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Homological stability for the ribbon Higman–Thompson groups

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We generalize the notion of asymptotic mapping class groups and allow them to surject to the Higman–Thompson groups, answering a question of Aramayona and Vlamis in the case of the Higman–Thompson groups. When the underlying surface is a disk, these new asymptotic mapping class groups can be identified with the ribbon and oriented ribbon Higman–Thompson groups. We use this model to prove that the ribbon Higman–Thompson groups satisfy homological stability, providing the first homological stability result for dense subgroups of big mapping class groups. Our result can also be treated as an extension of Szymik and Wahl's work on homological stability for the Higman–Thompson groups to the surface setting.

19D23, 20F36, 20J05, 57M07

Introduction

The family of Thompson's groups and the many groups in the extended Thompson family have long been studied for their many interesting properties. Thompson's group F is the first example of a type F_∞, torsion-free group with infinite cohomological dimension [9], while Thompson's groups T and V provided the first examples of finitely presented simple groups with infinitely many elements. More recently the braided and labeled braided Higman–Thompson groups have garnered attention in part due to their connections with big mapping class groups [8; 11; 4; 28]. In particular, Thumann constructed the ribbon version of Thompson's group V and proved that it is of type F_∞ [30]. We studied the ribbon Higman–Thompson groups $RV_{d,r}$ and their oriented version $RV_{d,r}^+$ in [28]. In fact, we identified them with the so-called labeled braided Higman–Thompson groups and proved that they are all of type F_∞.

The homology of Thompson's groups has also been well-studied. Brown and Geoghegan computed the homology of F in [9]; Ghys and Sergiescu calculated the homology of T in [18]. More recently Szymik and Wahl showed that V is acyclic [29], answering a question due to Brown [7]. One of the key ingredients for their proof was showing that the Higman–Thompson groups $V_{d,1} \hookrightarrow V_{d,2} \hookrightarrow \cdots \hookrightarrow V_{d,r} \hookrightarrow \cdots$ satisfy homological stability for any fixed d. Recall that a family of groups $G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \cdots$ is said to satisfy homological stability if the induced maps $H_i(G_n) \to H_i(G_{n+1})$ are isomorphisms for sufficiently large n. Classical examples of families of groups which satisfy homological stability include symmetric groups [25], general linear groups [24] and mapping class groups of surfaces [19].

Here we extend Szymik and Wahl's work to the class of ribbon Higman–Thompson groups. To accomplish this, we first build a geometric model for the ribbon Higman–Thompson groups using Funar–Kapoudjian's asymptotic mapping class groups [13]. These groups are defined using a rigid structure on a surface minus a Cantor set and they sit naturally inside the ambient big mapping class groups. More recently, Aramayona and Funar [2] generalized the definition to surfaces with nonzero genus. In fact, Aramayona and Funar showed that the half-twist version of their asymptotic mapping class group (see Definition 3.15) is dense in the big mapping class group [2, Theorem 1.3]. Another surprising result of Funar and Neretin says that the half-twist asymptotic mapping class group of a closed surface minus a standard ternary Cantor set is in fact isomorphic to its smooth mapping class group [15, Corollary 2]. Aramayona and Vlamis [3, Question 5.37] asked:

Question *Are there other geometrically defined subgroups of* $\mathrm{Map}(\Sigma_g)$ *which surject to other interesting classes of subgroups of homeomorphism group of the Cantor set, such as the Higman–Thompson groups, Neretin groups, etc?*

We proceed to construct two new classes of asymptotic mapping class groups, one of which answers their question in the case of Higman–Thompson groups while the other family surjects to the symmetric Higman–Thompson groups $V_{d,r}(\mathbb{Z}/2\mathbb{Z})$.

Proposition 3.18 and Theorem 3.20 *Let* Σ *be any compact surface and* $\mathscr{C}$ *be a Cantor set which lies in the interior of a disk in* Σ. *Then the mapping class group* $\mathrm{Map}(\Sigma \setminus \mathscr{C})$ *contains the following two families of dense subgroups: the asymptotic mapping class groups* $\mathscr{B}V_{d,r}(\Sigma)$, *which surject to the Higman–Thompson groups* $V_{d,r}$; *and the half-twist asymptotic mapping class groups* $\mathscr{H}V_{d,r}(\Sigma)$, *which surject to the symmetric Higman–Thompson groups* $V_{d,r}(\mathbb{Z}/2\mathbb{Z})$.

When Σ is the disk, we identify $\mathscr{H}V_{d,r}(\Sigma)$ with the ribbon Higman–Thompson group $RV_{d,r}$ and $\mathscr{B}V_{d,r}(\Sigma)$ with the oriented ribbon Higman–Thompson group $RV^+_{d,r}$ (see Theorem 3.24). Using this geometric model for the ribbon Higman–Thompson groups, we are able to prove the following.

Theorems 4.31 and 4.32 *Suppose* $d \geq 2$. *Then the inclusion maps induce isomorphisms*

$$\iota_{R,d,r}\colon H_i(RV_{d,r}, M) \to H_i(RV_{d,r+1}, M)$$

in homology in all dimensions $i \geq 0$, *for all* $r \geq 1$ *and for all* $H_1(RV_{d,\infty})$*-modules* M. *The same also holds for the oriented ribbon Higman–Thompson groups* $RV^+_{d,r}$.

Remark (1) Here we restrict our main result to the constant coefficient $\mathbb{Z}$ case. Nevertheless, the theorem also holds for some general coefficients by applying [27, Theorem A].

(2) The same method here can also be used to prove that the groups $\mathscr{B}V_{d,r}(\Sigma)$ and $\mathscr{H}V_{d,r}(\Sigma)$ satisfy homological stability. Still, it seems difficult to make it work directly for braided Higman–Thompson groups as we are lacking a good geometric model for them. Ideally, we would realize the braided

Higman–Thompson groups as some sort of mapping class groups of the disk minus a Cantor set. In fact, since the braided Higman–Thompson groups are subgroups of the oriented ribbon Higman–Thompson groups, we already have a geometric model in some sense. But it is less clear how one can tell when an element of the asymptotic mapping class group lies in the braided Higman–Thompson groups.

To the best of our knowledge, this is the first homological stability result for dense subgroups of big mapping class groups, although density will not play a role in our proof. Our proof uses a recent convenient framework given by Randal-Williams and Wahl [27]. The core of the proof is similar to [29], but with new technical difficulties arising from infinite-type surface topology. In particular, we take advantage of what we call the "mutual link trick", which we abstract from [10] and which we expect to be useful in a number of settings. We hope our result here can be further used to calculate the homology of ribbon Higman–Thompson groups and shed light on the question of whether braided V is acyclic. In fact, our Proposition 4.27 has already been used in [26] to prove that the mapping class groups of the disk minus a Cantor set is acyclic. It is also worth mentioning that the homology of the infinite genus version ribbon Thompson group has been calculated rationally in [14, Theorem 1.2] and integrally in [1, Theorem 1.16]. It in fact has the same homology as the stable homology of mapping class groups.

Outline of paper

In Section 1, we describe the connectivity tools that will be necessary for the remainder of the paper. In Section 2, we introduce the definition of the Higman–Thompson, ribbon Higman–Thompson, and oriented ribbon Higman–Thompson groups using paired forest diagrams to define the elements. In Section 3, we generalize the notion of asymptotic mapping class groups and allow them to surject to the Higman–Thompson groups. And finally, in Section 4, we prove homological stability for the ribbon Higman–Thompson groups and their oriented version.

Notation and conventions

All surfaces in this paper are assumed to be connected and orientable unless otherwise stated. Given a simplicial complex X and a cell $\sigma \in X$, we denote the link of σ in X by $\mathrm{Lk}_X(\sigma)$ and the star of σ by $\mathrm{St}_X(\sigma)$. When the situation is clear, we quite often omit X and simply denote the link by $\mathrm{Lk}(\sigma)$ and the star by $\mathrm{St}(\sigma)$. Recall that X is called n-connected if its homotopy groups are trivial up to dimension n. We also use the convention that (-1)-connected means nonempty and that every space is (-2)-connected. In particular, the empty set is (-2)-connected. Finally, we adopt the convention that elements in groups are multiplied from left to right.

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1 Connectivity tools

In this section, we review some of the connectivity tools that we need for calculating the connectivity of our spaces. A good reference is [21, Section 2], although not all the tools we use can be found there.

1.1 Complete join

The complete join is useful tool introduced by Hatcher and Wahl [22, Section 3] for proving connectivity results. We review the basics here.

Definition 1.1 A surjective simplicial map $\pi : Y \to X$ is called a *complete join* if it satisfies the following properties:

(1) π is injective on individual simplices.
(2) For each p-simplex $\sigma = \langle v_0, \dots, v_p \rangle$ of X, $\pi^{-1}(\sigma)$ is the join $\pi^{-1}(v_0) * \pi^{-1}(v_1) * \cdots * \pi^{-1}(v_p)$.

Definition 1.2 A simplicial complex X is called weakly Cohen–Macaulay of dimension n if X is $(n-1)$-connected and the link of each p-simplex of X is $(n-p-2)$-connected.

The main result regarding complete join that we will use is the following.

Proposition 1.3 [22, Propostion 3.5] *If Y is a complete join complex over a weakly Cohen–Macaulay complex X of dimension n, then Y is also weakly Cohen–Macaulay of dimension n.*

Remark 1.4 If $\pi: Y \to X$ is a complete join, then X is a retract of Y. In fact, we can define a simplicial map $s: X \to Y$ such that $\pi \circ s = \mathrm{id}_X$ by sending a vertex $v \in X$ to any vertex in $\pi^{-1}(v)$ and then extending it to simplices. The fact that s can be extended to simplices is granted by the condition that π is a complete join. In particular, we can also conclude that, if Y is n-connected, so is X.

1.2 Bad simplices argument

Let (X, Y) be a pair of simplicial complexes. We want to relate the n-connectedness of Y to the n-connectedness of X via a so-called bad simplices argument; see [21, Section 2.1] for more information. One identifies a set of simplices in $X \setminus Y$ as bad simplices, satisfying the following two conditions:

(i) Any simplex with no bad faces is in Y, where by a "face" of a simplex we mean a subcomplex spanned by any nonempty subset of its vertices, proper or not.

(ii) If two faces of a simplex are both bad, then their join is also bad.

We call simplices with no bad faces good simplices. Bad simplices may have good faces or faces which are neither good nor bad. If σ is a bad simplex, we say a simplex τ in $\mathrm{Lk}(\sigma)$ is good for σ if any bad face of $\tau * \sigma$ is contained in σ. The simplices which are good for σ form a subcomplex of $\mathrm{Lk}(\sigma)$, which we denote by GL_σ and call the good link of σ.

Proposition 1.5 [21, Proposition 2.1] *Let X, Y and GL_σ be as above. Suppose that, for some integer $n \geq 0$, the subcomplex GL_σ of X is $(n-\dim(\sigma)-1)$-connected for all bad simplices σ. Then the pair (X, Y) is n-connected, ie $\pi_i(X, Y) = 0$ for all $i \leq n$.*

We can apply the proposition in the following way.

Theorem 1.6 [21, Corollary 2.2] *Let Y be a subcomplex of a simplicial complex X and suppose the space $X \setminus Y$ has a set of bad simplices satisfying (i) and (ii) above; then:*

(1) *If X is n-connected and GL_σ is $(n-\dim(\sigma))$-connected for all bad simplices σ, then Y is n-connected.*

(2) *If Y is n-connected and GL_σ is $(n-\dim(\sigma)-1)$-connected for all bad simplices σ, then X is n-connected.*

1.3 The mutual link trick

In the proof of [10, Theorem 3.10], there is a beautiful argument for resolving intersections of arcs inspired by Hatcher's flow argument [20]. They attributed the idea to Andrew Putman. Recall Hatcher's flow argument allows one to "flow" a complex to its subcomplex. But in the process, one can only "flow" a vertex to a new one in its link. The mutual link trick will allow one to "flow" a vertex to a new one not in its link provided "the mutual link" is sufficiently connected.

To apply the mutual link trick, we first need a lemma that allows us to homotope a simplicial map to a simplexwise injective one [10, Lemma 3.9]. Recall a simplicial map is called *simplexwise injective* if its restriction to any simplex is injective. See also [16, Section 2.1] for more information.

Lemma 1.7 *Let Y be a compact m-dimensional combinatorial manifold. Let X be a simplicial complex and assume that the link of every p-simplex in X is $(m-p-2)$-connected. Let $\psi\colon Y \to X$ be a simplicial map whose restriction to ∂Y is simplexwise injective. Then, after possibly subdividing the simplicial structure of Y, ψ is homotopic relative ∂Y to a simplexwise injective map.*

Note that, as discussed in [17, Lemma 5.19], there is a mistake in the connectivity bound given in [10] that was corrected in an erratum.

Lemma 1.8 (the mutual link trick) *Let Y be a closed m-dimensional combinatorial manifold and $f\colon Y \to X$ be a simplexwise injective simplicial map. Let $y \in Y$ be a vertex and $f(y) = x$ for some $x \in X$. Suppose x' is another vertex of X satisfying the following conditions*:

(1) $f(\mathrm{Lk}_Y(y)) \le \mathrm{Lk}_X(x')$.

(2) *The mutual link* $\mathrm{Lk}_X(x) \cap \mathrm{Lk}_X(x')$ *is $(m-1)$-connected.*

Then we can define a new simplexwise injective map $g\colon Y \to X$ by sending y to x' and all the other vertices y' to $f(y')$ such that g is homotopic to f.

Proof The conditions that f is simplexwise injective and $f(\mathrm{Lk}_Y(y)) \le \mathrm{Lk}_X(x')$ guarantee that the definition of g can be extended over Y and g is again simplexwise injective.

