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Coarse cohomology of configuration space and coarse embedding

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We introduce a notion of equivariant coarse cohomology of the complement of a subspace in a metric space. We use this cohomology to define a notion of coarse cohomology of the two-points configuration space of a metric space and develop tools to compute this cohomology under various conditions. As an application of this theory, we show that certain classes in the coarse cohomology of two-points configuration space obstruct coarse embedding between two metric spaces.

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1 Introduction

Van Kampen [10] developed an obstruction theory for embeddings of n-dimensional simplicial complexes into $\mathbb{R}^{2n}$. A modern approach to his theory uses (co)homology of the two-points configuration space. In this article, we develop an analogous obstruction theory for coarse embedding by introducing a notion of coarse cohomology of two-points configuration space. For simplicity, throughout this article, we will say configuration space to mean two-points configuration space.

Let us first briefly describe the classical van Kampen obstruction for a topological space X. Let $\delta(X)$ be the diagonal set $\{(x,x) \mid x \in X\} \subset X \times X$. Consider the deleted product

$$\tilde{X} := X \times X - \delta(X) = \{(x,y) \in X \times X \mid x \neq y\},$$

where $\tilde{X}$ has a natural free action by $\mathbb{Z}_2$ by switching the coordinates. Consider the corresponding $\mathbb{Z}_2$ covering map $q\colon \tilde{X} \to \tilde{X}/\mathbb{Z}_2$. The space $\tilde{X}/\mathbb{Z}_2$, which we denote by $\mathrm{Conf}(X)$, is the unordered configuration space of two points in X. There exists a classifying map from the $\mathbb{Z}_2$-bundle $q\colon \tilde{X} \to \tilde{X}/\mathbb{Z}_2$ to the universal $\mathbb{Z}_2$-bundle $S^\infty \to \mathbb{R}P^\infty$ as follows:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \tilde{\phi}\ } & S^\infty \\ \Big\downarrow q & & \Big\downarrow \\ \mathrm{Conf}(X) & \xrightarrow{\ \phi\ } & \mathbb{R}P^\infty \end{array}$$

If there is an embedding $g\colon X \hookrightarrow \mathbb{R}^n$, then we can choose $\tilde{\phi}$ so that it factors through S^{n-1}. More precisely, we can choose $\tilde{\phi}$ to be the following map $\tilde{X} \to S^{n-1} \subset S^\infty$:

$$(x,y) \mapsto \frac{g(x)-g(y)}{|g(x)-g(y)|}.$$

In this case, ϕ maps $\mathrm{Conf}(X)$ to $\mathbb{R}P^{n-1} \subset \mathbb{R}P^\infty$.

So the induced map $\phi^*\colon \mathrm{H}^n(\mathbb{R}P^\infty) \to \mathrm{H}^n(\mathrm{Conf}(X))$ is trivial as it factors through $\mathrm{H}^n(\mathbb{R}P^{n-1})$. In particular, if $\eta^n \in \mathrm{H}^n(\mathbb{R}P^\infty; \mathbb{Z}_2)$ denotes the nonzero class, then $\phi^*(\eta^n)$ would be trivial. In other words, the cohomology class $\phi^*(\eta^n)$ gives an obstruction for the embedding of X into $\mathbb{R}^n$. We will call the class $\phi^*(\eta^n)$ the van Kampen obstruction class of degree n and denote it by $vk^n(X)$.

Let us now turn our attention to the coarse world. A map $f\colon X \to Y$ between two metric spaces is said to be a *coarse embedding* if there exist two proper nondecreasing maps $\rho_-, \rho_+\colon [0,\infty) \to [0,\infty)$ such that

$$\rho_-(d(x,y)) \le d(f(x), f(y)) \le \rho_+(d(x,y)) \quad \text{for all } x, y \in X.$$

Roe [13] defined the notion of coarse cohomology of a metric space which can be thought of as a coarse analog of Alexander–Spanier cohomology in the topological setting. The main motivation of this paper is to define a coarse version of vk^* that obstructs coarse embedding between metric spaces.

The study of obstruction to the coarse embedding of finitely generated groups into proper contractible n-manifold was initiated by Bestvina, Kapovich, and Kleiner [3]. For several interesting classes of metric spaces (for example CAT(0) space, Gromov-hyperbolic space) one can attach a boundary at infinity to the space. Suppose $\partial_\infty X$ denotes such a boundary of a space X. A popular theme in the study of large scale geometry is to find topological properties of $\partial_\infty X$ that provide information about the coarse geometry of X. A special case of [3] roughly says the following: if $\partial_\infty X$ does not embed into $\mathbb{R}^n$ due to the van Kampen embedding obstruction, then X does not coarsely embed into $\mathbb{R}^{n+1}$ with the Euclidean metric. One of the motivations behind the present work was to understand their obstruction in the language of Roe's coarse cohomology.

To this end, recall that the van Kampen obstruction class $vk^*(X)$ lives in the cohomology of the configuration space of X, which is the same as the $\mathbb{Z}_2$-equivariant cohomology of $X \times X - \delta(X)$ where the coefficients are $\mathbb{Z}_2$ with the trivial $\mathbb{Z}_2$ action. This suggests that a coarse version of the van Kampen obstruction class should live in some coarse version of the $\mathbb{Z}_2$-equivariant cohomology of the complement of $\delta(X)$ in $X \times X$. This motivates us to define a notion of equivariant coarse cohomology of the complement of a subspace in a metric space.

Building on Roe's theory, Banerjee and Okun [2] defined a notion of coarse cohomology of the complement of a subspace in a metric space. In this paper, we extend [2] to the equivariant setting. We then define the coarse cohomology of the configuration space of X simply to be the $\mathbb{Z}_2$-equivariant coarse cohomology of the complement of $\delta(X)$ in $X \times X$, where the action of $\mathbb{Z}_2$ on $X \times X$ is by switching coordinate.

Once we have a proper notion of the coarse cohomology of the configuration space, we can search for a coarse vk^* in that cohomology that obstructs coarse embeddings. Indeed, when X is a separable metric space, we find a class in the n^{th} degree of the coarse cohomology of the configuration space of X, which we denote by $\mathrm{cvk}^n(X)$, that obstructs coarse embedding of X into $\mathbb{R}^{n-1}$.[1] In general, the class $\mathrm{cvk}^n(X)$

[1]While the van Kampen obstruction vk^n is associated to $\mathbb{R}^n$, note that the coarse van Kampen obstruction cvk^n is related to $\mathbb{R}^{n-1}$. The reason for that is coarse cohomology of the configuration space of $\mathbb{R}^n$ is the same (except in degree 0 and 1) as the cohomology of the configuration space of $\mathbb{R}^{n-1}$ (cf. Example 5.2).

can be used to get obstruction to coarse embedding into any other metric space. We define the coarse obstruction dimension $\mathrm{cobdim}(X)$ of a separable metric space X to be the largest n such that $\mathrm{cvk}^n(X)$ is nonzero. One of our main theorems is the following:

Theorem 1.1 *If X admits a coarse embedding into Y, then* $\mathrm{cobdim}(X) \le \mathrm{cobdim}(Y)$.

So one way to determine when a given space does not admit coarse embedding into another space is to compare their cobdim. However, the equivariant coarse cohomology of the complement where cvk^* lives is hard to compute in general. A big part of this paper is devoted to the computation of equivariant coarse cohomology of the complement for certain spaces which may be of independent interest. We use these computations to estimate cobdim of certain spaces and obtain coarse embedding obstructions between certain classes of spaces. Below we highlight some of our results in this direction.

- We show that $\mathrm{cobdim}(\mathbb{R}^n) = n$. Hence Theorem 1.1 implies that X does not admit coarse embedding into $\mathbb{R}^{n-1}$ if $\mathrm{cobdim}(X) \ge n$.

- Suppose $X = K \times [0,\infty)/K \times \{0\}$ is the open cone on a finite simplicial complex K. A metric d on X is called expanding if for any two disjoint simplices $\sigma, \tau \in K$ and $S \ge 0$, there exists $r \ge 0$ such that $d(\sigma \times [r,\infty), \tau \times [r,\infty)) \ge S$. If there is a class $c \in \mathrm{H}_n(\mathrm{Conf}(X))$ such that $vk^n(X)(c) \ne 0$, then we show that $\mathrm{cobdim}(X) \ge n+1$ whenever X is equipped with a proper, expanding metric (Example 7.3). Hence such X does not admit coarse embedding into $\mathbb{R}^n$ by Theorem 1.1 and the previous example. This was initially proved in [3].

- If X is a proper, uniformly acyclic n-manifold with uniformly locally acyclic boundary, $\mathrm{cobdim}(X) \le n$ (Theorem 9.14). Examples of such spaces include universal cover of compact aspherical n-manifolds. Hence Theorem 1.1 implies that, if $\mathrm{cobdim}(X) \ge n$, then X does not admit coarse embedding into the universal cover of any compact aspherical $(n-1)$-manifold.

- If $\mathrm{H\,X}^*(X^2 - \delta(X)) = 0$ for $* \le n-1$, then $\mathrm{cobdim}(X) \ge n$ (Theorem 10.1). This implies, in particular, that any proper, uniformly contractible n-manifold has $\mathrm{cobdim} \ge n$. Hence it follows from the last example that any proper, uniformly contractible n-manifold does not admit coarse embedding into the universal cover of any compact aspherical $(n-1)$-manifold (see Corollary 10.2 for a more general version). This recovers a result of Yoon [16].

- If a finitely generated group G acts properly on X by isometries, then there exists a coarse embedding of G (equipped with a word metric coming from a finite generating set) into X by mapping G into an orbit of the action. That means, from the coarse point of view, any space X with a proper G-action has to be at least as large as G. More precisely, using Theorem 1.1 we show that if G acts properly, cocompactly on a contractible manifold M (possibly with boundary), then $\dim(M) \ge \mathrm{cobdim}(G)$ (Corollary 9.17).

Equivariant coarse (co)homology has been previously studied in [5; 15]. While our approach focuses specifically on metric spaces to address the obstruction theory and computations relevant to this article, the theories developed in [5; 15] apply to more general spaces (referred to as coarse spaces) and are motivated by certain coarse K-theoretic considerations (as discussed in the introduction of [15]). It would be intriguing to explore the extent to which our theory and computations relate to those in [5; 15].

Overview In Section 2 we describe several variations of Alexander–Spanier cochains and the corresponding cohomology theories. We also introduce coarse language and define coarse (co)homology. In Section 3, we define the equivariant coarse cohomology of the complement and give some examples using Theorem 3.8 that relate, for certain cases, equivariant coarse cohomology to the Alexander–Spanier cohomology of the quotient. In Section 4 we prove Theorem 3.8. In Section 5, we define the coarse cohomology of the configuration space. In Section 6, we give a class in the coarse cohomology of the configuration space that obstructs coarse embedding maps. Then we define the coarse obstruction dimension of a space and prove Theorem 6.4 which is a slightly stronger version of Theorem 1.1. In Section 7, we give a relation between classical van Kampen obstruction and coarse van Kampen obstruction. We use this relation to compute coarse van Kampen obstruction for certain Euclidean cones on simplicial complexes. In Section 8, we produce a coarse version of the Gysin sequence for the $\mathbb{Z}_2$-bundle to compute the coarse cohomology of configuration space. In Sections 9 and 10, we use the coarse Gysin sequence to estimate cobdim of certain spaces.

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2 Preliminaries

Alexander–Spanier complexes

We will refer to points in X^{n+1} as *n-simplices*. Let R be an abelian group. We will think of functions $X^{n+1} \to R$ as n-cochains or n-chains on X, depending on the context, and in the latter case will use additive notation $c = \sum_{\sigma \in X^{*+1}} c_\sigma \sigma$.

The basic complex is the complex of finitely supported chains

$$\mathrm{C}_*(X; R) := \left\{ c = \sum_{i=0}^{n} c_i \sigma_i \mid c_i \in R \text{ and } \sigma_i \in X^{*+1} \right\}$$

equipped with the usual boundary map, defined on the basis by

$$\partial(x_0, \dots, x_n) := \sum_{i=0}^{n} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_n).$$

The boundary map ∂ is well defined on a larger complex of *locally finite* chains $\mathrm{C}_*^{\mathrm{lf}}(X;R)$ which consists of chains c satisfying the following property: for any bounded $B \subset X$ only finitely many simplices in c have vertices in B.

The algebraic dual of $\mathrm{C}_*(X)$ is the complex of all Alexander–Spanier cochains

$$\mathrm{C}^*(X;R) = \{\phi\colon X^{*+1} \to R\}$$

with the coboundary operator

$$d(\phi)(x_0,\dots,x_n) = \sum_{i=0}^{n} (-1)^i \phi(x_0,\dots,\hat{x}_i,\dots,x_n).$$

Lemma 2.1 $\mathrm{C}^*(X;R)$ *is an acyclic complex, ie the homology of the complex is trivial in each degree except at degree zero where it is isomorphic to* R*.*

Proof In degree zero, the cohomology consists of all the constant functions $X \to R$ which are isomorphic to R.

To prove that the cohomology is trivial in degree ≥ 1, choose $a \in X$. Consider the following cone operator $D_*\colon \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$:

$$D_n\colon (x_0,x_1,\dots,x_n) \mapsto (a,x_0,\dots,x_n).$$

For any n-simplex σ with $n \geq 1$, we observe that $D_*\partial(\sigma) + \partial D_*(\sigma) = \sigma$. By considering the dual, we have that for any cochain $\phi \in \mathrm{C}^*(X;R)$ in degree ≥ 1,

$$dD^*(\phi) + D^*d(\phi) = \phi.$$

In particular, $dD^*(\phi) = \phi$ when ϕ is a cocycle. □

All the cochain complexes that we will consider in this paper are subcomplexes of the complex (C^*, d).

For a cochain $\phi \in \mathrm{C}^*(X)$, let $\|\phi\|$ be the intersection of the diagonal $\Delta = \{(x,x,\dots,x) \mid x \in X\} \subset X^{*+1}$ and the closure of the support of the function $\phi\colon X^{*+1} \to R$. Let $\mathrm{C}_0^*(X;R)$ be the complex of locally zero cochains:

$$\mathrm{C}_0^*(X;R) = \{\phi \in \mathrm{C}^*(X;R) \mid \|\phi\| = \varnothing\}.$$

The restriction of d gives a well-defined map $\mathrm{C}_0^*(X) \to \mathrm{C}_0^{*+1}(X)$. Consequently, d induces a well-defined map $\mathrm{C}_{\mathrm{as}}^*(X;R) \to \mathrm{C}_{\mathrm{as}}^{*+1}(X;R)$, where

$$\mathrm{C}_{\mathrm{as}}^*(X;R) = \mathrm{C}^*(X;R) / \mathrm{C}_0^*(X;R).$$

Alexander–Spanier cohomology, denoted by $\mathrm{H}^*(X;R)$, is the cohomology of the complex $(\mathrm{C}_{\mathrm{as}}^*(X;R), d)$. We will denote by $\tilde{\mathrm{H}}^*(X;R)$ the reduced Alexander–Spanier cohomology.

