from this decomposition and Remark 2.2.6. □

Lemma 2.4.2 *Let (V, ϕ) denote the $2n$-dimensional standard representation of Sp_{2n} with its canonical antisymmetric form. The exterior powers $\Lambda^i(V, \phi)$ for $i \in \{1, \ldots, n\}$ define homogeneous polynomial generators of the $\pm$-graded $GW^{\pm}(k)$-algebra $GW^{\pm}(\mathbf{Rep}\,Sp_{2n})$:*

$$GW^{\pm}(\mathbf{Rep}(Sp_{2n})) = GW^{\pm}(k)[(V, \phi), \Lambda^2(V, \phi), \ldots, \Lambda^n(V, \phi)].$$

The generator $\Lambda^i(V, \phi)$ is of degree $(-1)^i$.

Proof Applying Lemma 2.3.10 to the decomposition of $\Lambda^i(V)$ given in Lemma 2.4.1, we find that $\Lambda^i(V, \phi) = (E_{\omega_i}, \phi_{\omega_i}) + $ (smaller terms) for certain symmetric isomorphism ϕ_{ω_i} on E_{ω_i}. So the claim is immediate from Remark 2.3.16. □

Restriction to diagonal The group Sp_{2n} has a canonical subgroup isomorphic to $Sp_2^{\times n} = Sp_2 \times \cdots \times Sp_n$. This is most easily seen by replacing the antisymmetric form J from (2.4.1) by the isometric form $nH^- = H^- \perp \cdots \perp H^-$; then $Sp_2^{\times n}$ is simply given by n diagonally concatenated copies of Sp_2 inside Sp_{2n}. The inclusion of this subgroup induces a commutative diagram

$$
\begin{array}{ccc}
K_0(\mathbf{Rep}(Sp_{2n})) & \xrightarrow{\ \mathrm{res}\ } & K_0(\mathbf{Rep}(Sp_2^{\times n})) \\
& \searrow & \downarrow \\
& & K_0(\mathbf{Rep}(T))
\end{array}
$$

which shows that the restriction map $\mathrm{res}: K_0(\mathbf{Rep}(Sp_{2n})) \to K_0(\mathbf{Rep}(Sp_2^{\times n}))$ is injective. Its image can easily be identified, as follows (see explicit calculations in [25, Appendix D]). Consider the symmetric group S_n acting by permutation on $Sp_2^{\times n}$, and its induced action on $K_0(\mathbf{Rep}(Sp_2^{\times n}))$. The image of the restriction map is precisely the fixed ring of this action:

$$K_0(\mathbf{Rep}(Sp_{2n})) \xrightarrow[\mathrm{res}]{\cong} K_0(\mathbf{Rep}(Sp_2^{\times n}))^{S_n}$$

We will now show that the corresponding statement for $GW^{\pm}$ also holds.

Lemma 2.4.3 *The restriction map $GW^{\pm}(\mathbf{Rep}(Sp_{2n})) \to GW^{\pm}(\mathbf{Rep}(Sp_2^{\times n}))$ is injective, with image the invariant subring under the permutation action of the symmetric group S_n.*

Proof Note that $Sp_2^{\times n}$ is a simply connected reductive group, and that Proposition 2.3.12 applies just as it applies to Sp_{2n}. So $GW^{\pm}(\mathbf{Rep}(Sp_2^{\times n}))$ is a polynomial ring in n variables, which we can identify with $\prod_{i=1}^n GW^{\pm}(\mathbf{Rep}(Sp_2))$. Writing $v^{(i)} := (V^{(i)}, \phi^{(i)})$ for the standard representation of the ith factor in $Sp_2^{\times n}$, equipped with its canonical antisymmetric form, we thus obtain

$$(2.4.7) \qquad\qquad GW^{\pm}(\mathbf{Rep}(Sp_2^{\times n})) \cong \mathbb{Z}[v^{(1)}, \ldots, v^{(n)}].$$

Under this isomorphism, the permutation action of S_n on the left corresponds to the obvious permutation action on the generators $v^{(i)}$ on the right.

Now let (V, ϕ) denote the $2n$-dimensional standard representation of Sp_{2n}, equipped with its canonical antisymmetric form. As we have seen in Lemma 2.4.2 we can take the exterior powers $\Lambda^k(V, \phi)$ for $k = 1, \ldots, n$ as polynomial generators of $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2n}))$. Write $\sigma^k(v^{\bullet}) \in \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times n}))$ for the kth elementary symmetric function in the classes $v^{(i)} = (V^{(i)}, \phi^{(i)})$ introduced above. We claim that

$$(2.4.8) \qquad \mathrm{res}\,\Lambda^k(V, \phi) = \sigma_k(v^{\bullet}) + \text{a polynomial in } \sigma_j(v^{\bullet}) \text{ with } j < k.$$

To this end, recall that the operations λ^k defined in terms of the exterior powers Λ^k give $\mathrm{GW}^{\pm}(\mathbf{Rep}(G))$ the structure of a pre-λ-ring (for any affine group scheme G); see [17, Proposition 4.2.1; 49]. For $k = 1$, equation (2.4.8) simply follows from the fact that (V, ϕ) restricts to the direct sum of the representations $(V^{(i)}, \phi^{(i)})$:

$$\mathrm{res}(V, \phi) = \sum_{i=0}^{n} v(i) = \sigma_1(v^{\bullet}).$$

For higher k, consider the power series expansion $\lambda_t(x) := \sum_k \lambda^k(x)t^k$. For the two-dimensional standard representation (V_2, ϕ_2) of Sp_2, we find $\Lambda^2(V_2, \phi_2) = (k, \det(\phi)) = (k, \mathrm{id})$, and $\Lambda^k(V_2, \phi_2) = 0$ for $k > 2$, so

$$\lambda_t(V_2, \phi_2) = 1 + (V_2, \phi_2)t + t^2.$$

As the restriction commutes with the λ-operations, this implies

$$\mathrm{res}\,\lambda_t(V, \phi) = \lambda_t\left(\sum_i v^{(i)}\right) = \prod_{i=1}^{n} \lambda_t v^{(i)} = \prod_{i=1}^{n} (1 + v^{(i)}t + t^2).$$

Now $\mathrm{res}\,\lambda^k(V, \phi)$ is the coefficient of t^k in the above power series. As the power series is invariant under the permutation action of S_n, so is each coefficient. So $\mathrm{res}\,\lambda^k(V, \phi)$ is a polynomial in the symmetric polynomials $\sigma^i(v^{\bullet})$. Moreover, the highest-degree monomials in the $V^{(i)}$'s that occur in the coefficient of t^k are precisely the monomials that occur $\sigma^k(v^{\bullet})$, and they occur with multiplicity one. This proves (2.4.8), and hence the lemma. $\qquad \square$

For the completion at the augmentation ideal, it is again convenient to reformulate the above description in terms of the "first Borel classes" defined as in (2.4.3) above. So let us write $b^{(i)} := (V^{(i)}, \phi^{(i)}) - H^-$ for the first Borel class of the ith copy of Sp_2, and $\sigma_i(b^{\bullet})$ for the ith elementary symmetric polynomial in these classes. Clearly, the isomorphism (2.4.7) from the proof above can be rewritten as

$$(2.4.9) \qquad \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times n})) \cong \mathbb{Z}[b^{(1)}, \ldots, b^{(n)}].$$

Corollary 2.4.4 *The restriction induces an isomorphism between $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2n}))$ and the sub-ring $\mathrm{GW}^{\pm}[\sigma_1(b^{\bullet}), \ldots, \sigma_n(b^{\bullet})]$ of $\mathrm{GW}^{\pm}(\mathrm{Sp}_2^{\times n})$. Under this isomorphism, the graded augmentation ideal $\mathrm{IO}_{\mathrm{Sp}_{2n}}^{\pm}$ (see Definition 3.3.1) corresponds to the ideal generated by $\sigma_1(b^{\bullet}), \ldots, \sigma_n(b^{\bullet})$. In particular,*

$$\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2n}))^{\wedge}_{\mathrm{IO}_{\mathrm{Sp}_{2n}}^{\pm}} \cong \mathrm{GW}^{\pm}(k)[\![\sigma_1(b^{\bullet}), \ldots, \sigma_n(b^{\bullet})]\!].$$

3 Grothendieck–Witt rings of some classifying spaces

The main purpose of this chapter is to prove Theorem 1.0.2. First, we introduce the specific model category we will be working with to model motivic spaces. Next, in Section 3.2, we recall the notion of *acceptable gadgets* as introduced in [32, Definition 8.3] and establish some important properties of these. In Section 3.3, we finally use the results of Section 2.4 to construct the Atiyah–Segal completion map, and to prove the main theorem. In the final two subsections, we discuss the construction of a classifying space for the multiplicative group with nontrivial involution, and conjectural tools that may be used to generalize our main results to base schemes with nontrivial action and to arbitrary closed subgroups of Sp_{2n}.

