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**Correction to the article  
An algebraic model for finite loop spaces**

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## Correction to the article An algebraic model for finite loop spaces

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We correct here two errors in our earlier paper *An algebraic model for finite loop spaces*.

[55R35](#); [20D20](#), [20E22](#)

We correct two erroneous arguments found in [3].

In the proof of [3, Lemma A.8], we applied [2, Proposition 5.4] to orbit categories of transporter systems, while that proposition is stated only for orbit categories of fusion systems. After replacing that by [Proposition 2.3](#) below, Lemma A.8 can be proven as stated in [3].

In the proof of [3, Corollary A.10], we applied [3, Proposition A.9(b)] in a way that is not valid unless all objects in  $\mathcal{L}$  are  $\mathcal{F}$ -centric. That corollary is a special case of [Proposition 1.6](#) here.

### 1 Inclusions of linking systems for the same fusion system

We refer to [3, Definition 1.9] for the complete definition of a linking system associated to a fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$ . Very briefly, it consists of a category  $\mathcal{L}$  whose objects are subgroups of  $S$ , together with a pair of functors

$$\mathcal{T}_{\text{Ob}(\mathcal{L})}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}$$

satisfying certain conditions. Here,  $\mathcal{T}_{\text{Ob}(\mathcal{L})}(S)$  is the *transporter category*: the category with the same objects as  $\mathcal{L}$ , and where  $\text{Mor}_{\mathcal{T}_{\text{Ob}(\mathcal{L})}(S)}(P, Q)$  is the set of all  $g \in S$  such that  ${}^gP \leq Q$ . Also, the set  $\text{Ob}(\mathcal{L})$  is required to be closed under  $\mathcal{F}$ -conjugacy and overgroups, and must include all subgroups of  $S$  that are  $\mathcal{F}$ -centric and  $\mathcal{F}$ -radical.

As usual, when  $P \leq Q$  are objects in a linking system  $\mathcal{L}$ , we write  $\iota_{P,Q} = \delta_{P,Q}(1) \in \text{Mor}_{\mathcal{L}}(P, Q)$  for the inclusion morphism. By “extensions” and “restrictions” of morphisms we mean extensions and restrictions of source and target both, with respect to these inclusions.

**Proposition 1.1** *The following hold for each linking system  $\mathcal{L}$  associated to a saturated fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$ .*

- (a) For each  $P, Q \in \text{Ob}(\mathcal{L})$ , the homomorphism  $\pi_{P,Q}: \text{Mor}_{\mathcal{L}}(P, Q) \rightarrow \text{Hom}_{\mathcal{F}}(P, Q)$  is surjective. The group  $\text{Ker}(\pi_P)$  acts freely on  $\text{Mor}_{\mathcal{L}}(P, Q)$ , and  $\pi_{P,Q}$  induces a bijection

$$\text{Mor}_{\mathcal{L}}(P, Q)/\text{Ker}(\pi_P) \xrightarrow{\cong} \text{Hom}_{\mathcal{F}}(P, Q).$$

If  $P$  is fully centralized in  $\mathcal{F}$ , then  $\text{Ker}(\pi_P) = \delta_P(C_S(P))$ .

- (b) For every morphism  $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$ , and every  $P_*, Q_* \in \text{Ob}(\mathcal{L})$  such that  $P_* \leq P, Q_* \leq Q$ , and  $\pi(\psi)(P_*) \leq Q_*$ , there is a unique morphism  $\psi|_{P_*, Q_*} \in \text{Mor}_{\mathcal{L}}(P_*, Q_*)$  (the “restriction” of  $\psi$ ) such that  $\psi \circ \iota_{P_*, P} = \iota_{Q_*, Q} \circ \psi|_{P_*, Q_*}$ .
- (c) Let  $P \trianglelefteq \bar{P} \leq S$  and  $Q \leq \bar{Q} \leq S$  be objects in  $\mathcal{L}$ . Let  $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$  be such that for each  $g \in \bar{P}$ , there is  $h \in \bar{Q}$  satisfying  $\iota_{Q, \bar{Q}} \circ \psi \circ \delta_P(g) = \delta_{Q, \bar{Q}}(h) \circ \psi$ . Then there is a unique morphism  $\bar{\psi} \in \text{Mor}_{\mathcal{L}}(\bar{P}, \bar{Q})$  such that  $\bar{\psi}|_{P, Q} = \psi$ .
- (d) All morphisms in  $\mathcal{L}$  are monomorphisms and epimorphisms in the categorical sense.
- (e) A morphism  $\psi$  in  $\mathcal{L}$  is an isomorphism in  $\mathcal{L}$  if  $\pi(\psi)$  is an isomorphism in  $\mathcal{F}$ .

**Proof** Points (a)–(d) are shown in [3, Proposition A.4], while (e) follows from [3, Propositions A.2(c) and A.5]. □

We want to show that the geometric realizations of two linking systems associated to the same fusion system are homotopy equivalent. The next lemma is a first step towards doing that.

**Lemma 1.2** *Let  $\mathcal{F}$  be a saturated fusion system over a discrete  $p$ -toral group  $S$ . Let  $\mathcal{L}_0 \subseteq \mathcal{L}$  be linking systems associated to  $\mathcal{F}$  such that  $\text{Ob}(\mathcal{L}) \setminus \text{Ob}(\mathcal{L}_0) = \mathcal{P}$ , where  $\mathcal{P}$  is an  $\mathcal{F}$ -conjugacy class of subgroups of  $S$ . Then the inclusion of nerves  $|\mathcal{L}_0| \subseteq |\mathcal{L}|$  is a homotopy equivalence.*

**Proof** The following proof is essentially that given in [1, Proposition 3.11], modified for fusion systems over discrete  $p$ -toral groups. To simplify notation, for  $\varphi \in \text{Mor}_{\mathcal{L}}(Q, R)$ , we write  $\text{Im}(\varphi) = \text{Im}(\pi(\varphi)) \leq R$ , and  $\varphi(Q_0) = \pi(\varphi)(Q_0) \leq R$  if  $Q_0 \leq Q$ .

