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# The $\mathbb{S}_n$-equivariant Euler characteristic of the moduli space of graphs

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We prove a formula for the $\mathbb{S}_n$-equivariant Euler characteristic of the moduli space of graphs $\mathcal{MG}_{g,n}$. Moreover, we prove that the rational $\mathbb{S}_n$-invariant cohomology of $\mathcal{MG}_{g,n}$ stabilizes for large $n$. That means, if $n \geq g \geq 2$, then there are isomorphisms $H^k(\mathcal{MG}_{g,n};\mathbb{Q})^{\mathbb{S}_n} \to H^k(\mathcal{MG}_{g,n+1};\mathbb{Q})^{\mathbb{S}_{n+1}}$ for all $k$.

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## 1 Introduction

A graph $G$ is a finite, at most one-dimensional CW complex. It has *rank* $g$ if its fundamental group is a free group of rank $g$: $\pi_1(G) \cong F_g$. Here, graphs shall be *admissible*, that means they do not have vertices of degree 0 or 2. Univalent vertices have a special role and are called *legs* (often also *hairs*, *leaves* or *marked points*). Similarly, we will reserve the name *edge* to 1-cells that are only incident to nonleg vertices. Legs of graphs are uniquely labeled by integers $\{1,\dots,n\}$.

A *metric graph* is additionally equipped with a length $\ell(e) \geq 0$ for each edge $e$ such that all edge lengths sum to one, $\sum_e \ell(e) = 1$. Fix $g, n$ such that $g > 0$ and $2g-2+n > 0$. The moduli space of graphs, $\mathcal{MG}_{g,n}$, is the space of isometry classes of metric graphs of rank $g$ with $n$ legs, except for graphs that have a cycle in which all edges have length 0. It inherits the topology from the metric and by identifying each graph that has an edge $e$ of length 0 with the respective graph where $e$ is collapsed.

The moduli space of graphs was introduced in [15], where it was shown that $\mathcal{MG}_{g,0}$ serves as a rational classifying space for $\mathrm{Out}(F_g)$, the outer automorphism group of the free group of rank $g$. Further, $\mathcal{MG}_{g,n}$ can be seen as the moduli space of *pure tropical curves* [1] and it is relevant for *Feynman amplitudes*, central objects in quantum field theory [5].

A partition $\lambda \vdash n$ gives rise to both an irreducible representation $V_\lambda$ of the symmetric group $\mathbb{S}_n$ and a Schur polynomial $s_\lambda$, a symmetric polynomial in $\Lambda_n = \mathbb{Q}[x_1,\dots,x_n]^{\mathbb{S}_n}$. The symmetric group acts on the cohomology of $\mathcal{MG}_{g,n}$ by permuting the leg-labels. So, $H^k(\mathcal{MG}_{g,n};\mathbb{Q})$ is an $\mathbb{S}_n$-representation that we can decompose into irreducibles, ie there are integers $c^k_{g,\lambda}$ such that

$$H^k(\mathcal{MG}_{g,n};\mathbb{Q}) \cong \bigoplus_{\lambda\vdash n} c^k_{g,\lambda} V_\lambda.$$

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The multiplicities $c^k_{g,\lambda}$ are known explicitly if $g \leq 2$ [13]. The $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{MG}_{g,n}$ is the symmetric polynomial

$$e_{\mathbb{S}_n}(\mathcal{MG}_{g,n}) = \sum_{\lambda \vdash n} s_\lambda \sum_k (-1)^k c^k_{g,\lambda}. \tag{1}$$

Our first main result is an effective formula for $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$. It is stated as Theorem 2.18. Its proof in Section 2 is based on prior work by Borinsky and Vogtmann [8].

Analogous formulas exist, eg, for the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ [17] and for the $\mathbb{S}_n$-equivariant Euler characteristic of the moduli space of stable tropical curves [10]. The latter moduli space is a compactification of $\mathcal{MG}_{g,n}$ and its cohomology injects into the cohomology of $\mathcal{M}_{g,n}$ [11].

There is another important invariant of moduli spaces such as $\mathcal{MG}_{g,n}$, which is in general a rational number: the *virtual* Euler characteristic. It has more convenient properties while studying maps between moduli spaces than the *classical* Euler characteristic, the alternating sum of the Betti numbers. The virtual Euler characteristic of $\mathcal{MG}_{g,n}$ was studied, for instance, in [7]. The quantitative relation between the virtual and classical Euler characteristic of $\mathcal{MG}_{g,n}$ is discussed in Section 2.8.

To obtain the classical Euler characteristic, $e(\mathcal{MG}_{g,n}) = \sum_k (-1)^k \dim H^k(\mathcal{MG}_{g,n};\mathbb{Q})$, we replace $s_\lambda$ with $\dim V_\lambda$ in (1). Another specialization of $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ is the Euler characteristic for the $\mathbb{S}_n$-*invariant* part of $\mathcal{MG}_{g,n}$'s cohomology, denoted as $e(\mathcal{MG}^{\mathbb{S}_n}_{g,n})$. We obtain it from $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ by setting $s_{(n)} = 1$, (which corresponds to the trivial representation), and $s_\lambda = 0$ for all $\lambda \neq (n)$:

$$e(\mathcal{MG}^{\mathbb{S}_n}_{g,n}) = \sum_k (-1)^k \dim H^k(\mathcal{MG}_{g,n};\mathbb{Q})^{\mathbb{S}_n} = \sum_k (-1)^k c^k_{g,(n)}.$$

We denote $\Gamma_{g,n}$ as the group of homotopy classes of self-homotopy equivalences of a connected graph $G$ of rank $g$ with $n$ legs that fix the legs pointwise. Equivalently, $\Gamma_{g,n} = \pi_0(\mathrm{HE}(G, \text{fix } \partial G))$. We have $\Gamma_{g,0} \cong \mathrm{Out}(F_g)$ and $\Gamma_{g,1} \cong \mathrm{Aut}(F_g)$. The moduli space $\mathcal{MG}_{g,n}$ is a rational classifying space of $\Gamma_{g,n}$, implying that the rational cohomology of $\mathcal{MG}_{g,n}$ is the same as that of $\Gamma_{g,n}$ (see, eg, [13]).

Kontsevich made the observation that the homology of $\mathcal{MG}_{g,n}$ is computed by what he termed the *Lie graph complex* [19]. In Section 2.1, we utilize a related concept known as the *forested graph complex*, which was introduced in [14], for the purpose of computing this homology.

In his work, Kontsevich also proposed an *odd* version of the Lie graph complex. This differs from the original graph complex by the notion of the graph's orientation. Detailed discussions regarding this are given in Sections 2.1 and 2.3. He also indicated that, in the absence of legs, this odd graph complex computes the cohomology of $\mathrm{Out}(F_g)$ with coefficients in $\widetilde{\mathbb{Q}}$, the representation of $\mathrm{Out}(F_g)$ that results from the composition of the natural map $\mathrm{Out}(F_g) \to \mathrm{GL}_g(\mathbb{Z})$ with the determinant. A formula for this "odd" Euler characteristic, denoted as $e^{\mathrm{odd}}(\mathrm{Out}(F_g)) = \sum_k (-1)^k \dim H^k(\mathrm{Out}(F_g), \widetilde{\mathbb{Q}})$, was previously presented in [8]. Similar formulas can be obtained for all groups $\Gamma_{g,n}$ due to the existence of the natural

surjection $\Gamma_{g,n} \to \mathrm{Out}(F_g)$ that comes from forgetting the legs. The odd Euler characteristic is equivalent to the Euler characteristic of the moduli space of graphs $\mathcal{MG}_{g,n}$ that is fibered by the local system resulting from the det-representation of $\mathrm{Out}(F_n)$ as pointed out in [19]. Here, we will prove a formula for the $\mathbb{S}_n$-equivariant odd Euler characteristic, denoted as $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$. It is defined analogously to $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ in (1) with the coefficients in $\widetilde{\mathbb{Q}}$ instead of $\mathbb{Q}$.

Morita, Sakasai, and Suzuki previously calculated the Euler characteristic, $e(\mathcal{MG}_{g,0}) = e(\mathrm{Out}(F_g))$ for $g \leq 11$ [21]. The outcome of recent work of Borinsky and Vogtmann was the asymptotic formula $e(\mathrm{Out}(F_g)) \sim -e^{-1/4}(g/e)^g/(g \log g)^2$ for large $g$, implying that the total dimension of the cohomology of $\mathrm{Out}(F_g)$ grows superexponentially with $g$ [8]. In this work, we enhance this result with explicit computations.

We leverage our effective formulas with the `FORM` programming language by Vermaseren [27], for instance, to compute $e(\mathrm{Out}(F_g))$ for all $g \leq 100$. The resulting values of $e(\mathrm{Out}(F_g))$ and $e^{\mathrm{odd}}(\mathrm{Out}(F_g))$ for $g \leq 15$ were published with [8]. The ancillary files in the online supplement include these numbers for all $g \leq 100$, together with tables of $e(\mathcal{MG}_{g,n})$, $e^{\mathrm{odd}}(\mathcal{MG}_{g,n})$, $e(\mathcal{MG}^{\mathbb{S}_n}_{g,n})$, $e^{\mathrm{odd}}(\mathcal{MG}^{\mathbb{S}_n}_{g,n})$ for all $g+n \leq 60$, and the polynomials $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$, $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ for all $g+n \leq 30$. Subsets of these data are given in Tables 2–5. In Section 2.7, we comment on our implementation of Theorem 2.18 in `FORM`, which is included in the online supplement. Based on our computed data, we give some empirical observations on the asymptotic growth rate of the Euler characteristic $e(\mathcal{MG}_{g,n})$ and its variations for large $g$ in Section 2.8.

Our data in Tables 4–5 exhibit an interesting pattern. The $\mathbb{S}_n$-invariant Euler characteristics of $\mathcal{MG}_{g,n}$ *stabilize* for large $n$. The search for an explanation of this pattern led to our second main finding: The $\mathbb{S}_n$-invariant cohomologies $H^\bullet(\mathcal{MG}_{g,n};\mathbb{Q})^{\mathbb{S}_n}$ and $H^\bullet(\mathcal{MG}_{g,n};\widetilde{\mathbb{Q}})^{\mathbb{S}_n}$ *stabilize* for large $n$. That means if $n \geq g \geq 2$ and $\mathbb{Q}_\rho \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$, then there are isomorphisms $H^k(\mathcal{MG}_{g,n};\mathbb{Q}_\rho)^{\mathbb{S}_n} \to H^k(\mathcal{MG}_{g,n+1};\mathbb{Q}_\rho)^{\mathbb{S}_n}$ for all $k$. This statement is proven as Theorem 3.1 in Section 3, where we also comment on an analogy to the cohomology of the braid group. Theorem 3.1 complements previous stability results for $\mathcal{MG}_{g,n}$ (see [13]). Here, in contrast to these previous results, the range of stabilization does not depend on the cohomological degree.

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# 2 The $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{MG}_{g,n}$

## 2.1 Forested graph complexes

In this section we will describe chain complexes that compute the homology of $\mathcal{MG}_{g,n}$. These chain complexes are generated by certain graphs.

A subgraph is a subcomplex of a graph that consists of all its vertices, its nonedge 1-cells and a subset of its edges. A subforest is an acyclic subgraph. A pair $(G, \Phi)$ of a graph $G$ and a subforest $\Phi \subset G$ is a *forested graph*. We write $|\Phi|$ for the number of edges in the forest $\Phi$. Figure 1 shows some examples of forested graphs with different rank, forest edge and leg numbers. The forest edges are drawn thicker and in blue. Legs are drawn as labeled half-edges.

A $(+)$-marking $\sigma_\Phi$ of $(G, \Phi)$ is an ordering of the forest edges, ie a bijection $\sigma_\Phi \colon E_\Phi \to \{1, \dots, |\Phi|\}$. A $(-)$-marking $(\sigma_\Phi, \sigma_{H_1(G,\mathbb{Z})})$ of $(G, \Phi)$ is such an ordering of the forest edges $\sigma_\Phi$ together with a basis for the first homology of the graph, ie a bijection $\sigma_{H_1} \colon H_1(G, \mathbb{Z}) \to \mathbb{Z}^{h_1(G)}$, where $h_1(G) = \operatorname{rk} H_1(G, \mathbb{Z})$ is the rank of $H_1(G, \mathbb{Z})$ or equivalently the rank of $G$.

**Definition 2.1** For given $g, n \geq 0$ with $2g - 2 + n \geq 0$, we define two $\mathbb{Q}$-vector spaces:

- $\mathcal{F}^+_{g,n}$ is generated by tuples $(G, \Phi, \sigma_\Phi)$ of a connected admissible forested graph $(G, \Phi)$ of rank $g$ with $n$ legs, which is $(+)$-marked with $\sigma_\Phi$, modulo the relation

$$(G, \Phi, \pi \circ \sigma_\Phi) \sim \operatorname{sign}(\pi) \cdot (G, \Phi, \sigma_\Phi) \quad \text{for all } \pi \in \mathbb{S}_{|\Phi|},$$

and modulo isomorphisms of $(+)$-marked forested graphs.