Coarse inclusion

We adopt the notation from [12]. Let (X, d) be a metric space. For $A \subset X$ and $r \geq 0$, we define $N_r(A) = \{x \in X \mid d(x, A) \leq r\}$. We will call such neighborhoods *metric neighborhoods* of A. We will say that A is *r-contained* in B, $A \overset{r}{\subset} B$, if $A \subset N_r(B)$. We will say that A is *coarsely contained* in B, $A \overset{c}{\subset} B$, if $A \overset{r}{\subset} B$ for some $r \geq 0$. Two subsets are *coarsely equal*, $A \overset{c}{=} B$, if $A \overset{c}{\subset} B$ and $B \overset{c}{\subset} A$.

Coarse intersection

Now we recall from [12] the notion of *coarse intersection*: $A \overset{c}{\cap} B \overset{c}{=} C$ if for all sufficiently large $r \geq 0$, $N_r(A) \cap N_r(B) \overset{c}{=} C$. The coarse intersection is not always well defined, it may happen that the coarse type of $N_r(A) \cap N_r(B)$ does not stabilize as r goes to infinity. However the notion "coarse intersection is coarsely contained in" is well defined. $A \overset{c}{\cap} B \overset{c}{\subset} C$ means that for any $r \geq 0$, $N_r(A) \cap N_r(B) \overset{c}{\subset} C$. It is not hard to see that this condition is equivalent to the condition that for any $r \geq 0$, $A \cap N_r(B) \overset{c}{\subset} C$.

Notation From now on, all spaces will be assumed to be metric spaces unless stated otherwise. We will use the letter r to represent a nonnegative real number, while R will denote an abelian group unless specified otherwise.

Coarse (co)homology

In what follows, we will need to measure distances between simplices of different dimensions. A convenient way to do this is to stabilize simplices by repeating the last coordinate, as follows. Denote by X^∞ the subset of the product of countably many copies of X, consisting of eventually constant sequences. Equip X^∞ with the sup metric. Let $i\colon X^{n+1} \to X^\infty$ denote the map $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, x_n, x_n, \ldots)$. For a function $\phi\colon X^{n+1} \to R$ define its stabilized support

$$|\phi| = \{i(\sigma) \mid \sigma \in X^{n+1} \text{ and } \phi(\sigma) \neq 0\} \subset X^\infty.$$

Let $\Delta = i(X)$ denote the diagonal of X^∞. Define the support of ϕ at scale r to be $|\phi|_r = |\phi| \cap N_r(\Delta)$.

We now define coarse (co)homology theories, following Roe [13] and Hair [8] using our language. The coarse cochain complex is

$$\mathrm{C\,X}^*(X; R) := \{\phi \in \mathrm{C}^*(X; R) \mid |\phi|_r \text{ is bounded for all } r\}.$$

An equivalent way to define coarse cochains is to require the support to be coarsely disjoint from the diagonal:

$$\mathrm{C\,X}^*(X; R) = \{\phi \in \mathrm{C}^*(X; R) \mid |\phi| \overset{c}{\cap} \Delta \overset{c}{=} *\}.$$

The coboundary operator d preserves this property, and the coarse cohomology $\mathrm{H\,X}^*(X;R)$ is defined to be the cohomology of the complex $(\mathrm{C\,X}^*(X;R), d)$. The coarse homology[2] $\mathrm{H\,X}_*(X;G)$ is the homology of the subcomplex of $\mathrm{C}^{\mathrm{lf}}_*(X;G)$,

$$\mathrm{C\,X}_*(X;R) := \{c \in \mathrm{C}^{\mathrm{lf}}_*(X;R) \mid |c| \overset{\mathrm{c}}{\subset} \Delta\},$$

equipped with the restriction of the boundary operator ∂. In the presence of the support condition $|c| \overset{\mathrm{c}}{\subset} \Delta$ local finiteness is equivalent to $|c|$ having finite intersections with bounded subsets of X^∞.

Example 2.2 Roe [13] showed that the coarse cohomology is isomorphic to the compactly supported Alexander–Spanier cohomology if the space is uniformly contractible. In particular, this applies to the universal cover of finite aspherical complexes. In this case, the coarse homology is isomorphic to the locally finite homology [13, Chapter 2]. For example,

$$\mathrm{H\,X}_*(\mathbb{R}^n;R) = \mathrm{H\,X}^*(\mathbb{R}^n;R) = \begin{cases} R & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

3 Equivariant coarse cohomology of the complement

We start by briefly reviewing the notion of coarse cohomology of the complement from [2]. Roughly the idea is that the coarse complement of a subset A in X is determined by the collection of subsets $\mathcal{S}$ of X which are *coarsely disjoint* from A:

$$\mathcal{S} := \{B \subset X \mid N_r(B) \cap N_r(A) \text{ is bounded for any } r \geq 0\}.$$

Coarse cohomology of the complement of A, denoted by $\mathrm{H\,X}^*(X - A)$, is defined to be the cohomology of the following complex with the usual coboundary operator d mentioned in the previous section:

$$\mathrm{C\,X}^n(X - A;R) = \{\phi \in \mathrm{C}^n(X;R) \mid \phi|_B \in \mathrm{C\,X}^n(B;R) \text{ for all } B \in \mathcal{S}\}.$$

Recall that $i\colon X \to X^\infty$ denotes the map $x \mapsto (x, \dots, x, \dots)$ and $\Delta = i(X)$. For a subset $A \subset X$, we denote the set $i(A) \subset \Delta$ by Δ_A. For our purpose, we will work with the following equivalent description of $\mathrm{C\,X}^*(X - A)$ (see [2], Lemma 3.2):

$$\mathrm{C\,X}^n(X - A;R) = \{\phi \in \mathrm{C}^n(X;R) \mid |\phi| \overset{\mathrm{c}}{\cap} \Delta \overset{\mathrm{c}}{\subset} \Delta_A\}$$

This description of coarse cochains of the complement is closer to the spirit of Roe's original definition of coarse cochains discussed in the previous section.

[2] We will need coarse homology only in Section 9 for a brief discussion of coarse PD(n) spaces. For the rest of the paper, we will work with coarse cohomology.

Example 3.1 When $X \overset{c}{=} A$, we can see that $\mathrm{C\,X}^*(X - A; R)$ coincides with $\mathrm{C}^*(X; R)$ because in this case $\Delta \overset{c}{\subset} \Delta_A$ and therefore the support condition $|\phi| \overset{c}{\cap} \Delta \overset{c}{\subset} \Delta_A$ holds for any $\phi \in \mathrm{C}^*(X; R)$. Since $\mathrm{C}^*(X; R)$ is an acyclic complex, in this case $\mathrm{H\,X}^*(X - A; R)$ is trivial in all degrees except at degree 0 where it is isomorphic to R.

Example 3.2 If A is bounded, then $\mathrm{C\,X}^*(X - A; R)$ coincides with $\mathrm{C\,X}^*(X; R)$. Hence,

$$\mathrm{H\,X}^*(X - A, R) = \mathrm{H\,X}^*(X; R)$$

whenever A is bounded.

Example 3.3 As explained in [2, Theorem 5.9], the elements of $\mathrm{H\,X}^1(X - A; \mathbb{Z}_2)$ correspond to the "coarse complementary components" of A inside X. More precisely, if X is a geodesic space and $A \subset X$ and $k = \dim_{\mathbb{Z}/2} \mathrm{H\,X}^1(X - A; \mathbb{Z}/2)$ is finite, then there exists $r \geq 0$ such that for any $L \geq r$, $X - N_L(A)$ has exactly $k + 1$ deep path components, where deep path components are those path components which are not coarsely contained in A.

Equivariant coarse cohomology of the complement

We now define an equivariant version of the coarse cohomology of the complement. Let G be a group acting on X by isometries and R be an abelian group with a G-action. Suppose G acts on X^{n+1} by the diagonal action, $g(x_0, \dots, x_n) := (gx_0, \dots, gx_n)$. G-equivariant coarse cohomology of the complement of A in X, denoted by $\mathrm{H\,X}^*_G(X - A; R)$, is defined to be the cohomology of the cochain complex

$$\mathrm{C\,X}^*_G(X - A; R) := \{\phi \in \mathrm{C\,X}^*(X - A; R) \mid \phi \text{ is } G\text{-equivariant}\}$$

with the usual coboundary operator d.

In particular, if G is acting trivially on X and R, then $\mathrm{C\,X}^*_G(X - A; R)$ coincides with $\mathrm{C\,X}^*(X - A; R)$ and therefore $\mathrm{H\,X}^*_G(X - A; R) = \mathrm{H\,X}^*(X - A; R)$. While $\mathrm{H\,X}^*_G$ is hard to compute in general, we can relate $\mathrm{H\,X}^*_G$ to a more computable cohomology under certain acyclicity conditions which we define below.

Definition 3.4 X is locally acyclic with coefficients in R if for any $x \in X$ and a neighborhood U of x, there exists another open neighborhood $V \subset U$ of x such that the inclusion $V \hookrightarrow U$ induces the trivial map in the singular homology with coefficients in R.

X is called locally acyclic away from A with coefficients in R if for some r, $X - N_r(A)$ is locally acyclic with coefficients in R.

Example 3.5 Any locally finite simplicial complex is locally acyclic. If a space is locally acyclic, then it is locally acyclic away from any of its subset. However, the converse is not true. For instance, any bounded metric space is locally acyclic away from any subset.

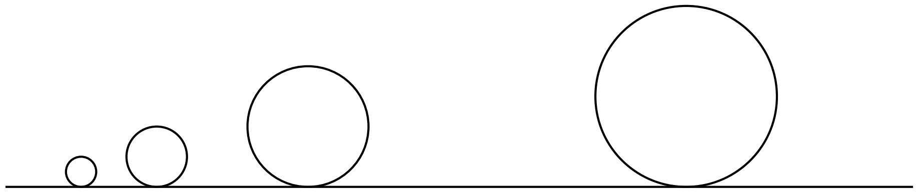

Figure 1: A subspace of $\mathbb{R}^2$ that consists of a countable union of circles $\{C_i\}_{i\in\mathbb{N}}$ and the ray $\mathcal{R} := [0,\infty)\times\{0\}$ such that the i^{th} circle has radius i and touches $\mathcal{R}$ at $(i^2, 0)$ so that the distance between two consecutive circles grows to infinity. This space is uniformly acyclic away from any bounded set.

Definition 3.6 X is uniformly acyclic with coefficients in R if there exists a nondecreasing function $\rho\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that for any $B\subset X$ the inclusion map $B\hookrightarrow N_{\rho(\operatorname{diam}(B))}(B)$ induces the trivial map in the singular homology with coefficients in R.

X is called uniformly acyclic away from $A\subset X$ with coefficients in R if there exist two nondecreasing functions $\rho,\mu\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that for any $B\subset X$ the inclusion map $B\hookrightarrow N_{\rho(\operatorname{diam}(B))}(B)$ induces the trivial map in the singular homology with coefficients in R if $d(B,A)\geq\mu(r)$.

Example 3.7 Universal covers of compact aspherical complexes are uniformly acyclic. Any uniformly acyclic space is uniformly acyclic away from any of its subset. Moreover, if we remove a bounded set from a uniformly acyclic space, then the resulting space is uniformly acyclic away from any point. In general, uniform acyclicity away from a point can be very far from being uniformly acyclic. Figure 1 describes such an example.

For the rest of the paper $G\curvearrowright X$ will mean that G is acting on X by isometries and $G\curvearrowright (X,A)$ will mean $G\curvearrowright X$ and $GA=A$. We let

$$\operatorname{Fix}_G(X) := \{x\in X \mid gx = x \text{ for all } g\in G\}.$$

We now state a theorem that relates the equivariant coarse cohomology of the complement and the reduced Alexander–Spanier cohomology for certain spaces.

Theorem 3.8 *Suppose $G\curvearrowright (X,A)$ and $\operatorname{Fix}_G(X)\neq\varnothing$. Let R be an abelian group with trivial G-action. Suppose X is uniformly acyclic away from $A\subset X$ with coefficients in R and locally acyclic away from A with coefficients in R.*

(1) *If $X\overset{c}{\subset} A$, then*
$$\mathrm{HX}^*_G(X-A;R)=\begin{cases} R & \text{if } *=0,\\ 0 & \text{otherwise.}\end{cases}$$

(2) *If X is not coarsely contained in A, then*
$$\mathrm{HX}^*_G(X-A;R)=\begin{cases} 0 & \text{if } *=0,\\ \varinjlim \tilde{\mathrm{H}}^{*-1}\big((X-N_r(A))/G;R\big) & \text{otherwise,}\end{cases}$$
where $\tilde{\mathrm{H}}^*(-)$ is the reduced Alexander–Spanier cohomology.

We will postpone the proof of Theorem 3.8 until the next section. We conclude this section with an example.

Example 3.9 Consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ by the antipodal map. Let M be a codimension-k vector subspace. Then for any abelian group R with a trivial action of $\mathbb{Z}_2$ and $i \geq 1$, we have

$$\begin{aligned} \mathrm{HX}^i_G(\mathbb{R}^n - M; R) &= \varinjlim \tilde{\mathrm{H}}^{i-1}\big((\mathbb{R}^n - N_r(M))/\mathbb{Z}_2; R\big) \quad \text{(by Theorem 3.8)} \\ &= \tilde{\mathrm{H}}^{i-1}((\mathbb{R}^n - M)/\mathbb{Z}_2; R) \\ &= \tilde{\mathrm{H}}^{i-1}(\mathbb{R}P^{k-1}; R). \end{aligned}$$

The second equality holds because $\mathbb{R}^n - N_r(M)$ is $\mathbb{Z}_2$-equivariantly homotopic to $\mathbb{R}^n - M$ and the last equality follows because $\mathbb{R}^n - M$ is $\mathbb{Z}_2$-equivariantly homotopic to S^{k-1} with the antipodal $\mathbb{Z}_2$-action.

4 Computation of equivariant coarse cohomology

In this section, our main goal is to prove Theorem 3.8. The proof is similar to the proof of a nonequivariant version of Theorem 3.8 proved in [1, Corollary 3.5]. The key is to relate the coarse cohomology of the complement to the boundedly supported cohomology of the complement which we introduce next.

Boundedly supported cohomology of the complement

Recall that $i\colon X \to X^\infty$ denotes the map $x \mapsto (x, \dots, x, \dots)$ and $\Delta = i(X)$, For a subset $A \subset X$, we denote the set $i(A) \subset \Delta$ by Δ_A. Also recall that for a cochain $\phi \in \mathrm{C}^*(X)$, $\|\phi\|$ denotes the intersection of the diagonal $\Delta = \{(x, x, \dots, x) \mid x \in X\} \subset X^{*+1}$ and the closure of the support of the function $\phi\colon X^{*+1} \to R$. Boundedly supported cohomology of the complement of $A \subset X$, denoted by $\mathrm{HB}^*(X - A)$, is the cohomology of the following cochain complex with d being the coboundary operator:

$$\mathrm{CB}^*(X - A; R) := \{\phi \in \mathrm{C}^*(X; R) \mid \|\phi\| \overset{c}{\subset} \Delta_A\}$$

Suppose R is an abelian group with a G-action. We define the equivariant boundedly supported cohomology of the complement with coefficients in R to be the cohomology of the cochain complex

$$\mathrm{CB}^*_G(X - A; R) := \{\phi \in \mathrm{CB}^*(X - A; R) \mid \phi \text{ is } G\text{-equivariant}\}.$$

We denote the cohomology of the above complex by $\mathrm{HB}^*_G(X - A; R)$.