We must show that the inclusion functor  $\mathcal{I}: \mathcal{L}_0 \rightarrow \mathcal{L}$  induces a homotopy equivalence  $|\mathcal{L}_0| \simeq |\mathcal{L}|$ . By Quillen’s Theorem A (see [5]), it will be enough to prove that the undercategory  $P \downarrow \mathcal{I}$  is contractible (ie  $|P \downarrow \mathcal{I}| \simeq *$ ) for each  $P$  in  $\mathcal{L}$ . This is clear when  $P \notin \mathcal{P}$  (since  $P \downarrow \mathcal{I}$  has initial object  $(P, \text{Id})$  in that case), so it suffices to consider the case  $P \in \mathcal{P}$ . Since all subgroups in  $\mathcal{P}$  are isomorphic in the category  $\mathcal{L}$ , we can assume that  $P$  is fully normalized.

Set

$$\hat{P} = \{x \in N_S(P) \mid c_x \in O_p(\text{Aut}_{\mathcal{F}}(P))\}.$$

Recall that  $P \notin \text{Ob}(\mathcal{L}_0)$ , and hence by definition of a linking system cannot be both  $\mathcal{F}$ -centric and  $\mathcal{F}$ -radical. If  $P$  is not  $\mathcal{F}$ -centric, then  $\hat{P} \geq P \cdot C_S(P) > P$ . If  $P$  is not  $\mathcal{F}$ -radical, then  $\text{Inn}(P) < O_p(\text{Aut}_{\mathcal{F}}(P)) \leq$

$\text{Aut}_S(P)$ , the last inclusion since  $P$  is fully normalized, and so  $\hat{P} > P$ . Thus  $\hat{P} \in \text{Ob}(\mathcal{L}_0)$  (and  $P \leq \hat{P}$ ) in either case.

Set  $N_{\mathcal{L}_0}(P) = \mathcal{L}_0 \cap N_{\mathcal{L}}(P)$ , and let

$$\mathcal{I}_N : N_{\mathcal{L}_0}(P) \rightarrow N_{\mathcal{L}}(P)$$

be the inclusion (thus the restriction of  $\mathcal{I}$ ). Consider the functors

$$\hat{P} \downarrow \mathcal{I}_N \xrightarrow{j_2} P \downarrow \mathcal{I}_N \xrightarrow{j_1} P \downarrow \mathcal{I},$$

where  $j_1$  is the inclusion of undercategories, induced by the inclusions  $N_{\mathcal{L}_0}(P) \rightarrow \mathcal{L}_0$  and  $N_{\mathcal{L}}(P) \rightarrow \mathcal{L}$ , and where  $j_2$  sends an object  $(Q, \alpha)$  to  $(Q, \alpha \circ \iota_{P, \hat{P}})$  (for  $\alpha \in \text{Mor}_{N_{\mathcal{L}}(P)}(\hat{P}, Q)$ ). For each  $i = 1, 2$ , we will construct a retraction  $r_i$  such that  $r_i \circ j_i = \text{Id}$ , together with a natural transformation of functors between  $j_i \circ r_i$  and the identity. It then follows that  $|P \downarrow \mathcal{I}| \simeq |P \downarrow \mathcal{I}_N| \simeq |\hat{P} \downarrow \mathcal{I}_N|$ , and the last space is contractible since  $(\hat{P}, \text{Id}_{\hat{P}})$  is an initial object in  $\hat{P} \downarrow \mathcal{I}_N$  (recall  $\hat{P} \in \text{Ob}(N_{\mathcal{L}_0}(P))$ ).

**Step 1** We first construct the retraction functor  $r_1 : P \downarrow \mathcal{I} \rightarrow P \downarrow \mathcal{I}_N$ , together with a natural transformation  $(j_1 \circ r_1 \xrightarrow{\eta} \text{Id}_{P \downarrow \mathcal{I}})$ . By [3, Lemma 1.7(c)], for each  $R \in \mathcal{P}$  (the  $\mathcal{F}$ -conjugacy class of  $P$ ), there is a morphism in  $\mathcal{F}$  from  $N_S(R)$  to  $N_S(P)$  which sends  $R$  isomorphically to  $P$ . Since  $\pi$  is surjective on morphism sets by Proposition 1.1(a), we can choose a morphism

$$\Phi_R \in \text{Mor}_{\mathcal{L}}(N_S(R), N_S(P)),$$

for each  $R \in \mathcal{P}$  such that  $\Phi_R(R) = P$ . We also require that  $\Phi_P = \text{Id}_{N_S(P)}$ .