- $\mathcal{F}^-_{g,n}$ is generated by tuples $(G, \Phi, \sigma_\Phi, \sigma_{H_1})$ of a connected admissible forested graph $(G, \Phi)$ of rank $g$ with $n$ legs, which is $(-)$-marked with $\sigma_\Phi$, $\sigma_{H_1}$ modulo the relation

$$(G, \Phi, \pi \circ \sigma_\Phi, \rho_{H_1} \circ \sigma_{H_1}) \sim \operatorname{sign}(\pi) \cdot \det \rho_{H_1} \cdot (G, \Phi, \sigma_\Phi) \quad \text{for all } \pi \in \mathbb{S}_{|\Phi|} \text{ and } \rho_{H_1} \in \mathrm{GL}_{h_1(G)}(\mathbb{Z})$$

and modulo isomorphisms of $(-)$-marked forested graphs.

We will discuss the relations imposed in this definition and how they give rise to two different ways to impose *orientations* on graphs in more detail in Section 2.3.

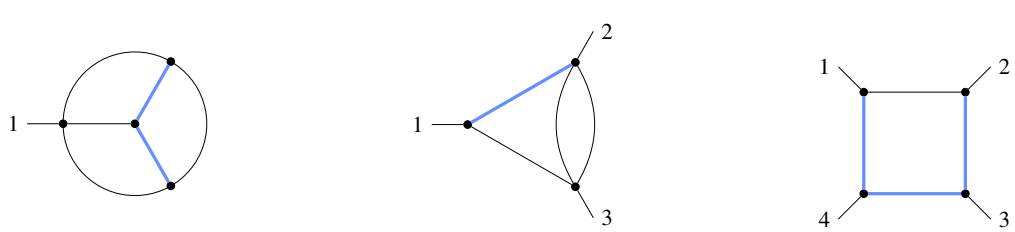


Figure 1: Examples of forested graphs. Left: forested graph of rank 3 with 1 leg and 2 forest edges. Middle: forested graph of rank 2 with 3 legs and 1 forest edge. Right: forested graph of rank 1 with 4 legs and 3 forest edges.

In what follows, we will often cover both the $(+)$- and $(-)$-cases analogously in the same sentence using the $\pm$-notation. Notice that the vector spaces $\mathcal{F}^{\pm}_{g,n}$ are defined for a slightly larger range of pairs $(g,n)$ then the moduli spaces $\mathcal{MG}_{g,n}$. This generalization makes the generating functions for the Euler characteristics in Section 2.6 easier to handle. There are no (admissible) graphs of rank 1 without legs, so $\mathcal{F}^{+}_{1,0} = \mathcal{F}^{-}_{1,0} = \varnothing$, but there is one admissible graph of rank 0 with two legs and no vertices or edges. It just consists of a 1-cell that connects the legs.

**Lemma 2.2** *For all $g,n \geq 0$ with $2g-2+n>0$, the vector spaces $\mathcal{F}^{\pm}_{g,n}$ are finite dimensional.*

**Proof** Let $G$ be a connected admissible graph of rank $g$ with $n$ legs, $|E_G|$ edges, and $|V_G|$ vertices. The Euler characteristic of such a graph equals $|V_G|-|E_G| = 1-g$. If we exclude the aforementioned graph with two legs and no vertices or edges, then each leg of $G$ is incident to a vertex. Each vertex has at most 3 incident edges or legs, so $3\,|V_G| \leq n+2\,|E_G|$. It follows that $|E_G| \leq 3g-3+n$. There are only finitely many isomorphism classes of graphs of bounded edge number. Hence, there are also only finitely many isomorphism classes of forested graphs of rank $g$ with $n$ legs. The relations in Definition 2.1 guarantee that each such isomorphism class contributes at most one generator to $\mathcal{F}^{\pm}_{g,n}$. □

The vector spaces $\mathcal{F}^{\pm}_{g,n}$ are graded by the number of edges in the forest. Let $C_k(\mathcal{F}^{\pm}_{g,n})$ be the respective subspace restricted to generators with $k$ forest edges. These spaces form a chain complex, so we will refer to $\mathcal{F}^{\pm}_{g,n}$ as the *forested graph complex* with $(\pm)$ orientation. We will not explicitly state the boundary maps, $\partial_k : C_k(\mathcal{F}^{\pm}_{g,n}) \to C_{k-1}(\mathcal{F}^{\pm}_{g,n})$ here (see [14]), as knowledge of the dimensions of these chain groups suffices for our Euler characteristic considerations.

The following theorem is a consequence of the works of Culler and Vogtmann [15], Kontsevich [19; 20] and Conant and Vogtmann [14] (see in particular [14, Sections 3.1–3.2]):

**Theorem 2.3** *For $g>0$, $n\geq 0$ and $2g-2+n>0$, the chain complexes $\mathcal{F}^{+}_{g,n}$ ($\mathcal{F}^{-}_{g,n}$) compute the homology of $\mathcal{MG}_{g,n}$ with trivial coefficients $\mathbb{Q}$ (with twisted coefficients $\widetilde{\mathbb{Q}}$). The $\mathbb{S}_n$-action on $H^\bullet(\mathcal{MG}_{g,n},\mathbb{Q})$ ($H^\bullet(\mathcal{MG}_{g,n},\widetilde{\mathbb{Q}})$) descents obviously to $\mathcal{F}^{+}_{g,n}$ ($\mathcal{F}^{-}_{g,n}$) by permuting leg-labels.*

In what follows, we will use this theorem to prove a formula for $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ and $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$.

## 2.2 Equivariant Euler characteristics

Recall that a permutation $\pi \in \mathbb{S}_n$ factors uniquely as a product of disjoint cycles. If the orders of these cycles are $\lambda_1,\dots,\lambda_\ell$, then $(\lambda_1,\dots,\lambda_\ell)$ is a partition of $n$ called the *cycle type* of $\pi$. For such a permutation $\pi \in \mathbb{S}_n$, we define the *power sum symmetric polynomial*, $p^\pi = p_{\lambda_1}\cdots p_{\lambda_\ell} \in \Lambda_n$ with $p_k = \sum_{i=1}^n x_i^k$. The *Frobenius characteristic* is a symmetric polynomial associated uniquely to a $\mathbb{S}_n$-representation $V$. It is defined by

$$\mathrm{ch}(\chi_V) = \frac{1}{n!}\sum_{\pi\in\mathbb{S}_n} \chi_V(\pi)\,p^\pi,$$

where $\chi_V$ is the *character* associated to $V$ (see, eg, [25, §7.18]). The $\mathbb{S}_n$*-equivariant Euler characteristic* of $\mathcal{F}^{\pm}_{g,n}$ is the alternating sum over the Frobenius characteristics of $H_k(\mathcal{F}^{\pm}_{g,n};\mathbb{Q})$. Note that this is consistent with the definition of $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ in (1) since $\mathrm{ch}(\chi_{V_\lambda}) = s_\lambda$.

As the spaces $\mathcal{F}^{\pm}_{g,n}$ are finite by Lemma 2.2, we can compute the equivariant Euler characteristic on the chain level:

**Proposition 2.4** *The $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{F}^{\pm}_{g,n}$ is given by*

$$e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n}) = \sum_k (-1)^k \,\mathrm{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}),$$

*where $\mathrm{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})})$ is the Frobenius characteristic of $C_k(\mathcal{F}^{\pm}_{g,n})$ as an $\mathbb{S}_n$-representation,*

$$\mathrm{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}) = \frac{1}{n!}\sum_{\pi\in\mathbb{S}_n} \chi_{C_k(\mathcal{F}^{\pm}_{g,n})}(\pi)\, p^\pi,$$

*and $\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}$ is the* ***character*** *associated to $C_k(\mathcal{F}^{\pm}_{g,n})$.*

**Corollary 2.5** *For $g>0$, $n\geq 0$ and $2g-2+n>0$,*

$$e_{\mathbb{S}_n}(\mathcal{MG}_{g,n}) = e_{\mathbb{S}_n}(\mathcal{F}^{+}_{g,n}) \quad \text{and} \quad e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n}) = e_{\mathbb{S}_n}(\mathcal{F}^{-}_{g,n}).$$

**Proof** Follows directly from Theorem 2.3. □

The other discussed Euler characteristics of $\mathcal{MG}_{g,n}$ can be obtained by evaluating the polynomials $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ for certain values of $\overline{p} = p_1, p_2, \ldots$. We write $\overline{p} = \overline{1}$ for the specification $p_1 = 1$, $p_2 = p_3 = \ldots = 0$ and $\overline{p} = \mathbb{1}$ for the specification $p_1 = p_2 = \ldots = 1$.

**Proposition 2.6** *For $g>0$, $n\geq 0$ and $2g-2+n>0$,*

$$e(\mathcal{MG}_{g,n}) = n!\cdot e_{\mathbb{S}_n}(\mathcal{F}^{+}_{g,n})|_{\overline{p}=\overline{1}}, \quad e^{\mathrm{odd}}(\mathcal{MG}_{g,n}) = n!\cdot e_{\mathbb{S}_n}(\mathcal{F}^{-}_{g,n})|_{\overline{p}=\overline{1}},$$
$$e(\mathcal{MG}^{\mathbb{S}_n}_{g,n}) = e_{\mathbb{S}_n}(\mathcal{F}^{+}_{g,n})|_{\overline{p}=\mathbb{1}}, \qquad e^{\mathrm{odd}}(\mathcal{MG}^{\mathbb{S}_n}_{g,n}) = e_{\mathbb{S}_n}(\mathcal{F}^{-}_{g,n})|_{\overline{p}=\mathbb{1}}.$$

**Proof** Recall the relationship between the character of the $\mathbb{S}_n$-representation $C_k(\mathcal{F}^{\pm}_{g,n})$ and the symmetric polynomials $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ from Proposition 2.4. Observe that $\chi_V(\mathrm{id}) = \dim V$, where id is the trivial permutation. To verify the first two equations, observe that substituting $p_1 = 1$ and $p_2 = p_3 = \ldots = 0$, in the formula for $\mathrm{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})})$ of Proposition 2.4 amounts to restricting the sum over $\pi$ to $\pi = \mathrm{id}$, as only the trivial permutation has cycle-type $(1,1,\ldots,1)$. For the second line recall that $\frac{1}{n!}\sum_{\pi\in\mathbb{S}_n}\chi_V(\pi) = \dim V^{\mathbb{S}_n}$, where $V^{\mathbb{S}_n}$ is the $\mathbb{S}_n$-invariant subspace of $V$. Additionally, use Corollary 2.5. □

To explicitly compute $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$, we hence need to compute the Frobenius characteristic of the chain groups and to compute those we need to know the character $\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}$. To compute these characters, we need a more detailed description of the chain groups $C_k(\mathcal{F}^{\pm}_{g,n})$ that are generated only by *orientable forested graphs*.

## 2.3 Orientable forested graphs

Roughly, the relations in Definition 2.1 have the effect that a chosen ordering of the forest edges or a chosen homology basis, *only matters up to its sign or its orientation*. We can therefore think of the generators as *oriented* forested graphs. Due to the relations, every forested graphs gives rise to at most one generator of $\mathcal{F}^+_{g,n}$ or $\mathcal{F}^-_{g,n}$. However, not every forested graph is *orientable* in this way, ie gives rise to a generator. Even though Definition 2.1 only involves connected forested graphs, we will define the notion of orientability here also for disconnected graphs. We will need it later.

Let $\mathrm{Aut}(G,\Phi)$ be the group of automorphisms of the forested graph $(G,\Phi)$. Each automorphism is required to fix the leg-labels and the forest. For instance, the graph in Figure 1, left, has one automorphism of order two that mirrors the graph on the horizontal axis. The graph in Figure 1, middle, has one automorphism that flips the two doubled edges and the graph in Figure 1, right, has no (leg-label-preserving) automorphisms.

Each $\alpha \in \mathrm{Aut}(G,\Phi)$ induces a permutation, $\alpha_\Phi\colon E_\Phi \to E_\Phi$ on the set of forest edges and an automorphism on the homology groups $\alpha_{H_0}\colon H_0(G,\mathbb{Z}) \to H_0(G,\mathbb{Z})$ and $\alpha_{H_1}\colon H_1(G,\mathbb{Z}) \to H_1(G,\mathbb{Z})$. For each (possibly disconnected) forested graph $(G,\Phi)$ with an automorphism $\alpha \in \mathrm{Aut}(G,\Phi)$, we define

$$\xi^+(G,\Phi,\alpha) = \mathrm{sign}(\alpha_\Phi) \text{ and } \xi^-(G,\Phi,\alpha) = \det(\alpha_{H_0})\det(\alpha_{H_1})\mathrm{sign}(\alpha_\Phi).$$

As connected graphs come with a canonical basis for $H_0(G,\mathbb{Z}) = \mathbb{Z}$ that cannot be changed by automorphisms, we always have $\det(\alpha_{H_0}) = 1$ for them. The other sign factors capture the signs of the relations in Definition 2.1 if the $(\pm)$-marking of the graph $(G,\Phi)$ is acted upon using $\alpha$ in the obvious way.

