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A minimality property for knots without Khovanov 2-torsion

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A conjecture of Shumakovitch states that every nontrivial knot has 2-torsion in its Khovanov homology. We show that if a knot K has no 2-torsion in its Khovanov homology, then the rank of its reduced Khovanov homology is minimal among all knots obtainable from K by a proper rational tangle replacement. It follows, for example, that unknotting number 1 knots have 2-torsion in their Khovanov homology.

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Shumakovitch conjectured that every nontrivial knot has an element of order 2 in its Khovanov homology [10, Conjecture 1]. The conjecture has been verified for some infinite families of knots (see, for example, [2; 9; 10]) and has withstood large computational searches. In this note, we provide topological evidence for the conjecture, and we verify the conjecture for a large class of knots that include all unknotting number 1 knots.

Two links differ by a *rational tangle replacement* if they agree outside of a ball, and if within the ball, each is a rational tangle. A rational tangle replacement is *proper* if the arcs of the two rational tangles connect the same end points [4; 8]. Changing a crossing is an example of a proper rational tangle replacement, while resolving a crossing is an example of a nonproper rational tangle replacement. In the following statement, $\mathrm{Kh}(K)$ and $\overline{\mathrm{Kh}}(K)$ denote the unreduced and reduced Khovanov homology groups of K, respectively, thought of as abelian groups with bigradings suppressed.

Theorem 1 *Suppose K is a knot such that there is no 2-torsion in $\mathrm{Kh}(K)$. If J is a knot that differs from K by a proper rational tangle replacement, then*

$$\operatorname{rank} \overline{\mathrm{Kh}}(K) \leq \operatorname{rank} \overline{\mathrm{Kh}}(J).$$

Corollary 2 *Any knot whose unknotting number is 1 has 2-torsion in its Khovanov homology. More generally, if K is a nontrivial knot that can be obtained from the unknot or a trefoil by a proper rational tangle replacement, then $\mathrm{Kh}(K)$ contains 2-torsion.*

Proof of Corollary 2 Let J be the unknot or a trefoil, and let K be obtained from J by a proper rational tangle replacement. If there is no 2-torsion in $\mathrm{Kh}(K)$, then $\operatorname{rank} \overline{\mathrm{Kh}}(K) \leq \operatorname{rank} \overline{\mathrm{Kh}}(J) \leq 3$ by Theorem 1. The rank of $\overline{\mathrm{Kh}}(K)$ cannot be 3 since then K would be a trefoil [3, Theorem 1.4], which has 2-torsion in its Khovanov homology. Since the rank of $\overline{\mathrm{Kh}}(K)$ is odd, it must be 1, and so K is the unknot [6]. □

Our proof of Theorem 1 combines the main result of Iltgen, Lewark, and Marino [4] with an observation of Kotelskiy, Watson, and Zibrowius [5, Proposition 9.3] using the following lemma.

Lemma 3 *Let $\mathbb{F}$ be a field, and suppose M and N are finitely generated modules over the polynomial ring $\mathbb{F}[X]$ of the form*

$$M = (\mathbb{F}[X])^r \oplus \bigoplus_{i=1}^{m} \frac{\mathbb{F}[X]}{X^{a_i}}, \quad N = (\mathbb{F}[X])^s \oplus \bigoplus_{i=1}^{n} \frac{\mathbb{F}[X]}{X^{b_i}},$$

where $r, m, s, n \geq 0$ and $a_1, \dots, a_m, b_1, \dots, b_n \geq 1$. Furthermore, suppose $f\colon M \to N$ and $g\colon N \to M$ are $\mathbb{F}[X]$-module maps for which $f \circ g = X$ and $g \circ f = X$. If the numbers $a_1, \dots, a_m$ are all at least 2, then $m \leq n$.

Proof Let X_M and X_N denote the structural maps $X\colon M \to M$ and $X\colon N \to N$, respectively. Our aim is to establish $m = \dim_{\mathbb{F}} \ker X_M \leq \dim_{\mathbb{F}} \ker X_N = n$.

Setting $C := g^{-1}(\ker X_M)$, we first claim that $g|_C\colon C \to \ker X_M$ is surjective. Since the numbers $a_1, \dots, a_m$ are all at least two, any element y in the kernel of X_M lies in the image of X_M, and therefore may be written as $y = X_M z = g(f(z))$, which proves the claim. Next, note that g sends $X_N C$ to zero, and so $g|_C$ induces a surjection $C/X_N C \to \ker X_M$. Thus

$$\dim_{\mathbb{F}} \ker X_M \leq \dim_{\mathbb{F}} C - \dim_{\mathbb{F}} X_N C = \dim_{\mathbb{F}} \ker(X_N|_C) \leq \dim_{\mathbb{F}} \ker X_N. \qquad \square$$

Proof of Theorem 1 Let $\overline{\mathrm{BN}}(K)$ denote the reduced Bar-Natan homology of K with rational coefficients. It is a rank-1 finitely generated graded module over $\mathbb{Q}[H]$ where H has nonzero degree, so we may write

$$\overline{\mathrm{BN}}(K) \cong \mathbb{Q}[H] \oplus \bigoplus_{i=1}^{m} \frac{\mathbb{Q}[H]}{H^{a_i}}, \quad \overline{\mathrm{BN}}(J) \cong \mathbb{Q}[H] \oplus \bigoplus_{i=1}^{n} \frac{\mathbb{Q}[H]}{H^{b_i}},$$

where $a_1, \dots, a_m, b_1, \dots, b_n$ are positive. By hypothesis, there is no 2-torsion in $\mathrm{Kh}(K)$, so Proposition 9.3 of [5] implies that the numbers $a_1, \dots, a_m$ are all at least 2. Furthermore, [5, Proof of Proposition 9.3] also gives $\mathrm{rk}\,\overline{\mathrm{Kh}}(K) = 1 + 2m$ and $\mathrm{rk}\,\overline{\mathrm{Kh}}(J) = 1 + 2n$.

By [4, Proof of Theorem 1.1], there are $\mathbb{Q}[H]$-module maps $f\colon \overline{\mathrm{BN}}(K) \to \overline{\mathrm{BN}}(J)$ and $g\colon \overline{\mathrm{BN}}(J) \to \overline{\mathrm{BN}}(K)$ satisfying $f \circ g = H$ and $g \circ f = H$. We note that the complex $[\![D]\!]$ over $\mathbb{Z}[G]$ associated to a diagram D considered in [4] recovers the reduced Bar-Natan complex as $[\![D]\!] \otimes_{\mathbb{Z}[G]} \mathbb{Q}[H]$ where $\mathbb{Z}[G] \to \mathbb{Q}[H]$ sends G to $-H$. By Lemma 3, we obtain

$$\mathrm{rk}\,\overline{\mathrm{Kh}}(K) = 1 + 2m \leq 1 + 2n = \mathrm{rk}\,\overline{\mathrm{Kh}}(J). \qquad \square$$

Remark 4 Suppose K is a knot such that $\overline{\mathrm{BN}}(K)$ does not contain $\mathbb{Q}[H]/H$ as a direct summand. Our proof of Theorem 1 implies that the conclusion of Theorem 1 holds for K. This observation gives evidence in favor of the affirmative for [5, Question 9.4], which suggests that the reduced Bar-Natan homology of any nontrivial knot contains $\mathbb{Q}[H]/H$ as a direct summand. We note that the existence of such a summand in reduced Bar-Natan homology implies the existence of 2-torsion in Khovanov homology [5, Proposition 9.3]. An analogous question is raised in [1; 7] in the context of the Floer homology of rational homology spheres.

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