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Linear linkless embeddings: proof of a conjecture by Sachs

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In 1983, Sachs conjectured that every linklessly embeddable graph has a linear linkless embedding. We prove a stronger statement: every flat embedding of a linkless graph can be linearized.

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1 Introduction

A spatial graph is an embedding of a graph into $\mathbb{R}^3$. Conway and Gordon [2], and Sachs [6], introduced the theory of spatial graphs by showing every embedding of the complete graph K_6 into $\mathbb{R}^3$ contains two cycles which are linked, and every embedding of K_7 into $\mathbb{R}^3$ contains a cycle which is a nontrivial knot. A graph which has a nontrivial link in every embedding is called *intrinsically linked*, and if a graph is not intrinsically linked, it is called *linklessly embeddable*. A significant amount of work followed, including the characterization of linklessly embeddable graphs by Robertson, Seymour, and Thomas [5]: A graph is linklessly embeddable if and only if it does not have a Petersen family minor. These graphs are obtained from the complete graph K_6 by performing a sequence of $Y-\Delta$ and $\Delta-Y$ transforms, as shown in Figure 1.

We are concerned with *linear embeddings*, which are embeddings into $\mathbb{R}^3$ where every edge is a straight line segment. In 1948, Fáry showed the following:

Theorem 1.1 (Fáry's Theorem; see Fáry [3], Stein [8], and Wagner [9]) *All planar graphs have a planar embedding with all edges straight line segments.*

From this, we can easily provide linear linkless embeddings of all apex graphs, which are planar after the removal of some vertex. In 1983, Sachs conjectured every linklessly embeddable graph has a linear linkless embedding [6]. Here we show a stronger statement: every flat embedding can be linearized.

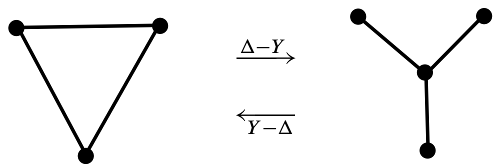

Figure 1: The $Y\Delta$ and ΔY transforms.

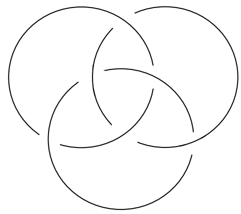

Figure 2: Borromean rings.

Definition 1.2 An embedding ϕ of a graph G is *flat* (*or paneled*) if every cycle $C \in \phi$ bounds a disk with interior disjoint from ϕ.

It is clear from this definition that every flat embedding ϕ must be linkless, as every cycle bounds a disk disjoint from ϕ, but the converse is not true. For example, consider three cycles embedded as the Borromean rings, shown in Figure 2. No two cycles are linked, so this is a linkless embedding. However, it is not flat, since no cycle can bound a disk disjoint from the graph.

However, Robertson, Seymour, and Thomas [5] have shown the following result:

Proposition 1.3 [5] *Every linklessly embeddable graph admits a flat embedding.*

Because of this, we will only work with flat embeddings.

It is very important to clarify how deletions and contractions apply to embeddings. For an embedding ϕ, the edge deletion $\phi\backslash e$ equals $\phi|_{G\backslash e}$. For a vertex deletion, $\phi\backslash v = \phi|_{G\backslash v}$. When we contract an edge e in the graph G, we delete double edges since we want our graph to remain simple.

However, for a topological contraction in the embedding ϕ, we move the ends of the edge towards some point in e and route all edges from its endpoints along the path of the edge e. A contraction of an embedding may contain double edges, and further contractions may create loops. See Figure 3 for an example of a contraction of an embedding which contains double edges. We denote this new embedding by ϕ/e.

In their survey paper, Robertson, Seymour, and Thomas show the following important result:

Proposition 1.4 [5] *An embedding ϕ is flat if and only if, for e a nonloop edge, both embeddings ϕ/e and $\phi\backslash e$ are flat.*

We will call back to this result later. The proof is discussed in the referenced survey paper.

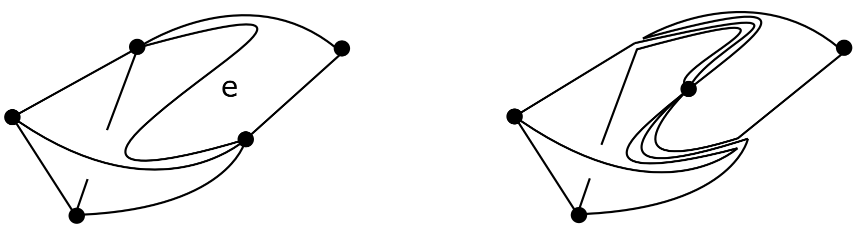

Figure 3: Topological contraction of the edge e.

2 Preliminary lemmas and constructions

The following lemma of Böhme is essential for our further constructions:

Lemma 2.1 [1] *Let ϕ be a flat embedding of a graph G, and $C_1, C_2, \dots, C_k$ be cycles in ϕ. If $C_i \cap C_j$ is connected or empty for all i and j, then there exist disks $\Gamma_1, \dots, \Gamma_k$ such that*

(1) *Γ_i is bounded by C_i,*

(2) *$\Gamma_i \cap \phi = C_i$, and*

(3) *Γ_i and Γ_j have disjoint interiors for all i and j.*

From this point on we will fix G, a linklessly embeddable graph. For an edge $e = xy \in G$, let $M(e) = \{v \in V(G) : v \text{ is adjacent to } x \text{ and } y\}$. We make the following claim:

Lemma 2.2 *Let ϕ be a flat embedding of G. Then for any edge $xy \in G$, there exists an embedded closed ball B_{xy} such that*

(1) *x and y are contained in the interior of B_{xy},*

(2) *all other neighbors of x and y are in the boundary of B_{xy},*

(3) *all edges incident to x or y are contained in B_{xy}, and*

(4) *B_{xy} is otherwise disjoint from $\phi(G)$.*

Proof We can construct this as follows. First, begin with an ϵ-tube around the edge xy, and add a small ball around x and one around y.