3.1 Čech localization

Let S be a scheme of finite type over a field k. Our standing assumption that $\mathrm{char}(k) \neq 2$ is not necessary for any of the results of this section. Let $\mathbf{sPre}(\mathbf{Sm}_S)$ be the model category of simplicial presheaves on $\mathbf{Sm}_S$ with the global injective model structure, and $L_{\mathrm{mot}}\,\mathbf{sPre}(\mathbf{Sm}_S)$ its motivic localization, which is defined as $L_{\mathrm{mot}} = (L_{\mathbb{A}^1}L_{\mathrm{Nis}})^\infty$, where the repeated localizations are necessary to ensure that $L_{\mathrm{mot}}F$ is both Nisnevich local and $\mathbb{A}^1$-local for any simplicial presheaf F. This is a presheaf variant of the model category constructed in [30], with homotopy category the unstable motivic homotopy category $\mathcal{H}(S)$. The weak equivalences in $L_{\mathrm{mot}}\,\mathbf{sPre}(\mathbf{Sm}_S)$ will be called *motivic weak equivalences*; these are precisely the maps that become isomorphisms in $\mathcal{H}(S)$.

More generally, for a subcanonical topology τ on $\mathbf{Sm}_S$, let $L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S)$ be the (left) Bousfield localization with respect to covering sieves for τ, as considered in [1, Section 3.1]. This localization is also considered in [15, Appendix A], where it is called the Čech localization because it is a localization with respect to Čech covers in the topology τ; see [15, Theorem A5]. This is the naming convention we will follow.

The fibrant objects of $L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S)$ are those fibrant simplicial presheaves in $\mathbf{sPre}(\mathbf{Sm}_S)$ that satisfy τ-descent; we will refer to these as τ-fibrant. The weak equivalences in $\mathbf{sPre}(\mathbf{Sm}_S)$ will be called *objectwise weak equivalences* and the weak equivalences in $L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S)$ will be called τ-*local weak equivalences*. We denote by L_τ the τ-fibrant replacement functor, viewed as endofunctor on $\mathbf{sPre}(\mathbf{Sm}_S)$.

Remark 3.1.1 (relation to model structure used by Morel and Voevodsky) Note that $L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S)$ is *not* the same as the Jardine model structure $L_{\mathrm{hyp}\text{-}\tau}\,\mathbf{sPre}(\mathbf{Sm}_S)$, which is Quillen equivalent to the Joyal model structure on simplicial sheaves used in [30]. In general, we only have successive Bousfield localizations

$$\mathbf{sPre}(\mathbf{Sm}_S) \to L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S) \to L_{\mathrm{hyp}\text{-}\tau}\,\mathbf{sPre}(\mathbf{Sm}_S).$$

However, the Bousfield localization $L_\tau\,\mathbf{sPre}(\mathbf{Sm}_S) \to L_{\mathrm{hyp}\text{-}\tau}\,\mathbf{sPre}(\mathbf{Sm}_S)$ is a Quillen equivalence for τ the Nisnevich topology and S any reasonable base scheme. See [1, Remark 3.1.4] for more details.

Remark 3.1.2 (∞-language) We may also view $\mathbf{sPre}(\mathbf{Sm}_S)$ as an ∞-category as in, eg, [20]. The fibrant replacement functor L_τ is analogous to the localization endofunctor defined in [20, Proposition 3.4] and has the same formal properties. Lemma 2.1 from [21] will play a key role in our discussion of classifying spaces (see the proof of Proposition 3.2.16 below).

We will need very few specifics about the τ-local model structure. The proof of the following proposition relies on the two subsequent lemmas, both presumably well known.

Proposition 3.1.3 (a) *For any subcanonical topology τ, every S-scheme is τ-fibrant.*

(b) *For any topology τ at least as coarse as the fppf topology, any filtered colimit in $\mathbf{sPre}(\mathbf{Sm}_S)$ of S-schemes is τ-fibrant.*

In particular, for any scheme X as in (a) *or any filtered colimit X as in* (b), *the τ-fibrant replacement map $X \to L_\tau(X)$ is an objectwise weak equivalence.*

Proof For (a), let X be an S-scheme. We need to show that X is fibrant in the injective model structure on $\mathbf{sPre}(\mathbf{Sm}_S)$ and τ-local. For the first assertion, see Lemma 3.1.4 below. For the second assertion, see [1, Example 3.1.2]: a simplicially constant presheaf is τ-local if and only if it is a τ-sheaf. The presheaf X in question is a τ-sheaf precisely because τ is assumed to be subcanonical. For the filtered colimit in (b), the claim follows by the same argument plus the observation, spelled out in Lemma 3.1.5 below, that such filtered colimits are indeed sheaves. The final assertion is just an application of Ken Brown's lemma. $\square$

As remarked above, the following two lemmas are probably well known to the experts. Lemma 3.1.4 is stated without proof in [36, Section 2.6]. We include full details here for future reference.

Lemma 3.1.4 *Let $\mathcal{C}$ be a small category. A simplicially constant presheaf $F \in \mathbf{sPre}(\mathcal{C})$ is fibrant in any model structure on $\mathbf{sPre}(\mathcal{C})$ whose weak equivalences are objectwise weak equivalences. In particular, constant simplicial presheaves are fibrant in the injective model structure.*

Proof Let $i: A \to B$ be an objectwise weak equivalence in $\mathbf{sPre}(\mathcal{C})$ and $a: A \to F$ a map of simplicial presheaves. To prove the statement, it suffices to prove the existence of a map $b: B \to F$ such that $bi = a$. Note that F is an objectwise Kan complex and therefore projective fibrant, so that a factors through a functorial projective fibrant replacement RA of A, as in the diagram

$$\begin{array}{ccccc} A & \overset{\sim}{\longrightarrow} & RA & \overset{a'}{\longrightarrow} & F \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle Ri} & \nearrow & \\ B & \overset{\sim}{\longrightarrow} & RB & {\scriptstyle b'} & \end{array}$$

and a lift b of a along i exists if a lift $b': RB \to F$ of a' along Ri exists. Therefore, we may and do assume both A and B to be projective fibrant, that is, they are both objectwise Kan complexes.

We first produce a collection of maps $b_C \colon B(C) \to F(C)$ such that $a_C = b_C i_C$ for each $C \in \mathcal{C}$. Let $C \in \mathcal{C}$. Since $i_C \colon A(C) \to B(C)$ is a weak equivalence of Kan complexes, it is a simplicial homotopy equivalence. Let $j_C \colon B(C) \to A(C)$ be a homotopy inverse to i_C and define $b_C \colon B(C) \to F(C)$ as the composition $a_C j_C$. Then the composition $a_C j_C i_C \colon A(C) \to F(C)$ is homotopic to the map a_C, and since homotopic maps into a constant simplicial set are equal, $a_C = b_C i_C$. Thus, for each $C \in \mathcal{C}$ we have $b_C = a_C j_C$ for a fixed choice of homotopy inverse j_C of i_C.

We now need to show that the square

$$
\begin{array}{ccc}
B(D) & \xrightarrow{\ b_D\ } & F(D) \\
\big\downarrow{\scriptstyle B(f)} & & \big\downarrow{\scriptstyle F(f)} \\
B(C) & \xrightarrow{\ b_C\ } & F(C)
\end{array}
$$

commutes for any arrow $f \colon C \to D$ in $\mathcal{C}$. Consider the diagram

$$
\begin{array}{ccccc}
B(D) & \xrightarrow{\ j_D\ } & A(D) & \xrightarrow{\ a_D\ } & F(D) \\
\big\downarrow{\scriptstyle B(f)} & & \big\downarrow{\scriptstyle A(f)} & & \big\downarrow{\scriptstyle F(f)} \\
B(C) & \xrightarrow{\ j_C\ } & A(C) & \xrightarrow{\ a_C\ } & F(C)
\end{array}
$$

and note that the right square commutes. Again because homotopic maps into a constant simplicial set are equal, it suffices to show that the left square commutes up to homotopy. Note that

$$
j_C i_C A(f) j_D = j_C B(f) i_D j_D
$$

because i is a map of simplicial presheaves. Moreover, the left-hand side is homotopic to $A(f) j_D$ and the right-hand side is homotopic to $j_C B(f)$ because $j_C i_C \sim \mathrm{id}_{A(C)}$ and $i_D j_D \sim \mathrm{id}_{B(D)}$ by definition. Hence, $F(f) b_D = b_C B(f)$. $\qquad\square$

Lemma 3.1.5 *Consider a topology τ coarser than the fppf topology. Given a family $(F_i)_{i \in I}$ of representable τ-sheaves on a quasicompact scheme S, where I is a filtered index category, the filtered colimit $F_\infty := \operatorname{colim}_{i \in I} F_i$ is also a τ-sheaf.*

Proof As S is quasicompact, we have to check the sheaf condition for F_∞ only for finite τ-covers $\mathcal{U}$ of a scheme U, by the following argument. Let $\mathcal{U}$ be an arbitrary τ-cover of U. As U is quasicompact, there exists a finite Zariski cover $\mathcal{V}$ of U. Note that $\mathcal{U} \cap \mathcal{V}$ is a refinement of $\mathcal{U}$. For $V \in \mathcal{V}$, we consider the τ-cover $\mathcal{U} \cap V$ of V. By (the proof of) [44, Tag 021P], this refines to a finite τ-cover of V. Hence $\mathcal{U} \cap \mathcal{V}$ refines to a finite τ-cover of U, and from now on we assume that $\mathcal{U}$ is a finite τ-cover of U. We denote the associated covering sieve by $h_{\mathcal{U}}$, which is a subpresheaf of the representable presheaf h_U. We are reduced to showing that if the canonical map $\operatorname{Hom}(h_U, F_i) \to \operatorname{Hom}(h_{\mathcal{U}}, F_i)$ is a bijection for all $i \in I$, then it also is a bijection for $i = \infty$. The representable sheaf h_U is compact, hence commutes with filtered colimits.