For each  $(Q, \varphi)$  in  $P \downarrow \mathcal{I}$ , set  $N_{(Q, \varphi)}(P) = \Phi_{\varphi(P)}(N_Q(\varphi(P)))$ . Thus  $P < N_{(Q, \varphi)}(P) \leq N_S(P)$ , since  $N_Q(\varphi(P)) > \varphi(P)$  by [2, Lemma 1.8]. Consider the diagram

$$(1-1) \quad \begin{array}{ccccc} P & \xrightarrow{\varphi_*} & N_Q(\varphi(P)) & \xrightarrow{\iota} & Q \\ & \searrow \varphi' & \cong \downarrow (\Phi_{\varphi(P)})_* & \nearrow \varphi'' & \\ & & N_{(Q, \varphi)}(P) & & \end{array}$$

where  $\iota$  is the inclusion (in  $\mathcal{L}$ ),  $\varphi_*$  and  $(\Phi_{\varphi(P)})_*$  denote the restriction morphisms of Proposition 1.1(b) (restricting source and/or target as indicated), and  $\varphi' = (\Phi_{\varphi(P)})_* \circ \varphi_*$  and  $\varphi'' = \iota \circ (\Phi_{\varphi(P)})_*^{-1}$  (so the triangles commute). Define  $r_1 : P \downarrow \mathcal{I} \rightarrow P \downarrow \mathcal{I}_N$  on objects by setting

$$r_1(Q, \varphi) = (N_{(Q, \varphi)}(P), \varphi').$$

We still need to define  $r_1$  on morphisms. For each morphism  $\beta \in \text{Mor}_{P \downarrow \mathcal{I}}((Q, \varphi), (R, \psi))$ , ie for each commutative square of the form

$$(1-2) \quad \begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \text{Id} \downarrow & & \beta \downarrow \\ P & \xrightarrow{\psi} & R \end{array}$$

we get the following commutative diagrams:

$$(1-3) \quad \begin{array}{ccc} P & \xrightarrow{\varphi_*} & N_Q(\varphi(P)) \xrightarrow{\iota} Q \\ \text{Id} \downarrow & & \beta_* \downarrow \quad \beta \downarrow \\ P & \xrightarrow{\psi_*} & N_R(\psi(P)) \xrightarrow{\iota} R \end{array} \quad \begin{array}{ccc} N_Q(\varphi(P)) & \xrightarrow[\cong]{(\Phi_{\varphi(P)})_*} & N_{(Q,\varphi)}(P) \\ \beta_* \downarrow & & \hat{\beta} \downarrow \\ N_R(\psi(P)) & \xrightarrow[\cong]{(\Phi_{\psi(P)})_*} & N_{(R,\psi)}(P) \end{array}$$

Again,  $\beta_*$ ,  $\varphi_*$ , and  $(\Phi_{\varphi(P)})_*$  denote the morphisms after restricting source and target as shown, while  $\hat{\beta} = (\Phi_{\psi(P)})_* \circ \beta_* \circ (\Phi_{\varphi(P)})_*^{-1}$ .

Since each of the three squares in (1-3) commutes, it follows that  $\psi' = \hat{\beta} \circ \varphi'$  and  $\psi'' \circ \hat{\beta} = \varphi''$ , where  $\varphi'$ ,  $\psi'$ ,  $\varphi''$ , and  $\psi''$  are as in (1-1). So we can define  $r_1$  on morphisms by setting

$$r_1(\beta) = \hat{\beta}: (N_{(Q,\varphi)}(P), \varphi') \rightarrow (N_{(R,\psi)}(P), \psi').$$

Define a natural transformation  $\eta: j_1 \circ r_1 \rightarrow \text{Id}_{P \downarrow \mathcal{I}}$  by sending an object  $(Q, \varphi)$  to the morphism

$$\varphi'': (j_1 \circ r_1)(Q, \varphi) = (N_{(Q,\varphi)}(P), \varphi') \rightarrow (Q, \varphi)$$

(a natural transformation since  $\psi'' \circ \hat{\beta} = \varphi''$ ). Since  $r_1 \circ j_1 = \text{Id}_{P \downarrow \mathcal{I}_N}$ , this finishes the proof that  $|P \downarrow \mathcal{I}| \simeq |P \downarrow \mathcal{I}_N|$ .

**Step 2** Recall that  $j_2: \hat{P} \downarrow \mathcal{I}_N \rightarrow P \downarrow \mathcal{I}_N$  is induced by precomposing with the inclusion  $\iota_{P, \hat{P}} \in \text{Mor}_{\mathcal{L}}(P, \hat{P})$ . We now construct a retraction functor  $r_2: P \downarrow \mathcal{I}_N \rightarrow \hat{P} \downarrow \mathcal{I}_N$ , together with a natural transformation of functors from  $\text{Id}_{P \downarrow \mathcal{I}_N}$  to  $j_2 \circ r_2$ .

Since  $P$  is fully normalized in  $\mathcal{F}$  by assumption, it is also fully centralized by the Sylow axiom for a saturated fusion system [3, Definition 1.4(I)]. So by Proposition 1.1(a), the structure homomorphism  $\pi_P: \text{Aut}_{\mathcal{L}}(P) \rightarrow \text{Aut}_{\mathcal{F}}(P)$  is surjective and  $\text{Ker}(\pi_P) = \delta_P(C_S(P))$ . Since  $\pi_P \circ \delta_P: N_S(P) \rightarrow \text{Aut}_{\mathcal{F}}(P)$  sends  $g$  to  $c_g$ , we have  $\hat{P} = \delta_P^{-1}(\pi_P^{-1}(O_P(\text{Aut}_{\mathcal{F}}(P))))$ , and so  $\delta_P(\hat{P}) = \pi_P^{-1}(O_P(\text{Aut}_{\mathcal{F}}(P)))$ . Since  $\pi_P$  is surjective, this proves that  $\delta_P(\hat{P}) \trianglelefteq \text{Aut}_{\mathcal{L}}(P)$ .