Next, we can add ϵ-tubes around all of the edges from x to its neighbors not in $M(xy)$, and the same for y. We do not do this for the vertices in $M(xy)$ because adding both tubes from x and from y to a vertex $m \in M(xy)$ would create genus, and then B_{xy} would not be homeomorphic to a closed ball.

Instead, if $M(xy) = \{m_1, m_2, \dots, m_k\}$, for the cycles $xym_1, xym_2, \dots, xym_k$, the intersection $C_i \cap C_j$ equals $\phi(xy)$ for all $i \neq j$. This is connected, so by Lemma 2.1 there exist disks Γ_i for $i = 1, \dots, k$ bounded by C_i with interiors disjoint from the embedding and each other. Then we can take a small 3D neighborhood around each disk, except m_i is on the boundary of this neighborhood. Union this into the ball B_{xy}. Because their interiors are disjoint, this does not add any genus or interior boundary into B_{xy}, and so the B_{xy} is still homeomorphic to a closed ball.

The ball B_{xy} constructed in this way satisfies properties (1)–(4) by construction. For examples of this construction, see Figures 11, 12, and 14–16 in Section 3. □

We further this construction with the following:

Lemma 2.3 *Given B_{xy} constructed as above, there exists a disk D_{xy} with boundary on the boundary of B_{xy} such that*

(1) *D_{xy} has the vertices of $M(xy)$ on its boundary,*

(2) *D_{xy} intersects the edge xy exactly once, and*

(3) *One component of $B_{xy} - D_{xy}$ contains x and all edges incident to x except xy, and the other contains y and all edges incident to y except xy.*

Proof If $M(xy)$ is empty, take a disk transverse to xy with boundary on the surface of B_{xy}. This disk can be constructed so that condition (3) is satisfied.

If $M(xy)$ is nonempty, we have disks Γ_i for $i = 1, \dots, k$ from before, with a thickening of these disks making up part of B_{xy}. Take a path along these disks from m_i to the midpoint of xy. Thicken this path transverse to Γ_i to meet the surface of B_{xy}. Paste these subdisks together to form a disk D_{xy}. □

We will refer to D_{xy} as the *separating disk* of xy. This disk has a nice property with how it interacts with disks bounded by cycles in the contraction ϕ/xy.

2.1 Intersection of the separating disk with cycle bounded disks

Let ϕ be a flat embedding of a graph G and D_{xy} be constructed as above. Let ϕ/xy be a contraction of this embedding where $v \in D_{xy}$, for v the vertex resulting from the contraction of xy. Note D_{xy} still separates the edges originally incident to x from the ones originally incident to y. We say a cycle *crosses* the disk D_{xy} if the cycle contains the sequence avb, where the edges av and vb are in separate components of $B_{xy} - D_{xy}$.

Proposition 2.4 *In ϕ/xy, for every cycle C not crossing D_{xy}, there exists a disk Γ disjoint from ϕ/xy bounded by C whose interior has no intersection with the interior of D_{xy}.*

Proof The figures for this lemma are accurate up to ambient isotopy, meaning B_{xy} will be shown as a sphere, and D_{xy} as an equatorial disk.

First, consider a cycle C which does not include the vertex v. Let Γ be a disk bounded by C disjoint from the rest of the embedding. The cycle does not intersect the interior of B_{xy}. If the interior of Γ has intersection with the interior of D_{xy}, Γ must pass into the ball B_{xy}, dividing B_{xy} into at least two components. If a region bounded by $\partial B_{xy} \cup \Gamma$ is disjoint from the embedding, push Γ out of B_{xy} through this component without intersecting the graph.

This creates a new disk Γ' which is disjoint from the graph, bounded by C, and disjoint from the interior of B_{xy}. Because the interior of D_{xy} is within the interior of B_{xy}, the disk Γ' has no intersection with the interior of D_{xy}.

If every component bounded by $\partial B_{xy} \cup \Gamma$ contains a portion of the embedding, every component must contain a vertex on the surface of B_{xy}. Then every component bounded by $\partial B_{xy} \cup \Gamma$ contains an edge going to v. Since v sits in one of the components only, Γ cannot be disjoint from the graph, which is a contradiction.

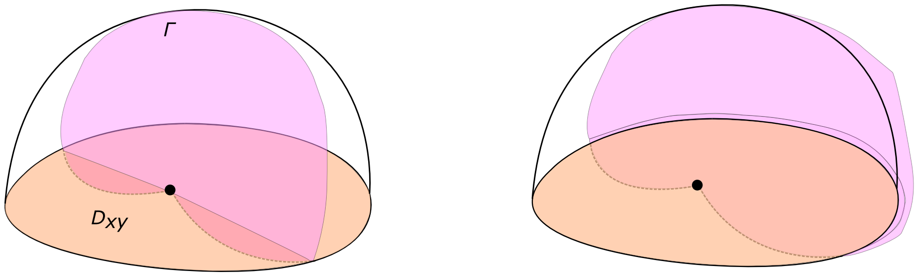

Figure 4: If a region bounded by $\Gamma \cup \partial B_{xy} \cup D_{xy}$ contains no edges, then Γ can be moved to the outside of this sphere without forcing it to intersect the graph. Here the disk is shown moving through the region in the back.