**Definition 2.7** A forested graph $(G,\Phi)$ is $(\pm)$-*orientable* if it has no automorphism $\alpha \in \mathrm{Aut}(G,\Phi)$ for which $\xi^\pm(G,\Phi,\alpha) = -1$.

For example, the automorphism of the forested graph in Figure 1, left, switches two forest edges and flips the orientation of a chosen homology basis. Hence, the graph is $(-)$-, but not $(+)$-orientable. The automorphism of the graph in Figure 1, middle, does not affect its subforest, but flips the orientation of its homology basis. So, it is $(+)$-, but not $(-)$-orientable. The forested graph in Figure 1, right, is both $(+)$- and $(-)$-orientable as it has no nontrivial automorphism. There are also forested graphs that are neither $(+)$- nor $(-)$-orientable.

**Proposition 2.8** *The chain groups $C_k(\mathcal{F}^\pm_{g,n})$ are freely generated by isomorphism classes of connected $(\pm)$-orientable forested graphs of rank $g$ with $n$ legs and $k$ forest edges.*

**Proof** Definition 2.1 guarantees that each isomorphism class of a connected forested graph $(G,\Phi)$ contributes at most one generator, as all different $(\pm)$-markings of $(G,\Phi)$ are related via a permutation $\pi \in \mathbb{S}_{|\Phi|}$ (and $\rho_{H_1} \in \mathrm{GL}_{h_1(G)}(\mathbb{Z})$).

It remains to be proven that $(G,\Phi)$ does contribute a free generator if and only if it is $(\pm)$-orientable. We will only prove this for the $(+)$-orientation, as the $(-)$-orientation follows analogously. Recall

that for any isomorphism class of a $(+)$-marked graph $(G,\Phi,\sigma_\Phi)$, Definition 2.1 imposes the relation $(G,\Phi,\pi\circ\sigma_\Phi)\sim \operatorname{sign}(\pi)\cdot(G,\Phi,\sigma_\Phi)$ for each $\pi\in\mathbb{S}_{|\Phi|}$. As $(G,\Phi,\sigma_\Phi)$ is given up to isomorphism, we have $(G,\Phi,\sigma_\Phi\circ\alpha_\Phi)=(G,\Phi,\sigma_\Phi)$ for each $\alpha\in\operatorname{Aut}(G,\Phi)$ with $\alpha_\Phi$ the permutation that $\alpha$ induces on $\Phi$. If there is an automorphism $\alpha\in\operatorname{Aut}(G,\Phi)$ such that $\alpha_\Phi$ is an odd permutation on $\Phi$, then the permutation $\pi:=\sigma_\Phi\circ\alpha_\Phi\circ\sigma_\Phi^{-1}\in\mathbb{S}_{|\Phi|}$ is odd as well and we find that $(G,\Phi,\sigma_\Phi)=(G,\Phi,\sigma_\Phi\circ\alpha_\Phi)=(G,\Phi,\pi\circ\sigma_\Phi)\sim-(G,\Phi,\sigma_\Phi)\sim 0$. If there is no such automorphism, then for any $\alpha\in\operatorname{Aut}(G,\Phi)$, $\operatorname{sign}(\sigma_\Phi\circ\alpha_\Phi\circ\sigma_\Phi^{-1})=1$. Hence, the sign of a $(+)$-orientable and $(+)$-marked forested graph is fixed by any automorphism which therefore gives a generator. □

## 2.4 The action of $\mathbb{S}_n$ on orientable graphs

The next step for the explicit evaluation of $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ is to quantify the action of $\mathbb{S}_n$ that permutes the legs of the generators of $\mathcal{F}^{\pm}_{g,n}$.

Let $\operatorname{UAut}(G,\Phi)$ be the set of automorphisms of a forested graph $(G,\Phi)$ with $n$ legs that are *allowed* to permute the leg-labels. For instance, for the graph in Figure 1, right, which has a trivial leg-label-preserving automorphism group, the group $\operatorname{UAut}(G,\Phi)$ is generated by the automorphism that mirrors the graph vertically and permutes the leg-labels by $(12)(34)$. We have a map $\pi_G\colon\operatorname{UAut}(G,\Phi)\to\mathbb{S}_n$, given by only looking at the induced permutation on the leg-labels. The kernel of this map is equal to $\operatorname{Aut}(G,\Phi)$, the group of leg-label-fixing automorphisms of $(G,\Phi)$.

The $\mathbb{S}_n$-action gives a representation $\rho\colon\mathbb{S}_n\to\operatorname{GL}(C_k(\mathcal{F}^{\pm}_{g,n}))$. For a specific $\pi\in\mathbb{S}_n$, the linear map $\rho(\pi)\in\operatorname{GL}(C_k(\mathcal{F}^{\pm}_{g,n}))$ replaces each generator of $C_k(\mathcal{F}^{\pm}_{g,n})$ with the respective generator where the leg labels are permuted by $\pi$. The character of $C_k(\mathcal{F}^{\pm}_{g,n})$ is the composition of $\rho$ with the trace, $\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}=\operatorname{Tr}\circ\rho$. As $\rho(\pi)$ maps each generator to a multiple of another generator, it is sufficient to look at generators that happen to be Eigenvectors of $\rho(\pi)$ to compute $\operatorname{Tr}(\rho(\pi))$. Let $(G,\Phi,\sigma^{\pm})$ be a connected $(\pm)$-orientable forested graph with a $(\pm)$-marking $\sigma^{\pm}$ corresponding to a generator of $C_k(\mathcal{F}^{\pm}_{g,n})$. This generator is an Eigenvector of $\rho(\pi)$ if the forested graph $(G,\Phi)$ has a non-leg-label-fixing automorphism $\alpha\in\operatorname{UAut}(G,\Phi)$ such that $\pi_G(\alpha)=\pi$. The following lemma describes the Eigenvalue corresponding to such an Eigenvector. It is either $+1$ or $-1$.

**Lemma 2.9** *If, for given $\pi\in\mathbb{S}_n$, the generator $(G,\Phi,\sigma^{\pm})\in C_k(\mathcal{F}^{\pm}_{g,n})$ is an Eigenvector of $\rho(\pi)$, then the corresponding Eigenvalue is $\xi^{\pm}(G,\Phi,\alpha)$, where $\alpha$ is some representative $\alpha\in\pi_G^{-1}(\pi)$.*

**Proof** By Proposition 2.8, $(G,\Phi)$ cannot have a leg-label-fixing automorphism that flips the orientation. However, an automorphism from the larger group $\operatorname{UAut}(G,\Phi)$ can change the sign of the orientation (ie the sign of the ordering and basis given by the $(\pm)$-marking in Definition 2.1 can be flipped). The map $\rho(\pi)$ does so if $\xi^{\pm}(G,\Phi,\alpha)=-1$ for a representative of the kernel $\alpha\in\pi_G^{-1}(\pi)$. As $\alpha\mapsto\xi^{\pm}(G,\Phi,\alpha)$ gives a group homomorphism $\operatorname{UAut}(G,\Phi)\to\mathbb{Z}/2\mathbb{Z}$ and as $\xi^{\pm}(G,\Phi,\alpha)=1$ for all $\alpha\in\ker\pi_G$, it does not matter which representative we pick. □

Summing over all such Eigenvalues of $\rho(\pi)$ gives the value of $\mathrm{Tr}(\rho(\pi)) = \chi_{C_k(\mathcal{F}^{\pm}_{g,n})}(\pi)$:

**Corollary 2.10** *For each $\pi \in \mathbb{S}_n$, the character of $C_k(\mathcal{F}^{\pm}_{g,n})$ is*

$$\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}(\pi) = \sum \xi^{\pm}(G, \Phi, \alpha_{(G,\Phi,\pi)}),$$

*where we sum over all isomorphism classes of $(\pm)$-orientable forested graphs $(G, \Phi)$ with $k = |\Phi|$ forest edges, for which the preimage $\pi_G^{-1}(\pi) \subset \mathrm{UAut}(G, \Phi)$ is nonempty (ie $(G, \Phi)$ has at least one non-leg-label-fixing automorphism that permutes the leg labels by $\pi$), and $\alpha_{(G,\Phi,\pi)}$ is some representative of such an automorphism in $\pi_G^{-1}(\pi) \subset \mathrm{UAut}(G, \Phi)$.*

To continue, it is convenient to pass to a sum over all connected forested graphs without restrictions on the orientability to get better combinatorial control over the expression:

**Corollary 2.11** *For $g, n \geq 0$ with $2g - 2 + n \geq 0$ and $k \geq 0$, we have*

$$\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}(\pi) = \sum_{[G,\Phi]} \frac{1}{|\mathrm{Aut}(G, \Phi)|} \sum_{\alpha \in \pi_G^{-1}(\pi)} \xi^{\pm}(G, \Phi, \alpha),$$

*where we sum over all isomorphism classes of (not necessarily orientable) connected forested graphs $[G, \Phi]$ of rank $g$, $n$ legs and $k = |\Phi|$ forest edges.*

**Proof** As $\alpha \mapsto \xi^{\pm}(G, \Phi, \alpha)$ also gives a map $\mathrm{Aut}(G, \Phi) \to \mathbb{Z}/2\mathbb{Z}$, we have by Definition 2.7

$$\sum_{\alpha \in \ker \pi_G} \xi^{\pm}(G, \Phi, \alpha) = \begin{cases} |\mathrm{Aut}(G, \Phi)| & \text{if } (G, \Phi) \text{ is } (\pm)\text{-orientable,} \\ 0 & \text{otherwise.} \end{cases}$$

The statement follows, as $\sum_{\alpha \in \pi_G^{-1}(\pi)} \xi^{\pm}(G, \Phi, \alpha) = \xi^{\pm}(G, \Phi, \alpha') \sum_{\alpha \in \ker \pi_G} \xi^{\pm}(G, \Phi, \alpha)$, for any representative $\alpha' \in \pi_G^{-1}(\pi)$ and the formula from Corollary 2.10 for $\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}(\pi)$. □

With this we finally obtain our first explicit formula for $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$:

**Theorem 2.12** *For $g, n \geq 0$ with $2g - 2 + n \geq 0$,*

$$e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n}) = \sum_{[G,\Phi]_U} \frac{(-1)^{|\Phi|}}{|\mathrm{UAut}(G, \Phi)|} \sum_{\alpha \in \mathrm{UAut}(G,\Phi)} \xi^{\pm}(G, \Phi, \alpha)\, p^{\pi_G(\alpha)},$$

*where we sum over all isomorphism classes of connected forested graphs $[G, \Phi]_U$ of rank $g$ with $n$* ***unlabeled*** *legs.*

**Proof** We can plug the statement of Corollary 2.11 into the definition of the Frobenius characteristic in Proposition 2.4 to get

$$\operatorname{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}) = \frac{1}{n!} \sum_{\pi \in \mathbb{S}_n} p^{\pi} \sum_{[G,\Phi]} \frac{1}{|\operatorname{Aut}(G,\Phi)|} \sum_{\alpha \in \pi_G^{-1}(\pi)} \xi^{\pm}(G,\Phi,\alpha).$$

Next, we can merge the first and the third sum into a summation over the whole group $\operatorname{UAut}(G,\Phi)$, as the preimages of different $\pi \in \mathbb{S}_n$ partition $\operatorname{UAut}(G,\Phi)$:

$$\operatorname{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}) = \frac{1}{n!} \sum_{[G,\Phi]} \frac{1}{|\operatorname{Aut}(G,\Phi)|} \sum_{\alpha \in \operatorname{UAut}(G,\Phi)} \xi^{\pm}(G,\Phi,\alpha) p^{\pi_G(\alpha)}.$$

Let $L_S(G,\Phi)$ be the set of isomorphism classes of forested graphs $(G,\Phi)$ that only differ by a relabeling of the legs. Obviously, $\mathbb{S}_n$ acts transitively on $L_S(G,\Phi)$ by permuting the leg-labels. The stabilizer of this action is the image of $\pi_G\colon \operatorname{UAut}(G,\Phi) \to \mathbb{S}_n$. By the orbit-stabilizer theorem, $n! = |\mathbb{S}_n| = |\operatorname{im}\pi_G||L_S(G,\Phi)|$. By the short exact sequence, $1 \to \operatorname{Aut}(G,\Phi) \to \operatorname{UAut}(G,\Phi) \to \operatorname{im}\pi_G \to 1$, we have $|\operatorname{im}\pi_G| = |\operatorname{UAut}(G,\Phi)|/|\operatorname{Aut}(G,\Phi)|$. Hence,

$$\operatorname{ch}(\chi_{C_k(\mathcal{F}^{\pm}_{g,n})}) = \sum_{[G,\Phi]} \frac{1}{|L_S(G,\Phi)||\operatorname{UAut}(G,\Phi)|} \sum_{\alpha \in \operatorname{UAut}(G,\Phi)} \xi^{\pm}(G,\Phi,\alpha) p^{\pi_G(\alpha)}.$$

The terms in this sum do not depend on the leg-labeling of the graphs, so we can just sum over non-leg-labeled graphs and remove the $|L_S(G,\Phi)|$ in the denominator. □

## 2.5 Disconnected forested graphs

The expression for $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ in Theorem 2.12 involves a sum over *connected* forested graphs without leg-labels. To eventually get an effective generating function for $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$, it is convenient to pass to an analogous formula that sums over *disconnected* forested graphs. Moreover, it is helpful to change from grading the graphs by rank to grading them by their negative Euler characteristic. For connected graphs, this is just a trivial shift as the negative Euler characteristic of such graphs fulfills $\operatorname{rk} H_1(G,\mathbb{Z}) - \operatorname{rk} H_0(G,\mathbb{Z}) = g-1$. So, we define for all $t \in \mathbb{Z}$ and $n \geq 0$,

$$\hat{e}^{\pm}_{t,n} = \sum_{[G,\Phi]_U} \frac{1}{|\operatorname{UAut}(G,\Phi)|} \sum_{\alpha \in \operatorname{UAut}(G,\Phi)} \xi^{\pm}(G,\Phi,\alpha) p^{\pi_G(\alpha)}, \tag{2}$$

where we sum over all isomorphism classes of (possibly disconnected) forested graphs $[G,\Phi]_U$ of Euler characteristic $\operatorname{rk} H_0(G,\mathbb{Z}) - \operatorname{rk} H_1(G,\mathbb{Z}) = -t$ and $n$ unlabeled legs. There is only a finite number of such graphs, so the sum is finite and $\hat{e}^{\pm}_{t,n}$ is a symmetric polynomial in $\Lambda_n$.

