Algebraic & Geometric Topology

Volume 25 (2025)

Classification of metric fibrations

Yasuhiko Asao

msp

Algebraic & Geometric Topology **25**:7 (2025) 4257–4285
DOI: 10.2140/agt.2025.25.4257
Published: 29 October 2025

Classification of metric fibrations

YASUHIKO ASAO

We study a notion of "fibration for metric spaces", called *metric fibration*, that was originally introduced by Leinster (Doc. Math. 18 (2013) 857–905) in the study of *magnitude*. He showed that the magnitude of a metric fibration splits into the product of those of the fiber and the base, which is analogous to the case for the Euler characteristic and topological fiber bundles. His idea and our approach are based on Lawvere's suggestion of viewing a metric space as an enriched category (Rend. Sem. Mat. Fis. Milano 43 (1973) 135–166). Actually, the metric fibrations are the restriction of the enriched *Grothendieck fibrations* (Séminaire Bourbaki 1959/1960 (1966) exposé 190) to metric spaces (arXiv 2303.05677). We give a complete classification of metric fibrations by several means, which are parallel to those used for topological fiber bundles. That is, the classification of metric fibrations is reduced to that of "principal fibrations", which is done by the "1-Čech cohomology" in an appropriate sense. Here we introduce the notion of *torsors* in the category of metric spaces, and the discussions are analogous to those in sheaf theory. Further, we can define the "fundamental group" $\pi_1^m(X)$ of a metric space X, which is a group-like object in metric spaces, such that the conjugation classes of homomorphisms $\mathsf{Hom}(\pi_1^m(X), \mathcal{G})$ correspond to the isomorphism classes of "principal $\mathcal{G}$-fibrations" over X. In other words, the latter are classified like topological covering spaces.

18F15, 51F30

1 Introduction

The idea of *metric fibration* was first introduced by Leinster in the study of magnitude [5]. The magnitude theory that he coined can be considered as a promotion of Lawvere's suggestion of viewing a metric space as a $[0, \infty]$-enriched category. The magnitude of a metric space was defined as a special case of the "Euler characteristic of enriched categories". In fact, he showed that the magnitude of a metric fibration splits into the product of those of the fiber and the base (Theorem 2.3.11 of [5]), which is analogous to the case of topological fiber bundles. Later, the author [1] pointed out that metric fibration can actually be seen as enriched *Grothendieck fibrations*, see [2], when restricted to metric spaces. Here we deal with small categories and metric spaces from a unified viewpoint, namely as *filtered set enriched categories*. By this approach, we can expect to obtain novel ideas for the study of metric spaces by transferring well-understood concepts in category theory, and vice versa.

As an example, Figure 1 is one of the simplest nontrivial metric fibrations. Note that we consider connected graphs as metric spaces by taking the shortest path metric (see also Proposition 2.8). Both graphs are

© 2025 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

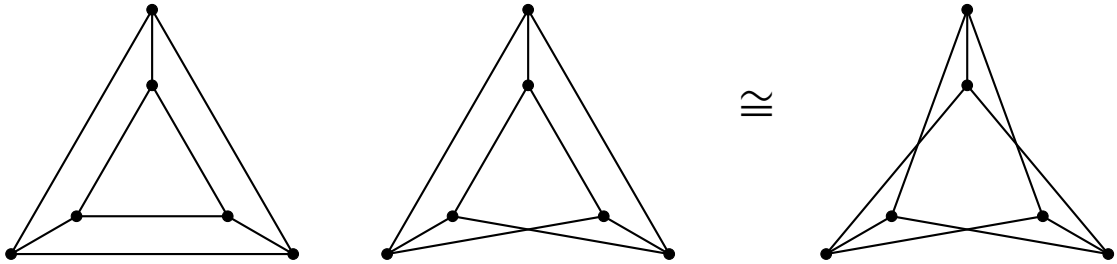

Figure 1: The left is $K_3 \times K_2$, and the right is isomorphic to $K_{3,3}$. They both have magnitude equal to $6/(1+3q+2q^2)$.

metric fibrations over the complete graph K_3 with fiber K_2 as shown in Example 5.29 of [1]. Further, they have the same magnitude as pointed out in Example 3.7 of [6]. In Proposition 5.30 of [1], it is shown that the right one is the only nontrivial metric fibration over K_3 with fiber K_2. Here, "trivial" means that it is the cartesian product of graphs. On the other hand, any metric fibration over a four-cycle graph C_4, or more generally an even-cycle graph, is shown to be trivial in the same proposition.

In this paper, we give a complete classification of metric fibrations by several means, which are parallel to those used to classify topological fiber bundles. Namely, we define "principal fibrations", "fundamental groups" and "a 1-Čech cohomology" for metric spaces, and obtain an equivalence between categories of these objects. Roughly speaking, we obtain an analogy of the following correspondence in the case of topological fiber bundles with a discrete structure group:

$$\begin{array}{c}
\text{Fiber bundles over } X \text{ with structure group } G \\
\updownarrow \\
\text{Principal } G\text{-bundles over } X \ (G\text{-torsors}) \\
\updownarrow \\
[X, BG] \cong \mathsf{Hom}(\pi_1(X), G)/\mathsf{conjugation} \\
\updownarrow \\
\mathsf{H}^1(X, G)
\end{array}$$

We give more details below. First recall that any Grothendieck fibration (in the usual sense) over a small category C can be obtained from a lax functor $C \to \mathsf{Cat}$, by a procedure known as the *Grothendieck construction* [3]. In [1], it is shown that any metric fibration over a metric space X can be obtained from a "lax functor" $X \to \mathsf{Met}$ that is called *metric action* (Definition 3.1). Here Met is the category of metric spaces and Lipschitz maps. Whereas metric fibrations can be defined by "a lifting property" in a model categorical spirit, metric actions are best understood by "transformation functions". The following shows that these two viewpoints are in fact equivalent.

Theorem 1.1 (Proposition 3.9) *For a metric space X, the Grothendieck construction gives an equivalence of categories*

$$\mathsf{Met}_X \simeq \mathsf{Fib}_X,$$

where we denote the category of metric actions $X \to \mathsf{Met}$ by Met_X and the category of metric fibrations over X by Fib_X (Definitions 3.1 and 3.2).

We can define a category $\mathsf{Tors}_X^{\mathcal{G}}$ that consists of "principal $\mathcal{G}$-fibrations" (Definition 5.7). We call it the *category of $\mathcal{G}$-torsors*. We can also define a subcategory $\mathsf{PMet}_X^{\mathcal{G}}$ of Met_X, that is the counterpart of $\mathsf{Tors}_X^{\mathcal{G}}$ (Definition 5.1). The objects of the category $\mathsf{PMet}_X^{\mathcal{G}}$ consist of a metric action $X \to \mathsf{Met}$ taking a group $\mathcal{G}$, not just a metric space, as the value. Then we have the following.

Theorem 1.2 (Proposition 5.11) *For a "metric group" $\mathcal{G}$, the Grothendieck construction gives an equivalence of categories*

$$\mathsf{PMet}_X^{\mathcal{G}} \simeq \mathsf{Tors}_X^{\mathcal{G}}.$$

Here, the group $\mathcal{G}$ is not just a group but is a group-like object in Met, which we call a *metric group* (Definition 4.1). As an example of a metric group, we construct the *fundamental group $\pi_1^m(X)$ of a metric space X* (Definition 4.8). We also define a category $\mathsf{Hom}(\pi_1^m(X), \mathcal{G})$ of homomorphisms $\pi_1^m(X) \to \mathcal{G}$, where a morphism between homomorphisms is defined as a conjugation relation (Definition 5.12). Then we have the following.

Theorem 1.3 (Proposition 5.15) *We have an equivalence of categories*

$$\mathsf{Hom}(\pi_1^m(X, x_0), \mathcal{G}) \simeq \mathsf{PMet}_X^{\mathcal{G}}.$$

As a corollary, we reprove Proposition 5.30 of [1] in the following form. We note that the notion of a metric group is equivalent to that of a "*normed group*" (Proposition 4.5). For a metric group $\mathcal{G}$, we denote the corresponding norm of an element $g \in \mathcal{G}$ by $|g| \in \mathbb{Z}_{\geq 0}$.

Proposition 1.4 (Proposition 5.16) *Let C_n be the undirected n-cycle graph. Then we have*

$$\pi_1^m(C_n) \cong \begin{cases} \mathbb{Z} \text{ with } |1| = 1 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Hence we have that

$$\mathsf{PMet}_{C_n}^{\mathcal{G}} \simeq \begin{cases} \mathsf{Hom}(\mathbb{Z}, \mathcal{G}) & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

for all metric groups $\mathcal{G}$, which implies that there is only a trivial metric fibration over C_{2n} and that there is at most one nontrivial metric fibration over C_{2n+1}.

Now, similarly to the topological case, we can define an "associated bundle construction" from a torsor and a metric space Y (Corollary 6.10). This construction gives the following.

Theorem 1.5 (Corollary 6.11) *Suppose that Y is a bounded metric space. Then we have an equivalence of categories*

$$\mathsf{PMet}_X^{\mathsf{Aut}\,Y} \simeq \mathsf{core}\,\mathsf{Fib}_X^Y,$$

where Fib_X^Y is the full subcategory of Fib_X that consists of metric fibrations with fiber Y (Definition 6.5), and we denote the core of a category by core (Definition 2.1(4)).

Here, we equip the group $\mathsf{Aut}\,Y$ of isometries of Y with a metric group structure by $d_{\mathsf{Aut}\,Y}(f,g) = \sup_{y\in Y} d_Y(fy, gy)$ (Example 4.3). However, for this we should suppose that Y is a bounded metric space so that $d_{\mathsf{Aut}\,Y}$ is indeed a distance function. For the case of general metric fibrations, we must extend our arguments to *extended metric groups*, allowing ∞ as the value of the distance function (Definition 6.12). For these we obtain an essentially similar but extended result (Proposition 6.18).

Finally, we define a "1-Čech cohomology category" $\mathsf{H}^1(X,\mathcal{G})$ of a $\mathcal{G}$-torsor X (Definition 7.2). This is analogous to the Čech cohomology constructed from the local sections of a principal bundle. Similarly to the topological case, we can construct a cocycle from a family of local sections (Proposition 7.9), and conversely we can construct a $\mathcal{G}$-torsor by pasting copies of $\mathcal{G}$ along a cocycle (Proposition 7.5). From this correspondence we have the following:

Theorem 1.6 (Corollary 7.11) *We have an equivalence of categories*

$$\mathsf{H}^1(X;\mathcal{G}) \simeq \mathsf{Tors}_X^{\mathcal{G}}.$$

Acknowledgements The author is grateful to Luigi Caputi and Emily Roff for fruitful and helpful comments and feedback on the first draft of the paper. In particular, Emily pointed out that the earlier version of the definition of metric groups is incorrect. She also checked the author's poor English and gave many very helpful suggestions. The author also would like to thank Masahiko Yoshinaga and Shun Wakatsuki for valuable discussions and comments. Finally, he expresses his deep appreciation to the anonymous referees and the editor for careful readings and important suggestions.

2 Conventions

In this section, we review terms for categories, graphs, weighted graphs and metric spaces that are well known but may not be commonly used. Those who are not familiar with the language of categories and functors may refer to [8]. For categories and metric spaces, [7] may be a good reference.

2.1 Categories

In this article, we suppose that categories are locally small. We denote the object class of a category C by $\mathrm{Ob}\,C$, and the set of all morphisms from a to b by $C(a,b)$ for each pair of objects $a,b \in \mathrm{Ob}\,C$. We denote the class of all morphisms in C by $\mathrm{Mor}\,C$.

Definition 2.1 Let C and D be categories, and $F\colon C \to D$ be a functor.

(1) We say that F is *faithful* if the map $F\colon C(a,b) \to D(Fa, Fb)$ is injective for all objects $a, b \in \operatorname{Ob} C$. We say that F is *full* if the map $F\colon C(a,b) \to D(Fa, Fb)$ is surjective for all objects $a, b \in \operatorname{Ob} C$. We say that F is *fully faithful* if it is faithful and full.

(2) We say that F is *split essentially surjective* if a family of isomorphisms $\{Fc \cong d \mid c \in \operatorname{Ob} C\}_{d \in \operatorname{Ob} D}$ exists.

(3) We say that F is a *category equivalence* if there exists a functor $G\colon D \to C$ and natural isomorphisms $GF \cong \mathrm{id}_C$ and $FG \cong \mathrm{id}_D$. When there exists an equivalence of categories $C \to D$, we say that *C and D are equivalent*.

(4) We define a groupoid $\mathsf{core}\, C$ by $\operatorname{Ob} \mathsf{core}\, C = \operatorname{Ob} C$ and $\mathsf{core}\, C(a,b) = \{f \in C(a,b) \mid f \text{ is an isomorphism}\}$ for all $a, b \in \operatorname{Ob} C$.

The following are standard.

Lemma 2.2 *If a functor $F\colon C \to D$ is fully faithful and split essentially surjective, then it is an equivalence of categories.* □

Lemma 2.3 *An equivalence of categories $F\colon C \to D$ induces an equivalence of categories $\mathsf{core}\, F\colon \mathsf{core}\, C \to \mathsf{core}\, D$.* □

Remark 2.4 For a classification of objects of a category, we often want to consider "isomorphism classes of objects" and compare it with another category. However, in general, we cannot do that since the class of objects is not necessarily a set. Instead, we consider an equivalence of categories $\mathsf{core}\, C \to \mathsf{core}\, D$ that implies a bijection between isomorphism classes of objects if they are small.

2.2 Metric spaces

Definition 2.5 (1) A *pseudometric space* (X, d) is a set X equipped with a function $d\colon X \to \mathbb{R}_{\geq 0}$ satisfying that, for all $x, x', x'' \in X$,

- $d(x,x) = 0$,
- $d(x,x') = d(x',x)$,
- $d(x,x') + d(x',x'') \geq d(x,x'')$.

(2) A *Lipschitz map* $f\colon X \to Y$ between pseudometric spaces X and Y is a map satisfying that $d_Y(fx, fx') \leq d_X(x,x')$ for all $x, x' \in X$. We denote the category of pseudometric spaces and Lipschitz maps by $\psi\mathsf{Met}$. We call an isomorphism in $\psi\mathsf{Met}$ an *isometry*.

(3) A *metric space* (X, d) is a pseudometric space satisfying that

- $d(x,x') = 0$ if and only if $x = x'$.

We denote the full subcategory of $\psi\mathsf{Met}$ that consists of metric spaces by Met.

Definition 2.6 (1) A *graph* G is a pair of sets $(V(G), E(G))$ such that $E(G) \subset \{e \in 2^{V(G)} \mid \#e = 2\}$, where we denote the cardinality of a set by #. We call an element of $V(G)$ a *vertex*, and an element of $E(G)$ an *edge*. A *graph homomorphism* $f : G \to H$ between graphs G and H is a map $f : V(G) \to V(H)$ such that $fe \in E(H)$ or $\#fe = 1$ for all $e \in E(G)$. We denote the category of graphs and graph homomorphisms by Grph.