Second, let C be a cycle containing v which does not cross D_{xy}. Assume the interior of any disk Γ bounded by C has intersection with the interior of D_{xy}. We will show this would require the existence of a nontrivial link in ϕ, which is a contradiction. To make the presentation more fluent, we refer to the two components of $B_{xy} - D_{xy}$ as the "top" and "bottom", though they may be differently positioned with respect to a horizontal plane.

The cycle C contains avb, with av and vb both in the same component of $B_{xy} - D_{xy}$. Without loss of generality, assume they are within the bottom of $B_{xy} - D_{xy}$. By assumption, any disk bounded by C and disjoint from $\phi/xy \backslash C$ must intersect D_{xy}. Consider such a disk Γ. The union of surfaces $D_{xy} \cup \Gamma \cup \partial(B_{xy})$ determines disjoint regions within the interior of the top of B_{xy}. Any of these regions which is disjoint from the graph can be used to remove intersections between Γ and D_{xy}, by "pushing" Γ through the region, as in Figure 4. If all interior intersections between Γ and D_{xy} can be removed this way, we get the desired Γ. If not all intersections can be removed, there must be at least two regions in the top of $B_{xy} - D_{xy}$ with boundary included in $D_{xy} \cup \Gamma \cup \partial(B_{xy})$ each containing edges not in C. These edges are on the opposite side of D_{xy} from av and vb, since they are contained in the top. A diagram of the situation is presented in Figure 5. We have a cycle C with edges to v, both on one side of D_{xy}, and two edges to v on the other side of D_{xy} which are separated by the disk bounded by C.

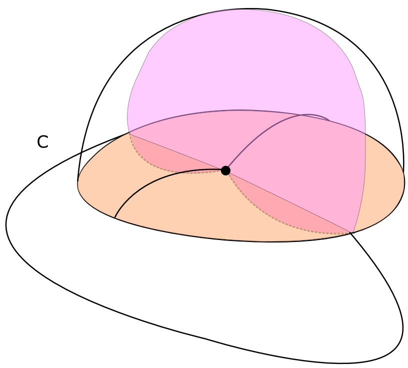

Figure 5: If Γ must have interior intersection with D_{xy}, then there must be at least 2 edges above D_{xy}.

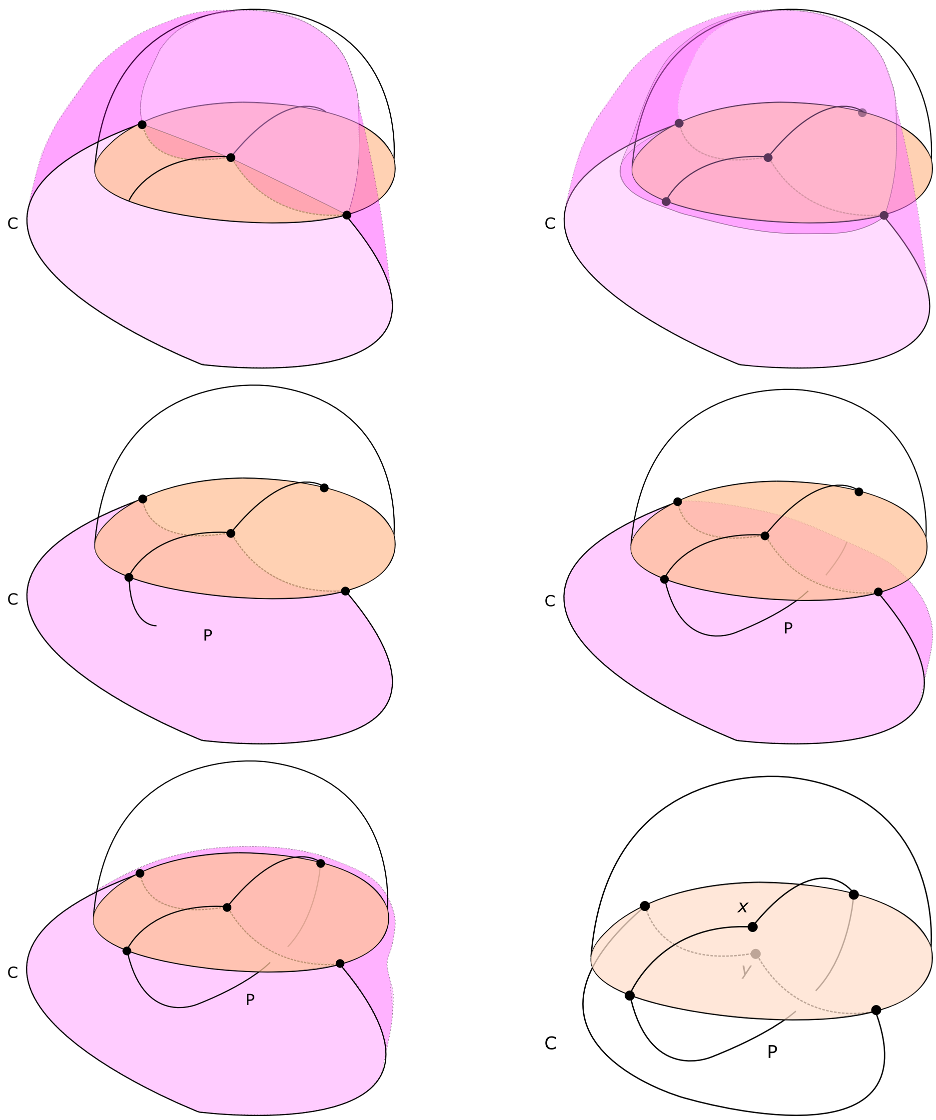

Figure 6: Relative position of Γ, D_{xy}, and $\partial(B_{xy})$. The disk Γ is pink and the disk D_{xy} is orange.