Our next goal is to relate HB^*_G to the equivariant Alexander–Spanier cohomology. We start by briefly recalling the notion of equivariant Alexander–Spanier cohomology.

Equivariant Alexander–Spanier cohomology

Honkasalo defined a notion of equivariant Alexander–Spanier cohomology in [9]. Let R be an abelian group with a G-action. Consider the cochain complex

(†) $$\mathcal{C}^*_G(X; R) := \{\phi \in \mathrm{C}^*(X; R) \mid \phi \text{ is } G \text{ equivariant}\}.$$

The equivariant Alexander–Spanier cochain complex is defined as

$$\mathrm{C}^*_G(X;R) := \mathcal{C}^*_G(X;R)/\mathcal{C}^*_G(X;R) \cap \mathrm{C}^*_0(X;R)$$

with the usual coboundary operator d. We will denote the cohomology of this complex by $\mathrm{H}^*_G(X;R)$.

The following theorem of Honkasalo relates the G-equivariant Alexander–Spanier cohomology with the Alexander–Spanier cohomology of the quotient by the G-action.

Theorem 4.1 (see [9, Corollary 6.8]) *Let R be an abelian group with the trivial G-action. There is a natural isomorphism*

$$\mathrm{H}^*_G(X;R) \cong \mathrm{H}^*(X/G;R),$$

where the right-hand side is the ordinary Alexander–Spanier cohomology of X/G.

Remark 4.2 To define C^*_G in general, one needs a contravariant coefficient system — a contravariant functor from the category of G-spaces G/H $(H \le G)$ and G-maps between them to the category of abelian groups. If R is an abelian group with a G-action, it defines a contravariant coefficient system $G/H \mapsto R$, each G-map $G/H \to G/K$ inducing the identity $R \to R$. With this coefficient system, the equivariant Alexander–Spanier cochain complex takes the form of $\mathrm{C}^*_G(X;R)$ discussed above.

Relation between HB^*_G and H^*_G

To relate the equivariant boundedly supported cohomology with the equivariant Alexander–Spanier cohomology, we will use the cohomology of the cochain complex $\mathcal{C}^*_G(X;R)$ as in †. We will denote this cohomology by $\mathcal{H}^*_G(X;R)$. We start with the following lemma.

Lemma 4.3 *Suppose $G \curvearrowright X$ and $\mathrm{Fix}_G(X) \neq \varnothing$. Then*

$$\mathcal{H}^*_G(X;R) = \begin{cases} R & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof The proof is basically the equivariant version of Lemma 2.1. $\mathcal{H}^0_G(X;R)$ consists of all the constant functions from X to R, and hence $\mathcal{H}^0_G(X;R) = R$.

To prove that the cohomology is trivial in degree ≥ 1, choose $a \in \mathrm{Fix}_G(X)$. Consider the following G-equivariant cone operator $D_*\colon \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$:

$$D_n\colon (x_0, x_1, \dots, x_n) \mapsto (a, x_0, \dots, x_n).$$

Proceeding similar to the proof of Lemma 2.1, we obtain that $dD^*(\phi) = \phi$ when ϕ is a cocycle where D^* is the dual of D_*. Since D_* is G-equivariant, we have $D^*(\phi) \in \mathcal{C}^{*+1}_G(X;R)$. Hence, $\mathcal{H}^*_G(X) = 0$ for $* \ge 1$. □

We now state our main proposition that relates HB_G^* and $\tilde{\mathrm{H}}_G^*$ under the assumption that $\mathrm{Fix}_G(X) \neq \varnothing$.

Proposition 4.4 *Suppose $G \curvearrowright (X, A)$ and $\mathrm{Fix}_G(X) \neq \varnothing$.*

(1) *If $X \overset{c}{\subset} A$, then*

$$\mathrm{HB}_G^*(X - A; R) = \begin{cases} R & \text{if } * = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If X is not coarsely contained in A, then*

$$\mathrm{HB}_G^*(X - A; R) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}_G^{*-1}(X - N_r(A); R) & \text{otherwise.} \end{cases}$$

Proof (1) If $X \overset{c}{\subset} A$, then $\mathrm{CB}_G^*(X - A; R)$ contain all G-equivariant cochains on X, in other words, $\mathrm{CB}_G^*(X - A; R) = \mathcal{C}_G^*(X; R)$. The claim now follows from Lemma 4.3.

(2) Elements in $\mathrm{HB}_G^0(X - A; R)$ are constant functions on X with support contained in $N_r(A)$ for some r. Therefore, $\mathrm{HB}_G^0(X - A; R) = 0$ if X is not coarsely contained in A.

To calculate $\mathrm{HB}_G^*(X - A; R)$ for $* \geq 1$, we first observe that we have a short exact sequence of G-equivariant cochain complexes,

$$(\star) \qquad 0 \to \mathrm{CB}_G^*(X - A; R) \xrightarrow{j} \mathcal{C}_G^*(X; R) \xrightarrow{i} \varinjlim \mathrm{C}_G^*(X - N_r(A); R) \to 0,$$

where j is the inclusion map and i is induced by the composition of, for each r, the canonical maps

$$\mathcal{C}_G^*(X; R) \xrightarrow{\text{restriction}} \mathcal{C}_G^*(X - N_r(A); R) \xrightarrow{\text{quotient}} \mathrm{C}_G^*(X - N_r(A); R).$$

To prove $(\star)$ is a short exact sequence, we observe

$$\begin{aligned} \ker(i) &= \{\phi \in \mathcal{C}_G^*(X; R) \mid \phi \in \mathrm{C}_0^*(X - N_r(A); R) \text{ for some r}\} \\ &= \{\phi \in \mathcal{C}_G^*(X; R) \mid |\phi| \cap \Delta \subset N_r(\Delta_A) \text{ for some } r\} \\ &= \{\phi \in \mathcal{C}_G^*(X; R) \mid \|\phi\| \overset{c}{\subset} \Delta_A\} \\ &= \mathrm{im}(j). \end{aligned}$$

The short exact sequence $(\star)$ induces a long exact sequence of corresponding reduced cohomologies. The reduced cohomology of the middle cochain complex $\mathcal{C}_G^*(X; R)$ is trivial in all degrees by Lemma 4.3. Hence, the long exact sequence implies that

$$\mathrm{HB}_G^*(X - A; R) \cong \varinjlim \tilde{\mathrm{H}}_G^{*-1}(X - N_r(A); R) \quad \text{for } * \geq 1. \qquad \square$$

Combining Proposition 4.4 and Theorem 4.1 we get the following.

Corollary 4.5 *Suppose X is not coarsely contained in A for some $A \subset X$, $G \curvearrowright (X, A)$ and $\mathrm{Fix}_G(X) \neq \varnothing$. Then*

$$\mathrm{HB}_G^*(X - A; R) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}\big((X - N_r(A))/G; R\big) & \text{otherwise.} \end{cases}$$

Relation between $\mathrm{H\,X}^*_G$ and $\mathrm{H\,B}^*_G$

The main reason for defining $\mathrm{H\,B}^*_G(X-A;R)$ is the following theorem.

Theorem 4.6 *Suppose $G \curvearrowright (X,A)$ and R is an abelian group with a G-action. Suppose X is uniformly acyclic away from A, and is locally acyclic away from A with coefficients in R. Then the inclusion $\mathrm{C\,X}^*_G(X-A;R) \hookrightarrow \mathrm{C\,B}^*_G(X-A;R)$ induces an isomorphism on the cohomology*

$$\mathrm{H\,X}^*_G(X-A;R) \cong \mathrm{H\,B}^*_G(X-A;R).$$

The proof of the above theorem follows a sketch provided by Roe in [14], which outlines a proof of a related result connecting coarse cohomology with compactly supported cohomology. The core idea of the proof involves a standard "connect the dots" construction, which allows us to attach a singular chain to each simplex while remaining close to it, thanks to the uniform acyclicity of the space. We will state this more precisely in Lemma 4.8.

Let us first fix some notation. For the rest of the section, we suppress the coefficient R from the notation unless it is important. Let $\mathrm{C}^{\mathrm{s}}_*(X)$ be the singular chain complex on X.

For the upcoming lemmas, we need to measure the distance between the support of a singular simplex and an n-simplex. Support of a singular simplex is in X while an n-simplex is in X^{n+1}. However, recall from Section 2 that X^n can be realized as a subset of X^∞ for any $n \in \mathbb{N}$ by stabilizing the last coordinate. This way both the support of a singular chain, and more generally any subset of X and n-simplices live in X^∞. Consequently, we can measure the distance between the support of a singular simplex and an n-simplex in X^∞.

In what follows, we will need a bound on the distance between a simplex and its boundary. This is the purpose of the following lemma.

Lemma 4.7 *We have $\max_{\tau\in|\partial\sigma|} d(\sigma,\tau) \le \mathrm{diam}(\sigma)$ for any simplex σ.*

Proof Let $\sigma = (x_0, x_1, \dots, x_n)$ and $\tau \in |\partial\sigma|$. The stabilization of both σ and τ in X^∞ has x_i in each coordinate for some i. It follows that

$$d(\sigma,\tau) \le \max_{i\neq j}\{|x_i, x_j|\} = \mathrm{diam}(\sigma). \qquad \square$$

Lemma 4.8 *Suppose X is uniformly acyclic away from A and locally acyclic away from A and $G \curvearrowright (X,A)$. Then there exist two nondecreasing sequences of functions $\mu_n, \rho_n \colon [0,\infty) \to [0,\infty)$ and a G-equivariant chain map $M \colon \mathrm{C}^F_*(X) \to \mathrm{C}^{\mathrm{s}}_*(X)$ where*

$$\mathrm{C}^F_n(X) = \langle \sigma^n \mid d(\sigma^n, A) \ge \mu_n(\mathrm{diam}\,\sigma^n)\rangle \subset \mathrm{C}_n(X)$$

such that:

1. $|M(\sigma^n)| \subset N_{\rho_n(\mathrm{diam}(\sigma^n))}(\sigma^n)$.
2. *There exists an $L \ge 0$ such that for every $x \in X - N_L(A)$, and a neighborhood U of x, there is a neighborhood $W \subset U$ of x such that $|M(\sigma^n)| \subset U$ for all $\sigma^n \in W^{n+1}$.*

Proof We will define μ_i, ρ_i, and the chain map $M\colon \mathrm{C}_i^F \to \mathrm{C}_i^{\mathrm{s}}$ by induction on i.

For $i=0$, define μ_0 and ρ_0 to be the constant functions $[0,\infty)\to[0,\infty)$ with image $\{0\}$. That means, $\mathrm{C}_0^F(X)=\mathrm{C}_0(X)$ and we define $M\colon \mathrm{C}_0^F(X)\to \mathrm{C}_0^{\mathrm{s}}(X)$ to be the identity map.

Since X is uniformly acyclic away from A, there exist $\rho,\mu\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that the map $i_*\colon \mathrm{H}_*(B)\to \mathrm{H}_*(N_{\rho(\mathrm{diam}(B))}(B))$ is trivial if $d(B,A)\geq \mu(\mathrm{diam}(B))$. By the induction hypothesis, suppose μ_i, ρ_i, and $M\colon \mathrm{C}_i^F \to \mathrm{C}_i^{\mathrm{s}}$ are already defined with the desired properties for all $i\leq n$. For $i=n+1$, define

$$\mu_{n+1}(x)=\max\bigl\{\mu[2\rho_n(x)+x]+\rho_n(x)+x,\mu_n(x)\bigr\}. \tag{4-1}$$

By construction, $\mu_{n+1}\geq\mu_n$.

Next, we define M on $\mathrm{C}_{n+1}^F(X)$. Take $\sigma\in \mathrm{C}_{n+1}^F(X)$ with $\mathrm{diam}(\sigma)=r$. By the induction hypothesis, M satisfies property (1) when applied to $\partial\sigma$ and therefore

$$|M(\partial\sigma)|)\leq N_{\rho_n(r)}(|\partial\sigma|). \tag{4-2}$$

Consequently we obtain

$$d\bigl(\sigma,|M(\partial\sigma)|\bigr)\leq d\bigl(|\partial\sigma|,|M(\partial\sigma)|\bigr)+d(\sigma,|\partial\sigma|)\leq \rho_n(r)+r, \tag{4-3}$$

where the first inequality is due to the triangle inequality and the second inequality follows from (4-2) and Lemma 4.7.

Since $\sigma\in \mathrm{C}_{n+1}^F(X)$, we have $d(\sigma,A)\geq\mu_{n+1}(r)$. It follows that

$$\begin{aligned}
d\bigl(|M(\partial\sigma)|,A\bigr)&\geq d(\sigma,A)-d\bigl(\sigma,|M(\partial\sigma)|\bigr) && \text{(by the triangle inequality)}\\
&\geq \mu_{n+1}(r)-\rho_n(r)-r && \text{(by (4-3))}\\
&\geq \mu(2\rho_n(r)+r)+\rho_n(r)+r-\rho_n(r)-r && \text{(by (4-1))}\\
&=\mu(2\rho_n(r)+r).
\end{aligned}$$

Note that, $|M(\partial\sigma)|\subset N_{\rho_n(r)}(|\partial\sigma|)$ and $\mathrm{diam}(|\partial d\sigma|)\leq r$ implies that

$$\mathrm{diam}\bigl(|M(\partial\sigma)|\bigr)\leq 2\rho_n(r)+r.$$

Since $d\bigl(|M(\partial\sigma)|,A\bigr)\geq\mu(2\rho_n(r)+r)$, and X is (μ,ρ)-uniformly acyclic away from A, it follows that $M(\partial\sigma)$ is a boundary of some singular chain of diameter at most $\rho[2\mu_n(r)+r]$. Let

$$k(\sigma):=\inf\{\mathrm{diam}(c)\mid \partial c=M(\partial\sigma)\}.$$

We define $M(\sigma)$ to be a singular chain whose boundary is $M(\partial\sigma)$ and has diameter at most $2k(\sigma)$. To make M a G-equivariant chain, we can first define M on a set of simplices from $\mathrm{C}_{n+1}^F(M)$ that contains one element from each orbit of simplices under the action of G and then extend the map G-equivariantly.

Next we define ρ_{n+1}, so that M satisfies (1). Let σ be as before. By construction, $\partial M(\sigma) = M(\partial\sigma)$ and $k(\sigma) \le \rho[2\mu_n(r) + r]$. Therefore we have

$$\begin{aligned} |M(\sigma)| \subset N_{2k(\sigma)}|M(\partial\sigma)| &\subset N_{2k(\sigma)+\rho_n(r)}(|\partial\sigma|) && \text{(by (4-2))} \\ &\subset N_{2k(\sigma)+\rho_n(r)+r}(\sigma) && \text{(by Lemma 4.7)} \\ &\subset N_{2\rho[(2\mu_n(r)+r)]+\rho_n(r)+r}(\sigma). \end{aligned}$$

Finally if we define

$$\rho_{n+1}(x) := 2\rho[2\mu_n(x) + x] + \rho_n(x) + x,$$

then by construction

$$M(\sigma) \subset N_{\rho_{n+1}(r)}(\sigma) = N_{\rho_{n+1}(\mathrm{diam}(\sigma))}(\sigma),$$

which is the desired property (1).