(2) A *path* in a graph G is a tuple $(x_0, \dots, x_n) \in V(G)^{n+1}$ for some $n \geq 0$ such that $\{x_i, x_{i+1}\} \in E(G)$ for all $0 \leq i \leq n-1$. A *connected graph* G is a graph such that for every $x, x' \in V(G)$ there exists a path $(x_0, \dots, x_n)$ with $x_0 = x$ and $x_n = x'$. We denote the full subcategory of Grph that consists of connected graphs by $\mathsf{Grph}_{\mathsf{conn}}$.

(3) A *weighted graph* (G, w_G) is a graph G equipped with a function $w_G : E(G) \to \mathbb{R}_{\geq 0}$. A *weighted graph homomorphism* $f : G \to H$ between weighted graphs G and H is a graph homomorphism such that $w_H(fe) \leq w_G(e)$ for all $e \in E(G)$, where we stipulate that $w_H(fe) = 0$ if $\#fe = 1$. We denote the category of weighted graphs and weighted graph homomorphisms by wGrph. We also denote the full subcategory of wGrph that consists of weighted graphs (G, w_G) such that the graph G is connected by $\mathsf{wGrph}_{\mathsf{conn}}$.

Definition 2.7 We define functors $\mathsf{Met} \to \psi\mathsf{Met}$ and $\mathsf{wGrph}_{\mathsf{conn}} \to \mathsf{Grph}_{\mathsf{conn}}$ by forgetting additional properties. We also define the functor $\psi\mathsf{Met} \to \mathsf{wGrph}_{\mathsf{conn}}$ that sends a pseudometric space (X, d) to a weighted graph (X, w_X) defined by $V(X) = X$, $E(X) = \{e \in 2^X \mid \#e = 2\}$ and $w_X\{x, x'\} = d(x, x')$.

Proposition 2.8 *The above functors have left adjoints.*

Proof We describe each functor F in the following, and they are the left adjoint functors of each functor G of the above since the unit and the counit give that $FGF = F$ and $GFG = G$.

- We define a functor $\mathsf{Grph}_{\mathsf{conn}} \to \mathsf{wGrph}_{\mathsf{conn}}$ by sending a connected graph to a weighted graph with $w = 0$.
- We define a functor $\mathsf{wGrph}_{\mathsf{conn}} \to \psi\mathsf{Met}$ by sending a weighted graph (G, w_G) to a pseudometric space $(V(G), d_G)$ defined by

$$d_G(x, x') = \inf \bigcup_{n \geq 0} \Big\{ \sum_{i=0}^{n-1} w_G\{x_i, x_{i+1}\} \mid (x = x_0, \dots, x_n = x') \text{ is a path on } G \Big\}.$$

- We define a functor $\psi\mathsf{Met} \to \mathsf{Met}$ by sending a pseudometric space (X, d) to a metric space $(\mathsf{KQ}X, \widetilde{d})$ defined as follows. We define an equivalence relation $\sim$ on X by $x \sim x'$ if and only if $d(x, x') = 0$. We also define a function $\mathsf{KQ}X := X/\!\sim \to \mathbb{R}_{\geq 0}$ by $\widetilde{d}([x], [x']) = d(x, x')$. □

Definition 2.9 For a pseudometric space X, we call the metric space $\mathsf{KQ}X$ the *Kolmogorov quotient* of X.

Definition 2.10 (1) For pseudometric spaces (X, d_X) and (Y, d_Y), we define a metric space called the L^1*-product* $(X \times Y, d_{X \times Y})$ by $d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$ for all $x, x' \in X$ and $y, y' \in Y$.

(2) For graphs G and H, we define a graph called the *cartesian product* $G \times H$ by $V(G \times H) = V(G) \times V(H)$, and $\{(x, y), (x', y')\} \in E(G \times H)$ if and only if one of the following holds:

- $x = x'$ and $\{y, y'\} \in E(H)$,
- $\{x, x'\} \in E(G)$ and $y = y'$,

for all $x, x' \in V(G)$ and $y, y' \in V(H)$.

(3) For weighted graphs (G, w_G) and (H, w_H), we define a weighted graph $(G \times H, w_{G\times H})$ by $w_{G\times H}\{(x, y), (x', y')\} = w_G\{x, x'\} + w_H\{y, y'\}$ for all $\{(x, y), (x', y')\} \in E(G \times H)$, where $G \times H$ is the cartesian product of graphs and we stipulate that $w_G\{x, x\} = w_H\{y, y\} = 0$.

These products make each category a symmetric monoidal category.

Proposition 2.11 *The functors* $\mathsf{Met} \to \psi\mathsf{Met} \to \mathsf{wGrph}_{\mathsf{conn}} \to \mathsf{Grph}_{\mathsf{conn}}$ *and their left adjoints are strong monoidal except for the functor* $\psi\mathsf{Met} \to \mathsf{wGrph}_{\mathsf{conn}}$ *that is lax monoidal.*

Proof For the functors $\mathsf{Met} \to \psi\mathsf{Met}$ and $\mathsf{wGrph}_{\mathsf{conn}} \to \mathsf{Grph}_{\mathsf{conn}}$, the claim is obvious since they are inclusions. The claim is also obvious for the functor $\mathsf{Grph}_{\mathsf{conn}} \to \mathsf{wGrph}_{\mathsf{conn}}$ by the definition. For the functor $\psi\mathsf{Met} \to \mathsf{Met}$, we define a map $\mathsf{KQ}(X \times Y) \to \mathsf{KQ}X \times \mathsf{KQ}Y$ by $[(x, y)] \mapsto ([x], [y])$. This is obviously natural and is an isometry since we have that $[(x, y)] \sim [(x', y')]$ if and only if $[x] \sim [x']$ and $[y] \sim [y']$. For the functor $F\colon \mathsf{wGrph}_{\mathsf{conn}} \to \psi\mathsf{Met}$, the identity on the set $F(G \times H) = F(G) \times F(H)$ is an isometry since

$$\begin{aligned}
d_{w_{G\times H}}((x, y), (x', y')) &= \inf \bigcup_{n\geq 0} \Big\{ \sum_{i=0}^{n-1} w_{G\times H}\{(x_i, y_i), (x_{i+1}, y_{i+1})\} \mid \\
&\qquad\qquad ((x, y) = (x_0, y_0), \dots, (x_n, y_n) = (x', y')) \text{ is a path on } G \times H \Big\} \\
&= \inf \bigcup_{n\geq 0} \Big\{ \sum_{i=0}^{n-1} w_G\{x_i, x_{i+1}\} + w_H\{y_i, y_{i+1}\} \mid \\
&\qquad\qquad ((x, y) = (x_0, y_0), \dots, (x_n, y_n) = (x', y')) \text{ is a path on } G \times H \Big\} \\
&= \inf \bigcup_{n\geq 0} \Big\{ \sum_{i=0}^{n-1} w_G\{x_i, x_{i+1}\} \mid (x = x_0, \dots, x_n = x') \Big\} \\
&\qquad\qquad + \inf \bigcup_{m\geq 0} \Big\{ \sum_{i=0}^{m-1} w_H\{y_i, y_{i+1}\} \mid (y = y_0, \dots, y_m = y') \Big\} \\
&= d_{w_G}(x, x') + d_{w_H}(y, y') \\
&= d_{F(G)\times F(H)}((x, y), (x', y')),
\end{aligned}$$

for all $x, x' \in V(G)$ and $y, y' \in V(H)$. It is obviously natural. Finally, for the functor $G\colon \psi\mathsf{Met} \to \mathsf{wGrph}_{\mathsf{conn}}$, the identity on the set $G(X) \times G(Y) = G(X \times Y)$ is a weighted graph homomorphism since it is an inclusion of graphs and preserves weightings. It is obviously natural. □

Definition 2.12 (1) An *extended pseudometric space* is a set X equipped with a function $d\colon X \to [0,\infty]$ that satisfies the same conditions for pseudometric spaces. In other words, it is a pseudometric space admitting ∞ as a value of distance. A *Lipschitz map* between extended pseudometric spaces is a distance nonincreasing map. We denote the category of extended pseudometric spaces and Lipschitz maps by $\mathsf{E}\psi\mathsf{Met}$. We similarly define *extended metric spaces* and we denote the full subcategory of $\mathsf{E}\psi\mathsf{Met}$ that consists of them by EMet.

(2) We define the L^1-product of extended pseudometric spaces similarly to that of pseudometric spaces. It makes the category $\mathsf{E}\psi\mathsf{Met}$ a symmetric monoidal category.

(3) We define functors $\mathsf{EMet} \to \mathsf{E}\psi\mathsf{Met}$ and $\mathsf{wGrph} \to \mathsf{Grph}$ by forgetting additional properties. We also define the functor $\mathsf{E}\psi\mathsf{Met} \to \mathsf{wGrph}$ similarly to the functor $\psi\mathsf{Met} \to \mathsf{wGrph}_{\mathsf{conn}}$ except that $\{x, x'\}$ does not span an edge for $x, x' \in X$ with $d(x, x') = \infty$.

The following is immediate.

Proposition 2.13 (1) *The functors* $\mathsf{EMet} \to \mathsf{E}\psi\mathsf{Met} \to \mathsf{wGrph} \to \mathsf{Grph}$ *have left adjoints. Further, the following diagram is commutative, where the vertical functors are all inclusions*:

$$\begin{tikzcd}
\mathsf{EMet} \arrow[r] & \mathsf{E}\psi\mathsf{Met} \arrow[r] & \mathsf{wGrph} \arrow[r] & \mathsf{Grph} \\
\mathsf{Met} \arrow[r] \arrow[u] & \psi\mathsf{Met} \arrow[r] \arrow[u] & \mathsf{wGrph}_{\mathsf{conn}} \arrow[r] \arrow[u] & \mathsf{Grph}_{\mathsf{conn}} \arrow[u]
\end{tikzcd}$$

(2) *The functors* $\mathsf{EMet} \to \mathsf{E}\psi\mathsf{Met}$ *and* $\mathsf{wGrph} \to \mathsf{Grph}$ *are strong monoidal and the functor* $\mathsf{E}\psi\mathsf{Met} \to \mathsf{wGrph}$ *is lax monoidal.* □

3 $\mathsf{Met}_X \simeq \mathsf{Fib}_X$

In this section, we introduce two notions, the *metric action* and the *metric fibration*, and show the equivalence between them. The notion of metric fibration is originally introduced by Leinster [5] in the study of magnitude. The metric action was introduced by the present author in [1], and is the counterpart of *lax functors* in category theory, while the metric fibration is a generalization of the *Grothendieck fibration*. As written in the introduction, we can think of the Grothendieck (or metric) fibration as the definition of fibrations by "the lifting property", while the lax functor is the one by "the transformation functions".

Definition 3.1 Let X be a metric space.

(1) A *metric action* $F\colon X \to \mathsf{Met}$ consists of metric spaces $Fx \in \mathsf{Met}$ for all $x \in X$ and isometries $F_{xx'}\colon Fx \to Fx'$ for all $x, x' \in X$ satisfying the following for all $x, x', x'' \in X$:

- $F_{xx} = \mathrm{id}_{Fx}$ and $F_{x'x} = F_{xx'}^{-1}$.
- $d_{Fx''}(F_{x'x''}F_{xx'}a, F_{xx''}a) \le d_X(x, x') + d_X(x', x'') - d_X(x, x'')$ for every $a \in Fx$.

(2) A *metric transformation* $\theta\colon F \Rightarrow G$ consists of Lipschitz maps $\theta_x\colon Fx \to Gx$ for all $x \in X$ satisfying that $G_{xx'}\theta_x = \theta_{x'}F_{xx'}$ for all $x, x' \in X$. We can define the composition of metric transformations θ and θ' by $(\theta'\theta)_x = \theta'_x\theta_x$. We denote the category of metric actions $X \to \mathsf{Met}$ and metric transformations by Met_X.

Definition 3.2 (1) Let $\pi\colon E \to X$ be a Lipschitz map between metric spaces. We say that π is a *metric fibration over* X if it satisfies the following: for all $\varepsilon \in E$ and $x \in X$, there uniquely exists $\varepsilon_x \in \pi^{-1}x$ such that

- $d_E(\varepsilon, \varepsilon_x) = d_X(\pi\varepsilon, x)$,
- $d_E(\varepsilon, \varepsilon') = d_E(\varepsilon, \varepsilon_x) + d_E(\varepsilon_x, \varepsilon')$ for all $\varepsilon' \in \pi^{-1}x$.

We call the point ε_x the *lift of* x *along* ε.

(2) For metric fibrations $\pi\colon E \to X$ and $\pi'\colon E' \to X$, a *morphism* $\varphi\colon \pi \to \pi'$ is a Lipschitz map $\varphi\colon E \to E'$ such that $\pi'\varphi = \pi$. We denote the category of metric fibrations over X and their morphisms by Fib_X.

Example 3.3 For a product of metric spaces $E = X \times Y$, the projection $X \times Y \to X$ is a metric fibration. We call it a *trivial metric fibration*.

Lemma 3.4 *Let $\pi\colon E \to X$ be a metric fibration, and $x, x' \in X$. Then the correspondence $\pi^{-1}x \ni a \mapsto a_{x'} \in \pi^{-1}x'$ is an isometry, where we equip the sets $\pi^{-1}x$ and $\pi^{-1}x'$ with the induced metric from E.*

Proof The statement is obviously true if $E = \varnothing$. We suppose that $E \neq \varnothing$ in the following; then every fiber $\pi^{-1}x$ is nonempty. For $a \in \pi^{-1}x$, we have

$$d_E(a_{x'}, a) = d_E(a_{x'}, (a_{x'})_x) + d_E((a_{x'})_x, a) = d_X(x', x) + d_E((a_{x'})_x, a).$$

We also have $d_E(a, a_{x'}) = d_X(x, x')$. Hence we obtain that $d_E((a_{x'})_x, a) = 0$, and thus $(a_{x'})_x = a$ for all $x, x' \in X$. This implies that the correspondence is a bijection. Further, we have

$$d_E(a, b_{x'}) = d_E(a, a_{x'}) + d_E(a_{x'}, b_{x'}) = d_X(x, x') + d_E(a_{x'}, b_{x'})$$

and

$$d_E(b_{x'}, a) = d_E(b_{x'}, b) + d_E(b, a) = d_X(x', x) + d_E(b, a)$$

for all $a, b \in \pi^{-1}x$. We thereby obtain that $d_E(a, b) = d_E(a_{x'}, b_{x'})$ for all $x, x' \in X$ and $a, b \in \pi^{-1}x$, which implies that the correspondence is an isometry. □

Lemma 3.5 *Let $\varphi\colon \pi \to \pi'$ be a morphism of metric fibrations. For all $x, x' \in X$ and $a \in \pi^{-1}x$, we have $(\varphi a)_{x'} = \varphi a_{x'}$.*

Proof We have

$$d_{E'}((\varphi a)_{x'}, \varphi a_{x'}) = d_{E'}(\varphi a, \varphi a_{x'}) - d_X(x, x') \leq d_E(a, a_{x'}) - d_X(x, x') = 0,$$

and hence we obtain that $(\varphi a)_{x'} = \varphi a_{x'}$. □

Definition 3.6 Let $F\colon X \to \mathsf{Met}$ be a metric action. We define a metric fibration $\pi_F\colon E(F) \to X$ as follows:

(1) $E(F) = \{(x,a) \mid a \in Fx,\ x \in X\}$.