Fix subgroups  $Q, R \leq N_S(P)$  containing  $P$ , and let  $\varphi \in \text{Mor}_{N_{\mathcal{L}}(P)}(Q, R)$  be a morphism. For  $g \in Q\hat{P}$ , write  $g = g'x$  for  $g' \in Q$  and  $x \in \hat{P}$ , and let  $y \in \hat{P}$  be such that  $\delta_P(y) = (\varphi|_P)\delta_P(x)(\varphi|_P)^{-1}$ . Then  $\varphi\delta_Q(g') = \delta_R(\varphi(g'))\varphi$  by condition (C) in the definition of a linking system ([3, Definition 1.9]), so after restriction of source and target to  $P$ , we get  $(\varphi|_P)\delta_P(g')(\varphi|_P)^{-1} = \delta_P(\varphi(g'))$ . Set  $h = \varphi(g')y$ . Then

$$(\varphi|_P)\delta_P(g)(\varphi|_P)^{-1} = (\varphi|_P)\delta_P(g')(\varphi|_P)^{-1} \circ (\varphi|_P)\delta_P(x)(\varphi|_P)^{-1} = \delta_P(\varphi(g')y) = \delta_P(h)$$

in  $\text{Aut}_{\mathcal{L}}(P)$ , and hence

$$\begin{aligned} \iota_{R, R\hat{P}} \circ \varphi \circ \delta_Q(g) \circ \iota_{P, Q} &= \iota_{P, R\hat{P}} \circ (\varphi|_P) \circ \delta_P(g) \\ &= \iota_{P, R\hat{P}} \circ \delta_P(h) \circ (\varphi|_P) = \delta_{R, R\hat{P}}(h) \circ \varphi \circ \iota_{P, Q} \in \text{Mor}_{N_{\mathcal{L}_0}(P)}(P, R\hat{P}). \end{aligned}$$

So  $\iota_{R,R\hat{P}} \circ \varphi \circ \delta_Q(g) = \delta_{R,R\hat{P}}(h) \circ \varphi$  since  $\iota_{P,Q}$  is an epimorphism by Proposition 1.1(d). Hence by Proposition 1.1(c), there is a unique morphism  $\hat{\varphi} \in \text{Mor}_{\mathcal{L}_0}(Q\hat{P}, R\hat{P})$  (the “extension” of  $\varphi$ ) such that the following diagram commutes in  $\mathcal{L}$ :

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & R \\ \downarrow \iota_{Q,Q\hat{P}} & & \downarrow \iota_{R,R\hat{P}} \\ Q\hat{P} & \xrightarrow{\hat{\varphi}} & R\hat{P} \end{array}$$

Note that  $Q\hat{P}, R\hat{P} \in \text{Ob}(\mathcal{L}_0)$  (and hence  $\hat{\varphi}$  lies in  $N_{\mathcal{L}_0}(P)$ ) since  $\hat{P} > P$ .

The functor  $r_2: P \downarrow \mathcal{I}_N \rightarrow \hat{P} \downarrow \mathcal{I}_N$  is defined on objects by setting

$$r_2(Q, \alpha) = (Q\hat{P}, \hat{\alpha}).$$

If  $\beta: (Q, \alpha) \rightarrow (R, \gamma)$  is a morphism in  $P \downarrow \mathcal{I}_N$ , that is  $\beta \in \text{Mor}_{\mathcal{L}}(Q, R)$  is such that  $\beta \circ \alpha = \gamma$ , then we define  $r_2(\beta) = \hat{\beta}$ . Because of the uniqueness of the extension  $\hat{\beta}$ , this construction defines a functor from  $P \downarrow \mathcal{I}_N$  to  $\hat{P} \downarrow \mathcal{I}_N$ . Moreover,  $r_2 \circ j_2 = \text{Id}_{\hat{P} \downarrow \mathcal{I}_N}$ , and  $j_2 \circ r_2 \simeq \text{Id}_{P \downarrow \mathcal{I}_N}$ , where the homotopy is induced by the natural transformation given by the inclusions  $\iota_{Q,Q\hat{P}}$ .  $\square$

Lemma 1.2 says that under certain conditions, we can remove one class of objects from a linking system without changing the homotopy type of its nerve. We need to combine this with the “bullet construction”, defined in [2], to show that we can remove infinitely many classes of objects without changing the homotopy type.

**Definition 1.3** [2, Definition 3.1] Let  $\mathcal{F}$  be a fusion system over a discrete  $p$ -toral group  $S$ , and let  $T \trianglelefteq S$  be its identity component. Let  $m \geq 0$  be such that  $S/T$  has exponent  $p^m$ ; thus  $g^{p^m} \in T$  for all  $g \in S$ . Set  $W = \text{Aut}_{\mathcal{F}}(T)$ .

- (a) For each  $A \leq T$ , set  $I(A) = C_T(C_W(A))$ , and let  $I(A)_0$  be its identity component.
- (b) For each  $P \leq S$ , set  $P^{[m]} = \langle g^{p^m} \mid g \in P \rangle \leq T$ , and set  $P^\bullet = P \cdot I(P^{[m]})_0$ .
- (c) Let  $\mathcal{F}^\bullet \subseteq \mathcal{F}$  be the full subcategory with  $\text{Ob}(\mathcal{F}^\bullet) = \{P^\bullet \mid P \leq S\}$ .

Some of the key properties of the bullet construction are listed in the following proposition:

**Proposition 1.4** [2] Let  $\mathcal{F}$  be a saturated fusion system over a discrete  $p$ -toral group  $S$ .

- (a) The set  $\text{Ob}(\mathcal{F}^\bullet) = \{P^\bullet \mid P \leq S\}$  contains only finitely many  $S$ -conjugacy classes of subgroups.
- (b) For each  $P \leq S$ , we have  $(P^\bullet)^\bullet = P^\bullet$ .
- (c) For each  $P \leq Q \leq S$ , we have  $P^\bullet \leq Q^\bullet$ .
- (d) For each  $P, Q \leq S$  and each  $\varphi \in \text{Hom}_{\mathcal{F}}(P, Q)$ , there is a unique map  $\varphi^\bullet \in \text{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$  such that  $\varphi^\bullet|_P = \varphi$ .