Moreover, for a finite covering $\mathcal{U} = (U_j \to U)_{j \in J}$, $h_{\mathcal{U}}$ can be described by the explicit small coequalizer

$$\coprod_{j,j'} U_j \times_U U_{j'} \rightrightarrows \coprod_j U_j$$

of representables, hence is also compact. $\qquad\square$

3.2 Acceptable gadgets and classifying spaces

Let S be a scheme of finite type over a field k. Our standing assumption that $\mathrm{char}(k) \neq 2$ is not necessary for any of the results of this section.

Classifying spaces We now fix a τ-sheaf of groups G over S. In applications, G will often be a linear algebraic group over $\mathrm{Spec}\, k$ base changed to S.

The *simplicial classifying space* $\mathrm{B}G$ is defined in [30] as the nerve of G viewed as a presheaf of groupoids with a single object (so the n-simplices are $\mathrm{B}G_n := G^{\times n}$ and face and degeneracy maps are given by composition and inserting identities, as usual).

Definition 3.2.1 For a topology τ at least as coarse as the fppf topology, the τ-*classifying space* $\mathrm{B}_\tau G$ of G is defined as $\mathrm{B}_\tau G := L_\tau(\mathrm{B}G)$.

Note that $\mathrm{B}_\tau G$ is well defined — irrespective of any particular choice of fibrant replacement functor — up to objectwise weak equivalence. We now briefly discuss the dependency of $\mathrm{B}_\tau G$ on the topology τ. For the definition of τ-locally trivial G-torsors, we refer to [2, Definition 2.3.1].

Lemma 3.2.2 *The following conditions on G and τ are equivalent*:

(1) *Every (fppf-locally trivial) G-torsor in $\mathbf{Sm}_S$ is already τ-locally trivial.*

(2) *The canonical map $\mathrm{B}_\tau G \to \mathrm{B}_{\mathrm{fppf}}G$ is an objectwise weak equivalence.*

Proof This equivalence follows from [2, Lemma 2.3.2(i) and (ii)]. (See also [30, Lemma 4.1.18]). $\quad\square$

Example 3.2.3 (smooth affine groups) For smooth affine group schemes G, every G-torsor is already a étale-locally trivial, so $\mathrm{B}_{\mathrm{fppf}}G$ is objectwise weakly equivalent to $\mathrm{B}_{\mathrm{\acute{e}t}}G$. Indeed, smoothness is preserved by faithfully flat descent, so any G-torsor is smooth over the base, and smooth morphisms admit sections étale locally.

Example 3.2.4 (special groups) A linear algebraic group G over S is called *special* if every étale-locally trivial G-torsor over a (not necessarily smooth) S-scheme is already Zariski-locally trivial. Thus, for a special group G, $\mathrm{B}_{\mathrm{Zar}}G$ is objectwise weakly equivalent to $\mathrm{B}_{\mathrm{\acute{e}t}}G$, and to $\mathrm{B}_\tau G$ for any intermediate topology such as $\tau = \mathrm{Nis}$. Special groups include, in particular, split tori, GL_r, SL_{2r}, Sp_{2r}, and finite products of these [35, Lemmas 3.1 and 3.2]. (In [35; 42], the notion is introduced and discussed only for groups defined over a field k, but note that the defining property is stable under base change along $S \to \mathrm{Spec}(k)$. Special groups over fields have been fully classified [18; 23; 29].)

Of course, for $\tau = \mathrm{Nis}$ or any coarser topology, $\mathrm{B}_\tau G$ is *motivically* equivalent to the simplicial classifying space $\mathrm{B}G$, for any G, as τ-local equivalences are motivic equivalences in this case. Only for finer choices of topologies τ, $\mathrm{B}_\tau G$ may be motivically distinct.

Remark 3.2.5 (comparison with [30]) By construction, $\mathrm{B}_\tau G$ only satisfies τ-descent. However, it also satisfies τ-hyperdescent for dimension reasons by the argument given in the proof of [2, Lemma 2.3.2], so by Ken Brown's lemma, the canonical map $\mathrm{B}_\tau G \to L_{\mathrm{hyp}\text{-}\tau} \mathrm{B}G$ is an objectwise weak equivalence. In view of Remark 3.1.1, this shows, in particular, that the definition of $\mathrm{B}_{\text{ét}}G$ used here agrees with the definition in [30, page 130].

Outline Using the *admissible gadgets with a nice G-action* of [30, Definitions 4.2.1 and 4.2.4], one constructs a geometric approximation of a universal G-torsor over $\mathrm{B}_{\text{ét}}G$, namely a colimit of G-torsors in $\mathbf{Sm}_S$. We will instead use the more flexible concept of *acceptable gadgets*, following [32, Definition 8.3], and adding a version of a nice G-action in this setting. Suppose that we have a commutative diagram of G-torsors

(3.2.1)
$$
\begin{array}{ccc}
X_1 & \longrightarrow X_2 \longrightarrow \cdots \\
\downarrow & \quad \downarrow \\
Y_1 & \longrightarrow Y_2 \longrightarrow \cdots
\end{array}
$$

in $\mathbf{Sm}_S$, where the horizontal arrows are closed immersions, such that the colimit of this diagram should be thought of a motivic approximation of $EG \to \mathrm{B}_{\text{ét}}G$, with EG being a motivically contractible space with free G-action. We will make this idea precise further in a more general setting for any Grothendieck topology τ at least as coarse as the fppf topology, and for τ-locally trivial G-torsors and τ-classifying spaces $\mathrm{B}_\tau G$.

Namely, we will establish a chain of weak equivalences

$$
Y_\infty \underset{(1)}{\simeq} L_\tau(X_\infty /\!/ G) \underset{(2)}{\simeq} L_\tau(S /\!/ G) \underset{(3)}{\cong} B_\tau G.
$$

All quotients $-/\!/ G$ appearing here are stacky; see Definition 3.2.9. The objectwise weak equivalence (1) is established below in Lemma 3.2.13. Equivalence (2) is a motivic weak equivalence; see Proposition 3.2.16. The isomorphism (3) is completely general and clear from the definitions; see Lemma 3.2.10. Combined, these equivalences yield *a motivic weak equivalence $Y_\infty \simeq \mathrm{B}_\tau G$ for any sequence of τ-locally trivial G-torsors as in* (3.2.1) *in which the sequence of X_i's forms an acceptable gadget.* This result, which is stated as Theorem 3.2.17 below, will be used in the next section to show that the products $(\mathrm{HP}^n)^{\times r}$ approximate $\mathrm{BSp}_2^{\times r}$, and similarly certain quaternionic Grassmannians approximate BSp_{2r}. These are crucial steps in the proof of Atiyah–Segal completion for BSp_{2r}.

We begin by recalling the definition of an acceptable gadget.

Definition 3.2.6 An *acceptable gadget* $(X_n)_{n \in I}$ over a scheme S is a countable totally ordered set I together with a diagram $X: I \to \mathbf{Sm}_S$ such that each $X_n = X(n)$ is quasiprojective and for each $n < m$ in I, the map $X(n < m): X_n \to X_m$ is a closed immersion of S-schemes, and such that for any Henselian regular local ring R and any commutative square

$$\begin{array}{ccc} \partial\Delta^i_R & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \Delta^i_R & \longrightarrow & S \end{array}$$

there exists an $m \geq n$ and a map $\Delta^i_R \to X_m$ making this diagram commute:

(3.2.2)
$$\begin{array}{ccccc} \partial\Delta^i_R & \longrightarrow & X_n & \longrightarrow & X_m \\ \downarrow & & & \nearrow & \downarrow \\ \Delta^i_R & & \longrightarrow & & S \end{array}$$

Here is a lemma that shows that acceptable gadgets yield contractible motivic spaces, which will make them useful to construct contractible spaces with free actions, whose quotients are classifying spaces as we will see in Proposition 3.2.16. Some version of Lemma 3.2.7 is implied in [32, page 954 after Proposition 8.5].

Lemma 3.2.7 *Let $(X_i)_{i \in I}$ be an acceptable gadget over a scheme S. Then $X_\infty = \operatorname{colim}_i X_i$ is contractible in the category $\mathcal{H}(S)$ of motivic spaces over S.*

Proof By [30, Lemmas 2.3.8 and 3.1.11], it suffices to show that the simplicial set $\operatorname{Sing}(X_\infty)(R)$ is contractible for every regular Henselian local ring R over S, in other words, for every Nisnevich point of S; see also Remark 3.1.1.

