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Christian Millichap

Rolland Trapp

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Flat fully augmented links are determined by their complements

CHRISTIAN MILLICHAP
ROLLAND TRAPP

We show that two flat fully augmented links with homeomorphic complements must be equivalent as links in $\mathbb{S}^3$. This requires a careful analysis of how totally geodesic surfaces and cusps intersect in these link complements and behave under homeomorphism. One consequence of this analysis is a complete classification of flat fully augmented link complements that admit multiple reflection surfaces. In addition, our work classifies those symmetries of flat fully augmented link complements which are not induced by symmetries of the corresponding link.

57K10, 57K32, 57M50

1 Introduction

Two links, L_1 and L_2, in $\mathbb{S}^3$ are *equivalent* if there exists an orientation-preserving homeomorphism of pairs from $(\mathbb{S}^3, L_1)$ to $(\mathbb{S}^3, L_2)$. An equivalence of links induces a homeomorphism between the link complements $\mathbb{S}^3 \setminus L_1$ and $\mathbb{S}^3 \setminus L_2$, which shows that links determine their complements. However, the converse of this statement is generally not true. For instance, if a link contains an unknotted component, then a Dehn twist along this component is a homeomorphism of the complement that will frequently produce a nonequivalent link. Whitehead used this technique in [25] to show that there are infinitely many distinct links with the same complement as the Whitehead link. Some other known constructions for producing distinct links with the same complement were found by Berge [4] and Gordon [10, Section 6]. In contrast to links, the Gordon–Luecke theorem [11] shows that knots are determined by their complements in $\mathbb{S}^3$. Knots in certain closed, oriented 3-manifolds other than $\mathbb{S}^3$ are also determined by their complements; see the work of Rong [24], Matignon [18], Gainullin [9], and Ichihara–Saito [14] for some examples. Moving forward, we will always assume knots and links are embedded in $\mathbb{S}^3$. This contrast between links with multiple components and knots motivated the following question raised by Mangum and Stanford [17]:

Question Is there a set of links $\mathcal{S}$ such that if $L_1, L_2 \in \mathcal{S}$ and $\mathbb{S}^3 \setminus L_1$ is homeomorphic to $\mathbb{S}^3 \setminus L_2$, then L_1 is equivalent to L_2?

In the same paper, Mangum–Stanford show that the set of homologically trivial and Brunnian links (called HTB links in their paper) provide an affirmative answer to this question [17, Theorem 3]. As far as the authors know this is the only example in the literature of an infinite set of links, other than the set of knots, with this property. This motivates the main goal of our paper, which is to show that the family of *flat fully augmented links* (flat FALs) also have this property.

Flat FALs are a family of hyperbolic links which can be obtained from link diagrams, meeting certain diagrammatic conditions, in the following way. Given a link diagram add a trivial component (called a crossing circle) enclosing each twist region, then remove all twists so the strands of the diagram run parallel through the crossing circles (see Figure 2). Knot circles of the resulting flat FAL are components that lie in the projection plane of the resulting diagram. Flat FALs, and more generally, FALs, are a rich family of hyperbolic links, which have received much attention in the last twenty years due to the explicit combinatorial descriptions of their geometric structures and connections to highly twisted knots via Dehn surgery; see [5; 6; 7; 8; 13; 15; 19; 20; 21; 22; 23] for some examples from the literature. We direct the reader to the beginning of Section 2 for more details on essential properties of flat FALs used in this paper. We can now formally state our main theorem, recalling that an isotopy of links in $\mathbb{S}^3$ induces an equivalence of links.

Theorem 1.1 *Let $\mathcal{A}$ and $\mathcal{A}'$ be flat FALs. Then $(\mathbb{S}^3, \mathcal{A})$ is isotopic to $(\mathbb{S}^3, \mathcal{A}')$ if and only if $\mathbb{S}^3 \setminus \mathcal{A}$ is homeomorphic to $\mathbb{S}^3 \setminus \mathcal{A}'$.*

Theorem 1.1 provides a new infinite set of links that are determined by their complements, distinct from knots and HTB links. By definition, every flat FAL contains at least three components, and so, no flat FALs are knots. At the same time, there are infinitely many flat FALs that are not HTB links. Specifically, any flat FAL with at least two knot circles is not an HTB link since it will contain a Hopf sublink. Such a sublink violates the homologically trivial property that the linking number is 0 for any two components.

Another way our work differs from both Gordon–Luecke and Mangum–Stanford is in the techniques used. The proof of Theorem 1.1 greatly leverages the hyperbolic structure of flat FAL complements and relies on analyzing the behavior of totally geodesic surfaces and cusps under isometries induced by homeomorphism. This geometric approach differs from the purely topological approaches used by Gordon–Luecke and Magnum–Stanford, where these authors determine which Dehn surgeries on a knot or HTB link produce $\mathbb{S}^3$.

The *flat* hypothesis of Theorem 1.1 is necessary, making the result as general as possible within the class of FALs. To see this, consider the twisted FALs of Figure 1. They differ by a Dehn twist on the top circle, so their complements are homeomorphic. The links are distinct, however, since all components are unknots in Figure 1(a) while one component is a trefoil in Figure 1(b).

The proof of Theorem 1.1 breaks into two cases, depending on the number of reflection surfaces contained in a flat FAL complement. Intuitively, a reflection surface for a flat FAL complement $\mathbb{S}^3 \setminus \mathcal{A}$ is a totally

(a) all components unknotted (b) one trefoil component

Figure 1: Distinct twisted FALs with homeomorphic complements.

geodesic surface that corresponds with the projection plane for an FAL diagram of $\mathcal{A}$; see Section 2 for more explicit details and properties of reflection surfaces. As an intermediary step in the proof of Theorem 1.1, we classify flat FAL complements with multiple reflection surfaces and show that homeomorphisms between flat FAL complements preserve reflection surfaces. For emphasis, we now state the classification of flat FALs with multiple reflection surfaces (see Figure 6 and the beginning of Section 3.2 for descriptions of the links P_n and O_n).

Theorem 1.2 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with multiple distinct reflection surfaces. Then either*

- *$\mathcal{A}$ is equivalent to the Borromean rings and M contains exactly three distinct reflection surfaces*, *or*
- *$\mathcal{A}$ is equivalent to P_n with $n \geq 3$, or O_n with $n \geq 2$, and M contains exactly two distinct reflection surfaces.*

A slightly more general version of this classification result is given in Theorem 3.11, along with several useful corollaries on how reflection surfaces behave under homeomorphism.

The proof of Theorem 1.2 relies on an analysis of how cusps and totally geodesic surfaces behave relative to different reflection surfaces in a flat FAL complement. With Theorem 1.2 in hand, basic properties of the links P_n and O_n show that two flat FALs whose complements admit multiple reflection surfaces are equivalent as links if and only if their respective complements are homeomorphic. This is highlighted in Corollary 3.12.

Theorem 1.2 also implies that a flat FAL with multiple reflection surfaces cannot be homeomorphic to a flat FAL with a single reflection surface; see Corollary 3.13. As a result, we now only need to consider the case where there exists a homeomorphism $h\colon M \to M'$ between two flat FAL complements, each with unique reflection surfaces. This is a far more challenging task, and relies on the topology and geometry of thrice-punctured spheres in a flat FAL complement that are not contained in the reflection surface, which were classified by Morgan–Ransom–Spyropoulos–Trapp–Ziegler in [20]. In particular, we make extensive use of sets of thrice-punctured spheres that separate M, whose homeomorphic images in M' are greatly restricted by the classification in [20]. These restrictions help us describe how the knot and crossing circles which intersect such thrice-punctured spheres must behave under homeomorphism.

This, in turn, allows us to show that any flat FAL complement that admits a unique reflection surface and a nontrivial homeomorphism must have a particular link structure, which we call a signature link; see Definition 5.1. Furthermore, only one type of nontrivial homeomorphism is possible in this case, which we call a full-swap. We carefully describe full-swap homeomorphisms of signature links in terms of compositions of Dehn twists along sets of Hopf sublinks; see Definition 5.4. In addition, the induced action of any such full-swap homeomorphism on the corresponding flat FAL (which must be a signature link) can be made explicit on a diagrammatic level, which makes it easy to construct a specific isotopy between any such pair of flat FALs with homeomorphic complements.

As partially noted in the previous paragraph, our work not only shows that flat FALs are determined by their complements, but classifies the types of homeomorphisms that can exist between flat FAL complements and what types of flat FAL complements can admit certain types of homeomorphisms. These severe geometric restrictions on self-homeomorphisms of flat FAL complements lead to a concise comparison between their symmetry groups and those of their complements. Recall that the symmetry group of a link $L \subset \mathbb{S}^3$ is the group of homeomorphisms of pairs $(\mathbb{S}^3, L)$ up to isotopy, which we denote by $\mathrm{Sym}(\mathbb{S}^3, L)$. Similarly, the symmetry group of the corresponding link complement is the group of self-homeomorphisms of $\mathbb{S}^3 \setminus L$ up to isotopy, denoted by $\mathrm{Sym}(\mathbb{S}^3 \setminus L)$. Any self-homeomorphism of $(\mathbb{S}^3, L)$ induces a self-homeomorphism of $\mathbb{S}^3 \setminus L$, and so, $\mathrm{Sym}(\mathbb{S}^3, L) \subseteq \mathrm{Sym}(\mathbb{S}^3 \setminus L)$. However, this can be a strict containment; see [12] for some examples. The following theorem classifies symmetries of flat FAL complements that are not induced by symmetries of the corresponding link.

Theorem 1.3 *Let $\mathcal{A}$ be a flat FAL. Then either*

- *$\mathcal{A}$ is not a signature link and both $\mathcal{A}$ and its complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ have the same symmetry group, or*
- *$\mathcal{A}$ is a signature link and full-swaps on $\mathcal{A}$ generate symmetries of $M = \mathbb{S}^3 \setminus \mathcal{A}$ which are not restrictions of symmetries of $\mathcal{A}$ to M.*

Part of Theorem 1.3 is proved in Theorem 3.14 at the end of Section 3.1. The rest of the proof of this theorem is completed at the end of Section 7.

We now describe the organization of the rest of this paper. In Section 2 we introduce flat FALs, the necessary terminology related to cusps and totally geodesic surfaces contained in flat FAL complements, and review some essential facts from the literature on the geometry of FAL complements. In Section 3, we prove Theorem 1.2. This allows us to focus the rest of the paper on homeomorphisms between flat FAL complements, each with a unique reflection surface and this transition is discussed in Section 4. Then Section 5 discusses "signature link" complements, a special class of flat FAL complements, along with a "full-swap" homeomorphism that can be performed on any signature link complement. The image of a full-swap homeomorphism is another signature link complement, and it is straightforward to construct an explicit isotopy between their corresponding links. In Section 6, we prove some useful facts about sets of thrice-punctured spheres that separate a flat FAL complement (with a unique reflection surface) and how they behave under homeomorphisms. Finally, in Section 7 we build off the tools from Section 6 to show that any homeomorphism between flat FAL complements (each with a unique reflection surface) must essentially be a "full-swap" homeomorphism between "signature link" complements. This allows us to construct an isotopy between the corresponding links and prove our main theorem.

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2 Totally geodesic surfaces and cusps in flat FALs

This section provides a brief introduction to flat FALs, compiles some known results about totally geodesic surfaces in their complements, and introduces two concepts used extensively in Section 3: reflection-like surfaces and their induced structures on a flat FAL complement. Aside from reflection-like surfaces and their induced structures, this section is a review of results in the literature.

We first describe how to construct a flat FAL. For this process, start with a link L and a diagram $D(L)$. We can build a diagram $D(\mathcal{F})$ for an FAL $\mathcal{F}$ corresponding to L by augmenting each twist region in $D(L)$ with a circle and undoing all full-twists from each respective twist region. After this procedure, twist regions of $D(L)$ that had contained an odd number of crossings will still contain a single crossing in $D(\mathcal{F})$. If we remove all of these remaining single crossings, then we will have constructed a diagram $D(\mathcal{A})$ for a flat FAL $\mathcal{A}$, as illustrated in Figure 2.

In our work, we will solely be interested in the case where a flat FAL $\mathcal{A}$ is hyperbolic in the sense that the complement $\mathbb{S}^3 \setminus \mathcal{A}$ admits a complete metric of constant negative curvature. As noted in [23, Theorem 2.5], a (flat) FAL $\mathcal{A}$ is hyperbolic if and only if it there exists a corresponding link diagram $D(L)$ that is nonsplittable, prime, twist-reduced, and contains at least two twist regions, where $D(\mathcal{A})$ is obtained

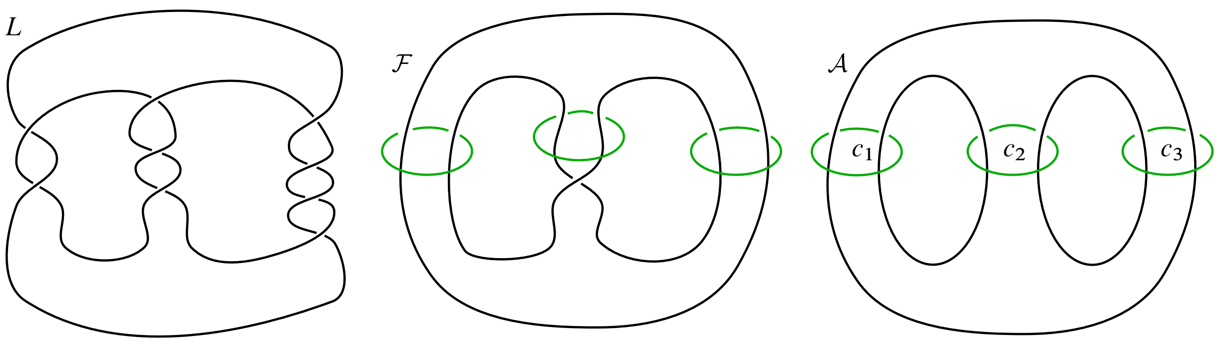


Figure 2: On the left is a diagram of a link L with three twist regions. The middle diagram shows the corresponding FAL $\mathcal{F}$ obtained from fully augmenting L. The right diagram shows the corresponding flat FAL $\mathcal{A}$. Crossing circles of $\mathcal{A}$ are labeled by c_i, for $i = 1, 2, 3$.

from $D(L)$ by the augmentation process described above. We refer the reader to [8, Section 1] for these diagrammatic definitions. Moving forward, we will assume any flat FAL is hyperbolic, and refer to $D(L)$ as the corresponding diagram that was augmented to construct $D(\mathcal{A})$.

This augmentation process partitions the components of $\mathcal{A}$ into *crossing circles*, the trivial components added via augmentation, and *knot circles*, components coming from the original link L. Observe that each crossing circle in a flat FAL bounds a *crossing disk*, a disk in $\mathbb{S}^3$ punctured twice by parallel knot circle arcs that replace an original twist region of $D(L)$. We define a flat FAL diagram to be a link diagram together with this additional structure. More precisely, we have:

Definition 2.1 A link diagram $D(\mathcal{A})$ is a *flat FAL diagram* for a (hyperbolic) link $\mathcal{A}$ if $D(\mathcal{A})$ was constructed by fully augmenting a corresponding diagram $D(L)$ and removing all crossings from each twist region. In addition, we assume $D(\mathcal{A})$ carries with it the partition of components of $\mathcal{A}$ into crossing- and knot-circles, as well as the choice of crossing disks, relative to this augmentation of $D(L)$. A link $\mathcal{A}$ is a *flat FAL* if it admits a flat FAL diagram $D(\mathcal{A})$.

The geometric structures of augmented links were first studied in Adams [2]. A particularly nice geometric decomposition of (flat) FAL complements into pairs of identical right-angled ideal polyhedra was described by Agol–Thurston in the appendix of [16]. This geometric decomposition has proved to be a fruitful tool for analyzing geometric and topological properties of FALs; see [6; 7; 8; 13; 15; 23] for a few examples. Infinite subclasses of FALs have also been examined in the literature. For instance, Meyer–Millichap–Trapp [19] studied the arithmeticity, invariant trace fields, symmetries, and hidden symmetries of FALs obtained by fully augmenting pretzel links, and Purcell examined FALs whose complements admit a decomposition into regular ideal hyperbolic octahedra in [23]. In both cases, each subclass contains an infinite number of flat FALs. Thus the family of flat FALs is a large set of links with many interesting properties.

We now collect some important facts about totally geodesic surfaces and cusps contained inside a flat FAL complement. These properties will be essential for proving both Theorems 1.1 and 1.2. Most of the results stated here are known in the literature and we refer the reader to [8; 20; 23] for more details on the geometric properties of FAL complements discussed here.

First, we describe reflection surfaces, which are an important type of totally geodesic surface contained in every flat FAL complement. Given a flat FAL $\mathcal{A}$ in $\mathbb{S}^3 \cong \mathbb{R}^3 \cup \{\infty\}$, position the crossing circles and disks so that they are orthogonal to the projection plane and embed the knot circles in the projection plane. Such an isotopy is always possible based on the diagrammatic definition of a flat FAL. Then reflection in the projection plane maps every component of $\mathcal{A}$ to itself and fixes the projection plane pointwise. Mostow–Prasad rigidity implies that this projection plane corresponds with a totally geodesic surface $R \subset M = \mathbb{S}^3 \setminus \mathcal{A}$ and there exists an orientation-reversing involution $\iota_R \colon M \to M$ that fixes R pointwise corresponding with reflection in the projection plane. Since $\mathcal{A}$ could admit FAL diagrams where different surfaces play the role of the projection plane, we provide the following definition.

Definition 2.2 Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement and let $R \subset M$ be an embedded totally geodesic surface. If there exists an FAL diagram $D(\mathcal{A})$ in the projection plane P for which $R = P \setminus \mathcal{A}$, then we say that R is a *reflection surface* of M (relative to the diagram $D(\mathcal{A})$).