**Proof** Points (a)–(c) are stated as Lemma 3.2 in [2], and (d) is stated as Proposition 3.3. □

In particular, points (b)–(d) in Proposition 1.4 imply that there is a well defined retraction functor  $\mathcal{F} \rightarrow \mathcal{F}^\bullet$  that sends an object  $P$  to  $P^\bullet$  and a morphism  $\varphi$  to  $\varphi^\bullet$ .

The following lemma is shown in [2, Proposition 4.5] when  $\mathcal{L}$  is a centric linking system, but we need it in a more general situation.

**Lemma 1.5** *Let  $\mathcal{F}$  be a saturated fusion system over a discrete  $p$ -toral group  $S$ , and let  $\mathcal{L}$  be a linking system associated to  $\mathcal{F}$ .*

- (a) *For each  $P, Q \in \text{Ob}(\mathcal{L})$  and each  $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$ , there is a unique morphism  $\psi^\bullet \in \text{Mor}_{\mathcal{L}}(P^\bullet, Q^\bullet)$  that restricts to  $\psi$ , and is such that  $\pi(\psi^\bullet) = (\pi(\psi))^\bullet$ .*
- (b) *The space  $|\mathcal{L}^\bullet|$  is a deformation retract of  $|\mathcal{L}|$ , with retraction  $|\mathcal{L}| \rightarrow |\mathcal{L}^\bullet|$  induced by the functor that sends  $P$  to  $P^\bullet$  and  $\psi$  to  $\psi^\bullet$ .*

**Proof** (a) It suffices to show this when  $P$  is fully normalized in  $\mathcal{F}$ . Set  $\varphi = \pi(\psi) \in \text{Hom}_{\mathcal{F}}(P, Q)$ . By Proposition 1.4(d), there is a unique  $\varphi^\bullet \in \text{Hom}_{\mathcal{F}}(P^\bullet, Q^\bullet)$  that extends  $\varphi$ . By [4, Proposition 1.13(v)], we have  $C_S(P) = C_S(P^\bullet)$ , and by Proposition 1.1(a), this group acts freely and transitively on the sets  $\pi_{P,Q}^{-1}(\varphi)$  and  $\pi_{P^\bullet,Q^\bullet}^{-1}(\varphi^\bullet)$ . The restriction map from  $\pi_{P^\bullet,Q^\bullet}^{-1}(\varphi^\bullet)$  to  $\pi_{P,Q}^{-1}(\varphi)$  commutes with this action, and hence is a bijection.

(b) Point (a) implies that there is a well defined functor  $(-)^{\bullet}: \mathcal{L} \rightarrow \mathcal{L}^\bullet$  that sends  $P$  to  $P^\bullet$  and  $\psi$  to  $\psi^\bullet$ . This defines a map  $r: |\mathcal{L}| \rightarrow |\mathcal{L}^\bullet|$ , which is a retraction by Proposition 1.4(b). Let  $i$  denote the inclusion; then  $i \circ r$  homotopic to the identity on  $|\mathcal{L}|$  since there is a natural transformation of functors from  $(-)^{\bullet}$  to the identity that sends each object  $P$  to the inclusion  $\iota_{P,P^\bullet}$  from  $P$  to  $P^\bullet$ . □

We now combine Lemmas 1.2 and 1.5 to get the result we need.

**Proposition 1.6** *Let  $\mathcal{F}$  be a saturated fusion system over a discrete  $p$ -toral group  $S$ , and let  $\mathcal{L}_0 \subseteq \mathcal{L}$  be a pair of linking systems associated to  $\mathcal{F}$ . Then the inclusion of  $|\mathcal{L}_0|$  in  $|\mathcal{L}|$  is a homotopy equivalence.*

**Proof** Let  $\mathbf{L}$  be the set of all linking subsystems  $\mathcal{L}' \subseteq \mathcal{L}$  containing  $\mathcal{L}_0$  such that the inclusion  $|\mathcal{L}_0| \subseteq |\mathcal{L}'|$  is a homotopy equivalence. We must show that  $\mathcal{L} \in \mathbf{L}$ .

Assume otherwise, and choose  $\mathcal{L}_1 \in \mathbf{L}$  for which  $\text{Ob}(\mathcal{L}_1^\bullet)$  contains the largest possible number of  $\mathcal{F}$ -conjugacy classes. Let  $\mathcal{L}_2 \subseteq \mathcal{L}$  be the full subcategory with  $\text{Ob}(\mathcal{L}_2) = \{P \in \text{Ob}(\mathcal{L}) \mid P^\bullet \in \text{Ob}(\mathcal{L}_1)\}$ . By Lemma 1.5(b),  $|\mathcal{L}_1^\bullet| = |\mathcal{L}_2^\bullet|$  is a deformation retract of  $|\mathcal{L}_1|$  and of  $|\mathcal{L}_2|$ , so  $|\mathcal{L}_1| \simeq |\mathcal{L}_2|$ , and  $\mathcal{L}_2 \in \mathbf{L}$ . Since  $\mathcal{L}_2 \subsetneq \mathcal{L}$  by the assumption that  $\mathcal{L} \notin \mathbf{L}$ , we have  $\text{Ob}(\mathcal{L}^\bullet) \not\subseteq \text{Ob}(\mathcal{L}_2)$ . Let  $P$  be maximal among objects in  $\mathcal{L}^\bullet$  not in  $\mathcal{L}_2$ . By definition of  $\text{Ob}(\mathcal{L}_2)$ ,  $P$  is maximal among all objects of  $\mathcal{L}$  not in  $\mathcal{L}_2$ . Let  $\mathcal{L}_3 \subseteq \mathcal{L}$  be the full subcategory with  $\text{Ob}(\mathcal{L}_3) = \text{Ob}(\mathcal{L}_2) \cup P^\mathcal{F}$ .