We need to prove g is homotopic to f. The homotopy will be the identity outside $\mathrm{St}_Y(y)$. Note that, since f is simplexwise injective, $f(\mathrm{Lk}_Y(y)) \le \mathrm{Lk}_X(x)$. Together with condition (1), this gives $f(\mathrm{Lk}_Y(y)) \le \mathrm{Lk}_X(x) \cap \mathrm{Lk}_X(x')$. Since $\mathrm{Lk}_Y(y)$ is an $(m-1)$-sphere and $\mathrm{Lk}_X(x) \cap \mathrm{Lk}_X(x')$ is $(m-1)$-connected, there exists an m-disk B with $\partial B = \mathrm{Lk}_Y(y)$ and a simplicial map $\varphi\colon B \to \mathrm{Lk}_X(x) \cap \mathrm{Lk}_X(x')$ such that φ restricted to ∂B coincides with ψ restricted to $\mathrm{Lk}_Y(y)$. Since the image of B under φ is contained in $\mathrm{St}_X(x)$, which is contractible, we can homotope g, replacing $g|_{\mathrm{St}_Y(y)}$ with φ. Since the image of B under f is also contained in $\mathrm{Lk}_X(x')$, we can similarly homotope f, replacing $f|_{\mathrm{St}_Y(y)}$ with φ. These both yield the same map, so g is homotopic to f. □

2 Higman–Thompson groups and their braided versions

In this section, we first give an introduction to the Higman–Thompson groups and then define their ribbon version.

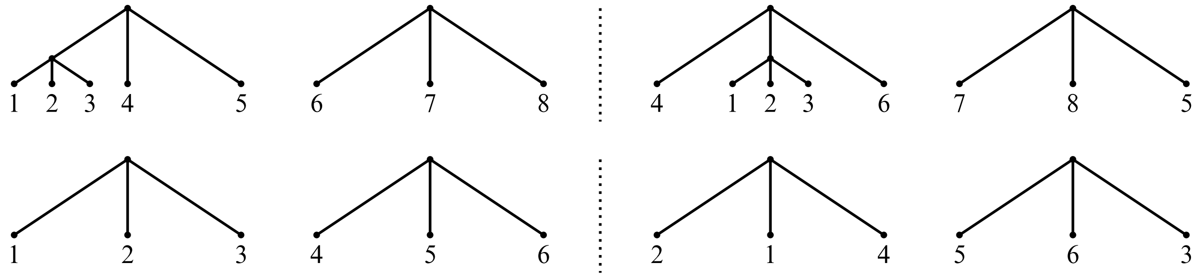

Figure 1: Reduction of the top paired $(3,2)$-forest diagram to the bottom one.

2.1 Higman–Thompson groups

The Higman–Thompson groups were first introduced by Higman as a generalization of the groups [23] given earlier in handwritten, unpublished notes of Richard Thompson. First let us recall the definition of the Higman–Thompson groups. Although there are a number of equivalent definitions of these groups, we will use the notion of paired forest diagrams. First we define a *finite rooted d-ary tree* to be a finite tree such that every vertex has degree $d+1$ except the *leaves*, which have degree 1, and the *root*, which has degree d (or degree 1 if the root is also a leaf). Usually we draw such trees with the root at the top and the nodes descending from it down to the leaves. A vertex v of the tree along with its d adjacent descendants will be called a *caret*. If the leaves of a caret in the tree are leaves of the tree, we will call the caret *elementary*. A collection of r d-ary trees will be called a *(d,r)-forest*. When d is clear from the context, we may just call it an r-forest.

Define a *paired (d,r)-forest diagram* to be a triple (F_-, ρ, F_+) consisting of two (d,r)-forests F_- and F_+ both with l leaves for some l, and a permutation $\rho \in S_l$, the symmetric group on l elements. We label the leaves of F_- with $1, \dots, l$ from left to right, and, for each i, the $\rho(i)^{\text{th}}$ leaf of F_+ is labeled i.

Define a *reduction* of a paired (d,r)-forest diagram to be the following: Suppose there is an elementary caret in F_- with leaves labeled by $i, \dots, i+d-1$ from left to right, and an elementary caret in F_+ with leaves labeled by $i, \dots, i+d-1$ from left to right. Then we can "reduce" the diagram by removing those carets, renumbering the leaves and replacing ρ with the permutation $\rho' \in S_{l-d+1}$ that sends the new leaf of F_- to the new leaf of F_+, and otherwise behaves like ρ. The resulting paired forest diagram (F'_-, ρ', F'_+) is then said to be obtained by *reducing* (F_-, ρ, F_+). See Figure 1 for an idea of reduction of paired $(3,2)$-forest diagrams. The reverse operation to reduction is called *expansion*, so (F_-, ρ, F_+) is an expansion of (F'_-, ρ', F'_+). A paired forest diagram is called *reduced* if there is no reduction possible. Define an equivalence relation on the set of paired (d,r)-forest diagrams by declaring two paired forest diagrams to be equivalent if one can be reached by the other through a finite series of reductions and expansions. Thus an equivalence class of paired forest diagrams consists of all diagrams having a common reduced representative. Such reduced representatives are unique.

There is a binary operation $*$ on the set of equivalence classes of paired (d,r)-forest diagrams. Let $\alpha = (F_-, \rho, F_+)$ and $\beta = (E_-, \xi, E_+)$ be reduced paired forest diagrams. By applying repeated expansions

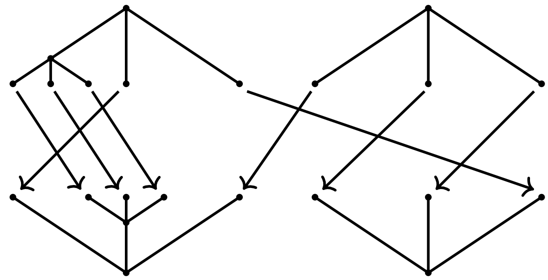

Figure 2: An element of $V_{3,2}$.

to α and β, we can find representatives (F'_-, ρ', F'_+) and (E'_-, ξ', E'_+) of the equivalence classes of α and β, respectively, such that $F'_+ = E'_-$. Then we declare $\alpha * \beta$ to be $(F'_-, \rho'\xi', E'_+)$. This operation is well defined on the equivalence classes and is a group operation.

Definition 2.1 The *Higman–Thompson group $V_{d,r}$* is the group of equivalence classes of paired (d,r)-forest diagrams with the multiplication $*$.

The usual Thompson group V is a special case of Higman–Thompson groups. In fact, $V = V_{2,1}$.

2.2 Ribbon Higman–Thompson groups

For convenience, we will think of the forest F_+ drawn beneath F_- and upside down, ie with the root at the bottom and the leaves at the top. The permutation ρ is then indicated by arrows pointing from the leaves of F_- to the corresponding paired leaves of F_+. See Figure 2 for this visualization of (the unreduced representation of) the element of $V_{3,2}$ from Figure 1.

Now, in the ribbon version of the Higman–Thompson groups, the permutations of leaves are simply replaced by ribbon braids which can twist between the leaves.

Definition 2.2 Let $\mathcal{I} = \coprod_{i=1}^{d} I_i : [0,1] \times \{1,\dots,l\} \to \mathbb{R}^2$ be an embedding, which we refer to as the *marked bands*. A *ribbon braid* is a map $R\colon ([0,1]\times\{0,1,\dots,l\})\times[0,1] \to \mathbb{R}^2$ such that, for any $0 \le t \le 1$, $R_t : [0,1]\times\{1,\dots,l\} \to \mathbb{R}^2$ is an embedding, $R_0 = \mathcal{I}$ and there exists $\sigma \in S_l$ such that $R_1(t)|_{I_i} = I_{\sigma(i)}(t)$ or $R_1(t)|_{I_i} = I_{\sigma(i)}(1-t)$. The usual product of paths defines a group structure on the set of ribbon braids up to homotopy among ribbon braids. This group, denoted by RB_l, does not depend on the choice of the marked bands and it is called the ribbon braid group with l bands. A ribbon braid is *pure* if σ is trivial and we define PRB_l to be the *pure ribbon braid group* with l bands. If we further assume $R_1(t)|_{I_i} = I_{\sigma(i)}(t)$, this subgroup is called the *oriented ribbon braid group* RB_l^+. Similarly, we have the *oriented pure ribbon braid group* PRB_l^+.

Remark 2.3 Note that $\mathrm{RB}_l \cong \mathbb{Z}^l \rtimes B_l$, where the action of the braid group B_l with l strings is induced by the symmetric group action on the coordinates of $\mathbb{Z}^l$. In particular, for the pure ribbon braid group

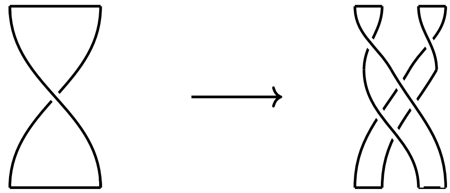

Figure 3: Splitting a ribbon into two ribbons.

PRB_l, $\mathrm{PRB}_l \cong \mathbb{Z}^l \times \mathrm{PB}_l$, where PB_l is the pure braid group with l strings. Under this isomorphism, $\mathrm{RB}_l^+ \cong (2\mathbb{Z})^l \rtimes B_l$ and $\mathrm{PRB}_l^+ \cong (2\mathbb{Z})^l \times \mathrm{PB}_l$.

Definition 2.4 A *ribbon braided paired (d,r)-forest diagram* is a triple $(F_-, \mathfrak{r}, F_+)$ consisting of two (d,r)-forests F_- and F_+ both with l leaves for some l and a ribbon braid $\mathfrak{r} \in \mathrm{RB}_l$ connecting the leaves of F_- to the leaves of F_+.

The expansion and reduction rules for the ribbon braids just come from the natural way of splitting a ribbon band into d components and the inverse operation to this. See Figure 3 for how to split a half twisted band when $d = 2$. Note that not only are the two bands themselves twisted, but the bands are also braided. Everything else will be the same as in the braided case, so we omit the details here. As usual, we define two ribbon braided paired forest diagrams to be equivalent if one is obtained from the other by a sequence of reductions or expansions. The multiplication operation $*$ on the equivalence classes is defined the same way as for $bV_{d,r}$. We direct the reader to [28, Section 2].

Definition 2.5 The *ribbon Higman–Thompson group $RV_{d,r}$* (resp. *oriented ribbon Higman–Thompson group $RV_{d,r}^+$*) is the group of equivalence classes of (resp. oriented) ribbon braided paired (d,r)-forests diagrams with the multiplication $*$.

3 Asymptotic mapping class groups related to the ribbon Higman–Thompson groups

The purpose of this section is to generalize the notion of asymptotic mapping class groups and allow them to surject to the Higman–Thompson groups. In particular, we will build a geometric model for the ribbon Higman–Thompson groups which will be crucial for proving homological stability in Section 4. Our construction is largely based on the ideas in [13, Section 2; 2, Section 3].

3.1 d-rigid structure

In this subsection, we generalize the notion of a rigid structure to that of a d-rigid structure.

Definition 3.1 A *d-leg pants* is a surface which is homeomorphic to a $(d+1)$-holed sphere.

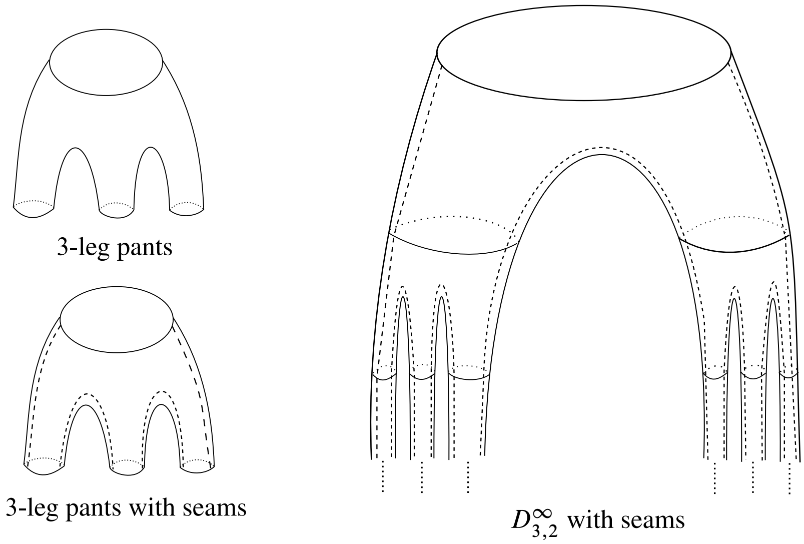

Figure 4: 3-leg pants and the surface $D^\infty_{3,2}$ with canonical seams.

Recall that the usual pair of pants is a 2-leg pants. We will draw a d-leg pants with one boundary component at the top. In this way, we can conveniently put a counterclockwise total order on the boundary components, making the top component the minimal one. See Figure 4 for an example of a 3-leg pants.

We proceed to build some infinite-type surfaces using some basic building blocks.

Definition 3.2 Let Σ be an compact oriented surface. Call the boundary components of Σ the *based boundary components*. Then $\Sigma^\infty_{d,r}$ is the infinite surface, built up as an inductive limit of infinite surfaces $\Sigma_{d,r,m}$ with $m \geq 0$:

(1) $\Sigma_{d,r,0}$ is obtained from Σ by deleting the interior of a disk in Σ. When Σ is a disk D, we declare $D_{d,r,0} = \partial D$.

(2) $\Sigma_{d,r,1}$ is obtained from $\Sigma_{d,r,0}$ attaching a copy of r-leg pants along the newly created boundary of $\Sigma_{d,r,0}$.

(3) For $m \geq 1$, $\Sigma_{d,r,m+1}$ is obtained from $\Sigma_{d,r,m}$ by gluing a pair of d-leg pants to every nonbased boundary circle of $\Sigma_{d,r,m}$ along the top boundary of the pants.

The surface $\Sigma_{d,r,1}$ is called the *base* of $\Sigma^\infty_{d,r}$ and the boundary components of $\Sigma^\infty_{d,r}$ coming from the base are the *based boundary components*. For each $m \geq 1$, the nonbased boundary components of $\Sigma_{d,r,m}$ naturally embed in $\Sigma^\infty_{d,r}$, and we call these the *admissible loops*. We call the admissible loops coming from $\Sigma_{d,r,1}$ the *rooted loops*. The surface $\Sigma^\infty_{d,r}$ has a natural induced orientation.

Remark 3.3 The two indices d, r in the definition of $\Sigma^\infty_{d,r}$ will be used later to define the Higman–Thompson version of the asymptotic mapping class group (see Definition 3.15), where d is related to the valence of the rooted trees and r is the number of roots in the definition of the Higman–Thompson groups.