By construction, a map $\partial\Delta^n \to \operatorname{Sing}(X_\infty)(R)$ is given by a morphism $\partial\Delta^n_R \to X_\infty$. Let f be such a morphism. Since $\partial\Delta^n_R$ is representable by an object of $\mathbf{Sm}_S$, it is compact, and f factors through a finite stage $f': \partial\Delta^n_R \to X_i$ for some $i \in I$. Since $(X_i)_{i \in I}$ is an acceptable gadget, there exists $j > i$ in I such that the composition $\partial\Delta^n_R \to X_i \to X_j$ extends to a morphism $\Delta^n_R \to X_j$, which shows that $\operatorname{Sing}(X_\infty)(R)$ is contractible. $\square$

We prove a lemma that allows us to construct acceptable gadgets from existing ones. The first of these shows that acceptable gadgets are stable under base change.

Lemma 3.2.8 *Let $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ be acceptable gadgets over a scheme S.*

  (a) *For any $Y \in \mathbf{Sm}_S$, the base change $(X_n \times_S Y)_{n \in \mathbb{N}}$ is an acceptable gadget over Y.*

  (b) *For any cofinal $I \subset \mathbb{N}$, the sequence $(X_n)_{n \in I}$ is an acceptable gadget over S.*

  (c) *The product $(X_n \times_S Y_n)_{n \in \mathbb{N}}$ is an acceptable gadget over S.*

Proof We only prove (c); the proofs of (a) and (b) are similar. Consider a commutative diagram

$$
\begin{array}{ccc}
\partial\Delta^i_R & \longrightarrow & X_n \times_S Y_n \\
\downarrow & & \downarrow \\
\Delta^i_R & \longrightarrow & S
\end{array}
$$

The projection maps $X_n \times_S Y_n \to X_n$ and $X_n \times_S Y_n \to Y_n$ yield commutative diagrams

$$
\begin{array}{ccccc}
\partial\Delta^i_R & \longrightarrow & X_n & \longrightarrow & X_{m_1} \\
\downarrow & \nearrow & & & \downarrow \\
\Delta^i_R & & \longrightarrow & & S
\end{array}
\qquad
\begin{array}{ccccc}
\partial\Delta^i_R & \longrightarrow & Y_n & \longrightarrow & Y_{m_2} \\
\downarrow & \nearrow & & & \downarrow \\
\Delta^i_R & & \longrightarrow & & S
\end{array}
$$

as $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are acceptable gadgets. Let $m = \max\{m_1, m_2\}$. Then the compositions $\Delta^i_R \to X_{m_1} \to X_m$ and $\Delta^i_R \to Y_{m_2} \to Y_m$ yield a map $\Delta^i_R \to X_m \times_S Y_m$ making this diagram commute:

$$
\begin{array}{ccccc}
\partial\Delta^i_R & \longrightarrow & X_n \times_S Y_n & \longrightarrow & X_m \times_S Y_m \\
\downarrow & \nearrow & & & \downarrow \\
\Delta^i_R & & \longrightarrow & & S
\end{array}
\qquad \square
$$

All quotients $-/\!/ G$ used in the following will be stacky quotients, defined as follows. Recall that the action groupoid of a set X with a group action is the groupoid with objects the elements of X, and morphisms from $x_1 \in X$ to $x_2 \in X$ given by the group elements that send x_1 to x_2.

Definition 3.2.9 For a presheaf of sets X on $\mathbf{Sm}_S$ with G-action, the *stacky quotient $X /\!/ G$* is the simplicial presheaf given by the nerve of the action groupoid of X.

We will mainly be interested in two extremal cases: the stacky quotient of the (trivial) G-action on the base scheme S, and the stacky quotient of the (free) G-action on a G-torsor. These are discussed in the following two lemmas, respectively.

Lemma 3.2.10 *For any subcanonical topology τ, we have a canonical isomorphism of simplicial presheaves $\mathrm{B}_\tau G \cong L_\tau(S /\!/ G)$.*

Proof For any $U \in \mathbf{Sm}_S$, the action groupoid of the $G(U)$-action on the one-point set $S(U)$ is isomorphic to the groupoid $G(U)$ used in the definition of BG. $\qquad \square$

Lemma 3.2.11 *For any τ-locally trivial G-torsor $\pi\colon X \to Y$ in $\mathbf{Sm}_S$, we have a canonical objectwise weak equivalence $L_\tau(X /\!/ G) \to Y$.*

Proof For any presheaf of sets X with a free G-action, the stacky quotient $X /\!\!/ G$ is objectwise weakly equivalent to the presheaf quotient $U \mapsto X(U)/G(U)$, which we temporarily denote by $X/_{\mathrm{pre}} G$. Indeed, when the $G(U)$-action on $X(U)$ is free, the action groupoid $X(U)$ is canonically equivalent to the orbit set $X(U)/G(U)$, viewed as a discrete groupoid, so we obtain a weak equivalence when passing to nerves.

For a smooth S-scheme X with a free G-action, we hence have an objectwise weak equivalence from $L_\tau(X /\!\!/ G)$ to the τ-sheafification $(X/_{\mathrm{pre}} G)^\tau$ of the presheaf quotient. Indeed, note first that the objectwise weak equivalence $X /\!\!/ G \to X/_{\mathrm{pre}} G$ described above induces an objectwise weak equivalence $L_\tau(X /\!\!/ G) \to L_\tau(X/_{\mathrm{pre}} G)$ (by two-out-of-three and Ken Brown's lemma). Secondly, as $(X/_{\mathrm{pre}} G)^\tau$ is τ-fibrant (by Lemma 3.1.4 and [1, Example 3.1.2]), the canonical map $X/_{\mathrm{pre}} G \to (X/_{\mathrm{pre}} G)^\tau$ factorizes through a morphism $L_\tau(X/_{\mathrm{pre}} G) \to (X/_{\mathrm{pre}} G)^\tau$, which also is an objectwise weak equivalence by [1, Example 3.1.2]. So altogether we obtain the objectwise weak equivalence claimed above. Finally, in the situation at hand, we have an isomorphism $(X/_{\mathrm{pre}} G)^\tau \cong Y$. $\square$

We now provide a variant of [30, Definition 4.2.4].

Definition 3.2.12 An *acceptable G-gadget* $(X_n)_{n \in I}$ over S is an acceptable gadget over S satisfying

  (i) for each $n \in I$, X_n is endowed with a G-action; and

  (ii) for each $n < m$ in I, the corresponding closed immersion $X_n \to X_m$ is G-equivariant.

The following result generalizes Lemma 3.2.11 from a single torsor to a sequences of torsors.

Lemma 3.2.13 *Let $(X_n \to Y_n)_{n \in \mathbb{N}}$ be a sequence of G-torsors as in* (3.2.1), *with all torsors τ-locally trivial and all maps $X_n \to X_{n+1}$ monomorphisms. Then the induced map*

$$L_\tau(X_\infty /\!\!/ G) \to Y_\infty$$

is an objectwise weak equivalence. In particular, this holds if $(X_n)_{n \in \mathbb{N}}$ is an acceptable G-gadget with a τ-locally trivial G-torsor $X_n \to Y_n$ for each $n \in \mathbb{N}$.

Proof All objects are cofibrant, and the maps $X_n \to X_m$, $X_n /\!\!/ G \to X_m /\!\!/ G$ and $Y_n \to Y_m$ are monomorphisms, hence cofibrations in our model structure. So X_∞, $\mathrm{colim}(X_n /\!\!/ G)$ and Y_∞ are colimits of cofibrations and thus homotopy colimits. The canonical maps $X_n /\!\!/ G \to Y_n$, which are τ-local weak equivalences by Lemma 3.2.11, therefore induce a τ-local weak equivalence $\mathrm{colim}_n(X_n /\!\!/ G) \to Y_\infty$. Finally, we observe that the action groupoid of X_∞ is the strict colimit of the action groupoids of the X_n, so that we have an isomorphism

$$\mathrm{colim}_n(X_n /\!\!/ G) \cong X_\infty /\!\!/ G.$$

Hence, we obtain a τ-local weak equivalence $X_\infty /\!\!/ G \to Y_\infty$. As Y_∞ is τ-fibrant by Proposition 3.1.3(b), this τ-local weak equivalence factors through $L_\tau(X_\infty /\!\!/ G)$, and by Ken Brown's lemma we thus obtain the desired objectwise weak equivalence

$$L_\tau(X_\infty /\!\!/ G) \to Y_\infty.$$

If $(X_n)_{n \in \mathbb{N}}$ is an acceptable G-gadget with a τ-locally trivial G-torsor $X_n \to Y_n$ for each $n \in \mathbb{N}$, the maps $X_n \to X_m$ are closed immersions and therefore monomorphisms by assumption. As $X_n(U) \to X_m(U)$ is a $G(U)$-equivariant monomorphism for every $U \in \mathbf{Sm}_X$, the induced maps $X_n(U)/G(U) \to X_m(U)/G(U)$ are also monomorphisms. We argue at the end of the proof of Lemma 3.2.11 that Y_n and Y_m differ from the presheaf quotients only by sheafification. As sheafification is exact, this shows that $Y_n \to Y_m$ is also a monomorphism. $\square$

We now verify a crucial generalized contractibility condition for acceptable G-gadgets. Given a τ-locally trivial torsor $T \to Y$ in $\mathbf{Sm}_S$, consider the map $L_\tau(X_\infty \times_S T) \to Y$ given by applying first $(- /\!\!/ G)$ and then L_τ to the G-equivariant projection map $X_\infty \times_S T \to T$ and then composing with the map $L_\tau(T /\!\!/ G) \to Y$ of Lemma 3.2.11. This map is canonical up to the choice of L_τ.