(2) $d_{E(F)}((x,a),(x',b)) = d_X(x,x') + d_{Fx'}(F_{xx'}a, b)$.

(3) $\pi_F(x,a) = x$.

We call the above construction the *Grothendieck construction*.

Proposition 3.7 *The Grothendieck construction gives a functor $E\colon \mathsf{Met}_X \to \mathsf{Fib}_X$.*

Proof Let $\theta\colon F \Rightarrow G$ be a metric transformation. Then we construct Lipschitz maps $\varphi_\theta\colon E(F) \to E(G)$ by $\varphi_\theta(x,a) = (x, \theta_x a)$ for all $x \in X$ and $a \in Fx$. To see that φ_θ is a Lipschitz map, observe that

$$\begin{aligned} d_{E(G)}(\varphi_\theta(x,a), \varphi_\theta(x',b)) &= d_{E(G)}((x,\theta_x a),(x',\theta_{x'} b)) \\ &= d_X(x,x') + d_{Gx'}(G_{xx'}\theta_x a, \theta_{x'} b) \\ &= d_X(x,x') + d_{Gx'}(\theta_{x'} F_{xx'} a, \theta_{x'} b) \\ &\le d_X(x,x') + d_{Fx'}(F_{xx'}a, b) \\ &= d_{E(F)}((x,a),(x',b)). \end{aligned}$$

It remains to see that the correspondence $\theta \mapsto \varphi_\theta$ is functorial — that is we have $\varphi_{\mathrm{id}_F} = \mathrm{id}_{E(F)}$ and $\varphi_{\theta'\theta} = \varphi_{\theta'}\varphi_\theta$ for all metric transformations $\theta\colon F \Rightarrow G$ and $\theta'\colon G \Rightarrow H$. The former is obvious and the latter is checked as follows:

$$\varphi_{\theta'\theta}(x,a) = (x, (\theta'\theta)_x a) = (x, \theta'_x \theta_x a) = \varphi_{\theta'}\varphi_\theta(x,a).$$

Finally, φ_θ is obviously a morphism of the metric fibration. □

Proposition 3.8 *We have a functor $F\colon \mathsf{Fib}_X \to \mathsf{Met}_X$ sending a metric fibration π to a metric action F_π with $F_\pi x = \pi^{-1}x$.*

Proof Let $\pi\colon E \to X$ be a metric fibration. We define a metric action $F_\pi\colon X \to \mathsf{Met}$ by $F_\pi x = \pi^{-1}x$ and $(F_\pi)_{xx'}a = a_{x'}$ for all $x, x' \in X$ and $a \in \pi^{-1}x$, where we equip the set $\pi^{-1}x$ with the induced metric from E. It follows that $(F_\pi)_{xx} = \mathrm{id}_{F_\pi x}$ by the uniqueness of the lifts, and that $(F_\pi)_{xx'}$ defines an isometry $F_\pi x \to F_\pi x'$ with $(F_\pi)^{-1}_{xx'} = (F_\pi)_{x'x}$ by Lemma 3.4. Further, we have that

$$\begin{aligned} d_{F_\pi x''}((F_\pi)_{x'x''}(F_\pi)_{xx'}a, (F_\pi)_{xx''}a) &= d_{F_\pi x''}((a_{x'})_{x''}, a_{x''}) \\ &= d_E(a, (a_{x'})_{x''}) - d_X(x,x'') \\ &\le d_E(a, a_{x'}) + d_E(a_{x'}, (a_{x'})_{x''}) - d_X(x,x'') \\ &= d_X(x,x') + d_X(x',x'') - d_X(x,x''), \end{aligned}$$

for all $x, x', x'' \in X$ and $a \in F_\pi x$. Hence F_π defines a metric action $X \to \mathsf{Met}$. Next, let $\varphi\colon \pi \to \pi'$ be a morphism of metric fibrations. We define a metric transformation $\theta_\varphi\colon F_\pi \Rightarrow F_{\pi'}$ by $(\theta_\varphi)_x a = \varphi a$ for all $x \in X$ and $a \in F_\pi x$. Then we have that

$$(F_{\pi'})_{xx'}(\theta_\varphi)_x a = (F_{\pi'})_{xx'}\varphi a = (\varphi a)_{x'} = \varphi a_{x'} = (\theta_\varphi)_{x'}(F_\pi)_{xx'},$$

where the third line follows from Lemma 3.5. Thus, θ_φ defines a metric transformation $F_\pi \Rightarrow F_{\pi'}$. Note that we have $\theta_{\mathrm{id}_\pi} = \mathrm{id}_{F_\pi}$ and $(\theta_{\psi\varphi})_x a = \psi\varphi a = (\theta_\psi)_x(\theta_\varphi)_x a$ for morphisms φ and ψ, which implies the functoriality of F. □

The following is the counterpart of the correspondence between lax functors and the Grothendieck fibrations [4, B1], and enhances Corollary 5.26 of [1].

Proposition 3.9 *The Grothendieck construction functor $E\colon \mathsf{Met}_X \to \mathsf{Fib}_X$ is an equivalence of categories.*

Proof We show that $FE \cong \mathrm{id}_{\mathsf{Met}_X}$ and $EF \cong \mathrm{id}_{\mathsf{Fib}_X}$. It is immediate to verify $FE \cong \mathrm{id}_{\mathsf{Met}_X}$ by the definition. We show that $EF_\pi \cong \pi$ for a metric fibration $\pi\colon E \to X$. Note that EF_π is a metric space consisting of points (x, a) with $x \in X$ and $a \in \pi^{-1}x$, and we have $d_{EF_\pi}((x,a),(x',a')) = d_X(x,x') + d_{\pi^{-1}x'}(a_{x'}, a')$. We define a map $f\colon EF_\pi \to E$ by $f(x,a) = a$ for all $x \in X$ and $a \in \pi^{-1}x$. Then it is obviously an isometry and preserves fibers, and hence is an isomorphism of metric fibrations. The naturality of this isomorphism is obvious. □

Remark 3.10 The trivial metric fibration corresponds to the constant metric action, that is $F_{xx'} = \mathrm{id}$ for all $x, x' \in X$.

4 The metric fundamental group of a metric space

In this section, we give a concise introduction to *metric groups*. We also give a definition of *metric fundamental group*, which plays a role of π_1 for metric space in the classification of metric fibrations.

4.1 Metric groups

Definition 4.1 (see [10, Definition 6.1]) (1) A *metric group* is a monoid object in Met that is a group when we forget the metric space structure. That is, a metric space $\mathcal{G}$ equipped with a Lipschitz map $\cdot\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, a function $(-)^{-1}\colon \mathcal{G} \to \mathcal{G}$ and a point $e \in \mathcal{G}$ satisfying the suitable conditions of monoids and groups.

(2) For metric groups $\mathcal{G}$ and $\mathcal{H}$, a *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a Lipschitz map $\mathcal{G} \to \mathcal{H}$ that commutes with the group structure.

(3) We denote the category of metric groups and homomorphisms by MGrp.

Lemma 4.2 *Let $(\mathcal{G}, d)$ be a metric group.*

(1) *We have $d(kg, kh) = d(g, h) = d(gk, hk)$ for all $g, h, k \in \mathcal{G}$.*

(2) *We have $d(g, h) = d(g^{-1}, h^{-1})$ for all $g, h \in \mathcal{G}$.*

Proof (1) Since the map $\mathcal{G} \to \mathcal{G}$, $g \mapsto kg$, is a Lipschitz map for every $k \in \mathcal{G}$, we have $d(kg, kh) \leq d(g, h)$ and $d(k^{-1}(kg), k^{-1}(kh)) \leq d(kg, kh)$. Hence we obtain that $d(kg, kh) = d(g, h)$. The other identity can be proved similarly.

(2) By (1), we have $d(g^{-1}, h^{-1}) = d(e, gh^{-1}) = d(h, g) = d(g, h)$. □

Example 4.3 Let (X, d) be a metric space, and let $\mathsf{Aut}^u X$ be the set of isometries f on X such that $\sup_{x \in X} d_X(x, fx) < \infty$. We equip $\mathsf{Aut}^u X$ with a group structure by compositions. We also define a distance function on $\mathsf{Aut}^u X$ by $d_{\mathsf{Aut}^u X}(f, g) = \sup_{x \in X} d_X(fx, gx)$. It is straightforward to verify that $(\mathsf{Aut}^u X, d_{\mathsf{Aut}^u X})$ is a metric group. Note that, if the metric space X is bounded, meaning that

$$\sup_{x, x' \in X} d_X(x, x') < \infty,$$

then the group $\mathsf{Aut}^u X$ consists of all isometries of X. In this case we denote it by $\mathsf{Aut}\, X$.

Definition 4.4 (1) A *normed group* is a group G equipped with a map $|-|\colon G \to \mathbb{R}_{\geq 0}$ called a norm, satisfying that

- $|g| = 0$ if and only if $g = e$,
- $|gh| \leq |g| + |h|$ for all $g, h \in G$.

(2) A norm on G is called *conjugation invariant* if it satisfies that $|h^{-1}gh| = |g|$ for all $g, h \in G$.

(3) A norm on G is called *inverse invariant* if it satisfies that $|g^{-1}| = |g|$ for all $g \in G$.

(4) For normed groups G and H, a *normed homomorphism* from G to H is a group homomorphism $\varphi\colon G \to H$ satisfying that $|\varphi g| \leq |g|$.

(5) We denote the category of conjugation and inverse invariant normed groups and normed homomorphisms by $\mathsf{NGrp}^{-1}_{\mathrm{conj}}$.

Proposition 4.5 (E Roff, [9, Chapter 6]) *The categories MGrp and $\mathsf{NGrp}^{-1}_{\mathrm{conj}}$ are equivalent.*

Proof Given a metric group $\mathcal{G}$, we can define a conjugation and inverse invariant normed group $\mathsf{N}\mathcal{G}$ by

- $\mathsf{N}\mathcal{G} = \mathcal{G}$ as a group,
- $|g| = d_{\mathcal{G}}(e, g)$ for all $g \in \mathsf{N}\mathcal{G}$.

This construction is functorial. Conversely, we can define a metric group $\mathsf{M}G$ given a conjugation and inverse invariant normed group G by

- $\mathsf{M}G = G$ as a group,
- $d_{\mathsf{M}G}(g,h) = |h^{-1}g|$.

This construction is also functorial. It is straightforward to verify that the compositions of these functors are naturally isomorphic to the identities. □

4.2 The metric fundamental group

Definition 4.6 Let X be a metric space and $x \in X$.

(1) For each $n \geq 0$, we define a set $P_n(X,x)$ by

$$P_n(X,x) := \{(x, x_1, \dots, x_n, x) \in X^{n+2}\}.$$

We also define that $P(X,x) := \bigcup_n P_n(X,x)$.

(2) We define a connected graph $G(X,x)$ with the vertex set $P(X,x)$ as follows. For $u, v \in P(X,x)$, an unordered pair $\{u,v\}$ spans an edge if and only if it satisfies both of the following:

- There is an $n \geq 0$ such that $u \in P_n(X,x)$ and $v \in P_{n+1}(X,x)$.
- There is a $0 \leq j \leq n$ such that $u_i = v_i$ for $1 \leq i \leq j$ and $u_i = v_{i+1}$ for $j+1 \leq i \leq n$, where we have $u = (x, u_1, \dots, u_n, x)$ and $v = (x, v_1, \dots, v_{n+1}, x)$.

(3) We equip the graph $G(X,x)$ with a weighted graph structure by defining a function $w_{G(X,x)}$ on edges by

$$w_{G(X,x)}\{u,v\} = \begin{cases} d_X(v_j, v_{j+1}) + d_X(v_{j+1}, v_{j+2}) - d_X(v_j, v_{j+2}) & \text{if } v_j \neq v_{j+2}, \\ 0 & \text{if } v_j = v_{j+2}, \end{cases}$$

where we use the notation in (2).

(4) We denote the quasimetric space obtained from the weighted graph $G(X,x)$ by $Q(X,x)$. We denote the Kolmogorov quotient of $Q(X,x)$ by $\pi_1^m(X,x)$.

Lemma 4.7 *Let X be a metric space and $x \in X$.*

(1) *The metric space $\pi_1^m(X,x)$ has a metric group structure with multiplication defined by the concatenation defined as*

$$[(x, u_1, \dots, u_n, x)] \bullet [(x, v_1, \dots, v_k, x)] = [(x, u_1, \dots, u_n, v_1, \dots, v_k, x)].$$

The unit is given by $[(x,x)] \in \pi_1^m(X,x)$.

(2) *For all $x' \in X$, we have an isomorphism $\pi_1^m(X,x) \cong \pi_1^m(X,x')$ given by*

$$[(x,u_1,\dots,u_n,x)] \mapsto [(x',x,u_1,\dots,u_n,x,x')].$$

Proof (1) We first show that concatenation makes the weighted graph $G(X,x)$ into a monoid object in $\mathsf{wGrph}_{\mathsf{conn}}$. Let $(u,v),(u',v') \in G(X,x)\times G(X,x)$, and suppose that $\{(u,v),(u',v')\}$ spans an edge. Then we have that $u=u'$ and $v\in P_n(X,x)$, $v'\in P_{n+1}(X,x)$, or $v=v'$ and $u\in P_n(X,x)$, $u'\in P_{n+1}(X,x)$ for some n. We also have that $w_{G(X,x)\times G(X,x)}\{(u,v),(u',v')\} = w_{G(X,x)}\{u,u'\} + w_{G(X,x)}\{v,v'\}$. Note that $\{u\bullet v, u'\bullet v'\}$ spans an edge in $G(X,x)$. Further, we have $w_{G(X,x)}\{u\bullet v, u'\bullet v'\} = w_{G(X,x)}\{u,u'\} + w_{G(X,x)}\{v,v'\}$. Hence the concatenation map $\bullet\colon G(X,x)\times G(X,x)\to G(X,x)$ is a weighted graph homomorphism. It is immediate to verify that the identity is the element (x,x) and that the multiplication is associative. Thus the weighted graph $G(X,x)$ is a monoid object in $\mathsf{wGrph}_{\mathsf{conn}}$, and by Proposition 2.11, $\pi_1^m(X,x)$ is a monoid object in Met.