We push the disk out to the surface of the ball. In Figure 6, top row, we show Γ being pushed to the front of B_{xy}. It should be noted we are intentionally forcing our disk to intersect with the graph. By our assumption, a disk Γ bounded by C and disjoint from $\phi/xy\backslash C$ must have interior intersection with D_{xy}. Because D_{xy} is contained in B_{xy}, this means Γ must have interior intersection with B_{xy}. Then by the contrapositive, if we have a disk Γ bounded by C with no interior intersection with B_{xy}, it must have nonempty intersection with $\phi/xy\backslash C$.

Then Γ must intersect an edge going to the boundary of the top of B_{xy}. Push the disk Γ out along every intersection it has with the embedding away from the component of $B_{xy} - D_{xy}$ not containing the edges

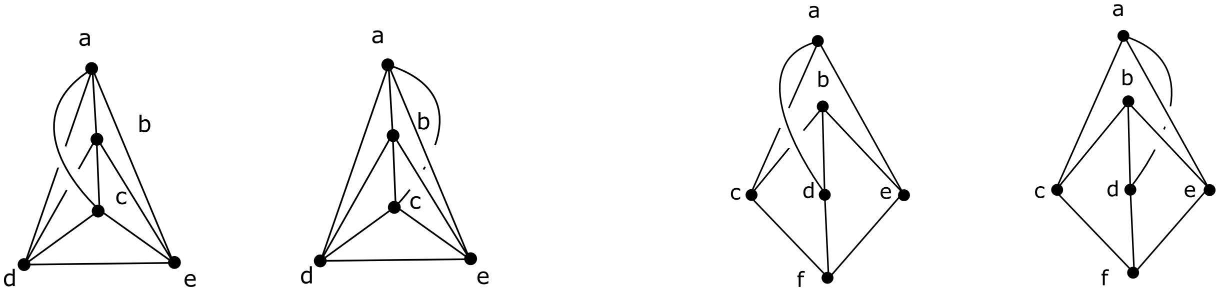

Figure 7: Two flat embeddings of the Kuratowski graphs. Left: K_5. Right: $K_{3,3}$.

in C. Trace these intersections. Eventually, we will no longer have any intersection with the graph. If no path P between vertices on the half of $B_{xy} - D_{xy}$ away from C is traced by intersecting with Γ, the final position for Γ is disjoint from this component of B_{xy}. This would mean Γ is disjoint from the graph, bounded by C, and the interior of Γ is disjoint from the interior of D_{xy}. This is a contradiction of our assumption, so this path P must exist. See Figure 6 for an example of such a path P being traced. But this necessitates the existence of a cycle C' and a cycle containing the path P' through x in ϕ. These must form a nontrivial link in ϕ, shown in Figure 6, bottom right. Then ϕ is not a flat embedding, which is a contradiction. □

We begin with some facts about embeddings of each of K_5 and $K_{3,3}$. These graphs are referred to as the Kuratowski graphs. From Robertson, Seymour, and Thomas's work, we know the following:

Lemma 2.5 [5] *The graphs K_5 and $K_{3,3}$ have exactly two nonambient isotopic flat embeddings.*

The two flat embeddings of K_5 and $K_{3,3}$ are shown in Figure 7. It can be seen that the two differ by the edge not in the maximal planar subgraph.

Lemma 2.6 [5] *If two flat embeddings of a graph G are not ambient isotopic, they must disagree on a K_5 or $K_{3,3}$ subdivision.*

We use the above to show the following simple lemma:

Lemma 2.7 *Let G be a K_5 or $K_{3,3}$ subdivision, and let ϕ_1 and ϕ_2 be the two not ambient isotopic flat embeddings of G. By taking ambient isotopy, ϕ_1 and ϕ_2 may be assumed as in Figure 7. Identify the labeled embeddings together along maximal planar subgraphs, a identified to a and so on, and e_j identified to e_j for the edges in the maximal planar subgraphs. Then the nonagreeing edges, $\phi_1(ax) \cup \phi_2(ax)$, and the cycle disjoint from those vertices form a nontrivial link.*

Proof Assume the embeddings of a K_5 or $K_{3,3}$ subdivision do not form a nontrivial link in the graph obtained by identification. Then the two paths not in the maximal planar graph must bound a union of disks disjoint from the graph. Then one path can be moved to the other through these disks, giving an ambient isotopy between the two distinct embeddings of K_5 or $K_{3,3}$, a contradiction. □

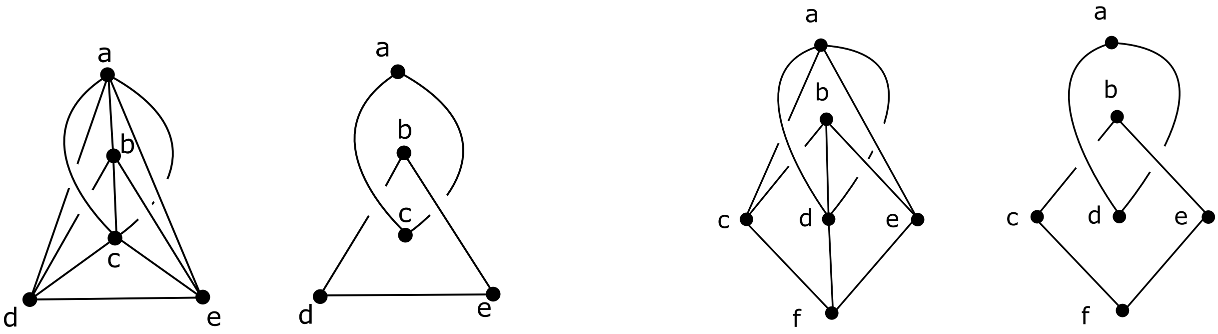

Figure 8: Linked cycles in the identification of the Kuratowski graphs. Left: K_5. Right $K_{3,3}$.