To prove (2), recall that X is locally acyclic away from A and hence there exists some $L > 0$ such that $X - N_L(A)$ is locally acyclic. Recall that $k(\sigma)$ is the infimum of the diameter of chains bounding $M(\partial\sigma)$. By induction on the dimension of σ, we observe that $k(\sigma)$ goes to zero as $\mathrm{diam}(\sigma)$ goes to 0, given that $d(\sigma, A) \ge L$ because of the local acyclicity of $X - N_L(A)$. By construction, $M(\sigma)$ is of diameter at most $2k(\sigma)$. Therefore $\mathrm{diam}(|M(\sigma)|)$ goes to zero as $\mathrm{diam}(\sigma)$ goes to 0, given that $d(\sigma, A) \ge L$. This gives us (2). □

Lemma 4.9 *Assume that $A \subset X$ and $G \curvearrowright (X, A)$. Let $f : \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ be a G-equivariant chain map and $\rho_n : [0,\infty) \to [0,\infty)$ is some nondecreasing sequence of nondecreasing functions such that*:

- $|f(\sigma^n)| \subset N_{\rho_n(\mathrm{diam}(\sigma^n))}(\sigma^n)$ *for any n-simplex σ^n.*
- *There exists an $L \ge 0$ such that for every $x \in X - N_L(A)$, and a neighborhood U of x, there is a neighborhood $W \subset U$ of x such that $|f(\sigma^n)| \subset U^{n+1}$ whenever $\sigma^n \in W^{n+1}$.*

Then f and the identity map on $\mathrm{C}_*(X)$ are chain homotopic via a G-equivariant chain homotopy

$$H_n : \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$$

with an associated nondecreasing sequence of nondecreasing functions $\rho'_n : [0,\infty) \to [0,\infty)$ such that:

(1) $|H_n(\sigma)| \subset N_{\rho'_n(\mathrm{diam}(\sigma))}(\sigma)$ *for any n-simplex σ^n.*

(2) *There exists an $L' \ge 0$ such that for every $x \in X - N_{L'}(A)$, and a neighborhood U of x, there is a neighborhood $W \subset U$ of x such that $|H_n(\sigma^n)| \subset U^{n+2}$ for all $\sigma^n \in W^{n+1}$.*

Proof We can define H_n by induction on the dimension n. Define $H_0(x) := (x, f(x))$. Note that H_0 is G-equivariant and $f(x) - x = \partial H_0(x)$. Since $f(x) \subset N_{\rho_0(\mathrm{diam}(x))}(x)$, we get $H_0(x) \subset N_{\rho_0(\mathrm{diam}\{x\})}(x)$. We define $\rho'_0 := \rho_0$.

Suppose we have already defined $H_m : \mathrm{C}_m(X) \to \mathrm{C}_{m+1}(X)$ and the nondecreasing map ρ'_m such that H_m is G-equivariant, $H_m(\sigma) \subset N_{\rho'_m(\mathrm{diam}(\sigma))}(\sigma)$ and $\partial H_m(\sigma) + D_{m-1}\partial(\sigma) = i(\sigma) - f(\sigma)$ for any $m \leq n$. To define $H_{n+1}(\sigma)$ for an $(n+1)$-simplex σ, consider the chain

$$c_\sigma := \sigma - f(\sigma) - H_n\partial(\sigma).$$

By induction hypothesis on H_n, c is a cycle. Take a vertex b from the chain c, and consider the cone operator

$$T_b : \mathrm{C}_{n+1}(X) \to \mathrm{C}_{n+2}(X), \quad (x_0, \dots, x_{n+1}) \mapsto (b, x_0, \dots, x_{n+1}).$$

Define $H_{n+1}(\sigma) := T_b(c_\sigma)$. It follows that

$$\partial H_{n+1}(\sigma) = c_\sigma - T_b(\partial c_\sigma) = c_\sigma = \sigma - f(\sigma) - H_n\partial(\sigma).$$

To make H_{n+1} equivariant, we first define it on elements from each orbit class of $(n+1)$-simplices and then extend it G-equivariantly.

We now focus on the support of $H_{n+1}(\sigma)$. Suppose $\mathrm{diam}(\sigma) = r$. By the induction hypothesis, $H_n(\tau) \subset N_{\rho'_n}(\tau)$ for any n-simplex τ. Since ρ'_n is a nondecreasing function, $\rho'_n(r) \geq \rho'_n(\mathrm{diam}(\tau))$ for any $\tau \in |\partial\sigma|$. We have

$$|H_n(\partial\sigma)| \subset \bigcup_{\tau \in |\partial\sigma|} N_{\rho'_n(\mathrm{diam}(\tau))}(\tau) \subset N_{\rho'_n(r)}(|\partial\sigma|) \subset N_{[\rho'_n(r)+r]}(\sigma),$$

where the last containment is by Lemma 4.7. Also, σ and $|f(\sigma)|$ are subsets of $N_{\rho_{n+1}(r)}(\sigma)$. If we define

$$q(x) := \max\{\rho'_n(x) + x, \rho_{n+1}(x)\}$$

then $|c_\sigma| \subset N_{q(r)}(\sigma)$ and consequently

$$\mathrm{diam}(|c_\sigma|) \leq \mathrm{diam}(\sigma) + 2q(r) \leq r + 2L(r).$$

Since $|T_b(c_\sigma)| \subset N_{\mathrm{diam}(|c_\sigma|)}(|c_\sigma|)$, we obtain

$$\begin{aligned} |H_{n+1}(\sigma)| = |T_b(c_\sigma)| &\subset N_{\mathrm{diam}(|c_\sigma|)}(|c_\sigma|) \\ &\subset N_{\mathrm{diam}(|c_\sigma|)}[N_{q(r)}(\sigma)] \\ &\subset N_{[\mathrm{diam}(|c_\sigma|)+q(r)]}(\sigma) \subset N_{[r+3q(r)]}(\sigma). \end{aligned}$$

Letting $\rho'_{n+1}(r) := r + 3q(r)$, we get property (1).

To get property (2), note that vertex of any simplex in $|H_n(\sigma)|$ is either a vertex of σ or a vertex of some simplices in $|f(\tau)|$ where τ is some subsimplex of σ. The claim then follows from the analogous property of the map f. □

Let $\mathcal{U}$ denotes an open cover X. We say $\mathcal{U}$ is G-invariant, if for any $U \in \mathcal{U}$, $gU \in \mathcal{U}$ for all $g \in G$. Let $\mathrm{C}^{\mathcal{U}}_*(X)$ be the chain complex generated by singular simplices supported in some open set in $\mathcal{U}$. Let $V : \mathrm{C}^{\mathrm{s}}_*(X) \to \mathrm{C}_*(X)$ be the forgetful map, which maps a singular simplex to its vertices.

To prove Theorem 4.6, we will need to fill in simplices by singular chains in $C_*^{\mathcal{U}}(X)$ for some G-invariant cover $\mathcal{U}$. More precisely, we need the following.

Proposition 4.10 *Suppose X is uniformly acyclic away from A and locally acyclic away from A and $G \curvearrowright (X, A)$. Let $\mathcal{U}$ be a G-invariant open cover of X. Then there exist two nondecreasing sequences of functions $\mu_n, \rho_n\colon [0,\infty) \to [0,\infty)$, a G-equivariant chain map $S\colon C_*^F(X) \to C_*^{\mathcal{U}}(X)$ where*

$$C_n^F(X) = \langle \sigma^n \mid d(\sigma^n, A) \geq \mu_n(\operatorname{diam} \sigma^n)\rangle \subset C_n(X)$$

and a G-equivariant chain homotopy $H\colon C_*^F(X) \to C_{*+1}(X)$ between $VS\colon C_*^F(X) \to C_*(X)$ and the inclusion map so that:

1. $|VS(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$ *for any n-simplex σ^n.*
2. $|H(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$ *for any n-simplex σ^n.*
3. *There exists $r > 0$ so that for every $x \notin N_r(A)$, there is a neighborhood W of x such that for all $\sigma^n \in W^{n+1}$, we have $H(\sigma^n) \in C_{n+1}^{\mathcal{U}}(X)$.*

Proof Because of the assumptions on X, we can invoke Lemma 4.8, which outputs a G-equivariant map $M\colon C_*^F(X) \to C_*^{s}(X)$ satisfying (1) and (2) from Lemma 4.8.

Next, we choose a barycentric subdivision map $P\colon C_*^{s}(X) \to C_*^{\mathcal{U}}(X)$. We can choose P in a G-equivariant way by the same trick as before: in each dimension, first define it on an element from each orbit class of simplices and then extend G-equivariantly. Note that P satisfies

$$(*) \qquad |P(\sigma)| \subset N_{\operatorname{diam}(\sigma)}(|\sigma|) \quad \text{for any } \sigma.$$

We define $S := PM$. Since both P and M are G-equivariant, so is S.

Since M satisfies Lemma 4.8(1) and P satisfies $(*)$, we obtain that

$$|VS(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$$

for some function nondecreasing sequence of function $\rho'_n\colon [0,\infty) \to [0,\infty)$.

Moreover, since M satisfies (2) from Lemma 4.8 and P satisfies $(*)$, there exists an $L \geq 0$ such that for every $x \in X - N_L(A)$, and a neighborhood U of x, there is a neighborhood $W \subset U$ of x such that $|VS(\sigma^n)| \subset U^{n+1}$ whenever $\sigma^n \in W^{n+1}$.

In conclusion, VS satisfies the hypothesis of Lemma 4.9. Applying Lemma 4.9 to the map VS we obtain a chain homotopy H_* between VS and the id that satisfies properties (2) and (3). □

Remark 4.11 If X is uniformly acyclic, then we can take $C_*^F(X)$ to be $C_*(X)$ in Proposition 4.10. Also, the local acyclicity away from A was only used to ensure property (3). So, if the space is just uniformly acyclic away from A, we still get H and S satisfying the properties (1) and (2).

Proof of Theorem 4.6 Consider the long exact sequence

$$\cdots \to \mathrm{H}\,\mathrm{B}_G^{*-1}(X-A) \to \mathrm{H}^*(\mathrm{C}\,\mathrm{B}_G^*(X-A)/\,\mathrm{C}\,\mathrm{X}_G^*(X-A)) \to \mathrm{H}\,\mathrm{X}_G^*(X-A) \to \mathrm{H}\,\mathrm{B}_G^*(X-A) \to .$$

Therefore it is enough to show that

$$\mathrm{H}^*(\mathrm{C}\,\mathrm{B}_G^*(X-A)/\,\mathrm{C}\,\mathrm{X}_G^*(X-A)) = 0.$$

In other words, for $\phi \in \mathrm{C}\,\mathrm{B}_G^n(X-A)$ with $d\phi \in \mathrm{C}\,\mathrm{X}_G^{n+1}(X-A)$ we need to find $\psi \in \mathrm{C}\,\mathrm{B}_G^{n-1}(X-A)$ so that $\phi - d\psi \in \mathrm{C}\,\mathrm{X}_G^n(X-A)$.

Our goal is to apply Proposition 4.10. In order to do that, we first need to choose a G-invariant cover of X. Let $r > 0$ and let U be union of all the balls of radius r that are centered at points in $A \cap \|\phi\|$. Since both A and $\|\phi\|$ are G-invariant, U is G-invariant: $gU = U$ for all $g \in G$. For each $x \in X - \|\phi\|$, choose a metric neighborhood U_x with diameter ≤ 1 such that $U_x^{n+1} \cap |\phi| = \varnothing$. Since ϕ is G-equivariant, we can choose the association $x \mapsto U_x$ so that $U_{gx} = gU_x$ for all $g \in G$. Let $\mathcal{U}$ denote the collection of U_x together with U. By construction, this is a G-invariant cover.

Now we apply Proposition 4.10 to this setup which outputs a complex $\mathrm{C}_*^F(X)$, a G-equivariant chain map $S\colon \mathrm{C}_*^F(X) \to \mathrm{C}_*^{\mathcal{U}}(X)$ and a G-equivariant chain homotopy $H\colon \mathrm{C}_*^F(X) \to \mathrm{C}_{*+1}(X)$ between VS and the inclusion map that satisfy properties (1), (2), and (3) from Proposition 4.10.

We define a linear map $D\colon \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$ by setting

$$D(\sigma^n) = \begin{cases} H(\sigma^n) & \text{if } \sigma^n \in \mathrm{C}_n^F(X), \\ 0 & \text{otherwise.} \end{cases}$$

We define $\tau\colon \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$ as

$$\tau = id + \partial D + D\partial.$$

If $\sigma \in \mathrm{C}_*^F(X)$, then by construction $D(\sigma) = H(\sigma)$ and therefore $\tau(\sigma) = VS(\sigma)$. Suppose τ^* denote the dual of τ. By applying τ^* on ϕ, we get

$$\tau^*\phi = \phi + dD^*\phi + D^*d\phi.$$

We claim that $\tau^*\phi \in \mathrm{C}\,\mathrm{X}_G^n(X-A)$. If $\operatorname{diam}(\sigma^n) \leq k$ and $d(\sigma^n, A) > \mu_n(k)$, then $\sigma^n \in \mathrm{C}_n^F(X)$ and hence $\tau(\sigma^n) = VS(\sigma^n)$. Moreover, if σ^n is outside of the $\rho_n(k)$-neighborhood of U^{n+1}, then $|\tau(\sigma^n)|$ does not touch U^{n+1} because $|\tau(\sigma^n)| = |VS\sigma^n| \subset N_{\rho_n(k)}(\sigma^n)$ by property (1) from Proposition 4.10. This implies $(\tau^*\phi)(\sigma^n) = \phi(\tau(\sigma^n)) = 0$. In other words, $\sigma^n \notin |\tau^*\phi|$. It follows that

$$|\tau^*\phi| \cap N_k(\Delta) \subset N_{\mu_n(k)}(\Delta_A) \cup N_{\rho_n(k)}(U^{n+1}).$$

Since $U \overset{c}{\subset} A$, we have

$$|\tau^*\phi| \cap N_k(\Delta) \overset{c}{\subset} \Delta_A.$$ □

Next, we claim that $D^*(\phi) \in \mathrm{C}\,\mathrm{B}_G^{n-1}(X-A)$. Since H satisfies property (3) from Proposition 4.10, we can choose a set $N_r(A)$ containing U such that for any $x \notin N_r(A)$, there is a neighborhood W_x of x

so that $|H(\sigma^n)| \subset U_x^{n+2}$ for any $\sigma^n \in W_x^{n+1}$. Hence for any $x \notin N_r(A) \cup \|\phi\|$, we have $|H(\sigma^n)| \notin |\phi|$ and hence $|D(\sigma^n)| \notin |\phi|$ for all $\sigma^n \in W_x^{n+1}$. Therefore, $\|D^*(\phi)\| \subset \|\phi\| \cup N_r(A)$. The claim follows since $\phi \in \mathrm{C\,B}^*(X)$.

Finally, we claim that $D^*d(\phi) \in \mathrm{C\,X}_G^n(X - A)$. By construction of D, if $d(\sigma^n, A) \geq \mu_n(k)$ then $|D\sigma^n| \subset N_{\rho_n(k)}(\sigma^n)$ where k is the diameter of σ^n. Since $d(\phi) \in \mathrm{C\,X}_G^*(X - A)$, the claim follows.

Now setting $\psi = -D^*(\phi)$ we get what we want.