Now we show that any element $[(x,x_0,\dots,x_n,x)]$ has an inverse, namely the element $[(x,x_n,\dots,x_0,x)]$. It is enough to show that

$$d_{Q(X,x)}((x,x_n,\dots,x_0,x_0,\dots,x_nx),(x,x)) = 0.$$

And indeed, it is obvious that the elements $(x,x_n,\dots,x_0,x_0,\dots,x_nx)$ and (x,x) can be connected by a path that consists of edges with weight 0 in $G(X,x)$, which implies the desired equality.

(2) This is straightforward. □

Definition 4.8 Let X be a metric space and $x\in X$. We call the metric group $\pi_1^m(X,x)$ the *metric fundamental group of* X with respect to the base point x. We sometimes omit the base point and denote it by $\pi_1^m(X)$.

Remark 4.9 As a group, $\pi_1^m(X)$ can be obtained as the fundamental group of a simplicial complex S_X whose n-simplices are subsets $\{x_0,\dots,x_n\}\subset X$ such that any distinct 3 points x_i,x_j,x_k satisfy that $|\Delta(x_i,x_j,x_k)|=0$ (see Definition 7.1).

Remark 4.10 Our fundamental group $\pi_1^m(X)$ is not functorial with respect to Lipschitz maps. However, it is functorial with respect to Lipschitz maps that preserve collinearity ($|\Delta(x_i,x_j,x_k)|=0$), including every embedding of metric spaces.

5 $\mathsf{PMet}_X^{\mathcal{G}} \simeq \mathsf{Tors}_X^{\mathcal{G}} \simeq \mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G})$

In this section, we introduce the notion of "principal $\mathcal{G}$-bundles" for metric spaces. We define it from two different viewpoints, namely as a metric action and as a metric fibration, which turn out to be equivalent. As a metric action, we call it a *$\mathcal{G}$-metric action*, and as a metric fibration, we call it a *$\mathcal{G}$-torsor*. Then we show that they are classified by the conjugation classes of homomorphisms $\pi_1^m(X,x_0)\to\mathcal{G}$.

5.1 $\mathsf{PMet}_X^{\mathcal{G}} \simeq \mathsf{Tors}_X^{\mathcal{G}}$

Definition 5.1 Let X be a metric space and $\mathcal{G}$ be a metric group.

(1) A *$\mathcal{G}$-metric action* $F\colon X \to \mathsf{Met}$ is a metric action satisfying the following:

- For all $x \in X$, $F_x = \mathcal{G}$.
- For all $x, x' \in X$, $F_{xx'}$ is a left multiplication by some $f_{xx'} \in \mathcal{G}$.

(2) Let $F, G\colon X \to \mathsf{Met}$ be $\mathcal{G}$-metric actions. A *$\mathcal{G}$-metric transformation* $\theta\colon F \Rightarrow G$ is a metric transformation such that each component $\theta_x\colon Fx \to Gx$ is a left multiplication by an element $\theta_x \in \mathcal{G}$. We denote the category of $\mathcal{G}$-metric actions $X \to \mathsf{Met}$ and $\mathcal{G}$-metric transformations by $\mathsf{PMet}_X^{\mathcal{G}}$.

Remark 5.2 Obviously, $\mathsf{PMet}_X^{\mathcal{G}}$ is a subcategory of Met_X and is also a groupoid.

Definition 5.3 Let G be a group and X be a metric space. We say that X is a *right G-torsor* if G acts on X from the right and satisfies the following:

- It is free and transitive.
- For every $g \in G$, the map $g\colon X \to X$ is an isometry.
- For all $x, x' \in X$ and every $g \in G$, we have $d_X(x, xg) = d_X(x', x'g)$.

Lemma 5.4 *Let (X, d_X) be a metric space and G be a group. Suppose that X is a right G-torsor. Then there exist a distance function d_G on G and a metric group structure $\cdot_x$ on X for each $x \in X$ such that the map*

$$G \to X, \quad g \mapsto xg,$$

gives an isomorphism of metric groups $(G, d_G) \cong (X, \cdot_x)$. Furthermore, the unit of the metric group $(X, \cdot_x)$ is x.

Proof Fix a point $x \in X$. We define a map $d_G\colon G \times G \to \mathbb{R}_{\geq 0}$ by $d_G(f, g) = d_X(xf, xg)$, which is independent from the choice of $x \in X$. It is immediate to check that (G, d_G) is a metric space. Further,

$$\begin{aligned} d_G(ff', gg') &= d_X(xff', xgg') \leq d_X(xff', xgf') + d_X(xgf', xgg') \\ &\leq d_X(xf, xg) + d_X(xf', xg') = d_G(f, g) + d_G(f', g'), \end{aligned}$$

and

$$\begin{aligned} d_G(f^{-1}, g^{-1}) &= d_X(xf^{-1}, xg^{-1}) = d_X(x, xg^{-1}f) \\ &= d_X(xg, (xg)g^{-1}f) = d_X(xg, xf) \\ &= d_X(xf, xg) = d_G(f, g), \end{aligned}$$

for all $f, f', g, g' \in G$. Hence (G, d_G) is a metric group.

Now we define a map $G \to X$ by $g \mapsto xg$. This map is an isometry by the definition. Hence we can transfer the metric group structure on G to X by this map. With respect to this group structure $\cdot_x$ on X, we have $x \cdot_x x' = eg' = x'$ and $x' \cdot_x x = g'e = x'$, where we put $x' = xg'$. Hence $x \in X$ is the unit of the group $(X, \cdot_x)$. □

Definition 5.5 Let G be a group. A metric fibration $\pi : E \to X$ is a *G-torsor over X* if it satisfies the following:

- G acts isometrically on E from the right, and preserves each fiber of π.
- Each fiber of π is a right G-torsor with respect to the above action.

Lemma 5.6 *Let $\pi : E \to X$ be a G-torsor, and $x, x' \in X$. Then the metric group structures on G induced from the fibers $\pi^{-1}x$ and $\pi^{-1}x'$ are identical.*

Proof For all $\varepsilon \in \pi^{-1}x$ and $f \in \mathcal{G}$, we have

$$\begin{aligned} d_E((\varepsilon f)_{x'}, \varepsilon_{x'} f) &= d_E(\varepsilon f, \varepsilon_{x'} f) - d_E(\varepsilon f, (\varepsilon f)_{x'}) \\ &= d_E(\varepsilon, \varepsilon_{x'}) - d_E(\varepsilon f, (\varepsilon f)_{x'}) \\ &= d_X(x, x') - d_X(x, x') \\ &= 0, \end{aligned}$$

and hence we obtain that $(\varepsilon f)_{x'} = \varepsilon_{x'} f$. Let d_x and $d_{x'}$ be the distance function on G induced from the fibers $\pi^{-1}x$ and $\pi^{-1}x'$, respectively. Explicitly, for $\varepsilon \in \pi^{-1}x$ and $f, g \in G$, we have $d_x(f, g) = d_E(\varepsilon f, \varepsilon g)$ and $d_{x'}(f, g) = d_E(\varepsilon_{x'} f, \varepsilon_{x'} g)$. Therefore we obtain that

$$d_{x'}(f, g) = d_E(\varepsilon_{x'} f, \varepsilon_{x'} g) = d_E((\varepsilon f)_{x'}, (\varepsilon g)_{x'}) = d_E(\varepsilon f, \varepsilon g) = d_x(f, g),$$

by Lemma 3.4. □

For a G-torsor $\pi : E \to X$, we can consider the group G as a metric group that is isometric to a fiber of π by Lemma 5.4. Further, such a metric structure is independent of the choice of the fiber by Lemma 5.6. Hence, in the following, we write "$\mathcal{G}$-torsors" instead of "G-torsors", where $\mathcal{G}$ denotes the group G equipped with this metric structure.

Definition 5.7 Let $\pi : E \to X$ and $\pi' : E' \to X$ be $\mathcal{G}$-torsors. A *$\mathcal{G}$-morphism* $\varphi : \pi \to \pi'$ is a G-equivariant map $E \to E'$ that is also a morphism of metric fibrations. We denote the category of $\mathcal{G}$-torsors over X and $\mathcal{G}$-morphisms by $\mathsf{Tors}_X^{\mathcal{G}}$.

Remark 5.8 We can show that any $\mathcal{G}$-morphism is an isomorphism as follows: For all $\varepsilon \in E$, $x \in X$ and $g \in \mathcal{G}$, we have $d_E(\varepsilon, \varepsilon_x g) = d_X(\pi\varepsilon, x) + |g|$ by the definitions. Then the $\mathcal{G}$-equivariance of φ and Lemma 3.5 imply that

$$\begin{aligned} d_{E'}(\varphi\varepsilon, \varphi(\varepsilon_x g)) &= d_{E'}(\varphi\varepsilon, (\varphi\varepsilon)_x g) \\ &= d_X(\pi'\varphi\varepsilon, x) + |g| \\ &= d_X(\pi\varepsilon, x) + |g| \\ &= d_E(\varepsilon, \varepsilon_x g), \end{aligned}$$

which says that φ preserves distances. The invertibility of φ is immediate from the G-equivariance.

Next we show the equivalence of $\mathcal{G}$-metric actions and $\mathcal{G}$-torsors.

Proposition 5.9 *The Grothendieck construction determines a functor* $E\colon \mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{Tors}_X^{\mathcal{G}}$.

Proof Let $F\colon X \to \mathsf{Met}$ be a $\mathcal{G}$-metric action. Let $E(F)$ be the metric fibration given by the Grothendieck construction. We have $d_{E(F)}((x,g),(x',g')) = d_X(x,x') + d_{\mathcal{G}}(g_{xx'}g, g')$. We define a $\mathcal{G}$ action on $E(F)$ by $(x,g)h = (x,gh)$ for all $g,h \in \mathcal{G}$ and $x \in X$. This is obviously compatible with the projection, and also free and transitive on each fiber. We also have that

$$\begin{aligned} d_{E(F)}((x,g)h,(x',g')h) &= d_{E(F)}((x,gh),(x',g'h)) \\ &= d_X(x,x') + d_{\mathcal{G}}(g_{xx'}gh, g'h) \\ &= d_X(x,x') + d_{\mathcal{G}}(g_{xx'}g, g') \\ &= d_{E(F)}((x,g),(x',g')), \end{aligned}$$

and hence it acts isometrically. Further, we have that

$$d_{E(F)}((x,g),(x,g)h) = d_{E(F)}((x,g),(x,gh)) = d_{\mathcal{G}}(g,gh) = d_{\mathcal{G}}(e,h),$$

and hence each fiber is a right $\mathcal{G}$-torsor. Therefore, we obtain that $E(F)$ is a $\mathcal{G}$-torsor.

Now let $\theta\colon F \Rightarrow F'$ be a $\mathcal{G}$-metric transformation. The Grothendieck construction gives a map $\varphi_\theta\colon E(F) \to E(F')$ by $\varphi_\theta(x,g) = (x, \theta_x g)$, which is a morphism of metric fibrations. It is checked that φ_θ is $\mathcal{G}$-equivariant as follows:

$$(\varphi_\theta(x,g))h = (x, \theta_x gh) = \varphi_\theta(x,gh).$$

Hence it is a $\mathcal{G}$-morphism. □

Proposition 5.10 *We have a functor* $F\colon \mathsf{Tors}_X^{\mathcal{G}} \to \mathsf{PMet}_X^{\mathcal{G}}$ *sending a* $\mathcal{G}$*-torsor* π *to a* $\mathcal{G}$*-metric action* F_π *with* $F_\pi x = \pi^{-1}x$.

Proof Let $\pi\colon E \to X$ be a $\mathcal{G}$-torsor. We fix points $x_0 \in X$ and $\varepsilon \in \pi^{-1}x_0$. For each $x \in X$, we equip the set $\pi^{-1}x$ with a metric group structure isomorphic to $\mathcal{G}$ with the unit ε_x by Lemma 5.4. Hence we can identify each fiber with $\mathcal{G}$ by the map $g \mapsto \varepsilon_x g$. Now we put $(\varepsilon_x)_{x'} = \varepsilon_{x'} g_{xx'} \in \pi^{-1}x'$ for $x, x' \in X$ and $g_{xx'} \in \mathcal{G}$. Then, for each $h \in \mathcal{G}$, we have

$$\begin{aligned} d_X(x,x') &= d_E(\varepsilon_x h, (\varepsilon_x h)_{x'}) \\ &= d_E(\varepsilon_x, (\varepsilon_x h)_{x'} h^{-1}) \\ &= d_E(\varepsilon_x, \varepsilon_{x'} g_{xx'}) + d_E(\varepsilon_{x'} g_{xx'}, (\varepsilon_x h)_{x'} h^{-1}) \\ &= d_X(x,x') + d_E(\varepsilon_{x'} g_{xx'}, (\varepsilon_x h)_{x'} h^{-1}), \end{aligned}$$

and hence we obtain that $(\varepsilon_x h)_{x'} = \varepsilon_{x'} g_{xx'} h$. This implies that the map $\pi^{-1}x \to \pi^{-1}x'$ given by lifts $\varepsilon_x h \mapsto (\varepsilon_x h)_{x'}$ is the left multiplication by $g_{xx'}$ when we identify each fiber with $\mathcal{G}$ as above. Hence the functor F gives a $\mathcal{G}$-metric action.

Next, let $\varphi\colon \pi \to \pi'$ be a $\mathcal{G}$-morphism between $\mathcal{G}$-torsors $\pi\colon E \to X$ and $\pi'\colon E' \to X$. It induces a Lipschitz map $\varphi_x\colon \pi^{-1}x \to \pi'^{-1}x$. Since fibers $\pi^{-1}x$ and $\pi'^{-1}x$ are identified with $\mathcal{G}$ and φ_x is $\mathcal{G}$-equivariant, we can identify φ_x with the left multiplication by $\varphi_x\varepsilon_x$. This implies that the functor F sends the $\mathcal{G}$-morphism φ to a $\mathcal{G}$-metric transformation between F_π and $F_{\pi'}$. □

Proposition 5.11 *The Grothendieck construction functor* $\mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{Tors}_X^{\mathcal{G}}$ *is an equivalence of categories.*

Proof This is similar to the proof of Proposition 3.9. □

5.2 $\mathsf{PMet}_X^{\mathcal{G}} \simeq \mathsf{Hom}(\pi_1^m(X, x_0), \mathcal{G})$

First we define the category of homomorphisms of metric groups $\mathcal{G} \to \mathcal{G}'$.