Let $\hat{\Lambda}$ be the *ring of formal symmetric power series* $\hat{\Lambda} = \lim_{\leftarrow n} \mathbb{Q}[\![x_1,\dots,x_n]\!]^{\mathbb{S}_n}$. Later in this section we will prove a generating function, expressed as a power series with coefficients in $\hat{\Lambda}$, for the polynomials $\hat{e}^{\pm}_{t,n}$. Before that, we explain how we can translate them into our desired $\mathbb{S}_n$-equivariant Euler characteristics:

**Proposition 2.13** *We define, over the ring of formal symmetric power series, the Laurent series*

$$\boldsymbol{e}^{\pm}(\hbar,\bar{p}) = \sum_{\substack{g,n\geq 0\\ 2g-2+n\geq 0}} e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})(\pm\hbar)^{g-1}, \quad \boldsymbol{E}^{\pm}(\hbar,\bar{p}) = \sum_{t\in\mathbb{Z}}\sum_{n\geq 0} \hat{e}^{\pm}_{t,n}(\pm\hbar)^t,$$

*which are elements of $\widehat{\Lambda}((\hbar))$. Both are related by the* ***plethystic exponential***

$$\boldsymbol{E}^{\pm}(\hbar,\bar{p}) = \exp\Big(\sum_{k\geq 1}\frac{\boldsymbol{e}^{\pm}(\hbar^k,\bar{p}_{[k]})}{k}\Big),$$

*where $\boldsymbol{e}^{\pm}(\hbar^k,\bar{p}_{[k]})$ denotes the power series $\boldsymbol{e}^{\pm}$ with the substitutions $\hbar\to\hbar^k$ and $p_i\to p_{ik}$.*

**Proof** The combinatorial argument for [8, Proposition 3.2] applies to translate between the sum over *connected* forested graphs in Theorem 2.12 to a sum over disconnected forested graphs in (2). The strategy goes back to Pólya [22] (see also [4, Chapter 4.3]). Briefly, each summand $\boldsymbol{e}^{\pm}(\hbar^k,\bar{p}_{[k]})/k$ in the exponent of the stated formula counts pairs consisting of a $k$-tuple of mutually isomorphic forested graphs and an automorphism that cyclically permutes the different graphs. Such automorphisms give rise to different sign factors depending on the orientation of the graphs. Accounting for these signs gives the minus signs in front of $\hbar$ in the $(-)$-orientation case (see [8, Theorem 5.1]). □

We can solve the equation in the statement above for the generating functions $\boldsymbol{e}^{\pm}(\hbar,\bar{p})$ and therefore obtain the value of $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ if we know sufficiently many coefficients of $\boldsymbol{E}^{\pm}(\hbar,\bar{p})$. The required inverse transformation of power series is the plethystic logarithm:

**Corollary 2.14** *We have*

$$\boldsymbol{e}^{\pm}(\hbar,\bar{p}) = \sum_{k\geq 1}\frac{\mu(k)}{k}\log \boldsymbol{E}^{\pm}(\hbar^k,\bar{p}_{[k]}),$$

*where $\mu(k)$ is the number-theoretic Möbius function.*

**Proof** Let $g_1, g_2, \ldots \in \widehat{\Lambda}((\hbar))$ such that $f_n = \sum_{k\geq 1} g_{nk}/(nk)$ converges uniformly in $n$. It follows from the definition of the Möbius function, $\sum_{d|n}\mu(d) = 0$ for all $n\geq 2$ and $\mu(1)=1$, that

$$g_1 = \sum_{k\geq 1}\mu(k)\frac{f_k}{k}.$$

With $g_n = \boldsymbol{e}^{\pm}(\hbar^n,\bar{p}_{[n]})$ and $f_n = \log \boldsymbol{E}^{\pm}(\hbar^n,\bar{p}_{[n]})$, the statement now follows form Proposition 2.13. □

## 2.6 Generating functions

In this section we will give the desired generating function for the polynomials $\hat{e}^{\pm}_{t,n}$. Together with the discussion in the last section, this generating function will give us an effective formula for the $\mathbb{S}_n$-equivariant Euler characteristic $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$.

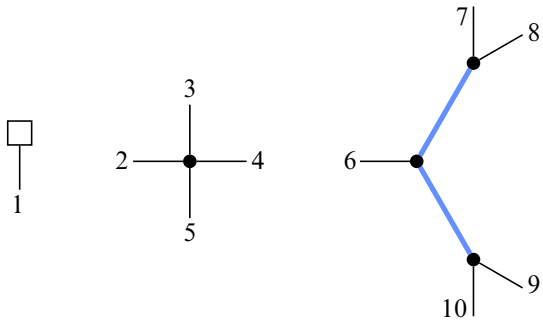


Figure 2: Extended forest with one special component.

The generating function is closely related to the one given in [8, Theorem 3.12] and we refer to the argument given there. As in [8], we define the following power series in $\overline{q} = q_1, q_2, \dots$:

$$V(\overline{q}) = q_1 + \tfrac{1}{2}q_1^2 - \tfrac{1}{2}q_2 - (1+q_1)\sum_{k\geq 1}\frac{\mu(k)}{k}\log(1+q_k). \tag{3}$$

See [8] for the first coefficients of $V(\overline{q})$. Moreover, we define two power series $F^+$ and $F^-$ both in two infinite sets of variables $\overline{q} = q_1, q_2, \dots$, $\overline{p} = p_1, p_2, \dots$ and a single variable $u$,

$$F^{\pm}(u, \overline{q}, \overline{p}) = \exp\Bigg(\sum_{k\geq 1}(\pm 1)^{k+1}u^{-2k}\frac{V((u\cdot\overline{q})_{[k]}) + u^k q_k p_k}{k}\Bigg), \tag{4}$$

where $V((u\cdot\overline{q})_{[k]})$ means that we replace each variable $q_i$ in $V(\overline{q})$ with $u^{ki}q_{ki}$.

Recall that a forest is an acyclic graph. As graphs, also forests are required to be admissible that means they have no vertices of degree 0 and 2. Again, the univalent vertices are interpreted as the legs of the forest. An *extended* forest is a forests that is additionally allowed to have *special* univalent vertices that are always connected by a 1-cell to a leg. In contrast to graphs, we do not allow extended forests to have components that consist of two adjacent legs or two adjacent special vertices. See also [8, Section 3.7] for a discussion of extended forests. The legs of a forest are labeled by integers $\{1, \dots, s\}$. The special vertices remain unlabeled. Figure 2 depicts an extended forest with three components; one of which is special. The special vertex is marked as a box. Legs are again drawn as labeled half-edges.

We write $k(\Phi)$, $s(\Phi)$ and $n(\Phi)$ for the total number of connected components, the number of legs and the number of special vertices of an extended forest $\Phi$. An automorphism $\gamma \in \mathrm{UAut}(\Phi)$ that is allowed to permute the leg-labels gives rise to a permutation of the edges of $\Phi$. We write $e_\gamma(\Phi)$ for the numbers of orbits of this permutation. An automorphism $\gamma \in \mathrm{UAut}(\Phi)$ also gives rise to a permutation $\pi_G(\gamma) \in \mathbb{S}_{s(\Phi)}$ of the legs, a permutation $\pi_G^\star(\gamma) \in \mathbb{S}_{n(\Phi)}$ of the special vertices and a permutation $\gamma_{H_0(\Phi)} \in \mathbb{S}_{h_0(\Phi)}$ of the connected components of $\Phi$.

**Proposition 2.15** *The generating functions $\boldsymbol{F}^{\pm}$ count signed pairs of an extended forest $\Phi$ and an automorphism $\gamma$ of $\Phi$. Explicitly,*

$$\boldsymbol{F}^{+}(u,\overline{q},\overline{p}) = \sum_{(\Phi,\gamma)} (-1)^{e_\gamma(\Phi)} u^{s(\Phi)-2k(\Phi)} p^{\pi_G^\star(\gamma)} \frac{q^{\pi_G(\gamma)}}{s(\Phi)!},$$

$$\boldsymbol{F}^{-}(u,\overline{q},\overline{p}) = \sum_{(\Phi,\gamma)} \operatorname{sign}(\gamma_{H_0(\Phi)})(-1)^{e_\gamma(\Phi)} u^{s(\Phi)-2k(\Phi)} p^{\pi_G^\star(\gamma)} \frac{q^{\pi_G(\gamma)}}{s(\Phi)!},$$

*where we sum over all pairs of an extended forest $\Phi$ and all $\gamma \in \mathrm{UAut}(\Phi)$.*

**Proof** This statement is a slight generalization of [8, Propositions 3.10 and 5.5]. Here, we also allow forests to have special vertices and modify the generating function accordingly. The term $(\pm 1)^{k+1} u^{-k} q_k p_k / k$ in (4) accounts for these special components. Explicitly, it stands for $k$ special components that are cyclically permuted by the overall automorphism that is acting on the forested graph (see [8, Lemma 3.3] or [4]). Each component contributes two negative powers of $u$, because it adds one component, and one positive power of $u$ as it adds one leg. So, we mark a cycle of $k$ special components with $u^{-k}$. The different signs for the odd case are a consequence of [8, Lemma 5.3] and the fact that each special vertex counts as a new connected component of $\Phi$. □

We will use the coefficient extraction operator notation. That means, we denote the coefficient of a power series $f(\overline{q})$ in front of $q^\lambda$ as $[q^\lambda] f(\overline{q})$.

**Corollary 2.16** *The coefficient $[u^{2t} q^\mu p^\lambda] \boldsymbol{F}^{\pm}(u,\overline{q},\overline{p})$ vanishes if $|\mu| = \sum_i \mu_i > 6t + 4|\lambda|$.*

**Proof** A *maximal* $(r,n)$-forest is an extended forest that consists of $r$ copies of a degree-3 vertex with three legs, and $n$ special components of a special vertex and one leg. All extended forests with $n$ special components can be obtained by first starting with a maximal $(r,n)$-forest, subsequently creating new edges by gluing together pairs of legs of different components and finally by contracting edges. The difference $s(\Phi) - 2k(\Phi)$ is left invariant by these gluing and contracting operations, but the number of legs $s(\Phi)$ decreases each time we glue together a pair of legs. It follows that maximal $(r,n)$-forests have the maximal number of legs for fixed $n$. Such a forest contributes a power $u^{3r-2r+n-2n} q^\mu p^\lambda = u^{r-n} q^\mu p^\lambda$ to the generating function, for some partitions $\mu$, $\lambda$ with $|\mu| = 3r+n$ and $|\lambda| = n$. Fixing $2t = r-n$ gives $|\mu| = 6t + 4n$.

Alternatively, the statement can also be verified by expanding $\boldsymbol{V}$ and $\boldsymbol{F}^{\pm}$ from (3)–(4). □

We define two sets of numbers, $\eta_\lambda^+$ and $\eta_\lambda^-$, that are indexed by an integer partition, $\lambda \vdash s$. These numbers combine the definitions in Corollaries 3.5 and 5.8 of [8], where $\eta_\lambda^+$ is denoted as $\eta_\lambda$ and $\eta_\lambda^-$ as $\eta_\lambda^{\mathrm{odd}}$. The discussion around these corollaries also includes a detailed combinatorial interpretation of these numbers: they count (signed) fixed-point free involutions that commute with a given permutation of cycle type $\lambda$.

