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Infinitely many homeomorphic hyperbolic plugs with the same basic sets

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Using Ghys' work on the geodesic flow of the modular surface, we construct infinitely many smooth flows on $T^2 \times [0,1]$ transverse to the boundary, whose maximal invariant sets are the same saddle hyperbolic set, but whose global dynamics are not topologically equivalent pairwise. As a corollary of the previous construction, we prove that there exist infinitely many simple Smale flows on S^3 with the same basic sets but different global dynamics.

37D40, 37D45

1 Introduction

1.1 Background and main results

A *hyperbolic plug* is a pair (W, ψ_t), where W is a compact 3-manifold with boundary and ψ_t is a smooth flow on W transverse to ∂W such that the maximal invariant set of ψ_t is a saddle hyperbolic set. Hyperbolic plugs play a fundamental role in the study of structurally stable flows in dimension 3 as they can be used to construct large families of structurally stable flows and also to decompose complex structurally stable flows into simpler ones that can be studied separately.

In [3], Béguin, Bonatti and Yu constructed transitive Anosov flows in dimension 3 by gluing hyperbolic plugs with filling laminations. Here, a *hyperbolic plug with filling laminations* is a hyperbolic plug (W, ψ_t) such that for the maximal invariant set Λ of ψ_t, the lamination $\partial W \cap (W^s(\Lambda) \cup W^u(\Lambda))$ can be filled into a foliation on ∂W. In this context, $W^s(\Lambda)$ (resp $W^u(\Lambda)$) denotes the union of stable (resp unstable) manifolds of the orbits in Λ. Therefore, exploring the topologically equivalent classes of homeomorphic hyperbolic plugs with filling laminations could provide an answer to the classical question of whether a 3-manifold can admit infinitely many Anosov flows up to topological equivalence.

Nowadays, many experts in the area of 3-dimensional Anosov flows tend to believe that any 3-manifold admits at most finitely many Anosov flows, up to topological equivalence. So, it is highly likely that:

Conjecture 1.1 *There are at most finitely many homeomorphic hyperbolic plugs with filling laminations, any two of which are not topologically equivalent.*

However, for the hyperbolic plugs without filling laminations, our result (Theorem 1.3), contrary to the claims of Conjecture 1.1, shows that there are infinitely many homeomorphic hyperbolic plugs without filling laminations, whose maximal invariant sets (even with their small neighborhoods) are the same, but whose global dynamics are not topologically equivalent pairwise.

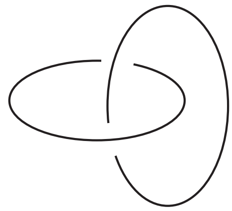

Figure 1: Hopf link.

Definition 1.2 Let φ_t (resp ϕ_t) be a smooth flow on a 3-manifold M (resp N) with a compact invariant set K (resp L). The pair (φ_t, K) is *equivalent* to (ϕ_t, L) if and only if there is a neighborhood U_K of K in M and a neighborhood U_L of L in N such that $\varphi_t|_{U_K}$ and $\phi_t|_{U_L}$ are topologically equivalent via a homeomorphism sending K to L. The *germ* $[\varphi_t, K]$ is the equivalence class represented by (φ_t, K).

Theorem 1.3 *There are infinitely many homeomorphic hyperbolic plugs* $\{(T^2 \times [0,1], \psi_t^i)\}_{i\in\tau}$*, such that*

(1) *for any* $i \in \tau$*, the maximal invariant set* Λ_i *of* ψ_t^i *is a* 1*-dimensional saddle basic set containing infinitely many closed orbits*;

(2) *for any* $j \in \tau$ *and* $j \neq i$, $[\psi_t^i, \Lambda_i] = [\psi_t^j, \Lambda_j]$, *but* ψ_t^i *is not topologically equivalent to* ψ_t^j.

Building on Theorem 1.3, we can explore its implications for Smale flows on S^3. A *Smale flow* is a smooth structurally stable flow, whose chain recurrent set is hyperbolic, at most 1-dimensional and satisfies the transversality condition [19]. Sullivan introduced a special class of Smale flows in [20], called *simple Smale flows*. A simple Smale flow is a Smale flow whose chain recurrent set consists of an attracting closed orbit, a repelling closed orbit and a *nontrivial* saddle basic set: a saddle basic set containing infinitely many closed orbits. In [25], Yu showed that every closed orientable 3-manifold admits a simple Smale flow.

Previous studies on simple Smale flows are focused on the templates related to saddle basic sets, including [1; 13; 20; 21; 22] by Adhikari, Haynes, Sullivan, and Yu. Utilizing Theorem 1.3, we can approach this topic from a different perspective (the germs of saddle basic sets) and obtain a surprising result (Corollary 1.4): there exist infinitely many simple Smale flows on S^3 with the same basic sets but different global dynamics.

Corollary 1.4 *There are infinitely many simple Smale flows* $\{\phi_t^i\}_{i\in\tau}$ *on* S^3 *up to topological equivalence, such that*

(1) *for any* $i \in \tau$, $A_i \sqcup R_i$ *is a Hopf link (see Figure 1)*;

(2) *for any* $j \in \tau$, $[\phi_t^i, \Lambda_i] = [\phi_t^j, \Lambda_j]$.

Here, A_i, R_i *and* Λ_i *are the attracting closed orbit, the repelling closed orbit and the saddle basic set of* ϕ_t^i, *respectively.*

Remark 1.5 A paper in preparation by Fan, Lai, and Yu shows that the Whitehead link exterior cannot be the background manifold of a hyperbolic plug (personal communication, 2024). Therefore, the Whitehead link cannot be realized as the attractor and repeller of a simple Smale flow on S^3.