By Lemma 1.2, the inclusion of  $|\mathcal{L}_2|$  into  $|\mathcal{L}_3|$  is a homotopy equivalence. Hence  $\mathcal{L}_3 \in \mathbf{L}$ , contradicting our maximality assumption on  $\mathcal{L}_1^\bullet$ . □

## 2 $\Lambda$ -functors

For any group  $G$ , let  $\mathcal{O}_p(G)$  be the  $p$ -subgroup orbit category of  $G$ : the category whose objects are the  $p$ -subgroups of  $G$ , and where

$$\text{Mor}_{\mathcal{O}_p(G)}(P, Q) = \{Qg \in Q \setminus G \mid {}^gP \leq Q\} = Q \setminus \{g \in G \mid {}^gP \leq Q\}.$$

Note that there is a bijection

$$\text{Mor}_{\mathcal{O}_p(G)}(P, Q) \cong \text{map}_G(G/P, G/Q)$$

that sends a coset  $Qg$  to the  $G$ -map ( $xP \mapsto xg^{-1}Q$ ).

**Definition 2.1** Let  $G$  be a locally finite group, and let  $M$  be a  $\mathbb{Z}G$ -module. Define a functor

$$F_M: \mathcal{O}_p(G)^{\text{op}} \rightarrow \text{Ab}$$

by setting

$$F_M(P) = \begin{cases} M & \text{if } P = 1, \\ 0 & \text{if } P \neq 1, \end{cases}$$

for each  $p$ -subgroup  $P \leq G$ . Here,  $\text{Aut}_{\mathcal{O}_p(G)}(1) \cong G$  has the given action on  $M$ . Set

$$\Lambda^*(G; M) = \varprojlim_{\mathcal{O}_p(G)}^*(F_M).$$

We refer to [3, Definition A.1] for the definition of a transporter system. As in Section 1, when  $S$  is a group and  $\mathcal{H}$  is a set of subgroups of  $S$ , we let  $\mathcal{T}_{\mathcal{H}}(S)$  be the transporter category of  $S$  with object set  $\mathcal{H}$ , where  $\text{Mor}_{\mathcal{T}_{\mathcal{H}}(S)}(P, Q)$  is the set of all  $g \in S$  such that  ${}^gP \leq Q$ . A transporter system associated to a fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$  consists of a category  $\mathcal{T}$  whose objects are subgroups of  $S$ , together with a pair of functors

$$\mathcal{T}_{\text{Ob}(\mathcal{T})}(S) \xrightarrow{\varepsilon} \mathcal{T} \xrightarrow{\rho} \mathcal{F},$$

such that  $\varepsilon$  is the identity on objects and injective on morphism sets,  $\rho$  is the inclusion on objects and surjective on morphism sets, and several other conditions are satisfied. The only requirements on the set  $\text{Ob}(\mathcal{T})$  are that it be nonempty and closed under  $\mathcal{F}$ -conjugacy and overgroups.

As with linking systems, when  $P \leq Q$  are objects in a transporter system  $\mathcal{T}$ , we write  $\iota_{P,Q} = \varepsilon_{P,Q}(1) \in \text{Mor}_{\mathcal{T}}(P, Q)$  for the inclusion morphism. By “extensions” and “restrictions” of morphisms we mean extensions and restrictions of source and target both, with respect to these inclusions. We will only need to refer to the following properties:

**Proposition 2.2** *The following hold for any transporter system  $\mathcal{T}$  associated to a saturated fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$ .*

- (a) *For each  $P, Q \in \text{Ob}(\mathcal{T})$ , the composite  $\rho_{P,Q} \circ \varepsilon_{P,Q}$  sends  $g \in N_S(P, Q)$  to the homomorphism  $c_g = (x \mapsto {}^g x) \in \text{Hom}_{\mathcal{F}}(P, Q)$ .*

- (b) Let  $P \trianglelefteq \bar{P} \leq S$  and  $Q \leq \bar{Q} \leq S$  be objects in  $\mathcal{T}$ . Let  $\psi \in \text{Iso}_{\mathcal{T}}(P, Q)$  be such that for each  $g \in \bar{P}$ , there is  $h \in \bar{Q}$  satisfying  $\iota_{Q, \bar{Q}} \circ \psi \circ \varepsilon_P(g) = \varepsilon_{Q, \bar{Q}}(h) \circ \psi$ . Then  $\psi$  extends to a unique morphism  $\bar{\psi} \in \text{Mor}_{\mathcal{T}}(\bar{P}, \bar{Q})$  such that  $\bar{\psi}|_{P, Q} = \psi$ .
- (c) All morphisms in  $\mathcal{T}$  are monomorphisms and epimorphisms in the categorical sense.