An alternative notation for an integer partitions is $\lambda = [1^{m_1} 2^{m_2} \cdots]$, where $m_k$ denotes the number of parts of size $k$ in $\lambda$. Let

$$\eta^{\pm}_{\lambda} = \prod_{k=1}^{s} \eta^{\pm}_{k,m_k}, \quad \text{where } \eta^{\pm}_{k,\ell} = \begin{cases} 0 & \text{if } k \text{ and } \ell \text{ are odd,} \\ k^{\ell/2}(\ell-1)!! & \text{if } k \text{ is odd and } \ell \text{ is even,} \\ \sum_{r=0}^{\lfloor \ell/2 \rfloor} (\pm 1)^{\ell k/2 + r} \binom{\ell}{2r} k^r (2r-1)!! & \text{otherwise.} \end{cases}$$

With this we get an effective expression for the polynomials $\widehat{e}^{\pm}_{t,n}$ from (2):

**Theorem 2.17** *We have*

$$\widehat{e}^{\pm}_{t,n} = \sum_{\mu} \eta^{\pm}_{\mu} \sum_{\lambda \vdash n} p^{\lambda} [u^{2t} q^{\mu} p^{\lambda}] \boldsymbol{F}^{\pm}(u, \overline{q}, \overline{p}),$$

*where we sum over all integer partitions $\mu$ and all partitions $\lambda$ of $n$. The terms in the sum are only nonzero for a finite number of such partitions. Moreover, all terms vanish if $t < -\frac{2}{3}n$.*

**Proof** By Corollary 2.16, the sum over $\mu$ has finite support and all terms vanish if $t < -\frac{2}{3}n$.

Matching all legs of an extended forest in pairs gives a graph with a marked forest and special univalent vertices. We get this graph by gluing together the legs as described by a matching. Gluing together two legs creates a new 1-cell between the vertices that are adjacent to the legs and forgets about the legs and the 1-cells that are incident to them. The special vertices are subsequently promoted to (unlabeled) legs of the resulting forested graph. All forested graphs with unlabeled legs can be obtained this way.

As an example consider the extended forest in Figure 2. We can match the legs in the pairs $(1,2)$, $(3,7)$, $(4,6)$, $(5,10)$ and $(8,9)$. Gluing together these legs and promoting the special vertex to a new leg recreates the forested graph in Figure 1, left, without the leg-label.

In general, if the extended forest that we start with has $k$ connected components and $s$ legs, then the graph that we obtain after the gluing has Euler characteristic $k - s/2$. By Proposition 2.15 and the definition of the polynomials $\widehat{e}_{t,n}$ in (2) we therefore extract the correct coefficient as all forests $\Phi$ with $s(\Phi) - 2k(\Phi) = 2t$ contribute to the coefficient of $u^{2t}$. See the proof of [8, Proposition 3.10] and its odd version [8, Proposition 5.5] for the discussions of automorphisms and sign factors which apply also in our generalized case. □

Together Corollaries 2.5–2.14 and Theorem 2.17 give an effective algorithm for the computation of the polynomials $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ and $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$. In summary:

**Theorem 2.18** *Fix $\chi > 1$. To compute $e(\mathcal{F}^{\pm}_{g,n})$ for all $2g-2+n > 0$ and $g+1+n \le \chi$:*

(1) *Compute the coefficients of $\boldsymbol{V}$ up to homogeneous order $6\chi$ in the $\overline{q}$-variables by expanding the power series defined in (3).*

(2) *Compute the coefficients of $\boldsymbol{F}^{\pm}$ up to homogeneous order $\chi$ in $u^2$ and in the $\overline{p}$ variables by expanding the power series defined in (4).*

(3) *Compute the polynomials $\hat{e}^{\pm}_{t,n}$ for all pairs $(t,n)$ with $n \geq 0$, $t+n \leq \chi$ and $t \geq -\frac{2}{3}n$ using Theorem 2.17.*

(4) *Compute the polynomials $e_{\mathbb{S}_n}(\mathcal{F}^{\pm}_{g,n})$ using the formula from Corollary 2.14.*

The formulas in Corollary 2.5 and Proposition 2.6 can be used to translate the result into the numbers $e(\mathcal{MG}_{g,n})$, $e^{\mathrm{odd}}(\mathcal{MG}_{g,n})$, $e(\mathcal{MG}^{\mathbb{S}_n}_{g,n})$ and $e^{\mathrm{odd}}(\mathcal{MG}^{\mathbb{S}_n}_{g,n})$. To compute the integers $\sum_k (-1)^k c^k_{g,\lambda}$ that are the alternating sums over the multiplicities of the irreducible representations in the respective cohomology of $\mathcal{MG}_{g,n}$, we have to write the polynomials $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ and $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ in terms of Schur polynomials as in (1). We can do so using the Murnaghan–Nakayama rule.

## 2.7 Implementation of Theorem 2.18 in FORM

The most demanding computational step in Theorem 2.18 is the expansion of the power series $\boldsymbol{F}^{\pm}$ as defined in (4) in the formal variables $u$, $\overline{p}$ and $\overline{q}$. Conventional computer algebra struggles with such expansions; usually only a small number of terms are accessible. To be able to apply Theorem 2.18 at moderately large values of $\chi$, we use the FORM programming language. FORM is designed to deal with large analytic expressions that come up in high-energy physics.

A FORM program that implements Theorem 2.18 can be found in the online supplement in the file `eMGgn.frm`. It can be run with the command `form eMGgn.frm` after downloading and installing FORM from https://github.com/vermaseren/form.git. The syntax and details of the code are described in a FORM tutorial [28]. We used FORM version 5 *beta* for our computations.

The output of the program is a power series in which each coefficient is a symmetric function that describes the respective $\mathbb{S}_n$-equivariant Euler characteristic. These coefficients are given in the power sum basis of the ring of symmetric functions. To translate the output into Schur symmetric function via the Murnaghan–Nakayama rule, we used `Sage` [23].

**Remark 2.19** By employing the *Feynman transform* introduced by Getzler and Kapranov [16], modified versions of this program can effectively compute the Euler characteristics of *modular operads*. Furthermore, by combining Joyal's theory of species (see, eg, [4]) with findings from Borinsky's thesis [6], the program can be adapted to count various combinatorial objects, such as the number of isomorphism classes of admissible graphs of rank $n$ [18].

## 2.8 Large-$g$ asymptotics of the Euler characteristics of $\mathcal{MG}_{g,n}$

The *virtual* or *rational* Euler characteristic $\chi(G)$ is an invariant of a group $G$ that is often better behaved than the usual Euler characteristic.

Recall that the notation "$f(g) \sim h(g)$ for large $g$" means that $\lim_{g\to\infty} f(g)/h(g) = 1$. By the short exact sequence $1 \to F^n_g \to \Gamma_{g,n} \to \mathrm{Out}(F_g) \to 1$ [13], we have $\chi(\Gamma_{g,n}) = (1-g)^n \chi(\mathrm{Out}(F_g))$. The

numbers $\chi(\mathrm{Out}(F_g))$ can be computed using [7, Proposition 8.5] and the asymptotic growth rate of $\chi(\mathrm{Out}(F_g))$ is known explicitly by [7, Theorem A]. Using this together with the formula for $\chi(\Gamma_{g,n})$ and Stirling's approximation, we find that for fixed $n \geq 0$,

$$\chi(\Gamma_{g,n}) \sim \frac{(-1)^{n+1} g^n \left(\frac{g}{e}\right)^g}{(g \log g)^2} \quad \text{for large } g.$$

In [8], it was proven that $e(\mathcal{MG}_{g,0}) \sim e^{-1/4}\chi(\mathrm{Out}(F_g))$ and $e^{\mathrm{odd}}(\mathcal{MG}_{g,0}) \sim e^{1/4}\chi(\mathrm{Out}(F_g))$ for large $g$. It would be interesting to make similar statements about $\mathcal{MG}_{g,n}$ for $n \geq 1$. Using our data on the Euler characteristics of $\mathcal{MG}_{g,n}$, we empirically verified the following conjecture which generalizes the known asymptotic behavior of $\mathcal{MG}_{g,0}$ to $\mathcal{MG}_{g,n}$ for all $n \geq 0$.

**Conjecture 2.20** *For fixed $n \geq 0$, we have, for large $g$,*

$$e(\mathcal{MG}_{g,n}) \sim e^{-1/4}\chi(\Gamma_{g,n}), \quad e^{\mathrm{odd}}(\mathcal{MG}_{g,n}) \sim e^{1/4}\chi(\Gamma_{g,n}),$$
$$e(\mathcal{MG}_{g,n}^{\mathbb{S}_n}) \sim \frac{e^{-1/4}\chi(\Gamma_{g,n})}{n!}, \quad e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n}) \sim \frac{e^{1/4}\chi(\Gamma_{g,n})}{n!}.$$

Proving this conjecture should be feasible by generalizing the analytic argument in [8, Section 4].

## 3 The large-$n$ $\mathbb{S}_n$-invariant cohomological stability of $\mathcal{MG}_{g,n}$

Our data in the Tables 2–5 exhibit an obvious pattern: for fixed $g$ the $\mathbb{S}_n$-*invariant* Euler characteristics $e(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ and $e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ appear to be constant for all $n \geq g$. Unfortunately, this is not manifest from our formulas (ie from Theorem 2.18). To explain this pattern, we will prove the stabilization of the associated cohomologies.

**Theorem 3.1** *Fix $\mathbb{Q}_\rho \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$. The cohomology $H^\bullet(\mathcal{MG}_{g,n}; \mathbb{Q}_\rho)^{\mathbb{S}_n}$ stabilizes for $n \to \infty$: if $n \geq g \geq 2$, then there are isomorphisms $H^k(\mathcal{MG}_{g,n}; \mathbb{Q}_\rho)^{\mathbb{S}_n} \to H^k(\mathcal{MG}_{g,n+1}; \mathbb{Q}_\rho)^{\mathbb{S}_n}$ for all $k$.*

This statement is a refinement of the known *representational stability* [12] of $\mathcal{MG}_{g,n}$, which was shown to hold in [13]. In contrast to those previous results, Theorem 3.1 holds independently of the cohomological degree $k$. We will prove this theorem below in Section 3.2 using an argument that is based on the Lyndon–Hochschild–Serre spectral sequence and closely related to lines of thought in [13; 24].

Unfortunately, our proof of Theorem 3.1 gives neither a concrete description of the stable cohomologies $H^\bullet(\mathcal{MG}_{g,\infty}; \mathbb{Q})^{\mathbb{S}_\infty}$ and $H^\bullet(\mathcal{MG}_{g,\infty}; \widetilde{\mathbb{Q}})^{\mathbb{S}_\infty}$ nor gives explicit stabilization maps. A candidate for such a map is a generalization of the injection $H^\bullet(\mathrm{Out}(F_g); \mathbb{Q}) \to H^\bullet(\mathrm{Aut}(F_g); \mathbb{Q})$ from Theorem 1.4 of [13] to arbitrarily many legs. The first few terms of the Euler characteristics associated to these stable cohomologies are tabulated in Table 1. The values are remarkably small in comparison to the value of the Euler characteristic of $\mathrm{Out}(F_g)$ for the respective rank (see the top and bottom row of Table 4 for

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e(\mathcal{MG}_{g,\infty}^{\mathbb{S}_\infty})$ | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 | 295 |
| $e^{\mathrm{odd}}(\mathcal{MG}_{g,\infty}^{\mathbb{S}_\infty})$ | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 | −295 |

Table 1: Euler characteristics for $H^\bullet(\mathcal{MG}_{n,\infty};\mathbb{Q})^{\mathbb{S}_\infty}$ and $H^\bullet(\mathcal{MG}_{n,\infty};\widetilde{\mathbb{Q}})^{\mathbb{S}_\infty}$.

a direct comparison). Empirically, the Euler characteristics of the stable cohomologies appear to grow exponentially and not superexponentially. Our argument for Theorem 3.1 does not give any hint why these Euler characteristics are so small. It would also be interesting to prove Theorem 3.1 using graph cohomology methods as this kind of stabilization appears to be a distinguished feature of Lie- and forest graph cohomology. For instance, commutative graph cohomology with legs does not stabilize in the strong sense observed here [26].

## 3.1 Analogy to Artin's braid group

The observed large-$n$ stabilization carries similarities with the cohomology of Artin's *braid group* $B_n$ of equivalence classes of $n$-braids.