An FAL diagram $D(\mathcal{A})$ partitions the cusps of $M = \mathbb{S}^3 \setminus \mathcal{A}$ into *crossing circle cusps* and *knot circle cusps*, corresponding with crossing circles and knot circles of $\mathcal{A}$, respectively. We also say that this partition is induced by a reflection surface R, though this ultimately depends on the FAL diagram since reflection surfaces are defined relative to the projection plane for a diagram. As we will see in Section 3, it is possible for a flat FAL complement to admit distinct reflection surfaces that induce different partitions on the components of $\mathcal{A}$. At the same time, it is possible for a flat FAL complement to admit one reflection surface that induces different partitions on the components of $\mathcal{A}$; this phenomenon will be examined in Section 5. Here, we say a cusp of M is an R*-knot circle cusp* (respectively, R*-crossing circle cusp*) if the corresponding link component of $\mathcal{A}$ is a knot circle (respectively, crossing circle) relative to R. In this paper, we frequently use the same notation to refer to a component of $\mathcal{A}$ and the corresponding cusp on M. In addition, each R-crossing circle C of $\mathcal{A}$ bounds an R*-crossing disk* D, which is a disk twice punctured by the two (not necessarily distinct) knot circles going through C. As noted in [23, Lemma 2.1], each such R-crossing disk is an embedded totally geodesic thrice-punctured sphere in M. Furthermore, R intersects D orthogonally in the set of three simple, nonseparating geodesics on D; see Figure 3 for a visual of simple geodesics on a thrice-punctured sphere.

A flat FAL diagram $D(\mathcal{A})$ (or, alternatively, its reflection surface R) determines more structure than the partitioning of cusps of M into knot and crossing circle cusps. It also determines a meridian and longitude on each boundary torus of a cusp neighborhood. In this context, meridians and longitudes will be considered *slopes*, or unoriented isotopy classes of simple closed curves on this boundary torus.

Each component L of a flat FAL is topologically an unknot in $\mathbb{S}^3$, so the torus boundary T_L of a tubular neighborhood V_L of L has a natural choice of meridional and longitudinal slopes. The unknotted torus $T_L \subset \mathbb{S}^3$ bounds a solid torus on each side. A meridian m is the slope of T_L that bounds a disk in V_L, while a longitude ℓ is a slope of T_L that bounds a disk in $\mathbb{S}^3 \setminus V_L$. The unoriented curves m and ℓ are well defined up to ambient isotopy on T_L, so give well-defined slopes on T_L.

From a geometric perspective, the slopes m and ℓ can be described as the intersection of T_L with totally geodesic surfaces. More precisely, let $\mathcal{A}$ be a flat FAL with reflection surface R. If K is a knot circle of $\mathcal{A}$ with torus boundary T_K of a cusp neighborhood of K, then $R \cap T_K$ is a pair of simple closed curves that represent longitudes on T_K. Similarly, for any crossing circle C of $\mathcal{A}$, the set $R \cap T_C$ is a pair of simple closed curves that represent meridians on T_C. Crossing disks relative to the reflection surface R are orthogonal to it, and so intersect T_K in meridians and T_C in longitudes. See [8, Lemma 2.3] for more details.

The longitudes and meridians just described will be referred to as R*-meridians* and R*-longitudes* when we want to emphasize the FAL diagram (and corresponding reflection surface) used to determine them.

Since each torus boundary of a cusp T of M is rectangular (see [8, Lemma 2.3]), curves orthogonal to $R \cap T$ determine the second generator for the fundamental group of each such torus. We refer to the basis for each $\pi_1(T)$ described above as the *peripheral structure* relative to the FAL diagram $D(\mathcal{A})$.

The following proposition summarizes the essential features of a reflection surface in a flat FAL complement that we just discussed.

Proposition 2.3 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with an FAL diagram $D(\mathcal{A})$ which partitions the components of $\mathcal{A}$ into crossing circles and knot circles. Then M contains an embedded totally geodesic R with the following features*:

- *R corresponds with $P \setminus \mathcal{A}$, where P is the projection plane for $D(\mathcal{A})$.*
- *R is fixed pointwise by an orientation reversing involution $i_R\colon M \to M$.*
- *R intersects each R-crossing disk of M orthogonally in its three simple nonseparating geodesics.*
- *R intersects every cusp of M in two parallel curves.*
- *R intersects the boundary torus T_C of each crossing circle cusp C in a pair of meridians and provides a peripheral structure on $\pi_1(T_C)$ relative to $D(\mathcal{A})$.*
- *R intersects the boundary torus T_K of each knot circle cusp K in a pair of longitudes and provides a peripheral structure on $\pi_1(T_K)$ relative to $D(\mathcal{A})$.*

Our main goal is to show that flat FALs are determined by their complements among the set of all flat FALs. For this reason we will need to consider homeomorphic images of reflection surfaces and the structures associated with them that are highlighted in the previous proposition. By Mostow–Prasad rigidity, any homeomorphism between flat FALs induces an isometry between them, and so, we can restrict our analysis to the isometric image of a reflection surface.

Let M, M' be flat FAL complements with reflection surfaces R, R', respectively, and let $\rho\colon M' \to M$ be an isometry between these hyperbolic 3-manifolds. Since the definition of a reflection surface is diagram dependent, we can not immediately assume that $\rho(R')$ is a reflection surface for M. This motivates the following definition.

Definition 2.4 We say that S is a *reflection-like* surface for a flat FAL complement M if there exists an isometry $\rho\colon M' \to M$ between flat FAL complements such that $S = \rho(R')$, where R' is a reflection surface for M'.

Any reflection surface for M is also a reflection-like surface for M via the identity map. In addition, any homeomorphic flat FAL complements have the same number of reflection-like surfaces. Indeed, a homeomorphism $h\colon M' \to M$ between flat FAL complements M' and M induces a unique isometry $\rho_h\colon M' \to M$. By Definition 2.4, the image of each reflection-like surface in M' is a reflection-like surface in M. At the same time, ρ_h^{-1} preserves the property of being reflection-like, so M' and M have the same number of reflection-like surfaces. Similarly, equivalent flat FALs have the same number of reflection-like surfaces since an equivalence of links induces a homeomorphism between their respective complements.

Many important topological and geometric properties of reflection surfaces will be preserved under isometry, which we highlight below. These directly follow from Proposition 2.3.

Proposition 2.5 *Let $\rho\colon M' \to M$ be an isometry between flat FAL complements, let $R' \subset M'$ be a reflection surface, and let $S = \rho(R')$ denote the corresponding reflection-like surface in M. Then:*

- *S is fixed pointwise by an orientation reversing involution $i_S\colon M \to M$.*
- *S intersects the image of each R'-crossing disk of M' orthogonally in its three simple nonseparating geodesics.*
- *S intersects every cusp of M in two parallel curves.*
- *Let T be the boundary torus of a cusp of M. Then S determines a peripheral structure on T, namely, the image of the peripheral structure that R' determines on $\rho^{-1}(T)$.*

Our work in Section 3 will show that every reflection-like surface in an FAL complement is a reflection surface. This is highlighted in Corollary 3.13. After proving this result, we will drop the term reflection-like surface, and instead, only use reflection surface.

A reflection-like surface $S \subset M$ determines a partition of the components of $\mathcal{A}$ into S-crossing circles and S-knot circles, coming from the partition of $\mathcal{A}'$ into R'-crossing circles and R'-knot circles. We use the term *S-structure* on M to refer to this partition of the components of $\mathcal{A}$, the peripheral structures on each cusp determined by S, and a fixed choice of images of R'-crossing disks in M. Suppose M contains distinct reflection-like surfaces R and S. A component L of $\mathcal{A}$ *changes type* if L is an R-knot circle and an S-crossing circle (or vice versa). This terminology will be useful when discussing flat FAL complements that admit multiple reflection surfaces.

In addition to reflection surfaces, (totally geodesic) thrice-punctured spheres contained in flat FAL complements will serve as powerful tools for analyzing homeomorphisms between these manifolds. We now review properties of, and results regarding, embedded, totally geodesic, thrice-punctured spheres in flat FAL complements.

Any thrice-punctured sphere has precisely six simple geodesics, three of which are separating, three of which are nonseparating, and none of which are closed, as depicted in Figure 3. Any intersection of a totally geodesic thrice-punctured sphere D with another totally geodesic surface in a flat FAL complement must be some subset of these six simple geodesics on D that are pairwise disjoint. This puts strong restrictions on the behavior of intersections between D and other totally geodesic surfaces, which we shall exploit.

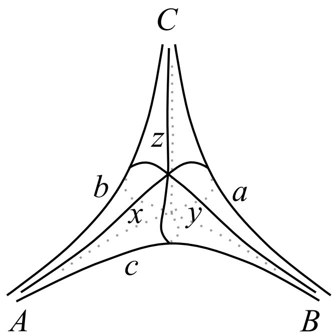


Figure 3: A thrice-punctured sphere with separating geodesics labeled x, y, z and nonseparating geodesics labeled a, b, c.

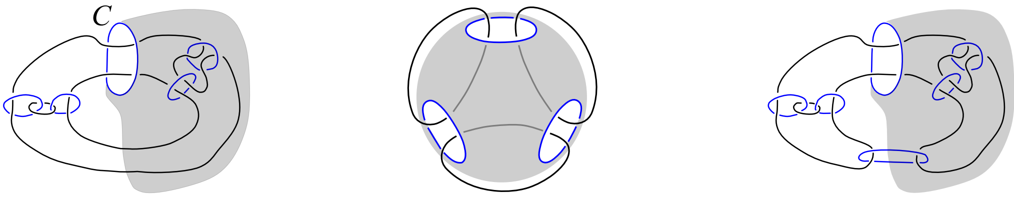
Every FAL complement contains many totally geodesic thrice-punctured spheres that are not contained in a reflection surface, which we will call *nonreflection*, *thrice-punctured spheres* (relative to a designated reflection surface). Crossing disks in M, for example, constitute one category of such spheres. Results in [20] imply there are three types of nonreflection thrice-punctured spheres in flat FAL complements, classified in terms of their intersections with R and their punctures. Here, a *puncture* refers to the intersection of a totally geodesic surface in M with a torus boundary of a cusp of M, which will produce a set of simple closed curves, each of which represents a slope on this torus. We sometimes use the term longitudinal puncture to refer to a puncture that is a representative for a longitude on this torus and we also use the term meridional puncture for a puncture that is a representative for a meridian on this torus (here meridian and longitude refer to the peripheral structure induced by a reflection surface $R \subset M$). We now define the three types of nonreflection, thrice-punctured spheres in flat FAL complements.

Definition 2.6 Let D be a nonreflection thrice-punctured sphere in the flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ with reflection surface R.

- D is a *crossing disk* if $D \cap R$ consists of the three nonseparating geodesics of D and the punctures of D are one crossing circle longitude and two knot circle meridians.
- D is a *longitudinal disk* if $D \cap R$ consists of the three nonseparating geodesics of D and the punctures of D are longitudes of three distinct crossing circles.
- D is a *singly separated disk* if $D \cap R$ consists of one separating geodesic of D and the punctures of D are a longitude of one crossing circle C and two meridians of another crossing circle C'.

Collectively, we refer to crossing disks and longitudinal disks as *N-disks* since their intersections with R both consist of a nonseparating geodesics of D.

This definition of crossing disk broadens the typical meaning of the term. The crossing circle C illustrated in Figure 4(a) bounds two distinct crossing disks, the one interior to C as well as the shaded disk. The



(a) a crossing disk (b) a longitudinal disk (c) a singly separated disk

Figure 4: Types of nonreflection, thrice-punctured spheres.

reason for extending the meaning of "crossing disk" is that any such disk D for a crossing circle C could be chosen as a crossing disk in $\mathcal{A}$ by replacing the current crossing disk with D. The two crossing disk structures come from fully augmenting diagrams of the same link obtained by flyping one twist region to a different part of the link diagram (see [20]).

Parts (b) and (c) of Figure 4 illustrate a longitudinal and singly separated disk, respectively. The following theorem from [20] classifies nonreflection thrice-punctured spheres in flat FAL complements.

Theorem 2.7 [20, Theorem 3.11] *Let D be a nonreflection thrice-punctured sphere in the flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ with reflection surface R. Then D is orthogonal to R in M and D is either a crossing, longitudinal, or singly separated disk.*

In our work, we will frequently make use of Theorem 2.7 and the qualifications of nonreflection thrice-punctured spheres given in Definition 2.6. The properties of separating and nonseparating are topological, so if a reflection surface intersects a thrice-punctured sphere in its separating geodesic, so will any homeomorphic image of them. Thus, if D is singly separated by the reflection surface R, then $h(D)$ is also separated by $h(R)$ along one separating geodesic. Similarly, the homeomorphic image of an N-disk relative to the reflection surface R must intersect the reflection-like surface $h(R)$ in its nonseparating geodesics. In the case when $h(R)$ is again a reflection surface, we say that a homeomorphism preserves the type of a singly separated disk, and preserves the property of being an N-disk. The only type changes possible when $h(R)$ is a reflection surface, then, are between crossing and longitudinal disks. These can be recognized by analyzing the images of punctures under this homeomorphism.

Finally, we define a *separating pair* to be a pair of disjoint thrice-punctured spheres that separate M. The following theorem from [20] tells us exactly which pairs of thrice-punctured spheres are separating pairs in FAL complements.

Theorem 2.8 [20, Theorem 4.8] *Let $\{S_1, S_2\}$ be a pair of disjoint essential thrice-punctured spheres in the complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ of the FAL $\mathcal{A}$. The pair $\{S_1, S_2\}$ is a separating pair if and only if each is either a crossing disk or a singly separated disk and their longitudinal slopes coincide.*

It is clear that any two crossing and/or singly separated disks that share longitudinal crossing-circle punctures separate a flat FAL complement. Theorem 2.8 states that these are the only separating pairs in FAL complements.

3 Flat FAL complements with multiple reflection surfaces

Proposition 2.3 shows that a reflection surface determines a lot of the geometric structure of a flat FAL complement. In this section, we will determine exactly how many reflection surfaces can exist in a flat FAL complement and exactly which flat FAL complements admit multiple reflection surfaces. This will be a useful first step towards our main goal of showing that flat FALs are determined by their complements. We begin with some preliminary observations which specify how distinct reflection-like surfaces can intersect boundary tori of cusp neighborhoods.

As described in Section 2, reflection-like surfaces induce a peripheral structure on boundary tori of cusp neighborhoods. Recall that the peripheral structure consists of two orthogonal, unoriented isotopy classes of simple closed curves (slopes), one labeled meridian and the other longitude. Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ denote a flat FAL complement with distinct reflection-like surfaces R and S, and let T denote the boundary torus of a cusp of M. Our first lemma shows that the meridian-longitude pairs determined by R and S are the same setwise.

Lemma 3.1 *Let M be a flat FAL complement with two distinct reflection-like surfaces R and S and let T be the boundary torus of a cusp of M. Then an R-meridian of T is either an S-meridian or S-longitude. Likewise, an R-longitude of T is either an S-meridian or S-longitude.*

Proof The result will follow from the fact that these meridian-longitude pairs are orthogonal.

Let T denote the boundary torus of a cusp neighborhood of M for some cusp expansion of M. On this torus, let m, ℓ denote the R-meridian and R-longitude, with geodesic lengths μ, λ, respectively. Likewise, let m', ℓ' denote the S-meridian and S-longitude with lengths μ', λ', respectively. Since meridian-longitude pairs are orthogonal

$$\mu\lambda = \text{Area}(T) = \mu'\lambda'.$$

Given an arbitrary orientation on the slopes $\{m, \ell\}$ and $\{m', \ell'\}$, there are integers p, q, r, s with $m' = pm + q\ell$ and $\ell' = rm + s\ell$. By orthogonality we have

$$\mu' = \sqrt{(p\mu)^2 + (q\lambda)^2} \quad \text{and} \quad \lambda' = \sqrt{(r\mu)^2 + (s\lambda)^2}.$$

At least one of p or q is nonzero, which implies $\mu' \geq \min\{\mu, \lambda\}$, with equality exactly when m' is the shortest curve of the set $\{m, \ell\}$. Moreover, when neither p nor q are zero, $\mu' > \max\{\mu, \lambda\}$. Similar inequalities hold for λ'. If both p and q are nonzero, then

$$\mu'\lambda' > \max\{\mu, \lambda\} \min\{\mu, \lambda\} = \mu\lambda.$$

Since $\mu\lambda = \mu'\lambda'$, however, this implies the oriented and simple m' is one of $\pm m$ or $\pm\ell$. The slope m', then, is one of the slopes m or ℓ. Similarly, ℓ' equals the other slope and the sets $\{m', \ell'\}$ and $\{m, \ell\}$ are equal. □

Thus, if a flat FAL complement admits multiple reflection-like surfaces $\{R_i\}$, each such R_i-structure still induces the same basis on a boundary torus of a cusp, though their induced peripheral structures may differ since meridians and longitudes might switch roles (ie some components of $\mathcal{A}$ could change type).

Remark The proof of Lemma 3.1 would not work if we assumed M was just an FAL complement. If the corresponding FAL has some number of half twists, then certain meridian-longitude pairs might no longer be orthogonal, and so, our arguments used above would no longer apply. We point this out since multiple essential results for this paper rely on Lemma 3.1, and we want to make sure the reader knows why our work doesn't immediately apply more broadly to the class of FALs and not just flat FALs.

The next two propositions clarify how a different reflection-like surface S behaves relative to a given R-structure by considering R-knot circle and R-crossing circle cusps separately.

Proposition 3.2 *Let M be a flat FAL complement with two distinct reflection-like surfaces R and S. If K is an R-knot circle and T_K the boundary torus of a cusp corresponding to K, then $S \cap T_K$ is a pair of R-meridians of K.*

Proof Lemma 3.1 implies that $S \cap T_K$ is either a pair of R-meridians or R-longitudes. For the sake of contradiction, suppose that $S \cap T_K$ is a pair of R-longitudes.