Remark 1.6 If we replace the word "nontrivial" in the definition of simple Smale flows with "trivial," then simple Smale flows will transform into Morse–Smale flows with three basic sets. For certain classes of 3-manifolds, complete classifications of Morse–Smale systems have already been achieved, for instance:

- In [24], Yu proved that up to topological equivalence, there are at most a finite number of nonsingular Morse–Smale flows on S^3 with three closed orbits. However, due to the complexity of heteroclinic trajectories, there are infinitely many nonsingular Morse–Smale flows on S^3 with four closed orbits up to topological equivalence.
- Pochinka and Shubin introduced an invariant to determine the topological equivalence classes of nonsingular Morse–Smale flows on orientable 3-manifolds with three closed orbits [18].
- The problem of a topological classification of Morse–Smale cascades on 3-manifolds either without heteroclinic points or without heteroclinic curves was solved in [5; 6; 7; 8; 10] by Bonatti, Grines, Laudenbach, Medvedev, Pécou, and Pochinka.

1.2 The key ideas

At the 2006 International Congress of Mathematicians, Ghys constructed a compact hyperbolic orbifold Σ by deforming the modular surface [11]. Bonatti and Pinsky extended the geodesic flow g_t on the unit tangent bundle $T^1\Sigma$ of Σ into a Smale flow ϕ_t on S^3 [9]. Let Λ be the saddle basic set of ϕ_t related to g_t. A hyperbolic plug (N, η_t) is called a *hyperbolic plug for* Λ if the germ of η_t along the maximal invariant set coincides with the germ $[\phi_t, \Lambda]$. The key idea of this paper is to construct infinitely many homeomorphic hyperbolic plugs for Λ, such that no two plugs can be topologically equivalent.

In [9], Bonatti and Pinsky constructed a hyperbolic plug (W, ψ_t) for Λ where $W \cong S^2 \times [0,1]$. Let Λ_0 be the maximal invariant set of ψ_t and $\partial^+ W$ be the connected components of ∂W on which ψ_t points inward. We will prove that $\partial^+ W \setminus W^s(\Lambda_0)$ consists of infinitely many connected components. By performing a surgery-type operation along the connected components of $\partial^+ W \setminus W^s(\Lambda_0)$, we will produce infinitely many homeomorphic hyperbolic plugs for Λ. By using hyperbolic geometry, we will prove that the topological equivalence of these plugs distinguishes between the different connected components of $\partial^+ W \setminus W^s(\Lambda_0)$ on which we performed surgery, thus proving Theorem 1.3.

1.3 Organization of the paper

In Section 2, we introduce some definitions and elementary properties of models for saddle basic sets. In Section 3, we define Ghys' compact orbifold Σ from the modular surface and discuss the geodesic flow g_t on the unit tangent bundle $T^1\Sigma$ of Σ. In Section 4, we discuss the mapping class group of $T^1\Sigma$. In Section 5, we prove Theorem 1.3. In Section 6, we introduce some interesting open questions.

2 Preliminaries

2.1 Models and templates

Let ϕ_t be a Smale flow on a closed orientable 3-manifold and Λ be a nontrivial saddle basic set of ϕ_t.

Definition 2.1 A *hyperbolic plug for* Λ is a hyperbolic plug (W, ψ_t) such that $[\psi_t, \Lambda_0] = [\phi_t, \Lambda]$ where Λ_0 is the maximal invariant set of ψ_t.

To describe the dynamical behavior of Λ, Béguin and Bonatti defined the following concept in [2].

Definition 2.2 A *model* for Λ is a hyperbolic plug (W, ψ_t) for Λ, such that if Λ_0 is the maximal invariant set of ψ_t then

(1) any embedded circle in $\partial^+ W \setminus W^s(\Lambda_0)$ bounds a disk in $\partial^+ W \setminus W^s(\Lambda_0)$, where $\partial^+ W$ denotes the connected components of ∂W on which ψ_t points inward;

(2) any connected component of W contains at least one point of Λ_0.

The following theorem was proved by Béguin and Bonatti in [2, Theorem 0.3].

Theorem 2.3 *There exists a unique model for* Λ *up to topological equivalence.*

Let (W, ψ_t) be a model for Λ and Λ_0 be the maximal invariant set of ψ_t. In [2], Béguin and Bonatti introduced the following method to construct hyperbolic plugs for Λ. Let D_1 and D_2 be two disjoint closed disks in $\partial^+ W \setminus W^s(\Lambda_0)$. Since the orbit of any point in $\partial^+ W \setminus W^s(\Lambda_0)$ goes from the entrance to the exit boundary, the orbits of D_1, D_2 form two cylinders $D^2 \times [0, 1]$ endowed with the vector field $\frac{\partial}{\partial t}$. Remove these two cylinders from W and then glue the two resulting tangent boundary annuli together by preserving flowlines. By doing so, we obtain a new hyperbolic plug for Λ (see Figure 2). This dynamical surgery is called *handle attachment*.

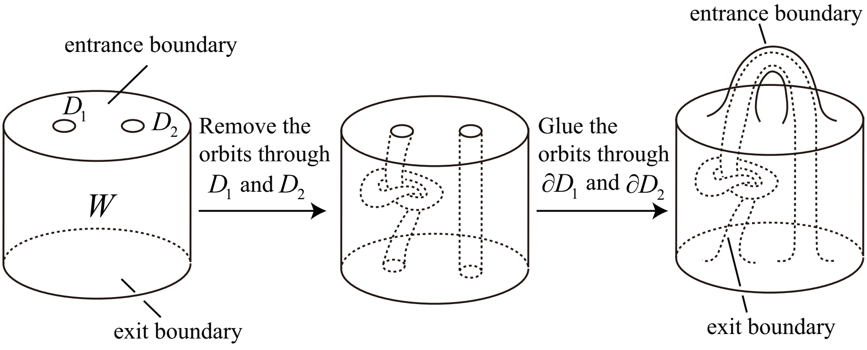

Figure 2: Handle attachment.