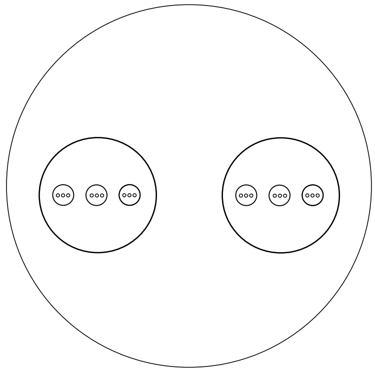

Figure 5: Disk model for the surface $D^{\infty}_{3,2}$.

Remark 3.4 To define our d-rigid structure, we do not really need $\Sigma_{d,r,0}$. But it will be convenient to have $\Sigma_{d,r,0}$ later, in Definition 3.17, for defining the map from $\Sigma^{\infty}_{d,r}$ to the tree $\mathcal{T}_{d,r}$.

Definition 3.5 A compact subsurface $A \subset \Sigma^{\infty}_{d,r}$ is *admissible* if $\Sigma_{d,r,1} \subseteq A$ and all of its nonbased boundaries are admissible. The subsurfaces $\Sigma_{d,r,m}$ are called the *standard admissible subsurfaces* of $\Sigma^{\infty}_{d,r}$.

Remark 3.6 In the special case where the starting surface is a disk, we will use the notation $\Sigma = D$, $\Sigma_{d,r,m} = D_{d,r,m}$, and $\Sigma^{\infty}_{d,r} = D^{\infty}_{d,r}$. See Figure 4 for a picture of the surface $D^{\infty}_{3,2}$. In this case, we can think of $D^{\infty}_{d,r}$ as a subsurface of a disk D. More specifically, let $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$ and $x_i = (2i - r - 1)/(r+1)$ for $1 \leq i \leq r$. We place r disks with center at each $(x_i, 0)$ of radius $r_0 = 1/(4(r+1))$. Denote these disks by $D_1, \dots, D_r$. The complement of the interior of these r disks in D is homeomorphic to the r-leg pants $D_{d,r,1}$. Now, for each disk D_i, $1 \leq i \leq r$, we can equally distribute d points in the x-axis inside D_i and place a disk with radius r_0/d^2 centered at each. The complements of the interiors of these d disks in D_i are all d-leg pants. We can continue the process inductively. At the end, the disks converge to a Cantor set which we denote by $\mathscr{C}$. In particular $D^{\infty}_{d,r}$ is homeomorphic to $D \setminus \mathscr{C}$. We will refer to this as the *puncture model* for $D^{\infty}_{d,r}$. See Figure 5 for a picture of $D^{\infty}_{3,2}$ with this model. The advantage of this model is we can view $D^{\infty}_{d,r}$ and all its admissible subsurfaces directly as subsurfaces of D.

Remark 3.7 Now $\Sigma^{\infty}_{d,r}$ can be obtained from Σ by attaching a copy of $D^{\infty}_{d,r}$ to the nonbased boundary component of $\Sigma_{d,r,0}$. In particular, $\Sigma^{\infty}_{d,r}$ is obtained from Σ by deleting a copy of the Cantor set, and any admissible subsurface of $\Sigma^{\infty}_{d,r}$ can be viewed directly as a subsurface of Σ using the puncture model. Recall that any two Cantor sets are homeomorphic; hence, by the classification of infinite surfaces [3, Theorem 2.2], $\Sigma^{\infty}_{d,r}$ is homeomorphic to $\Sigma \setminus \mathscr{C}$, where $\mathscr{C}$ is the standard ternary Cantor set sitting inside some disk in Σ regardless of the choice of d and r.

Definition 3.8 (1) A *suited d-pants decomposition* of the infinite surface $\Sigma^{\infty}_{d,r}$ is a maximal collection of distinct nontrivial simple closed curves in the interior of $\Sigma^{\infty}_{d,r} \setminus \Sigma_{d,r,1}$ which are not isotopic to the boundary, pairwise disjoint and pairwise nonisotopic, with the additional property that the complementary regions in $\Sigma^{\infty}_{d,r} \setminus \Sigma_{d,r,1}$ are all d-leg pants.

(2) A *d-rigid structure* on $\Sigma^{\infty}_{d,r}$ consists of two pieces of data:

- a suited d-pants decomposition; and
- a *d-prerigid structure*, ie a countable collection of disjoint line segments embedded into $\Sigma^{\infty}_{d,r} \setminus \Sigma_{d,r,1}$ such that the complement of their union in each component of $\Sigma^{\infty}_{d,r} \setminus \Sigma_{d,r,1}$ has two connected components.

These pieces must be *compatible* in the following sense: firstly, the traces of the d-prerigid structure on each d-leg pants (ie the intersections with pants) are made up of $d+1$ connected components, called *seams*; secondly, each boundary component of the pants intersects with exactly two components of the seams at two distinct points; thirdly, the seams cut each pants into two components. Note that these conditions imply that each component is homeomorphic to a disk. One says then that the suited d-pants decomposition and the d-prerigid structure are *subordinate* to the d-rigid structure.

(3) By construction, $\Sigma^{\infty}_{d,r}$ is naturally equipped with a suited d-pants decomposition, which will be referred to below as the *canonical suited d-pants decomposition*. We also fix a d-prerigid structure on $\Sigma^{\infty}_{d,r}$ (called the *canonical d-prerigid structure*) compatible with the canonical suited d-pants decomposition. See Figure 4. Using the puncture model, the seams of the canonical d-prerigid structure are just the intersections of $[-1,1]\times\{0\}$ with each d-leg pants. The resulting d-rigid structure is called the *canonical d-rigid structure* on $\Sigma^{\infty}_{d,r}$. Very importantly, for each admissible subsurface, the canonical d-rigid structure induces an order on the admissible boundaries. In Figure 4, the induced order on the admissible loops are counterclockwise. Using the puncture model, the admissible loops are ordered from left to right.

(4) The seams cut each component of $\Sigma^{\infty}_{d,r} \setminus \Sigma_{d,r,1}$ into two pieces; we choose the front piece in each component, and these r pieces together form the *visible side* of $\Sigma^{\infty}_{d,r}$.

(5) A suited d-pants decomposition (resp. d-(pre)rigid structure) is *asymptotically trivial* if, outside a compact subsurface of $\Sigma^{\infty}_{d,r}$, it coincides with the canonical suited d-pants decomposition (resp. canonical d-(pre)rigid structure).

Remark 3.9 It is important that the seams cut each d-pants into two components and each component is homeomorphic to a disk, as the mapping class group of a disk is trivial.

Definition 3.10 Let $\Sigma^{\infty}_{d,r}$ and $\overline{\Sigma}^{\infty}_{d,r'}$ be two surfaces with d-rigid structure and let $\varphi\colon \Sigma^{\infty}_{d,r} \to \overline{\Sigma}^{\infty}_{d,r'}$ be a homeomorphism. One says that φ is *asymptotically rigid* if there exists an admissible subsurface $A \subset \Sigma^{\infty}_{d,r}$ such that

(1) $\varphi(A)$ is also admissible in $\overline{\Sigma}^{\infty}_{d,r'}$;

(2) $\varphi|_A$ maps the based boundaries to based boundaries, admissible loops to admissible loops; and

(3) the restriction of $\varphi\colon \Sigma^{\infty}_{d,r} \setminus A \to \overline{\Sigma}^{\infty}_{d,r'} \setminus \varphi(A)$ is *rigid*, meaning that it respects the traces of the canonical d-rigid structure, mapping the suited d-pants decomposition into the suited d-pants decomposition, the seams into the seams, and the visible side into the visible side.

If we drop the condition that φ should map the visible side into the visible side, φ is called *asymptotically quasirigid*. The surface A is called a *support* for φ.

Remark 3.11 We are not using the word "support" in the usual sense, as the map outside the support defined above might well not be the identity, but the map is uniquely determined up to isotopy by Remark 3.9.

Remark 3.12 In [13, Definition 2.3], they do not actually require that the support contain the base. This will not make a difference, as one can always enlarge the support so that it contains the base.

Remark 3.13 The surface $\Sigma^{\infty}_{d,r+d-1}$ can be identified with the surface $\Sigma^{\infty}_{d,r}$ such that $\Sigma_{d,r+d-1,m} = \Sigma_{d,r,m+1}$ for any $m \geq 1$, and the d-rigid structure of $\Sigma^{\infty}_{d,r}$ coincides with d-rigid on $\Sigma^{\infty}_{d,r+d-1}$ outside $\Sigma_{d,r,2}$. In this way, $\Sigma^{\infty}_{d,r}$ is asymptotically rigid homeomorphic to $\Sigma^{\infty}_{d,r+d-1}$ through the identity map.

Remark 3.14 Let Σ' be a subsurface of $\Sigma^{\infty}_{d,r}$ such that there exist an admissible subsurface A of $\Sigma^{\infty}_{d,r}$ satisfying:

(1) $A \cap \Sigma'$ is a compact surface.

(2) The boundaries of Σ' are disjoint from the admissible boundary components of A.

(3) If an admissible boundary component L of A is contained in Σ', then the punctured disk component of $\Sigma^{\infty}_{d,r}$ cutting along L is also contained in Σ'.

Then Σ' has a naturally induced d-rigid structure. In fact, we can take $A \cap \Sigma'$ to be the base surface and the d-rigid structure can simply be inherited from $\Sigma^{\infty}_{d,r}$. One, of course, can choose different A here, which may give different induced d-rigid structure, but it is unique up to asymptotically rigid homeomorphism.

3.2 Asymptotic mapping class groups surjecting to Higman–Thompson groups

Given a (possibly noncompact) surface Σ, recall the mapping class group of Σ is defined to be the group of isotopy classes of orientation preserving homeomorphisms of Σ that fixes $\partial\Sigma$ pointwise, ie

$$\mathrm{Map}(\Sigma) = \mathrm{Map}(\Sigma, \partial\Sigma) := \mathrm{Homeo}^{+}(\Sigma, \partial\Sigma)/\mathrm{Homeo}_0(\Sigma, \partial\Sigma).$$

With this, we can now define the asymptotic mapping class group and the half-twist asymptotic mapping class group.

Definition 3.15 The *asymptotic mapping class group* $\mathcal{B}V_{d,r}(\Sigma)$ (resp. *half-twist asymptotic mapping class group* $\mathcal{H}V_{d,r}(\Sigma)$) is the subgroup of $\mathrm{Map}(\Sigma^\infty_{d,r})$ consisting of isotopy classes of asymptotically rigid (resp. quasirigid) self-homeomorphisms of $\Sigma^\infty_{d,r}$. When Σ is the disk, we sometimes simply denote the group by $\mathcal{B}V_{d,r}$ (resp. $\mathcal{H}V_{d,r}$).

Definition 3.16 Let A be an admissible subsurface of $\Sigma^\infty_{d,r}$, and $\mathrm{Map}(A)$ be its mapping class group which fixes the each boundary component pointwise. Each inclusion $A \subseteq A'$ of admissible surfaces induces an injective embedding $j_{A,A'}\colon \mathrm{Map}(A) \to \mathrm{Map}(A')$. The collection forms a direct system, whose direct limit we call the *compactly supported pure mapping class group*, denoted by $\mathrm{PMap}_c(\Sigma^\infty_{d,r})$. The group $\mathrm{PMap}_c(\Sigma^\infty_{d,r})$ is naturally a subgroup of $\mathcal{B}V_{d,r}(\Sigma)$ and we denote the inclusion map by j.

Definition 3.17 Let $\mathcal{F}_{d,r}$ be the forest with r copies of a rooted d-ary tree and $\mathcal{T}_{d,r}$ be the rooted tree obtained from $\mathcal{F}_{d,r}$ by adding an extra vertex to $\mathcal{F}_{d,r}$ and r extra edges each connecting this vertex to a root of a tree in $\mathcal{F}_{d,r}$. There is a natural projection $q\colon \Sigma^\infty_{d,r} \to \mathcal{T}_{d,r}$, such that the pullback of the root is $\Sigma_{d,r,0}$ and the pullbacks of the midpoints of any edges are admissible loops.

Any element in $\mathcal{B}V_{d,r}(\Sigma)$ can be represented by an asymptotically rigid homeomorphism $\varphi\colon \Sigma^\infty_{d,r} \to \Sigma^\infty_{d,r}$. In particular, we have an admissible subsurface A of $\Sigma^\infty_{d,r}$ such that $\varphi|_A\colon (A, \partial_b A) \to (\varphi(A), \varphi(\partial_b A))$ is a homeomorphism. Let F_- be the smallest subforest of $\mathcal{F}_{d,r}$ which contains $q(A) \cap \mathcal{F}_{d,r}$, and F_+ be the smallest subforest of $\mathcal{F}_{d,r}$ which contains $q(\varphi(A)) \cap \mathcal{F}_{d,r}$. Note that F_- and F_+ have the same number of leaves and their leaves are in one-to-one correspondence with the admissible loops of A and $\varphi(A)$. Now let ρ be the map from leaves of F_- to F_+ induced by φ. Together, these define an element $[(F_-, \rho, F_+)] \in V_{d,r}$. We call this map π. One can show π is well defined. Similarly to [13, Proposition 2.4; 2, Propositions 4.2 and 4.6], we now have the following proposition.

Proposition 3.18 *We have the short exact sequences*

$$1 \to \mathrm{PMap}_c(\Sigma^\infty_{d,r}) \xrightarrow{j} \mathcal{B}V_{d,r}(\Sigma) \xrightarrow{\pi} V_{d,r} \to 1, \quad 1 \to \mathrm{PMap}_c(\Sigma^\infty_{d,r}) \xrightarrow{j} \mathcal{H}V_{d,r}(\Sigma) \xrightarrow{\pi} V_{d,r}(\mathbb{Z}/2\mathbb{Z}) \to 1.$$

Remark 3.19 Here, as in [2], $V_{d,r}(\mathbb{Z}/2\mathbb{Z})$ is the twisted version of the Higman–Thompson group where one allows flipping the subtree below every leaf. See for example [5] for more information.