Proposition 2.8 *Let ϕ and ψ be flat embeddings of the simple flat graph G. If there is a B_{xy} such that it is valid in both ϕ and ψ, and the subembeddings $\phi - B_{xy}$ and $\psi - B_{xy}$ are ambient isotopic, then $\phi \cong \psi$.*

Proof By assumption, $\phi - B_{xy} \cong \psi - B_{xy}$. Then we may replace ψ by the embedding after this ambient isotopy, so outside of B_{xy} the embeddings are identically equal.

If ϕ and ψ are not ambient isotopic, they must disagree by some K_5 or $K_{3,3}$ subdivision by Lemma 2.6. Because the embeddings are identical outside of B_{xy}, without loss of generality, the disagreement must involve an edge incident to x. From Lemma 2.7, $\phi(xa) \cup \psi(xa)$ links with the other cycle in our subdivision. However, we know both $\phi(xa)$ and $\psi(xa)$ are contained within B_{xy}. Because B_{xy} is disjoint from the rest of the graph, the linking cycle must contain an edge from y, or the second link component would have to pierce the ball. This is a contradiction, since the only edges inside B_{xy} are those adjacent to x or y. This means the link must be of the form shown in Figure 9.

However, because D_{xy} separates the surface of B_{xy}, if the link as shown existed, then x and the vertex a connected to it are on opposite sides of D_{xy}. The edges from x to a must pass through D_{xy}, which we know does not happen. Then the cycle made with disagreeing edges incident to x from our two embeddings cannot link with a cycle containing an edge incident to y, and so there is no way that $\phi(xa) \cup \psi(xa)$ can link with anything in the graph. Therefore ϕ and ψ cannot differ by any K_5 or $K_{3,3}$ subdivisions, and so $\phi \cong \psi$. □

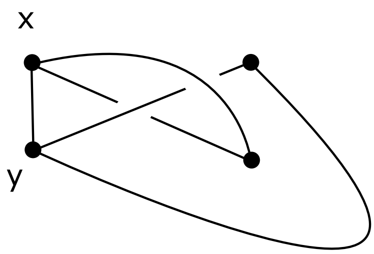

Figure 9: Linking forced by disagreeing embeddings.

2.2 Proof of main result

The following is our main result:

Theorem 2.9 *Every flat embedding of a linklessly embeddable graph G can be linearized.*

Proof We can restrict to connected graphs, since for disconnected graphs the components can be embedded separately. Graphs with up to four vertices are planar. By Fáry's theorem, these graphs have a straight-edge planar embedding. By Lemma 2.6, these graphs have a unique flat embedding. Because the planar embedding is flat, our straight-edge planar embedding gives a linear flat embedding. Assume the statement is true for all graphs with fewer than n vertices.

Let G be a linklessly embeddable graph with n vertices. Let ϕ be a flat embedding of G. Pick an edge $xy \in G$, and construct B_{xy} and D_{xy} in ϕ as described previously. Contract the edge xy so that the new vertex v lies in D_{xy}. This can be done so that the two components of $B_{xy} - D_{xy}$ still completely contain all of the edges they contained before, ie one component containing all edges to x, and the other containing all edges to y. For the vertices which have two edges to v we can assume the edges of the underlying simple graph are within the disk D_{xy}.

Then ϕ/xy is a flat embedding of a graph of order $n-1$, and so the underlying simple graph is linearizable by induction. By induction, there is an ambient isotopy $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $F_0(\phi/xy) = \phi/xy$ and $F_1(\phi/xy)$ is a linear embedding of the underlying simple graph. Because ambient isotopy is continuous, we can move the double edges as close to the linear edges in the disk as we wish.

Because $\phi \cong F_1(\phi)$, we replace ϕ with $F_1(\phi)$, which is linear everywhere except inside B_{xy}. In $\mathbb{R}^3$, an ambient isotopy is an orientation-preserving homeomorphism. This means that $F_1(B_{xy})$ still contains all the edges incident to v. Moreover, the edges incident to x and to y are still separated by $F_1(D_{xy})$, since a single component of $B_{xy} - D_{xy}$ unioned with D_{xy} forms a closed ball. For this reason, we still refer to the images of the original ball and disk under the ambient isotopy F_1 by B_{xy} and D_{xy}.

Let ψ be a linear embedding of G such that $\psi(G - \{x, y\}) = (\phi/xy)(G - v)$, and x and y are placed very close to $(\phi/xy)(v)$ on the appropriate side of D_{xy}. This can be linear because x and y being close to v ensures all the edges are close enough to the straight edges in ϕ/xy.

We proceed to show ψ is flat. Recall Proposition 1.4.

By construction, $\psi/xy = \phi/xy$, and is therefore flat. We show now that $\psi \backslash xy$ is flat.

Consider a cycle $C \in \psi \backslash xy$. If C does not contain either x or y, then it corresponds exactly with a cycle in ϕ/xy. By Proposition 2.4, there exists a disk Γ bounded by C, disjoint from $\phi/xy \backslash C$, and disjoint from D_{xy}. Then Γ does not interact with the neighborhood around D_{xy}, which is the only change between the embeddings. Then the same disk Γ from ϕ/xy will also bound C in $\psi \backslash xy$, and is disjoint from the rest of $\psi \backslash xy$.