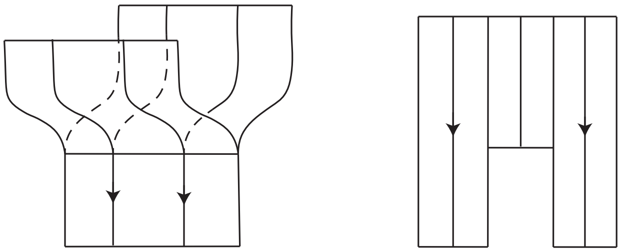

Figure 3: Joining and splitting charts.

Definition 2.4 A *template* is a compact branched 2-manifold with boundary and smooth expansive semiflow built locally from two types of charts: *joining* and *splitting* (see Figure 3). The gluing maps between charts must respect the semiflow and act linearly on the edges.

In [4], Birman and Williams proved that collapsing the strong stable manifolds of a suitable neighborhood of Λ yields a template. By reversing the construction of Birman and Williams, we extend a template T in the direction perpendicular to its surface to obtain a *thickened template* $\overline{T}$. Since the semiflow on T is expanding, we can extend the semiflow on each chart, and produce thickened charts as in Figure 4. Figure 5 provides a sectional view of thickened charts. See Meleshuk [15] for more details on thickened templates.

A *dividing curve* of $\overline{T}$ is a closed curve c in $\partial\overline{T}$ such that the natural flow on $\overline{T}$ is tangent to $\partial\overline{T}$ at the curve c. For the thickened charts in Figure 4, the dividing curves correspond to the bold curves. We can attach a 2-handle to a dividing curve c of $\overline{T}$ as follows: First, we blow up c to obtain $c \times [0, 1]$ and endow $c \times [0, 1]$ with the vector field $\frac{\partial}{\partial t}$. Next, we glue a 2-handle $D^2 \times [0, 1]$ with the vector field $\frac{\partial}{\partial t}$ to $c \times [0, 1]$ by

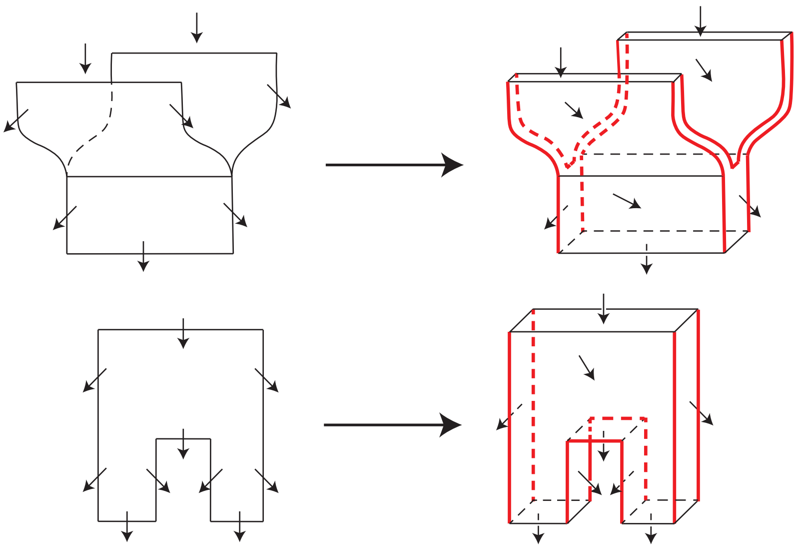

Figure 4: Thickened charts. Top: joining chart thickening. Bottom: splitting chart thickening.

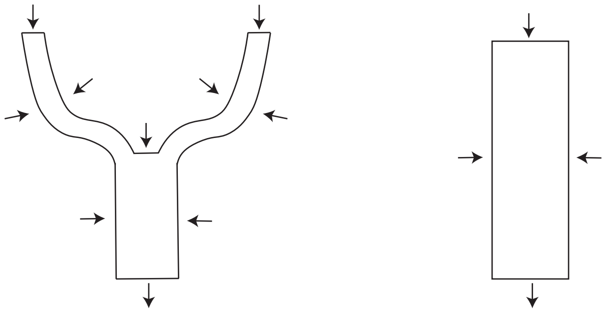

Figure 5: Sectional view of thickened charts.

preserving the flowlines. Figure 6 provides a sectional view of the attachment of a 2-handle. The following theorem of Yu [23, Theorem 4.3], indicates the relationship between thickened templates and models.

Theorem 2.5 *Let $\overline{T}$ be a thickened template related to Λ. Then up to topological equivalence, the model for Λ can be obtained by attaching a 2-handle to each dividing curve of $\overline{T}$.*

2.2 The entrance boundary of models

Suppose that Λ is a nontrivial saddle basic set of a Smale flow ϕ_t on a closed orientable 3-manifold. Let (W, ψ_t) be a model for Λ and Λ_0 be the maximal invariant set of ψ_t.

Definition 2.6 A *free stable separatrix* of a closed orbit $O \subset \Lambda_0$ is a connected component of $W^s(O) \setminus O$ that is disjoint from Λ_0.

The following lemma was proved by Béguin, Bonatti, and Yu in [3, Remark 3.5 and Proposition 3.8].