Proof We will prove the proposition for $\mathcal{B}V_{d,r}(\Sigma)$. The other case is essentially the same. First we show the map π is surjective. Given any element $[(F_-, \rho, F_+)] \in V_{d,r}$, let T_- (resp. T_+) be the tree obtained from F_- (resp. F_+) by adding a single root on the top and r edges connecting to each root of the trees in F_- (resp. F_+). Furthermore, let T'_- (resp. T'_+) be the tree obtained from F_- (resp. F_+) by throwing away the leaves and the open half edge connecting to the leaves. Then let $A_- = q^{-1}(T'_+)$ and $A_+ = q^{-1}(T'_+)$, which are both admissible subsurfaces of $\Sigma^\infty_{d,r}$. Now one can produce a homeomorphism $\varphi_0\colon A_- \to A_+$ which is the identity on the based boundary and maps the admissible loops of A_- to the admissible loops of A_+ following the information from ρ, mapping the visible part to the visible part for each admissible loop. From here, we extend φ_0 to a map $\varphi\colon \Sigma^{\infty}_{d,r} \to \Sigma^{\infty}_{d,r}$ such that φ is an asymptotically rigid homeomorphism.

If an element $g \in \mathcal{B}V_{d,r}(\Sigma)$ is mapped to a trivial element $\pi(g) = [(F_-, \rho, F_+)] \in V_{d,r}$, then the two forests F_- and F_+ are the same and the induced map ρ is trivial. This means we can assume the support A for the asymptotically rigid homeomorphism φ_g corresponding to g is the same as $\varphi(A)$, and φ induces identity map on the admissible boundary components. Thus $g \in \mathrm{PMap}_c(\Sigma^{\infty}_{d,r})$. Finally, given any element $g \in \mathrm{PMap}_c(\Sigma^{\infty}_{d,r})$, it is clear that $\pi \circ j(g) = 1$. □

The mapping class group $\mathrm{Map}(\Sigma^{\infty}_{d,r})$ has a natural quotient topology coming from the compact–open topology on $\mathrm{Homeo}^+(\Sigma^{\infty}_{d,r}, \partial\Sigma^{\infty}_{d,r})$. See [3, Sections 2.3 and 4.1] for more information. Aramayona and Funar [2, Theorem 1.3] showed that, when Σ is a closed surface, $\mathcal{H}V_{2,1}(\Sigma)$ is dense in $\mathrm{Map}(\Sigma^{\infty}_{2,1})$. We improve their result to the following.

Theorem 3.20 *The groups $\mathcal{B}V_{d,r}(\Sigma)$ and $\mathcal{H}V_{d,r}(\Sigma)$ are dense in the mapping class group* $\mathrm{Map}(\Sigma^{\infty}_{d,r})$.

Proof The proof in [2, Section 6] adapts directly to show that $\mathcal{H}V_{d,r}(\Sigma)$ is dense in $\mathrm{Map}(\Sigma^{\infty}_{d,r})$ and so we will not repeat it here. To show $\mathcal{B}V_{d,r}(\Sigma)$ is also dense in $\mathrm{Map}(\Sigma^{\infty}_{d,r})$, it suffices to show any element in $\mathcal{H}V_{d,r}(\Sigma)$ can be approximated by a sequence of elements in $\mathcal{B}V_{d,r}(\Sigma)$. Note first that any half-Dehn twists around admissible loops in $\Sigma^{\infty}_{d,r}$ lies in $\mathcal{H}V_{d,r}(\Sigma)$. In fact, given an admissible loop α, we can choose an admissible subsurface A such that α is an admissible loop of A. Then the half-Dehn twist around α is asymptotic quasirigid with support A; in fact, it is the identity on all the components of $\Sigma^{\infty}_{d,r} \setminus A$ except at the component containing α, where it rotates 180 degree. Now given an asymptotic quasirigid homeomorphism f of $\Sigma^{\infty}_{d,r}$ with support A', we can first compose f with half-Dehn twists around those admissible loops of A where f restricted to the component below them switches the front and back. The composition now is an asymptotic rigid homeomorphism. Thus $\mathcal{H}V_{d,r}(\Sigma)$ can be generated by $\mathcal{B}V_{d,r}(\Sigma)$ and half-Dehn twists around the admissible loops in $\Sigma^{\infty}_{d,r}$; it suffices now to show that any half-Dehn twists around admissible loops in $\Sigma^{\infty}_{d,r}$ can be approximated by a sequence of elements in $\mathcal{B}V_{d,r}(\Sigma)$. Given an admissible loop L, let h_L be a half-Dehn twist at L. We will construct a sequence of elements $x_i \in \mathcal{B}V_{d,r}(\Sigma)$ such that, for any compact subset K of $\Sigma^{\infty}_{d,r}$, there exists N such that, for any $j \geq N$, x_j and h_L coincide on K. Recall we have the map $q\colon \Sigma^{\infty}_{d,r} \to \mathcal{T}_{d,r}$ (see Definition 3.17) that maps the admissible loops to edge middle points in $\mathcal{T}_{d,r}$. Now consider those admissible loops whose images under q lying below $q(L)$ have distance i to $q(L)$. Note that there are d^i such admissible loops. We list them as $L_{i,1}, \dots, L_{i,d^i}$. Let $h_{L_{i,k}}$ be the half-Dehn twists around $L_{i,k}$ and let $x_i = h_L h_{L_{i,1}} \cdots h_{L_{i,d^i}}$; then $x_i \in \mathcal{B}V_{d,r}(\Sigma)$ and the sequence $\{x_i\}$ has the desired property. □

Now recall that, by Remark 3.7, $\Sigma^{\infty}_{d,r}$ is homeomorphic to $\Sigma \setminus \mathcal{C}$ for any d and r; hence, we have our first result stated in the introduction.

3.3 The asymptotic mapping class group of the disk punctured by the Cantor set

In this last subsection, we want to identify the asymptotic mapping class group $\mathcal{B}V_{d,r}(D)$ with the oriented ribbon Higman–Thompson groups $RV_{d,r}^{+}$ and the half-twist asymptotic mapping class group $\mathcal{H}V_{d,r}(D)$ with the ribbon Higman–Thompson group $RV_{d,r}$. The following lemma appears in [6, Section 2] without a proof, so we provide the details here. Note that what they call the pure ribbon braid group is the oriented pure ribbon braid group in Definition 2.2.

Lemma 3.21 *Let D_k be the $(k+1)$-holed sphere. Then $\mathrm{Map}(D_k)$ can be naturally identified with the pure oriented ribbon braid group PRB_k^{+}.*

Proof Note that D_k can be identified with a disk with k holes. Let ∂_b denote the boundary of the disk. Let $\overline{D}_k$ be a disk with k punctures obtained from D_k by attaching one punctured disk to each hole. The induced map $\mathcal{C}ap\colon \mathrm{Map}(D_k) \to \mathrm{PMap}(\overline{D}_k)$ is the capping homomorphism. Note that $\mathrm{PMap}(\overline{D}_k) \cong \mathrm{PB}_k$. Now, applying [12, Proposition 3.19] and the fact that the Dehn twists around the holes of D_k commute, one sees that the kernel K is a free abelian group of rank k generated by these k Dehn twists. Here the capping homomorphism splits. To prove this, we first embed PB_k into PRB_k^{+} by viewing the pure braid group of k strings as the set of ribbon braids on k bands that have no twists. We can think of D_k as being embedded into $\mathbb{R}^2$ with ∂_b as the unit circle and the k holes in D_k equally distributed inside ∂_b along the x-axis. The intersections of these holes with the x-axis gives k subintervals of the x-axis, denoted by $I_1, \dots, I_k$. We now put the bands representing a pure braid $x \in \mathrm{PB}_k \le \mathrm{PRB}_k^{+}$ in $D \times [0,1]$ which starts and ends at $I_1, \dots, I_k$. Note that the bands here will not twist at all. Now we comb the bands straight from bottom to top. This induces a homeomorphism of $D_k \times \{0\}$ and hence an element in the mapping class group $\mathrm{Map}(D_k)$. One checks that this map is a group homomorphism and injective. Since PB_k acts on K trivially, we have $\mathrm{Map}(D_k) \cong K \times \mathrm{PB}_k \cong \mathbb{Z}^k \times \mathrm{PB}_k \cong \mathrm{PRB}_k^{+}$, where the number of Dehn twists around each boundary component is naturally identified with the number of full twists on each bands. □

To promote Lemma 3.21 so that it works for the ribbon braid group, we need some extra terminology. As in the proof of Lemma 3.21, we identify D_k with the unit disk in $\mathbb{R}^2$ with k small disks removed whose centers are equally distributed on the x-axis. The x-axis cuts the boundary loops of each deleted disk into two components, providing a cell structure on the loops. We will call the part that lies above the x-axis the *visible part*. We define the *rigid mapping class group* $\mathrm{RMap}_{+}(D_k)$ of D_k to be the isotopy classes of homeomorphisms of D_k which fix $\partial_b D_k$ pointwise and map the visible part of the holes to the visible part of the holes. Note elements in $\partial_b D_k$ are allowed to map one boundary hole to another just as in the definition of the asymptotic mapping class group. If we only assume the cell structure on the loops has to be preserved, the resulting group is called *quasirigid mapping class group D_k* and denoted by $\mathrm{RMap}(D_k)$. With these preparations, the following lemma is now clear.

Lemma 3.22 *There is a natural isomorphism between the oriented ribbon braid group RB_k^{+} and $\mathrm{RMap}_{+}(D_k)$ (resp. between the ribbon braid group RB_k and $\mathrm{RMap}(D_k)$).*

Proof As in the proof of Lemma 3.21, we put the element in the (oriented) ribbon braid group between $D \times [0,1]$, then we comb the bands straight from bottom to top, which gives the corresponding element in $\mathrm{RMap}_+(D_k)$ (resp. $\mathrm{RMap}(D_k)$). □

Given two admissible subsurfaces A and A' of $D^\infty_{d,r}$ (possibly with different r) with k admissible boundary components, we want to fix a canonical way to identify a homeomorphism $f\colon A \to A'$ as an element in the ribbon braid group. Note that each boundary loop except the base one inherits a visible side from $D^\infty_{d,r}$. We will use the puncture model for $D^\infty_{d,r}$ going forward.

As above, let D_k be the subsurface of D which is the complement of k disjoint open disks with centers at $a_i = (2i-k-1)/(k+1)$ of radius 2^{-k} for $1 \le i \le k$. Now, given any admissible subsurface A_k of $D^\infty_{d,r}$ with k admissible boundaries, denote the centers from left to right by $c_i \in [0,1] \times \{0\}$ for $1 \le i \le k$ with corresponding radii $r_1, r_2, \dots, r_k$. Now we define an isotopy $\mathcal{N}_{A_k}\colon D \times [0,1] \to D$ such that $\mathcal{N}_{A_k,0} = \mathrm{id}_D$ and $\mathcal{N}_{A_k,1}$ maps A_k to D_k via a homeomorphism. We first shrink the admissible boundary loops of A_k so that they have radius r, where $r = \min\{r_1, \dots, r_k, 2^{-k}\}$. Then we isotope A_k by moving the centers c_i to a_i along $[0,1] \times \{0\}$ in D. And in the last step we enlarge the radii one by one to 2^{-k}. The following lemma is now immediate.

Lemma 3.23 *Let $\phi\colon D^\infty_{d,r} \to D^\infty_{d,r}$ be an asymptotically rigid (resp. quasirigid) homeomorphism which is supported on the admissible subsurface A_k. Write $A'_k = \phi(A_k)$; then:*

1. *$\mathfrak{r}_\phi = \mathcal{N}_{A'_k,1} \circ \phi|_{A_k} \circ \mathcal{N}^{-1}_{A_k,1}\colon D_k \to D_k$ gives an element in the oriented ribbon braid group RB^+_k (resp. the ribbon braid group RB_k). Conversely, given an element $\mathfrak{r} \in \mathrm{RB}^+_k$ (resp. RB_k), we have an asymptotic rigid (resp. quasirigid) homeomorphism which is unique up to isotopy, supported on A_k, and maps A_k to A'_k.*
2. *Let A_{k+d} be the admissible subsurface of $D^\infty_{d,r}$ obtained from A_k by adding a d-leg pants and $\phi(A_{k+d}) = A'_{k+d}$. Then the associated oriented ribbon braid (resp. the ribbon braid) of ϕ can be obtained from $\mathfrak{r}_\phi$ by splitting the corresponding band into d bands. Conversely, if we split one band of the ribbon braids into d bands, the isotopy class of the corresponding asymptotic rigid (resp. quasirigid) homeomorphism does not change.*

Note that, for any $d \ge 2$ and $r \ge 1$, we have a natural embedding $\iota_{d,r}\colon D^\infty_{d,r} \to D^\infty_{d,r+1}$ that maps the rooted boundaries of $D^\infty_{d,r}$ to the first r rooted boundaries according to the order. This induces embeddings of groups $i_{\mathcal{H},d,r}\colon \mathcal{H}V_{d,r} \to \mathcal{H}V_{d,r+1}$ and $i_{\mathcal{B},d,r}\colon \mathcal{B}V_{d,r} \to \mathcal{B}V_{d,r+1}$. On the other hand, we also have natural embeddings $i_{R,d,r}\colon RV_{d,r} \to RV_{d,r+1}$ and $i_{R+,d,r}\colon RV^+_{d,r} \to RV^+_{d,r+1}$ induced by inclusion of roots. We have the following.

Theorem 3.24 *There exist isomorphisms $f_{d,r}\colon \mathcal{H}V_{d,r} \to RV_{d,r}$ such that $f_{d,r+1} \circ i_{\mathcal{H},d,r} = i_{\mathcal{H},d,r+1} \circ f_{d,r}$. Restricting to the subgroups $\mathcal{B}V_{d,r}$, one gets isomorphisms $f_{d,r}\colon \mathcal{B}V_{d,r} \to RV^+_{d,r}$ with the same property.*

Proof Since the two cases are parallel, we will only prove the theorem for $\mathcal{B}V_{d,r}$. We will define two maps $f_{d,r}\colon \mathcal{B}V_{d,r} \to RV^+_{d,r}$ and $g_{d,r}\colon RV^+_{d,r} \to \mathcal{B}V_{d,r}$ such that $f_{d,r} \circ g_{d,r} = \mathrm{id}$ and $g_{d,r} \circ f_{d,r} = \mathrm{id}$.