Since K is an R-knot circle, $R \cap T_K$ is a pair of R-longitudes. First suppose that $R \cap T_K = S \cap T_K$. Then the composition $\iota_R \circ \iota_S$ acts as the identity on T_K and preserves the normal direction. Therefore $\iota_R \circ \iota_S$ is the identity on M by [3, Proposition A.2.1], and $R = S$. This is a contradiction, which implies $S \cap T_K \neq R \cap T_K$.

Now consider the case where $S \cap T_K$ are R-longitudes distinct from $R \cap T_K$, and let D be an R-crossing disk punctured by K. Note that D intersects T_K in R-meridians, which are orthogonal to the R-longitudes $S \cap T_K$. Thus S intersects D orthogonally. Moreover, R intersects D in the nonseparating geodesics of D, which implies that $S \cap D$ consists of a single separating geodesic with both "ends" on K. The other two punctures of D must be on distinct cusps because one is an R-knot circle puncture, and the other an R-crossing circle puncture. Since S intersects D orthogonally along a separating geodesic, the reflection ι_S preserves D and interchanges the two punctures on distinct cusps. This contradicts the fact that ι_S preserves all cusps, and $S \cap T_K$ cannot be R-longitudes distinct from $R \cap T_K$.

Since S cannot meet T_K in R-longitudes, the pair of curves $S \cap T_K$ are R-meridians. □

Proposition 3.3 *Let M be a flat FAL complement with two distinct reflection-like surfaces R and S. Let C be an R-crossing circle bounding the R-crossing disk D. Then either*

(i) *D is a component of S, or*

(ii) *S intersects D orthogonally along the separating geodesic of D with punctures on C. In this case the R-crossing circle C is also an S-crossing circle.*

Proof Let C be the R-crossing circle bounding D. By Lemma 3.1 we know that $S \cap T_C$ is either two R-meridians or two R-longitudes of C.

Consider the case where $S \cap T_C$ consists of two R-meridians of C. Since C is an R-crossing circle the intersection $R \cap T_C$ consists of R-meridians as well. If the surfaces R and S intersect T_C in the same curves then, as in the proof of Proposition 3.2, one arrives at the contradiction that $R = S$. Thus the R-meridians $S \cap T_C$ are distinct from those of $R \cap T_C$. Now S and D are an embedded totally geodesic surfaces in M, so their intersection is a union of disjoint, complete, simple geodesics. This follows from the observation that the local picture of $S \cap D$, as seen in the universal cover, consists of two planes intersecting along a geodesic. The only complete, simple geodesics of D that intersect T_C are two nonseparating geodesics, say γ_1, γ_2, and one separating geodesic, label it γ. Since R contains the geodesics γ_1, γ_2, their intersection with T_C lies in $R \cap T_C$. This implies that S cannot contain γ_1, γ_2, since S and R intersect T_C in distinct curves. Thus $S \cap D$ contains the separating geodesic γ and neither γ_1 nor γ_2. All other complete, simple geodesics on D intersect γ, which implies $\gamma = S \cap D$ and we are in case (ii). Orthogonality follows from the fact that the R-meridians $S \cap T_C$ are orthogonal to the R-longitude $D \cap T_C$.

It remains to prove that C is an S-crossing circle as well. Since S is reflection-like, there is a flat FAL M' with reflection surface R' and an isometry $h\colon M' \to M$ with $S = h(R')$. Note that $D' = h^{-1}(D)$ is not contained in R' since D is not contained in S. Moreover, since S intersects D in a separating geodesic, the geodesic $R' \cap D'$ separates D' as well. The classification of nonreflection thrice punctured spheres given in Theorem 2.7 shows that D' must be a singly separated disk, and all punctures of singly separated disks are crossing-circle punctures (see Figure 4). Thus all punctures of D are S-crossing circles, and C must be an S-crossing circle.

Now suppose $S \cap T_C$ consists of two R-longitudes of T_C. We will show $D \subset S$. Since C is an R-crossing circle we know $R \cap T_C$ is an R-meridian, and since D is an R-crossing disk for C it intersects T_C in a single R-longitude of T_C. Thus $D \cap T_C$ is a single curve parallel to, or equal to one of, the curves in $S \cap T_C$.

First consider the case where $D \cap T_C$ equals one of the curves of $S \cap T_C$, call it γ. Then S and D are embedded totally geodesic surfaces whose intersection contains γ which, despite being geodesic in the induced Euclidean metric on T_C, is not a geodesic in M. If S and D met transversally, their intersection could only contain geodesics. As D is connected and intersects S in a nongeodesic curve, it must be a subset of S.

Now consider the case where $D \cap T_C$ is parallel to, and distinct from, the curves of $S \cap T_C$. We will show this case leads to a contradiction. As above, let M' be a flat FAL with reflection surface R' that admits an isometry $h\colon M' \to M$ for which $S = h(R')$. Further, let $D' = h^{-1}(D)$ and $T' = h^{-1}(T_C)$.

Since we are assuming $D \cap T_C$ is disjoint from $S \cap T_C$, so also $D' \cap T'$ and $R' \cap T'$ are disjoint and D' is not a subset of R'. Then D' is a nonreflection, thrice-punctured sphere in M' and must be one of the types described in Theorem 2.7 (see Figure 4). The classification shows that punctures of nonreflection, thrice-punctured spheres are either knot meridians, crossing longitudes or crossing meridians. Of these, only the crossing meridians of singly separated disks are disjoint from the reflection surface (see Figure 4(c)).

Thus D' is a singly separated disk in M' and $D' \cap T'$ consists of two crossing circle meridians. This, however, contradicts the fact that $D \cap T_C$ is a single component.

Therefore, if $S \cap T_C$ consists of two R-longitudes of T_C, then D is a component of S and we are in case (i) of the theorem. □

3.1 Three reflection surfaces

In this subsection we classify flat FAL complements that contain more than two distinct reflection-like surfaces.

Theorem 3.4 *Suppose a flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ admits at least three distinct reflection-like surfaces. Then $\mathcal{A}$ is equivalent to the Borromean rings and M contains exactly three reflection-like surfaces, all of which are reflection surfaces.*

Proof First, we show that a flat FAL complement M can admit at most three distinct reflection-like surfaces. This follows from the observation that at most one reflection-like surface can satisfy each case of Proposition 3.3. To see this, let D be a crossing disk with respect to a reflection-like surface R, and suppose S_1 and S_2 are reflection-like surfaces distinct from R in M. If both S_1 and S_2 contain D (so satisfy Proposition 3.3(i)), then $\iota_{S_2} \circ \iota_{S_1}$ is the identity on D and preserves the normal direction. By [3, Proposition A.2.1] we have $S_1 = S_2$. On the other hand, if S_1 and S_2 both satisfy Proposition 3.3(ii), then both intersect D orthogonally along the same separating geodesic. This implies $\iota_{S_2} \circ \iota_{S_1}$ fixes any point on this geodesic as well as the tangent space there, so again $S_1 = S_2$.

Thus a flat FAL complement has at most three distinct reflection-like surfaces, and now suppose M has three distinct reflection-like surfaces: R, S, and P. Moreover, let D be an R-crossing disk bounded by the R-crossing circle C. Then the argument given above shows that (up to relabeling) P contains D while S and D satisfy Proposition 3.3(ii). In particular, R and S intersect T_C in disjoint pairs of R-meridians. Our goal is to show that P consists of exactly two (disjoint) thrice-punctured spheres, which allows us to quickly classify all such flat FAL complements with such a reflection surface.

Since S and D satisfy Proposition 3.3(ii), C is an S-crossing circle. We will let D' be an S-crossing disk for C. Note that $D' \neq D$ since S intersects D in a separating geodesic so D cannot be an S-crossing disk. Now consider the surfaces R and P relative to D'. Since D' intersects S orthogonally, and $R \cap T_C$ is parallel to $S \cap T_C$, we have R intersects D' orthogonally. Proposition 3.3 applied to D' and R implies that they satisfy case (ii). Hence P and D' must satisfy case (i). Thus D' is a component of P, and $D \cup D' \subset P$.

Now consider $D \cup D'$ relative to the reflection-like surface S. The disk D is an S-singly-separated disk which shares a longitudinal crossing-circle puncture with the S-crossing disk D'. Thus D' and D form a separating pair by Theorem 2.8. Since no proper subset of a reflection-like surface separates, $P = D \cup D'$.

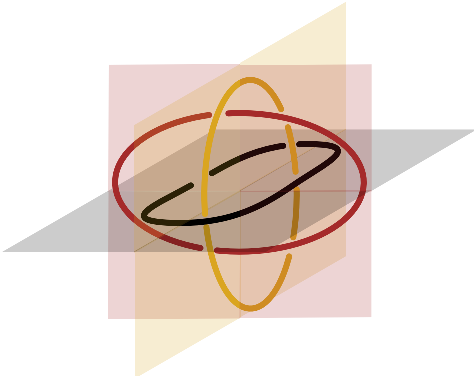


Figure 5: Three reflection surfaces in the Borromean rings complement.

The reflection-like surface $P = D \cup D'$ has a total of six punctures, and must intersect the boundary of each cusp in two curves; so there are three cusps in M. The only three-component flat FAL is the Borromean rings, so $\mathcal{A}$ must be the Borromean rings.

Finally, note that the Borromean rings complement contains three distinct reflection surfaces, as depicted in Figure 5. Each shaded plane is a reflection surface with the link component it contains serving as the one knot circle. □

3.2 Two reflection surfaces

Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with reflection-like surface R and an additional (distinct) reflection-like surface R'. Our goal in this subsection is to show any such flat FAL must be equivalent to either the Borromean rings, P_n with $n \geq 3$, or O_n with $n \geq 2$. See Figure 6 for FAL diagrams of P_4 and O_4.

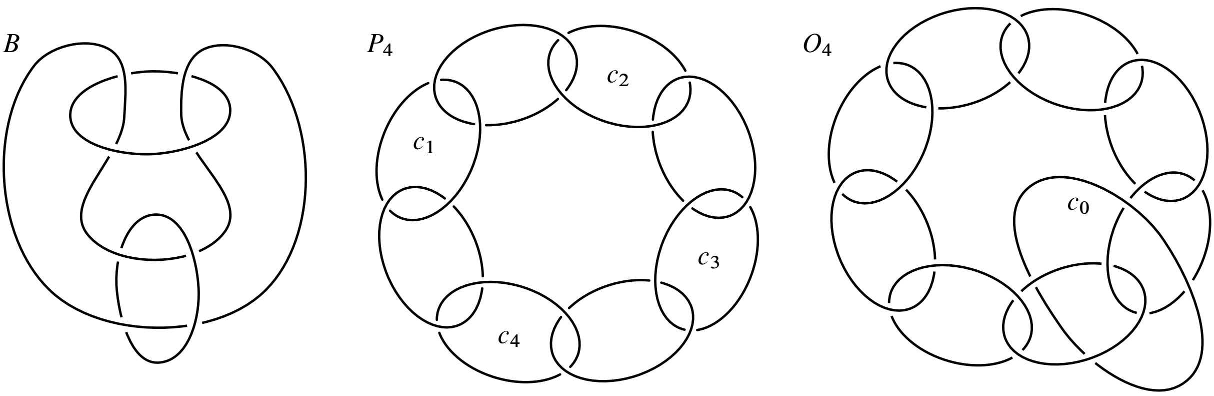


Figure 6: On the left, the Borromean rings, B, is depicted as a flat FAL. In the middle, P_4 is depicted with its four crossing circles labeled. On the right, O_4 is depicted, which can be constructed by adding the crossing circle C_0 to P_4.

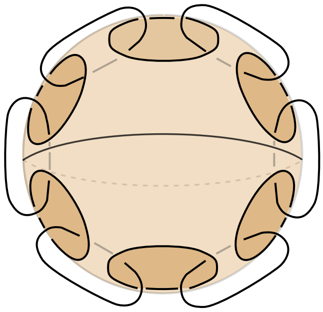


Figure 7: The second reflection-like surface in P_6.

More generally, P_n is the minimally twisted chain with $2n$ components. Thus P_n is a flat FAL that admits an FAL diagram with n crossing circles and n knot circles, linked together in a chain alternating between crossing circles and knot circles. This link can also be described as fully augmenting a pretzel link with n twist regions with an even number of crossings in each twist region, and performing a homeomorphism of the complement to undo all twists from each twist region. The link O_n is a flat FAL that admits an FAL diagram with $n+1$ crossing circles and n knot circles, where this FAL diagram can be obtained by adding a single crossing circle to the FAL diagram for P_n, just as depicted in Figure 6.

Each of P_n and O_n contain (at least) two distinct reflection surfaces: one corresponding with the projection plane in Figure 6 and one corresponding with a 2-sphere containing $C_1, \dots, C_n$ and meeting $K_1, \dots, K_n$ orthogonally in these figures; in the case of O_n, this 2-sphere meets the link component C_0 orthogonally. A $90°$ rotation of $\mathbb{S}^3$ about the axis of the chain interchanges the two reflection surfaces (see Figure 7 for the second reflection-like surface in P_6).

We now prove a useful fact about thrice-punctured spheres contained in a reflection surface, which we will make use of throughout this subsection.

Lemma 3.5 *Let M be a flat FAL complement with reflection-like surface R, and let D be a thrice-punctured sphere component of R. Then D has an odd number of R-knot circle punctures.*

Proof The properties involved are topological, so without loss of generality we assume that R is the reflection surface of the flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$. Further let $\mathbb{S}^2$ be the projection two-sphere in M, so that $R = \mathbb{S}^2 \setminus \mathcal{A}$ is the reflection surface. Finally, let D be a thrice-punctured sphere component of R. Since each component of R has at least one knot circle puncture, we must show that D does not have two knot circle punctures.

Suppose, on the contrary, that exactly two knot circles, labeled J and K, puncture D. We will show that one of J or K has at most one crossing circle linking it, contradicting the fact that every knot circle is linked by at least two crossing circles. Some notation will be useful.

Note that $\mathbb{S}^2 \setminus (J \cup K)$ consists of two disks and an annulus. Let D_J and D_K denote the disk components bounded by J and K, respectively, and note that D is the annular component. Then D is a once-punctured annulus and the additional puncture, label it the point p, comes from a crossing circle C. Now C punctures $\mathbb{S}^2$ twice, and only once in D, so (possibly after relabeling) C also punctures D_K. Now D is a thrice-punctured sphere, so there is a unique geodesic γ_{pK} in D joining the punctures p and K. Thus, if D_C is a crossing disk bounded by C then $\gamma_{pK} = D_C \cap D$.

Now let C' be a crossing circle other than C with a crossing disk D' punctured by J. As C is the only crossing circle puncturing D, we know C' is disjoint from D. However, since J punctures D' we know $D' \cap D$ is a nonempty subset of geodesics on D with at least one endpoint on J. The separating geodesic γ_J of D with both endpoints on J intersects γ_{pK} nontrivially, since γ_{pK} is the nonseparating geodesic of D opposite J. Since $\gamma_{pK} = D_C \cap D$ and crossing disks are disjoint, D' cannot intersect D in γ_J. Further, $D' \cap D$ cannot be the nonseparating geodesic of D joining J and p since p is on the crossing circle C and $C' \neq C$. Thus $D' \cap D$ must be the nonseparating geodesic γ_{JK} of D joining J and K.

We have shown that any crossing disk punctured by J intersects D in the geodesic γ_{JK}. Since crossing disks are orthogonal to the reflection surface, at most one crossing disk intersects D along a given geodesic. Thus at most one crossing circle bounds a crossing disk punctured by J, contradicting the fact that each knot circle in an FAL is linked by at least two crossing circles. □

Each reflection-like surface partitions the cusps corresponding to components of $\mathcal{A}$ into a set of knot circle cusps and a set of crossing circle cusps. If some component of $\mathcal{A}$ changes type, then the geometry of M is greatly restricted. It is also possible that no "swapping" occurs, ie, R and R' induce the same partition on the components of $\mathcal{A}$ into knot circle (cusps) and crossing circle (cusps). We now show that the latter case can not happen when M has two distinct reflection surfaces.

Lemma 3.6 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with two reflection-like surfaces R and R' that induce the same partitions on the cusps of M into crossing circle cusps and knot circle cusps. Then $R = R'$.*

Proof Proposition 4.6 of [13] shows that there exists some R-crossing circle C of $\mathcal{A}$ such that the corresponding R-crossing disk D is the unique totally geodesic thrice-punctured sphere in M that intersects the torus boundary of the cusp corresponding to C in a longitude and intersects the boundary of two R-knot circle cusps (not necessarily distinct) in a meridian on each of these cusps. By assumption, C is also an R'-crossing circle and the two R-knot circle cusps intersecting D are also R'-knot circle cusps. By Lemma 3.5, D can not be a subset of R', and so, D is a nonreflection thrice-punctured sphere for R'. Since D has two R'-knot circle and one R'-crossing circle punctures, Theorem 2.7 implies D must also be the corresponding R'-crossing disk for C. Let T_K be the boundary torus of a cusp neighborhood for one of the knot circles that intersect D. Then $D \cap T_K$ is orthogonal to both $R \cap T_K$ and $R' \cap T_K$. However, if $R \neq R'$, then Proposition 3.2 implies that $R \cap T_K$ and $R' \cap T_K$ are orthogonal, which provides a contradiction. Thus, $R = R'$ under these conditions. □

In the rest of this subsection, we consider the case where a component of $\mathcal{A}$ changes type. First, we give a useful lemma.

Lemma 3.7 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with two distinct reflection-like surfaces R and R'. Suppose L is an R-knot circle and an R'-crossing circle. If D' is an R'-crossing disk for L, then $D' \subset R$; moreover, D' is unique.*

Proof To see this, note that since L is an R-knot circle, the reflection surfaces R' and R intersect T_L in orthogonal pairs of curves by Proposition 3.2. As an R-knot circle there are at least two R-crossing disks that link L. Then $\partial D'$ is a single curve on T_L orthogonal to $R' \cap T_L$, so it must be an R-longitude parallel to $R \cap T_L$. As such D' intersects R-crossing disks linking L, and that intersection must be in a nonseparating geodesic with one puncture on L. To see this let D be an R-crossing disk punctured by L. If D' intersected D in a separating geodesic of D with endpoints on L then $\partial D'$ would consist of two R-longitudes of L, but $\partial D' \cap T_L$ is a single R-longitude since D' is an R'-crossing disk.