Lemma 2.7 (1) *For each connected component C of $W^s(\Lambda_0) \setminus \Lambda_0$, there are two possible situations:*

- *Either C is a free stable separatrix of some closed orbit. Then $C \cap \partial^+ W$ is homeomorphic to a circle.*
- *Or C is bounded by two orbits in Λ_0. Then $C \cap \partial^+ W$ is homeomorphic to a line.*

(2) *In $W^s(\Lambda_0) \cap \partial^+ W$, each half line is asymptotic to a circle.*

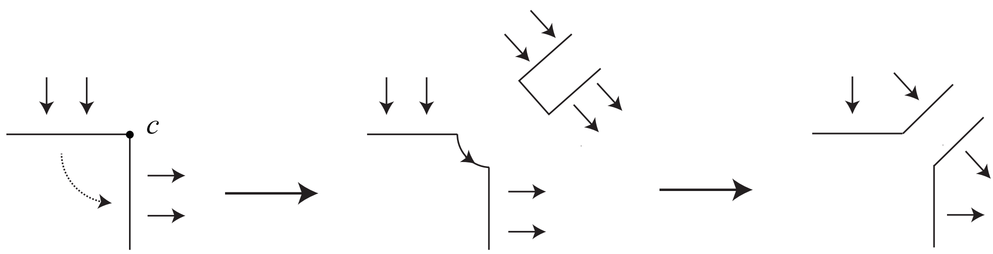

Figure 6: Attaching a 2-handle to a dividing curve c.

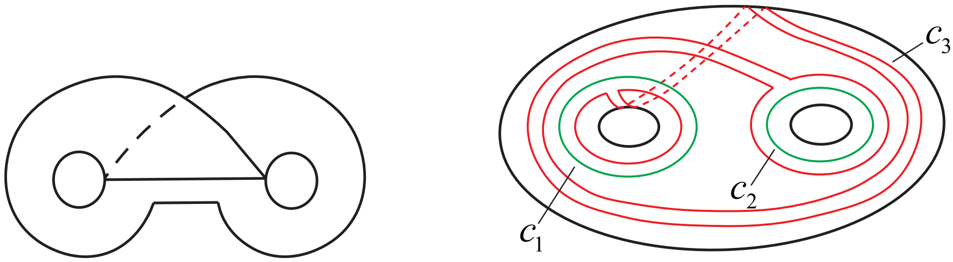

Figure 7: Left: Lorenz template. Right: thickened Lorenz template.

In this paper, a 2-manifold homeomorphic to $\mathbb{R}^2$ is called an *open disk* if its accessible boundary is a circle. A 2-manifold homeomorphic to $\mathbb{R}^2$ is called a *strip* if its accessible boundary consists of exactly two lines which are asymptotic to each other at both ends.

Proposition 2.8 *Suppose that the template related to* Λ *is a Lorenz template* T (*see* Figure 7). *Then*

(1) $W \cong S^2 \times [0,1]$;

(2) $\partial^+ W \cap W^s(\Lambda_0)$ *consists of two circles and infinitely many lines*;

(3) $\partial^+ W \setminus W^s(\Lambda_0)$ *consists of two open disks and infinitely many strips.*

Proof The thickened Lorenz template $\overline{T}$ has three dividing curves c_1, c_2, c_3, as shown in Figure 7, right. By Theorem 2.5, we obtain a model (W^T, ψ_t^T) from $\overline{T}$ by attaching three 2-handles to the dividing curves. By attaching two 2-handles to c_1, c_2, the 3-ball is obtained. Thus, W^T is homeomorphic to $S^2 \times [0,1]$.

Let Λ_0^T be the maximal invariant set of ψ_t^T. Bonatti and Pinsky showed in [9] that Λ_0^T has a section with a first return map Γ, called the fake horseshoe map (see Figure 8). The map Γ was introduced by Smale in [19]. Utilizing Markov partitions and symbolic dynamics, we find that there are only two closed orbits in Λ_0^T admitting a free stable separatrix, which correspond to the two fixed points p_1, p_2 of Γ. By Lemma 2.7, $\partial^+ W^T \cap W^s(\Lambda_0^T)$ consists of two circles and infinitely many lines, where each half line is asymptotic to a circle. Therefore, $\partial^+ W^T \setminus W^s(\Lambda_0^T)$ consists of two open disks and infinitely many strips. By Theorem 2.3, (W^T, ψ_t^T) is topologically equivalent to (W, ψ_t). □

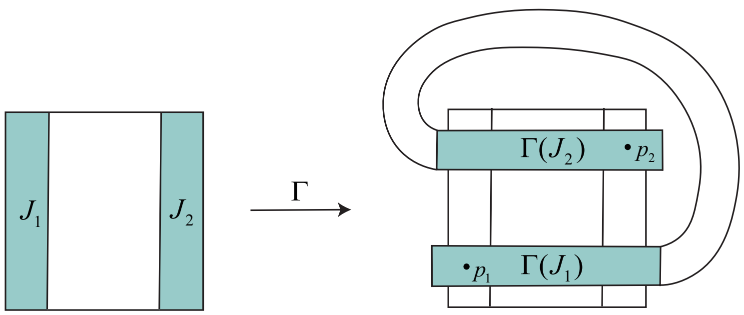

Figure 8: Fake horseshoe map.

3 The geodesic flow of the modular surface

Recall that the group $\mathrm{PSL}(2,\mathbb{Z})$ is generated by

$$U = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad V = \pm\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

As in the discussion of Ghys in [11], we choose two points in the Poincaré disk at distance $\rho > 0$, and take U_ρ to be the rotation of angle π around the one and V_ρ to be the rotation of angle $\frac{2\pi}{3}$ around the other. Recall that the group of positive isometries of the Poincaré disk is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. Thus, we define a homomorphism $i_\rho\colon \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{R})$, such that $i_\rho(U) = U_\rho$ and $i_\rho(V) = V_\rho$. If ρ_0 is corresponding to the hyperbolic distance between $\sqrt{-1}$ and $\frac{1}{2}\big(-1+\sqrt{-3}\big)$ in Poincaré's upper half plane, then we call the orbifold $\mathbb{H}^2/i_{\rho_0}\mathrm{PSL}(2,\mathbb{Z})$ the *modular surface*, which admits two cone points and a cusp (see Figure 9, left).