Given an element $x \in \mathcal{B}V_{d,r}$, we can define $f_{d,r}$ as follows. Let φ_x be an asymptotically rigid homeomorphism of $D^\infty_{d,r}$ representing x with support A_k, where k is the number of admissible loops. By Proposition 3.18, $\pi(x)$ provides an element $[F_-, \sigma, F_+]$ in the Higman–Thompson group $V_{d,r}$, where F_- and F_+ are (d,r)-forests with k leaves. But what we want is a ribbon braid connecting the k leaves. For this we simply apply Lemma 3.23(1) to the map φ_x with support A_k; denote the corresponding element in RB^+_k by $\mathfrak{r}_{\varphi_x}$. We define $f_{d,r}(x) = [F_-, \mathfrak{r}_{\varphi_x}, F_+]$.

Now, given $y \in RV^+_{d,r}$, one can define an element in $\mathcal{B}V_{d,r}$ as follows. Suppose $(F_-, \mathfrak{r}_y, F_+)$ is a representative for y, where F_- and F_+ are (d,r)-forests and $\mathfrak{r}$ is a ribbon braid between the leaves of F_- and F_+. Add a root to F_- (resp. F_+) with an edge connecting to the root of each tree in F_- (resp. F_+) and then throw away the open half edge connecting to the leaves. Denote the resulting tree by T_- (resp. T_+). Now $q^{-1}(T_-)$ and $q^{-1}(T_+)$ give us two admissible subsurfaces A_k and A'_k in $D^\infty_{d,r}$, where k is the number of leaves for F_-. And, by Lemma 3.23(1), the ribbon element $\mathfrak{r}_y$ in RB^+_k gives us an asymptotic rigid homeomorphism ψ_y with support A_k and maps A_k to A'_k.

Now one can check that $f_{d,r} \circ g_{d,r} = \mathrm{id}$ and $g_{d,r} \circ f_{d,r} = \mathrm{id}$. Therefore, the two groups are isomorphic. The fact that the diagram commutes is immediate from the definition. □

4 Homological stability of ribbon Higman–Thompson groups

In this section, we show the homological stability for oriented ribbon Higman–Thompson groups and explain at the end how the same proof applies to the ribbon Higman–Thompson groups.

4.1 Homogeneous categories and homological stability

In this subsection, we review the basics of homogeneous categories and refer the reader to [27] for more details. Note that we adopt their convention of identifying objects of a category with their identity morphisms.

Definition 4.1 [27, Definition 1.3] A monoidal category $(\mathscr{C}, \oplus, 0)$ is called *homogeneous* if 0 is initial in $\mathscr{C}$ and if the following two properties hold:

(H1) $\mathrm{Hom}(A, B)$ is a transitive $\mathrm{Aut}(B)$-set under postcomposition.

(H2) The map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ taking f to $f \oplus \mathrm{id}_B$ is injective with image

$$\mathrm{Fix}(B) := \{\phi \in \mathrm{Aut}(A \oplus B) \mid \phi \circ (\iota_A \oplus \mathrm{id}_B) = \iota_A \oplus \mathrm{id}_B\},$$

where $\iota_A\colon 0 \to A$ is the unique map.

Definition 4.2 [27, Definition 1.5] Let $(\mathscr{C}, \oplus, 0)$ be a monoidal category with 0 initial. We say that $\mathscr{C}$ is *prebraided* if its underlying groupoid is braided and, for each pair of objects A and B in $\mathscr{C}$, the groupoid braiding $b_{A,B} \colon A \oplus B \to B \oplus A$ satisfies

$$b_{A,B} \circ (A \oplus \iota_B) = \iota_B \oplus A \colon A \to B \oplus A.$$

Definition 4.3 [27, Definition 2.1] Let $(\mathscr{C}, \oplus, 0)$ be a monoidal category with 0 initial and (A, X) a pair of objects in $\mathscr{C}$. Define $W_n(A, X)_\bullet$ to be the semisimplicial set with set of p-simplices

$$W_n(A, X)_p := \mathrm{Hom}_{\mathscr{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$$

and with face map

$$d_i \colon \mathrm{Hom}_{\mathscr{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n}) \to \mathrm{Hom}_{\mathscr{C}}(X^{\oplus p}, A \oplus X^{\oplus n})$$

defined by precomposing with $X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}$.

Also say the category $\mathscr{C}$ satisfies (LH3) at a pair of objects (A, X) with *slope* $k \geq 2$ if:

(LH3) For all $n \geq 1$, $W_n(A, X)_\bullet$ is $((n-2)/k)$-connected.

Quite often, we can reduce the semisimplicial complex to a simplicial complex.

Definition 4.4 [27, Definition 2.8] Let A, X be objects of a homogeneous category $(\mathscr{C}, \oplus, 0)$. For $n \geq 1$, let $S_n(A, X)$ denote the simplicial complex whose vertices are the maps $f \colon X \to A \oplus X^{\oplus n}$ and whose p-simplices are $p+1$ sets $\{f_0, \ldots, f_p\}$ such that there exists a morphism $f \colon X^{\oplus p+1} \to A \oplus X^{\oplus n}$ with $f \circ i_j = f_j$ for some order on the set, where

$$i_j = \iota_{X^{\oplus j}} \oplus \mathrm{id}_X \oplus \iota_{X^{\oplus p-j}} \colon X = 0 \oplus X \oplus 0 \to X^{\oplus p+1}.$$

Definition 4.5 Let $\mathrm{Aut}(A \oplus X^{\oplus \infty})$ be the colimit of

$$\cdots \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+1}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+2}) \xrightarrow{-\oplus X} \cdots.$$

Then any $\mathrm{Aut}(A \oplus X^{\oplus \infty})$-module M may be considered as an $\mathrm{Aut}(A \oplus X^{\oplus n})$-module for any n, by restriction, which we continue to call M. We say that the module M is abelian if the action of $\mathrm{Aut}(A \oplus X^{\oplus \infty})$ on M factors through the abelianizations of $\mathrm{Aut}(A \oplus X^{\oplus \infty})$, or in other words if the derived subgroup of $\mathrm{Aut}(A \oplus X^{\oplus \infty})$ acts trivially on M.

We are now ready to quote the theorem that we will use.

Theorem 4.6 [27, Theorem 3.4] *Let $(\mathscr{C}, \oplus, 0)$ be a prebraided homogeneous category satisfying* (LH3) *for a pair (A, X) with slope $k \geq 3$. Then, for any abelian $\mathrm{Aut}(A \oplus X^{\oplus \infty})$-module M, the map*

$$H_i(\mathrm{Aut}(A \oplus X^{\oplus n}); M) \to H_i(\mathrm{Aut}(A \oplus X^{\oplus n+1}); M)$$

induced by the natural inclusion map is surjective if $i \leq (n-k+2)/k$, and injective if $i \leq (n-k)/k$.

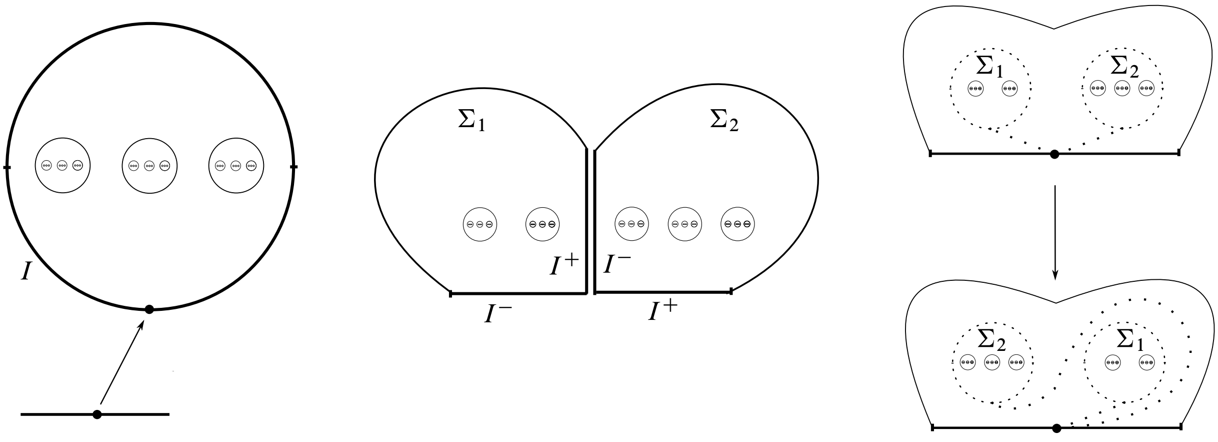

Figure 6: The braided monoidal structure for the category $\mathcal{G}_d$.

4.2 Homogeneous category for the groups $RV^+_{d,r}$

The purpose of this section is to produce a homogeneous category for proving homological stability of the ribbon Higman–Thompson groups $RV^+_{d,r}$. Note that, by Theorem 3.24, this is the same as proving the asymptotic mapping class groups $\mathcal{B}V_{d,r}$ have homological stability. This allows us to define our homogeneous category geometrically. The category is similar to the ones produced in [27, Section 5.6]. Essentially, we replace the annulus or Möbius band by the infinite surface $D^\infty_{d,1}$.

Recall $D^\infty_{d,r}$ is an infinite surface equipped with a canonical asymptotic rigid structure with boundary component denoted by $\partial_b D^\infty_{d,r}$. Let $I = [-1, 1] \subset \partial_b D^\infty_{d,r}$ be an embedded interval as in Figure 6, left. Let $I^- = [-1, 0]$ and $I^+ = [0, 1]$ be subintervals of I. Let $D^\infty_{d,1} \oplus D^\infty_{d,1}$ be the boundary sum of two copies of $D^\infty_{d,1}$ obtained by identifying I^+ of the first copy with I^- of the second copy. Inductively, we could define similarly $\bigoplus_r D^\infty_{d,1}$ for any $r \geq 0$. Here $\bigoplus_0 D^\infty_{d,1}$ is just the standard disk D. Abusing notation, when referring to I^- and I^+ on $\bigoplus_r D^\infty_{d,1}$, we will mean the two copies of I^- and I^+ which remain on the boundary. Thus we have an operation $\oplus$ on the set $\bigoplus_r D^\infty_{d,1}$ for any $r \geq 0$. See Figure 6, center, for a picture of $\left(\bigoplus_2 D^\infty_{d,1}\right) \oplus \left(\bigoplus_3 D^\infty_{d,1}\right)$. In fact, $\left(\left(\bigoplus_r D^\infty_{d,1}\right), \oplus\right)$ is the free monoid generated by $D^\infty_{d,1}$. Note that $\bigoplus_r D^\infty_{d,1}$ has a naturally induced d-rigid structure and we can identify it with $D^\infty_{d,r}$, which will be of use to us later.

We can now define the category $\mathcal{G}_d$ to be the monoidal category with objects $\bigoplus_r D^\infty_{d,1}$ for $r \geq 0$, $\oplus$ as the operation, and D as the 0 object. So far this is the same as defining the objects as the natural numbers and addition as the operation. When $r = s$, we define the morphisms $\mathrm{Hom}\left(\bigoplus_r D^\infty_{d,1}, \bigoplus_s D^\infty_{d,1}\right) = \mathcal{B}V_{d,r}$, which is the group of isotopy classes of asymptotically rigid homeomorphisms of $D^\infty_{d,r}$; when $r \neq s$, let $\mathrm{Hom}\left(\bigoplus_r D^\infty_{d,1}, \bigoplus_s D^\infty_{d,1}\right) = \varnothing$. Note that we did not universally define the morphisms to be the sets of isotopy classes of asymptotically rigid homeomorphisms as we want our category to satisfy cancellation, ie if $A \oplus C = A$ then $C = 0$; see [27, Remark 1.11] for more information. The category $\mathcal{G}_d$ has a natural braiding as in the usual braid group case; see Figure 6, right.

Now, applying [27, Theorem 1.10], we have a prebraided homogeneous category $U\mathcal{G}_d$, which has the same objects as $\mathcal{G}_d$ and morphisms defined as follows: for any $s \le r$, a morphism in $\mathrm{Hom}\big(\bigoplus_s D^{\infty}_{d,1}, \bigoplus_r D^{\infty}_{d,1}\big)$ is an equivalence class of pairs $\big(\bigoplus_{r-s} D^{\infty}_{d,1}, f\big)$, where $f\colon \big(\bigoplus_{r-s} D^{\infty}_{d,1}\big) \oplus \big(\bigoplus_s D^{\infty}_{d,1}\big) \to \bigoplus_r D^{\infty}_{d,1}$ is a morphism in $\mathcal{G}_d$ and $\big(\bigoplus_{r-s} D^{\infty}_{d,1}, f\big) \sim \big(\bigoplus_{r-s} D^{\infty}_{d,1}, f'\big)$ if there exists an isomorphism $g\colon \bigoplus_{r-s} D^{\infty}_{d,1} \to \bigoplus_{r-s} D^{\infty}_{d,1} \in \mathcal{G}_d$ making the following diagram commute up to isotopy:

$$\begin{tikzcd}
\big(\bigoplus_{r-s} D^{\infty}_{d,1}\big) \oplus \big(\bigoplus_s D^{\infty}_{d,1}\big) \arrow[r, "f"] \arrow[d, "g\oplus \mathrm{id}_{\bigoplus_s D^{\infty}_{d,1}}"'] & \bigoplus_r D^{\infty}_{d,1} \\
\big(\bigoplus_{r-s} D^{\infty}_{d,1}\big) \oplus \big(\bigoplus_s D^{\infty}_{d,1}\big) \arrow[ur, "f'"'] &
\end{tikzcd}$$

We write $\big[\bigoplus_{r-s} D^{\infty}_{d,1}, f\big]$ for such an equivalence class. Now, by Theorem 4.6, to prove homological stability for the oriented ribbon Higman–Thompson groups, we only need to verify that the category $\mathcal{G}_d$ satisfies (LH3) at the pair $(D, D^{\infty}_{d,1})$. In fact, by Theorem 3.24, proving the oriented ribbon Higman–Thompson groups satisfy homological stability is the same as proving that the asymptotic mapping class groups $\mathcal{B}V_{d,r}$ satisfy homological stability. Now consider the family of groups

$$\mathrm{Aut}(A \oplus X) \hookrightarrow \mathrm{Aut}(A \oplus X^{\oplus 2}) \hookrightarrow \mathrm{Aut}(A \oplus X^{\oplus 2}) \hookrightarrow \cdots \hookrightarrow \mathrm{Aut}(A \oplus X^{\oplus n}) \hookrightarrow \cdots,$$

where $A = D$ and $X = D^{\infty}_{d,1}$. By definition, this gives rise to the family of groups $\mathcal{B}V_{d,1} \hookrightarrow \mathcal{B}V_{d,2} \hookrightarrow \cdots \hookrightarrow \mathcal{B}V_{d,n} \hookrightarrow \cdots$. Now we have shown that the category $(\mathcal{G}_d, \oplus, D)$ is a prebraided homogeneous category, so, by Theorem 4.6, it suffices to verify (LH3) at the pair $(D, D^{\infty}_{d,1})$ to prove our homological stability result. As a matter of fact, we will show that $W_r(D, D^{\infty}_{d,1})_\bullet$ is $(r-3)$-connected in the next subsection. First, let us further characterize the morphisms in $U\mathcal{G}_d$. Call $0 = I^- \cap I^+$ the basepoint of $\bigoplus_r D^{\infty}_{d,1}$.