Then R and D' have a common intersection with R-crossing disks that link L, and both are orthogonal to such crossing disks, so $D' \subset R$.

To see that D' is unique, assume there is a second R'-crossing disk D^* for L. By the above argument, $D^* \subset R$ as well. Then $D' \cup D^*$ form a separating pair contained in R. Since a proper subset of R cannot separate, R must equal $D' \cup D^*$. As in the proof of Theorem 3.4, a cusp count shows manifold has three cusps and must be Borromean rings. The Borromean rings, however, do not have a crossing circle that bounds two crossing disks in same R'-structure; therefore, D' is unique. □

Suppose K is a component of $\mathcal{A}$ that corresponds with an R-knot circle cusp and an R'-crossing circle cusp. Let D' be the R'-crossing disk bound by K. Then Lemma 3.7 guarantees that D' is a component of R. Since K is an R-knot circle, there exists some R-crossing circle C that links K, along with an R-crossing disk D which K punctures. We now consider the intersection patterns for $D \cap D'$. The work of Yoshida [26, Proposition 3.1] shows that there are three possible cases to consider:

(i) $D \cap D'$ is a single nonseparating geodesic on both D and D',

(ii) $D \cap D'$ is a pair of nonseparating geodesics on both D and D', and

(iii) $D \cap D'$ is a separating geodesic on one and nonseparating on the other.

Since $D' \subset R$ and $D \cap R$ consists of nonseparating geodesics on D, the curve $D \cap D'$ must be nonseparating on D. Hence, in case (iii), the separating geodesic must be on D'. See Figure 8 for corresponding diagrams. These cases are considered separately in the next three propositions.

Proposition 3.8 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with two distinct reflection-like surfaces R and R'. Let K be a component of $\mathcal{A}$ that is an R'-crossing circle and an R-knot circle, and let D' be the R'-crossing disk K bounds. Let C be an R-crossing circle bounding the R-crossing disk D, with K puncturing D. If the intersection $D' \cap D$ is a single nonseparating geodesic on each thrice-punctured sphere* (case (i) *of* Figure 8)*, then $\mathcal{A}$ is either P_n with $n \geq 3$, or O_n with $n \geq 2$.*

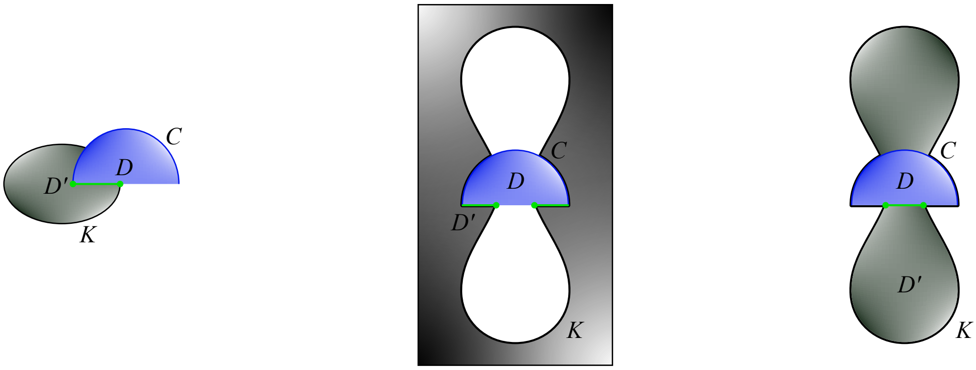


Figure 8: The three cases for intersection patterns for $D' \cap D$ where the projection plane here is determined by R. In each figure, the thrice-punctured spheres D and D' are shaded blue and gray, respectively. Only the top half of the crossing disk D is drawn in each diagram. Intersections between D and D' are highlighted in green.

Proof Let K, D', C and D be as in the statement of the proposition. By Lemma 3.7 the disk D' is a component of R and note that, since $D' \cap D$ is a single nonseparating geodesic on both, D' punctured once each by K and C. Lemma 3.5 implies that the remaining puncture of D' comes from an R-crossing circle, say J.

Since D' has punctures along three distinct cusps, the R-crossing disk D_J that J bounds intersects D' in a single nonseparating geodesic for both disks (case (i) of Figure 8). In the R-structure, then, the cusps J, K, C form a chain of three components.

Now consider J, K and C in the R'-structure. We know that K is an R'-crossing circle and that D' is its corresponding R'-crossing disk. Thus the other two punctures, J and C, must be R'-knot circles. By Lemma 3.7, then, the R-crossing disks D and D_J are components of the reflection surface R'.

Applying Lemma 3.5 to the disks D and D_J in the R'-structure implies the third puncture in each is an R'-crossing circle puncture. Translating the picture to the R-structure, and applying Lemma 3.7 to each end of the chain produces a chain of four or five components. There are four components if the punctures unaccounted for in D and D_J correspond to the same R'-crossing circle.

Since $\mathcal{A}$ has finitely many components, iterating this argument terminates in a sublink of $\mathcal{A}$ isotopic to P_n, and which we denote by $\mathcal{P}$. The R-crossing disks for $\mathcal{P}$ are components of R', and vice versa. For convenience, let $K_1, \dots, K_n$ denote the R-knot circles of $\mathcal{P}$, and let $S_1, \dots, S_n$ be the corresponding thrice-punctured sphere components of R that they bound (so the S_i are R'-crossing disks). Further let $C_1, \dots, C_n$ denote the R-crossing circles of $\mathcal{P}$, and let $D_1, \dots, D_n$ be the respective R-crossing disks that they bound.

Consider the action of $\iota_{R'}$ on the components of the reflection surface R. Each of the S_i reflect to themselves since they are R'-crossing disks. Now let $\widehat{R} = R \setminus \left(\bigcup_{i=1}^n S_i\right)$. Our goal is to show that the K_i are the only R-knot circle punctures of $\widehat{R}$. Cutting $\widehat{R}$ along its intersection with the D_i separates $\widehat{R}$ into two subsets $\widehat{R}_0$ and $\widehat{R}_1$, which are R'-reflections of each other. Let N be an R-knot circle distinct from the K_i that punctures $\widehat{R}$. Then N cannot pass through the D_i, which are already punctured twice by the K_i. Thus N is contained entirely in one of $\widehat{R}_0$ or $\widehat{R}_1$, and $\iota_{R'}(N)$ is in the other. This contradicts the fact that $\iota_{R'}$ preserves each cusp; therefore, no such N exists and the K_i are the only R-knot circle punctures of $\widehat{R}$. Since $R = \widehat{R} \cup \left(\bigcup_{i=1}^{n} S_i\right)$ and the only R-knot circle puncturing S_i is K_i for $i = 1, \dots, n$, we have that the K_i are the only R-knot circles puncturing R, and so, they are also the only R-knot circles of $\mathcal{A}$.

At this stage, any components of $\mathcal{A}$ that are not in $\mathcal{P}$ must be R-crossing circles that puncture $\widehat{R}$ twice. Our goal is to show there is at most one such R-crossing circle. Let $\widetilde{C}$ be an R-crossing circle of $\mathcal{A}$ that is not a component of $\mathcal{P}$, and let $\widetilde{D}$ be an R-crossing disk bounded by $\widetilde{C}$. Since the K_i are the only R-knot circles of $\mathcal{A}$, the disk $\widetilde{D}$ intersects some K_j and the thrice-punctured sphere $S_j \subset R$ that it bounds. The crossing disks of $\mathcal{P}$ intersect S_j in the two nonseparating geodesics on S_j with endpoints on K_j, so $\gamma_j = S_j \cap \widetilde{D}$ must be a separating geodesic on S_j since R-crossing disks must be disjoint.

Suppose there are at least two R-crossing circles in $\mathcal{A} \setminus \mathcal{P}$, and label a second one $\widetilde{C}'$ with R-crossing disk $\widetilde{D}'$. We now describe how to construct an open, embedded, essential annulus $A \subset M$, with boundary (in $\mathbb{S}^3$) curves $\widetilde{C}$ and $\widetilde{C}'$. Consider the standard genus one Heegaard decomposition of $\mathbb{S}^3 = T_1 \cup T_2$, where T_1 and T_2 are solid tori. A sufficiently small closed neighborhood of $\bigcup_{i=1}^{n}(K_i \cup S_i \cup C_i \cup D_i)$ produces a solid torus in $\mathbb{S}^3$ that is isotopic to one of these solid tori, say T_1, and disjoint from $\widetilde{C}$, $\widetilde{C}'$, as well as any additional R-crossing circles not in $\mathcal{P}$. In addition, by taking appropriate neighborhoods, we can assume each of $\widetilde{D}$ and $\widetilde{D}'$ intersects T_1 in a meridional disk of T_1. Note that $\partial T_1 \setminus (\widetilde{D} \cup \widetilde{D}')$ produces two embedded annuli in M, and label one of these A'. Then $A = A' \cup (\widetilde{D} \setminus \operatorname{int}(T_1)) \cup (\widetilde{D}' \setminus \operatorname{int}(T_1))$ provides the desired annulus, contradicting hyperbolicity. Thus, $\mathcal{A} \setminus \mathcal{P}$ is either empty or contains exactly one R-crossing circle that intersects $\widehat{R}$ twice, once in $\widehat{R}_0$ and once in $\widehat{R}_1$, as needed. □

Proposition 3.9 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with two distinct reflection-like surfaces R and R'. Let K be a component of $\mathcal{A}$ that is an R'-crossing circle and R-knot circle, and let D' be the R'-crossing circle disk K bounds. Let C be an R-crossing circle bounding the R-crossing disk D, with K puncturing D. If the intersection $D' \cap D$ is a pair of nonseparating geodesics on each thrice-punctured sphere* (*case* (ii) *in Figure 8*), *then $\mathcal{A}$ is the Borromean rings.*

Proof By the work of Yoshida [26, Lemma 3.7], any hyperbolic 3-manifold with two thrice-punctured spheres intersecting in this manner must be a (possibly empty) Dehn filling of one of three manifolds: a certain double cover of the Whitehead link complement, the Borromean rings complement, or the minimally twisted hyperbolic 4-chain link complement. See Figure 17 in [26] for diagrams of these links. All three of these manifolds have the common hyperbolic volume of $2v_8$, where v_8 denotes the volume of a regular ideal hyperbolic octahedra. At the same time, the work of Purcell [23, Proposition 3.6] shows that the Borromean rings complement is the unique minimal volume flat FAL complement. Since nonempty Dehn filling strictly decreases volume, the only flat FAL complement with thrice-punctured spheres intersecting in this manner is the Borromean rings complement, as needed. □

Proposition 3.10 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with two distinct reflection-like surfaces R and R'. Let K be a component of $\mathcal{A}$ that is an R'-crossing circle and R-knot circle, and let D' be the R'-crossing circle disk K bounds. Let C be an R-crossing circle bounding the R-crossing disk D, with K puncturing D. If the intersection $D' \cap D$ is a separating geodesic on D' and a nonseparating geodesic on D (case (iii) in Figure 8), then $\mathcal{A}$ is either the Borromean rings or O_n with $n \geq 2$.*

Proof Let K, D', C and D be as in the statement of the proposition. By Lemma 3.7 the disk D' is a component of R, and note that it is punctured once by K. Since D' is a thrice-punctured sphere, we know that it must have two more punctures. We break down this proof into cases depending on whether those punctures come from R-crossing circles or R-knot circles. Since $D \cap D'$ is a separating geodesic γ_K on D', the thrice-punctured sphere D partitions D' into two regions separated by γ_K.

Case I Suppose the other two punctures of D' come from R-crossing circles C_1 and C_2. Further, suppose $C_1 = C_2$. Then C_1 and C_2 must puncture different regions of $D' \setminus D$ since D' must have a puncture on each side of the separating geodesic γ_k. In this case, $\mathcal{A}$ contains a Borromean rings sublink, $L_B = K \cup C \cup C_1$. Let D_1 designate the R-crossing disk corresponding to C_1. As an R-crossing disk, D_1 intersects R in the three nonseparating geodesics on D_1. Two of these geodesics must be contained in $D' \subset R$ since C_1 punctures D' in two different regions and D_1 can not intersect γ_K. Since $D' \cap D_1$ is a pair of nonseparating geodesics on D_1, it follows that this intersection is also a pair of nonseparating geodesics on D' by the work of Yoshida [26, Proposition 3.1]. Following the proof of Proposition 3.9 with D_1 replacing D implies that $A = L_B$, ie, A is the Borromean rings.

Now suppose that C_1 and C_2 are distinct R-crossing circles, each of which puncture D'. Then C_1 has an R-crossing disk D_1, which is punctured by K and some distinct R-knot circle K_1. This implies that the intersection $D' \cap D_1$ is a single nonseparating geodesic on each of these thrice-punctured spheres. Then Proposition 3.8 with D_1 replacing D shows that L is either P_n or O_n. However, P_n does not contain a crossing circle C whose crossing disk D separates a region of R, which implies that $\mathcal{A}$ must be O_n here.

Case II Suppose that at least one of the other two punctures of D' comes from an R-knot circle, K_1. We will show this case leads to a contradiction. By Lemma 3.5, D' must have an odd number of R-knot circle punctures, and so, the third puncture of D' comes from an R-knot circle distinct from K and K_1, which we label K_2. Furthermore, since γ_K is a separating geodesic on D', K_1 and K_2 must puncture different regions of $D' \setminus D$. Any R-crossing disk punctured by K_1, call it D_1, is disjoint from D and so must intersect D' in geodesic(s) disjoint from $\gamma_K = D \cap D'$. The only geodesics of D' disjoint from $D \cap D'$ are the nonseparating geodesics $\{\gamma_1, \gamma_2\}$ joining K to the other punctures K_1 and K_2, respectively. Since D_1 is punctured by K_1, it's one other R-knot circle puncture must come from K, and so, $\gamma_1 = D_1 \cap D'$. Now there is at most one connected, embedded, totally geodesic surface orthogonal to R and containing γ_1; therefore, at most one crossing disk intersects D' along γ_1. This implies at most one crossing circle links K_1, contradicting the fact that every knot circle must be linked by at least two crossing circles in an FAL. □

We can now give the following classification of flat FAL complements that admit multiple reflection surfaces. A slightly less general version of this theorem was originally stated in Theorem 1.2 in Section 1.

Theorem 3.11 *Suppose $M = \mathbb{S}^3 \setminus \mathcal{A}$ is a flat FAL complement with multiple distinct reflection-like surfaces. Then either*

- *$\mathcal{A}$ is equivalent to the Borromean rings, and M contains exactly three distinct reflection-like surfaces, all of which are reflection surfaces, or*
- *$\mathcal{A}$ is equivalent to P_n with $n \geq 3$, or O_n with $n \geq 2$, and M contains exactly two distinct reflection-like surfaces, both of which are reflection surfaces.*

Proof Suppose M is a flat FAL complement with at least two distinct reflection-like surfaces. Then Lemma 3.6 shows that some R'-crossing circle K of $\mathcal{A}$ must switch to become an R-knot circle (or vice versa). Let D' be the R'-crossing disk corresponding to K. By Lemma 3.7, $D' \subset R$. At the same time, since K is an R-knot circle, it punctures at least two R-crossing disks, one of which we label D. Then D and D' are both totally geodesic thrice-punctured spheres in M that intersect nontrivially since D must intersect R on each side of K (thinking of K as a simple closed curve in the projection plane), and one of these components is D'. The work of Yoshida [26, Proposition 3.1] shows that there are exactly three possibilities for such intersection patterns, which are covered in Propositions 3.8, 3.9, and 3.10. Combined, these propositions tell us that $\mathcal{A}$ is equivalent to either the Borromean rings, P_n with $n \geq 3$, or O_n with $n \geq 2$. Theorem 3.4 distinguishes the Borromean rings as the only flat FAL whose complement admits at least three distinct reflection-like surfaces. In this case, this FAL complement has exactly three reflection-like surfaces, all of which are reflection surfaces, as noted in Theorem 3.4; see Figure 5 for a visualization of the three reflection surfaces. If an FAL complement has exactly two reflection-like surfaces, then the corresponding link is either P_n with $n \geq 3$ or O_n with $n \geq 2$. The discussion at the beginning of Section 3.2 shows that the corresponding FAL complements for these links each admit two distinct reflection surfaces, and so, every reflection-like surface is also a reflection surface in all of these cases. □

This classification theorem implies that within the family of flat FALs, those whose complements admit multiple reflection surfaces are determined by their complements. We highlight this result in the following corollary.

Corollary 3.12 *Let $\mathcal{A}$ and $\mathcal{A}'$ be flat FALs, and suppose the complement of $\mathcal{A}$ admits multiple distinct reflection surfaces. Then $\mathbb{S}^3 \setminus \mathcal{A}$ is homeomorphic to $\mathbb{S}^3 \setminus \mathcal{A}'$ if and only if $\mathcal{A}$ and $\mathcal{A}'$ are equivalent links.*

Proof If $\mathcal{A}$ and $\mathcal{A}'$ are equivalent links, the orientation-preserving homeomorphism between the pairs $(\mathbb{S}^3, \mathcal{A})$ and $(\mathbb{S}^3, \mathcal{A}')$ induces one between their complements. So, suppose $\mathbb{S}^3 \setminus \mathcal{A}$ and $\mathbb{S}^3 \setminus \mathcal{A}'$ are homeomorphic flat FAL complements where $\mathbb{S}^3 \setminus \mathcal{A}$ admits multiple reflection surfaces. Then these flat FAL complements are isometric and $\mathbb{S}^3 \setminus A'$ also contains multiple reflection-like surfaces. By Theorem 3.11, A' is equivalent to either the Borromean rings B, P_n with $n \geq 3$, or O_n with $n \geq 2$. Thus, we just need to consider the cases where $\mathbb{S}^3 \setminus \mathcal{A}$ is homeomorphic to the complement of one of these links. Note that B has three components, each P_n has $2n$ components with $n \geq 3$, and O_n has $2n+1$ components with $n \geq 2$. Thus the number of components distinguishes the links B, $\{P_n\}_{n=3}^{\infty}$ and $\{O_n\}_{n=2}^{\infty}$. Since there is a one-to-one correspondence between components of a link and cusps of the corresponding link complement, this shows that the number of cusps of one of these link complements determines the corresponding link, completing the proof. □

The work from this section places an important restriction on the behavior of homeomorphisms between flat FAL complements.