We can now prove Theorem 3.8.

Proof of Theorem 3.8 We observe that $\mathrm{C\,X}_G^*(X - A; R) = \mathcal{C}_G^*(X; R)$ when X is coarsely contained in A. Hence Theorem 3.8(1) follows from Lemma 4.3.

Theorem 3.8(2) follows immediately from Proposition 4.4 and Theorem 4.6. □

5 Coarse cohomology of the configuration space

Let (X, d) be a metric space. Equip $X^2 = X \times X$ with the sup metric. Consider the $\mathbb{Z}_2$-action on X^2 that flips the coordinates. Since the fixed point set for this action is the diagonal subspace

$$\delta(X) = \{(x, x) \mid x \in X\} \subset X^2,$$

we have $\mathbb{Z}_2 \curvearrowright (X^2, \delta(X))$. Let R be an abelian group with a $\mathbb{Z}_2$-action. We define *coarse cohomology of the two-points configuration space of* X with coefficients in R to be the cohomology of the complex $\mathrm{C\,X}_{\mathbb{Z}_2}^*(X^2 - \delta(X); R)$. From now on, for the sake of simplicity, we will omit the term "two-points" from our terminology and refer to it simply as the coarse cohomology of the configuration space.

Theorem 3.8 immediately gives us the following.

Proposition 5.1 *If X is unbounded, uniformly acyclic, and locally acyclic with coefficients in R where R is an abelian group with trivial $\mathbb{Z}_2$-action, then*

$$\mathrm{H\,X}_{\mathbb{Z}_2}^*(X^2 - \delta(X); R) = \begin{cases} 0 & \text{if } * = 0, \\ \varinjlim \tilde{\mathrm{H}}^{*-1}\big((X^2 - N_r(\delta(X)))/\mathbb{Z}_2; R\big) & \text{otherwise.} \end{cases}$$

Example 5.2 $\mathbb{R}^n$ satisfies the hypothesis of Proposition 5.1. Moreover, for any r, there is a $\mathbb{Z}_2$-equivariant deformation retraction of $(\mathbb{R}^n)^2 - \delta(\mathbb{R}^n)$ to $(\mathbb{R}^n)^2 - N_r(\delta(\mathbb{R}^n))$. Therefore, applying Proposition 5.1 we obtain the following where the coefficients group is $\mathbb{Z}_2$ with the trivial $\mathbb{Z}_2$-action:

$$\begin{aligned} \mathrm{H\,X}_{\mathbb{Z}_2}^*((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n); \mathbb{Z}_2) &= \begin{cases} 0 & \text{if } * = 0, \\ \tilde{\mathrm{H}}^{*-1}\big(((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n))/\mathbb{Z}_2; \mathbb{Z}_2\big) & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0 & \text{if } * = 0, \\ \tilde{\mathrm{H}}^{*-1}(\mathbb{R}P^{n-1}; \mathbb{Z}_2) & \text{otherwise,} \end{cases} \\ &= \begin{cases} \mathbb{Z}_2 & \text{if } 2 \leq * \leq n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Example 5.3 Recall that ℓ^∞ is the space of bounded sequences of real numbers with the sup-norm metric. The space $(\ell^\infty)^2 - \delta(\ell^\infty)$ $\mathbb{Z}_2$-equivariantly deformation retracts to $(\ell^\infty)^2 - N_r(\delta(\ell^\infty))$. Since $(\ell^\infty)^2 - \delta(\ell^\infty)$ is acyclic, $((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2$ is a classifying space for $\mathbb{Z}_2$ and hence is homotopy equivalent to $\mathbb{R}P^\infty$. Hence arguing as in Example 5.2 we obtain

$$\mathrm{HX}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } * \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Next we describe the maps between two metric spaces that induce map between the corresponding coarse cohomology of the configuration spaces. In the topological setting, an injective continuous map induces a map between the cohomology of the corresponding configuration spaces. In the coarse setting, the role of injective continuous maps are played by *coarse expanding* maps which we define next.

Definition 5.4 A map $f : (X, A) \to (Y, B)$ between pairs is called relatively proper if $f^{-1}(N_r(B)) \overset{c}{\subset} A$ for any r.

A map $f : (X, A) \to (Y, B)$ between pairs is called relatively coarse if f is relatively proper and there exists a nondecreasing function $\mu : [0, \infty) \to [0, \infty)$ such that $d(f(x), f(y)) \leq \mu(d(x, y))$ for all $x, y \in X$.

A map $f : X \to Y$ is a coarse expanding map if the induced map $(x, y) \mapsto (f(x), f(y))$ from $(X^2, \delta(X))$ to $(Y^2, \delta(Y))$ is a relatively coarse map.

Example 5.5 Recall that a map $f : X \to Y$ between two metric spaces is said to be a *coarse embedding* if there exist two proper nondecreasing maps $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ such that

$$\rho_-(d(x, y)) \leq d(f(x), f(y)) \leq \rho_+(d(x, y)) \quad \text{for all } x, y \in X.$$

One can see that any coarse embedding map is a coarse expanding map.

Example 5.6 A map that is not coarse expanding is the map $x \mapsto |x|$ between real numbers. The reason is that the map $(x, y) \mapsto (|x|, |y|)$ from $(\mathbb{R}^2, \delta(\mathbb{R}))$ to itself is not relatively proper.

We now recall the following from [2].

Lemma 5.7 *For any abelian group R, a relatively coarse map $f : (X, A) \to (Y, B)$ between pairs induces the chain map $f^* : \mathrm{CX}^*(Y - B; R) \to \mathrm{CX}^*(X - A; R)$ by the canonical formula*

$$(f^*\phi)(x_0, x_1, \ldots, x_n) = \phi(f(x_0), f(x_1), \ldots, f(x_n)).$$

The following is immediate from Lemma 5.7.

Lemma 5.8 *If $f : X \to Y$ is a coarse expanding map and R is an abelian group, then the map $(x, y) \mapsto (f(x), f(y))$ induces a map $f^* : \mathrm{HX}^*_{\mathbb{Z}_2}(Y^2 - \delta(Y); R) \to \mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X); R)$.*

6 Coarse van Kampen obstruction

In this section, we find an obstruction to the existence of coarse expanding maps between two metric spaces. Our key observation is the next proposition.

Proposition 6.1 *Any two coarse expanding maps from the space X to ℓ^∞ induce the same map from* $\mathrm{H\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty);R)$ *to* $\mathrm{H\,X}^*_{\mathbb{Z}_2}(X^2-\delta(X);R)$.

Proof For convenience, we will suppress the coefficient R from the notation in the proof.

Suppose $f,g\colon X^2\to(\ell^\infty)^2$ are two maps induced by two coarse expanding maps from X to ℓ^∞. For convenience, we will denote the induced map between $\mathrm{C}_*(X^2)$ and $\mathrm{C}_*((\ell^\infty)^2)$ by f and g as well. Let f^* and g^* be the corresponding map from $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty))$ to $\mathrm{C\,X}^*_{\mathbb{Z}_2}(X^2-\delta(X))$. We will show that there is a chain homotopy $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty))\to\mathrm{C\,X}^{*+1}_{\mathbb{Z}_2}(X^2-\delta(X))$ between f^* and g^*.

Since $(\ell^\infty)^2$ is uniformly contractible, the proof of Proposition 4.10 (see Remark 4.11) gives us a $\mathbb{Z}_2$-equivariant chain map $S\colon \mathrm{C}_*((\ell^\infty)^2)\to\mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$ such that $|S(\sigma^n)|\subset N_{\rho_n(\mathrm{diam}(\sigma^n))}(\sigma^n)$ for some nondecreasing sequence of functions $\rho_n\colon[0,\infty)\to[0,\infty)$. Moreover, by Proposition 4.10, the composition VS is chain homotopic to the identity map by a $\mathbb{Z}_2$-equivariant chain homotopy H such that $|H(\sigma^n)|\subset N_{\rho_n(\mathrm{diam}(\sigma^n))}(\sigma^n)$.

We now construct a chain homotopy $D_*\colon\mathrm{C}_*(X^2)\to\mathrm{C}^{\mathrm{s}}_{*+1}((\ell^\infty)^2)$ between the two maps

$$S\circ f,\, S\circ g\colon \mathrm{C}_*(X^2)\to\mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$$

with certain properties: in particular we want the homotopy to avoid $\delta(\ell^\infty)$.

For convenience let us fix some notation. For a singular chain $c\in\mathrm{C}^{\mathrm{s}}_*(X)$, let

$$\mathrm{size}(c):=\sup_{\tau\in|c|}\{\mathrm{diam}(\tau)\}.$$

For a simplex $\sigma\in(X^2)^*$, let

$$r_\sigma:=\min\bigl\{d\bigl(|S(f(\sigma))|,\delta(\ell^\infty)\bigr),d\bigl(|S(g(\sigma))|,\delta(\ell^\infty)\bigr)\bigr\}$$

and

$$R_\sigma:=\max\bigl\{d\bigl(|S(f(\sigma))|,\delta(\ell^\infty)\bigr),d\bigl(|S(g(\sigma))|,\delta(\ell^\infty)\bigr)\bigr\}.$$

Let $C_\sigma\subset(\ell^\infty)^2$ be the following annulus around $\delta(\ell^\infty)$:

$$\{z\mid r_\sigma\le d(z,\delta(\ell^\infty))\le R_\sigma\}.$$

We note that C_σ is acyclic because it is homotopy equivalent to S^∞.

We claim that there exists a $D_i\colon\mathrm{C}_i(X^2)\to\mathrm{C}^{\mathrm{s}}_{i+1}((\ell^\infty)^2)$ such that for any i-simplex σ the following hold.

(1) $S \circ f(\sigma) - S \circ g(\sigma) = \partial D_i(\sigma) + D_{i-1}\partial(\sigma)$.

(2) $|D_i(\sigma)| \subset C_\sigma^{i+2}$.

(3) $\operatorname{size}(D_i(\sigma)) \leq \operatorname{size}\big(S(f(\sigma)) - S(g(\sigma)) - D_{i-1}\partial(\sigma)\big)$.

(4) D_i is $\mathbb{Z}_2$-equivariant.

We will defer the construction of D_i to Lemma 6.2. Assuming the existence of such D_i, we now complete the proof of the proposition.

Let $V\colon \mathrm{C}^{\mathrm{s}}_* \to \mathrm{C}_*$ be the map that sends a singular simplex to its vertices. Since $V\partial = \partial V$, applying V on both sides of (1), we have

$$(**) \qquad VS(f-g) = \partial VD_i + VD_{i-1}\partial.$$

Recall that VS is homotopic to the identity map with a $\mathbb{Z}_2$-equivariant chain homotopy H such that $|H(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$. So, we have

$$(***) \qquad VS = id + \partial H_i + H_{i-1}\partial.$$

Combining $(**)$ and $(***)$, we obtain

$$f - g = \partial[VD_i + H_i(f-g)] + [VD_{i-1} + H_{i-1}(f-g)]\partial.$$

Dualizing the above we get f^* and g^* are chain homotopic via the cochain homotopy $(VD)^* + (f-g)^* H^*$. To complete the proof, we need to show that this cochain homotopy maps $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^{*-1}_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$.

Since $|H(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$ and H is $\mathbb{Z}_2$-equivariant, the dual H^* maps $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^{*-1}_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$. Consequently, $(f-g)^* H^*$ maps $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^{*-1}_{\mathbb{Z}_2}(X^2 - \delta(X))$. It is therefore enough to prove that $(VD)^*$ maps $\mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ to $\mathrm{C\,X}^{*-1}_{\mathbb{Z}_2}(X^2 - \delta(X))$. Let $\phi \in \mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ and $\sigma \in |(VD)^*(\phi)| \cap N_r(\Delta)$ for some $r \geq 0$. Since $\sigma \in N_r(\Delta)$, and f and g are coarse expanding maps, we have $|S(f(\sigma))| \subset N_s(\Delta)$ and $|S(g(\sigma))| \subset N_s(\Delta)$ for some s that depends only on r. Property (3) then implies that $|VD(\sigma)| \subset N_s(\Delta)$ for some s that depends only on r. Since $\phi \in \mathrm{C\,X}^*_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ and $\phi(VD(\sigma)) \neq 0$, it then follows that $|VD(\sigma))| \subset N_t(\Delta_{\delta(\ell^\infty)})$ where t depends only on s and hence depends only on r. It now follows from property (2) that $\sigma \in N_p(\Delta_{\delta(X)})$ for some p that depends only on t and hence only on r. Hence, we proved that for each $r \geq 0$, there exists $p \geq 0$ such that $|(VD)^*(\phi)| \cap N_r(\Delta) \subset N_p(\Delta_{\delta(X)})$. Finally, $(VD)^*(\phi)$ is $\mathbb{Z}_2$-equivariant by property (4). Hence, $(VD)^*(\phi) \in \mathrm{C\,X}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X))$. □

Lemma 6.2 *Let f and g are two coarse expanding maps from X to ℓ^∞. Let $S\colon \mathrm{C}_*((\ell^\infty)^2) \to \mathrm{C}^{\mathrm{s}}_*((\ell^\infty)^2)$ be an $\mathbb{Z}_2$-equivariant chain map such that $|S(\sigma^n)| \subset N_{\rho_n(\operatorname{diam}(\sigma^n))}(\sigma^n)$ for some sequence of functions $\rho_n\colon [0,\infty) \to [0,\infty)$. Then, for each i, there exists $D_i\colon \mathrm{C}_i(X^2) \to \mathrm{C}^{\mathrm{s}}_{i+1}((\ell^\infty)^2)$ such that for any i-simplex σ the following hold.*

(1) $S \circ f(\sigma) - S \circ g(\sigma) = \partial D_i(\sigma) + D_{i-1}\partial(\sigma)$.

(2) $|D_i(\sigma)| \subset C_\sigma^{i+2}$.

(3) $\mathrm{size}(D_i(\sigma)) \leq \mathrm{size}\big(S(f(\sigma)) - S(g(\sigma)) - D_{i-1}\partial(\sigma)\big)$.

(4) D_i *is* $\mathbb{Z}_2$*-equivariant.*

Proof We first define $D_0\colon \mathrm{C}_0(X^2) \to \mathrm{C}_1^{\mathrm{s}}((\ell^\infty)^2)$. For a 0-simplex $\sigma \in X^2$, join $f(\sigma)$ and $g(\sigma)$ by a path γ in C_σ. Such a path exists because the annulus C_σ is contractible. Next, we subdivide γ so that each subarc is of diameter ≤ 1. We define $D_0(\sigma)$ to be this subdivided path. By construction, D_0 satisfies the first three desired properties. To make it G-equivariant, we first define D_0 on a simplex from each G-orbit of simplices and then move it $\mathbb{Z}_2$-equivariantly. It is straightforward to see that D_0 still satisfies the first property because ∂ commutes with the G-action. It satisfies the second property because $gC_\sigma = C_{g\sigma}$ for any $g \in \mathbb{Z}_2$. D_0 satisfies the third property because isometric action preserves size of chains.