Definition 4.7 Given $s < r$, an injective map $\varphi\colon \big(\bigoplus_s D^{\infty}_{d,1}, I^+\big) \to \big(\bigoplus_r D^{\infty}_{d,1}, I^+\big)$ is called an *asymptotically rigid embedding* if it satisfies the following properties:

1. $\varphi(\partial D^{\infty}_{d,s}) \cap \partial D^{\infty}_{d,r} = I^+$.
2. φ maps $\bigoplus_s D^{\infty}_{d,1}$ homeomorphically to $\varphi\big(\bigoplus_s D^{\infty}_{d,1}\big)$ and there exists an admissible surface $A \subset \bigoplus_s D^{\infty}_{d,1}$ such that $\varphi\colon \bigoplus_s D^{\infty}_{d,1} \setminus A \to \varphi\big(\bigoplus_s D^{\infty}_{d,1}\big) \setminus \varphi(A)$ is rigid.
3. The closure of the complement of $\varphi\big(\bigoplus_s D^{\infty}_{d,1}\big)$ in $\bigoplus_r D^{\infty}_{d,1}$ with its induced d-rigid structure is asymptotically rigidly homeomorphic to $\bigoplus_{r-s} D^{\infty}_{d,1}$.

Lemma 4.8 *For $s < r$, the equivalence classes of pairs $\big[\bigoplus_{r-s} D^{\infty}_{d,1}, f\big]$ are in one-to-one correspondence with the isotopy classes of asymptotically rigid embeddings of $\big(\bigoplus_s D^{\infty}_{d,1}, I^+\big)$ into $\big(\bigoplus_r D^{\infty}_{d,1}, I^+\big)$.*

Remark 4.9 Here isotopies are carried out among asymptotically rigid embeddings.

Proof Given an equivalence class of a pair $\left[\bigoplus_{t-s} D_{d,1}^\infty, f\right]$, the restriction map $f|_{\bigoplus_s D_{d,1}^\infty}$ is an asymptotically rigid embedding. Any two equivalence classes of pairs will induce the same map $f|_{\bigoplus_s D_{d,1}^\infty}$; hence, we have a well-defined map from the set of equivalence pairs to the set of isotopy classes of asymptotically rigid embeddings.

We produce the inverse of the restriction map as follows. If we have an asymptotically rigid embedding $\varphi\colon \left(\bigoplus_s D_{d,1}^\infty, I^+\right) \to \left(\bigoplus_r D_{d,1}^\infty, I^+\right)$, by part (3) of Definition 4.7, we also have an asymptotically rigid homeomorphism $\phi\colon C \to \bigoplus_{r-s} D_{d,1}^\infty$, where C is the closure of the complement of $\varphi\left(\bigoplus_s D_{d,1}^\infty\right)$ in $\bigoplus_r D_{d,1}^\infty$. Up to isotopy, we can assume $\phi^{-1}|_{I^+}$ coincides with $\varphi|_{I^-}$. Now define a map $\bar{f}\colon \left(\bigoplus_{r-s} D_{d,1}^\infty\right) \oplus \left(\bigoplus_s D_{d,1}^\infty\right) \to \bigoplus_r D_{d,1}^\infty$ by $\bar{f}|_{\bigoplus_{r-s} D_{d,1}^\infty} = \phi^{-1}$ and $\bar{f}|_{\bigoplus_s D_{d,1}^\infty} = \varphi$. One can check that $\bar{f}$ is an asymptotically rigid homeomorphism. Then $\left(\bigoplus_{r-s} D_{d,1}^\infty, \bar{f}\right)$ gives a representative of an equivalence class of pairs. □

4.3 Higher connectivity of the complex $W_r(D, D_{d,1}^\infty)_\bullet$

We want to prove that the complex $W_r(D, D_{d,1}^\infty)_\bullet$ is highly connected; see Figure 8 and the paragraph following the proof of Lemma 4.24 for an outline of our general strategy.

Remark 4.10 As explained in the proof of [27, Lemma 5.21], a simplex of $S_r(D, D_{d,1}^\infty)$ has a canonical ordering on its vertices induced by the local orientation of the surfaces near the parametrized interval in their based boundary. Thus the geometric realization $|W_r(D, D_{d,1}^\infty)_\bullet|$ is homeomorphic to $S_r(D, D_{d,1}^\infty)$.

Our first step now is to simplify the complex $S_r(D, D_{d,1}^\infty)$ further.

Definition 4.11 Given $r \geq 2$, we call a loop $\alpha\colon (I, \partial I) = ([0,1], \{0,1\}) \to \left(\bigoplus_r D_{d,1}^\infty, 0\right)$ an *asymptotically rigidly embedded loop* if there exists an asymptotically rigid embedding $\varphi\colon (D_{d,1}^\infty, I^+) \to \left(\bigoplus_r D_{d,1}^\infty, I^+\right)$ with $\varphi|_{(\partial D_{d,1}^\infty, 0)} = \alpha$ up to based isotopy. See Figure 7.

Remark 4.12 When $r = 1$, we just call a loop asymptotically rigidly embedded if it is isotopic to the boundary.

Lemma 4.13 *When $r \geq 2$, a loop $\alpha\colon (I, \partial I) \to \left(\bigoplus_r D_{d,1}^\infty, 0\right)$ is isotopic to an asymptotically rigidly embedded loop if and only if there exists an admissible surface $A \subseteq \bigoplus_r D_{d,1}^\infty$ such that the admissible loops of A are disjoint from α, the number of admissible loops of A that lie in the disk bounded by α is $1 + a(d-1)$ for some $a \geq 0$, and there exist some admissible loops which do not lie inside the disk bounded by α up to isotopy.*

Proof It is clear that a loop which is isotopic to an asymptotically rigidly embedded loop has the properties given in the lemma.

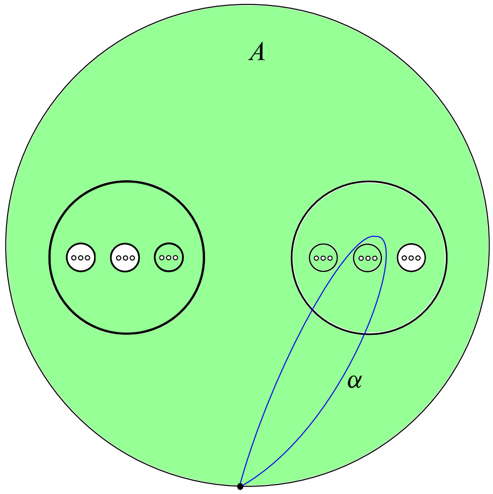

Figure 7: The curve α is an asymptotically rigidly embedded loop, with the green shaded surface A the corresponding admissible subsurface.

For the other direction, we can assume up to isotopy that $\alpha(I) \cap \partial\big(\bigoplus_r D^{\infty}_{d,1}\big) = I^+$. We know that $D^{\infty}_{d,r}$ is asymptotically rigidly homeomorphic to $D^{\infty}_{d,r+d-1}$; thus, the surface bounded by the loop α is asymptotically rigidly homeomorphic to $D^{\infty}_{d,1}$. Therefore, the number of boundary components of A bounded by the complement disk is $r-1 \bmod d-1$ and thus $D^{\infty}_{d,r-1}$ is asymptotically rigidly homeomorphic to the complement surface. These two facts together imply α is an asymptotically rigidly embedded loop. $\square$

Now we define the complex $U_r(D, D^{\infty}_{d,1})$ which is the surface version of the complex U_r given in [29, Section 2.4].

Definition 4.14 For $r \geq 1$, let $U_r(D, D^{\infty}_{d,1})$ denote the simplicial complex whose vertices are isotopy classes of asymptotically rigidly embedded loops and a set of vertices $\alpha_0, \dots, \alpha_p$ forms a p-simplex if and only if any corresponding asymptotically rigid embeddings $\phi_0, \dots, \phi_p$ form a p-simplex in $S_r(D, D^{\infty}_{d,1})$.

We denote the canonical map from $S_r(D, D^{\infty}_{d,1})$ to $U_r(D, D^{\infty}_{d,1})$ by π. The next lemma follows directly from the definition. In fact, given a set of vertices $\alpha_0, \dots, \alpha_p$, if they form a p-simplex, then any corresponding asymptotically rigid embeddings $\phi_0, \dots, \phi_p$ form a p-simplex in $S_r(D, D^{\infty}_{d,1})$. But this means that, for any $\psi_i \in \pi^{-1}(\alpha_i)$ for $0 \leq i \leq p$, the collection $\psi_0, \psi_1, \dots, \psi_p$ forms a p-simplex.

Lemma 4.15 *The map π is a complete join.*

By Proposition 1.3, we need only show that $U_r(D, D^{\infty}_{d,1})$ is highly connected. Similar to [29, Section 2.4], we will produce several other complexes closely related to $U_r(D, D^{\infty}_{d,1})$. We first have the following complex, which is analogous to the complex U^{∞}_r in [29, Definition 2.12].

Definition 4.16 Let $U_r^\infty(D, D_{d,1}^\infty)$ be the simplicial complex with vertices given by asymptotically rigidly embedded loops in $\bigoplus_r D_{d,1}^\infty$, where $\alpha_0, \alpha_1, \dots, \alpha_p$ form a p-simplex if the punctured disks bounded by them are pairwise disjoint (outside of the basepoint) and there exists at least one admissible loop that does not lie in those disks.

Remark 4.17 (1) The $(r-2)$-skeleton of $U_r^\infty(D, D_{d,1}^\infty)$ is the same as that of $U_r(D, D_{d,1}^\infty)$. Notice though that $U_r^\infty(D, D_{d,1}^\infty)$ is in fact infinite-dimensional.

(2) Since $\bigoplus_r D_{d,1}^\infty$ is asymptotically rigidly homeomorphic to $\bigoplus_{r+d-1} D_{d,1}^\infty$, $U_r^\infty(D, D_{d,1}^\infty)$ is isomorphic to $U_{r+d-1}^\infty(D, D_{d,1}^\infty)$ as a simplicial complex.

We also need another complex, which is the surface version of the complex T_r^∞ of [29, Defintion 2.14]. For convenience, we will orient the admissible loops in $\bigoplus_r D_{d,1}^\infty$ so that they bound the punctured disk according to the orientation.

Definition 4.18 An *almost admissible loop* is a loop $\alpha\colon (I, \partial I) \to \left(\bigoplus_r D_{d,1}^\infty, 0\right)$ which is freely isotopic to one of the nonbased admissible loops.

Note that, by Lemma 4.13, an almost admissible loop is an asymptotically rigidly embedded loop.

Definition 4.19 Define the simplicial complex $T_r^\infty(D, D_{d,1}^\infty)$ to be the full subcomplex of $U_r^\infty(D, D_{d,1}^\infty)$ all of whose vertices are almost admissible loops.

Just as discussed in Remark 4.17, $T_r^\infty(D, D_{d,1}^\infty)$ is in fact isomorphic to $T_{r+d-1}^\infty(D, D_{d,1}^\infty)$ as a simplicial complex.

We now want to further characterize the almost admissible loops by building a connection to the usual arc complex. We let A be the quotient $[0,2]/1\sim 2$. This corresponds to identifying the endpoint 1 of the interval $[0,1]$ with the basepoint 1 of the circle given by $[1,2]/1\sim 2$.

Definition 4.20 An injective continuous map $L\colon (A, 0) \to (D_{d,r}^\infty, 0)$ is called a *lollipop* on the surface $D_{d,r}^\infty$ if $\alpha|_{[1,2]}$ is isotopic to an admissible loop in $D_{d,r}^\infty$ and $L|_{[0,1]}$ is an arc connecting the basepoint 0 to the loop $L([1,2])$. The map $L|_{[0,1]}$ is called the *arc part* of the lollipop L and $L|_{[1,2]}$ is called the *loop part*. See Figure 9, where the blue curve is a lollipop.

Lollipops are examples of what Hatcher and Vogtmann [21] refer to as tethered curves.

Lemma 4.21 *The set of isotopy classes of almost admissible loops is in one-to-one correspondence with the set of isotopy classes of lollipops.*

Proof We define a map g from the isotopy classes of lollipops to the isotopy classes of almost admissible loops and show that the map is bijective.

Given a lollipop $L\colon (A,0)\to(D^{\infty}_{d,r},0)$, we can map it to an almost admissible loop $\alpha\colon[0,1]\to(D^{\infty}_{d,r},0)$ as follows. We define $\alpha(0)=0$ and let $\alpha(t)$ run parallel to L outside the region bounded by L. The orientation of α is simply the one that coincides with the loop part of L. Since α can be freely homotoped to the admissible loop $L|_{[1,2]}$, α is almost admissible. Any isotopy of L induces an isotopy of α; hence, the map is well defined.

Now we show g is surjective. Given any almost admissible loop $\alpha\colon[0,1]\to(D^{\infty}_{d,r},0)$, let A be the admissible loop which is freely isotopic to α. Up to isotopy, we can assume that A lies in the interior of the surface bounded by α. Then the surface bounded by α and A must be an annulus. From here one can produce an arc connecting the basepoint 0 to a point in A. Together with A, this provides the lollipop.