Corollary 3.13 *Let M and M' be flat FAL complements and suppose there exists a homeomorphism $h\colon M \to M'$, which induces an isometry ρ_h. Then R is a reflection-like surface for M if and only if R is a reflection surface for M. In particular, ρ_h provides a one-to-one correspondence between reflection surfaces.*

Proof By the comments immediately following Definition 2.4, the isometry ρ_h produces a one-to-one correspondence between reflection-like surfaces. If we show that every reflection-like surface is actually a reflection surface, we will be done.

Theorem 3.11 covers the multiple reflection-like surface case. On the other hand, the reflection surface in a flat FAL is one reflection-like surface. Therefore, if a flat FAL has a unique reflection-like surface it must be the reflection surface, completing the proof. □

Before moving on, we make two useful observation that follow from Corollary 3.13. First off, we will no longer use the term reflection-like surface since a reflection-like surface is a reflection surface. In addition, if $h\colon M \to M'$ is a homeomorphism, M has a unique reflection surface R, and M' contains a reflection surface R', then M' also has a unique reflection surface and $\rho_h(R) = R'$.

Our final result from this section shows that in most cases, every symmetry of a flat FAL complement with multiple reflection surfaces is induced by a symmetry of that link. In particular, the following theorem proves the first statement from Theorem 1.3 in the introduction.

Theorem 3.14 *Let $\mathcal{A}$ be a flat FAL, other than P_3, whose complement admits multiple reflection surfaces. Then both $\mathcal{A}$ and its complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ have the same symmetry group.*

Proof Since every symmetry of $(\mathbb{S}^3, \mathcal{A})$ induces one of its complement, it's enough to show that every homeomorphism $h\colon M \to M$ extends to an isotopy of $\mathbb{S}^3$.

Since $\mathcal{A}$ is not P_3, Theorem 3.11 implies it is either P_n with $n \geq 4$, O_n with $n \geq 2$, or the Borromean rings, and we consider the cases separately.

Let $\mathcal{A} = O_n$, with $n \geq 2$, and let R be a reflection surface in $M = \mathbb{S}^3 \setminus O_n$. Note that R consists of n thrice-punctured spheres and one $(n+2)$-punctured sphere S^2_{n+2} whose punctures are n longitudes along the R-knot circles of O_n and 2 punctures by the same R-crossing circle C_0. Since $n \geq 2$, S^2_{n+2} is the only component of R with more than three punctures.

Now let $h\colon M \to M$ be a homeomorphism with induced isometry $\rho_h\colon M \to M$. Corollary 3.13, applied to the case where $M' = M$, implies that $R' = \rho_h(R)$ is a reflection surface for O_n. Since R' is a reflection surface for O_n, it has a unique $(n+2)$-punctured sphere component ${S^2_{n+2}}'$. The unique cusp punctured twice by ${S^2_{n+2}}'$ corresponds to an R'-crossing circle while the remaining n cusps punctured by ${S^2_{n+2}}'$ correspond to the R'-knot circles of O_n. Since ρ_h preserves the topology of components of R we have $\rho_h(S^2_{n+2}) = {S^2_{n+2}}'$, and R-knot circles must map to R'-knot circles. Moreover, $C_0' = \rho_h(C_0)$ must be the R'-crossing circle of O_n puncturing ${S^2_{n+2}}'$ twice. The remaining components of O_n are R-crossing circles and, since all R'-knot circles are accounted for, ρ_h must map them to R'-crossing circles.

Thus ρ_h preserves the type of each component of O_n-crossing circles map to crossing circles, and similarly with knot circles. Let L be a component of O_n with image $L' = \rho_h(L)$, and let $\{m, \ell\}$ and $\{m', \ell'\}$ denote the respective R- and R'- meridian-longitude pairs. Lemma 3.1 implies ρ_h maps the set $\{m, \ell\}$ to the $\{m', \ell'\}$, and we must show it takes meridians to meridians.

If K is an R-knot circle of O_n, then $T_K \cap R$ is a pair of simple closed curves, each representing an R-longitude, which we denoted by ℓ earlier. As above, let $\{m', \ell'\}$ denote the R' meridian and longitude for $T_{K'} = \rho_h(T_K)$. Then $\rho_h(T_K \cap R) = T_{K'} \cap R'$, and so maps longitudinal slopes of K to those of K'. As above, Lemma 3.1 implies that ρ_h preserves meridians as well. Thus ρ_h preserves peripheral structures on all R-knot circles.

To see that h preserves peripheral structures on R-crossing circles, let C be an R-crossing circle with R'-crossing circle image $C' = \rho_h(C)$. Using notation similar to the above, we have the calculation $\rho_h(T_C \cap R) = T_{C'} \cap R'$ so ρ_h preserves meridional slopes, and Lemma 3.1 implies it preserves peripheral structures on crossing circles of O_n as well.

Thus ρ_h, and therefore our original $h\colon M \to M$, preserves peripheral structures on all components and extends to an isotopy of $\mathbb{S}^3$. This implies, of course, that a symmetry of M is the restriction of a symmetry of $(\mathbb{S}^3, O_n)$ to its complement.

The proof for P_n, with $n \ge 4$, follows similarly, but is a little more direct. In this case the reflection surface R consists of n thrice punctured spheres and one n-punctured sphere S^2_n. Since $n \ge 4$, the sphere S^2_n is the unique component of R with more than three punctures. The proof follows as above, with the simplification that punctures of S^2_n correspond to the distinct R-knot circles of P_n.

We have proven the theorem for flat FALs with two reflection surfaces and the only case remaining is the flat FAL with three reflection surfaces – the Borromean rings. In this case, a quick check using SnapPy confirms the symmetry group of the Borromean rings and its complement coincide, with common symmetry group $\mathbb{Z}_2 \times G$, where G represents the group of symmetries of the octahedron. □

The proof of Theorem 3.14 does not extend to P_3. For P_3 all components of the reflection surface R are thrice-punctured spheres, with one punctured by three knot circles. A simple puncture count, then, does not guarantee that the isometry ρ_h preserves the component of R punctured by knot circles. Section 5 will show that this is more than just a shortcoming of the above proof. In fact P_3 is a signature link, and it will be seen that signature link complements have more symmetries than the links themselves.

4 Transition to the unique reflection surface case

Corollary 3.13 tells us that homeomorphic flat FAL complements must have the same number of reflection surfaces. This allows us to break down the proof of our main result, Theorem 1.1, into two cases:

(1) There exists a homeomorphism between flat FAL complements, each with multiple reflection surfaces. This case is already covered by Corollary 3.12.

(2) There exists a homeomorphism between flat FAL complements, each with a unique reflection surface.

We can actually put a more narrow focus on the homeomorphisms we need to analyze in case (2). Let M and M' be the complements of the flat FALs $\mathcal{A}$ and $\mathcal{A}'$, respectively, each containing unique reflection surfaces denoted by R and R'. If there exists a homeomorphism $h\colon M \to M'$, which induces isometry ρ_h, then we must have $R' = \rho_h(R)$. Recall that the reflection surfaces determine peripheral structures on each component of $\mathcal{A}$ and $\mathcal{A}'$, respectively. Thus if h preserves both knot circles and crossing circles, then it preserves peripheral structures and extends to an isotopy of $\mathbb{S}^3$, making the links $\mathcal{A}$ and $\mathcal{A}'$ equivalent. For this reason we will be mainly interested in homeomorphisms that change a knot circle K into a crossing circle, or vice versa. In this case we will say that h *changes the type* of a component K of $\mathcal{A}$, and will call h a *type-changing homeomorphism*.

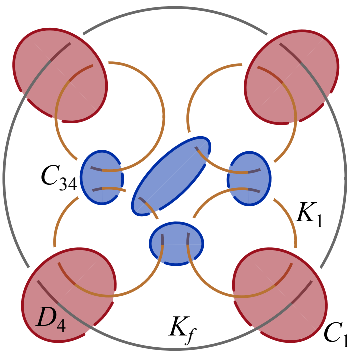
The rest of this paper is dedicated to analyzing type-changing homeomorphisms of flat FAL complements containing unique reflection surfaces. Section 5 will introduce a particular class of type-changing homeomorphisms where we can easily find an isotopy in $\mathbb{S}^3$ between the corresponding FALs. Section 6 examines how separating sets (partially introduced in Section 2) behave under type-changing homeomorphisms to help restrict the behavior of such homeomorphisms. Combining the work of these two sections, we then prove our main result in Section 7.

5 Signature links and full-swap homeomorphisms

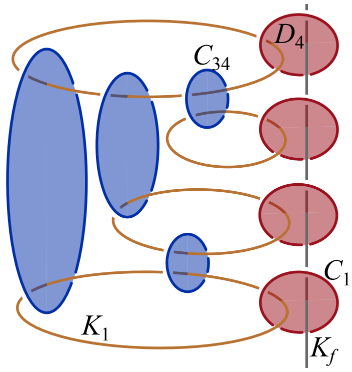
In this section we define signature links, which are a special class of FALs, and show that they admit a particular type-changing homeomorphisms, which we call a full-swap, on their complements. Furthermore, we will show that signature links whose complements are homeomorphic via a full-swap exhibit an explicit isotopy in $\mathbb{S}^3$.

Definition 5.1 A *signature link* is a flat FAL $\mathcal{L}$ whose components can be partitioned into four nonempty sets

$$\mathcal{L} = \{K_f\} \cup \mathcal{K} \cup \mathcal{C} \cup \mathcal{C}_{\mathcal{K}},$$



(a) aesthetic diagram



(b) pragmatic diagram

Figure 9: Two diagrams of a signature link $\mathcal{L}$.

with the following properties. The first set consists solely of a knot circle K_f which cuts the projection plane into two disks, one of which contains the remaining knot circles $\mathcal{K} = \{K_1, \dots, K_n\}$, which we call the inside of K_f. Each $K_i \in \mathcal{K}$ is linked with K_f by a unique crossing circle C_i, and $\mathcal{C} = \{C_1, \dots, C_n\}$. The set of remaining crossing circles is denoted by $\mathcal{C}_{\mathcal{K}}$, and components in $\mathcal{C}_{\mathcal{K}}$ link two distinct knot circles in $\mathcal{K}$. A crossing circle of $\mathcal{C}_{\mathcal{K}}$ that links $K_i, K_j \in \mathcal{K}$ will be denoted by C_{ij}.

Throughout this section, we let D_i designate a crossing disk for $C_i \in \mathcal{C}$ and we let D_{ij} designate a crossing disk for $C_{ij} \in \mathcal{C}_{\mathcal{K}}$.

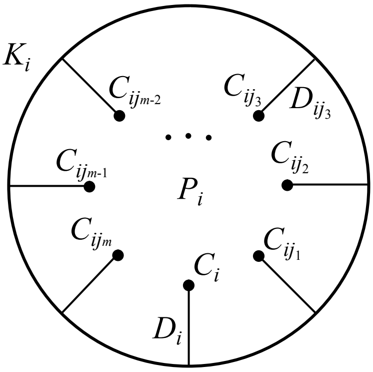
Figure 9(a) depicts a signature link. The knot circles in $\mathcal{K}$ are all inside K_f, and are numbered counterclockwise around K_f (only K_1 is labeled in Figure 9(a)). The diagram of Figure 9(b) will be helpful in visualizing a full-swap, and is obtained by isotoping K_f until it is vertical. We remark that there can be more than one way to decompose $\mathcal{L}$ as a signature link. The knot circle K_4 (unlabeled in Figure 9) could have been designated K_f instead since it is linked to every other knot circle.

According to Definition 5.1 the link P_3 is a signature link, but a quick check verifies that this is the only link whose complement contains multiple reflection surfaces which is a signature link. Thus all other signature link complements have a unique reflection surface.

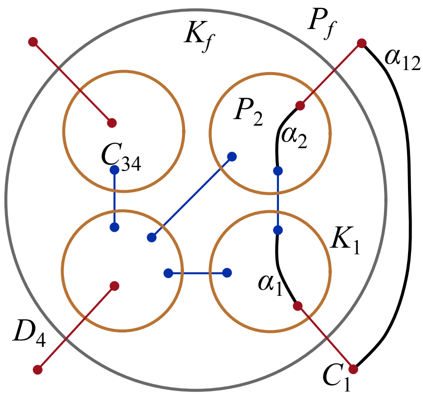
The fact that $\mathcal{L}$ is hyperbolic places restrictions on the set $\mathcal{C}_{\mathcal{K}}$. The set $\mathcal{C}_{\mathcal{K}}$, for example, can not be empty. More can be said, of course, about properties of $\mathcal{C}_{\mathcal{K}}$ resulting from the hyperbolicity of $\mathcal{L}$, but we content ourselves with a result about longitudinal disks which requires the following technical lemma.

Lemma 5.2 *Two N-disks in an FAL complement are either identical or disjoint.*

Proof To see this, we show that if two N-disks intersect, then they are identical. Let D_1, D_2 be N-disks in an FAL complement M, and so, they each intersect the reflection surface R in their nonseparating geodesics. Lemma 3.4 of [26] states that thrice-punctured spheres in orientable three-manifolds cannot



(a) crossing disks in $\mathcal{P}_i$



(b) α curves for D_ℓ

Figure 10: Crossing disks and a longitudinal disk intersecting the reflection surface.

intersect along a geodesic that is separating in both. Thus, if $\gamma \in D_1 \cap D_2$, then γ is nonseparating in at least one disk, say D_1. The disk D_1 is an N-disk so γ is contained in the reflection surface R and D_2 intersects D_1 along a geodesic in R. By Theorem 2.7 both D_1 and D_2 are orthogonal to R along γ. Therefore, since D_1 and D_2 are both embedded totally geodesic surfaces that intersect in a common geodesic and are orthogonal to R, we can conclude that they must be equal, completing the proof. □

Lemma 5.3 *Let $\mathcal{L}$ be a signature link. Then every crossing circle $C_{ij} \in \mathcal{C}_\mathcal{K}$ bounds a totally geodesic, longitudinal disk with crossing circles C_i and C_j.*

Proof Let $\mathcal{L}$ be a signature link with complement $M = \mathbb{S}^3 \setminus \mathcal{L}$. Given a crossing circle $C_{ij} \in \mathcal{C}_\mathcal{K}$, we will construct a longitudinal disk with punctures C_i, C_j, C_{ij} by gluing two disks, which are essentially topological descriptions of the geodesic disks described in [20]. Afterwards, we will show this longitudinal disk is totally geodesic.

A knot circle $K_i \in \mathcal{K}$ of a signature link $\mathcal{L}$ bounds two disks in the projection plane, and we define the *inside* of K_i to be the disk $\mathcal{P}_i$ that does not contain K_f. Thus $\mathcal{P}_i$ is punctured once by C_i and once by each crossing circle of $\mathcal{C}_\mathcal{K}$ that links K_i.

Now consider how crossing disks intersect $\mathcal{P}_i$. Since all crossing circles in $\mathcal{L}$ link distinct knot circles, and since K_i is the only knot circle puncture of $\mathcal{P}_i$, no crossing disk intersects $\mathcal{P}_i$ in a geodesic arc with both endpoints on K_i. Crossing disks that intersect $\mathcal{P}_i$, then, do so in a geodesic joining a crossing circle puncture to the boundary curve K_i. Further, since crossing disks are disjoint they intersect $\mathcal{P}_i$ in disjoint arcs (see Figure 10(a)). The complement of crossing disks in $\mathcal{P}_i$ is then connected and there is an embedded arc, disjoint from crossing disks, between any two crossing circle punctures of $\mathcal{P}_i$.

Similarly, the knot circle K_f divides the projection plane into two topological disks, one punctured by the knot circles of $\mathcal{K}$ and the other by the crossing circles of $\mathcal{C}$. The *outside* of K_f, denoted by $\mathcal{P}_f$, refers to the disk punctured by the crossing circles of $\mathcal{C}$. The argument of the previous paragraph shows that there is an embedded arc in $\mathcal{P}_f$ joining any two crossing circle punctures that is disjoint from crossing disks of $\mathcal{L}$.

Given a crossing circle $C_{ij} \in \mathcal{C}_\mathcal{K}$ we construct a longitudinal disk D_ℓ, with punctures C_i, C_j, C_{ij}, by gluing a disk D_+ above the reflection surface to its reflection D_-. We begin by describing the boundary of D_+. Let $\alpha_i \subset \mathcal{P}_i$ be an embedded arc disjoint from crossing disks that joins C_i and C_{ij} punctures, and define α_j similarly. Also let α_{ij} denote an embedded arc in $\mathcal{P}_f$ which is disjoint from crossing disks and joins the C_i and C_j punctures. Figure 10(b), for example, illustrates the α arcs corresponding to the crossing circle C_{12} of the signature link in Figure 9.