Now, we choose a distance $\rho > \rho_0$. Then $\mathbb{H}^2/i_\rho\mathrm{PSL}(2,\mathbb{Z})$ is a noncompact orbifold with a "funnel". By cutting the orbifold $\mathbb{H}^2/i_\rho\mathrm{PSL}(2,\mathbb{Z})$ along the geodesic γ of Figure 9, right, we get a compact orbifold Σ. In fact, the geodesic flow g_t of Σ contains the same chain recurrent set as the geodesic flow of $\mathbb{H}^2/i_\rho\mathrm{PSL}(2,\mathbb{Z})$.

Let $M = T^1\Sigma$, the unit tangent bundle of Σ. See Montesinos [17] for the definition of the unit tangent bundle of orbifolds. M is the complement of the trefoil ξ in S^3 (see Milnor [16]). Let α_1 and α_2 be the orbits of g_t (endowed with the natural orientation defined by g_t) corresponding to γ and $-\gamma$, respectively. In [11], Ghys proved that α_1 is a meridian of ξ, and α_2 is isotopic to $-\alpha_1$. Due to the hyperbolicity of the geodesic flow of $\mathbb{H}^2$, g_t is a Smale flow on M with one nontrivial saddle basic set Λ. In fact, Λ is the maximal invariant set of g_t.

It is easy to observe that g_t is tangent to α_1, α_2, and that $\partial M \setminus (\alpha_1 \sqcup \alpha_2)$ consists of two open annuli A^+, A^-. Here g_t points inward on A^+ and points outward on A^-. In the following lines, we will explain how one can distinguish the connected components of $A^+ \setminus W^s(\Lambda)$.

For the convenience of description, we provide the following definitions. Let l_0, l_1 be two orbits of g_t through $A^+ \setminus W^s(\Lambda)$, then $l_i \cap A^- \neq \varnothing$ for $i = 0, 1$. A topological embedding $L_i\colon [0,1] \to M$ is called

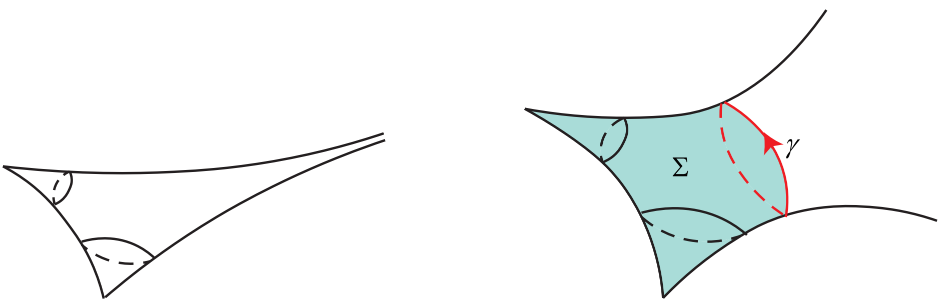

Figure 9: Deforming the modular surface.

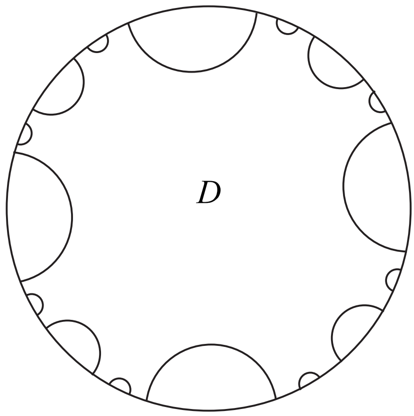

Figure 10: The universal cover of Σ.

an *orbit-path* of l_i if the image of L_i is the orbit l_i and $L_i(0) \in A^+ \setminus W^s(\Lambda)$. We say that l_0 *is homotopic to* l_1 *preserving* ∂M if there is a continuous map $H: [0,1] \times [0,1] \to M$ such that H_0 (resp H_1) is an orbit-path of l_0 (resp l_1), and $H_s(0), H_s(1) \in \partial M$ where $H_s = H(\cdot, s)$ for any s.

Lemma 3.1 *Let l_0, l_1 be two orbits of g_t through $A^+ \setminus W^s(\Lambda)$ (endowed with the natural orientations defined by g_t). The starting points of l_0 and l_1 are connected in $A^+ \setminus W^s(\Lambda)$ if and only if l_0 is homotopic to l_1 preserving ∂M.*

Proof **Necessity** Suppose that the starting points of l_0 and l_1 are connected in $A^+ \setminus W^s(\Lambda)$. We can connect a path in $A^+ \setminus W^s(\Lambda)$ from the starting point of l_0 to the starting point of l_1. Note that each positive orbit through this path can reach A^-. Therefore, along the orbits through this path, we obtain that l_0 is homotopic to l_1 preserving ∂M.

Sufficiency Suppose that l_0 is homotopic to l_1 preserving ∂M. Then there is a continuous map $H: [0,1] \times [0,1] \to M$ such that H_0 (resp H_1) is an orbit-path of l_0 (resp l_1), and $H_s(0), H_s(1) \in \partial M$ where $H_s = H(\cdot, s)$ for any s.

Recall that Σ is obtained from a noncompact orbifold $\mathbb{H}^2 / i_\rho \mathrm{PSL}(2,\mathbb{Z})$ by cutting the funnel, where $\rho > \rho_0$. Then we get the universal cover D of the orbifold Σ by removing infinitely many disjoint open semidisks from the Poincaré's disk, as shown in Figure 10. The orbifold-covering map $p: D \to D / i_\rho \mathrm{PSL}(2,\mathbb{Z}) = \Sigma$ is defined by $a \mapsto [a]$. Here, $[a'] = [a]$ if and only if there is an action $B \in i_\rho \mathrm{PSL}(2,\mathbb{Z})$ such that $B(a) = a'$. Obviously, p is a branched covering map.