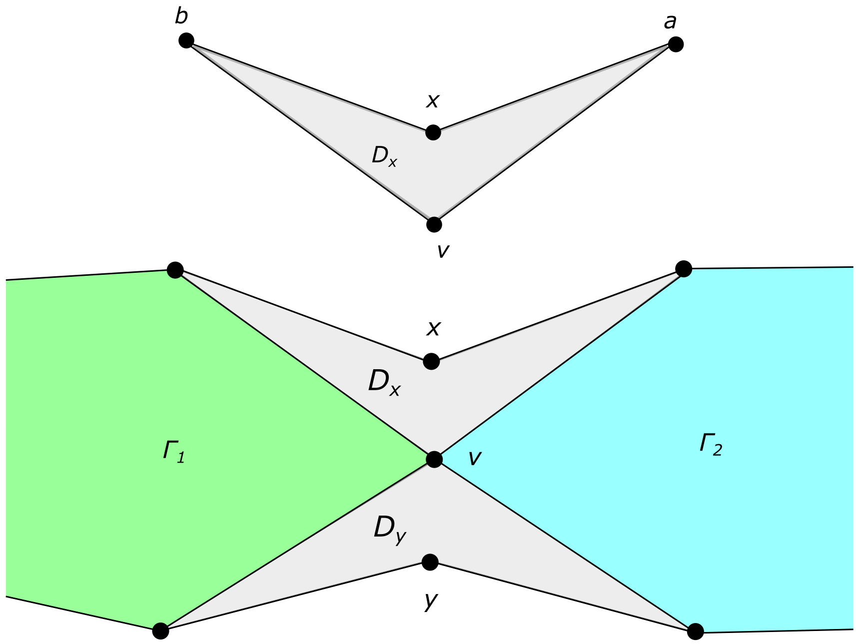

Figure 10: A cycle C in $\phi\backslash xy$ containing both x and y corresponds to two smaller cycles, C_1 and C_2 in ϕ/xy. The top part shows a disk bounded by $xavb$. This is disjoint from $\phi\backslash xy$ by construction. C_1 and C_2 bound disks disjoint from ϕ/xy, Γ_1 and Γ_2. These disks can be extended as shown in the bottom part through disks D_x and D_y to a disk disjoint from $\phi\backslash xy$

If C contains only one of x and y, then C corresponds to some cycle C' in ϕ/xy not crossing D_{xy}, with the x or y replaced by v. Without loss of generality, assume $x \in C$, and ax and xb are part of C. By Proposition 2.4, C' bounds a disk Γ' disjoint from ϕ/xy which has no interior intersection with D_{xy}. Then in $\psi\backslash xy$, Γ' does not intersect any edges incident to y, since they are separated from Γ' by the bounding disk D_{xy}. Because x is so close to the position of v, Γ' is also disjoint from any edges incident to x, since it was disjoint from the edges of v. We can extend Γ' from its edges avb to axb through a disk D bounded by $avbxa$. This is disjoint from the graph by construction, since x is very close to the position of v, and because $xy \notin \psi\backslash xy$. Then after extending Γ' along D to form a disk Γ, C bounds a disk Γ disjoint from the rest of $\psi\backslash xy$.

If C contains both x and y, then C corresponds to two cycles C_1 and C_2 in ϕ/xy meeting at v. By Lemma 2.1, as these cycles have connected intersection, there exist disks Γ_1 and Γ_2 bounded by C_1 and C_2 with interior disjoint from ϕ/xy, such that $\Gamma_1 \cap \Gamma_2 = v$. Then we can mold the edges of $C_1 \cup C_2$ to the edges in C in a similar process to before. These sections are all disjoint from the graph and each other. The union $\Gamma_1 \cup \Gamma_2$ can be extended through D_x and D_y, forming a disk Γ bounded by C with interior disjoint from $\phi\backslash xy$; see Figure 10. Then $\psi\backslash xy$ is flat. By Proposition 1.4, ψ is a flat linear embedding of G.

By Proposition 2.8, ϕ and ψ are exactly the same outside of D_{xy}, and are both flat. Then $\phi \cong \psi$. □

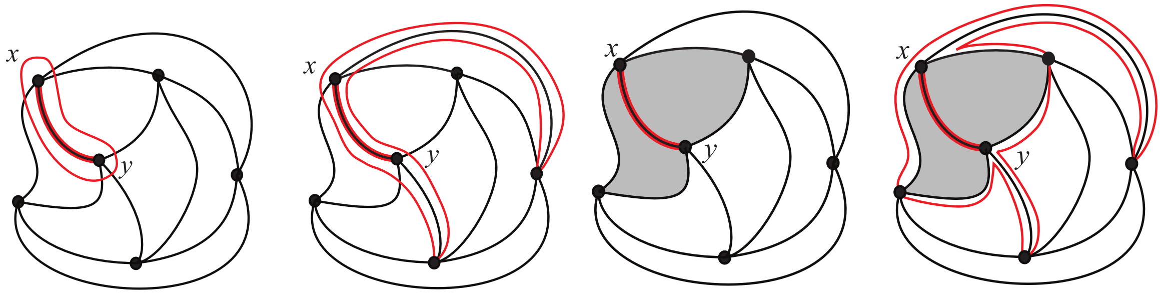

Figure 11: The construction for a planar graph on six vertices, part 1.

3 Examples

3.1 Example 1: planar graph

In Figures 11 and 12, we present an example of our construction. We start with a planar graph on six vertices. Around the highlighted edge xy, we form B_{xy}, whose intersection with the plane is drawn in red. The disks bounded by x, y, and their common neighbors are both contained in the plane, and shaded in gray. Figure 12, center left, shows the contraction to the midpoint of xy, forming the new vertex v. In Figure 12, center right, we have linearized the contraction so that the straight edges from v to the common vertices are the dashed paths in Figure 12, far and center left. Then Figure 12, far right, shows a linear flat embedding of our graph constructed by placing x and y on opposite sides of our separating disk.