Finally, we argue that g is injective. Suppose L_1 and L_2 are two lollipops such that $g(L_1)$ and $g(L_2)$ are isotopic; denote the isotopy by f. By the isotopy extension theorem (see for example [12, Proposition 1.11]) there exists an isotopy $F\colon D^{\infty}_{d,r}\times[0,1]\to D^{\infty}_{d,r}$ such that $F|_{D^{\infty}_{d,r}\times 0}=\mathrm{id}_{D^{\infty}_{d,r}}$ and $F|_{g(L_1)\times[0,1]}=f$. In particular, $F|_{D^{\infty}_{d,r}\times 1}$ maps the almost admissible loop $g(L_1)$ to the almost admissible loop $g(L_2)$. Hence L_1 is isotoped through F to a lollipop which lies in a small neighborhood of L_2 and is bounded by the loop $g(L_2)$. Therefore, one can then isotope L_1 to L_2. □

We now have the following definition of lollipop complex.

Definition 4.22 The *lollipop complex* $L^{\infty}_r(D,D^{\infty}_{d,1})$ has vertices as lollipops, and $p+1$ lollipops $L_0,L_1,\dots,L_p$ form a p-simplex if they are pairwise disjoint outside the basepoint 0 and there exists at least one admissible loop which does not lie inside the disks bounded by the L_i.

The following lemma is immediate from Lemma 4.21.

Lemma 4.23 *The complex $L^{\infty}_r(D,D^{\infty}_{d,1})$ is isomorphic to $T^{\infty}_r(D,D^{\infty}_{d,1})$ as a simplicial complex.*

Lemma 4.24 *Given a p-simplex σ in $L^{\infty}_r(D,D^{\infty}_{d,1})$, its link* $\mathrm{Lk}(\sigma)$ *is isomorphic to $L^{\infty}_{r_\sigma}(D,D^{\infty}_{d,1})$ for some $r_\sigma>0$.*

Proof By Lemma 4.23, we can just prove the lemma for $T^{\infty}_r(D,D^{\infty}_{d,1})$. Let $\alpha_0,\alpha_1,\dots,\alpha_p$ be the vertices of σ, which are almost admissible loops. Up to isotopy, we can assume they are pairwise disjoint except at the basepoint 0. Now let C be the complement surface of σ, whose based boundary is the concatenation of $\alpha_p,\alpha_{p-1},\dots,\alpha_0$ and ∂D. The surface C has a naturally induced d-rigid structure. In particular, C is asymptotically rigidly homeomorphic to D^{∞}_{d,r_σ} for some $r_\sigma>0$. Thus link $\mathrm{Lk}(\sigma)$ is isomorphic to $T^{\infty}_{r_\sigma}(D,D^{\infty}_{d,1})$. □

$$\begin{tikzcd}
 & & & U_r^{\infty}(D, D_{d,1}^{\infty}) & T_r^{\infty}(D, D_{d,1}^{\infty}) \arrow[l, "\supseteq"'] \arrow[r, "\cong"] & L_r^{\infty}(D, D_{d,1}^{\infty}) \\
W_r(D, D_{d,1}^{\infty})_{\bullet} \arrow[r, "\simeq"] & S_r(D, D_{d,1}^{\infty}) \arrow[r, "\pi"] & U_r(D, D_{d,1}^{\infty}) & & & \\
 & & U_r^{(r-2)}(D, D_{d,1}^{\infty}) \arrow[u, "\subseteq"'] \arrow[r, "\cong"] & (U_r^{\infty}(D, D_{d,1}^{\infty}))^{(r-2)} \arrow[uu, "\subseteq"'] & &
\end{tikzcd}$$

Figure 8: A summary of the relationships between the complexes defined so far.

Let us summarize the relationships we have so far between our various complexes, which are illustrated in Figure 8. The leftmost homeomorphism between $W_r(D, D_{d,1}^{\infty})_{\bullet}$ and $S_r(D, D_{d,1}^{\infty})$ comes from Remark 4.10. By Lemma 4.15, there is a complete join map π from the complex $S_r(D, D_{d,1}^{\infty})$ to the complex of isotopy classes of asymptotically rigidly embedded loops, $U_r(D, D_{d,1}^{\infty})$, which implies that both complexes have exactly the same connectivity properties. Thus we can choose to work with $U_r(D, D_{d,1}^{\infty})$. Next, Remark 4.17(1) demonstrates that the $(r-2)$-skeleton of $U_r(D, D_{d,1}^{\infty})$ is the same as the $(r-2)$-skeleton of the complex of asymptotically rigidly embedded loops in $\bigoplus_r D_{d,1}^{\infty}$, denoted by $U_r^{\infty}(D, D_{d,1}^{\infty})$. Since our goal is to show $W_r(D, D_{d,1}^{\infty})_{\bullet}$ is weakly Cohen–Macauley of dimension $r-2$ (see Corollary 4.30), this implies we can again shift our focus to $U_r^{\infty}(D, D_{d,1}^{\infty})$. Next, by Definition 4.19, $T_r^{\infty}(D, D_{d,1}^{\infty})$ is a subcomplex of $U_r^{\infty}(D, D_{d,1}^{\infty})$. In the next pages, we will show that this complex is isomorphic to the lollipop complex $L_r^{\infty}(D, D_{d,1}^{\infty})$ as a simplicial complex (Lemma 4.23), and that the lollipop complex (and hence $T_r^{\infty}(D, D_{d,1}^{\infty})$) is contractible with a bad simplices argument (Proposition 4.27). We then use the contractibility of $T_r^{\infty}(D, D_{d,1}^{\infty})$ and a bad simplices argument to prove the complex $U_r^{\infty}(D, D_{d,1}^{\infty})$ is contractible and weakly Cohen–Macaulay of dimension $r-2$ (Proposition 4.28 and Corollary 4.29), implying ultimately that our initial complex is weakly Cohen–Macaulay of dimension $r-2$, as needed.

In Proposition 4.28, we will deduce the connectivity of $U_r^{\infty}(D, D_{d,1}^{\infty})$ using the connectivity of the lollipop complex $L_r^{\infty}(D, D_{d,1}^{\infty})$ by applying a bad simplices argument. Our goal now is to show that $L_r^{\infty}(D, D_{d,1}^{\infty})$ is highly connected. Let us make some definitions first.

Definition 4.25 Given any lollipop $L\colon (A, 0) \to (D_{d,r}^{\infty}, 0)$, we define the *free height* $\mathfrak{h}_L$ to be the minimal number m such that $L([1,2])$ is contained in $D_{d,r,m}$ up to free isotopy. We also define the height of an admissible loop to be the minimal number m such that it is contained in $D_{d,r,m}$ (see Definition 3.2).

To analyze the connectivity of $L_r^{\infty}(D, D_{d,1}^{\infty})$, we need the following lemma, which is a direct translation of [29, Lemma 3.8].

Lemma 4.26 *For any $r, p, N \geq 1$, there exists a number $\mathfrak{h}_{r,p,N} \geq 0$, such that, for any p-simplex σ in $L_r^{\infty}(D, D_{d,1}^{\infty})$ and any $\mathfrak{h} \geq \mathfrak{h}_{r,p,N}$, there are at least N lollipops of free height $\mathfrak{h}$ in $L_r^{\infty}(D, D_{d,1}^{\infty})$ that are in* $\mathrm{Lk}(\sigma)$.

Proof Note that, for any vertex L in $L_r^\infty(D, D_{d,1}^\infty)$, $L|_{[1,2]}$ is an admissible loop in $\bigoplus_r D_{d,1}^\infty$. Recall the function q defined in Definition 3.17, which maps an admissible loop to an edge midpoint in the tree $\mathcal{T}_{d,r}$. Since each edge has a unique descendent vertex, we can instead map the loop to this vertex which lies in the forest $\mathcal{F}_{d,r}$. Using this connection, we can now choose $\mathfrak{h}_{r,p,N}$ to be the same as in [29, Lemma 3.8]. Then we have at least N admissible loops of height $\mathfrak{h} \geq \mathfrak{h}_{r,p,N}$ which lie in the complement of the surface corresponding to σ in $\bigoplus_r D_{d,1}^\infty$. Connecting each of these admissible loops to the basepoint in the complement surface, we get a set of lollipops in $\mathrm{Lk}(\sigma)$. $\square$

We now show that the complex $L_r^\infty(D, D_{d,1}^\infty)$ is in fact contractible. The idea of proof is similar to that of [29, Proposition 3.1] but with significantly more technical difficulty. Intuitively, to define their complex, one only needs information from the loop parts of the lollipops, which are much easier to "make" disjoint in general, but for us, we also have to deal with the arc parts, which could potentially cause more problems.

Proposition 4.27 *The complex $L_r^\infty(D, D_{d,1}^\infty)$ is contractible for any $r \geq 1$.*

Proof The complex $L_r^\infty(D, D_{d,1}^\infty)$ is obviously nonempty. We will show by induction that, for all $k \geq 0$, any map $S^k \to L_r^\infty(D, D_{d,1}^\infty)$ is null-homotopic. Assume $L_r^\infty(D, D_{d,1}^\infty)$ is $(k-1)$-connected.

Let $f\colon S^k \to L_r^\infty(D, D_{d,1}^\infty)$ be a map. As usual, we can assume that the sphere S^k comes with a triangulation such that the map f is simplicial. We first use Lemma 1.7 to homotope f to a map that is simplexwise injective. For that we need that, for every p-simplex σ in $L_r^\infty(D, D_{d,1}^\infty)$, its link $\mathrm{Lk}\,(\sigma)$ is $(k-p-2)$-connected. But, by Lemma 4.24, $\mathrm{Lk}(\sigma)$ can be identified with $L_{r_\sigma}^\infty(D, D_{d,1}^\infty)$ for some $r_\sigma \geq 1$, so it is $(k-1)$-connected and the conditions of Lemma 1.7 are satisfied.

Now, since S^k is a finite simplicial complex, the free height of the vertices of S^k has a maximum value. We first want to homotope f to a new map such that all the vertices have free height at least $\mathfrak{h} = \mathfrak{h}_{r,k,N}$, where $N = v_0 + v_1 + \cdots + v_k + 2$, v_i is the number of i-simplices of S^k, and $\mathfrak{h}_{r,k,N}$ is determined by Lemma 4.26. For that we use a bad simplices argument.

We call a simplex of the sphere S^k bad if all of its vertices are mapped to vertices in $L_r^\infty(D, D_{d,1}^\infty)$ that have free height less than $\mathfrak{h}$. We will modify f by removing the bad simplices inductively, starting with those of the highest dimension. Let σ be a bad simplex of maximal dimension p among all bad simplices. We will modify f and the triangulation of S^k in the star of σ in a way that does not add any new bad simplices. In the process, we will increase the number of vertices by at most 1 in each step, and not at all if σ is a vertex. This implies that, after doing this for all bad simplices, we will have increased the number of vertices of the triangulation of S^k by at most $v_1 + \cdots + v_k$. As S^k originally had v_0 vertices, at the end of the process its new triangulation will have at most $v = v_0 + v_1 + \cdots + v_k$ vertices. There are two cases.

Case 1 $(p = k)$ If the bad simplex σ is of the dimension k of the sphere S^k, then its image $f(\sigma)$ has a complement loop which bounds a surface C asymptotically rigidly homeomorphic to D^∞_{d,r_σ} for some $r_\sigma \geq 1$ by Lemma 4.24. Now we can choose a lollipop y in C with free height at least $\mathfrak{h}+1$. In particular, $f(\sigma) \cup y$ forms a $(k+1)$-simplex. We can then add a vertex a in the center of σ, replacing σ by $\partial\sigma * a$ and replacing f by the map $(f|_{\partial\sigma}) * (a \mapsto y)$ on $\partial\sigma * a$. This map is homotopic to f through the simplex $f(\sigma) \cup \{y\}$. We have added a single vertex to the triangulation. Because L has free height $\mathfrak{h}+1$, we have not added any new bad simplices, and we have removed one bad simplex, namely σ. Moreover, f remains simplexwise injective.

Case 2 $(p < k)$ If the bad simplex σ is a p-simplex for some $p < k$, by maximality of its dimension, the link of σ is mapped to vertices of free height at least $\mathfrak{h}$ in the complement of the subsurface $f(\sigma)$. The simplex σ has $p+1$ vertices, whose images are pairwise disjoint outside the basepoint up to based isotopy. By Lemma 4.26 and our choice of $\mathfrak{h}$, there are at least $N = v+2$ lollipops $y_1, \dots, y_N$ of free height $\mathfrak{h}$ such that each $f(\sigma) \cup \{y_i\}$ forms a $(p+1)$-simplex. As there are fewer vertices in the link than in the whole sphere S^k, and S^k has at most v vertices, by the pigeonhole principle, the loop parts of the vertices in $f(\mathrm{Lk}(\sigma))$ are contained in at most v punctured disks bounded by the corresponding admissible loops with free height $\mathfrak{h}$. As $N = v+2$, there are at least two of the above vertices y_i and y_j of free height $\mathfrak{h}$ such that any loop parts of vertices in $f(\mathrm{Lk}(\sigma))$ are disjoint from the loop parts of y_i and y_j. We can further assume that the arc parts of y_i and y_j never intersect with any loop part of a vertex in $f(\mathrm{Lk}(\sigma))$. And, up to replacing the loop parts of y_i and y_j with an admissible loop lying inside the disk bounded by the loop parts of y_i and y_j (note that this may increase the free height of y_i and y_j), we can further assume that the arc parts of vertices in $f(\mathrm{Lk}(\sigma))$ are disjoint from the loop parts of y_i and y_j. But, unlike the situation in the proof of [29, Proposition 3.1], a new problem we are facing here is that the arc parts of y_i or y_j might intersect the arc parts of the vertices in $f(\mathrm{Lk}(\sigma))$ even up to isotopy. In particular, given a simplex τ lying in the link of σ, $f(\sigma) \cup f(\tau) \cup y_i$ does not necessarily form a simplex now.