Inductively, let us assume that $D_i\colon \mathrm{C}_i(X^2) \to \mathrm{C}_{i+1}^{\mathrm{s}}((\ell^\infty)^2)$ is already defined for all $i \leq n$ with the desired properties. To define $D_{n+1}(\sigma)$, let $K = S(f(\sigma)) - S(g(\sigma)) - D_n(\partial(\sigma))$. By the induction hypothesis (1), $\partial K = D_{n-1}(\partial^2(\sigma)) = 0$ and hence K is a cycle. By (2), $|K| \subset C_\sigma$. Since C_σ is acyclic, there exists a singular chain c supported in C_σ such that $\partial c = K$. After applying the appropriate subdivision, we can make c to satisfy $\mathrm{size}(c) \leq \mathrm{size}(K)$ without changing its boundary. Define $D_{n+1}(\sigma)$ to be that c. By construction, D_{n+1} satisfies conditions (1), (2), and (3). To make D_{n+1} satisfy (4), we can use the same trick as before: first define it on a simplex from each $\mathbb{Z}_2$-orbit of $(n+1)$-simplices and then extend equivariantly. For the similar reason as D_0, this D_{n+1} has all the desired four properties. □

Coarse van Kampen obstruction class Let X be a separable metric space and $g\colon X \to \ell^\infty$ be a coarse expanding map. Such a map exists because any separable metric space admits an isometric embedding into ℓ^∞ by the work of Fréchet ([7]). We consider the induced map

$$g^*\colon \mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty); \mathbb{Z}_2) \to \mathrm{HX}^n_{\mathbb{Z}_2}(X^2 - \delta(X); \mathbb{Z}_2).$$

Recall from Example 5.3 that $\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty); \mathbb{Z}_2) = \mathbb{Z}_2$ if $n \geq 2$. Let e^n be the nontrivial element in $\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ for $n \geq 2$. Proposition 6.1 implies that $g^*(e^n)$ depends only on the space X, not on g. We call the class $g^*(e^n)$ to be the *n^{th}-degree coarse van Kampen obstruction class* of X and denote it by $\mathrm{cvk}^n(X)$ where $n \geq 2$.

Assumption From now on all our metric spaces will be separable. In the nonequivariant setting, our coefficient group will be $\mathbb{Z}_2$ and in the equivariant setting, the coefficient group will be $\mathbb{Z}_2$ with the trivial $\mathbb{Z}_2$-action unless stated otherwise. In those cases, we will omit the coefficient from the notation.

Definition 6.3 (coarse obstruction dimension) The *coarse obstruction dimension* of a space X, denoted by $\mathrm{cobdim}(X)$, is 0 if X is bounded, is 1 if $\mathrm{cvk}^n(X) = 0$ for all n, and otherwise it is the largest n such that $\mathrm{cvk}^n(X) \neq 0$.

Now we prove the main theorem of this section.

Theorem 6.4 *If X admits a coarse expanding map into Y, then* $\operatorname{cobdim}(X) \le \operatorname{cobdim}(Y)$.

Proof If $\operatorname{cobdim}(X) = 0$, then there is nothing to prove.

If $\operatorname{cobdim}(X) = 1$, then X is unbounded by definition. This means Y is also unbounded and hence $\operatorname{cobdim}(Y) \ge 1$ by definition.

Suppose $\operatorname{cobdim}(X) = n \ge 2$. Let $g\colon Y \to \mathbb{R}^\infty$ be a coarse expanding map. Consider the composition

$$X \times X \xrightarrow{f} Y \times Y \xrightarrow{g} \ell^\infty \times \ell^\infty$$

By Lemma 5.8, the above maps induce, between coarse cohomology of the configuration spaces, the maps

$$\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \xrightarrow{g^*} \mathrm{HX}^n_{\mathbb{Z}_2}(Y^2 - \delta(Y)) \xrightarrow{f^*} \mathrm{HX}^n_{\mathbb{Z}_2}(X^2 - \delta(X)).$$

Let $e^n \in \mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty))$ be the generator. Then

$$\operatorname{cvk}^n(Y) = g^*(e^n) \quad \text{and} \quad \operatorname{cvk}^n(X) = f^*g^*(e^n)) = f^*(\operatorname{cvk}^n(Y)).$$

By assumption $\operatorname{cvk}^n(X) \ne 0$ and hence $\operatorname{cvk}^n(Y) \ne 0$. So, we get $\operatorname{cobdim}(Y) \ge n$. □

Recall that X and Y are said to be *coarsely equivalent* if there exists a coarse embedding map $f\colon X \to Y$ such that $Y \overset{c}{\subset} f(X)$. One can observe that two coarsely equivalent spaces coarsely embed into each other. As a consequence, the above theorem immediately yields the following.

Corollary 6.5 *If X and Y are coarsely equivalent, then* $\operatorname{cobdim}(X) = \operatorname{cobdim}(Y)$.

In Example 5.2, we saw that $\mathrm{HX}^*_{\mathbb{Z}_2}((\mathbb{R}^n)^2 - \delta(\mathbb{R}^n)) = 0$ for all $* > n$. Hence $\operatorname{cobdim}(\mathbb{R}^n) \le n$. Using Theorem 6.4, we obtain:

Corollary 6.6 *If* $\operatorname{cobdim}(X) \ge n$, *then X does not admit a coarse expanding map into* $\mathbb{R}^{n-1}$.

7 Relation to the classical van Kampen obstruction

Let us recall the classical van Kampen obstruction class. Let X be any topological space. Any continuous embedding $f\colon X \hookrightarrow \mathbb{R}^\infty$ induces a map

$$f\colon \mathrm{H}^*((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)/\mathbb{Z}_2) \to \mathrm{H}^*((X^2 - \delta(X))/\mathbb{Z}_2).$$

This map depends only on X because the quotient map

$$q\colon (\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty) \to ((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty))/\mathbb{Z}_2$$

is a universal $\mathbb{Z}_2$-bundle. Let $\eta^* \in \mathrm{H}^*((\mathbb{R}^\infty)^2 - \delta(\mathbb{R}^\infty)/\mathbb{Z}_2)$ be the generator. The cohomology class $f(\eta^*)$ is called the van Kampen obstruction class in degree $*$ and will be denoted by $vk^*(X)$. The quotient map $(\ell^\infty)^2 - \delta(\ell^\infty) \to ((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2$ can also be considered as a universal $\mathbb{Z}_2$-bundle because $(\ell^\infty)^2 - \delta(\ell^\infty)$ is contractible. Hence we can use ℓ^∞ instead of $\mathbb{R}^\infty$ to define $vk^*(X)$. We use this viewpoint in the next proposition.

Proposition 7.1 *Let X be a metric space. Suppose*

$$i : \mathrm{H}^*\big((X^2 - \delta(X))/\mathbb{Z}_2\big) \to \varinjlim \mathrm{H}^*\big((X^2 - N_r(\delta(X)))/\mathbb{Z}_2\big)$$

is the map induced by inclusions $X^2 - N_r(\delta(X)) \hookrightarrow X^2 - \delta(X)$ for each $r > 0$. If $i(vk^{n-1}(X))$ is nontrivial for some $n \geq 2$, then $\mathrm{cvk}^n(X)$ is nontrivial.

Proof Let $f : X \to \ell^\infty$ be an isometry. Fix $n \geq 2$. Suppose $i(vk^{n-1}(X)) \neq 0$. We consider the following commutative diagram where all the horizontal maps are canonically induced by f. We want to show that the top horizontal map is nontrivial:

$$\begin{array}{ccc}
\mathrm{HX}^n_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) & \xrightarrow{\quad f^* \quad} & \mathrm{HX}^n_{\mathbb{Z}_2}(X^2 - \delta(X)) \\
\big\downarrow\cong & & \big\downarrow \\
\varinjlim \mathrm{H}^{n-1}\big(((\ell^\infty)^2 - N_r(\delta(\ell^\infty)))/\mathbb{Z}_2\big) & \xrightarrow{f^*} & \varinjlim \mathrm{H}^{n-1}\big((X^2 - N_r(\delta(X)))/\mathbb{Z}_2\big) \\
\big\uparrow & & i\big\uparrow \\
\mathrm{H}^{n-1}\big(((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2\big) & \xrightarrow{\quad f^* \quad} & \mathrm{H}^{n-1}\big((X^2 - \delta(X))/\mathbb{Z}_2\big)
\end{array}$$

The top left isomorphism map is due to the Proposition 5.1.

The image of the generator of $\mathrm{H}^{n-1}\big(((\ell^\infty)^2 - \delta(\ell^\infty))/\mathbb{Z}_2\big)$ under the bottom horizontal map is $vk^{n-1}(X)$ if $n \geq 2$. Since $i(vk^{n-1}(X)) \neq 0$, the commutativity of the above diagram implies that the middle horizontal map is nontrivial. Again using commutativity of the diagram, we conclude that the top horizontal map is nontrivial. □

Suppose $X = K \times [0, \infty)/K \times \{0\}$ is the open cone on a finite simplicial complex K. A metric d on X is called *expanding* if for any two disjoint simplices $\sigma, \tau \in K$ and $S \geq 0$, there exists $r \geq 0$ such that $d(\sigma \times [r, \infty), \tau \times [r, \infty)) \geq S$.

Proposition 7.2 *Let X be an open cone on a finite simplicial complex K and X is equipped with an expanding metric. If $vk^{n-1}(X) \neq 0$ for some $n \geq 2$, then $\mathrm{cvk}^n(X) \neq 0$.*

Proof Since the metric on X is expanding, any representative of $c \in \mathrm{H}_*(\mathrm{Conf}(X))$ can be homotoped to a cycle that lives in $\mathrm{H}^*\big((X^2 - N_r(\delta(X)))/\mathbb{Z}_2\big)$ for any r. Considering the dual, this implies that the restriction map $\tilde{\mathrm{H}}^*\big((X^2 - \delta(X))/\mathbb{Z}_2\big) \to \varinjlim \tilde{\mathrm{H}}^*\big((X^2 - N_r(\delta(X)))/\mathbb{Z}_2\big)$ sends $vk^{n-1}(X)$ to a nontrivial element for any $r \geq 0$. Hence, $i(vk^{n-1}(X))$ is nontrivial where i is as in Proposition 7.1. Hence Proposition 7.1 implies $\mathrm{cvk}^{n+1}(X) \neq 0$. □

Example 7.3 (Bestvina–Kapovich–Kleiner obstruction) One of the results in [3] can be stated as follows: if there is class $c \in \mathrm{H}_n(\mathrm{Conf}(X))$ such that $vk^n(X)(c) \neq 0$, then X with a proper, expanding metric cannot be coarsely embedded inside $\mathbb{R}^n$. One can see that Proposition 7.2 combined with Corollary 6.6 recovers Bestvina, Kapovich, and Kleiner's result.

Remark 7.4 Note that $vk^{n-1}(X) \neq 0$ does not necessarily imply $\mathrm{cvk}^n(X) \neq 0$. For example, take X to be a unit disk in $\mathbb{R}^2$, then $vk^1(X) \neq 0$. However, since X is bounded, Proposition 4.4 implies $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$, and therefore $\mathrm{cvk}^2(X) = 0$.

8 A coarse Gysin sequence

Recall that coarse van Kampen obstruction class lives in $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$. In this section, we relate $\mathrm{HX}^*_{\mathbb{Z}_2}$ to HX^* and apply that to compute $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$ for certain X.

Suppose $\mathbb{Z}_2$ is acting on some metric space X by isometries and A is the subset of X that is fixed by the action. We consider the exact sequence

$$0 \to \mathrm{CX}^*_{\mathbb{Z}_2}(X - A) \xrightarrow{i} \mathrm{CX}^*(X - A) \xrightarrow{p} \mathrm{CX}^*_{\mathbb{Z}_2}(X - A) \xrightarrow{r} \mathrm{C}^*(A) \to 0,$$

where i is the inclusion map and $p(\phi)\colon \sigma \mapsto \phi(\sigma) + \phi(g\sigma)$ where g is the generator of $\mathbb{Z}_2$ and r is the restriction map. The image of p consists of those cochains in $\mathrm{CX}^*_{\mathbb{Z}_2}(X - A)$ that send any simplex supported on A to zero. It is easy to see that collection of such cochains gives a subcomplex of $\mathrm{CX}^*_{\mathbb{Z}_2}(X - A)$. We denote this complex by $\mathrm{CX}^*_{\mathbb{Z}_2}(X - A, A)$, and the corresponding cohomology by $\mathrm{HX}^*_{\mathbb{Z}_2}(X - A, A)$. Hence, the above four-term short exact sequence splits into the two short exact sequences

$$0 \to \mathrm{CX}^*_{\mathbb{Z}_2}(X - A) \xrightarrow{i} \mathrm{CX}^*(X - A) \xrightarrow{p} \mathrm{CX}^*_{\mathbb{Z}_2}(X - A, A) \to 0,$$
$$0 \to \mathrm{CX}^*_{\mathbb{Z}_2}(X - A, A) \xrightarrow{i} \mathrm{CX}^*_{\mathbb{Z}_2}(X - A) \xrightarrow{r} \mathrm{C}^*(A) \to 0.$$

These two short exact sequences give us two long exact sequences in the cohomology which we record as our next lemma.

Lemma 8.1 *Let X be a metric space and $\mathbb{Z}_2 \curvearrowright (X, A)$ such that A is the fixed point set of the action. Then we have the two long exact sequences*

$$\cdots \to \mathrm{HX}^*_{\mathbb{Z}_2}(X - A) \to \mathrm{HX}^*(X - A) \to \mathrm{HX}^*_{\mathbb{Z}_2}(X - A, A) \to \mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X - A) \to \cdots, \tag{8-1}$$

$$\cdots \to \mathrm{HX}^*_{\mathbb{Z}_2}(X - A, A) \to \mathrm{HX}^*_{\mathbb{Z}_2}(X - A) \to \mathrm{H}^*(\mathrm{C}^*(A)) \to \mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X - A, A) \to \cdots. \tag{8-2}$$

Lemma 8.2 *Suppose $A \subset X$ and X is not coarsely contained in A. If $\mathbb{Z}_2 \curvearrowright (X, A)$ where A is the fixed point set of the action, then*

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X - A, A) = \begin{cases} 0 & \text{if } * = 0, \\ \mathrm{HX}^*_{\mathbb{Z}_2}(X - A) \oplus \mathbb{Z}_2 & \text{if } * = 1, \\ \mathrm{HX}^*_{\mathbb{Z}_2}(X - A) & \text{if } * \geq 2. \end{cases}$$

Proof Since X is not coarsely contained in A, we have

$$\mathrm{HX}^0_{\mathbb{Z}_2}(X-A,A)=\mathrm{HX}^0_{\mathbb{Z}_2}(X-A)=0.$$

By Lemma 2.1, the homology of the complex $\mathrm{C}^*(A)$ vanishes everywhere except in the degree 0 where it is $\mathbb{Z}_2$. Combining this with the second long exact sequence of the Lemma 8.1, we obtain

$$(8\text{-}3)\quad 0\to\mathbb{Z}_2\to\mathrm{HX}^1_{\mathbb{Z}_2}(X-A,A)\to\mathrm{HX}^1_{\mathbb{Z}_2}(X-A)\to 0\to \\ \cdots\to 0\to\mathrm{HX}^*_{\mathbb{Z}_2}(X-A,A)\to\mathrm{HX}^*_{\mathbb{Z}_2}(X-A)\to 0\cdots,$$

where $*\geq 2$.