Proposition 3.2.14 *Let $(X_n)_{n \in I}$ be an acceptable G-gadget over S. Then, for each τ-locally trivial G-torsor $T \to Y$ in $\mathbf{Sm}_S$, the map*

$$L_\tau((X_\infty \times_S T) /\!\!/ G) \to Y$$

is a motivic weak equivalence.

Proof Apply Lemma 3.2.13 to the sequence of τ-locally trivial G-torsors $X_n \times_S T \to X_n \times_S Y$ to obtain an objectwise weak equivalence

$$L_\tau((X_\infty \times_S T) /\!\!/ G) \to X_\infty \times_S Y.$$

This objectwise weak equivalence is, in particular, a motivic weak equivalence. As the map in the statement of the proposition can be obtained by composing this map with the projection map $p \colon X_\infty \times_S Y \to Y$, it suffices to show that p is a motivic equivalence over S.

We know that p is a motivic weak equivalence over Y by Lemmas 3.2.7 and 3.2.8. We now use the functor $u_\sharp \colon L_{\mathrm{mot}} \mathbf{sPre}(\mathbf{Sm}_Y) \to L_{\mathrm{mot}} \mathbf{sPre}(\mathbf{Sm}_S)$ from [5, Proposition 4.5.4] for the structure map $u \colon Y \to S$ to deduce that p is also a motivic weak equivalence over S. (These general results do not use the assumption on the "coefficients" discussed at the beginning of section 4.4 of loc cit. Compare also [20, Section 4.1] for a related discussion in the ∞-setting.) More precisely, $u_\sharp$ maps a smooth scheme over Y to the same smooth scheme over S by the proof of loc cit., and is a left Quillen functor for the projective model structures by [5, Theorem 4.5.14]. As left adjoints preserve colimits, $u_\sharp$ also maps the morphism $p \colon X_\infty \times_S Y \to Y$ over Y to the same morphism considered over S. Hence, as $u_\sharp$ is left Quillen, p is a motivic weak equivalence over S for the projective motivic model structure, and thus also for the injective model structure, which has the same weak equivalences. $\square$

Remark 3.2.15 There are definitions of G-torsors which are more general than [2, Definition 2.3.1], see, eg, [30, page 128], but which have in common that the conclusion of Lemma 3.2.11 holds. The proofs of Lemma 3.2.13 and Proposition 3.2.14 immediately generalize to these more general torsors provided the bases Y_n are representable.

Proposition 3.2.16 *For any acceptable G-gadget $(X_n)_{n \in I}$ over S, $L_\tau(X_\infty /\!\!/ G)$ is motivically equivalent to $L_\tau(S /\!\!/ G)$.*

Proof By [21, Lemma 2.1] it suffices to verify the condition established in Proposition 3.2.14. Though [21, Lemma 2.1] is only stated for the fppf-topology, it holds equally for any finer topology τ, as is evident from its (very short) proof. Also, while [21, Lemma 2.1] is stated in terms of general ∞-G-torsors, for a simplicially discrete group G any such torsor over a simplicially discrete base X is again discrete, as it is locally isomorphic to $X \times_S G$. So in our setting it suffices to verify the assumptions for simplicially discrete G-torsors as in [2, Definition 2.3.1]. (Alternatively, we could use Remark 3.2.15.) Finally, the ∞-quotients appearing in [21, Lemma 2.1] coincide with our stacky quotients. See Lemmas 3.2.10 and 3.2.11 above for special cases, and [31, Section 3] for details in the general case. $\qquad\square$

The following theorem is the promised precise version of the "outline" at the beginning of this section.

Theorem 3.2.17 *Let $(X_n)_{n \in \mathbb{N}}$ be an acceptable G-gadget such that we have a τ-locally trivial G-torsor $X_n \to Y_n$ and induced monomorphisms $Y_n \to Y_m$ for all $n, m \in \mathbb{N}$. Then we have motivic weak equivalences*

$$Y_\infty \simeq \mathrm{B}_\tau G \simeq \mathrm{B}_{\mathrm{fppf}} G.$$

Proof As already explained in the outline, Lemma 3.2.13, Proposition 3.2.16 and Lemma 3.2.10 yield a chain of motivic weak equivalences $Y_\infty \simeq L_\tau(X_\infty /\!\!/ G) \simeq L_\tau(S /\!\!/ G) \simeq \mathrm{B}_\tau G$. As the same chain of equivalences also applies for any topology finer than τ, we also obtain the motivic equivalence with $\mathrm{B}_{\mathrm{fppf}} G$. $\qquad\square$

As recalled in Example 3.2.4, for *special* affine algebraic group schemes G any torsor is Zariski-locally trivial, so that Theorem 3.2.17 will yield motivic equivalences $Y_\infty \simeq \mathrm{B} G \simeq \mathrm{B}_{\mathrm{fppf}} G$.

3.3 Hermitian ASC for symplectic groups

Let G be a linear algebraic group over a field k of characteristic not two. In particular, G is smooth and an étale sheaf on $\mathbf{Sm}_k$.

Definition 3.3.1 We define IO_G and $\mathrm{IO}_G^\pm$ as kernels of restriction maps:

$$\mathrm{IO}_G := \ker(\mathrm{GW}^+(\mathbf{Rep}(G)) \to \mathrm{GW}^+(k)) \quad \text{and} \quad \mathrm{IO}_G^\pm := \ker(\mathrm{GW}^\pm(\mathbf{Rep}(G)) \to \mathrm{GW}^\pm(k)).$$

Our definition of IO_G agrees with the definition given in [38]. The graded ideal $\mathrm{IO}_G^\pm$ is not considered there.

Lemma 3.3.2 *The $\mathrm{IO}_G^\pm$-adic topology on $\mathrm{GW}^\pm(\mathbf{Rep}(G))$ agrees with the IO_G-adic topology.*

Proof This is a general fact about graded ideals. Consider a $\pm$-graded ring $R = R^+ \oplus R^-$ and a graded ideal $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^- \subset R$. Clearly $(\mathfrak{a}^+)^i \cdot R \subset \mathfrak{a}^i$. On the other hand, as $(\mathfrak{a}^-)^2 \subset \mathfrak{a}^+$, we find that

$$\mathfrak{a}^{2i} = (\mathfrak{a}^+ + \mathfrak{a}^-)^{2i} \subseteq \sum_{j=0}^{2i} (a^+)^{2i-j}(a^-)^j \subseteq (\mathfrak{a}^+)^i \cdot R^+ \oplus (\mathfrak{a}^+)^i \cdot R^-.$$

So $\mathfrak{a}^{2i} \subset (\mathfrak{a}^+)^i \cdot R$. This shows that the $\mathfrak{a}$-adic topology on R agrees with the $\mathfrak{a}^+$-adic topology. $\qquad\square$

Recall from [32; 33] the Sp_{2r}-torsors $\mathrm{HU}(r, n)$ over the quaternionic Grassmannians $\mathrm{HGr}(r, n)$ associated with the tautological symplectic bundle, defined over a smooth quasiprojective base scheme S.

Proposition 3.3.3 *The sequence of* Sp_{2r}-*torsors* $\mathrm{HU}(r, n) \to \mathrm{HGr}(r, n)$ *with* $n \in \mathbb{N}$ *of* [32, Proposition 8.5] *defines an acceptable* Sp_{2r}-*gadget* $(\mathrm{HU}(r, n))_{n \in \mathbb{N}}$ *over* S.

Proof It is shown in [32, Proposition 8.5] that $(\mathrm{HU}(r, n))_{n \in \mathbb{N}}$ is an acceptable gadget over S. It obviously also satisfies the conditions of an acceptable Sp_{2r}-gadget, where the compatible actions come from the structure as Sp_{2r}-torsors. $\qquad\square$

Theorem 3.3.4 (Panin–Walter) *There are motivic weak equivalences*

$$\mathrm{B}_{\mathrm{Nis}} \mathrm{Sp}_{2r} \simeq \mathrm{B}_{\text{ét}} \mathrm{Sp}_{2r} \simeq \mathrm{HGr}(r, \infty).$$

Proof Both equivalences are stated in [32, after Proposition 8.5], along with a brief indication on how to modify the arguments of [30] to obtain a proof, using their concept of an acceptable gadget. Section 3.2 provides more details for this argument. The first equivalence is evident from the fact that Sp_{2r} is special (see Example 3.2.4). The second is immediate from Theorem 3.2.17 applied to the acceptable gadget of Proposition 3.3.3. $\qquad\square$

Remark 3.3.5 The motivic equivalence $\mathrm{B}_{\text{ét}} \mathrm{Sp}_{2r} \simeq \mathrm{HGr}(r, \infty)$ of Theorem 3.3.4 was proven in [41, Proposition 5] for $r = \infty$ using a different technique. However, [41, Proposition 3], which is an ingredient of this alternative proof, has no obvious analogue for finite r.