Define a map $\tilde{p}: T^1 D \to T^1 D / i_\rho \mathrm{PSL}(2,\mathbb{Z}) = M$ such that $(a, v) \mapsto [(a, v)]$. Here, $[(a', v')] = [(a, v)]$ if and only if there is an action $B \in i_\rho \mathrm{PSL}(2,\mathbb{Z})$ such that $B(a) = a'$ and $dB_a(v) = v'$. It is easy to prove that $\tilde{p}$ is a covering map. Let $\pi: M \to \Sigma$ and $\tilde{\pi}: T^1 D \to D$ be the projections of bundles. Then we get the commutative diagram

$$\begin{array}{ccc} T^1 D & \xrightarrow{\ \widetilde{p}\ } & M \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow[\ p\ ]{} & \Sigma \end{array}$$

Let $\widetilde{g}_t$ be the geodesic flow of D. According to the definition of the geodesic flow g_t,

$$\widetilde{p} \circ \widetilde{g}_t((a, v)) = g_t([(a, v)])$$

for any $(a, v) \in T^1 D$ and any t. Then each lift of H_i is an orbit-path in $T^1 D$ for $i = 0, 1$. Let $[(a_i, v_i)] = H_i(0)$, and $\widetilde{H}_0$ be the lift of H_0 starting at (a_0, v_0). By the homotopy lifting property (see Hatcher [12]), there is a homotopy $\widetilde{H}_s \colon [0, 1] \to \mathrm{T}^1 D$ of $\widetilde{H}_0$ lifts H_s $(s \in [0, 1])$. Without loss of generality, we assume that $\widetilde{H}_1(0) = (a_1, v_1)$.

Using the commutative diagram, $\widetilde{\pi} \circ \widetilde{H}_s(0)$ and $\widetilde{\pi} \circ \widetilde{H}_s(1)$ are two path in $p^{-1}(\partial\Sigma)$. It is not difficult to observe that there is a path α in $\widetilde{p}^{-1}(\partial M)$ from (a_0, v_0) to (a_1, v_1), such that the positive orbits of $\widetilde{g}_t$ through α can reach $\widetilde{p}^{-1}(\partial M)$. Then we get a path $\widetilde{p} \circ \alpha$ in ∂M from $[(a_0, v_0)]$ to $[(a_1, v_1)]$, such that the positive orbits of g_t through $\widetilde{p} \circ \alpha$ can reach ∂M. Namely, $[(a_0, v_0)]$ and $[(a_1, v_1)]$ are connected in $A^+ \setminus W^s(\Lambda)$. □

4 The mapping class group of M

In the following section, we define M, g_t, Σ, α_1, α_2, A^+ as in Section 3. Recall that α_1 is a meridian of ∂M and it is isotopic to $-\alpha_2$. Fix an orientation on M. This induces an orientation on ∂M. We choose a circle l in ∂M intersecting once α_1 and α_2. Then l is a longitude of ∂M. Choose an orientation for l such that (α_1, l) is positively oriented on ∂M. Without losing generality, we assume that the positive direction of $l \cap A^+$ is from α_1 to α_2, as shown in Figure 11.

Let h be a topological equivalence from (M, g_t) to (M, g_t). Then $h(\alpha_1 \sqcup \alpha_2) = \alpha_1 \sqcup \alpha_2$ and $h(A^+) = A^+$. If $h(\alpha_1) = \alpha_1$, then $h(l) \simeq x\alpha_1 + l$. Thus the action of $h|_{\partial M}$ on the homology group of ∂M corresponds

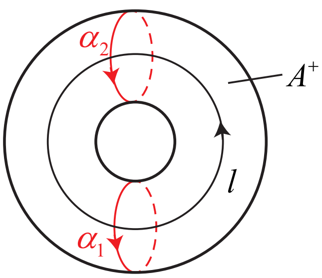

Figure 11: The positive direction of $l \cap A^+$ is from α_1 to α_2.

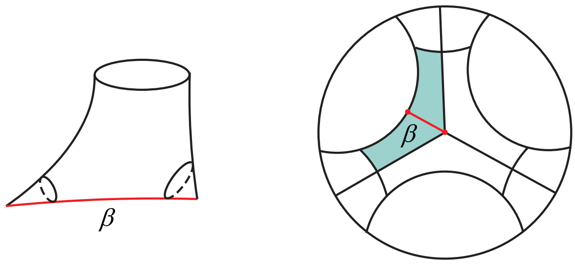

Figure 12: The orbifold Σ and the three fold cover of its fundamental domain (shaded area).

to the matrix $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ for some $x \in \mathbb{Z}$. Similarly, if $h(\alpha_1) = \alpha_2$, then the action of $h|_{\partial M}$ on the homology group of ∂M corresponds to the matrix $\left(\begin{smallmatrix}-1 & y\\ 0 & -1\end{smallmatrix}\right)$ for some $y \in \mathbb{Z}$. Therefore, $h|_{\partial M}$ preserves the induced orientation of ∂M, which implies that h preserves the orientation of M.

Next, let us prove that the image by h of α_1 determines the isotopy class of h. Denote by $\mathrm{Mod}^+(M)$ *the mapping class group* of M, ie, the group of isotopy classes of the orientation-preserving homeomorphisms of M. For the orbifold Σ, let $\mathrm{Mod}^{\pm}(\Sigma)$ be the group of homeomorphisms of Σ fixing the singular points, modulo isotopies fixing the singular points. Then $\mathrm{Mod}^{\pm}(\Sigma) \cong \mathbb{Z}_2$.