For a graph $G$ of rank $g$ with $n$ legs, let $\mathrm{U}\Gamma_{g,n}$ be the group of homotopy classes of self-homotopy equivalences of $G$ that only fix the legs as a set and not pointwise, ie $\mathrm{U}\Gamma_{g,n} = \pi_0(\mathrm{HE}(G,\partial G))$. By looking only at the action of $\mathrm{U}\Gamma_{g,n}$ on the leg-labels $\{1,\dots,n\}$, we get a surjective map to the symmetric group $\mathbb{S}_n$, ie

$$1 \to \Gamma_{g,n} \to \mathrm{U}\Gamma_{g,n} \to \mathbb{S}_n \to 1,$$

where the "pure" group, $\Gamma_{g,n}$, is the subgroup of $\mathrm{U}\Gamma_{g,n}$ that fixes the legs pointwise as defined in the introduction. Analogously, the braid group $B_n$ maps surjectively to $\mathbb{S}_n$. So,

$$1 \to P_n \to B_n \to \mathbb{S}_n \to 1,$$

where the kernel $P_n$ is the *pure braid group*. Whereas the braid groups $B_n$ exhibit *homological stability*, ie if $n \geq 3$, then $H_k(B_n;\mathbb{Q}) \cong H_k(B_{n+1};\mathbb{Q})$ for all $k$ [2], the pure braid groups $P_n$ only satisfy representational stability [12]. For instance, $H^\bullet(P_n;\mathbb{Q})$ is an exterior algebra on $\binom{n}{2}$ generators modulo a 3-term relation [2] and $H^\bullet(\Gamma_{1,n};\mathbb{Q})$ is the even-degree part of an exterior algebra on $n-1$ generators [13]. The numbers of generators of both algebras increase with $n$. The cohomology of $\mathrm{U}\Gamma_{g,n}$ is equal to the $\mathbb{S}_n$-invariant cohomology of $\mathcal{MG}_{g,n}$. So, analogously to the stabilization of $H^\bullet(B_n)$, Theorem 3.1 implies the large-$n$ cohomological stability of $\mathrm{U}\Gamma_{g,n}$.

## 3.2 The Lyndon–Hochschild–Serre spectral sequence

Our proof of Theorem 3.1 makes heavy use of tools from [13]. Following [13], we abbreviate $\mathrm{H} = H^1(F_g)$ and think of it as an $\mathrm{Out}(F_g)$-module. The action of $\mathrm{Out}(F_g)$ on H factors through the action of $\mathrm{GL}_g(\mathbb{Z})$ on H. The $q^{th}$ *exterior power* of H is denoted as $\bigwedge^q \mathrm{H}$. As $\mathrm{rk}\,\mathrm{H} = g$, the determinant representation is recovered by $\bigwedge^g \mathrm{H} = \widetilde{\mathbb{Q}}$. As before, we only work with modules over rational coefficients.

**Proposition 3.2** *Fix $g, n$ with $g \geq 2, n \geq 0$. There are two Lyndon–Hochschild–Serre spectral sequences with $E_2$ pages given by*

$$E_2^{p,q} = H^p\Big(\mathrm{Out}(F_g); \textstyle\bigwedge^q \mathrm{H}\Big), \quad \widetilde{E}_2^{p,q} = H^p\Big(\mathrm{Out}(F_g); \widetilde{\mathbb{Q}} \otimes \textstyle\bigwedge^q \mathrm{H}\Big)$$

*for $p, q \geq 0$ with $p \leq 2g-3$ and $q \leq \min(g,n)$, and $E_2^{p,q} = \widetilde{E}_2^{p,q} = 0$ for all other values of $p, q$. Both spectral sequences converge: $E_2^{p,q} \Rightarrow H^{p+q}(\Gamma_{g,n}; \mathbb{Q})^{\mathbb{S}_n}$ and $\widetilde{E}_2^{p,q} \Rightarrow H^{p+q}(\Gamma_{g,n}; \widetilde{\mathbb{Q}})^{\mathbb{S}_n}$.*

The argument works along the lines of Section 3.2 of [13] followed by an application of Schur–Weyl duality and the projection to the $\mathbb{S}_n$-invariant cohomology. A similar argument can be found in [24, Lemma 4.1] where it is used to prove representational stability of $H^\bullet(\Gamma_{g,n}; \mathbb{Q})$.

**Proof** For $g \geq 2$ and $n \geq 0$, the Lyndon–Hochschild–Serre spectral sequence associated to the group extension $1 \to F_g^n \to \Gamma_{g,n} \to \mathrm{Out}(F_g) \to 1$ is a first-quadrant spectral sequence with the second page given by $E_2^{p,q} = H^p(\mathrm{Out}(F_g); H^q(F_g^n; \mathbb{Q})) \Rightarrow H^{p+q}(\Gamma_{g,n}; \mathbb{Q})$ (see [13, Section 3.2]). The virtual cohomological dimension of $\mathrm{Out}(F_n)$ is $2g-3$ [15], so if $p > 2g-3$, then $E_2^{p,q} = \widetilde{E}_2^{p,q} = 0$. By the Künneth formula, the $\mathbb{S}_n$-module $H^q(F_g^n; \mathbb{Q})$ vanishes if $q > n$ and for $0 \leq q \leq n$ it is obtained by inducing the $\mathbb{S}_q \times \mathbb{S}_{n-q}$-module $\mathrm{H}^{\wedge q} \otimes V_{(n-q)}$ to $\mathbb{S}_n$, (see [13, Lemma 3.4] for details),

$$H^q(F_g^n; \mathbb{Q}) = \mathrm{Ind}_{\mathbb{S}_q \times \mathbb{S}_{n-q}}^{\mathbb{S}_n}(\mathrm{H}^{\wedge q} \otimes V_{(n-q)}),$$

where $V_{(n-q)}$ is the trivial representation of $\mathbb{S}_{n-q}$ and $\mathrm{H}^{\wedge q}$ is $\mathrm{H}^{\otimes q} \otimes V_{(1^q)}$ with $V_{(1^q)}$ the alternating representation of $\mathbb{S}_q$ and $\mathbb{S}_q$ acts on $\mathrm{H}^{\otimes q}$ by permuting the entries of the tensor product.

We can set up a similar spectral sequence with coefficients twisted by $\widetilde{\mathbb{Q}}$ with second page

$$\widetilde{E}_2^{p,q} = H^p(\mathrm{Out}(F_g); H^q(F_g^n; \widetilde{\mathbb{Q}})) \Rightarrow H^{p+q}(\Gamma_{g,n}; \widetilde{\mathbb{Q}}).$$

Note that $F_g^n$ acts trivially on $\widetilde{\mathbb{Q}}$. So, applying the Künneth formula to expand $H^q(F_g^n; \widetilde{\mathbb{Q}})$ as an $\mathbb{S}_n$-module gives

$$H^q(F_g^n; \widetilde{\mathbb{Q}}) = \widetilde{\mathbb{Q}} \otimes \mathrm{Ind}_{\mathbb{S}_q \times \mathbb{S}_{n-q}}^{\mathbb{S}_n}(\mathrm{H}^{\wedge q} \otimes V_{(n-q)}),$$

which only differs by the $GL_g(\mathbb{Z})$-action on $\widetilde{\mathbb{Q}}$. Schur–Weyl duality gives the irreducible decomposition of $\mathrm{H}^{\wedge q}$ as a module over $\mathrm{GL}(\mathrm{H}) \times \mathbb{S}_q$ (see, eg, [13, Section 3.1]):

$$\mathrm{H}^{\wedge q} = \bigoplus_{\lambda \vdash q} W_\lambda \otimes V_{\lambda'},$$

where we sum over all partitions $\lambda$ of $q$ with at most $g$ rows and $\lambda'$ is the transposed partition. Here, $W_\lambda$ and $V_\lambda$ are the irreducible $\mathrm{GL}(\mathrm{H})$- and $\mathbb{S}_q$-representations associated to $\lambda$.

As $\mathbb{S}_n$ only acts on the coefficients in $E_2^{p,q}$, we get

$$\begin{aligned} E_2^{p,q} &= H^p(\mathrm{Out}(F_g); \mathrm{Ind}_{\mathbb{S}_q \times \mathbb{S}_{n-q}}^{\mathbb{S}_n}(\mathrm{H}^{\wedge q} \otimes V_{(n-q)})) \\ &= \bigoplus_{\lambda \vdash q} H^p(\mathrm{Out}(F_g); W_\lambda) \otimes \mathrm{Ind}_{\mathbb{S}_q \times \mathbb{S}_{n-q}}^{\mathbb{S}_n}(V_{\lambda'} \otimes V_{(n-q)}). \end{aligned}$$

If we project to the trivial $\mathbb{S}_n$-representation, only the partition $\lambda = (1^q)$ contributes. Hence,

$$(E_2^{p,q})^{\mathbb{S}_n} = H^p(\mathrm{Out}(F_g); W_{(1^q)}).$$

The representation $W_{(1^q)}$ is the $q^{\text{th}}$ exterior power of the defining representation H. It vanishes if $q > \operatorname{rk} \mathrm{H} = g$. The argument works analogously for $\widetilde{E}_2^{p,q}$. □

**Proof of Theorem 3.1** For $n \geq g \geq 2$, the $E_2$ pages of the spectral sequences in Proposition 3.2 are independent of $n$. Hence, by the spectral sequence comparison theorem, also $H^k(\Gamma_{g,n}, \mathbb{Q})^{\mathbb{S}_n}$ and $H^k(\Gamma_{g,n}, \widetilde{\mathbb{Q}})^{\mathbb{S}_n}$ must be independent of $n$. □

In the stable regime, the Euler characteristics $e(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ and $e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ are equal up to a sign factor (see Table 1). This can be explained as another consequence of Proposition 3.2, and the following two technical lemmas.

For a graph $G$, the first cohomology $H^1(G) = H^1(G; \mathbb{Q})$ is a representation of its automorphism group $\mathrm{Aut}(G)$. Let $H^1(G)^*$ be the corresponding dual representation.

**Lemma 3.3** *$H^1(G) \cong H^1(G)^*$ as $\mathrm{Aut}(G)$-representations.*

**Proof** Each graph $G$ comes with an $\mathrm{Aut}(G)$-invariant symmetric positive definite bilinear form $H_1(G) \otimes H_1(G) \to \mathbb{Q}$, *the graph Laplacian* (see, eg, [3, 3.1.1]). Dually, we have an $\mathrm{Aut}(G)$-invariant pairing on $H^1(G)$ giving the stated isomorphism of $\mathrm{Aut}(G)$-representations. □

Let $\mathrm{H}^*$ be the $\mathrm{Out}(F_g)$-representation dual to H and $e(H^\bullet(G; M)) = \sum_p (-1)^p \dim(H^p(G; M))$.

**Lemma 3.4** *For $g \geq 2$ and $q \geq 0$, we have $e\big(H^\bullet\big(\mathrm{Out}(F_g); \bigwedge^q \mathrm{H}\big)\big) = e\big(H^\bullet\big(\mathrm{Out}(F_g); \bigwedge^q \mathrm{H}^*\big)\big)$.*

**Proof** Using standard arguments that relate Culler–Vogtmann outer space to the rational group cohomology of $\mathrm{Out}(F_n)$, we are looking for a cochain model for the stated cohomologies. Let $K_g$ be the *spine of outer space*, ie $K_g$ is a cubical complex in which each $k$-cube corresponds to an isomorphism class of a forested graph $(G, \Phi, \mu)$ which is equipped with a *marking*, an isomorphism $\mu\colon \pi_1(G) \to F_g$. Each cube is oriented by ordering the edges in $\Phi$. The group $\mathrm{Out}(F_g)$ acts on $K_g$ by composition with the marking. This action is not free, but it has finite stabilizers. Most importantly, $K_g$ is contractible [15]. Let $M$ be $\bigwedge^q \mathrm{H}$ or $\bigwedge^q \mathrm{H}^*$, then by standard arguments $H^\bullet(\mathrm{Out}(F_g); M) = H^\bullet(\mathrm{Hom}_{\mathrm{Out}(F_g)}(C_\bullet(K_g), M))$ (see, eg, [9, Chapter 7]). The action of $\mathrm{Out}(F_g)$ on the tuples $(G, \Phi, \mu)$ fixes the isomorphism class of the forested graph and acts transitively on the marking. The stabilizer of this action is $\mathrm{Aut}(G, \Phi)$. Hence,

$$\mathrm{Hom}_{\mathrm{Out}(F_g)}(C_\bullet(K_g), M) = \mathrm{Hom}(C_\bullet(K_g), M)^{\mathrm{Out}(F_g)} = \bigoplus_{[G,\Phi]} (\mathrm{sign}_{(G,\Phi)} \otimes M)^{\mathrm{Aut}(G,\Phi)},$$

where we take the sum over all isomorphism classes of forested graphs of rank $g$ or equivalently over all $\mathrm{Out}(F_g)$-orbits of isomorphism classes of marked forested graphs of rank $g$. Here, $\mathrm{sign}_{(G,\Phi)}$ is the

orientation module, the one-dimensional representation of $\mathrm{Aut}(G,\Phi)$ that is given by $\alpha \mapsto \mathrm{sign}(\alpha_\Phi)$ with $\alpha_\Phi$ the permutation that $\alpha \in \mathrm{Aut}(G,\Phi)$ induces on the forest edges. The group $\mathrm{Aut}(G,\Phi)$ acts on $M$ via the map $\mathrm{Aut}(G,\Phi) \to \mathrm{Out}(F_g)$ that any chosen marking representative $\pi_1(G) \to F_g$ induces. As a $\mathrm{Aut}(G,\Phi)$-representation, H is hence equal to $H^1(G)$. Therefore, due to Lemma 3.3, we may replace the right-most $M$ in the displayed formula above with either $\bigwedge^q H^1(G)$, or $\bigwedge^q (H^1(G)^*) \cong \big(\bigwedge^q H^1(G)\big)^*$. So, in the stated equality, both sides can be computed using cochain complexes with the same generators in each degree. The Euler characteristic only depends on the dimensions of the cochain spaces and not on the differentials. □