Definition 5.12 Let $\mathcal{G}$ and $\mathcal{G}'$ be metric groups, and let $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')$ be the set of all homomorphisms $\mathcal{G} \to \mathcal{G}'$. We equip $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')$ with a groupoid structure by defining $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')(\varphi, \psi) = \{h \in \mathcal{G}' \mid \varphi = h^{-1}\psi h\}$ for homomorphisms $\varphi, \psi : \mathcal{G} \to \mathcal{G}'$. The identity on $\varphi \in \mathsf{Hom}(\mathcal{G}, \mathcal{G}')$ is the unit $e \in \mathcal{G}'$, and the composition of morphisms $h \in \mathsf{Hom}(\mathcal{G}, \mathcal{G}')(\varphi, \psi)$ and $h' \in \mathsf{Hom}(\mathcal{G}, \mathcal{G}')(\psi, \xi)$ is defined by $h' \circ h = h'h$.

Lemma 5.13 *Let X be a metric space and $\mathcal{G}$ be a metric group. For each $x_0 \in X$, we have a functor $A\colon \mathsf{Hom}(\pi_1^m(X, x_0), \mathcal{G}) \to \mathsf{PMet}_X^{\mathcal{G}}$ sending a homomorphism $\varphi\colon \pi_1^m(X, x_0) \to \mathcal{G}$ to a $\mathcal{G}$-metric action F_φ with $F_\varphi x = \mathcal{G}$.*

Proof Let $\varphi\colon \pi_1^m(X, x_0) \to \mathcal{G}$ be a homomorphism. We define a $\mathcal{G}$-metric action $F_\varphi\colon X \to \mathsf{Met}$ by $F_\varphi x = \mathcal{G}$ and $(F_\varphi)_{xx'} = \varphi[(x_0, x', x, x_0)]\cdot\colon \mathcal{G} \to \mathcal{G}$, where we denote the left multiplication by $(-)\cdot$. It is verified that this defines a $\mathcal{G}$-metric action as follows.

For all $x, x' \in X$, we have $(F_\varphi)_{xx} = \varphi[(x_0, x, x, x_0)]\cdot = e\cdot = \mathrm{id}_{\mathcal{G}}$, and $(F_\varphi)_{x'x} = \varphi[(x_0, x, x', x_0)]\cdot = (\varphi[(x_0, x', x, x_0)])^{-1}\cdot = (F_\varphi)_{xx'}^{-1}$. Further, we have

$$\begin{aligned}
d_{\mathcal{G}}((F_\varphi)_{x'x''}(F_\varphi)_{xx'}g, (F_\varphi)_{xx''}g) &= d_{\mathcal{G}}\big(\varphi[(x_0, x'', x', x_0)]\varphi[(x_0, x', x, x_0)], \varphi[(x_0, x'', x, x_0)]\big)\\
&= d_{\mathcal{G}}\big(\varphi[(x_0, x'', x', x, x_0)], \varphi[(x_0, x'', x, x_0)]\big)\\
&= d_{\mathcal{G}}\big((\varphi[(x_0, x'', x, x_0)])^{-1}\varphi[(x_0, x'', x', x, x_0)], e\big)\\
&= d_{\mathcal{G}}\big(\varphi[(x_0, x, x'', x', x, x_0)], e\big)\\
&\le d_{\pi_1^m(X,x_0)}([(x_0, x, x'', x', x, x_0)], [x_0, x_0])\\
&\le d_X(x, x') + d_X(x', x'') - d_X(x, x''),
\end{aligned}$$

for all $x, x', x'' \in X$ and $g \in \mathcal{G}$. Let $h\colon \varphi \to \psi$ be a morphism in $\mathsf{Hom}(\pi_1^m(X, x_0), \mathcal{G})$; that is, we have $\varphi = h^{-1}\psi h$ with $h \in \mathcal{G}$. Then we can construct a $\mathcal{G}$-metric transformation $\theta\colon F_\varphi \Rightarrow F_\psi$ by $\theta_x = h\cdot\colon \mathcal{G} \to \mathcal{G}$. It satisfies that $(F_\psi)_{xx'}\theta_x = \theta_{x'}(F_\varphi)_{xx'}$ since we have $\psi[(x_0, x', x, x_0)]h = h\varphi[(x_0, x', x, x_0)]$. □

Lemma 5.14 *Let X be a metric space and $\mathcal{G}$ be a metric group. For each $x_0 \in X$, there is a functor $B\colon \mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G})$ sending a $\mathcal{G}$-metric action F to a homomorphism $\varphi_F\colon \pi_1^m(X,x_0) \to \mathcal{G}$ defined by*

$$\varphi_F[(x_0,x_1,\dots,x_n,x_0)] = F_{x_1x_0}F_{x_2x_1}\dots F_{x_nx_{n-1}}F_{x_0x_n},$$

for each $[(x_0,x_1,\dots,x_n,x_0)] \in \pi_1^m(X,x_0)$.

Proof It is immediate to check that this is well defined. Let $F, F'\colon X \to \mathsf{Met}$ be $\mathcal{G}$-metric actions and $\theta\colon F \Rightarrow F'$ be a $\mathcal{G}$-metric transformation. Then we have

$$\begin{aligned}\theta_{x_0}^{-1}\varphi_{F'}[(x_0,x_1,\dots,x_n,x_0)]\theta_{x_0} &= \theta_{x_0}^{-1}F'_{x_1x_0}F'_{x_2x_1}\dots F'_{x_nx_{n-1}}F'_{x_0x_n}\theta_{x_0}\\ &= F_{x_1x_0}F_{x_2x_1}\dots F_{x_nx_{n-1}}F_{x_0x_n}\\ &= \varphi_F[(x_0,x_1,\dots,x_n,x_0)],\end{aligned}$$

for each $[(x_0,x_1,\dots,x_n,x_0)] \in \pi_1^m(X,x_0)$. Hence $\theta_{x_0} \in \mathcal{G}$ gives a morphism $\theta_{x_0}\colon \varphi_F \to \varphi_{F'}$. This correspondence is obviously functorial. □

Proposition 5.15 *The functor $A\colon \mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G}) \to \mathsf{PMet}_X^{\mathcal{G}}$ of Lemma 5.13 is an equivalence of categories.*

Proof We show the natural isomorphisms $BA \cong \mathrm{id}_{\mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G})}$ and $AB \cong \mathrm{id}_{\mathsf{PMet}_X^{\mathcal{G}}}$. For a homomorphism $\varphi\colon \pi_1^m(X,x_0) \to \mathcal{G}$, we have

$$\begin{aligned}\varphi_{F_\varphi}[(x_0,x_1,\dots,x_n,x_0)] &= (F_\varphi)_{x_1x_0}(F_\varphi)_{x_2x_1}\dots(F_\varphi)_{x_0x_n}\\ &= \varphi[(x_0,x_0,x_1,x_0)]\varphi[(x_0,x_1,x_2,x_0)]\dots\varphi[(x_0,x_n,x_0,x_0)]\\ &= \varphi[(x_0,x_0,x_1,x_1,\dots,x_n,x_n,x_0,x_0)]\\ &= \varphi[(x_0,x_1,\dots,x_n,x_0)],\end{aligned}$$

for each $[(x_0,x_1,\dots,x_n,x_0)] \in \pi_1^m(X,x_0)$. Hence we obtain an isomorphism $BA\varphi = \varphi$ that is obviously natural. Conversely, let $F\colon X \to \mathsf{Met}$ be a $\mathcal{G}$-metric action. Then we have

$$(F_{\varphi_F})_x = \mathcal{G} \quad \text{and} \quad (F_{\varphi_F})_{xx'} = \varphi_F[(x_0,x',x,x_0)] = F_{x'x_0}F_{xx'}F_{x_0x}.$$

Now we define a $\mathcal{G}$-metric transformation $\theta\colon F_{\varphi_F} \Rightarrow F$ by $\theta_x = F_{x_0x}$. It is obvious that we have $F_{xx'}\theta_x = \theta_{x'}(F_{\varphi_F})_{xx'}$, and hence it is well defined and obviously an isomorphism. For a $\mathcal{G}$-metric transformation $\tau\colon F \Rightarrow F'$, we have $(AB\tau)_x = \tau_{x_0}\cdot\colon (F_{\varphi_F})_x \to (F'_{\varphi_{F'}})_x$ by the construction. Hence the condition $\tau_x F_{x_0x} = F'_{x_0x}\tau_{x_0}$ of the $\mathcal{G}$-metric transformation implies the naturality of this isomorphism. □

5.3 Example

We give the following example of a metric fundamental group.

Proposition 5.16 *Let C_n be the undirected n-cycle graph. Then we have*

$$\pi_1^m(C_n) \cong \begin{cases} \mathbb{Z} \text{ with } |1| = 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence we have that

$$\mathsf{PMet}^{\mathcal{G}}_{C_n} \simeq \begin{cases} \mathsf{Hom}(\mathbb{Z}, \mathcal{G}) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

for all metric groups $\mathcal{G}$, which implies that there is only a trivial metric fibration over C_{2n} and that there is at most one nontrivial metric fibration over C_{2n+1}.

Proof Let $V(C_n) = \{v_1, \dots, v_n\}$ be the vertex set whose numbering is anticlockwise. For C_{2n}, it reduces to show that $[(v_1, v_2, \dots, v_{2n}, v_1)] = [(v_1, v_1)]$. Since we have $d_{C_{2n}}(v_i, v_j) = d_{C_{2n}}(v_i, v_k) + d_{C_{2n}}(v_k, v_j)$ for all $i \le k \le j$ with $j - i \le n$, we obtain that

$$\begin{aligned} [(v_1, v_2, \dots, v_{2n}, v_1)] &= [(v_1, \dots, v_{n+1}, \dots, v_{2n}, v_1)] \\ &= [(v_1, v_{n+1}, v_1)] \\ &= [(v_1, v_1)]. \end{aligned}$$

For C_{2n+1}, the possible nontrivial element of $\pi_1^m(C_{2n+1})$ is a concatenation or its inverse of the element $[(v_1, \dots, v_{2n+1}, v_1)]$. Now we have $[(v_1, \dots, v_{2n+1}, v_1)] = [(v_1, v_{n+1}, v_{n+2}, v_1)]$, by the same argument as above, and

$$\begin{aligned} &d_{Q(C_{2n+1}, v_1)}((v_1, v_{n+1}, v_{n+2}, v_1), (v_1, v_{n+1}, v_1)) \\ &= d_{C_{2n+1}}(v_{n+1}, v_{n+2}) + d_{C_{2n+1}}(v_{n+2}, v_1) - d_{C_{2n+1}}(v_{n+1}, v_1) \\ &= d_{C_{2n+1}}(v_{n+1}, v_{n+2}) \\ &= 1. \end{aligned}$$

Hence we obtain that $|[(v_1, \dots, v_{2n+1}, v_1)]| = 1$. □

Remark 5.17 The cycle graph C_n is a metric group $\mathbb{Z}/n\mathbb{Z}$ with $|1| = 1$. Hence the examples in Figure 1 are $\mathbb{Z}/2\mathbb{Z}$-torsors, which are classified by $\mathsf{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

6 Classification of metric fibrations

In this section, we classify general metric fibrations by fixing the base and the fiber. It is analogous to that of topological fiber bundles, namely it reduces to classifying principal bundles whose fiber is the structure group of the concerned fibration. We divide it into two cases, whether the fiber is bounded or not, since we need to consider *expanded metric spaces* for the unbounded case.

6.1 The functor $\widehat{(-)}^{x_0}$

Before we show the classification, we introduce a technical functor that will be used later.

Definition 6.1 For all metric actions $F\colon X \to \mathsf{Met}$ and a point $x_0 \in X$, we define a metric action $\widehat{F}^{x_0}\colon X \to \mathsf{Met}$ as follows. We define that $\widehat{F}^{x_0}x = Fx_0$ and $\widehat{F}^{x_0}_{xx'} = F_{x'x_0}F_{xx'}F_{x_0x}\colon Fx_0 \to Fx_0$ for all $x, x' \in X$. We verify that this defines a metric action as follows. We have $\widehat{F}^{x_0}_{xx} = F_{xx_0}F_{xx}F_{x_0x} = \mathrm{id}_{Fx_0} = \mathrm{id}_{\widehat{F}^{x_0}x}$. We also have $(\widehat{F}^{x_0}_{x'x})^{-1} = (F_{xx_0}F_{x'x}F_{x_0x'})^{-1} = F_{x'x_0}F_{xx'}F_{x_0x} = \widehat{F}^{x_0}_{xx'}$ and

$$\begin{aligned} d_{\widehat{F}^{x_0}x''}(\widehat{F}^{x_0}_{x'x''}\widehat{F}^{x_0}_{xx'}a, \widehat{F}^{x_0}_{xx''}a) &= d_{Fx_0}(F_{x''x_0}F_{x'x''}F_{x_0x'}F_{x'x_0}F_{xx'}F_{x_0x}a, F_{x''x_0}F_{xx''}F_{x_0x}a) \\ &= d_{Fx''}(F_{x'x''}F_{xx'}F_{x_0x}a, F_{xx''}F_{x_0x}a) \\ &\le d_X(x,x') + d_X(x',x'') - d_X(x,x''), \end{aligned}$$

for all $x, x', x'' \in X$ and $a \in \widehat{F}^{x_0}x$.

Lemma 6.2 *The correspondence $F \mapsto \widehat{F}^{x_0}$ defines a fully faithful functor $\widehat{(-)}^{x_0}\colon \mathsf{Met}_X \to \mathsf{Met}_X$. Further, this restricts to a fully faithful functor $\mathsf{PMet}^{\mathcal{G}}_X \to \mathsf{PMet}^{\mathcal{G}}_X$ for each metric group $\mathcal{G}$.*

Proof Let $\theta\colon F \Rightarrow G$ be a metric transformation. We define a metric transformation $\widehat{\theta}^{x_0}\colon \widehat{F}^{x_0} \Rightarrow \widehat{G}^{x_0}$ by $\widehat{\theta}^{x_0}_x = \theta_{x_0}\colon \widehat{F}^{x_0}x \to \widehat{G}^{x_0}x, a \mapsto \theta_{x_0}a$. Then we have

$$\begin{aligned} \widehat{G}^{x_0}_{xx'}\widehat{\theta}^{x_0}_x &= G_{x'x_0}G_{xx'}G_{x_0x}\theta_{x_0} \\ &= G_{x'x_0}G_{xx'}\theta_x F_{x_0x} \\ &= G_{x'x_0}\theta_{x'}F_{xx'}F_{x_0x} \\ &= \theta_{x_0}F_{x'x_0}F_{xx'}F_{x_0x} \\ &= \widehat{\theta}^{x_0}_x\widehat{F}^{x_0}_{xx'}, \end{aligned}$$

and hence this defines a metric transformation. It is obvious that $\widehat{\mathrm{id}_F}^{x_0} = \mathrm{id}_{\widehat{F}^{x_0}}$ and $(\widehat{\theta'\theta})^{x_0} = \widehat{\theta}'^{x_0}\widehat{\theta}^{x_0}$. It is a faithful functor because $G_{xx_0}\theta_x = \theta_{x_0}F_{xx_0}$ implies that $\theta_x = \theta'_x$ for all $x \in X$ if two metric transformation θ, θ' satisfy $\theta_{x_0} = \theta'_{x_0}$. By definition, it restricts to a faithful functor $\mathsf{PMet}^{\mathcal{G}}_X \to \mathsf{PMet}^{\mathcal{G}}_X$ for each metric group $\mathcal{G}$. Next we show the fullness. Let $\eta\colon \widehat{F}^{x_0} \Rightarrow \widehat{G}^{x_0}$ be a metric transformation. Then we have $\widehat{G}^{x_0}_{x_0x}\eta_{x_0} = \eta_x\widehat{F}^{x_0}_{x_0x}$ and $\widehat{F}^{x_0}_{x_0x} = \mathrm{id}_{F_{x_0}}$, $\widehat{G}^{x_0}_{x_0x} = \mathrm{id}_{G_{x_0}}$. Hence we obtain that $\eta_{x_0} = \eta_x$ for all $x \in X$.