Johannson [14, Proposition 25.3] proposed a short exact sequence

$$1 \to \mathrm{H}_1(\Sigma, \partial\Sigma) \to \mathrm{Mod}^+(M) \to \mathrm{Mod}^{\pm}(\Sigma) \to 1.$$

Here, $\mathrm{H}_1(\Sigma, \partial\Sigma)$ denotes the first relative homology group, then it is a trivial group. This indicates that $\mathrm{Mod}^+(M) \cong \mathrm{Mod}^{\pm}(\Sigma) \cong \mathbb{Z}_2$.

Let β be a geodesic of Σ connecting the two singular points; see Figure 12. Let f be the symmetric map on Σ about β. Obviously, f is an isometry, which implies that the map $F = df : M \to M$ induced by f is a topological equivalence from (M, g_t) to (M, g_t). Thus, F preserves the orientation of M. In addition, $F(\alpha_1) = \alpha_2$ is isotopic to $-\alpha_1$, then F cannot be isotopic to id_M. Therefore, $\mathrm{Mod}^+(M) = \{[\mathrm{id}_M], [F]\}$. Then, we get the following proposition.

Proposition 4.1 *Any topological equivalence h from (M, g_t) to (M, g_t) is isotopic to either id_M or F, according to either $h(\alpha_1) = \alpha_1$ or $h(\alpha_1) = \alpha_2$.*

5 Proof of Theorem 1.3

Recall the definitions of (M, g_t), Λ, α_1 and α_2 in Section 3. In ∂M, g_t is tangent to $\alpha_1 \sqcup \alpha_2$, points inward on A^+, and points outward on A^-. Here, A^+ and A^- are two open annuli which are the connected components of $\partial M \setminus (\alpha_1 \sqcup \alpha_2)$. In [9], Bonatti and Pinsky extended the flow g_t into a Smale flow ϕ_t on S^3, as follows.

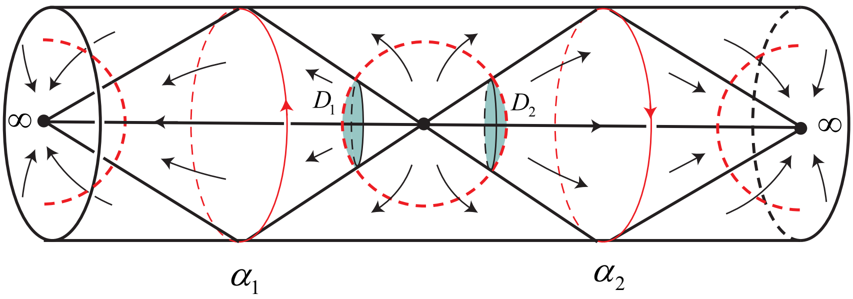

Figure 13: The solid torus is cut along a disk containing ∞. The cone-like disks are the stable and the unstable manifolds of the tangent orbits α_1, α_2. In the diamond shape regions between the stable and the unstable manifolds of α_1, α_2, each point in the past tends to the source and in the future to the sink.

(1) Construct a flow on a solid torus as shown in Figure 13, whose chain recurrent set consists of a source, a sink and two tangent orbits α_1, α_2.

(2) By suitably gluing the solid torus to M, we get a Smale flow ϕ_t on S^3, whose chain recurrent set consists of Λ, a source and a sink.

Then Λ is the saddle basic set of ϕ_t. By digging up two open ball neighborhoods of the source and the sink in the solid torus, we get a model (W, ψ_t) for Λ. Obviously, g_t is the restriction of ψ_t to $M \subset W$, and the maximal invariant set of ψ_t is also Λ. In the following, we denote by $W^s(\Lambda, W)$ the union of stable manifolds of the orbits in Λ for the system (W, ψ_t), and denote by $W^s(\Lambda, M)$ the union of stable manifolds of the orbits in Λ for the system (M, g_t).

In [11], Ghys proved that the template related to Λ is a Lorenz template. By Proposition 2.8 and Theorem 2.3, $W \cong S^2 \times [0,1]$ and $\partial^+ W \setminus W^s(\Lambda, W)$ consists of two open disks D_1, D_2 and infinitely many strips, where D_1 and D_2 are shown in Figure 13. Let $\mathcal{S} = \partial^+ W \setminus (W^s(\Lambda, W) \sqcup D_1 \sqcup D_2)$, and $\mathcal{A} = A^+ \setminus W^s(\Lambda, M)$. Along to the orbits of ψ_t, we can construct a homeomorphism $\mu\colon \mathcal{S} \to \mathcal{A}$ such that $\mu(x)$ and x lie in a same orbit of ψ_t for each $x \in \mathcal{S}$. Recall that in Section 4, we defined a topological equivalence F from (M, g_t) to (M, g_t). From now on, we distinguish the connected components of $\partial^+ W \setminus W^s(\Lambda, W)$.

Lemma 5.1 *Let h be a topological equivalence from (W, ψ_t) to (W, ψ_t). Then there are two possible situations*:

(1) *$h(D_1) = D_1, h(D_2) = D_2$, and $h(C) = C$ for each connected component C of $\mathcal{S}$.*

(2) *$h(D_1) = D_2, h(D_2) = D_1$, and $h(C) = \mu^{-1} \circ F \circ \mu(C)$.*

Proof Recall that $\partial^+ W \setminus W^s(\Lambda, W)$ consists of two open disks D_1, D_2 and infinitely many strips, where D_i is bounded by a circle in $W^s(\alpha_i, W)$ for $i = 1, 2$. Then $h(D_1 \sqcup D_2) = D_1 \sqcup D_2$ and $h(\alpha_1 \sqcup \alpha_2) = \alpha_1 \sqcup \alpha_2$.