The arcs $\alpha_i, \alpha_j, \alpha_{ij}$, together with the top halves of the crossing circles C_i, C_j, C_{ij}, form a simple closed curve γ in $\mathbb{S}^3$. Let M^+ be the region of M above the reflection surface and we wish to show that γ bounds a disk in M^+. First, M^+ is a handlebody, since it is a three-ball with arcs removed for each crossing circle. Further, the top half of each crossing disk is a meridional disk for each handle. Thus removing an open neighborhood of each crossing disk from M^+ results in a three-ball with γ in its boundary. The curve γ, then, bounds a disk D_+ in M^+. The disk D_+ is disjoint from crossing disks and intersects the reflection surface along the α arcs in its boundary. Let D_- be the reflection of D_+, and let $D_\ell = D_+ \cup D_-$. Then the crossing circles C_i, C_j, C_{ij} form the boundary of D_ℓ in $\mathbb{S}^3$, and the interior of D_ℓ is an embedded thrice-punctured sphere in M with longitudinal punctures along the crossing circles C_i, C_j, C_{ij}, as desired.

To see that D_ℓ is totally geodesic it is enough to show that it is incompressible and boundary incompressible, by Theorem 3.1 of [1]. The proof given here is essentially that of [23, Lemma 2.1], but slightly simpler because the punctures of D_ℓ are distinct. Suppose α is a curve in D_ℓ that bounds a compressing disk $D \subset M \setminus D_\ell$. Then α separates D_ℓ into two pieces, one of which contains a single puncture C of D_ℓ. Thus $\alpha \cup C$ bound an annulus in D_ℓ whose union with D is a boundary compressing disk for the crossing circle C, contradicting the fact that M is hyperbolic.

When discussing ∂-incompressibility it will be convenient to think of M as the interior of a orientable, closed three-manifold $\overline{M}$ with torus boundary components. In this case, D_ℓ is properly embedded in $\overline{M}$ with longitudinal boundary curves along T_i, T_j, T_{ij}, the boundary tori of $\overline{M}$ corresponding to cusps C_i, C_j, C_{ij} of M. For convenience, and by an abuse of notation, we let C_i, C_j, C_{ij} denote the boundary curves of D_ℓ as well.

Now suppose that D is a ∂-compressing disk for D_ℓ. Then $\partial D = \alpha \cup \beta$ where $\alpha = D \cap D_\ell$ is an arc in D_ℓ with both endpoints on the same boundary curve, say C_i, and β is an arc in T_i. The arc α is isotopic to a separating geodesic of D_ℓ, which decomposes D_ℓ into two annuli, and we let A denote the one containing C_j. The other boundary component of the annulus A consists of the arc α together with a subarc of C_i, call it δ. Gluing the ∂-compressing disk D to A along their intersection α yields another annulus $A \cup D$ with C_j as one boundary component. The other boundary component of $A \cup D$ is the simple closed curve $\beta \cup \delta$ on the boundary torus T_i.

If $\beta \cup \delta$ bounds a disk on T_i, a copy of it in M caps off one boundary component of $A \cup D$, creating a ∂-compressing disk for C_j, which is impossible. On the other hand, suppose $\beta \cup \delta$ is nontrivial on T_i. Then $A \cup D$ is an incompressible annulus which is not boundary parallel since its boundary curves are on separate boundary components of $\overline{M}$. Again, this contradicts the fact that M is hyperbolic.

Thus D_ℓ is incompressible and ∂-incompressible and it has a totally geodesic representative by [1, Theorem 3.1]. □

The remainder of this section is devoted to constructing a "full-swap", which is a type-changing homeomorphism of the complement of a signature link $\mathcal{L}$. Full-swaps, despite changing the types of some components of $\mathcal{L}$, will be shown to produce a link equivalent to $\mathcal{L}$.

To begin, we define an ml-swap homeomorphism on a flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ in terms of Dehn twists on a Hopf sublink $\mathcal{H} \subset \mathcal{A}$. In particular, the Hopf sublink will consist of a knot and crossing circle pair that are linked. Zevenbergen, in [27], first constructed a product that exchanged meridional and longitudinal slopes on each component of $\mathcal{H}$ and was able to analyze the effect on the other components of $\mathcal{A}$. We review his construction here, then provide an alternative cut-and-paste construction which highlights how ml-swaps effect reflection surfaces and crossing disks.

Figure 11 illustrates an ml-swap on the Hopf sublink determined by the knot- and crossing-circle pair labeled $\{K, C\}$. Before describing the Dehn twists we observe some features of $\mathcal{A}$ relative to $\mathcal{H}$. Since $\mathcal{H} = K \cup C$ is a Hopf link, the crossing circle C must link distinct knot circles and we label the other one J. Let D be a crossing disk for C and let N be an open regular neighborhood of the cell complex $K \cup C \cup D$ in $\mathbb{S}^3$. Then N is an unknotted open solid torus, so $W = \mathbb{S}^3 \setminus N$ is a closed unknotted solid torus in $\mathbb{S}^3$. The neighborhood N can be chosen so that $N \cap \mathcal{A}$ contains the components K and C, and an arc of J that intersects D (see Figure 11(a)). Then W contains all components of $\mathcal{A} \setminus (K \cup C \cup J)$ together with one arc of J.

We now describe Zevenbergen's Dehn twists that make up an ml-swap and, abusing notation, we refer to components by their original labels throughout the Dehn twist process. First perform a Dehn twist along K, which adds a full twist to W and links C around W as in Figure 11(b). For simplicity, isotope C so that it is flat, twisting J and K vertical in the process, to get Figure 11(c). Now perform a Dehn twist along C that untwists W so that it is returned to its original form. This twist unlinks K and W while linking the arc of J with both W and K, as in Figure 11(d). Finally, perform a Dehn twist on K that unlinks C and J, obtaining the link depicted in Figure 11(e). This composition of Dehn twists is an ml-swap. The result will not be another flat FAL in general, as observed in [27], but we will see that performing multiple ml-swaps on signature links can produce a flat FAL.

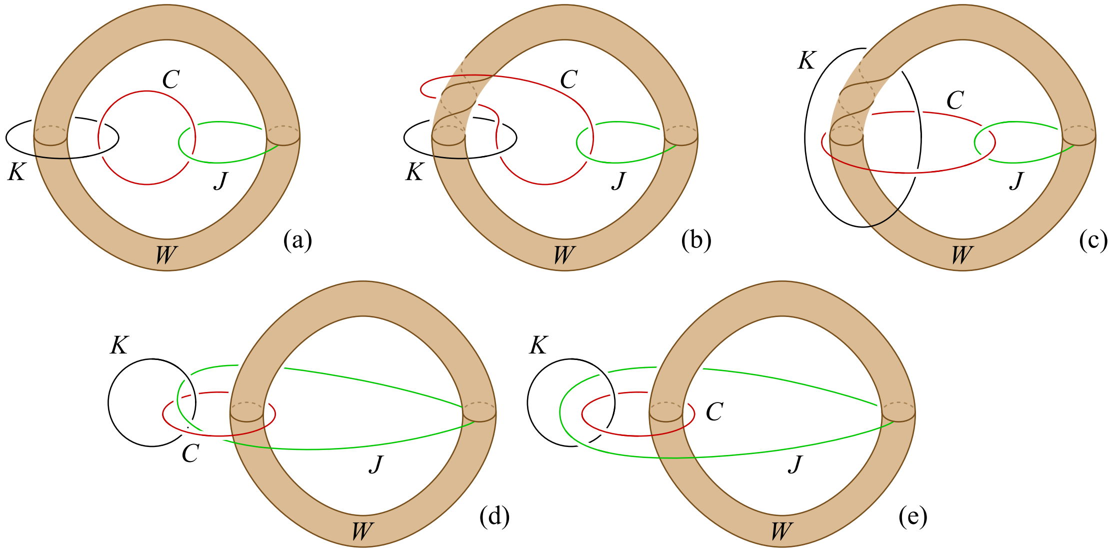


Figure 11: An ml-swap as a product of Dehn twists.

This product of Dehn twists is ultimately a local operation in the sense that changes to the link occur within a 3-ball containing the Hopf sublink. Moreover, if there are multiple Hopf sublinks that are contained in disjoint three-balls, then the result of performing Dehn twists on each Hopf sublink is independent of the order in which they are done. With this background in place we make the following definition.

Definition 5.4 An *ml-swap* on a Hopf sublink $\mathcal{H}$ of a flat FAL $\mathcal{A}$ is the homeomorphism resulting from performing the Dehn twists just described above on the components of $\mathcal{H}$. A *full-swap* on a signature link $\mathcal{L}$ is the composition of all ml-swaps on Hopf sublinks $K_i \cup C_i$, for $i = 1, \dots, n$.

Dehn twists provide a convenient description of an ml-swap, and we now consider an alternative description that highlights the effect of an ml-swap on the reflection surface and crossing disks involved. Figure 12(a) highlights a crossing circle C and knot circle K whose meridians and longitudes will be swapped. The other knot circle linked by C is included to emphasize how the reflection surface moves, but the rest of the link is not pictured. The homeomorphism maps C to the knot circle C' and K to the crossing circle K' depicted in Figure 12(g). An ml-swap preserves the reflection surface, while moving the location of the component R_0 to that of R_0'.

Let us walk through the homeomorphism one step at a time. In Figure 12(b), torus neighborhoods of C and K are pictured to emphasize what happens to meridians and longitudes. Now slice the manifold along the reflection surface and consider the top half pictured in Figure 12(c), which is a handlebody H_+ (the homeomorphism on the bottom half is the reflection of that pictured). In Figure 12(c), the half-tori around K and C, as well as the copy of R_0, form three annuli. The middle row of Figure 12 depicts an isotopy sliding the three annuli along a handle of H_+. In the process meridians and longitudes of K and

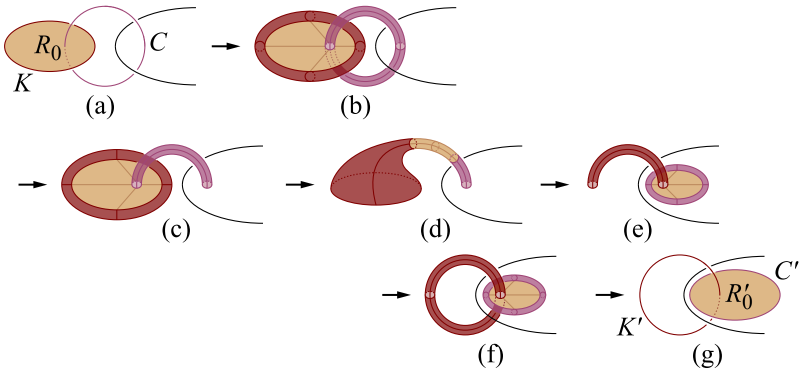


Figure 12: An ml-swap.

C are swapped. The final row depicts regluing the isotoped $H_\pm$ along R, and finally removing the torus neighborhoods of K' and C'. In terms of peripheral structures on cusps of M, an ml-swap swaps meridians and longitudes on cusps corresponding to C and K, and leaves the remaining structures the same.

It is instructive to consider the image under an ml-swap of crossing disks punctured by the knot circle involved. The image of a crossing disk D bounded by C is the natural first choice to consider. Figure 13 illustrates that D is sliced in half, each half rotated by "a third", then reglued along its nonseparating geodesics. A similar rotation is done on the bottom half, so swapping the types of C and K has the effect of "rotating" D.

Let C^* be a crossing circle other than C linking K. Let D^* a crossing disk bounded by C^*, and consider the image of D^* under the ml-swap. Since the K-puncture of D^* becomes a crossing circle puncture, its image $D^{*\prime}$ has two crossing circle punctures. This is illustrated in Figure 14. Since an N-disk in an FAL complement has either one or three crossing circle punctures, we know $D^{*\prime}$ is not a crossing disk, assuming the image of $M = \mathbb{S}^3 \setminus \mathcal{A}$ under this ml-swap is a flat FAL complement.

In fact, an ml-swap on a flat FAL can result in a link that is not an FAL (see [27]). Performing ml-swaps on all possible (C_i, K_i) pairs in a signature link (ie a full-swap), however, does produce another flat FAL. Consider, for example, the simplest signature link: the chain P_3 in Figure 15(a). A full-swap homeomorphism h_f is realized by performing successive ml-swaps on the Hopf sublinks $C_1 \cup K_1$ and $C_2 \cup K_2$, which yields the sequence of Figure 15.

The image of P_3 is isotopic to P_3. Proposition 5.5 will show that this holds more generally — that the image of a signature link under a full-swap is always isotopic to the original link. While full-swaps

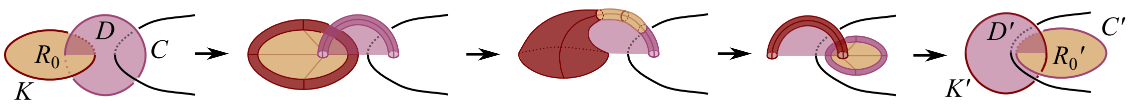


Figure 13: Rotating a crossing disk in an ml-swap.

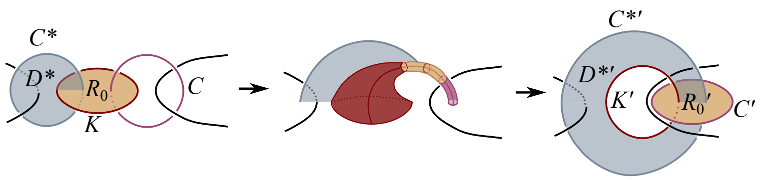


Figure 14: Sliding a crossing disk in an ml-swap.

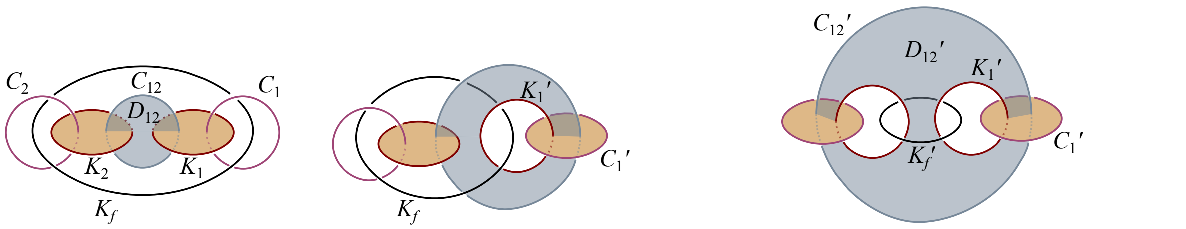
produce equivalent links, they do interchange some crossing- and longitudinal-disks. For example, in Figure 15(c), the image of the crossing disk D_{12} is the longitudinal disk D'_{12}. In addition, the longitudinal disk with punctures $\{C_1, C_2, C_{12}\}$ of Figure 15(a) becomes the crossing disk that C'_{12} bounds (neither are pictured, but reading Figure 15 backwards illustrates the change from longitudinal to crossing disk). The proof of Proposition 5.5 will show that a full-swap on an arbitrary signature link interchanges crossing- and longitudinal-disks for every crossing circle in $\mathcal{C}_{\mathcal{K}}$.

The pragmatic diagram of Figure 9(b) will be more convenient for the proof of Proposition 5.5, so we assume K_f is vertical and crossing circles of $\mathcal{C}$ are ordered from bottom to top. Figure 16 illustrates the effect of a full-swap on such a diagram of a signature link. Do note that h_f *does not* map the crossing disk for C_{24} to that of C'_{24} but to the longitudinal disk with punctures K'_2, K'_4, and C'_{24}. This happens on a more general scale and will be justified in the following proof.

Proposition 5.5 *Let $\mathcal{L}$ be a signature link and let the homeomorphism h_f designate the full-swap on all Hopf sublinks $C_i \cup K_i$. Then $\mathcal{L}$ and $h_f(\mathcal{L})$ are isotopic links. In particular, $h_f(\mathcal{L})$ is a signature link.*

Proof Let $\mathcal{L} = \{K_f\} \cup \mathcal{K} \cup \mathcal{C} \cup \mathcal{C}_{\mathcal{K}}$ be a signature link with $\mathcal{K}$ and $\mathcal{C}$ each containing n components. Since each ml-swap maps a link in $\mathbb{S}^3$ to a link in $\mathbb{S}^3$, we have that $h_f(\mathcal{L})$ is a link in $\mathbb{S}^3$ by construction. We will describe how to build $h_f(\mathcal{L})$ by performing a full-swap on the pragmatic diagram of $\mathcal{L}$, where K_f corresponds with the z-axis and so that each C_j is contained in a vertical translate of the xy-plane at height $z = j$, as depicted on the left side of Figure 16. Here, the yz-plane corresponds with the projection plane for an FAL diagram of $\mathcal{L}$ with each $C_{ij} \in \mathcal{C}_{\mathcal{K}}$ linking only K_i and K_j and meeting the yz-plane orthogonally.

Now, consider the sublink $\mathcal{L}_s = \{K_f\} \cup \mathcal{K} \cup \mathcal{C}$ and its image $h_f(\mathcal{L}_s)$. Recall that h_f is the composition of n ml-swaps, one performed on each Hopf sublink $(C_i, K_i) \in \mathcal{C} \times \mathcal{K}$ for $i = 1, \ldots, n$; see Figure 12 for the



(a) original FAL (b) non-FAL after one ml-swap (c) FAL isotopic to original after two ml-swaps

Figure 15: Two ml-swaps.

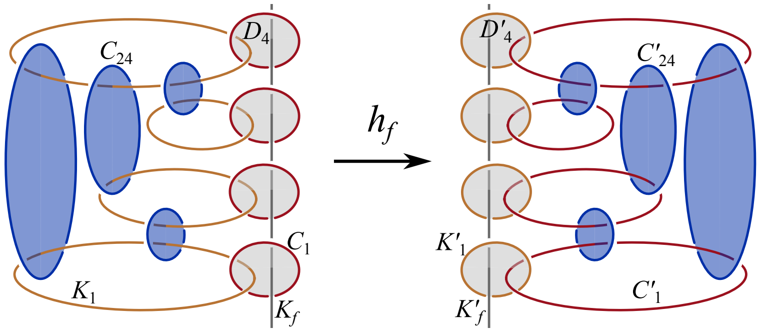


Figure 16: The type-changing homeomorphism h_f applied to a signature link.

local picture of a single ml-swap. Then $h_f(\mathcal{L}_s)$ is constructed from $\mathcal{L}_s$ by keeping K_f fixed as the z-axis, while each Hopf sublink $h_f(K_j \cup C_j)$ links $K'_f = h_f(K_f)$ via $K'_j = h_f(K_j)$. In addition, each K'_j is contained in the vertical translate of the xy-plane at height $z = j$, as depicted on the right in Figure 16.