For that we want to apply the mutual link trick (see Lemma 1.8) to remove the intersections of $f(\mathrm{Lk}(\sigma))$ with y_i via a sequence of homotopies. In the process, we will only modify f on $\mathrm{Lk}(\sigma)$ and the new map will still map $\mathrm{Lk}(\sigma)$ to $\mathrm{Lk}_{L_r^\infty(D,D^\infty_{d,1})}(f(\sigma))$. Recall that f is simplexwise injective. Up to isotopy, we can further choose representatives for vertices in $f(\mathrm{Lk}(\sigma))$ such that the intersection points of vertices in $f(\mathrm{Lk}(\sigma))$ and y_i are isolated. Moreover, we assume the number of intersection points is minimal for each vertex in $f(\mathrm{Lk}(\sigma))$. Now we choose an intersection point x_0 in the arc $y_i([0,1])$ that is closest to $y_i(1)$, and denote the corresponding lollipop by β, which is the image of some vertex $b \in \mathrm{Lk}(\sigma)$. We can choose β' to be a variation of β: β' coincides with β for the most part, except around the intersection point with y_i, we replace it by an arc going around the loop part of y_i. See Figure 9. Now we apply Lemma 1.8, for which we need to check the following two conditions:

(1) $f(\mathrm{Lk}_{S^k}(b)) \leq \mathrm{Lk}_{L_r^\infty(D,D^\infty_{d,1})}(\beta')$. This follows from our definition of β'. If a vertex v in $f(\mathrm{Lk}_{S^k}(b))$ is disjoint from β, using the fact that the intersection point x_0 is the closest one to $y_i(1)$ and $f(v)$ is disjoint from the loop part of y_i, β' is also disjoint from v.

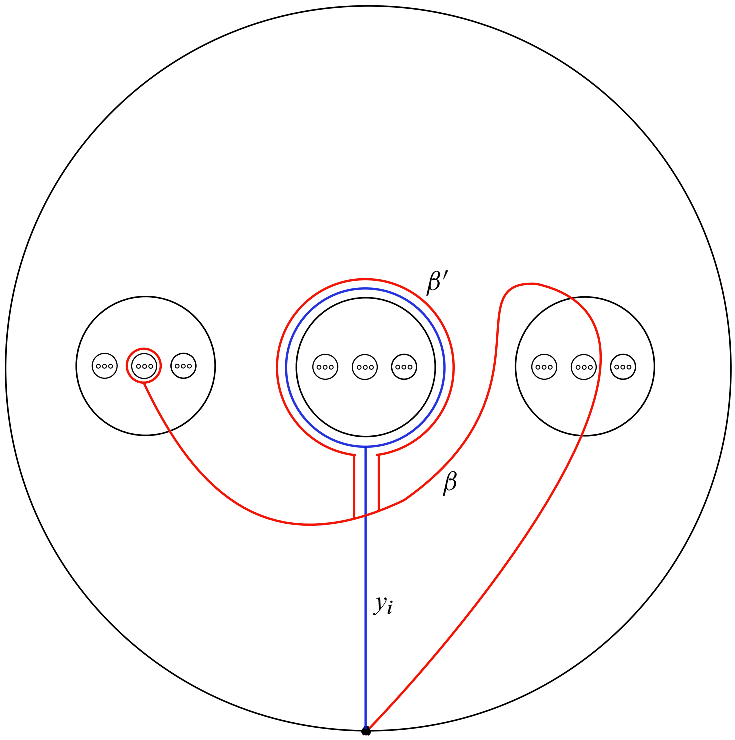

Figure 9: Replacing β by β' to reduce the number of intersection points with y_i.

(2) $\mathrm{Lk}(\beta) \cap \mathrm{Lk}(\beta')$ is $(k-1)$-connected. The lollipops β and β' together will bound a disk which contains the loop part of β and y_i. In any event, the complement of these is a surface asymptotically rigidly homeomorphic to some surface $D^{\infty}_{d,r'}$ for some $r' \geq 1$. By our induction, it is $(k-1)$-connected.

Now Lemma 1.8 says we can homotope f to a new map such that $f(b) = \beta'$ and $f(\mathrm{Lk}(\sigma))$ has fewer intersection points with y_i. Step by step, at the end we have a simplexwise injective map f such that any vertex in $f(\mathrm{Lk}(\sigma))$ only intersects with y_i at the basepoint. In particular, for any $\tau \in \mathrm{Lk}(\sigma)$, $f(\sigma) \cup f(\tau) \cup \{y_i\}$ forms a simplex in $L^{\infty}_r(D, D^{\infty}_{d,1})$.

We can then replace f inside the star

$$\mathrm{St}(\sigma) = \mathrm{Lk}(\sigma) * \sigma \simeq S^{k-p-1} * D^p$$

by the map $(f|_{\mathrm{Lk}(\sigma)}) * (a \mapsto y_i) * (f|_{\partial\sigma})$ on

$$\mathrm{Lk}(\sigma) * a * \partial\sigma \simeq S^{k-p-1} * D^0 * S^{p-1},$$

which agrees with f on the boundary $\mathrm{Lk}(\sigma) * \partial\sigma$ of the star and is homotopic to f through the map $(f|_{\mathrm{Lk}(\sigma)}) * (a \mapsto y_i) * (f|_{\sigma})$ defined on

$$\mathrm{Lk}(\sigma) * a * \sigma \simeq S^{k-p-1} * D^0 * D^p.$$

Now $\mathrm{Lk}(\sigma) * a * \partial(\sigma)$ has exactly one extra vertex a compared to the star of σ, unless σ is just a vertex, in which case $\partial(\sigma)$ is empty and it has the same number of vertices. As y_i has height at least $\mathfrak{h}$, we have not added any new bad simplices. Hence we have reduced the number of bad simplices by one by removing σ.

By induction, we can now assume that there are no bad simplices for f with respect to a triangulation with at most v vertices. With this assumption, we want to cone off f just as we coned off the links in the above argument. We have more than $N = v + 2$ vertices of free height $\mathfrak{h}$ in $L_r^\infty(D, D_{d,1}^\infty)$, and at most v vertices in the sphere. The loop parts of these vertices are admissible loops of height at least $\mathfrak{h}$. By the pigeonhole principle, we know that there are at least two lollipops z_i and z_j of free height $\mathfrak{h}$ such that the punctured disks bounded by their loop parts are disjoint from the punctured disk bounded by any loop part of the lollipops in the vertices of $f(S^k)$. Just as before, we can further assume that the arc parts of z_i and z_j never intersect with any loop part of the vertices in $f(S^k)$, and the arc parts of vertices in $f(S^k)$ are disjoint from the loop part of z_i and z_j. But the same problem appears again, as we want vertices of $f(S^k)$ to be disjoint from the whole lollipop z_i. For that we apply Lemma 1.8 again, and the same proof as before implies that we can homotope f so that its image is disjoint from z_i. In particular, $f(S^k)$ lies in the link of z_i. Hence we can homotope f to a constant map since $\mathrm{St}(z_i)$ is contractible. □

Proposition 4.28 *The complex $U_r^\infty(D, D_{d,1}^\infty)$ is contractible.*

Proof As $T_r^\infty(D, D_{d,1}^\infty)$ is a subcomplex of $U_r^\infty(D, D_{d,1}^\infty)$, we can use a bad simplices argument.

We call a vertex of $U_r^\infty(D, D_{d,1}^\infty)$ bad if it does not lie in $T_r^\infty(D, D_{d,1}^\infty)$ and a simplex bad if all of its vertices are bad. Given a bad p-simplex σ, we need to determine the connectivity of the good link GL_σ (see Section 1.2 for the definition of GL_σ). As in the proof of Lemma 4.24, we have a complement surface C_σ of σ in $D_{d,1}^\infty$. Note that C_σ inherits a d-rigid structure and it is asymptotically rigidly homeomorphic to $\bigoplus_{r_\sigma} D_{d,1}^\infty$ for some $r_\sigma > 0$. In particular, we can now identify GL_σ with $T_{r_\sigma}^\infty(D, D_{d,1}^\infty)$, which is contractible. Thus, by Proposition 1.5, the pair $(U_r^\infty(D, D_{d,1}^\infty), T_r^\infty(D, D_{d,1}^\infty))$ is i-connected for any $i \geq 0$. By Proposition 4.27, $T_r^\infty(D, D_{d,1}^\infty) \cong L_r^\infty(D, D_{d,1}^\infty)$ is contractible, so $U_r^\infty(D, D_{d,1}^\infty))$ is also contractible. □

Corollary 4.29 *The complex $U_r(D, D_{d,1}^\infty)$ is weakly Cohen–Macaulay of dimension $r-2$.*

Proof Note first that a simplicial complex is $(r-3)$-connected if and only if its $(r-2)$-skeleton is. Since $U_r(D, D_{d,1}^\infty)$ has the same $(r-2)$-skeleton as $U_r^\infty(D, D_{d,1}^\infty)$ and $U_r^\infty(D, D_{d,1}^\infty)$ is contractible, so in particular $(r-3)$-connected, indeed $U_r(D, D_{d,1}^\infty)$ is $(r-3)$-connected.

Now let σ be a p-simplex of $U_r(D, D_{d,1}^\infty)$, with vertices $\phi_0, \phi_1, \dots, \phi_p$. We need to check that the link $\mathrm{Lk}_{U_r(D, D_{d,1}^\infty)}(\sigma)$ is $(r-p-4)$-connected. We can assume $p \leq r-3$ as any space is (-2)-connected. Moreover, it suffices to show the $(r-p-3)$-skeleton of $\mathrm{Lk}_{U_r(D, D_{d,1}^\infty)}(\sigma)$ is $(r-p-4)$-connected. Since

$\phi_0, \phi_1, \dots, \phi_p$ form a p-simplex, similar to the proof of Lemma 4.24, the complement surface of σ is asymptotically rigidly homeomorphic to some d-rigid surface D^∞_{d,k_σ} for some $k_\sigma > 0$. Then we can identify the $(r-p-3)$-skeleton of $\mathrm{Lk}_{U_r(D,D^\infty_{d,1})}(\sigma)$ with the $(r-p-3)$-skeleton of $U^\infty_{k_\sigma}(D, D^\infty_{d,1})$. Since $U^\infty_{k_\sigma}(D, D^\infty_{d,1})$ is even contractible, we have the connectivity bound we need. $\square$

Now, by Lemma 4.15 and Proposition 1.3, we have the following.

Corollary 4.30 *The complexes $S_r(D, D^\infty_{d,1})$ and $W_r(D, D^\infty_{d,1})_\bullet$ are weakly Cohen–Macaulay of dimension $r-2$.*

4.4 Homological stability

We are finally ready to prove the homological stability result.

Theorem 4.31 *Suppose $d \geq 2$. Then the inclusion maps induce isomorphisms*

$$\iota_{R^+,d,r} \colon H_i(RV^+_{d,r}, M) \to H_i(RV^+_{d,r+1}, M)$$

in homology in all dimensions $i \geq 0$ for all $r \geq 1$ and for all $H_1(RV^+_{d,\infty})$-modules M.

Proof From Corollary 4.30, $W_r(D, D^\infty_{d,1})$ is $(r-2)$-connected; hence, in particular, the category $\mathcal{G}_d$ satisfies (LH3) at the pair of objects $(D, D^\infty_{d,1})$ with slope $k = 3$. By Theorem 4.6, for any abelian RV^+_∞-module M, the map

$$H_i(RV^+_{d,r}; M) \to H_i(RV^+_{d,r+1}; M)$$

induced by the natural inclusion map is an isomorphism if $r \geq 3i + 3$.

But we can improve the stability range as in the proof of [29, Theorem 3.6] by noticing that we have the same canonical isomorphism between $RV^+_{d,r}$ and $RV^+_{d,r+d-1}$. In fact, denoting this isomorphism by $I_{d,r}$, we have the commutative diagram

$$\begin{tikzcd} RV^+_{d,1+r-1} \arrow[r, "\iota_{R^+,d,r}"] \arrow[d, "I_{d,r}"] & RV^+_{d,1+(r-1)+1} \arrow[d, "I_{d,1+(r-1)+1}"] \\ RV^+_{d,d+r-1} \arrow[r, "\iota_{R^+,d,r+1}"] & RV^+_{d,d+r-1+1} \end{tikzcd}$$

Given that the vertical maps are isomorphisms and the bottom horizontal maps induce isomorphisms on the i^{th} homology when $d + r - 1 \geq 3i + 3$, the top map must also induce isomorphism on the homology as long as $r \geq 3i + 3$. This has improved the stable range by $d - 1$. Step by step, the map $\iota_{R^+,d,r}$ must induce isomorphisms on homology in all dimensions $i \geq 0$ and for all $r \geq 1$. $\square$

Theorem 4.32 *Suppose $d \geq 2$. Then the inclusion maps induce isomorphisms*

$$\iota_{R,d,r} \colon H_i(RV_{d,r}, M) \to H_i(RV_{d,r+1}, M)$$

in homology in all dimensions $i \geq 0$ for all $r \geq 1$ and for all $H_1(RV_{d,\infty})$-modules M.

Sketch of proof The proof will be exactly the same as that of Theorem 4.31. Note first that, by Theorem 3.24, this is the same as proving the half-twist asymptotic mapping class groups $\mathcal{H}V_{d,r}$ have homological stability. We define the braided monoidal category $\mathcal{G}'_d$ to be the category with objects $\bigoplus_r D^{\infty}_{d,1}$ for $r \geq 0$, $\oplus$ as the operation, and D as the 0 object. When $r = s$, we define the morphisms $\mathrm{Hom}\big(\bigoplus_r D^{\infty}_{d,1}, \bigoplus_s D^{\infty}_{d,1}\big) = \mathcal{H}V_{d,r}$, which can also be understood as the group of isotopy classes of asymptotically quasirigid homeomorphisms of $\bigoplus_r D^{\infty}_{d,1}$; when $r \neq s$, let $\mathrm{Hom}\big(\bigoplus_r D^{\infty}_{d,1}, \bigoplus_s D^{\infty}_{d,1}\big) = \varnothing$. We then have a homogeneous category $U\mathcal{G}'_d$, and, to prove homological stability for the sequence of groups $\mathcal{H}V_{d,1} \leq \mathcal{H}V_{d,2} \leq \cdots$, we only need to prove the associated space $W_r(D, D^{\infty}_{d,1})_\bullet$ — or, in fact, the associated simplicial complex $S_r(D, D^{\infty}_{d,1})$ — is highly connected. At this point, the complex is slightly different from the oriented case, but still the new complex $S_r(D, D^{\infty}_{d,1})$ is a complete join over the old complex $U_r(D, D^{\infty}_{d,1})$. Hence, the connectivity of $S_r(D, D^{\infty}_{d,1})$ again follows from Corollary 4.29 and Proposition 1.3. □

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