**Proof** Point (a) is axiom (B) in [3, Definition A.1], and point (c) is shown in [3, Proposition A.2(d)].

The existence of an extension in point (b) is axiom (II) in [3, Definition A.1], except that it is stated there under the additional assumption that  $Q$  is normal in  $\bar{Q}$ . But if  $Q \leq \bar{Q}$  is not normal, and  $\psi \in \text{Iso}_{\mathcal{T}}(P, Q)$ ,  $g \in \bar{P}$ , and  $h \in \bar{Q}$  are as above, then

$$Q = \text{Im}(\rho(\iota_{Q, \bar{Q}} \circ \psi \circ \varepsilon_P(g))) = \text{Im}(\rho(\varepsilon_{Q, \bar{Q}}(h) \circ \psi)) = {}^h Q,$$

the last equality by (a), and so  $h \in N_{\bar{Q}}(Q)$ . Hence  $\psi$  extends to  $\hat{\psi} \in \text{Mor}_{\mathcal{T}}(\bar{P}, N_{\bar{Q}}(Q))$  by axiom (II) as stated in [3], and we can take  $\bar{\psi} = \iota_{N_{\bar{Q}}(Q), \bar{Q}} \circ \hat{\psi}$ .

The extension in (b) is unique since inclusion morphisms in  $\mathcal{T}$  are epimorphisms by (c). □

If  $\mathcal{T}$  is a transporter system over a discrete  $p$ -toral group  $S$ , then its orbit category  $\mathcal{O}(\mathcal{T})$  is the category with the same objects, and where for each  $P, Q \in \text{Ob}(\mathcal{T})$ ,

$$\text{Mor}_{\mathcal{O}(\mathcal{T})}(P, Q) = \varepsilon_Q(Q) \backslash \text{Mor}_{\mathcal{T}}(P, Q).$$

Thus, for example, if  $\mathcal{T}$  is the transporter system of a group  $G$  (with some set of objects), then  $\mathcal{O}(\mathcal{T})$  is the orbit category of  $G$  in the above sense.

**Proposition 2.3** *Let  $\mathcal{T}$  be a transporter system associated to a saturated fusion system  $\mathcal{F}$  over a discrete  $p$ -toral group  $S$ . Fix  $P \in \text{Ob}(\mathcal{T})$ , and let*

$$\Phi: \mathcal{O}(\mathcal{T})^{\text{op}} \rightarrow \text{Ab}$$

*be a functor such that  $\Phi(Q) = 0$  for each  $Q \notin P^{\mathcal{F}}$ . Then*

$$\varprojlim_{\mathcal{O}(\mathcal{T})}^* (\Phi) \cong \Lambda^*(\text{Aut}_{\mathcal{O}(\mathcal{T})}(P); \Phi(P)).$$

**Proof** By axiom (I) for a transporter system (see [3, Definition A.1]), there is  $Q \in P^{\mathcal{F}}$  such that the index of  $\varepsilon_Q(N_S(Q))$  in  $\text{Aut}_{\mathcal{T}}(Q)$  is finite and prime to  $p$ . Thus  $\text{Aut}_{\mathcal{T}}(Q)$  is an extension of a discrete  $p$ -toral group by a finite group, and we write  $\varepsilon_Q(N_S(Q)) \in \text{Syl}_p(\text{Aut}_{\mathcal{T}}(Q))$  for short. Since  $\Lambda^*(\text{Aut}_{\mathcal{O}(\mathcal{T})}(Q); \Phi(Q)) \cong \Lambda^*(\text{Aut}_{\mathcal{O}(\mathcal{T})}(P); \Phi(P))$  for each  $Q \in P^{\mathcal{F}}$ , it suffices to prove the proposition when  $\varepsilon_P(N_S(P)) \in \text{Syl}_p(\text{Aut}_{\mathcal{T}}(P))$ . We will show that this is a special case of [2, Proposition 5.3].

Set  $\Gamma = \text{Aut}_{\mathcal{O}(\mathcal{T})}(P) = \varepsilon_P(P) \backslash \text{Aut}_{\mathcal{T}}(P)$ . To simplify the notation, we identify  $N_S(P)$  with its image  $\varepsilon_P(N_S(P)) \leq \text{Aut}_{\mathcal{T}}(P)$ , and identify  $N_S(P)/P$  with  $\varepsilon_P(N_S(P))/\varepsilon_P(P)$ .

Let  $\mathcal{H}$  be the set of all subgroups of  $N_S(P)/P \in \text{Syl}_p(\Gamma)$ . Define

$$\alpha: \mathcal{O}_{\mathcal{H}}(\Gamma) \rightarrow \mathcal{O}(\mathcal{T})$$

by setting  $\alpha(Q/P) = Q$ , and

$$\alpha(Q/P \xrightarrow{R\gamma} R/P) = [\bar{\gamma}] \in \text{Mor}_{\mathcal{O}(\mathcal{T})}(Q, R)$$

where  $\gamma \in \text{Aut}_{\mathcal{T}}(P)$  extends to a unique morphism  $\bar{\gamma} \in \text{Mor}_{\mathcal{T}}(Q, R)$  by [Proposition 2.2\(b\)](#).

The proposition will follow immediately from [\[2, Proposition 5.3\]](#) once we have shown that conditions (a)–(d) in the proposition all hold. Set  $c_0 = \alpha(1)$ , following the notation used in that proposition, and note that  $c_0 = P$ .

(a) For each  $P \in \text{Ob}(\mathcal{T})$  and each  $\psi \in \text{End}_{\mathcal{T}}(P)$ , the homomorphism  $\rho(\psi) \in \text{End}_{\mathcal{F}}(P)$  is an isomorphism of groups, and hence  $\psi \in \text{Aut}_{\mathcal{T}}(P)$  by [\[3, Proposition A.2\(c\)\]](#). So by construction,  $\alpha$  sends  $\Gamma = \text{Aut}_{\mathcal{O}_{\mathcal{H}}(\Gamma)}(1)$  isomorphically to  $\text{End}_{\mathcal{O}(\mathcal{T})}(P) = \text{Aut}_{\mathcal{O}(\mathcal{T})}(P)$ .