Since the coefficient group is $\mathbb{Z}_2$, the first five terms give us a split short exact sequence. Hence $\mathrm{HX}^1_{\mathbb{Z}_2}(X-A,A)=\mathrm{HX}^1_{\mathbb{Z}_2}(X-A)\oplus\mathbb{Z}_2$. It also follows from (8-3) that $\mathrm{HX}^*(X-A,A)=\mathrm{HX}^*(X-A)$ for all $*\geq 2$. □

Hence under the hypothesis of Lemma 8.2, we can rewrite the first long exact sequence of Lemma 8.1 as follows.

Lemma 8.3 (coarse Gysin sequence) *Suppose $\mathbb{Z}_2 \curvearrowright (X,A)$ where $A\subset X$ is the fixed point set of the action. Moreover, assume that A is unbounded and X is not coarsely contained in A. Then there is a long exact sequence of the form*

$$0\to\mathrm{HX}^0_{\mathbb{Z}_2}(X-A)\to\mathrm{HX}^0(X-A)\to 0 \\ \to\mathrm{HX}^1_{\mathbb{Z}_2}(X-A)\to\mathrm{HX}^1(X-A)\to\mathrm{HX}^1_{\mathbb{Z}_2}(X-A)\oplus\mathbb{Z}_2\to\cdots \\ \to\mathrm{HX}^*_{\mathbb{Z}_2}(X-A)\to\mathrm{HX}^*(X-A)\to\mathrm{HX}^*_{\mathbb{Z}_2}(X-A)\to\cdots,$$

where $*\geq 2$.

We are now ready to state our main result of this section.

Proposition 8.4 *Suppose X is unbounded and $\mathbb{Z}_2$ is acting on X^2 by permuting the coordinates. Then there is a long exact sequence of the form*

$$(8\text{-}4)\quad 0\to\mathrm{HX}^0_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathrm{HX}^0(X^2-\delta(X))\to 0 \\ \to\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathrm{HX}^1(X^2-\delta(X))\to\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X))\oplus\mathbb{Z}_2\to\cdots \\ \to\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathrm{HX}^*(X^2-\delta(X))\to\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))\to\cdots,$$

where $*\geq 2$.

Proof By hypothesis, $\mathbb{Z}_2 \curvearrowright (X^2,\delta(X))$ and $\delta(X)$ is the set of fixed points of the action. Since X is unbounded X^2 is not coarsely contained in $\delta(X)$. So, we can apply Lemma 8.3 on $(X^2,\delta(X))$, and the claim follows. □

Remark 8.5 If X is unbounded and $\mathrm{HX}^1(X^2 - \delta(X)) = 0$, then it follows from Proposition 8.4 that $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$. As a consequence, we can write the beginning part of the coarse Gysin sequence (8-4) as

$$0 \to \mathbb{Z}_2 \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{HX}^2(X^2 - \delta(X)) \to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \cdots . \tag{8-5}$$

This observation will be useful for us in our next theorem where we compute $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$ when $\mathrm{HX}^*(X^2 - \delta(X))$ is concentrated in some degree.

Theorem 8.6 *Suppose X is a metric space such that, for some $n \geq 1$,*

$$\mathrm{HX}^*(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, suppose that $\mathrm{HX}^i_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for some $i \geq n+1$. Then

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & \text{if } n \geq 2 \text{ and } 2 \leq * \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Elements of $\mathrm{HX}^0(X^2 - \delta(X))$ and $\mathrm{HX}^0_{\mathbb{Z}_2}(X^2 - \delta(X))$ are constant functions on X^2 with support contained in a neighborhood of $\delta(X)$. Since $\mathrm{HX}^0(X^2 - \delta(X)) = 0$, we have $\mathrm{HX}^0_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$.

Next, we will show that $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ if $* \geq n+1$. Consider the following part of the coarse Gysin sequence where $* \geq 2$:

$$\to \mathrm{HX}^*(X^2 - \delta(X)) \to \mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathrm{HX}^{*+1}(X^2 - \delta(X)) \to . \tag{8-6}$$

Since $\mathrm{HX}^*(X^2 - \delta(X)) = 0$ for $* \geq n+1$, the middle map is an isomorphism for $* \geq n+1$. Hence,

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \quad \text{for all } * \geq n+1.$$

That means if $\mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) \neq 0$ then $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) \neq 0$ for all $* \geq n+1$. By hypothesis, $\mathrm{HX}^i_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for some $i \geq n+1$. Therefore, $\mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ and consequently,

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0 \quad \text{for all } * \geq n+1.$$

We divide the rest of the calculations into three cases.

Case 1 ($n = 1$) We only have to show that $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$. By the hypothesis,

$$\mathrm{HX}^1(X^2 - \delta(X)) = \mathbb{Z}_2 \quad \text{and} \quad \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0.$$

Hence, we get the following from the coarse Gysin sequence:

$$0 \to \mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \to \mathbb{Z}_2 \to \mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \oplus \mathbb{Z}_2 \to 0.$$

Since the third map is surjective, $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X))$ cannot have more than one element and hence it is trivial.

Case 2 ($n=2$) By hypothesis, $\mathrm{HX}^1(X^2-\delta(X))=0$, Therefore, $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X))=0$ (see Remark 8.5). Furthermore, $\mathrm{HX}^2(X^2-\delta(X))=\mathbb{Z}_2$ by hypothesis and we already showed $\mathrm{HX}^3_{\mathbb{Z}_2}(X^2-\delta(X))=0$. So a part of the sequence (8-5) takes the form

$$0\to\mathbb{Z}_2\to\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathbb{Z}_2\to\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))\to 0.$$

So there is an injective map and a surjective map from $\mathbb{Z}_2$ into $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))$. It follows that $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))=\mathbb{Z}_2$.

Case 3 ($n>2$) In this case, $\mathrm{HX}^1(X^2-\delta(X))=0$. Remark 8.5 gives us $\mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X))=0$.

Since $\mathrm{HX}^2(X^2-\delta(X))=0$, the beginning part of the sequence (8-5) takes the form

$$0\to\mathbb{Z}_2\to\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))\to 0\to\cdots.$$

Hence $\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))=\mathbb{Z}_2$.

Since $\mathrm{HX}^*(X^2-\delta(X))=0$ for $*\le n-1$, it follows from (8-6) that

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))=\mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X^2-\delta(X))\quad\text{when } 2\le *\le n-2.$$

That implies

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))=\mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))=\mathbb{Z}_2\quad\text{when } 2\le *\le n-1.$$

Finally to compute $\mathrm{HX}^n_{\mathbb{Z}_2}(X^2-\delta(X))$, consider the following part of the coarse Gysin sequence (Lemma 8.3):

$$\cdots\to\mathrm{HX}^n_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathrm{HX}^n(X^2-\delta(X))\to\mathrm{HX}^n_{\mathbb{Z}_2}(X^2-\delta(X))\to\mathrm{HX}^{n+1}_{\mathbb{Z}_2}(X^2-\delta(X))\to\cdots.$$

Since $\mathrm{HX}^n(X^2-\delta(X))\neq 0$ by assumption, it follows from the above sequence that $\mathrm{HX}^n_{\mathbb{Z}_2}(X^2-\delta(X))\neq 0$. We already know that the fourth term is trivial in the above sequence. That means the third map is surjective. Since $\mathrm{HX}^n(X^2-\delta(X))=\mathbb{Z}_2$, we can conclude that $\mathrm{HX}^n_{\mathbb{Z}_2}(X^2-\delta(X))=\mathbb{Z}_2$. □

Remark 8.7 It follows from the first part of the proof of the above theorem that if $\mathrm{HX}^*(X^2-\delta(X))=0$ for all $*\ge n+1$ and $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))=0$ for some $*\ge n+1$, then $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X))=0$ for all $*\ge n+1$. In particular, for such X, we have $\mathrm{cvk}^*(X)$ is trivial for $*\ge n+1$ and hence $\mathrm{cobdim}(X)\le n$.

9 Upper bound of cobdim

In this section, our goal is to apply Theorem 8.6 to estimate cobdim for proper, uniformly acyclic n-manifolds. The key tool that we are going to exploit here is a coarse Alexander duality theorem that holds for such spaces. In fact, such duality holds for a more general class of metric spaces called coarse $\mathrm{PD}(n)$ spaces, first introduced by Kapovich and Kleiner in [11] where they proved a coarse Alexander duality theorem for these spaces. A different treatment of coarse $\mathrm{PD}(n)$ space and coarse Alexander duality using coarse cohomology is given in [2]. Let us now recall the definition of a coarse $\mathrm{PD}(n)$ space from [2].

Definition 9.1 A metric space X is a *coarse* PD(n) *space* if there exist chain maps

$$p\colon \mathrm{C}^*(X;\mathbb{Z}) \to \mathrm{C\,X}_{n-*}(X;\mathbb{Z}) \quad \text{and} \quad q\colon \mathrm{C\,X}_{n-*}(X;\mathbb{Z}) \to \mathrm{C}^*(X;\mathbb{Z}),$$

so that pq and qp are chain homotopic to identities via chain homotopies $G\colon \mathrm{C\,X}_*(X;\mathbb{Z}) \to \mathrm{C\,X}_{*+1}(X;\mathbb{Z})$ and $F\colon \mathrm{C}^*(X;\mathbb{Z}) \to \mathrm{C}^{*-1}(X;\mathbb{Z})$ which are controlled:

$$\begin{aligned}
&\forall \phi \in \mathrm{C}^*(X;\mathbb{Z})\ |p(\phi)| \overset{c}{\subset} |\phi|,\\
&\forall \phi \in \mathrm{C}^*(X;\mathbb{Z})\ |F(\phi)| \overset{c}{\cap} \Delta \overset{c}{\subset} |\phi|,\\
&\forall c \in \mathrm{C\,X}_*(X;\mathbb{Z})\ |q(c)| \overset{c}{\cap} \Delta \overset{c}{\subset} |c|,\\
&\forall c \in \mathrm{C\,X}_*(X;\mathbb{Z})\ |G(c)| \overset{c}{\subset} |c|.
\end{aligned}$$

Example 9.2 Any proper, uniformly acyclic n-manifold is a coarse PD(n) space [2, Corollary 8.3]. In particular, the universal cover of a closed, aspherical n-manifold is a coarse PD(n) space.

We now recall the coarse Alexander duality theorem from [2].

Theorem 9.3 (coarse Alexander duality [2]) *If X is a coarse* PD(n) *space, then, for any $A \subset X$ and finitely generated abelian group G,*

$$\mathrm{H\,X}^*(X - A; G) \cong \mathrm{H\,X}_{n-*}(A; G).$$

As a consequence we have the following.

Lemma 9.4 *Let X be a coarse* PD(n) *space. Then*

$$\mathrm{H\,X}_*(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \textit{if } * = n,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof By Theorem 9.3, $\mathrm{H\,X}_{n-*}(X;G) = \mathrm{H\,X}^*(X - X;G)$. Observe that $\mathrm{C\,X}^*(X-X;\mathbb{Z}_2) = \mathrm{C}^*(X;\mathbb{Z}_2)$. Since $\mathrm{C}^*(X;\mathbb{Z}_2)$ is acyclic by Lemma 2.1, we obtain

$$\mathrm{H\,X}_*(X;\mathbb{Z}_2) = \mathrm{H\,X}^{n-*}(X - X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = n,\\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

For the rest of the paper, the omitted coefficient will mean $\mathbb{Z}_2$.

Lemma 9.5 *If X is a coarse* PD(n) *space, then*

$$\mathrm{H\,X}^*(X^2 - \delta(X)) = \begin{cases} \mathbb{Z}_2 & \textit{if } * = n,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof If X is a coarse PD(n) space, then X^2 is a coarse PD$(2n)$ space. We obtain

$$\begin{aligned}
\mathrm{H\,X}^*(X^2 - \delta(X)) &= \mathrm{H\,X}_{2n-*}(\delta(X)) && \text{(by Theorem 9.3)}\\
&= \begin{cases} \mathbb{Z}_2 & \text{if } * = n,\\ 0 & \text{otherwise}\end{cases} && \text{(by Lemma 9.4).}
\end{aligned} \qquad \square$$

Lemma 9.6 *If X is a proper, uniformly acyclic n-manifold where $n \geq 1$, then $\mathrm{HX}^i_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for $i \geq 2n+2$.*

Proof If X is bounded, then $X^2 \overset{c}{=} \delta(X)$ and the claim follows from Theorem 3.8(1). So, we assume that X is unbounded. This implies that X^2 is not coarsely contained in $\delta(X)$. By hypothesis, X^2 is uniformly acyclic and locally acyclic. Hence we can apply Theorem 3.8 to the pair $(X^2, \delta(X))$ and obtain

$$\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = \varinjlim \tilde{\mathrm{H}}^{*-1}\big(X^2 - N_r(\delta(X))/\mathbb{Z}_2\big).$$

Since $\big(X^2 - N_r(\delta(X))\big)/\mathbb{Z}_2$ is a $2n$-manifold, $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for all $* \geq 2n+2$. □

Theorem 9.7 *If X is a proper, uniformly acyclic n-manifold, then $\mathrm{cobdim}(X) \leq n$.*

Proof For $n = 0$, the claim is trivial. For $n \geq 1$, Lemmas 9.5 and 9.6 imply that X satisfies the hypothesis of Theorem 8.6. Hence by Theorem 8.6, we have $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for $* \geq n+1$. This implies $\mathrm{cobdim}(X) \leq n$. □

Remark 9.8 To prove Theorem 9.7, we did not need the full strength of the Theorem 8.6. We needed to show $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X)) = 0$ for $* \geq n+1$ which only requires vanishing of $\mathrm{HX}^*(X^2 - \delta(X))$ for $* \geq n+1$ and vanishing of $\mathrm{HX}^*_{\mathbb{Z}_2}(X^2 - \delta(X))$ for some $* \geq n+1$ (see Remark 8.7).

Our next goal is to improve Theorem 9.7 to include manifolds with boundaries. For that, we need to impose a condition on the metric of the boundary. This is the purpose of the following definition which is inspired by the uniformly locally k-connected space defined in [6].

Definition 9.9 A metric space (X, d) is uniformly locally acyclic if for every $\epsilon > 0$, there is a $\delta > 0$ such that any ball of radius δ is acyclic inside a ball of radius ϵ.

Example 9.10 Any compact, locally acyclic space is uniformly locally acyclic. Similarly, any locally acyclic space that admits a cocompact group action by homeomorphisms is uniformly locally acyclic. In particular, the universal cover of a compact manifold is uniformly locally acyclic.

Any uniformly locally acyclic space is also locally acyclic. However, the converse is not true. For example, the set $\left\{\frac{1}{n}\right\}$ with the subspace metric from $\mathbb{R}$ is locally acyclic, but it is not uniformly locally acyclic.

Lemma 9.11 *Let (X, d) be a uniformly locally acyclic metric space. Then the space $X \times [1, \infty)$ has a metric that makes the space uniformly acyclic away from $X \times \{1\}$ and the map $x \mapsto (x, 1)$ is an isometric embedding of X into $X \times [1, \infty)$.*

Proof The construction of the metric follows the one appearing in Lemma 2.2 of [6]. Choose a continuous strictly increasing function $\phi\colon [1, \infty) \to [1, \infty)$ with $\phi(1) = 1$. Let d be the original metric on X and define a function ρ' by

(1) $\rho'((x,t),(x',t)) = \phi(t)d(x,x')$,

(2) $\rho'((x,t),(x,t')) = |t - t'|$.