We will now show that the images of the crossing circles in $\mathcal{C}_\mathcal{K}$ under h_f are unlinked unknots, which will help us determine how $h_f(\mathcal{C}_\mathcal{K})$ behaves. Every component of $\mathcal{L}$ is unknotted, and links those components used in the Dehn twists of an ml-swap at most once. In this situation, the Dehn twists never knot an unknotted component so the image of each component in $\mathcal{L}$ is an unknot as well. Further, an ml-swap does not link two crossing circles of $\mathcal{C}_\mathcal{K}$ because the linking introduced by the first Dehn twist along K in Figure 11 is undone by the following twist along C. In addition, since the ml-swaps that comprise a full-swap occur in disjoint 3-balls, the same applies to a full-swap.

Now unknots in $\mathbb{S}^3$ have a canonical peripheral structure in which a meridian links the component once and a longitude bounds an embedded disk in its complement. An ml-swap preserves this $\mathbb{S}^3$-peripheral structure on all components of $\mathcal{L}$ except K and C, for which it swaps meridians and longitudes. This observation allows us to discuss the *topology* of the images of thrice-punctured spheres under a full-swap by analyzing images of their punctures.

A crossing disk D_{ij} has a longitudinal puncture along C_{ij} and meridional punctures along K_i and K_j. Since meridians of K_i, K_j map to $\mathbb{S}^3$-longitudes of K'_i, K'_j, the image D'_{ij} of D_{ij} under a full-swap is a thrice-punctured sphere with longitudinal punctures along each of C'_{ij}, K'_i and K'_j. Similarly, the longitudinal disk D^{ℓ}_{ij} (guaranteed by Lemma 5.3) has longitudinal punctures along C_{ij}, C_i and C_j. A full-swap maps longitudes of C_i and C_j to meridians in the $\mathbb{S}^3$-peripheral structure of C'_i and C'_j, so $D^{\ell\,\prime}_{ij}$ has a $\mathbb{S}^3$-longitudinal puncture along C'_{ij}, and $\mathbb{S}^3$-meridional punctures along C'_i and C'_j. The component C'_{ij}, then, bounds an embedded disk in $\mathbb{S}^3$ punctured once by each of C'_i and C'_j. This implies C'_{ij} links only C'_i and C'_j.

Using these representatives for $\mathcal{L}$ and $h_f(\mathcal{L})$, we see that a rotation along the z-axis by 180° provides the necessary isotopy between $\mathcal{L}$ and $h_f(\mathcal{L})$ in $\mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3$. Furthermore, $h_f(\mathcal{L})$ is a signature link $\{K'_f\} \cup \mathcal{K}' \cup \mathcal{C}' \cup \mathcal{C}'_\mathcal{K}$, where $K'_f = h_f(K_f)$, $\mathcal{K}' = h_f(\mathcal{C})$, $\mathcal{C}' = h_f(\mathcal{K})$, and $\mathcal{C}'_\mathcal{K} = h_f(\mathcal{C}_\mathcal{K})$. □

6 Separating sets

In this section, we will utilize two important subsets of thrice-punctured spheres whose removal separates an FAL complement: separating pairs and separating quadruples. The behavior of type-changing homeomorphisms between flat FAL complements is significantly restricted by the existence of separating sets.

Separating pairs were introduced in Section 2 as a pair of disjoint thrice-punctured spheres whose union separates an FAL complement. Theorem 4.8 from [20] was also introduced in that section, which states that a pair of thrice-punctured spheres $\{S_1, S_2\}$ is a separating pair if and only if each is either a crossing disk or a singly separated disk and their longitudinal slopes coincide.

We now introduce a particular type of separating set consisting of four thrice-punctured spheres. Let D be a longitudinal disk with longitudinal punctures along the crossing circles C_1, C_2, C_3, and let D_i be a crossing disk with crossing circle puncture C_i. Then the set $Q = \{D, D_1, D_2, D_3\}$ is a separating set of four thrice-punctured spheres. The sets we are concerned with have one additional property.

Definition 6.1 Let D be a longitudinal disk in a flat FAL complement M with crossing circle punctures C_1, C_2, and C_3 that bound crossing disks D_1, D_2 and D_3. Then $Q = \{D, D_1, D_2, D_3\}$ is a *separating quadruple* if each crossing disk D_i is punctured by distinct knot circles.

The signature links of Section 5 contain separating quadruples. To see this, note that in a signature link every crossing circle links distinct knot circles. Moreover, Lemma 5.3 guarantees the existence of a longitudinal disk D^{ℓ}_{ij} for every $C_{ij} \in \mathcal{C}_{\mathcal{K}}$. Thus, each triple (C_i, C_j, C_{ij}) in a signature link generates a separating quadruple Q_{ij}. The remark below will show that Q_{ij} is unique up to a choice of crossing disks.

We introduce some terminology. A general separating quadruple Q is illustrated in Figure 17, which depicts only those components which puncture Q. In a flat FAL knot circles cannot cross each other, so knot circles puncture adjacent crossing disks. Let K_i denote the knot circle puncturing Q that is *opposite* the crossing circle C_i in the sense that they are not linked. By an abuse of terminology, a *component of* (*or in*) Q will refer to components that puncture disks in Q.

Remark We also point out that two separating quadruples with the same crossing circle punctures have the same knot circle punctures and longitudinal disk. Suppose Q and Q' are separating quadruples with the same crossing circle punctures. Lemma 4.5 of [20] shows that there is at most one longitudinal disk containing any two given crossing circle punctures, let alone three, so Q and Q' have the same longitudinal disk. Now recall that crossing circles in a separating quadruple link distinct knot circles, and only those components of $\mathcal{A}$. This implies that every disk they bound is punctured by the same two knot circles. Hence two separating quadruples sharing the same crossing circle punctures can differ only in their crossing disks.

We begin with the following lemma which immediately leads to a special case, Corollary 6.3, of our main result. This lemma will also be essential for establishing some technical results on how separating quadruples can behave under a type-changing homeomorphism.

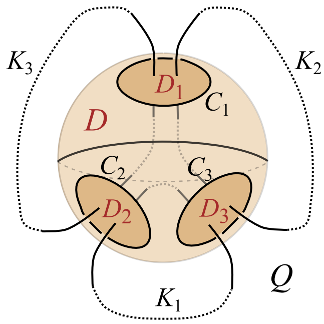


Figure 17: A separating quadruple.

Lemma 6.2 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with a unique reflection surface, and let C be a crossing circle in $\mathcal{A}$ that links the same knot circle K twice. If $h\colon M \to M'$ is a homeomorphism of flat FAL complements, then h preserves the types of both C and K.*

Proof Let C be a crossing circle in a flat FAL complement M containing a unique reflection surface R. Further, let C bound a crossing disk D which is punctured twice by the same knot circle K, and suppose $h\colon M \to M'$ is a homeomorphism of flat FAL complements. The assumption that the reflection surface $R \subset M$ is unique implies that M' has a unique reflection surface R', and that $R' = h(R)$.

We first prove that $K' = h(K)$ must be a knot circle in M'. Suppose, on the contrary, that K' is a crossing circle in M'. Then meridians of K map to longitudes of K' because both are perpendicular to reflection surfaces and $R' = h(R)$. The fact that $R' = h(R)$ further implies that $D' = h(D)$ is a nonreflection thrice-punctured sphere in M', since D is in M. The two meridional K-punctures of D map to two longitudinal K'-punctures of D'. However, by Theorem 2.7, nonreflection thrice-punctured spheres in a flat FAL complement do not have two longitudinal punctures along the same crossing circle. So, K' must be a knot circle in M'.

Since K' is a knot circle puncturing the disk D' twice in M', the remaining puncture of D' must be an M' crossing circle by the characterization of Theorem 2.7. The remaining puncture of D' is the image $C' = h(C)$ of C, and h preserves the types of both C and K. □

If a flat FAL (whose complement has a unique reflection surface) has a single knot circle, as is the case for FALs of two-bridge links with an even number of twist regions (other than the Borromean rings, which has three reflection surfaces), then every crossing circle links the same knot circle twice. Thus any homeomorphism preserves the type of all components and their peripheral structures and can be realized by an isotopy of $\mathbb{S}^3$. This observation leads to the following immediate corollary of Lemma 6.2:

Corollary 6.3 *A flat FAL with a single knot circle is determined by its complement among all flat FALs. In particular, flat FALs corresponding to 2-bridge links with an even number of twists are determined by their complements.*

Proof The only flat FAL with one knot circle and multiple reflection surfaces is the Borromean rings, which we've already seen to be determined by its complement. The unique reflection surface case follows from Lemma 6.2. □

Remark We would like to emphasize the necessity of the unique reflection surface hypothesis in Lemma 6.2. For instance, as noted in Theorem 3.4, the Borromean rings complement admits three distinct reflection surfaces and the flat FAL diagram for this link has two crossing circles and a single knot circle that links each crossing circle twice. However, there exists homeomorphisms of the Borromean rings complement where both a crossing circle and the knot circle switch types.

We now prove a technical lemma considering homeomorphisms that change a crossing disk to a longitudinal disk, or vice versa. It turns out that such a homeomorphism h implies the existence of a separating quadruple Q, and the action of h on Q can be made quite precise.

Lemma 6.4 *Let M and M' be homeomorphic flat FAL complements with unique reflection surfaces, and $h\colon M \to M'$ a homeomorphism that changes the type of an N-disk $D \subset M$. Then:*

(i) *The disk D is part of a separating quadruple Q in M whose image $Q' = h(Q)$ in M' is also a separating quadruple.*
(ii) *The homeomorphism h fixes the types of exactly one opposite knot- and crossing-circle pair K_f, C_f in Q.*
(iii) *The longitudinal disk and exactly one crossing disk in Q change type under h. These disks share the crossing circle puncture C_f.*

Proof We are given that M and M' are homeomorphic flat FAL complements with unique reflection surfaces, and that $h\colon M \to M'$ is a homeomorphism changing the type of an N-disk D. Since M and M' each have unique reflection surfaces R and R', we have $R' = h(R)$. Then the image of a nonreflection thrice-punctured sphere in M is nonreflection in M'. Further, N-disks and singly separated disks are distinguished by the topological property of separating, which implies that h maps N-disks to N-disks.

Consider first the case where D is a longitudinal disk whose image $D' = h(D)$ is a crossing disk. Let C_1, C_2, C_3 denote the crossing circle punctures of D, and let D_i be a choice of crossing disk bounded by C_i. To show D is part of a separating quadruple we must show that the D_i are punctured by distinct knot circle components.

Two of the crossing circle punctures of D, say C_2, C_3, change type because D' is a crossing disk. Thus Lemma 6.2 implies D_2, D_3 are punctured by distinct knot circle punctures. Now consider D_1, whose image D_1' must be a crossing disk or longitudinal disk since N-disks map to N-disks. If D_1' stays a crossing disk, then Theorem 2.8 implies that $\{D_1', D'\}$ is a separating pair since both are crossing disks sharing the crossing circle puncture C_1'. This cannot happen because the pair $\{D_1, D\}$ does not separate in M (again by Theorem 2.8). Therefore, D_1 changes type and D_1' must be a longitudinal disk. Then, by Lemma 6.2, D_1 has distinct knot circle punctures and $Q = \{D, D_1, D_2, D_3\}$ is a separating quadruple in M.

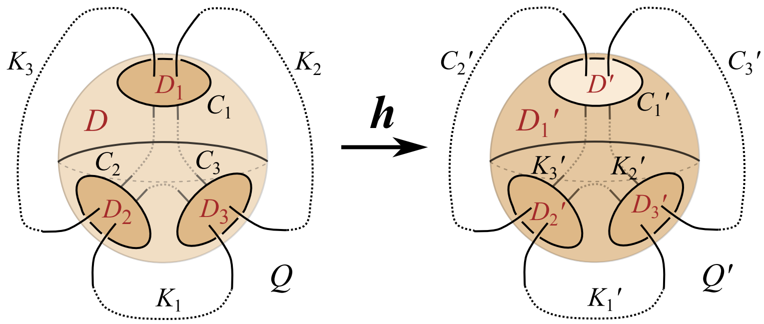


Figure 18: Homeomorphic image of Q when longitudinal disk changes type.

For convenience, label the knot circle punctures K_1, K_2, K_3, where K_i is the knot circle opposite C_i in that it does not puncture D_i.

To see that $Q' = \{D', D_1', D_2', D_3'\}$ is also a separating quadruple, we must show that it consists of three crossing disks, with distinct knot circle punctures, and one longitudinal disk. The disk D' is assumed to be a crossing disk and, in this case, the disk D_1' was shown to be longitudinal. Moreover, since C_2', C_3' are knot circles, the disks D_2', D_3' must be crossing disks in M' because these are the only nonreflection disks with knot circle punctures (Theorem 2.7). Further, each of the disks in Q are punctured by three distinct components, so their images are as well, and each crossing disk in Q' is punctured by distinct knot circle components. Thus Q' is a separating quadruple in M'.

This analysis proves the third conclusion as well, since D, D_1 change type while the crossing disks D_2, D_3 do not.

To see statement (ii) note that C_1', K_2', K_3' are the crossing circle punctures of Q' since they are the punctures of the longitudinal disk D_1'. The remaining punctures of Q' must be knot circles in M', so K_1' is a knot circle. Thus h preserves the type of the opposite knot- and crossing-circle pair C_1, K_1, while changing the type of all other components. Observe that C_1 is the crossing circle puncture shared by the disks that change type, namely D and D_1.

Now suppose D is a crossing disk in M that changes type to a longitudinal disk D' in M'. Apply the previous argument to h^{-1} and D', then note that if h^{-1} and D' satisfy the conclusions of the lemma, then so does h and D. □

Lemma 6.4 can be applied to the full-swap homeomorphisms discussed in Section 5, revealing some of the geometric structure inherent in such homeomorphisms. Before proceeding we remark that any homeomorphism between flat FAL complements with unique reflection surfaces that does not preserve peripheral structures must change the type of a knot circle. Indeed, if a crossing circle changes type, then one of the knot circles it links changes type as well since there are no (nonreflection) thrice-punctured spheres with three knot circle punctures. Thus if h changes the type of a component, there must be a knot circle that changes type.

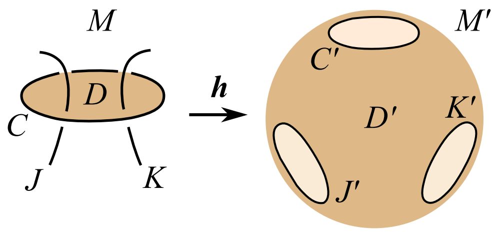


Figure 19: Crossing disk D maps to longitudinal disk D'.

Lemma 6.5 *Let $h\colon M \to M'$ be a homeomorphism between flat FAL complements with unique reflection surfaces, and suppose h changes the type of the knot circle K in M. Then h changes the type of exactly one crossing circle C_1 linked by K and, of all crossing disks punctured by K, h fixes the type of exactly those bounded by C_1.*

Proof First we show h changes the type of at most one crossing circle linking K. Suppose, on the contrary, that C_1, C_2 are distinct crossing circles linking K whose images C_1', C_2' are both knot circles in M'. Let D_1, D_2 be a choice of crossing disks they bound and note that, since C_1, C_2 are distinct, the disks D_1 and D_2 do not form a separating pair by Theorem 2.8. To determine if the image of an N-disk is a crossing disk, it is enough to show that it has a knot circle puncture because homeomorphisms between flat FAL complements with unique reflection surfaces map N-disks to N-disks. Since $D_i' = h(D_i)$ is punctured by the knot circle C_i', the disk D_i' must be a crossing disk in M'. The disks D_1', D_2' also share the crossing circle puncture K' and so form a separating pair in M', again by Theorem 2.8. The homeomorphic image of a nonseparating set, however, cannot be separating, and h changes the type of at most one crossing circle linking K.

Now we argue that the image of at least one crossing circle linking K is a knot circle in M'. Since K is linked by at least two crossing circles, at most one of which can change type, there is a crossing circle C linking K whose image C' is a crossing circle in M'. Let D be a crossing disk bounded by C and J be the knot circle other than K which punctures D. By Lemma 6.2, $J \neq K$. Since $D' = h(D)$ has two crossing circle punctures in K' and C', we find J' must also be a crossing circle. Hence D' is a longitudinal disk in M' (see Figure 19).

Thus D is a crossing disk in M that changes type, and Lemma 6.4 shows that D is part of a separating quadruple $Q = \{D, D_1, D_2, D_3\}$. Assume we've labeled disks so that D_3 is the longitudinal disk, and J punctures D_2 while K punctures D_1. Finally, let C_1 be crossing circle puncture of D_1 and K_f the final knot circle in Q. Figure 20 depicts a "schematic" diagram of this labeling in the sense that components of the FAL which are not in Q are not pictured.

Now h changes the crossing disk D to a longitudinal disk, so Lemma 6.4(iii) implies the images of D_1, D_2 are again crossing disks. Since D_1' is a crossing disk with crossing circle puncture K', the crossing circle C_1 changes type under h. Thus at least one crossing circle linking K maps to a knot circle.

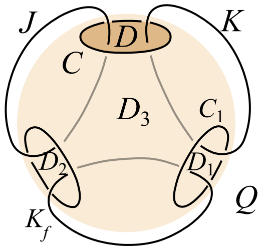


Figure 20: Schematic of Q.

Combining that with the first part of the proof shows that h changes the type of exactly one crossing circle linking K.