It is easy to observe that the set of orbits crossing A^+ (resp A^-) coincides with the set of orbits crossing $h(A^+)$ (resp $h(A^-)$). Then up to isotopy along the flowlines, $h(\partial M) = \partial M$. This implies that h induces a topological equivalence $h'\colon (M, g_t) \to (M, g_t)$ such that $h'(\alpha_1) = h(\alpha_1)$, $h'(\alpha_2) = h(\alpha_2)$, and $h' \circ \mu(C) = \mu \circ h(C)$ where C is a connected component of $\mathcal{S}$.

Let l be an orbit of g_t through $\mu(C)$ (endowed with the natural orientations defined by g_t). Since μ is a homeomorphism, $\mu(C)$ is a connected component of $\mathcal{A}$.

Case 1 ($h(D_1) = D_1$ and $h(D_2) = D_2$) Then $h'(\alpha_1) = \alpha_1$ and $h'(\alpha_2) = \alpha_2$. By Proposition 4.1, h' is isotopic to id_M. Then $h'(l)$ is homotopic to l preserving ∂M. By Lemma 3.1, the starting points of $h'(l)$ and l are connected in $\mathcal{A}$. Hence, $h' \circ \mu(C) = \mu(C)$ and $h(C) = C$.

Case 2 ($h(D_1) = D_2$ and $h(D_2) = D_1$) Then $h'(\alpha_1) = \alpha_2$ and $h'(\alpha_2) = \alpha_1$. By Proposition 4.1, h' is isotopic to F. Then $h'(l)$ is homotopic to $F(l)$ preserving ∂M. By Lemma 3.1, the starting points of $h'(l)$ and $F(l)$ are connected in $\mathcal{A}$. Hence, $h' \circ \mu(C) = F \circ \mu(C)$ and $h(C) = \mu^{-1} \circ F \circ \mu(C)$. □

Proof of Theorem 1.3 Let $\{C_i\}_{i\in\tau}$ be an infinite set of connected components of $\mathcal{S}$ such that if $C \in \{C_i\}_{i\in\tau}$, then $\mu^{-1} \circ F \circ \mu(C) \notin \{C_i\}_{i\in\tau}$. For each $i \in \tau$, choose two disjoint closed disks in C_i and then do handle attachment for (W, ψ_t) along them. We obtain a manifold N_i endowed with a flow ψ_t^i, such that (N_i, ψ_t^i) is a hyperbolic plug for Λ. Since $W \cong S^2 \times [0,1]$ and the orbits through C_i are isotopic, we have $N_i \cong T^2 \times [0,1]$.

Let Λ_i be the maximal invariant set of ψ_t^i. Then Λ_i is a 1-dimensional saddle basic set containing infinitely many closed orbits. Moreover, for any $j \in \tau$ and $j \neq i$, we have $[\psi_t^i, \Lambda_i] = [\psi_t^j, \Lambda_j]$. Now, we prove that ψ_t^i is not topologically equivalent to ψ_t^j by contradiction.

Suppose that there are two different elements $i_0, j_0 \in \tau$, such that $\psi_t^{i_0}$ is topologically equivalent to $\psi_t^{j_0}$ via a homeomorphism $h_0\colon (N_{i_0}, \psi_t^{i_0}) \to (N_{j_0}, \psi_t^{j_0})$. By our construction and the fact that ∂W consists of two spheres, there is only one component T_{i_0} (resp T_{j_0}) of $\partial^+ N_{i_0} \setminus W^s(\Lambda_{i_0})$ (resp $\partial^+ N_{j_0} \setminus W^s(\Lambda_{j_0})$) with genus one. Since h_0 is a topological equivalence, $h_0(T_{i_0}) = T_{j_0}$, which implies that h_0 sends an inseparable circle c_{i_0} of T_{i_0} (that is, $T_{i_0} \setminus c_{i_0}$ is connected) to an inseparable circle c_{j_0} of T_{j_0}. We cut N_{i_0} (resp N_{j_0}) along the orbits through c_{i_0} (resp c_{j_0}) and then glue two cylinders $D^2 \times [0,1]$ endowed with the vector field $\frac{\partial}{\partial t}$ by preserving the oriented orbits. Then, we get the system (W, ψ_t) up to topological equivalence. Thanks to h_0, we can construct a topological equivalence h_0' from (W, ψ_t) to (W, ψ_t) such that $h_0'(C_{i_0}) = C_{j_0}$. By Lemma 5.1, $C_{j_0} = C_{i_0}$ or $\mu^{-1} \circ F \circ \mu(C_{i_0})$, which contradicts the definition of $\{C_i\}_{i\in\tau}$. Therefore, ψ_t^i cannot be topologically equivalent to ψ_t^j. □

Proof of Corollary 1.4 Let η_t^a be the flow on $N' = S^1 \times D^2$ induced by the vector field

$$X(\theta, z) = \frac{\partial}{\partial\theta} - z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}.$$

Here, z_1, z_2 denote the standard coordinate functions on D^2 and $\frac{\partial}{\partial\theta}$ is a vector field along the factor S^1. Let N'' be a second copy of N' equipped with the flow η_t^r induced by $-X$. By suitably attaching (N', η_t^a) and (N'', η_t^r) to (N_i, ψ_t^i) for each $i \in \tau$, we can construct infinitely many simple Smale flows on S^3 that satisfy the conclusion of Corollary 1.4. □

6 Future research

In this paper, utilizing the theory on models of Béguin and Bonatti [2], we constructed infinitely many hyperbolic plugs that are homeomorphic to the Hopf link exterior, such that no two plugs can be topologically equivalent. This prompts a problem: can we do the same trick for the hyperbolic plugs whose background manifolds are other link exteriors?

The forthcoming work by Fan, Lai, and Yu presents a significant finding that the Whitehead link exterior cannot be the background manifold of a hyperbolic plug. Then, a potential starting point for this inquiry could involve investigating which two-component link can be realized as the attractor and repeller of a simple Smale flow on S^3, along with determining the model for the corresponding saddle basic set.

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