(b) Let  $U \in \text{Ob}(\mathcal{T})$  be such that  $U \notin P^{\mathcal{F}}$ . We must show that all isotropy subgroups of the  $\Gamma$ -action on  $\text{Mor}_{\mathcal{O}(\mathcal{T})}(P, U)$  are nontrivial and conjugate in  $\Gamma$  to subgroups in  $\mathcal{H}$ . Since  $\mathcal{H}$  is the set of all subgroups of a Sylow  $p$ -subgroup of  $\Gamma$ , where  $\Gamma$  has a discrete  $p$ -toral subgroup of finite index, this means showing that each isotropy subgroup is a nontrivial discrete  $p$ -toral subgroup.

Fix  $\psi \in \text{Mor}_{\mathcal{T}}(P, U)$ , and let  $[\psi]$  be its class in  $\text{Mor}(\mathcal{O}(\mathcal{T}))$ . Set  $Q = \rho(\psi)(P) < U$ . Then  $\psi = \iota_{Q,U} \circ (\psi|_{P,Q})$ , where  $\psi|_{P,Q} \in \text{Iso}_{\mathcal{T}}(P, Q)$ . So the isotropy subgroup for the action of  $\Gamma$  on  $\psi$  is isomorphic to the isotropy subgroup of  $\text{Aut}_{\mathcal{O}(\mathcal{T})}(Q)$  on  $\iota_{Q,U}$ .

For each  $\gamma \in \text{Aut}_{\mathcal{T}}(Q)$ , we have  $[\iota_{Q,U}\gamma] = [\iota_{P,U}]$  if and only if there is  $g \in U$  such that  ${}^g Q = Q$  and  $\gamma = \varepsilon_U(g)|_{Q,Q} = \varepsilon_Q(g)$ . Thus the isotropy subgroup is the group of all  $[\varepsilon_Q(g)]$  for  $g \in N_U(Q)$ , and hence is isomorphic to  $N_U(Q)/Q$ . This last group is a nontrivial discrete  $p$ -toral group by [\[2, Lemma 1.8\]](#) and since  $Q < U$  are both discrete  $p$ -toral groups.

(c) We claim that all morphisms in  $\mathcal{O}(\mathcal{T})$  are epimorphisms in the categorical sense. To see this, fix subgroups  $P, Q, R \in \text{Ob}(\mathcal{T})$  and morphisms  $\varphi \in \text{Mor}_{\mathcal{T}}(P, Q)$  and  $\alpha, \beta \in \text{Mor}_{\mathcal{T}}(Q, R)$  such that  $[\alpha][\varphi] = [\beta][\varphi]$ , where  $[-]$  denotes the class in  $\mathcal{O}(\mathcal{T})$  of a morphism in  $\mathcal{T}$ . Thus there is  $g \in R$  such that  $\alpha\varphi = \varepsilon_R(g)\beta\varphi$  in  $\mathcal{T}$ .

By [Proposition 2.2\(c\)](#), all morphisms in  $\mathcal{T}$  are epimorphisms. So  $\alpha = \varepsilon_R(g)\beta$ , and hence  $[\alpha] = [\beta]$ , proving that  $\varphi$  is an epimorphism in  $\mathcal{O}(\mathcal{T})$ .

(d) Fix  $Q/P \leq N_S(P)/P, U \in \text{Ob}(\mathcal{T})$ , and  $\varphi \in \text{Mor}_{\mathcal{T}}(P, U)$  such that  $[\varphi\varepsilon_P(g)] = [\varphi]$  in  $\text{Mor}_{\mathcal{O}(\mathcal{T})}(P, U)$  for each  $g \in Q$ . We must show that there is  $\bar{\varphi} \in \text{Mor}_{\mathcal{T}}(Q, U)$  such that  $[\bar{\varphi}|_{P,U}] = [\varphi]$  in  $\text{Mor}_{\mathcal{O}(\mathcal{T})}(P, U)$ .

By assumption, for each  $g \in Q$ , there is  $u \in U$  such that  $\varepsilon_U(u)\varphi = \varphi\varepsilon_P(g)$ . Set  $U_0 = \rho(\varphi)(P) \leq U$ . Thus  $\varphi = \iota_{U_0,U}\varphi_0$ , where  $\varphi_0 = \varphi|_{P,U_0} \in \text{Iso}_{\mathcal{T}}(P, U_0)$ . So

$$\varepsilon_U(u)\iota_{U_0,U}\varphi_0 = \iota_{U_0,U}\varphi_0\varepsilon_P(g).$$

Hence  $U_0 = \text{Im}(\rho(\varphi_0 \varepsilon_P(g))) = \text{Im}(\rho(\varepsilon_U(u) \iota_{U_0, U} \varphi_0)) = {}^u U_0$  (the last equality by [Proposition 2.2\(a\)](#)), and so  $u \in N_U(U_0)$ . Thus  $\iota_{U_0, U} \varphi_0 \varepsilon_P(g) \varphi_0^{-1} = \varepsilon_U(u) \iota_{U_0, U} = \iota_{U_0, U} \varepsilon_{U_0}(u)$ . Since morphisms in a transporter category are monomorphisms by [Proposition 2.2\(c\)](#), this implies that  $\varphi_0 \varepsilon_P(g) \varphi_0^{-1} \in \varepsilon_{U_0}(N_U(U_0))$ . So by [Proposition 2.2\(b\)](#), there is  $\bar{\varphi} \in \text{Mor}_{\mathcal{T}}(Q, U)$  such that  $\bar{\varphi}|_{P, U_0} = \varphi_0$ , and hence  $\bar{\varphi}|_{P, U} = \varphi$ .  $\square$

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