3.2 The graph $Q_{13,3}$

Figures 14–16 show another full example of our process for the graph $Q_{13,3}$, shown in Figure 13. This graph has 13 vertices, where every vertex is connected in a sequential cycle. Then, every vertex is connected to the two vertices distance 3 away within the cycle of length 13 — 4 connects to 1 and 7, 2 connects to 12 and 5, etc. Note this graph is triangle-free, so $M(e) = \varnothing$ for all e. In Figure 14, we have a flat embedding.

This embedding is shown to be flat as follows. The graph $Q_{13,3}$ has a maximal planar subgraph with the removal of the edges (1, 11), (11, 12), (9, 12), and (10, 13). By Lemma 2.6, planar graphs have unique

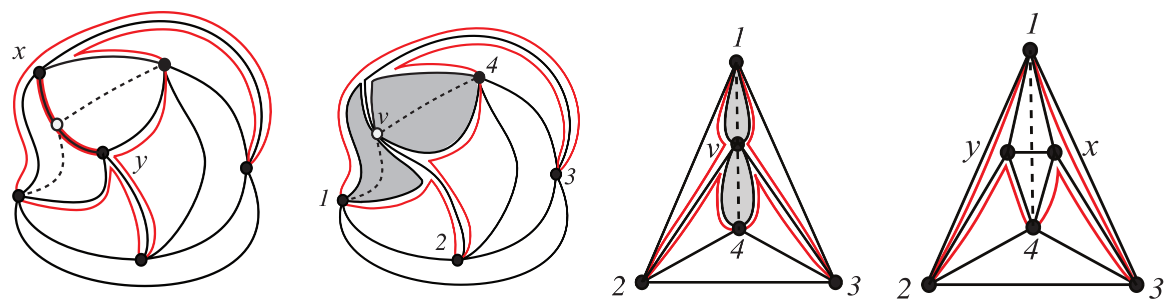

Figure 12: The construction for a planar graph on six vertices, part 2.

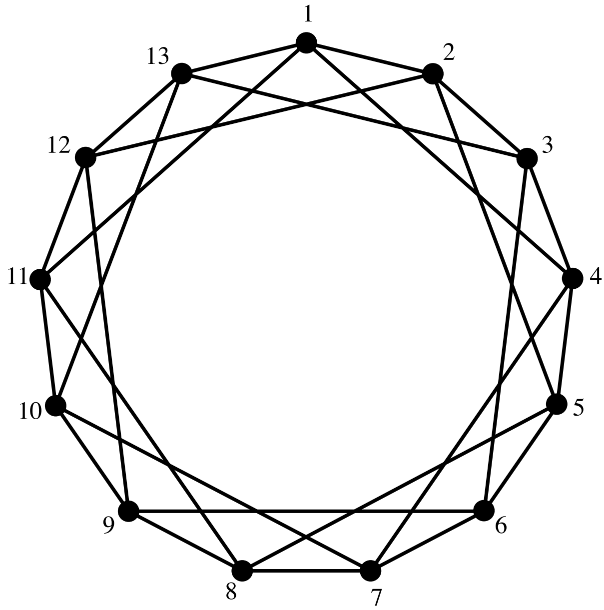

Figure 13: The maximal linklessly embeddable graph $Q_{13,3}$.

flat embeddings, since they do not have any K_5 or $K_{3,3}$ subdivisions. With some simple manipulations, it can be shown that this subgraph in Figure 14 is ambient isotopic to a planar embedding. If we embed one of our removed edges above this embedded plane and another below, we always create a nontrivial link. Then our flat embedding must have all four removed edges embedded above the plane or all four below the plane. In Figure 14, the edges are all embedded above.

We work by contracting the edge $e = (8, 9)$. Figure 15 shows the surface of B_e in green, and D_e in red. Because vertices 8 and 9 have no common neighbors, D_e is just a disk transverse to e placed at its

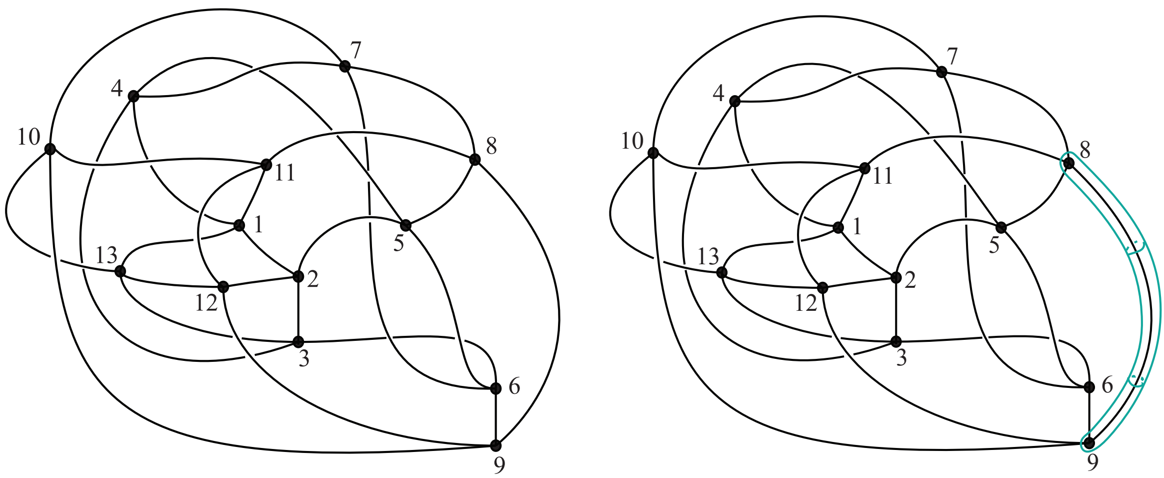

Figure 14: The construction for the graph $Q_{13,3}$, part 1.