Remark 3.3.6 For the reader's convenience, let us compare the discussion above to some arguments in [30]. Theorem 3.3.4 is an analogue for Sp_{2r} of [30, Proposition 4.3.7] for GL_r. In either case, the first equivalence is established by noting that GL_r and Sp_{2r}, respectively, are special, and by using [30, Proposition 4.1.18]. The second equivalence rests on [30, Proposition 4.2.6], which has its parallel in our Proposition 3.2.16. Both of these intermediate results rely on the contractibility of colimits of gadgets. For admissible gadgets, this is proved in [30, Proposition 4.2.3], using the ambient vector bundles, while for acceptable gadgets this is Lemma 3.2.7 above. In [30], this contractibility enters via [30, Lemma 4.2.9]; see the proof of [30, Proposition 4.2.6] spelled out below [30, Lemma 4.2.9]. In our case, the contractibility

enters via the corresponding Proposition 3.2.14. Despite all these parallels, the definition of acceptable G-gadget employed here is significantly simpler than the definition of nice admissible G-gadget used by Morel and Voevodsky: there is no analogue of part (iii) of [30, Definition 4.2.4] in our Definition 3.2.12. The main reason why this simplification is possible is [21, Lemma 2.1].

For Sp_2, Proposition 3.3.3 yields an acceptable gadget $(HU(1, n + 1))_{n \geq 1}$ built from Sp_2-torsors $HU(1, n + 1) \to HP^n$, where HP^n is the quaternionic projective space defined in [33]. In combination with Lemma 3.2.8(c), we moreover obtain acceptable gadgets $(HU(1, n + 1)^{\times r})_{n \geq 1}$ built from $Sp_2^{\times r}$-torsors over products of quaternionic projective spaces $(HP^n)^{\times r}$, for any $r \in \mathbb{N}$.

Proposition 3.3.7 *The following maps are isomorphisms*:

(i)   *the canonical map* $GW^{\pm}(B\,Sp_2) \to \lim_n GW^{\pm}(HP^n)$;

(ii) *the canonical map* $GW^{\pm}(B\,Sp_2^{\times r}) \to \lim_n GW^{\pm}((HP^n)^{\times r})$; *and*

(iii) *the canonical map* $GW^{\pm}(B\,Sp_{2r}) \to \lim_n GW^{\pm}(HGr(r, n))$.

Proof   By [32, Theorem 13.4], bigraded GW-theory is represented by a commutative ring spectrum BO, which is $(8, 4)$-periodic by [32, Theorem 7.5]. Restricting BO to bidegrees $(0, 0)$ and $(4, 2)$ yields the commutative ring $GW^{\pm}(X)$ for motivic spaces X which are not schemes. For an acceptable G-gadget $(X_n)_{n \in \mathbb{N}}$ such that we have compatible G-torsors $X_n \to Y_n$ for each n, we may consider the motivic Milnor exact sequences (see, eg, [34, Theorem 5.7]):

$$0 \to \lim_n^1 BO^{-1,0}(Y_n) \to GW^+(Y_\infty) \to \lim_n GW^+(Y_n) \to 0,$$

$$0 \to \lim_n^1 BO^{3,2}(Y_n) \to GW^-(Y_\infty) \to \lim_n GW^-(Y_n) \to 0.$$

These yield a canonical isomorphisms $GW^{\pm}(Y_\infty) \to \lim_n GW^{\pm}(Y_n)$ provided the $\lim^1$-terms vanish. When $Y_n = HGr(r, n)$, they do vanish, as [32, Theorem 9.5] provides the necessary surjections of BO-groups, thus proving (i) and (iii). To prove (ii), we argue similarly, using [32, Theorem 9.5] inductively by viewing $(HP^n)^{\times r}$ as a trivial quaternionic projective bundle over $(HP^n)^{\times r-1}$. $\qquad\square$

In view of this proposition, the reader preferring motivic spaces to actual varieties may replace several $\lim_n GW^{\pm}(HP^n)$ below by $GW^{\pm}(B_{\text{ét}}\,Sp_2)$ or $GW^{\pm}(BSp_2)$.

We are now ready to examine the case $G = Sp_2$ in detail. The gadget $(HU(1, n + 1))_{n \geq 1}$ has nice formal properties for *Borel classes*. Let $(\mathcal{U}_n, \psi_n)$ be the tautological rank-2 symplectic bundle on HP^n. It defines an element in $GW^-(HP^n)$. We define the universal first Borel class

$$b_1 = (b_{1,(n)})_{n \in \mathbb{N}} \in \lim_n GW^-(HP^n)$$

to be the limit of the first Borel classes $b_{1,(n)} \in GW^-(HP^n)$, where $b_{1,(n)} = b_1(\mathcal{U}_n, \psi_n)$ is the first Borel class of $(\mathcal{U}_n, \psi_n)$ on HP^n, as defined in [33, Definition 8.3]. By [32, Proposition 9.9], we have

$$b_{1,(n)} = [\mathcal{U}_n, \psi_n] - H^- \in GW^-(HP^n),$$

where H^- is the symplectic hyperbolic space of rank 2 with trivial Sp_2-action. Let (V, ϕ) be a trivial symplectic vector space of rank $2n + 2$ in which (V_0, ϕ_0) is a distinguished rank-2 symplectic subspace, and write $\mathrm{Sp}_{2n+2} = \mathrm{Sp}(V, \phi)$. In this case (V, ϕ) is the standard representation of Sp_{2n+2} discussed in Example 2.3.15. There is a decomposition

$$(V, \phi) \cong (V_0, \phi_0) \perp (V_0^\perp, \phi_0^\perp).$$

Then Sp_{2n+2} acts on $\mathrm{HP}^n = \mathrm{HP}(V, \phi)$, and the stabilizer of the distinguished point (V_0, ϕ_0) in HP^n is given by those $a \in \mathrm{Sp}(V, \phi)$ that fix (V_0, ϕ_0) and $(V_0^\perp, \phi_0^\perp)$, and is therefore given by $\mathrm{Sp}_{2n} \times \mathrm{Sp}_2$, where we make the identifications $\mathrm{Sp}_{2n} = \mathrm{Sp}(V_0^\perp, \phi_0^\perp)$ and $\mathrm{Sp}_2 = \mathrm{Sp}(V_0, \phi_0)$. Under the identification

$$\mathrm{HP}^n \cong \mathrm{Sp}_{2n+2} / (\mathrm{Sp}_{2n} \times \mathrm{Sp}_2),$$

the canonical Sp_2-torsor $\mathrm{HU}(1, n + 1)$ is given by $\mathrm{Sp}_{2n+2} / \mathrm{Sp}_{2n}$, which corresponds to the universal bundle $(\mathcal{U}_n, \psi_n)$ under the correspondence between Sp_2-torsors and rank-2 symplectic bundles as outlined in, for instance, [2, Section 3.3, page 1025]. We let X_n denote $\mathrm{HU}(1, n + 1)$ for ease of notation. Let $\pi_n \colon X_n \to \operatorname{Spec} k$ be the G-equivariant structure map. Since the pullback functor $\gamma_n^* \colon \mathrm{Vect}(\mathrm{HP}^n) \to \mathrm{Vect}^G(X_n)$ along the projection $\gamma_n \colon X_n \to \mathrm{HP}^n$ is an equivalence of exact categories with duality, γ_n^* induces a canonical isomorphism $\gamma_n^* \colon \mathrm{GW}^\pm(\mathrm{HP}^n) \to \mathrm{GW}_G^\pm(X_n)$. The following lemma studies the Atiyah–Segal map $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2)) \to \mathrm{GW}^\pm(B\mathrm{Sp}_2)$ induced by approximations $\pi_n \colon X_n \to \operatorname{Spec} k$ of the pullback along $E\mathrm{Sp}_2 \to \operatorname{Spec} k$.

Proposition 3.3.8 *Let H^- be the trivial symplectic plane bundle equipped with the trivial Sp_2-action and $[V_0, \phi_0]$ the Sp_2-representation from above. The composition*

$$(\gamma_n^*)^{-1} \pi_n^* \colon \mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2)) \to \mathrm{GW}_{\mathrm{Sp}_2}^\pm(X_n) \to \mathrm{GW}^\pm(\mathrm{HP}^n)$$

sends $[V_0, \phi_0] - H^-$ to the (first) Borel class $b_{1,(n)} = [\mathcal{U}_n, \psi_n] - H^-$.