**Proposition 3.5** *If $n \geq g \geq 2$, then $e(\mathcal{MG}_{g,n}^{\mathbb{S}_n}) = (-1)^g e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$.*

**Proof** There is an isomorphism of $\mathrm{GL}_g(\mathbb{Z})$-representations: $\widetilde{\mathbb{Q}} \otimes \bigwedge^q \mathrm{H} = \bigwedge^g \mathrm{H} \otimes \bigwedge^q \mathrm{H} \cong \bigwedge^{g-q} \mathrm{H}^*$. This can be seen by computing the characters of both $\bigwedge^g \mathrm{H} \otimes \bigwedge^q \mathrm{H}$ and $\bigwedge^{g-q} \mathrm{H}^*$ and by observing that they differ by a multiple of the character of $\big(\bigwedge^g\big)^{\otimes 2}$, which is the same as the trivial representation of $\mathrm{GL}_g(\mathbb{Z})$, because $\det(h)^2 = 1$ for each $h \in \mathrm{GL}_g(\mathbb{Z})$ (see, eg, [25, Chapter 7.A.2]).

Hence, for $n \geq g \geq 2$, we have, for the $E_2$ pages of the spectral sequences in Proposition 3.2,

$$\widetilde{E}_2^{p,q} = H^p\Big(\mathrm{Out}(F_g); \widetilde{\mathbb{Q}} \otimes \bigwedge^q \mathrm{H}\Big) \cong H^p\Big(\mathrm{Out}(F_g); \bigwedge^{g-q} \mathrm{H}^*\Big).$$

Therefore, it follows from Lemma 3.4 that $\sum_p (-1)^p \dim \widetilde{E}_2^{p,q} = \sum_p (-1)^p \dim E_2^{p,g-q}$. So,

$$e(\mathcal{MG}_{g,n}^{\mathbb{S}_n}) = \sum_{p,q} (-1)^{p+q} \dim E_2^{p,q} = \sum_{p,q} (-1)^{p+g-q} \dim E_2^{p,g-q} = (-1)^g e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n}). \qquad \square$$

Apart from the (up-to-sign) equality of the Euler characteristics, our argument does not tell us how even and odd $\mathbb{S}_n$-invariant stable $n \to \infty$ cohomology of $\mathcal{MG}_{g,n}$ are related.

Moreover, the $E_2$ page of the spectral sequences in Proposition 3.2 contains the whole (twisted) cohomology of $\mathrm{Out}(F_g)$ (eg $E_2^{p,0} = H^p(\mathrm{Out}(F_g);\mathbb{Q})$). So, we also do not explain why the Euler characteristics of the large-$n$ stable $\mathbb{S}_n$-invariant cohomologies of $\mathcal{MG}_{g,n}$ are so small in comparison to the Euler characteristic of $\mathrm{Out}(F_g)$.

## Appendix Tables of Euler characteristics of $\mathcal{MG}_{g,n}$

Larger versions of Tables 2–7 are included in the online supplement: values of the Euler characteristics $e(\mathcal{MG}_{g,n})$, $e^{\mathrm{odd}}(\mathcal{MG}_{g,n})$, $e(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ and $e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$ for $g+n \leq 60$ can be found in `eMGgn.tsv`, `eMGgn-odd.tsv`, `eMGgn-modSn.tsv` and `eMGgn-modSn-odd.tsv`. These larger tables are readable as plain text or with standard spreadsheet software. The files `eOutFn.txt` and `eOutFn-odd.txt` list the values $e(\mathrm{Out}(F_g))$ and $e^{\mathrm{odd}}(\mathrm{Out}(F_g))$ for all $g \leq 100$. The full $\mathbb{S}_n$-equivariant Euler characteristic is listed in the files `eMGgn-equiv.tsv` and `eMGgn-equiv-odd.tsv` that contain tables of the polynomials $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ and $e_{\mathbb{S}_n}^{\mathrm{odd}}(\mathcal{MG}_{g,n})$ for $g+n \leq 30$, respectively.

| $n$ | $g=$ 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | | 1 | 1 | 2 | 1 |
| 1 | 1 | 1 | 1 | 2 | 0 |
| 2 | 1 | 1 | 2 | 3 | −2 |
| 3 | 2 | 1 | 6 | 6 | 0 |
| 4 | 4 | 0 | 18 | 4 | −24 |
| 5 | 8 | −4 | 61 | −3 | 64 |
| 6 | 16 | −19 | 202 | −158 | −69 |
| 7 | 32 | −69 | 701 | −831 | 2905 |
| 8 | 64 | −230 | 2438 | −5135 | 9917 |
| 9 | 128 | −734 | 8721 | −25446 | 112518 |
| 10 | 256 | −2289 | 31602 | −134879 | 552339 |
| 11 | 512 | −7039 | 116821 | −670008 | 4149475 |
| 12 | 1024 | −21460 | 437758 | −3414254 | 22844193 |
| 13 | 2048 | −65064 | 1663481 | −17022549 | 149941792 |
| 14 | 4096 | −196559 | 6388202 | −85672220 | 864112247 |
| 15 | 8192 | −592409 | 24759741 | −427885725 | 5376583485 |
| 16 | 16384 | −1782690 | 96647478 | −2144390153 | 31618003029 |

| $n$ | $g=$ 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| 0 | 2 | 1 | 1 | −21 |
| 1 | 3 | 3 | 31 | 154 |
| 2 | 1 | −28 | −153 | −1486 |
| 3 | 36 | 152 | 1423 | 12072 |
| 4 | −86 | −1062 | −9474 | −103392 |
| 5 | 675 | 6421 | 72249 | 845821 |
| 6 | −3453 | −38028 | −506827 | −6971380 |
| 7 | 11182 | 253892 | 3616144 | 56742775 |
| 8 | −124467 | −1306559 | −25952916 | −459328520 |
| 9 | 47564 | 9957185 | 178180002 | 3726950202 |
| 10 | −4683027 | −42459898 | −1308692710 | −29852809180 |
| 11 | −9296152 | 399601667 | 8669878028 | 242171554559 |
| 12 | −188955568 | −1233422911 | −65810827609 | −1924630979085 |
| 13 | −725537667 | 16798999974 | 416483532392 | 15652884150733 |
| 14 | −8133442381 | −25768675818 | −3327304711052 | −123449090799389 |
| 15 | −41758325066 | 753773677302 | 19660027898985 | 1009591434254489 |
| 16 | −367416474589 | 247507159657 | −170405333570573 | −7887831186821342 |

Table 2: Euler characteristic $e(\mathcal{MG}_{g,n})$.

| $n$ | $g=$ 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 | | 0 | 0 | −1 | 0 |
| 1 | 0 | 0 | −1 | −1 | −1 |
| 2 | −1 | 0 | −3 | −3 | −5 |
| 3 | −2 | 0 | −8 | −7 | −9 |
| 4 | −4 | −1 | −24 | −27 | −66 |
| 5 | −8 | −5 | −71 | −81 | −137 |
| 6 | −16 | −20 | −228 | −364 | −1185 |
| 7 | −32 | −70 | −743 | −1394 | −3536 |
| 8 | −64 | −231 | −2544 | −6716 | −28509 |
| 9 | −128 | −735 | −8891 | −29974 | −118323 |
| 10 | −256 | −2290 | −32028 | −148035 | −837525 |
| 11 | −512 | −7040 | −117503 | −708621 | −4206038 |
| 12 | −1024 | −21461 | −439464 | −3528385 | −27300597 |
| 13 | −2048 | −65065 | −1666211 | −17361527 | −150529289 |
| 14 | −4096 | −196560 | −6395028 | −86682326 | −934471725 |
| 15 | −8192 | −592410 | −24770663 | −430902388 | −5383163340 |
| 16 | −16384 | −1782691 | −96674784 | −2153412834 | −32735332605 |

| $n$ | $g=$ 6 | 7 | 8 | 9 |
|---|---|---|---|---|
| 0 | −1 | −2 | −8 | −38 |
| 1 | 1 | 3 | 34 | 278 |
| 2 | −7 | −45 | −273 | −2143 |
| 3 | 20 | 172 | 1688 | 16279 |
| 4 | −184 | −1365 | −12037 | −127665 |
| 5 | 340 | 6153 | 80002 | 995425 |
| 6 | −5356 | −47862 | −568014 | −7861828 |
| 7 | 3081 | 210751 | 3814347 | 61973273 |
| 8 | −166300 | −1760727 | −27489461 | −492911760 |
| 9 | −144896 | 6985727 | 182953976 | 3900012883 |
| 10 | −5638835 | −67361954 | −1348453174 | −31209047062 |
| 11 | −13846788 | 214832720 | 8779760089 | 246950368646 |
| 12 | −211328036 | −2704794269 | −66861595436 | −1987593802313 |
| 13 | −834038152 | 5420792406 | 418887979427 | 15685889711601 |
| 14 | −8667253394 | −115615170716 | −3355504362873 | −127079507192857 |
| 15 | −44375995137 | 48464678629 | 19709309010324 | 997347475403633 |
| 16 | −380341928592 | −5327400148291 | −171167953029982 | −8150861890475894 |

Table 3: Euler characteristics $e^{\mathrm{odd}}(\mathcal{MG}_{g,n})$.

| $n$ | $g=$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | 1 | 1 | 2 | 1 | 2 | 1 | 1 | −21 | −124 | −1202 | −10738 |
| 1 | 1 | 1 | 1 | 2 | 0 | 3 | 3 | 31 | 154 | 1405 | 12409 | 128198 |
| 2 | 1 | 1 | 2 | 2 | 0 | 1 | −8 | −71 | −645 | −5916 | −60661 | −680524 |
| 3 | 1 | 1 | 2 | 2 | 1 | 4 | 22 | 148 | 1432 | 14933 | 173615 | 2170285 |
| 4 | 1 | 1 | 2 | 1 | 2 | 2 | −6 | −158 | −1911 | −24310 | −324161 | −4610023 |
| 5 | 1 | 1 | 2 | 1 | 2 | 4 | 14 | 114 | 1677 | 26129 | 415109 | 6837661 |
| 6 | 1 | 1 | 2 | 1 | 2 | 3 | 9 | −34 | −900 | −18622 | −368380 | −7231581 |
| 7 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 8 | 296 | 8548 | 223615 | 5456280 |
| 8 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | −20 | −2292 | −88814 | −2878337 |
| 9 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | 268 | 20992 | 1010696 |
| 10 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | −2154 | −212879 |
| 11 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | 20256 |
| 12 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 |
| 13 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 |
| 14 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 |
| 15 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 |
| 16 | 1 | 1 | 2 | 1 | 2 | 3 | 11 | 0 | 18 | −7 | 71 | −102 |

| $n$ | $g=$ 13 | 14 | 15 | 16 |
|---|---|---|---|---|
| 0 | −112901 | −1271148 | −15668391 | −208214777 |
| 1 | 1428208 | 17431842 | 229796854 | 3260731764 |
| 2 | −8319674 | −110000218 | −1564363190 | −23810497027 |
| 3 | 29302324 | 423870178 | 6547649971 | 107566687178 |
| 4 | −69547908 | −1112576568 | −18825687993 | −336214369334 |
| 5 | 117254253 | 2100407688 | 39335548893 | 770173833141 |
| 6 | −143971474 | −2934457032 | −61596805980 | −1335762223221 |
| 7 | 129717602 | 3072060291 | 73437715819 | 1786143030025 |
| 8 | −85129652 | −2410044793 | −66988401529 | −1856600124025 |
| 9 | 39693125 | 1399384996 | 46526543233 | 1500185772169 |
| 10 | −12481703 | −584609581 | −24221345542 | −934645878596 |
| 11 | 2377198 | 166410659 | 9165254242 | 440926365563 |
| 12 | −207026 | −28931603 | −2383055405 | −152471494842 |
| 13 | 295 | 2319534 | 381124388 | 36485839046 |
| 14 | 295 | −602 | −28285686 | −5402710802 |
| 15 | 295 | −602 | 1730 | 373201519 |
| 16 | 295 | −602 | 1730 | −3616 |

Table 4: Euler characteristics $e(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$.