We then define $\rho\colon (X\times[1,\infty))^2\to[0,\infty)$ to be

$$\rho((x,t),(x',t'))=\inf\sum_{i=1}^{l}\rho'((x_i,t_i),(x_{i-1},t_{i-1})),$$

where the sum is over all chains

$$(x,t)=(x_0,t_0),(x_1,t_1),\dots,(x_l,t_l)=(x',t')$$

and each segment is either horizontal or vertical. Also $\phi(1)=1$ implies that $X\times\{1\}$ with the subspace metric is isometric to X via the map $(x,1)\mapsto x$. Now we will describe a ϕ, so that the corresponding metric ρ makes $X\times[1,\infty)$ uniformly acyclic away from $X\times\{1\}$. Since X is uniformly locally acyclic, we have an infinite positive decreasing sequence $\{r_i\}$ with $r_1=1$ such that for every $x\in X$, the inclusions $\cdots\subset B^d_{r_i}(x)\subset B^d_{r_{i-1}}(x)\subset$ are nullhomotopic maps. Set $\phi(t)=1/r_t$ for $t\in\mathbb{N}$. For nonintegral values of t, we set

$$\phi(t)=\phi([t])+(t-[t])\phi([t]+1).$$

Suppose, $N_i=\phi(i)/\phi(i-1)$. Now we consider the ball $B^\rho_k(x,i)\subset X\times[1,\infty)$. Note that

$$B^\rho_k(x,i)\subset B^d_{k/N_{i-k}}(x)\times[i-k,i+k]$$

and that $B^\rho_k(x,i)$ contracts in itself to $B^\rho_k(x,i)\cap(X\times[i-k,i])\subset B^d_{k/N_{i-k}}(x)\times[i-k,i]$. Also, $B^d_{k/N_{i-k-1}}(x)\times\{i-k-1\}\subset B^\rho_{k+2}(x,i)$. So, $B^\rho_k(x,i)$ can be contracted inside $B^\rho_{k+2}(x,i)$ by pushing it down to the $(i-k-1)$-level and contracting it there. □

The following gluing lemma in a slightly different form can be found in [4].

Lemma 9.12 [4, Lemma I.5.24] *Let X_1 and X_2 be two proper metric spaces. Let $A_i\subset X_i$ be closed subsets and $f: A_1\to A_2$ be an isometry. Let Y be the space obtained by gluing (X_i,A_i) along A_i via the map f. Define $d: Y\times Y\to\mathbb{R}$ as*

$$d(x,y)=\begin{cases} d_i(x,y) & \text{if } x,y\in X_i,\\ \inf_{a\in A_1}\{d_1(x,a)+d_2(f(a),y)\} & \text{if } x\in X_1,\ y\in X_2.\end{cases}$$

Then:

(1) *d is a proper metric on Y.*

(2) *The canonical inclusions $X_i\hookrightarrow Y$ are isometric embedding.*

Proposition 9.13 *Any uniformly acyclic, proper n-manifold with uniformly locally acyclic boundary admits an isometric embedding into a uniformly acyclic, proper n-manifold.*

Proof Since ∂M is uniformly locally acyclic, Lemma 9.11 allows us to equip $\partial M \times [1,\infty)$ with a metric so that it is uniformly acyclic away from $\partial M \times \{1\}$. Let $\rho, \mu\colon \mathbb{R}_+ \to \mathbb{R}_+$ be two functions such that any ball $B(x,r)$ in $\partial M \times [1,\infty)$ is acyclic inside $B(x,\rho(r))$ whenever $d(x, \partial M \times \{1\}) \geq \mu(r)$. We glue $\partial M \times [1,\infty)$ to M along ∂M by the attaching map $(x,1) \mapsto x$. Let Y be the resulting space. By Lemma 9.12, there is a proper metric on Y such that the canonical maps $M \hookrightarrow Y$ and $\partial M \times [1,\infty) \hookrightarrow Y$ are isometric embedding. For the rest of the proof we will regard M and $\partial M \times [1,\infty)$ as subspaces of Y.

We claim that Y is uniformly acyclic. Since M is uniformly acyclic, there is a function $\tau\colon [0,\infty) \to [0,\infty)$ such that, for any $r \geq 0$, any ball of radius r in M is acyclic inside a concentric ball of radius $\tau(r)$. Take a ball $B_r(x)$ of radius r in Y. If $x \in M$ and $d(x,\partial M) \geq r$, then $B_r(x)$ is contained inside M because points in $\partial M \times [1,\infty)$ are at least as far from x as points in ∂M by the construction of the metric on Y. Hence, $B_r(x)$ is acyclic inside $B_{\tau(r)}(x)$. If $x \in \partial M \times [1,\infty)$ and $d(x,\partial M) \geq \mu(r) + r$, then $B(x,r) \subset \partial M \times [1,\infty)$ and hence is acyclic in $B(x,\rho(r))$. If $d(x,\partial M) \leq \mu(r) + r$, we can deformation retract $B_r(x) \cap (\partial M \times [1,\infty))$ inside $\partial M \times [1,\infty)$ by sliding it along $[1,\infty)$ until it lands on $\partial M \times \{1\}$. Note that this deformation takes place inside $\partial M \times [1, \mu(r) + 2r]$, and the diameter shrinks as one approaches $\partial M \times \{1\}$. Hence the deformation takes place in a set of diameter at most $\mu(r) + 2r + 2r$. Also, the deformed ball is now contained in M and has diameter at most $2r$ and hence it is acyclic inside a set of diameter at most $\tau(2r)$ by uniform acyclicity of M. Hence, we conclude that $B_r(x)$ is acyclic inside a set of diameter at most $\mu(r) + 4r + \tau(2r)$. Hence Y is uniformly acyclic. □

As a consequence of the above proposition, we get the following.

Theorem 9.14 *If X is a proper, uniformly acyclic n-manifold with uniformly locally acyclic boundary, then* $\operatorname{cobdim}(X) \leq n$.

Proof By Proposition 9.13, we have a uniformly acyclic proper n-manifold Y such that X embeds isometrically in Y. By Theorem 6.4, it follows that $\operatorname{cobdim}(X) \leq \operatorname{cobdim}(Y)$. By Theorem 9.7, we know $\operatorname{cobdim}(Y) \leq n$ and hence $\operatorname{cobdim}(X) \leq n$. □

Now we can state the following improvement of Corollary 6.6. The proof is immediate from Theorems 6.4 and 9.14.

Corollary 9.15 *If* $\operatorname{cobdim}(X) \geq n$, *then X cannot be coarsely embedded into a proper, uniformly acyclic $(n-1)$-manifold with a uniformly locally acyclic boundary.*

Definition 9.16 (cocompact action dimension) The *cocompact action dimension* $\operatorname{cadim}(G)$ of a group G is the least dimension of a contractible manifold (possibly with boundary) that admits a proper cocompact G-action.

Corollary 9.17
$$\operatorname{cadim}(G) \geq \operatorname{cobdim}(G).$$

Proof Suppose $\operatorname{cadim}(G) = n$. Then G admits a proper, cocompact action on an acyclic n-manifold M. Choose a point $x_0 \in M$. By the Milnor–Schwarz lemma, the map $g \mapsto g.x_0$ gives a coarse equivalence $f: G \to M$. Since M is acyclic and it admits a cocompact action, M is uniformly acyclic. Similarly, since ∂M is locally acyclic and it admits a cocompact action, it is uniformly locally acyclic. Since M is proper, by Corollary 9.15, we get $\operatorname{cobdim}(G) \le n = \operatorname{cadim}(G)$. □

Remark 9.18 The action dimension $\operatorname{actdim}(G)$ of a group G is the least dimension of a contractible manifold that admits a proper G-action. Note that $\operatorname{cadim}(G) \ge \operatorname{actdim}(G)$, however, we do not know of any groups where cadim > actdim. Nonetheless, we believe that $\operatorname{cobdim}(G)$ gives a lower bound to $\operatorname{actdim}(G)$. A naive approach to prove this might be as follows: Suppose, G admits a proper action on a contractible n-manifold M. Then the map $f: G \to M$, sending G to one of its orbit gives a coarse embedding. Furthermore, the image of f is uniformly contractible in M: for any r there exists s such that any ball of radius r in M centered at a point in $f(G)$ is uniformly contractible inside a concentric ball of radius s. This should imply that there is a bounded function $p: f(G) \to (0, \infty)$ such that the space $X = \bigcup_{x \in f(G)} N_{p(x)}(x)$ is a uniformly contractible manifold with boundary. Since G acts on X cocompactly, X has uniformly locally acyclic boundary. Theorem 9.14 then implies that $\operatorname{cobdim}(X) \le n$. Since G coarsely embed into X by the map f, we get $\operatorname{cobdim}(G) \le \operatorname{cobdim}(X) \le n$, proving the desired claim. However, the author does not know how to prove that there exists such X.

Question Is $\operatorname{cobdim}(G) \ge \operatorname{actdim}(G)$ true?

10 Lower bound of cobdim

Theorem 10.1 *If* $\mathrm{HX}^*(X^2 - \delta(X)) = 0$ *for* $* \le n-1$, *then* $\operatorname{cobdim}(X) \ge n$.

Proof For $n = 0$, the claim is trivial. If $n = 1$, the assumption says $\mathrm{HX}^0(X^2 - \delta(X)) = 0$. This means X is unbounded, otherwise, nonzero constant functions from X^2 to $\mathbb{Z}_2$ give nontrivial elements in $\mathrm{HX}^0(X^2 - \delta(X))$. Hence, $\operatorname{cobdim}(X) \ge 1$ in this case.

Suppose $n \ge 2$ and $f: X \to \ell^\infty$ is an isometry. We first show that

$$f^*: \mathrm{HX}^2_{\mathbb{Z}_2}(\ell^\infty)^2 - \delta(\ell^\infty)) \xrightarrow{f^*} \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X))$$

is a nontrivial map. To see that, consider the following part of the maps between the concerned coarse Gysin sequences. Our goal is to show that the second vertical map is nontrivial:

$$\begin{array}{ccccccc}
\mathrm{HX}^1((\ell^\infty)^2 - \delta(\ell^\infty)) & \longrightarrow & \mathrm{HX}^1_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) \oplus \mathbb{Z}_2 & \longrightarrow & \mathrm{HX}^2_{\mathbb{Z}_2}((\ell^\infty)^2 - \delta(\ell^\infty)) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \\
\mathrm{HX}^1(X^2 - \delta(X)) & \longrightarrow & \mathrm{HX}^1_{\mathbb{Z}_2}(X^2 - \delta(X)) \oplus \mathbb{Z}_2 & \xrightarrow{j} & \mathrm{HX}^2_{\mathbb{Z}_2}(X^2 - \delta(X)) & \longrightarrow & \dots
\end{array}$$

By the commutativity of the diagram, our claim follows if we can show that j is injective and the first vertical map is nontrivial.

Since $\mathrm{HX}^1(X^2-\delta(X))=0$, it follows from the long exact sequence (8-5) that j is injective.

Next, we show that the first vertical map in the above commutative diagram is nontrivial. It is equivalent to showing that the following map is nontrivial:

$$f^*\colon \mathrm{HX}^1_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty),\delta(\ell^\infty))\to \mathrm{HX}^1_{\mathbb{Z}_2}((X^2-\delta(X),\delta(X)).$$

We can show that using the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^0(\mathrm{C}^*(\delta(\ell^\infty)) & \to & \mathrm{HX}^1_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty),\delta(\ell^\infty)) & \to & \mathrm{HX}^1((\ell^\infty)^2-\delta(\ell^\infty)) & \to & 0 \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ 0 & \to & \mathrm{H}^0(\mathrm{C}^*(\delta(X))) & \longrightarrow & \mathrm{HX}^1_{\mathbb{Z}_2}(X^2-\delta(X),\delta(X)) & \xrightarrow{j} & \mathrm{HX}^1(X^2-\delta(X)) & \longrightarrow & 0 \end{array}$$

The long exact sequence is due to the second long exact sequence of Lemma 8.1 combined with the fact that $\mathrm{HX}^1((\ell^\infty)^2-\delta(\ell^\infty)), \mathrm{H}^1(\mathrm{C}^*(\delta(\ell^\infty))), \mathrm{HX}^1(X^2-\delta(X))$ and $\mathrm{H}^1(\mathrm{C}^*(\delta(X)))$ all are trivial. The second vertical map is an isomorphism as both the domain and the range are constant functions defined on the respective spaces. The fourth vertical map is an isomorphism because both the domain and the range are trivial. So, applying the five lemma on the above diagram, we conclude that the middle vertical map is an isomorphism, as desired.

Hence we obtain that the map $f^*\colon \mathrm{HX}^2_{\mathbb{Z}_2}(\ell^\infty)^2-\delta(\ell^\infty))\to \mathrm{HX}^2_{\mathbb{Z}_2}(X^2-\delta(X))$ is nontrivial.

Let us now consider the maps between the following parts of the coarse Gysin sequences where $*\geq 2$:

$$\begin{array}{ccccccc} \to \mathrm{HX}^*((\ell^\infty)^2-\delta(\ell^\infty)) & \to & \mathrm{HX}^*_{\mathbb{Z}_2}(\ell^\infty)^2-\delta(\ell^\infty)) & \to & \mathrm{HX}^{*+1}_{\mathbb{Z}_2}((\ell^\infty)^2-\delta(\ell^\infty)) & \to \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\ \longrightarrow \mathrm{HX}^*(X^2-\delta(X)) & \longrightarrow & \mathrm{HX}^*_{\mathbb{Z}_2}(X^2-\delta(X)) & \longrightarrow & \mathrm{HX}^{*+1}_{\mathbb{Z}_2}(X^2-\delta(X)) & \longrightarrow \end{array}$$

The first terms of both sequences above are trivial for $*\leq n-1$. That implies that the third horizontal maps in the above diagram are injective. Hence by commutativity of the diagram, if the second vertical map is nontrivial then so is the third vertical map when $*\leq n-1$. We saw previously that, when $*=2$, the second vertical map is injective. It now follows by induction that the third vertical map is injective when $*\leq n-1$. In particular, when $*=n-1$, injectivity of the third vertical map means $\mathrm{cvk}^n(X)\neq 0$. Hence, $\mathrm{cobdim}(X)\geq n$. □

As a consequence we get the following, which also follows from [16, Lemma 5.3].

Corollary 10.2 *If X is a coarse* $\mathrm{PD}(n)$ *space, then X does not coarsely embed into a proper, uniformly acyclic $(n-1)$-manifold with a uniformly locally acyclic boundary.*

Proof If X is a coarse $\mathrm{PD}(n)$ space, it satisfies the hypotheses of Theorem 10.1. Hence, $\mathrm{cobdim}(X)\geq n$. The claim now follows from Corollary 9.15. □

Example 10.3 If X is a proper, uniformly acyclic n-manifold, then it satisfies all the hypotheses of the above corollary. Hence $\mathrm{cobdim}(X) \geq n$. Theorem 9.14 implies that $\mathrm{cobdim}(X) \leq n$. Hence $\mathrm{cobdim}(X) = n$ whenever X is a proper, uniformly acyclic n-manifold.

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