Proof The pullback map $\pi_n^* \colon \mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2)) \to \mathrm{GW}_{\mathrm{Sp}_2}^\pm(X_n)$ sends $[V_0, \phi_0]$ to the class $[V_n, \phi_n]$ of the trivial rank-2 symplectic bundle on X_n with the standard Sp_2-action. Since $(\mathcal{U}_n, \psi_n)$ is a rank-2 symplectic bundle corresponding to the torsor $X_n = \mathrm{Symp}_{(\mathcal{U}_n, \psi_n)}$, its pullback to X_n is also the trivial symplectic bundle $[V_n, \phi_n]$ with the standard Sp_2-action. As $X_n \to \mathrm{HP}^n$ is an Sp_2-torsor, the map

$$\mathrm{Sp}_2 \times_k X_n \to X_n \times_{\mathrm{HP}^n} X_n$$

given on points by $(a, x) \mapsto (ax, x)$ is an isomorphism. Hence,

$$\mathrm{Aut}_{\mathrm{Symp}}(H_{X_n}^-) \cong \mathrm{Sp}_2 \times_k X_n \cong \gamma_n^* \mathrm{Symp}_{(\mathcal{U}_n, \psi_n)} \cong \mathrm{Symp}_{(f^* \mathcal{U}_n, f^* \psi_n)}$$

and we deduce

$$\pi_n^*([V_0, \phi_0] - H^-) = [V_n, \phi_n] - H^- = \gamma_n^*([\mathcal{U}_n, \psi_n] - H^-). \qquad \square$$

As an immediate corollary, we obtain Atiyah–Segal completion for $G = \mathrm{Sp}_2$ and $\mathrm{GW}^\pm$.

Corollary 3.3.9 *For $G = \mathrm{Sp}_2$, the map $\mathrm{GW}^{\pm}(\mathbf{Rep}(G)) \to \lim_n \mathrm{GW}^{\pm}(\mathrm{HP}^n)$ from Proposition 3.3.8 above is a completion of $\mathrm{GW}^{\pm}(\mathbf{Rep}(G))$ with respect to the Hermitian augmentation ideal IO_G.*

Proof The computation (2.4.3) immediately implies that $\mathrm{IO}_{\mathrm{Sp}_2}^{\pm}$ is generated by $[V_0, \phi_0] - H^-$. By [32, Theorem 9.5],

$$\lim_n \mathrm{GW}^{\pm}(\mathrm{HP}^n) \cong \mathrm{GW}^{\pm}(k)[\![b_1]\!].$$

Hence the claim follows from Proposition 3.3.8 and Lemma 3.3.2. $\qquad\square$

Now we prove Atiyah–Segal completion for general Sp_{2r}. We will use the diagram

(3.3.1)
$$\begin{array}{ccc}
\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r})) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times r})) \\
 & & \downarrow \\
\mathrm{GW}^{\pm}(\mathrm{BSp}_{2r}) & \longrightarrow & \mathrm{GW}^{\pm}(\mathrm{BSp}_2^{\times r})
\end{array}$$

in which the vertical arrow is a completion by Proposition 3.3.12. We start with the following result of Panin and Walter.

Proposition 3.3.10 *For $1 \le i \le r$, let $y_i \in \lim_n \mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r})$ be the element defined by the inverse limit*

$$\lim_{n \in \mathbb{N}} b_1(\mathcal{U}_n^{(i)}, \phi_n^{(i)})$$

of the first Borel classes $b_1(\mathcal{U}_n^{(i)}, \phi_n^{(i)})$ of the ith tautological rank-2 bundle on $(\mathrm{HP}^n)^{\times r}$. Then

$$\lim_n \mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r}) = \mathrm{GW}^{\pm}(k)[\![y_1, \ldots, y_r]\!].$$

Proof For $r = 1$, this is a consequence of [33, Section 11] or [32, Theorem 9.5] as already recalled above. Note that $(\mathrm{HP}^n)^{\times r} \to (\mathrm{HP}^n)^{\times r-1}$ is a trivial HP^n-bundle, so by [32, Theorem 9.4],

$$\mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r}) \cong \frac{\mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r-1})[b_1(\mathcal{U}_n^{(r)}, \phi_n^{(r)})]}{(b_1(\mathcal{U}_n^{(r)}, \phi_n^{(r)})^n)}.$$

Iterating, we obtain

$$\mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r}) \cong \frac{\mathrm{GW}^{\pm}(k)[b_1(\mathcal{U}_n^{(1)}, \phi_n^{(1)}), \ldots, b_1(\mathcal{U}_n^{(r)}, \phi_n^{(r)})]}{(b_1(\mathcal{U}_n^{(1)}, \phi_n^{(1)})^n, \ldots, b_1(\mathcal{U}_n^{(r)}, \phi_n^{(r)})^n)}$$

and it follows that

$$\lim_n \mathrm{GW}^{\pm}((\mathrm{HP}^n)^{\times r}) = \mathrm{GW}^{\pm}(k)[\![y_1, \ldots, y_r]\!]. \qquad\square$$

For $n \in \mathbb{N}$, let $f_n \colon (\mathrm{HP}^n)^{\times r} \to \mathrm{HGr}(r, rn)$ be the canonical map such that the pullback of the tautological rank-$2r$ symplectic bundle on $\mathrm{HGr}(r, rn)$ is the orthogonal sum of the rank-2 symplectic bundles $(\mathcal{U}_n^{(i)}, \phi_n^{(i)})$ on $(\mathrm{HP}^n)^{\times r}$; such a map exists by the universal property of quaternionic Grassmannians discussed in [33, 10–11]. Note that $(\mathrm{HGr}(r, rn))_{n \in \mathbb{N}}$ is an acceptable gadget by Lemma 3.2.8. Recall also that we have $\lim_n \mathrm{GW}^{\pm}(\mathrm{HGr}(r, rn)) \cong \mathrm{GW}^{\pm}(k)[\![b_1, \ldots, b_r]\!]$ by [33, Theorem 11.4].

Theorem 3.3.11 *Let $b_i \in \lim_n \mathrm{GW}^\pm(\mathrm{HGr}(r, rn))$ be the element defined by the inverse limit*

$$b_1(\mathcal{U}_{r,rn}, \phi_{r,rn})_{n \in \mathbb{N}}$$

of Borel classes of the tautological rank-$2r$ bundle on $\mathrm{HGr}(r, rn)$. The limit

$$f^* \colon \lim_n \mathrm{GW}^\pm(\mathrm{HGr}(r, rn)) \to \lim_n \mathrm{GW}^\pm((\mathrm{HP}^n)^{\times r})$$

of the pullback maps f_n^* sends b_i to the ith symmetric elementary polynomial in the variables y_j defined in Proposition 3.3.10.

Proof The ith tautological symplectic bundle of rank 2 on $(\mathrm{HP}^n)^{\times r}$ is an orthogonal direct summand of $f_n^*(\mathcal{U}_{r,rn}, \phi_{r,rn})$ by definition of f_n. We can now run the same argument as in the first part of the proof of [33, Theorem 10.2] to show that the image of b_i is the ith symmetric elementary polynomial in the variables y_j. $\qquad\square$

Recall from Proposition 3.3.7 that we have an isomorphism

$$\mathrm{GW}^\pm(\mathrm{BSp}_2^{\times r}) \cong \lim_n \mathrm{GW}^\pm((\mathrm{HP}^n)^{\times r})$$

with the ring structure given by Proposition 3.3.10. Considering the composition $(\gamma_n)^{-1} \circ \pi_n^*$ and $n \to \infty$ for $G = \mathrm{Sp}_2^{\times r}$ and $G = \mathrm{Sp}_{2r}$, we obtain morphisms of $\mathrm{GW}^\pm(k)$-algebras $\mathrm{GW}^\pm(\mathbf{Rep}(G)) \to \mathrm{GW}^\pm(BG)$ as well, both generalizing (take $r = 1$) the map of Corollary 3.3.9.

Proposition 3.3.12 *The map of $\mathrm{GW}^\pm(k)$-algebras*

$$\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2^{\times r})) \to \mathrm{GW}^\pm(\mathrm{BSp}_2^{\times r})$$

defined above exhibits $\mathrm{GW}^\pm(\mathrm{BSp}_2^{\times r})$ as the completion of $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ with respect to $\mathrm{IO}_{\mathrm{Sp}_2^{\times r}}$.

Proof By Proposition 3.3.8, the generator $b^{(i)} \in \mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ of (2.4.9) is mapped to the Borel class $y_i \in \mathrm{GW}^\pm(\mathrm{BSp}_2^{\times r})$ of Proposition 3.3.10 for each $1 \le i \le r$. Since $\mathrm{IO}_{\mathrm{Sp}_2^{\times r}} \subset \mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ is generated, as an ideal, by the classes $b^{(i)}$, the result follows. $\qquad\square$

We are now ready to prove the following Atiyah–Segal completion result for Hermitian K-theory and symplectic groups:

Corollary 3.3.13 *The map of $\mathrm{GW}^\pm(k)$-algebras*

$$\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_{2r})) \to \mathrm{GW}^\pm(\mathrm{BSp}_{2r})$$

defined above exhibits $\mathrm{GW}^\pm(\mathrm{BSp}_{2r})$ as the completion of $\mathrm{GW}^\pm(\mathbf{Rep}(\mathrm{Sp}_{2r}))$ with respect to $\mathrm{IO}_{\mathrm{Sp}_{2r}}$.