| $n$ \ $g=$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | 0 | 0 | −1 | 0 | −1 | −2 | −8 | −38 | −275 | −2225 | −20358 |
| 1 | 0 | 0 | −1 | −1 | −1 | 1 | 3 | 34 | 278 | 2285 | 20921 | 212777 |
| 2 | 0 | 1 | −1 | −1 | −2 | −1 | −17 | −114 | −918 | −8555 | −88885 | −1010798 |
| 3 | 0 | 1 | −2 | −1 | −2 | 2 | 11 | 158 | 1659 | 18615 | 223544 | 2878235 |
| 4 | 0 | 1 | −2 | 1 | −1 | 0 | −19 | −148 | −1929 | −26136 | −368451 | −5456382 |
| 5 | 0 | 1 | −2 | 1 | −2 | 1 | −8 | 71 | 1414 | 24303 | 415038 | 7231479 |
| 6 | 0 | 1 | −2 | 1 | −2 | 3 | −10 | −31 | −663 | −14940 | −324232 | −6837763 |
| 7 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | −1 | 136 | 5909 | 173544 | 4609921 |
| 8 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −39 | −1412 | −60732 | −2170387 |
| 9 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | 117 | 12338 | 680422 |
| 10 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −1273 | −128300 |
| 11 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | 10636 |
| 12 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 |
| 13 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 |
| 14 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 |
| 15 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 |
| 16 | 0 | 1 | −2 | 1 | −2 | 3 | −11 | 0 | −18 | −7 | −71 | −102 |

| $n$ \ $g=$ | 13 | 14 | 15 | 16 |
|---|---|---|---|---|
| 0 | −207321 | −2320136 | −28287416 | −373205135 |
| 1 | 2376903 | 28931001 | 381122658 | 5402707186 |
| 2 | −12481998 | −166411261 | −2383057135 | −36485842662 |
| 3 | 39692830 | 584608979 | 9165252512 | 152471491226 |
| 4 | −85129947 | −1399385598 | −24221347272 | −440926369179 |
| 5 | 129717307 | 2410044191 | 46526541503 | 934645874980 |
| 6 | −143971769 | −3072060893 | −66988403259 | −1500185775785 |
| 7 | 117253958 | 2934456430 | 73437714089 | 1856600120409 |
| 8 | −69548203 | −2100408290 | −61596807710 | −1786143033641 |
| 9 | 29302029 | 1112575966 | 39335547163 | 1335762219605 |
| 10 | −8319969 | −423870780 | −18825689723 | −770173836757 |
| 11 | 1427913 | 109999616 | 6547648241 | 336214365718 |
| 12 | −113196 | −17432444 | −1564364920 | −107566690794 |
| 13 | −295 | 1270546 | 229795124 | 23810493411 |
| 14 | −295 | −602 | −15670121 | −3260735380 |
| 15 | −295 | −602 | −1730 | 208211161 |
| 16 | −295 | −602 | −1730 | −3616 |

Table 5: Euler characteristics $e^{\mathrm{odd}}(\mathcal{MG}_{g,n}^{\mathbb{S}_n})$.

| $g$ | $n=$ 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | | | $s_3$ |
| 1 | | $s_1$ | $s_2$ | $s_{1,1,1}+s_3$ |
| 2 | 1 | $s_1$ | $s_2$ | $s_3$ |
| 3 | 1 | $s_1$ | $2s_2$ | $2s_{2,1}+2s_3$ |
| 4 | 2 | $2s_1$ | $s_{1,1}+2s_2$ | $2s_{1,1,1}+s_{2,1}+2s_3$ |
| 5 | 1 | 0 | $-2s_{1,1}$ | $-s_{1,1,1}+s_3$ |
| 6 | 2 | $3s_1$ | $s_2$ | $10s_{1,1,1}+11s_{2,1}+4s_3$ |

| $g$ | $n=$ 4 | 5 |
|---|---|---|
| 0 | $s_4$ | $s_5$ |
| 1 | $s_{2,1,1}+s_4$ | $s_{1,1,1,1,1}+s_{3,1,1}+s_5$ |
| 2 | $-s_{2,1,1}+s_{2,2}+s_4$ | $-s_{2,1,1,1}-s_{3,1,1}+s_{3,2}+s_5$ |
| 3 | $s_{2,1,1}+2s_{2,2}+3s_{3,1}+2s_4$ | $s_{2,1,1,1}+2s_{2,2,1}+3s_{3,1,1}+3s_{3,2}+3s_{4,1}+2s_5$ |
| 4 | $s_{1,1,1,1}+s_{2,2}+s_4$ | $2s_{1,1,1,1,1}+s_{2,1,1,1}-s_{3,1,1}-s_{4,1}+s_5$ |
| 5 | $-5s_{1,1,1,1}-4s_{2,1,1}-3s_{2,2}-s_{3,1}+2s_4$ | $4s_{2,1,1,1}+s_{2,2,1}+4s_{3,1,1}+s_{3,2}+3s_{4,1}+2s_5$ |
| 6 | $-5s_{1,1,1,1}-15s_{2,1,1}-4s_{2,2}-10s_{3,1}+2s_4$ | $34s_{1,1,1,1,1}+51s_{2,1,1,1}+41s_{2,2,1}+22s_{3,1,1}+16s_{3,2}+4s_{4,1}+4s_5$ |

Table 6: $\mathbb{S}_n$-equivariant Euler characteristic $e_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ in the Schur basis of $\Lambda_n$.

| $g$ | $n=$ 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | | | $s_3$ |
| 1 | | 0 | $-s_{1,1}$ | $-s_{2,1}$ |
| 2 | 0 | 0 | $-s_{1,1}+s_2$ | $-s_{1,1,1}+s_3$ |
| 3 | 0 | $-s_1$ | $-2s_{1,1}-s_2$ | $-2s_{1,1,1}-2s_{2,1}-2s_3$ |
| 4 | $-1$ | $-s_1$ | $-2s_{1,1}-s_2$ | $-2s_{1,1,1}-2s_{2,1}-s_3$ |
| 5 | 0 | $-s_1$ | $-3s_{1,1}-2s_2$ | $-s_{1,1,1}-3s_{2,1}-2s_3$ |
| 6 | $-1$ | $s_1$ | $-6s_{1,1}-s_2$ | $4s_{1,1,1}+7s_{2,1}+2s_3$ |

| $g$ | $n=$ 4 | 5 |
|---|---|---|
| 0 | $s_4$ | $s_5$ |
| 1 | $-s_{1,1,1,1}-s_{3,1}$ | $-s_{2,1,1,1}-s_{4,1}$ |
| 2 | $-s_{1,1,1,1}-s_{2,1,1}+s_{2,2}+s_4$ | $-s_{1,1,1,1,1}-s_{2,1,1,1}-s_{3,1,1}+s_{3,2}+s_5$ |
| 3 | $-2s_{1,1,1,1}-3s_{2,1,1}-s_{2,2}-3s_{3,1}-2s_4$ | $-2s_{1,1,1,1,1}-3s_{2,1,1,1}-2s_{2,2,1}-3s_{3,1,1}-3s_{3,2}-3s_{4,1}-2s_5$ |
| 4 | $-3s_{1,1,1,1}-4s_{2,1,1}-2s_{2,2}-3s_{3,1}+s_4$ | $-3s_{1,1,1,1,1}-4s_{2,1,1,1}-4s_{2,2,1}-4s_{3,1,1}-3s_{3,2}-s_{4,1}+s_5$ |
| 5 | $-7s_{1,1,1,1}-9s_{2,1,1}-5s_{2,2}-7s_{3,1}-s_4$ | $-2s_{1,1,1,1,1}-6s_{2,1,1,1}-5s_{2,2,1}-8s_{3,1,1}-4s_{3,2}-4s_{4,1}-2s_5$ |
| 6 | $-22s_{1,1,1,1}-32s_{2,1,1}-12s_{2,2}-14s_{3,1}$ | $19s_{1,1,1,1,1}+27s_{2,1,1,1}+26s_{2,2,1}+6s_{3,1,1}+10s_{3,2}-s_{4,1}+s_5$ |

Table 7: $\mathbb{S}_n$-equivariant Euler characteristic $e^{\mathrm{odd}}_{\mathbb{S}_n}(\mathcal{MG}_{g,n})$ in the Schur basis of $\Lambda_n$.

# References


[1] **D Abramovich**, **L Caporaso**, **S Payne**, *The tropicalization of the moduli space of curves*, Ann. Sci. Éc. Norm. Supér. 48 (2015) 765–809 MR

[2] **V I Arnold**, *The cohomology ring of the group of dyed braids*, Mat. Zametki 5 (1969) 227–231 MR In Russian; translated in Math. Notes 5 (1969) 138–140

[3] **O Baker**, *The Jacobian map on outer space*, PhD thesis, Cornell University (2011) Available at `https://ecommons.cornell.edu/handle/1813/30769`

[4] **F Bergeron**, **G Labelle**, **P Leroux**, *Combinatorial species and tree-like structures*, Encyclopedia of Mathematics and its Applications 67, Cambridge Univ. Press (1998) MR

[5] **M Berghoff**, *Feynman amplitudes on moduli spaces of graphs*, Ann. Inst. Henri Poincaré D 7 (2020) 203–232 MR

[6] **M Borinsky**, *Graphs in perturbation theory: algebraic structure and asymptotics*, Springer (2018) MR

[7] **M Borinsky**, **K Vogtmann**, *The Euler characteristic of* $\mathrm{Out}(F_n)$, Comment. Math. Helv. 95 (2020) 703–748 MR

[8] **M Borinsky**, **K Vogtmann**, *The Euler characteristic of the moduli space of graphs*, Adv. Math. 432 (2023) art. id. 109290 MR

[9] **K S Brown**, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer (1982) MR

[10] **M Chan**, **C Faber**, **S Galatius**, **S Payne**, *The $S_n$-equivariant top weight Euler characteristic of $\mathcal{M}_{g,n}$*, Amer. J. Math. 145 (2023) 1549–1585 MR

[11] **M Chan**, **S Galatius**, **S Payne**, *Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$*, J. Amer. Math. Soc. 34 (2021) 565–594 MR

[12] **T Church**, **B Farb**, *Representation theory and homological stability*, Adv. Math. 245 (2013) 250–314 MR

[13] **J Conant**, **A Hatcher**, **M Kassabov**, **K Vogtmann**, *Assembling homology classes in automorphism groups of free groups*, Comment. Math. Helv. 91 (2016) 751–806 MR

[14] **J Conant**, **K Vogtmann**, *On a theorem of Kontsevich*, Algebr. Geom. Topol. 3 (2003) 1167–1224 MR

[15] **M Culler**, **K Vogtmann**, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986) 91–119 MR

[16] **E Getzler**, **M M Kapranov**, *Modular operads*, Compositio Math. 110 (1998) 65–126 MR

[17] **E Gorsky**, *The equivariant Euler characteristic of moduli spaces of curves*, Adv. Math. 250 (2014) 588–595 MR

[18] **A Howroyd**, **W P Orrick**, *Number of unlabeled connected multigraphs with circuit rank n and degree $\geq 3$ at each node, loops allowed*, from "The on-line encyclopedia of integer sequences" (2023) A360864

[19] **M Kontsevich**, *Formal (non)commutative symplectic geometry*, from "The Gelfand Mathematical Seminars, 1990–1992" (L Corwin, I Gelfand, J Lepowsky, editors), Birkhäuser, Boston, MA (1993) 173–187 MR

[20] **M Kontsevich**, *Feynman diagrams and low-dimensional topology*, from "First European Congress of Mathematics, II" (A Joseph, F Mignot, F Murat, B Prum, R Rentschler, editors), Progr. Math. 120, Birkhäuser, Basel (1994) 97–121 MR

[21] **S Morita**, **T Sakasai**, **M Suzuki**, *Integral Euler characteristic of* $\mathrm{Out}\, F_{11}$, Exp. Math. 24 (2015) 93–97 MR

[22] **G Pólya**, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. 68 (1937) 145–254 MR

[23] *SageMath*, *version* 9.0 (2020) Available at https://www.sagemath.org

[24] **A Saied**, *On the FI-module structure of $H^i(\Gamma_{n,s})$*, preprint (2015) arXiv 1506.05861

[25] **R P Stanley**, *Enumerative combinatorics, I*, Cambridge Studies in Advanced Mathematics 49, Cambridge Univ. Press (1997) MR

[26] **V Turchin**, **T Willwacher**, *Commutative hairy graphs and representations of* $\mathrm{Out}(F_r)$, J. Topol. 10 (2017) 386–411 MR

[27] **J A M Vermaseren**, *New features of FORM*, preprint (2000) arXiv math-ph/0010025

[28] **J A M Vermaseren**, *FORM tutorial on the code that computes the Euler characteristic of $\mathcal{MG}_{g,n}$*, lecture notes (2020) Available at https://www.nikhef.nl/~form/maindir/courses/uam2022/six.pdf

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