Now we define a metric transformation $\widetilde{\eta}\colon F \Rightarrow G$ by $\widetilde{\eta}_x = G_{x_0x}\eta_{x_0}F_{xx_0}\colon Fx \to Gx$. Then we have

$$\begin{aligned} G_{xx'}\widetilde{\eta}_x &= G_{xx'}G_{x_0x}\eta_{x_0}F_{xx_0} \\ &= G_{x_0x'}\widehat{G}^{x_0}_{xx'}\eta_x F_{xx_0} \\ &= G_{x_0x'}\eta_{x'}\widehat{F}^{x_0}_{xx'}F_{xx_0} \\ &= G_{x_0x'}\eta_{x'}F_{x'x_0}F_{xx'}F_{x_0x}F_{xx_0} \\ &= G_{x_0x'}\eta_{x_0}F_{x'x_0}F_{xx'} \\ &= \widetilde{\eta}_{x'}F_{xx'}, \end{aligned}$$

and hence this defines a metric transformation. We obviously have $\widehat{(\widetilde{\eta})}^{x_0} = \eta$, which implies that the functor $\widehat{(-)}^{x_0}$ is full. The restriction to $\mathsf{PMet}^{\mathcal{G}}_X \to \mathsf{PMet}^{\mathcal{G}}_X$ is immediate. □

Lemma 6.3 *The functor* $\widehat{(-)}^{x_0}\colon \mathsf{Met}_X \to \mathsf{Met}_X$ *is split essentially surjective. Its restriction* $\mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{PMet}_X^{\mathcal{G}}$ *is also split essentially surjective for all metric groups* $\mathcal{G}$.

Proof Let $F\colon X \to \mathsf{Met}$ be a metric action. We define a metric transformation $\theta\colon \widehat{F}^{x_0} \Rightarrow F$ by $\theta_x = F_{x_0x}\colon \widehat{F}^{x_0}x \to Fx, a \mapsto F_{x_0x}a$. It satisfies that

$$\begin{aligned} F_{xx'}\theta_x &= F_{xx'}F_{x_0x} \\ &= F_{x_0x'}F_{x'x_0}F_{xx'}F_{x_0x} \\ &= \theta_{x'}\widehat{F}^{x_0}_{xx'}. \end{aligned}$$

Further, we define a metric transformation $\theta^{-1}\colon F \Rightarrow \widehat{F}^{x_0}$ by $\theta_x^{-1} = F_{xx_0}\colon Fx \to \widehat{F}^{x_0}x$ for all $x \in X$. Then we have $\widehat{F}^{x_0}_{xx'}\theta_x^{-1} = \theta_x^{-1}F_{xx'}$ similarly to the above, and hence it defines a metric transformation. It is obviously an isomorphism. The restriction to $\mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{PMet}_X^{\mathcal{G}}$ is immediate. □

Corollary 6.4 *The functor* $\widehat{(-)}^{x_0}\colon \mathsf{Met}_X \to \mathsf{Met}_X$ *and its restriction* $\mathsf{PMet}_X^{\mathcal{G}} \to \mathsf{PMet}_X^{\mathcal{G}}$ *for all metric groups* $\mathcal{G}$ *are category equivalences.* □

Definition 6.5 (1) We denote the image of the functor $\widehat{(-)}^{x_0}\colon \mathsf{Met}_X \to \mathsf{Met}_X$ by $\widehat{\mathsf{Met}}_X^{x_0}$.

(2) For each metric space Y, we denote by Met_X^Y the full subcategory of Met_X that consists of metric actions $F\colon X \to \mathsf{Met}$ such that $Fx \cong Y$ for all $x \in X$.

(3) We denote the image of $\widehat{(-)}^{x_0}$ restricted to Met_X^Y and $\mathsf{PMet}_X^{\mathcal{G}}$ by $\widehat{\mathsf{Met}}_X^{Y,x_0}$ and $\widehat{\mathsf{PMet}}_X^{\mathcal{G},x_0}$, respectively.

(4) For each metric space Y, we denote by Fib_X^Y the full subcategory of Fib_X that consists of metric fibrations $\pi\colon E \to X$ such that $\pi^{-1}x \cong Y$ for all $x \in X$.

Lemma 6.6 (1) *We have category equivalences* $\mathsf{Met}_X^Y \to \widehat{\mathsf{Met}}_X^{Y,x_0}$ *and* $\mathsf{PMet}_X^{\mathcal{G}} \to \widehat{\mathsf{PMet}}_X^{\mathcal{G},x_0}$.

(2) *The Grothendieck construction functor* $E\colon \mathsf{Met}_X \to \mathsf{Fib}_X$ *restricts to the category equivalence* $\mathsf{Met}_X^Y \to \mathsf{Fib}_X^Y$.

Proof (1) follows from Corollary 6.4, and (2) follows from the proof of Proposition 3.9. □

6.2 Classification for the case of bounded fibers

In this subsection, we suppose that X and Y are metric spaces and Y is bounded. In this case the group $\mathsf{Aut}\,Y$ of automorphisms is a metric group (Example 4.3).

Lemma 6.7 *We have a faithful functor*

$$- \curvearrowright Y\colon \mathsf{PMet}_X^{\mathsf{Aut}\,Y} \to \mathsf{Met}_X^Y.$$

Proof Let $F \in \mathsf{PMet}_X^{\mathsf{Aut}\,Y}$. We define a metric action $F \curvearrowright Y\colon X \to \mathsf{Met}$ by $(F \curvearrowright Y)x = Y$ and $(F \curvearrowright Y)_{xx'} = F_{xx'}\colon Y \to Y$. It is immediate to verify that this defines a metric action. For an $\mathsf{Aut}\,Y$-metric transformation $\theta\colon F \Rightarrow G$, we define a metric transformation $\theta \curvearrowright Y\colon F \curvearrowright Y \Rightarrow G \curvearrowright Y$ by $(\theta \curvearrowright Y)_x = \theta_x\colon Y \to Y$, $y \mapsto \theta_x y$. Then it is also immediate to verify that it is a metric transformation. Further, this obviously defines a faithful functor. □

Lemma 6.8 *The functor* $- \curvearrowright Y\colon \mathsf{PMet}_X^{\mathsf{Aut}\,Y} \to \mathsf{Met}_X^Y$ *is split essentially surjective.*

Proof Let $F \in \mathsf{Met}_X^Y$ and fix isometries $\varphi_x\colon Y \to Fx$ using the axiom of choice. We define an $\mathsf{Aut}\,Y$-metric action $\mathsf{Aut}\,F$ by $(\mathsf{Aut}\,F)x = \mathsf{Aut}\,Y$ and $(\mathsf{Aut}\,F)_{xx'} = \varphi_{x'}^{-1} F_{xx'}\varphi_x \cdot$ that is a left multiplication. We can verify that it is an $\mathsf{Aut}\,Y$-metric action as follows. We have $(\mathsf{Aut}\,F)_{xx} = \varphi_x^{-1} F_{xx}\varphi_x \cdot = \mathrm{id}_{\mathsf{Aut}\,Y}$ and $(\mathsf{Aut}\,F)_{xx'}^{-1} = \varphi_x^{-1} F_{x'x}\varphi_{x'} \cdot = (\mathsf{Aut}\,F)_{x'x}$. We also have that

$$\begin{aligned}
d_{\mathsf{Aut}\,Y}((\mathsf{Aut}\,F)_{x'x''}(\mathsf{Aut}\,F)_{xx'}, (\mathsf{Aut}\,F)_{xx''}) &= d_{\mathsf{Aut}\,Y}(\varphi_{x''}^{-1} F_{x'x''}\varphi_{x'}\varphi_{x'}^{-1} F_{xx'}\varphi_x, \varphi_{x''}^{-1} F_{xx''}\varphi_x) \\
&= d_{\mathsf{Aut}\,Y}(\varphi_{x''}^{-1} F_{x'x''} F_{xx'}\varphi_x, \varphi_{x''}^{-1} F_{xx''}\varphi_x) \\
&= \sup_{a \in Y} d_Y(\varphi_{x''}^{-1} F_{x'x''} F_{xx'}\varphi_x a, \varphi_{x''}^{-1} F_{xx''}\varphi_x a) \\
&= \sup_{a \in Fx} d_{Fx''}(F_{x'x''} F_{xx'} a, F_{xx''} a) \\
&\le d_X(x, x') + d_X(x', x'') - d_X(x, x'').
\end{aligned}$$

Now we define a metric transformation $\varphi\colon \mathsf{Aut}\,F \curvearrowright Y \Rightarrow F$ by $\varphi_x\colon (\mathsf{Aut}\,F \curvearrowright Y)x = Y \to Fx$. This satisfies that $F_{xx'}\varphi_x = \varphi_{x'}(\mathsf{Aut}\,F \curvearrowright Y)_{xx'}$ and is an isomorphism by the definition. □

Since the category $\mathsf{PMet}_X^{\mathsf{Aut}\,Y}$ is a groupoid, the image of the functor $- \curvearrowright Y$ is in $\mathsf{core}\,\mathsf{Met}_X^Y$. (Here core denotes the subcategory consisting of all isomorphisms, as in Definition 2.1(4).)

Lemma 6.9 *The functor* $\widehat{- \curvearrowright Y}^{x_0}\colon \mathsf{PMet}_X^{\mathsf{Aut}\,Y} \to \mathsf{core}\widehat{\mathsf{Met}}_X^{Y,x_0}$ *is full.*

Proof We have $\widehat{- \curvearrowright Y}^{x_0} = \widehat{(-)}^{x_0} \curvearrowright Y$ by the definitions. Since $\widehat{(-)}^{x_0}\colon \mathsf{PMet}_X^{\mathsf{Aut}\,Y} \to \mathsf{PMet}_X^{\mathsf{Aut}\,Y}$ is full by Lemma 6.2, it will suffice to show that the restriction

$$- \curvearrowright Y\colon \widehat{\mathsf{PMet}}_X^{\mathsf{Aut}\,Y,x_0} \to \mathsf{core}\widehat{\mathsf{Met}}_X^{Y,x_0}$$

is full. Let $\theta\colon \widehat{F}^{x_0} \curvearrowright Y \Rightarrow \widehat{G}^{x_0} \curvearrowright Y$ be an isomorphism in $\widehat{\mathsf{Met}}_X^{Y,x_0}$, where $F, G \in \mathsf{PMet}_X^{\mathsf{Aut}\,Y}$. Then we have an isometry $\theta_x\colon Y \to Y$ such that $G_{x'x_0}G_{xx'}G_{x_0x}\theta_x = \theta_{x'}F_{x'x_0}F_{xx'}F_{x_0x}$ for all $x, x' \in X$. Since we have $\theta_x \in \mathsf{Aut}\,Y$, we obtain a morphism

$$\theta'\colon \widehat{F}^{x_0} \Rightarrow \widehat{G}^{x_0} \in \widehat{\mathsf{PMet}}_X^{\mathsf{Aut}\,Y,x_0}$$

defined by $\theta'_x = \theta_x$. It is obvious that we have $\theta' \curvearrowright Y = \theta$. □

Corollary 6.10 *The functor* $\widehat{- \curvearrowright Y}^{x_0} : \mathsf{PMet}_X^{\mathsf{Aut}\,Y} \to \mathsf{core}\widehat{\mathsf{Met}}_X^{Y,x_0}$ *is an equivalence of categories.* □

Corollary 6.11 *The categories* $\mathsf{PMet}_X^{\mathsf{Aut}\,Y}$ *and* $\mathsf{core}\,\mathsf{Fib}_X^Y$ *are equivalent.*

Proof This follows from Corollary 6.10, with $\mathsf{core}\,\mathsf{Fib}_X^Y \simeq \mathsf{core}\,\mathsf{Met}_X^Y \simeq \mathsf{core}\widehat{\mathsf{Met}}_X^{Y,x_0}$ by Lemma 6.6. □

6.3 Classification for the case of unbounded fibers

To classify general metric fibrations, we generalize the discussions so far to *extended metric groups*.

Definition 6.12 (1) An *extended metric group* is a monoid object in EMet that is a group when we forget the metric structure.

(2) For extended metric groups $\mathcal{G}$ and $\mathcal{H}$, a *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ is a Lipschitz map $\mathcal{G} \to \mathcal{H}$ that commutes with the group structure.

(3) We denote the category of extended metric groups and homomorphisms by EMGrp. The category MGrp is a full subcategory of EMGrp.

Example 6.13 Let (X, d) be a metric space, and let $\mathsf{Aut}\,X$ be the group of isometries on X. We define a distance function on $\mathsf{Aut}\,X$ by $d_{\mathsf{Aut}\,X}(f, g) = \sup_{x \in X} d_X(fx, gx)$. Then it is immediate to verify that $(\mathsf{Aut}\,X, d_{\mathsf{Aut}\,X})$ is an extended metric group. The "unit component" of $\mathsf{Aut}\,X$, that is the set of isometries f such that $d_{\mathsf{Aut}\,X}(\mathrm{id}_X, f) < \infty$, is exactly $\mathsf{Aut}^u\,X$ (Example 4.3). If the metric space X has finite diameter, then we have $\mathsf{Aut}\,X = \mathsf{Aut}^u\,X$.

Definition 6.14 Let $\mathcal{G}$ and $\mathcal{G}'$ be extended metric groups, and let $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')$ be the set of homomorphisms. We equip $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')$ with a groupoid structure similarly to the metric group case by defining $\mathsf{Hom}(\mathcal{G}, \mathcal{G}')(\varphi, \psi) = \{h \in \mathcal{G}' \mid \varphi = h^{-1}\psi h\}$ for all homomorphisms $\varphi, \psi : \mathcal{G} \to \mathcal{G}'$.

Remark 6.15 A statement analogous to Lemma 4.2 holds for extended metric groups. Further, the relationship between extended metric spaces and normed groups similar to Lemma 4.2 holds if we replace the codomain of norms by $[0, \infty]$.