Proof By Corollary 2.4.4, the map $\mathrm{res}\colon \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r})) \to \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ from the upper line of (3.3.1) is injective and maps (higher) Borel classes to elementary symmetric polynomials in the generators $b^{(i)} \in \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$. Similarly, by Theorem 3.3.11 and Proposition 3.3.7, the map $\mathrm{GW}^{\pm}(\mathrm{BSp}_{2r}) \to \mathrm{GW}^{\pm}(\mathrm{BSp}_2^{\times r})$ from (3.3.1) is injective and maps (higher) Borel classes to elementary symmetric polynomials in the generators $y_i \in \mathrm{GW}^{\pm}(\mathrm{BSp}_2^{\times r})$. It follows that the image of $\mathrm{IO}_{\mathrm{Sp}_{2r}}$ in $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ is

$$\mathrm{IO}_{\mathrm{Sp}_{2r}} = \mathrm{IO}_{\mathrm{Sp}_2^{\times r}} \cap \mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r}))$$

if we consider $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r}))$ as a subalgebra of $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_2^{\times r}))$ via the restriction map. Thus it follows from Proposition 3.3.12 that the map

$$\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r})) \to \mathrm{GW}^{\pm}(\mathrm{BSp}_{2r})$$

exhibits $\mathrm{GW}^{\pm}(\mathrm{BSp}_{2r})$ as the completion of $\mathrm{GW}^{\pm}(\mathbf{Rep}(\mathrm{Sp}_{2r}))$ with respect to $\mathrm{IO}_{\mathrm{Sp}_{2r}}$. $\square$

3.4 Classifying space for multiplicative group with a nontrivial involution

Consider the multiplicative group $\mathbb{G}_m$ with the involution $\iota\colon t \mapsto t^{-1}$, as in Example 2.0.1. In this section, we show how to approximate the classifying space $\mathrm{B}\mathbb{G}_m$ in a way that is compatible with the involution. That is, we will construct $\mathbb{G}_m$-torsors $U_n \to B_n$ such that the torsors U_n form an acceptable gadget, and are equipped with involutions $\iota\colon U_n \to U_n$ that are compatible with the $\mathbb{G}_m$-action in the sense that

$$(3.4.1) \qquad\qquad \iota(t.x) = \iota(t).\iota(x)$$

for $t \in \mathbb{G}_m$, $x \in U_n$. This condition ensures, in particular, that the involution on U_n descends to an involution on B_n.

Remark 3.4.1 Usually, $\mathrm{B}\mathbb{G}_m$ is approximated by the projective spaces $\mathbb{P}^n$. We have $\mathbb{G}_m$-torsors $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$, and these torsors form an acceptable gadget. However, there seems to be no involution ι on $\mathbb{A}^{n+1} \setminus 0$ satisfying (3.4.1).

Concretely, we will use the principal $\mathbb{G}_m$-torsors

$$U_n := \{(x, y) \in \mathbb{A}^{n+1} \times \mathbb{A}^{n+1} \mid x^T y = 1\}$$

with $\mathbb{G}_m$-action $t.(x, y) := (tx, t^{-1}y)$ and involution $\iota\colon (x, y) \mapsto (y, x)$. This involution clearly satisfies (3.4.1). Quotienting by the $\mathbb{G}_m$-action, we obtain the following open subschemes of $\mathbb{P}^n \times \mathbb{P}^n$:

$$B_n := \{([x], [y]) \in \mathbb{P}^n \times \mathbb{P}^n \mid x^T y \neq 0\}$$

The induced involution on B_n is given by $\iota\colon ([x], [y]) \mapsto ([y], [x])$.

Remark 3.4.2 To verify that the obvious projection $U_n \to B_n$ is a principal $\mathbb{G}_m$-torsor, we can identify it with the canonical projection

$$\mathrm{GL}_{1+n} / (1 \times \mathrm{GL}_n) \to \mathrm{GL}_{1+n} / (\mathbb{G}_m \times \mathrm{GL}_n)$$

via the map that sends the left coset represented by a matrix A to the pair (x, y) consisting of the first column of A, and of the first row of A^{-1}:

$$
\begin{array}{ccc}
\frac{\mathrm{GL}_{1+n}}{1 \times \mathrm{GL}_n} & \xrightarrow{\ \cong\ } & U_n \\
\downarrow {\scriptstyle \mathbb{G}_m} & & \downarrow {\scriptstyle \mathbb{G}_m} \\
\frac{\mathrm{GL}_{1+n}}{\mathbb{G}_m \times \mathrm{GL}_n} & \xrightarrow{\ \cong\ } & B_n
\end{array}
$$

The involutions on U_n and B_n are induced by the involution $A \mapsto (A^{-1})^T$ on GL_{1+n}.

Remark 3.4.3 The fixed points of the scheme B_n under the involution can be identified with the scheme $\{[x] \in \mathbb{P}^n \mid x^T x \neq 0\}$, ie with the complement of a quadric in $\mathbb{P}^n$. This complement is often used as an algebro-geometric replacement for real projective space $\mathbb{RP}^n$, for example in [14] or [48]. It is a special case of the algebraic orthogonal Grassmannians of Schlichting and Tripathi [41].

Lemma 3.4.4 *The schemes U_n with the obvious inclusions $U_n \subset U_{n+1} \subset \cdots$ form an acceptable gadget in the sense of Definition 3.2.6.*

Proof Let R be an arbitrary commutative ring, and $g \in R$. As in the proof of [32, Proposition 8.5], it suffices to show that, given an arbitrary morphism $\mathrm{Spec}(R/g) \to U_n$, we can fill in the dashed arrow in the diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(R/g) & \longrightarrow & U_n \ \hookrightarrow \ U_{n+1} \\
\downarrow & {\nearrow} & \\
\mathrm{Spec}(R) & &
\end{array}
$$

The given arrow corresponds to a tuple $(\bar{a}_0, \ldots, \bar{a}_n, \bar{b}_0, \ldots, \bar{b}_n)$ in $(R/g)^{2n+2}$ such that $\sum_i \bar{a}_i \bar{b}_i = 1$ in R/g. Pick an arbitrary lift $(a_0, \ldots, a_n, b_0, \ldots, b_n) \in R^{2g+2}$. Then the previous equality tells us that there exists some element $r \in R$ such that $\sum_i a_i b_i = 1 + gr$ in R. The composition from $\mathrm{Spec}(R/g)$ to U_{n+1} corresponds to the tuple $(\bar{a}_0, \ldots, \bar{a}_n, \bar{0}, \bar{b}_0, \ldots, \bar{b}_n, \bar{0})$ in $(R/g)^{2n+4}$. In order to construct the dashed arrow in a way that the diagram commutes, we need to construct a lift of this tuple, say,

$$
(a_0', \ldots, a_n', gc, b_0', \ldots, b_n', gd)
$$

such that $\sum_i a_i' b_i' + g^2 cd = 1$. Pick $a_i' := a_i$, $b_i' := b_i - grb_i$, and $c := d := r$. A quick calculation shows that this choice fits the bill. $\square$

3.5 Tools for generalizations to base schemes with nontrivial group actions

Recall that if we have Atiyah–Segal completion for GL_n for schemes X with arbitrary GL_n-action, then applying it to $X = \mathrm{GL}_n / H$ yields the completion theorem for H over a point $\mathrm{Spec}(k)$. The same applies to Sp_{2n}, and hence either case would cover all H which are, eg, split reductive, and in particular all finite groups.

We now briefly recall two techniques that have been successfully used to generalize results on schemes X with trivial G-action to general G-schemes X. Both only apply to $G = T$ a torus, and both have the same underlying idea: under suitable assumptions, there is a big open G-subscheme U of X on which the action of X has a very simple product description. Using this product description, we may prove the desired result for U, and then using a finite number of induction steps also for X, assuming that the equivariant cohomology theory we care about (here: equivariant Hermitian K-theory) satisfies a suitable equivariant localization theorem.

The first technique is to work with T-filtrable schemes for $G = T$ an "algebraic torus", and has been used, eg, by Brion and Krishna, and more recently by [45]. This goes back to Białynicki-Birula. The main ingredient is probably [6, Theorem 2.5], which states that U is T-equivariantly isomorphic to $(U \cap X^T) \times V$, where V is a finite-dimensional T-module.

The second technique uses the "torus generic slice theorem" of Thomason; see [46, Proposition 4.10]. Here X is very general, and $G = T$ is a "diagonalizable torus". The main geometric result here is that we have a T-equivariant isomorphism $U \cong T/T' \times U/T$ with T' a diagonalizable subtorus, and the induction is then done essentially on page 804 of loc cit.

Both techniques would be useful if we could generalize them from tori T to GL_n or Sp_{2n}, or to products of Sp_2, or to tori with involution. We unfortunately don't have this yet. Still, the above method is expected to yield the completion theorem for subgroups of T, eg, products of groups of roots of unity μ_l. If we assume that these roots μ_l are contained in the base field, we could deduce the completion theorem for the corresponding direct sums of (constant) cyclic groups.

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