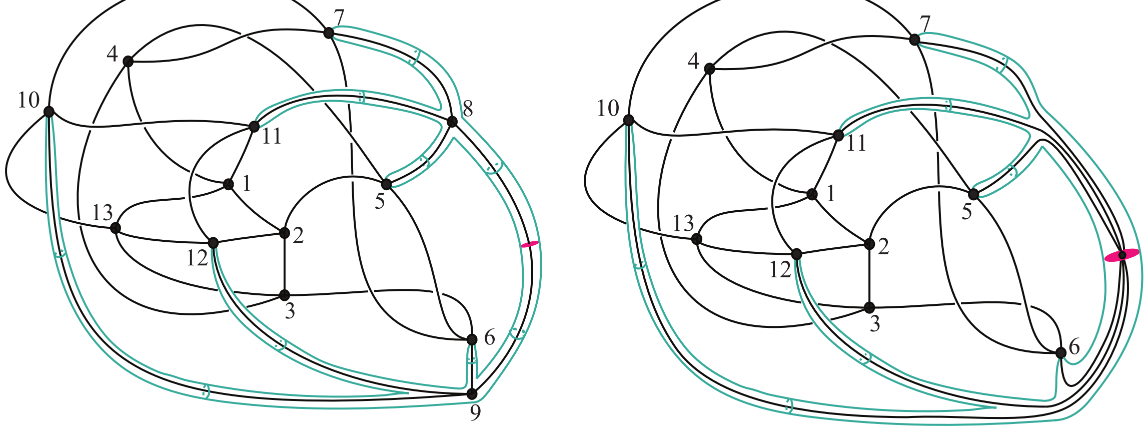

Figure 15: The construction for the graph $Q_{13,3}$, part 2.

midpoint. In Figure 15, we have performed the topological contraction of e to a vertex $v \in D_e$. Again, vertices 8 and 9 have no common neighbors in the graph, so ϕ/e is exactly an embedding of a simple graph. By assumption, there is an ambient isotopy of this embedding which produces a linear embedding. This linear embedding is shown in Figure 16.

We can see here that D_e still separates the neighbors of 9, $\{6, 10, 12\}$, from the neighbors of 8, $\{5, 7, 11\}$. Then in Figure 16, we have placed 8 and 9 on the appropriate sides of D_e, close to the location of v. By our proof, this is a linear flat embedding of $Q_{13,3}$ which is ambient isotopic to the original embedding.

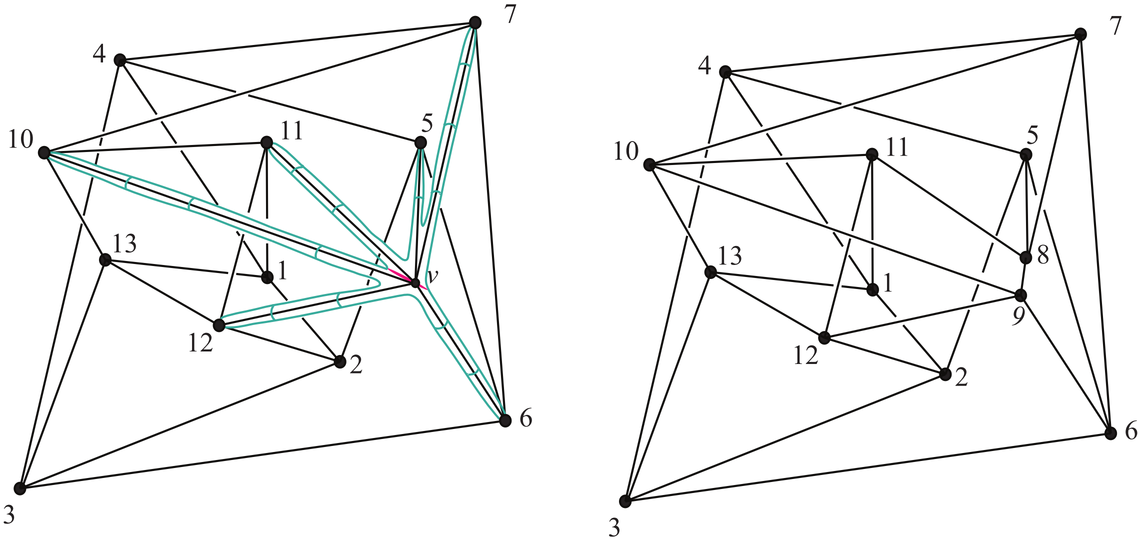

Figure 16: The construction for the graph $Q_{13,3}$, part 3.

4 Conclusion

We proved Sachs' 1983 conjecture on linklessly embeddable graphs, showing they all admit linear linkless embeddings. Further, we showed every flat embedding of a simple graph can be linearized. An obvious next class of embeddings to study is unknotted embeddings of nonintrinsically knotted graphs.

It has been shown by Hughes [4] that not every unknotted embedding of a knotlessly embeddable graph can be realized using straight edges. K_6 is an unknotted graph, and has unknotted embeddings with anywhere from 1 to 10 links. Hughes has shown there are only two nonambient isotopic knotless linear embeddings of K_6, which have 1 and 3 links respectively. Therefore not every knotless embedding can be linear.

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