Definition 6.16 Let $\mathcal{G}$ be an extended metric group and X be a metric space. An *extended $\mathcal{G}$-metric action* F is a correspondence $X \ni x \mapsto Fx = \mathcal{G}$ and $F_{xx'} \in \mathcal{G}$ such that

- $F_{xx} = e$, $F_{xx'} = F_{x'x}^{-1}$,
- $d_{\mathcal{G}}(F_{x'x''}F_{xx'}, F_{xx''}) \le d_X(x, x') + d_X(x', x'') - d_X(x, x'')$.

For extended $\mathcal{G}$-metric actions F and G, an *extended $\mathcal{G}$-metric transformation* $\theta : F \Rightarrow G$ is a family of elements $\{\theta_x \in \mathcal{G}\}_{x \in X}$ such that $G_{xx'}\theta_x = \theta_{x'}F_{xx'}$. We denote the category of extended $\mathcal{G}$-metric actions and extended $\mathcal{G}$-metric transformations by $\mathsf{EPMet}_X^{\mathcal{G}}$.

The following is obtained from the same arguments in Section 5.2 by replacing the term "metric group" by "extended metric group".

Proposition 6.17 *For an extended metric group $\mathcal{G}$ and a metric space X, the categories $\mathsf{EPMet}_X^{\mathcal{G}}$ and $\mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G})$ are equivalent.* □

Further, the arguments in Section 6.2 can be applied to the extended case, and we obtain the following.

Proposition 6.18 *For all metric spaces X and Y, the categories $\mathsf{EPMet}_X^{\mathsf{Aut}\,Y}$ and $\mathsf{core}\,\mathsf{Fib}_X^Y$ are equivalent. Hence metric fibrations with fiber Y are classified by $\mathsf{Hom}(\pi_1^m(X,x_0),\mathcal{G})$.* □

7 Cohomological interpretation

In this section, we give a cohomological classification of $\mathcal{G}$-torsors. It is an analogy of the 1-Čech cohomology. Before giving the definition, we introduce the following technical term.

Definition 7.1 Let X be a metric space, and $x_1,x_2,x_3\in X$. We denote the subset $\{x_1,x_2,x_3\}\subset X$ by $\Delta(x_1,x_2,x_3)$ and call it a *triangle*. We define the *degeneracy degree of the triangle* $\Delta(x_1,x_2,x_3)$ by

$$|\Delta(x_1,x_2,x_3)| := \min\bigl\{d_X(x_i,x_j)+d_X(x_j,x_k)-d_X(x_i,x_k)\mid \{i,j,k\}=\{1,2,3\}\bigr\}.$$

It is enough to consider i,j,k's running in the cyclic order to obtain the above minimum.

The following is the definition of our "1-Čech cohomology".

Definition 7.2 Let X be a metric space and suppose that points of X are indexed as $X=\{x_i\}_{i\in I}$. For a metric group $\mathcal{G}$, we define the 1-*cohomology of X with coefficients in $\mathcal{G}$* as the category $\mathsf{H}^1(X;\mathcal{G})$ by

$$\mathrm{Ob}\,\mathsf{H}^1(X;\mathcal{G}) = \bigl\{(a_{ijk})\in\mathcal{G}^{I^3}\mid a_{ijk}a_{kj\ell}=a_{ij\ell},\ |a_{ijk}a_{jki}a_{kij}|\le|\Delta(x_i,x_j,x_k)|\bigr\},$$

and

$$\mathsf{H}^1(X;\mathcal{G})((a_{ijk}),(b_{ijk})) = \{(f_{ij})\in\mathcal{G}^{I^2}\mid a_{ijk}f_{jk}=f_{ij}b_{ijk}\},$$

where we denote the conjugation invariant norm on $\mathcal{G}$ by $|-|$. We call an object of $\mathsf{H}^1(X;\mathcal{G})$ *a cocycle*. Obviously, the above constructions are independent from the choice of the index I.

Remark 7.3 For a cocycle $(a_{ijk})\in\mathsf{H}^1(X;\mathcal{G})$, the condition $a_{ijk}a_{kj\ell}=a_{ij\ell}$ implies that $a_{iji}=e$ and $a_{ijk}=a_{kji}^{-1}$ for all $i,j,k\in I$. Further, for a morphism (f_{ij}), we have $f_{ij}=f_{ji}$ from the condition $a_{ijk}f_{jk}=f_{ij}b_{ijk}$ and $a_{iji}=b_{iji}=e$.

Lemma 7.4 *The 1-cohomology of X with coefficients in $\mathcal{G}$ is well defined. That is, $\mathsf{H}^1(X;\mathcal{G})$ is indeed a category, and in fact a groupoid.*

Proof Let $(a_{ijk}),(b_{ijk}),(c_{ijk})\in\mathrm{Ob}\,\mathsf{H}^1(X;\mathcal{G})$, and $(f_{ij})\colon(a_{ijk})\to(b_{ijk})$ and $(f'_{ij})\colon(b_{ijk})\to(c_{ijk})$ be morphisms. Then $(f'\circ f)_{ij}:=f_{ij}f'_{ij}$ defines a morphism $((f'\circ f)_{ij})\colon(a_{ijk})\to(c_{ijk})$ since

$$a_{ijk}f_{jk}f'_{jk}=f_{ij}b_{ijk}f'_{jk}=f_{ij}f'_{ij}c_{ijk}.$$

It obviously satisfies associativity. The identity on a_{ijk} is obviously defined by $e_{ij}=e$, where e denotes the unit of $\mathcal{G}$. Further, (f_{ij}^{-1}) defines a morphism $(f_{ij}^{-1})\colon b_{ijk}\to a_{ijk}$ that is the inverse of (f_{ij}). □

Proposition 7.5 *We have a faithful functor $\beta\colon \mathsf{H}^1(X;\mathcal{G}) \to \mathsf{Tors}_X^{\mathcal{G}}$.*

Proof For $(a_{ijk}) \in \operatorname{Ob} \mathsf{H}^1(X;\mathcal{G})$, we define a $\mathcal{G}$-torsor $\beta(a_{ijk})$ as follows. Let $\mathcal{U} = \coprod_{(i,j)\in I^2} \mathcal{G}_{ij}$, where $\mathcal{G}_{ij} = \mathcal{G}_i^{ij} \amalg \mathcal{G}_j^{ij} = \mathcal{G} \amalg \mathcal{G}$. We write an element of $\mathcal{G}_\bullet^{ij}$ as $g_\bullet^{ij}$ and we denote the identification $\mathcal{G} = \mathcal{G}_\bullet^{ij}$ by the map $\mathcal{G} \to \mathcal{G}_\bullet^{ij}$, $g \mapsto g_\bullet^{ij}$, where $\bullet \in \{i, j\}$ for all $i \neq j \in I$. We define an equivalence relation $\sim$ on $\mathcal{U}$ generated by

$$g_j^{ij} \sim (g a_{ijk})_j^{jk}.$$

We have $g_j^{ij} \sim g_j^{ji}$ for all $i, j \in I$. We denote the quotient set $\mathcal{U}/\sim$ by $\beta(a_{ijk})$. Then we have a surjective map $\pi\colon \beta(a_{ijk}) \to X$ defined by $\pi[g_j^{ij}] = x_j$. For this map π, we have the following.

Lemma 7.6 *For all $i, j \in I$, the map $\mathcal{G} \to \pi^{-1}x_j$, $g \mapsto [g_j^{ij}]$, is a bijection.*

Proof The surjectivity is clear. We show the injectivity. Suppose that we have $[g_j^{ij}] = [h_j^{ij}]$ for $g, h \in \mathcal{G}$. That is, we have elements $a_{k_0jk_1}, a_{k_1jk_2}, \dots, a_{k_{N-1}jk_N} \in \mathcal{G}$ such that $g a_{k_0jk_1} \dots a_{k_{N-1}jk_N} = h$ and $k_0 = k_N = i$. Then the condition $a_{ijk}a_{kj\ell} = a_{ij\ell}$ implies that $g a_{iji} = h$, hence $g = h$. □

Lemma 7.6 ensures that $[g_j^{ij}] = [h_j^{jk}]$ implies $h = g a_{ijk}$. Now we can define a distance function $d_{\beta(a_{ijk})}$ on $\beta(a_{ijk})$ as follows. Let $\varepsilon_i \in \pi^{-1}x_i$ and $\varepsilon_j \in \pi^{-1}x_j$. Then there uniquely exist $g, h \in \mathcal{G}$ such that $[g_i^{ij}] = \varepsilon_i$ and $[h_j^{ij}] = \varepsilon_j$ by Lemma 7.6. Then we define that

$$d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon_j) = d_X(x_i, x_j) + d_{\mathcal{G}}(g, h).$$

The nondegeneracy is clear. The symmetry follows from that $[g_i^{ij}] = [g_i^{ji}]$. The triangle inequality is verified as follows. Let $\varepsilon_i \in \pi^{-1}x_i$, $\varepsilon_j \in \pi^{-1}x_j$ and $\varepsilon_k \in \pi^{-1}x_k$. Suppose that we have $[g_i^{ij}] = \varepsilon_i = [g_i'^{ik}]$, $[h_j^{ij}] = \varepsilon_j = [h_j'^{jk}]$, and $[m_k^{jk}] = \varepsilon_k = [m_k'^{ik}]$. Then we have $g = g' a_{kij}$, $h' = h a_{ijk}$ and $m = m' a_{ikj}$; hence we obtain that

$$\begin{aligned}
d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon_j) + d_{\beta(a_{ijk})}(\varepsilon_j, \varepsilon_k) &= d_X(x_i, x_j) + d_{\mathcal{G}}(g, h) + d_X(x_j, x_k) + d_{\mathcal{G}}(h', m) \\
&= d_X(x_i, x_j) + d_X(x_j, x_k) + d_{\mathcal{G}}(g' a_{kij}, h) + d_{\mathcal{G}}(h a_{ijk}, m' a_{ikj}) \\
&= d_X(x_i, x_j) + d_X(x_j, x_k) + d_{\mathcal{G}}(g' a_{kij}, h) + d_{\mathcal{G}}(h a_{ijk} a_{jki} a_{kij}, m' a_{kij}) \\
&\qquad + d_{\mathcal{G}}(h, h a_{ijk} a_{jki} a_{kij}) - d_{\mathcal{G}}(h, h a_{ijk} a_{jki} a_{kij}) \\
&\geq d_X(x_i, x_j) + d_X(x_j, x_k) + d_{\mathcal{G}}(g' a_{kij}, m' a_{kij}) - |a_{ijk} a_{jki} a_{kij}| \\
&\geq d_X(x_i, x_j) + d_X(x_j, x_k) + d_{\mathcal{G}}(g', m') - |\Delta(x_i, x_j, x_k)| \\
&\geq d_X(x_i, x_k) + d_{\mathcal{G}}(g', m') \\
&= d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon_k).
\end{aligned}$$

Now a map $\pi\colon \beta(a_{ijk}) \to X$ is obviously 1-Lipschitz. We verify that it is a metric fibration as follows. Let $x_i, x_j \in X$ and $\varepsilon_i \in \pi^{-1}x_i$. Suppose that $\varepsilon_i = [g_i^{ij}]$ for $g \in \mathcal{G}$. Then $\varepsilon_j := [g_j^{ij}] \in \pi^{-1}x_j$ is the unique element in $\pi^{-1}x_j$ such that $d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon_j) = d_X(x_i, x_j)$. Also, for $\varepsilon_j' := [h_j^{ij}] \in \pi^{-1}x_j$, we

have $d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon'_j) = d_X(x_i, x_j) + d_{\mathcal{G}}(g, h) = d_{\beta(a_{ijk})}(\varepsilon_i, \varepsilon_j) + d_{\beta(a_{ijk})}(\varepsilon_j, \varepsilon'_j)$. Finally, we equip the metric fibration $\pi\colon \beta(a_{ijk}) \to X$ with a right action by $\mathcal{G}$ as $[g^{ij}_\bullet]h = [(h^{-1}g)^{ij}_\bullet]$ for all $i, j \in I$ and $\bullet \in \{i, j\}$. This is well defined since we have that

$$[(ga_{ijk})^{jk}_j]h = [(h^{-1}ga_{ijk})^{jk}_j] = [(h^{-1}g)^{ij}_j] = [g^{ij}_j]h.$$

It is straightforward to verify that this is a $\mathcal{G}$-torsor.

Next we show the functoriality. Let $(f_{ij})\colon (a_{ijk}) \to (b_{ijk}) \in \mathsf{H}^1(X;\mathcal{G})$. We construct a map $f_*\colon \beta(a_{ijk}) \to \beta(b_{ijk})$ by $[g^{ij}_\bullet] \mapsto [(gf_{ij})^{ij}_\bullet]$ for all $i, j \in I$ and $\bullet \in \{i, j\}$. It is well defined since we have that

$$[(ga_{ijk})^{jk}_j] \mapsto [(ga_{ijk}f_{jk})^{jk}_j] = [(gf_{ij}b_{ijk})^{jk}_j] = [(gf_{ij})^{ij}_j].$$

The map f_* obviously preserves fibers, and is an isometry since we have that

$$\begin{aligned} d_{\beta(b_{ijk})}\big(f_*[g^{ij}_i], f_*[h^{ij}_j]\big) &= d_{\beta(b_{ijk})}\big([(gf_{ij})^{ij}_i], [(hf_{ij})^{ij}_j]\big) \\ &= d_X(x_i, x_j) + d_{\mathcal{G}}(gf_{ij}, hf_{ij}) \\ &= d_X(x_i, x_j) + d_{\mathcal{G}}(g, h) \\ &= d_{\beta(a_{ijk})}([g^{ij}_i], [h^{ij}_j]). \end{aligned}$$

Further, it is $\mathcal{G}$-equivariant since we have that

$$(f_*[g^{ij}_j])m = [(gf_{ij})^{ij}_j]m = [(m^{-1}gf_{ij})^{ij}_j] = f_*([g^{ij}_j]m).$$

The faithfulness is obvious from the construction. □

Proposition 7.7 *The functor $\beta\colon \mathsf{H}^1(X;\mathcal{G}) \to \mathsf{Tors}^{\mathcal{G}}_X$ is full.*

Proof Let $(a_{ijk}), (b_{ijk}) \in \operatorname{Ob} \mathsf{H}^1(X;\mathcal{G})$ be cocycles, and suppose that we have a morphism $\varphi\colon \beta(a_{ijk}) \to \beta(b_{ijk})$ in $\mathsf{Tors}^{\mathcal{G}}_X$. We denote the projections $\beta(a_{ijk}) \to X$ and $\beta(b_{ijk}) \to X$ by π_a and π_b, respectively. For all $i, j \in I$, we have bijections $A_{ij}\colon \mathcal{G} \to \pi_a^{-1}x_j$ and $B_{ij}\colon \mathcal{G} \to \pi_b^{-1}x_j$ given by $g \mapsto [g^{ij}_j]$ by Lemma 7.6. Define a map $\varphi_{ij} = B_{ij}^{-1}\varphi A_{ij}\colon \mathcal{G} \to \mathcal{G}$. We have $\varphi[g^{ij}_j] = [(\varphi_{ij}g)^{ij}_j]$. Now the $\mathcal{G}$-equivariance of φ implies

$$\varphi[g^{ij}_j] = \varphi[(ge)^{ij}_j] = (\varphi[e^{ij}_j])g^{-1} = [(\varphi_{ij}e)^{ij}_j]g^{-1} = [(g\varphi_{ij}e)^{ij}_j],$$

which implies that $\varphi_{ij}g = g\varphi_{ij}e$ by Lemma 7.6. From this, we obtain that

$$\varphi[(ga_{ijk})^{jk}_j] = \varphi[(ga_{ijk})^{kj}_j] = [(\varphi_{kj}(ga_{ijk}))^{kj}_j] = [(ga_{ijk}\varphi_{kj}e)^{kj}_j].$$

Since we have $[g^{ij}_j] = [(ga_{ijk})^{jk}_j]$, we obtain that $a_{ijk}\varphi_{kj}e = (\varphi_{ij}e)b_{ijk}$ by Lemma 7.6. Further, since the lift of x_j along $[g^{ij}_i]$ is $[g^{ij}_j]$ and φ preserves the lift, the conditions $\varphi[g^{ij}_j] = [(\varphi_{ij}g)^{ij}_j]$ and $\varphi[g^{ji}_i] = [(\varphi_{ji}g)^{ji}_i]$ implies that $\varphi_{ij} = \varphi_{ji}$. Hence we obtain a morphism $(\varphi_{ij}e)\colon (a_{ijk}) \to (b_{ijk})$ in $\mathsf{H}^1(X;\mathcal{G})$, which satisfies that $\beta(\varphi_{ij}e) = \varphi$ by construction. □

Definition 7.8 Let $\pi\colon E \to X$ be a $\mathcal{G}$-torsor. For $x_i, x_j \in X$, we define a *local section of π over a pair* (x_i, x_j) to be a pair of points $(\varepsilon_i, \varepsilon_j) \in E^2$ such that $\pi\varepsilon_i = x_i, \pi\varepsilon_j = x_j$ and ε_j is the lift of x_j along ε_i. We say that $((\varepsilon_i^{ij}, \varepsilon_j^{ij}))_{(i,j)\in I^2}$ is a *local section of* π if each $(\varepsilon_i^{ij}, \varepsilon_j^{ij})$ is a local section of π over a pair (x_i, x_j) and satisfies that $\varepsilon_i^{ij} = \varepsilon_i^{ji}$.

Proposition 7.9 *Let $\pi\colon E \to X$ be a $\mathcal{G}$-torsor. For a local section $s = ((\varepsilon_i^{ij}, \varepsilon_j^{ij}))_{(i,j)\in I^2}$ of π, we can construct a cocycle $\alpha_s\pi \in \operatorname{Ob}\mathsf{H}^1(X;\mathcal{G})$. Further, for all two local sections s, s' of π, the corresponding cocycles $\alpha_s\pi$ and $\alpha_{s'}\pi$ are isomorphic.*

Proof We define $a_{ijk} \in \mathcal{G}$ as the unique element such that $\varepsilon_j^{ij} a_{ijk} = \varepsilon_j^{jk}$. Then (a_{ijk}) satisfies that $a_{ijk}a_{kj\ell} = a_{ij\ell}$ since we have

$$\varepsilon_j^{ij} a_{ijk}a_{kj\ell} = \varepsilon_j^{jk} a_{kj\ell} = \varepsilon_j^{kj} a_{kj\ell} = \varepsilon_j^{j\ell}.$$

We have $\varepsilon_x g = (\varepsilon g)_x$ for all $\varepsilon \in E, x \in X$ and $g \in \mathcal{G}$. Hence we have that

$$\begin{aligned}\varepsilon_j^{ij} a_{ijk}a_{jki}a_{kij} &= \varepsilon_j^{jk} a_{jki}a_{kij} = (\varepsilon_k^{jk})_{x_j} a_{jki}a_{kij}\\ &= (\varepsilon_k^{jk} a_{jki})_{x_j} a_{kij} = (\varepsilon_k^{ki})_{x_j} a_{kij}\\ &= ((\varepsilon_i^{ki})_{x_k} a_{kij})_{x_j} = ((\varepsilon_i^{ki} a_{kij})_{x_k})_{x_j} = ((\varepsilon_i^{ij})_{x_k})_{x_j}.\end{aligned}$$

It follows that

$$\begin{aligned}|a_{ijk}a_{jki}a_{kij}| &= d_E(\varepsilon_j^{ij}, \varepsilon_j^{ij} a_{ijk}a_{jki}a_{kij})\\ &= d_E\big(\varepsilon_j^{ij}, ((\varepsilon_i^{ij})_{x_k})_{x_j}\big)\\ &= -d_E(\varepsilon_j^{ij}, \varepsilon_i^{ij}) + d_E\big(\varepsilon_i^{ij}, ((\varepsilon_i^{ij})_{x_k})_{x_j}\big)\\ &\le -d_E(\varepsilon_j^{ij}, \varepsilon_i^{ij}) + d_E(\varepsilon_i^{ij}, (\varepsilon_i^{ij})_{x_k}) + d_E\big((\varepsilon_i^{ij})_{x_k}, ((\varepsilon_i^{ij})_{x_k})_{x_j}\big)\\ &= -d_X(x_j, x_i) + d_X(x_i, x_k) + d_X(x_k, x_j).\end{aligned}$$

Since the norm $|-|$ on $\mathcal{G}$ is conjugation invariant, the value $|a_{ijk}a_{jki}a_{kij}|$ is invariant under the cyclic permutation on $\{i, j, k\}$, and hence we obtain that $|a_{ijk}a_{jki}a_{kij}| \le |\Delta(x_i, x_j, x_k)|$. Thus we obtain a cocycle $\alpha_s\pi := (a_{ijk}) \in \operatorname{Ob}\mathsf{H}^1(X;\mathcal{G})$. Suppose that we have local sections $s = ((\varepsilon_i^{ij}, \varepsilon_j^{ij}))_{(i,j)\in I^2}$ and $s' = ((\mu_i^{ij}, \mu_j^{ij}))_{(i,j)\in I^2}$. Then there exists an element $(f_{ij}) \in \mathcal{G}^{I^2}$ such that $(\varepsilon_i^{ij} f_{ij}, \varepsilon_j^{ij} f_{ij}) = (\mu_i^{ij}, \mu_j^{ij})$. Let $\alpha_s\pi = (a_{ijk})$ and $\alpha_{s'}\pi = (b_{ijk})$. Then we obtain that

$$\varepsilon_j^{ij} a_{ijk} f_{jk} b_{ijk}^{-1} = \varepsilon_j^{jk} f_{jk} b_{ijk}^{-1} = \mu_j^{jk} b_{ijk}^{-1} = \mu_j^{ij},$$

which implies that $f_{ij} = a_{ijk} f_{jk} b_{ijk}^{-1}$. Hence (f_{ij}) defines a morphism $(f_{ij})\colon (a_{ijk}) \to (b_{ijk})$ in $\mathsf{H}^1(X;\mathcal{G})$. Since $\mathsf{H}^1(X;\mathcal{G})$ is a groupoid, this is an isomorphism. □

Proposition 7.10 *The functor $\beta\colon \mathsf{H}^1(X;\mathcal{G}) \to \mathsf{Tors}_X^{\mathcal{G}}$ is split essentially surjective.*

Proof Let $\pi\colon E \to X$ be a $\mathcal{G}$-torsor. Fix a local section $s = ((\varepsilon_i^{ij}, \varepsilon_j^{ij}))_{(i,j)\in I^2}$ of π. Let $\alpha_s\pi = (a_{ijk})$ be the cocycle constructed in Proposition 7.9. We show that the $\mathcal{G}$-torsors $\beta(a_{ijk})$ and π are isomorphic.

We define a map $\varphi\colon \beta(a_{ijk}) \to E$ by $[g^{ij}_\bullet] \mapsto \varepsilon^{ij}_\bullet g^{-1}$. It is well defined since we have that

$$[(ga_{ijk})^{jk}_j] \mapsto \varepsilon^{jk}_j a^{-1}_{ijk} g^{-1} = \varepsilon^{ij}_j g^{-1}.$$

It obviously preserves fibers and is a bijection. Also, it is an isometry since we have that

$$\begin{aligned} d_E(\varphi[g^{ij}_i], \varphi[h^{ij}_j]) &= d_E(\varepsilon^{ij}_i g^{-1}, \varepsilon^{ij}_j h^{-1}) \\ &= d_E(\varepsilon^{ij}_i, \varepsilon^{ij}_j h^{-1} g) \\ &= d_E(\varepsilon^{ij}_i, \varepsilon^{ij}_j) + d_E(\varepsilon^{ij}_j, \varepsilon^{ij}_j h^{-1} g) \\ &= d_X(x_i, x_j) + d_{\mathcal{G}}(g^{-1}, h^{-1}) \\ &= d_{\beta(a_{ijk})}([g^{ij}_i], [h^{ij}_j]). \end{aligned}$$

Further, it is immediate that φ is $\mathcal{G}$-equivariant. Hence the map φ gives an isomorphism in $\mathsf{Tors}^{\mathcal{G}}_X$. □

Corollary 7.11 *The functor $\beta\colon \mathsf{H}^1(X;\mathcal{G}) \to \mathsf{Tors}^{\mathcal{G}}_X$ is an equivalence of categories.* □

References

[1] **Y Asao**, *Magnitude and magnitude homology of filtered set enriched categories*, preprint (2023) arXiv 2303.05677

[2] **A Grothendieck**, *Technique de descente et théorèmes d'existence en géométrie algébrique, I: Généralités: Descente par morphismes fidèlement plats*, from "Séminaire Bourbaki 1959/1960", W A Benjamin, Amsterdam (1966) exposé 190 Reprinted in Séminaire Bourbaki 5 (1958–1960) 299–327, Soc. Math. France, Paris, 1995

[3] **A Grothendieck**, *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Math. 224, Springer (1971) MR

[4] **P T Johnstone**, *Sketches of an elephant*: *a topos theory compendium*, *I*, Oxford Logic Guides 43, Oxford Univ. Press (2002) MR

[5] **T Leinster**, *The magnitude of metric spaces*, Doc. Math. 18 (2013) 857–905 MR

[6] **T Leinster**, *The magnitude of a graph*, Math. Proc. Cambridge Philos. Soc. 166 (2019) 247–264 MR

[7] **T Leinster**, **M Shulman**, *Magnitude homology of enriched categories and metric spaces*, Algebr. Geom. Topol. 21 (2021) 2175–2221 MR

[8] **S Mac Lane**, *Categories for the working mathematician*, Graduate Texts in Math. 5, Springer (1971) MR

[9] **E Roff**, *The size and shape of things*: *magnitude*, *diversity*, *homology*, PhD thesis, University of Edinburgh (2022) Available at `https://era.ed.ac.uk/handle/1842/39543`

[10] **E Roff**, *Iterated magnitude homology*, preprint (2023) arXiv 2309.00577

Department of Applied Mathematics, Fukuoka University
Fukuoka, Japan

`asao@fukuoka-u.ac.jp`

Received: 27 February 2024 Revised: 19 August 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr
Octav Cornea	Université' de Montreal cornea@dms.umontreal.ca
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx
David Futer	Temple University dfuter@temple.edu
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk
Matthew Hedden	Michigan State University mhedden@math.msu.edu
Kristen Hendricks	Rutgers University kristen.hendricks@rutgers.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu
Markus Land	LMU München markus.land@math.lmu.de
Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Jessica S Purcell	Monash University jessica.purcell@monash.edu
Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Zoltán Szabó	Princeton University szabo@math.princeton.edu
Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

© 2025 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 7 (pages 3789–4436) 2025

Algebras for enriched ∞-operads 3789

RUNE HAUGSENG

An overtwisted convex hypersurface in higher dimensions 3813

RIVER CHIANG and KLAUS NIEDERKRÜGER-EID

Presheaves of groupoids as models for homotopy types 3833

LÉONARD GUETTA

Product and coproduct on fixed point Floer homology of positive Dehn twists 3875

YUAN YAO and ZIWEN ZHAO

Multitwists in big mapping class groups 3921

GEORGE DOMAT, FEDERICA FANONI and SEBASTIAN HENSEL

Cofibrantly generated model structures for functor calculus 3931

LAUREN BANDKLAYDER, JULIA E BERGNER, RHIANNON GRIFFITHS, BRENDA JOHNSON and REKHA SANTHANAM

Endomorphisms of Artin groups of type D 3975

FABRICE CASTEL and LUIS PARIS

Linear bounds of the crosscap number of knots 4009

ROB MCCONKEY

Atiyah–Segal completion for the Hermitian K-theory of symplectic groups 4037

JENS HORNBOSTEL, HERMAN ROHRBACH and MARCUS ZIBROWIUS

A minimality property for knots without Khovanov 2-torsion 4073

ONKAR SINGH GUJRAL and JOSHUA WANG

Local indicability of groups with homology circle presentations 4077

AGUSTÍN NICOLÁS BARRETO and ELÍAS GABRIEL MINIAN

Cubulating drilled bundles over graphs 4095

MAHAN MJ and BISWAJIT NAG

Infinitely many homeomorphic hyperbolic plugs with the same basic sets 4147

FANGFANG CHEN

One-point compactifications of configuration spaces and the self duality of the little disks operad 4163

CONNOR MALIN

Braided multitwists 4185

RODRIGO DE POOL

Positive intermediate Ricci curvature on connected sums 4209

PHILIPP REISER and DAVID J WRAITH

The $\mathbb{S}_n$-equivariant Euler characteristic of the moduli space of graphs 4229

MICHAEL BORINSKY and JOS VERMASEREN

Classification of metric fibrations 4257

YASUHIKO ASAO

Spaces over BO are thickened manifolds 4287

HIRO LEE TANAKA

Representation-graded Bredon homology of elementary abelian 2-groups 4321

MARKUS HAUSMANN and STEFAN SCHWEDE

The $\mathbb{Z}/p$-equivariant cohomology of the genus-zero Deligne–Mumford space with $1+p$ marked points 4341

DAIN KIM and NICHOLAS WILKINS

Geometry of free extensions of free groups via automorphisms with fixed points on the complex of free factors 4357

PRITAM GHOSH and FUNDA GÜLTEPE

Coarse cohomology of configuration space and coarse embedding 4391

ARKA BANERJEE

Correction to the article An algebraic model for finite loop spaces 4427

CARLES BROTO, RAN LEVI and BOB OLIVER