To see the last statement, let C_1 be the unique crossing circle linking K that changes type and suppose D_1 is any crossing disk punctured by K which is bounded by C_1. Since C_1 changes type, the image $D_1' = h(D_1)$ has a knot circle puncture in M' and must be a crossing disk.

Conversely, suppose D is a crossing disk punctured by K and bounded by the crossing circle $C \neq C_1$. Then two punctures of D', both C' and K', are crossing circles in M'. Theorem 2.7 implies the third puncture of D' is also a crossing circle, and D' is a longitudinal disk. □

Lemmas 6.4 and 6.5 allow us to show, in the following lemma, that separating quadruples exist in the presence of type-changing homeomorphisms.

Lemma 6.6 *Let $h\colon M \to M'$ be a homeomorphism between flat FAL complements with unique reflection surfaces, and suppose K is a knot circle that changes type under h. Further, let C_1 be the unique crossing circle linking K that changes type under h. Then each crossing circle $C \neq C_1$ that links K is part of a separating quadruple that includes the punctures K, C_1 and C.*

Proof Since $C \neq C_1$, Lemma 6.5 implies that h fixes the type of C so $C' = h(C)$ is a crossing circle in M'. The image D' of a crossing disk D bounded by C, then, is punctured by the crossing circles K' and C'; therefore, D' must be a longitudinal disk. Thus D is an N-disk that changes type, and Lemma 6.4 implies it is part of a separating quadruple Q.

It remains to show that C_1 must be a crossing circle puncture in Q. First, since K punctures D it is a knot circle component of Q and must puncture one other crossing disk, say D_1, in Q. Moreover, since exactly one crossing disk of Q changes type, by Lemma 6.4(iii), h fixes the type of D_1. Lemma 6.5 then implies that D_1 is bounded by C_1, finishing the proof. □

Lemmas 6.5 and 6.6 demonstrate that flat FALs which admit a type-changing homeomorphism contain features similar to those of signature links. This motivates considering a certain sublink, the signature sublink, which we define in the next section.

7 Complements determine flat FALs

In this section we show that flat FALs are determined by their complements. We start by assuming that a flat FAL complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ has a unique reflection surface and admits a type-changing homeomorphism to another flat FAL complement, since all other cases are either trivial or covered by the work in Section 3. With these assumptions, we first show in Section 7.1 that any such $\mathcal{A}$ contains a sublink, $\mathcal{L}_h$, that features many of the properties of a signature link. Then in Section 7.2 we prove some technical tools involving separating quadruples, which are used to introduce an embedded two-sphere, S^2_α, in $\mathbb{S}^3$ that intersects any such $\mathcal{A}$ in only two points on $\mathcal{K}_f$ and separates the other components of $\mathcal{A}$ in a useful manner. From here, our next goal is to show that $\mathcal{L}^c = \mathcal{A} \setminus \mathcal{L}_h$ must be empty and in fact, $\mathcal{A}$ must be a signature link. This is all done in Section 7.3. Essentially, if $\mathcal{L}^c$ is nonempty or $\mathcal{A}$ fails to have any of the features necessary to be a signature link, then we can use our two-sphere S^2_α to show that $\mathcal{A}$ is either a connect-sum or a split link, contradicting hyperbolicity. Once we have proven that $\mathcal{A}$ must be a signature link, then we can quickly show that this type-changing homeomorphism is a full-swap, possibly pre- or post-composed with homeomorphisms that extend to isotopies of $\mathbb{S}^3$. At this point, we can use Proposition 5.5 to obtain the desired result.

7.1 Signature sublinks

The forthcoming lemma highlights the necessary properties for us to define a signature sublink.

Lemma 7.1 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with a unique reflection surface, and suppose there is a type-changing homeomorphism $h\colon M \to M'$ to another flat FAL complement M'. Then $\mathcal{A}$ contains a sublink*

$$\mathcal{L}_h = \{K_f\} \cup \mathcal{K} \cup \mathcal{C} \cup \mathcal{C}_{\mathcal{K}},$$

whose components satisfy the following properties:

(i) *The sets $\mathcal{K}$ and $\mathcal{C}$ contain $n \geq 2$ knot- and crossing-circles, respectively.*

(ii) *Each crossing circle of $\mathcal{C} = \{C_1, \dots, C_n\}$ links the corresponding knot circle of $\mathcal{K} = \{K_1, \dots, K_n\}$ and the knot circle K_f. Further, the homeomorphism h changes the types of the components in $\mathcal{C} \cup \mathcal{K}$ and fixes the type of K_f.*

(iii) *Let $\mathcal{C}_{\mathcal{K}}$ denote the set of all crossing circles of $\mathcal{A}$ that link distinct knot circles in $\mathcal{K}$. Then h fixes the type of each component in $\mathcal{C}_{\mathcal{K}}$. Moreover, each knot circle in $\mathcal{K}$ is linked with at least one crossing circle in $\mathcal{C}_{\mathcal{K}}$, and at most one crossing circle in $\mathcal{C}_{\mathcal{K}}$ links two given knot circles in $\mathcal{K}$.*

Proof We begin by showing that h changes the type of at least one knot circle in $\mathcal{A}$. Since M and M' are homeomorphic and M has a unique reflection surface R, Corollary 3.13 say that $R' = h(R)$, where R' is the unique reflection surface for M'. Suppose h only changes the type of crossing circles. Let C be a crossing circle that changes type and D a crossing disk bounded by C. Then $D' = h(D)$ would be a thrice-punctured sphere with three knot circle punctures, and Theorem 2.7 implies that D' must be part of the reflection surface R'. This cannot occur since D is a nonreflection thrice-punctured sphere in M and $R' = h(R)$; therefore, h changes the type of a knot circle, call it K_1.

Since h changes K_1 to a crossing circle, Lemma 6.5 implies there is a unique crossing circle C_1 that links K_1 and changes type. Lemma 6.2 implies that C_1 links distinct knot circles, K_1 and another which we denote by K_f. The type of K_f is fixed by h; otherwise, a crossing disk bounded by C_1 would map to a nonreflection thrice punctured sphere with exactly one knot circle puncture, contradicting Theorem 2.7.

The type-changing homeomorphism h, then, guarantees the existence of a knot circle K_f whose type is fixed, together with a Hopf sublink $\{K_1, C_1\}$ whose types change and for which K_f is linked by C_1.

Define $\mathcal{C}$ to be all crossing circles of $\mathcal{A}$ which link K_f *and* change type. Note that $\mathcal{C}$ contains at least two crossing circles. Indeed, since K_1 and C_1 satisfy the hypotheses of Lemma 6.6 we conclude that every crossing circle $C \neq C_1$ that links K_1 is part of a separating quadruple Q_C that includes the components K_1, C_1, and C. Moreover, since C_1 links K_f, the knot circle K_f is a puncture of Q_C as well. Lemma 6.4(ii) shows that both crossing circles of Q_C that link K_f change type, so that both are in $\mathcal{C}$. Thus $\mathcal{C}$ contains at least two crossing circles of $\mathcal{A}$.

Now each $C_i \in \mathcal{C}$ changes type so links distinct knot circles (Lemma 6.2), K_f and a second knot circle $K_i \in \mathcal{A}$. Moreover, since C_i changes type and the type of K_f is fixed, the argument above for the existence of K_1 shows that K_i must change type. Finally, since each knot circle that changes type is linked by a unique crossing circle that changes type (Lemma 6.5), the K_i are distinct. Let $\mathcal{K} = \{K_1, \dots, K_n\}$ denote the knot circles (other than K_f) linked by crossing circles in $\mathcal{C}$, and note that h changes the type of each knot circle in $\mathcal{K}$.

At this stage we have proven parts (i) and (ii) of the lemma.

Now define $\mathcal{C}_{\mathcal{K}}$ to be all crossing circles of $\mathcal{A}$ that link two knot circles of $\mathcal{K}$, and let C_{ij} denote a crossing circle linking $K_i, K_j \in \mathcal{K}$. To see that h fixes the type of C_{ij}, note that a crossing disk D_{ij} bounded by C_{ij} is punctured by both K_i and K_j. Since h changes K_i and K_j to crossing circles, the disk $h(D_{ij})$ must be a thrice-punctured sphere with at least two crossing circle punctures. Theorem 2.7 implies $h(D_{ij})$ must be a longitudinal disk, so h fixes the type of C_{ij}.

We now address the existence statements of part (iii) of the lemma. First fix a knot circle $K_i \in \mathcal{K}$. We must show there is at least one crossing circle in $\mathcal{C}_{\mathcal{K}}$ linking K_i. As in the above proof that $\mathcal{C}$ contains at least two crossing circles, Lemma 6.6 applies to the Hopf sublink $\{K_i, C_i\}$. Thus each crossing circle $C \neq C_i$ that links K_i is part of a separating quadruple Q_C that includes K_f as a puncture. Again as above, Lemma 6.4(ii) implies that C links two knot circles of $\mathcal{K}$, so that for each K_i there is at least one $C_{ij} \in \mathcal{C}_{\mathcal{K}}$. In this case let Q_{ij} denote the separating quadruple Q_C, and note that the crossing circles of Q_{ij} are C_i, C_j, and C_{ij}.

Now fix a pair of knot circles $K_i, K_j \in \mathcal{K}$, and suppose $C_{ij} \in \mathcal{C}_{\mathcal{K}}$ exists. We begin by showing C_{ij} is a puncture in a separating quadruple Q whose crossing circle punctures are C_i, C_j, and C_{ij}. Since $C_{ij} \neq C_i$ links K_i, Lemma 6.6 implies it is part of a separating quadruple Q containing the punctures K_i, C_i, and C_{ij}. The knot circles K_i, K_j, and K_f are the knot circle punctures of Q, since they are linked by C_i and C_{ij}. The final crossing circle of Q must link K_f and K_j, and must change type since C_{ij} is the only crossing circle of Q whose type is fixed by h. Therefore the final crossing circle is C_j, and Q has crossing circle punctures C_i, C_j, and C_{ij}.

Hence each crossing circle linking K_i and K_j forms a longitudinal disk with C_i and C_j. Now Lemma 4.2 of [20] shows there is at most one longitudinal disk with two given crossing circle punctures, so there is at most one C_{ij}. □

The sublink $\mathcal{L}_h$ of Lemma 7.1 satisfies many of the properties of a signature link. In particular, $\mathcal{L}_h$ consists of knot and crossing circles partitioned into nonempty sets that satisfy all linking requirements of Definition 5.1. The only properties of a signature link not yet verified are that all knot circles in $\mathcal{K}$ lie on the same side of K_f and that $\mathcal{L}_h$ is a flat FAL itself. Proofs of these properties appear in Theorems 7.10 and 7.11, respectively, but several preliminary results are required. We remark that Lemma 7.1 highlights a further similarity: the homeomorphism h changes types on components of $\mathcal{L}_h$ in precisely the same way that a full-swap does on a signature link. Finally, note that the sublink depends on the choice of a type-changing homeomorphism h together with a knot circle K_1 that changes type. The subscript of $\mathcal{L}_h$ is intended to emphasize this dependency.

Lemma 7.1 motivates the following definition.

Definition 7.2 Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with a unique reflection surface, and suppose M admits a type-changing homeomorphism h that changes the type of the knot circle K_1 of $\mathcal{A}$. Let K_f, $\mathcal{C}$, $\mathcal{K}$, and $\mathcal{C}_{\mathcal{K}}$ be as in Lemma 7.1.

The *signature sublink* $\mathcal{L}_h$ of $\mathcal{A}$ is the union of these components, so that

$$\mathcal{L}_h = \{K_f\} \cup \mathcal{K} \cup \mathcal{C} \cup \mathcal{C}_{\mathcal{K}},$$

endowed with a fixed choice of crossing disk D_i for each $C_i \in \mathcal{C}$. Let $\mathcal{D}$ denote the set of chosen crossing disks $\{D_1, \dots, D_n\}$.

Given a signature sublink there may be several choices for crossing disks bounded by crossing circles in $\mathcal{C}$. For example, the crossing circle C_4 of Figure 9 bounds the crossing disk D_4 pictured as well as a crossing disk passing between K_2 and K_3. Fixing an orientation on K_f, we make the convention that the ordering on $\mathcal{C}$ and $\mathcal{D}$ is determined by traversing K_f from D_1 in the chosen direction. Different choices for $\mathcal{D} = \{D_1, \dots, D_n\}$ can lead to different orderings, so for convenience we assume the choice is fixed throughout.

The next lemma shows that every crossing circle of $\mathcal{A} \setminus \mathcal{C}$ that bounds a crossing disk punctured by a knot circle of $\mathcal{K}$ is an element of $\mathcal{C}_{\mathcal{K}}$. The proof will show both that the crossing circle links distinct knot circles (rather than the same knot circle twice), and that both knot circles are from $\mathcal{K}$. The lemma also associates a unique separating quadruple Q_{ij} to each $C_{ij} \in \mathcal{C}$, and characterizes possible intersections of these separating quadruples. The separating quadruples guaranteed by Lemma 7.3 will be the main tool used in what follows, so we let $\mathcal{Q} = \bigcup_{\mathcal{C}_{\mathcal{K}}} Q_{ij}$ denote their union.

Lemma 7.3 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with a unique reflection surface, suppose there is a type-changing homeomorphism $h\colon M \to M'$ to another flat FAL complement M', and let $\mathcal{L}_h$ be the corresponding signature sublink of $\mathcal{A}$. If C is a crossing circle of $\mathcal{A} \setminus \mathcal{C}$, with a crossing disk punctured by some $K_i \in \mathcal{K}$, then*:

(i) *There is an index j with $C = C_{ij} \in \mathcal{C}_{\mathcal{K}}$.*

(ii) *The crossing circle C_{ij} bounds a unique crossing disk D_{ij} in M.*

(iii) *There is a separating quadruple Q_{ij} uniquely determined by C_i, C_j, C_{ij} and the chosen crossing disks in $\mathcal{D}$.*

(iv) *Distinct separating quadruples $Q_{ij}, Q_{kl} \in \mathcal{Q}$, are either disjoint or share a single crossing disk in $\mathcal{D}$.*

Proof Let C be a crossing circle in $\mathcal{A} \setminus \mathcal{C}$ that bounds a crossing disk D punctured by $K_i \in \mathcal{K}$. Since K_i changes type, C links distinct knot circles by Lemma 6.2, and we let K^* denote the other knot circle puncturing D. Further, again since K_i changes type, C_i is the unique crossing circle linking it that changes type by Lemma 6.5. Now C is not equal to C_i, so there is a separating quadruple Q with components K_i, C_i, and C by Lemma 6.6. The crossing circles C_i and C link all three knot circle components of Q, so K^* and K_f are the other knot circles of Q and must be linked by the final crossing circle, say C^*, of Q. Note that C and K_f are the only components of Q whose type is fixed by the homeomorphism h (by Lemma 6.4(ii)), so h changes the types of C^* and K^*. Thus C^* is a crossing circle linking K_f which changes type under h, implying there is an index j with $C^* = C_j \in \mathcal{C}$, and $K^* = K_j$ as well. The original crossing circle C is then $C_{ij} \in \mathcal{C}_{\mathcal{K}}$, verifying part (i) of the lemma.

Now consider statement (ii) of the lemma. Suppose C_{ij} bounded two distinct crossing disks, the original disk D and another D_1. Then D and D_1 form a separating pair by Theorem 2.8, and every knot circle of $\mathcal{A}$ intersects $D \cup D_1$ an even number of times (possibly zero). This implies both are punctured by $K_i, K_j \in \mathcal{K}$, both of which change to crossing circles under h. Since C_{ij} doesn't change type, the homeomorphism h maps D and D_1 to distinct longitudinal disks with the same punctures. This is impossible, however, because Lemma 4.5 of [20] shows that there is at most one longitudinal disk with two given punctures, let alone three. Hence C_{ij} bounds a unique crossing disk.

The existence portion of statement (iii) is guaranteed by Lemma 6.6, as noted above. To see uniqueness, note that two separating quadruples with the same crossing circle components have the same longitudinal disks by Lemma 4.5 of [20]. Thus two separating quadruples with the same crossing circle components can differ only in their crossing disks. Since C_{ij} bounds a unique crossing disk and, by convention, there is a fixed choice of disks $\mathcal{D}$ for crossing circles of $\mathcal{C}$, the Q_{ij} are unique.

Finally, we prove statement (iv) of the lemma. If Q_{ij} and Q_{kl} are disjoint we are done, so suppose their intersection is nonempty. All N-disks are identical or disjoint (Lemma 5.2), so if Q_{ij} intersects Q_{kl} nontrivially, they share some subset of disks. We will show that if the intersection is other than a single crossing disk in $\mathcal{D}$, the separating quadruples are equal.

An initial observation is that if $\{D_i, D_j\} \subset Q_{ij} \cap Q_{kl}$ then $Q_{ij} = Q_{kl}$. In this case both Q_{ij} and Q_{kl} contain crossing circles C_i and C_j. Lemma 7.1(iii) shows they also contain the unique crossing circle $C_{ij} \in \mathcal{C}_{\mathcal{K}}$, and the proof of statement (iii) above shows that the separating quadruples are equal.

Now if $Q_{ij} \cap Q_{kl}$ contains a longitudinal disk, they are equal by the above argument since they would share the punctures C_i and C_j. Similarly, Q_{ij} and Q_{kl} are equal if they share D_{ij} since this implies they share C_{ij}, and the definition of Q_{ij} implies they share C_i and C_j as well.

Thus the intersection $Q_{ij} \cap Q_{kl}$ of distinct separating quadruples from $\mathcal{Q}$ is either empty or a single disk from $\mathcal{D}$. □

One way to phrase Lemma 7.3(i) is to say that the signature sublink $\mathcal{L}_h$ contains all crossing circles of the FAL $\mathcal{A}$ that link a knot circle of $\mathcal{K}$. Since each knot circle in $\mathcal{K}$ changes type, Lemma 6.2 implies that if K_i punctures a crossing disk bounded by $C \in \mathcal{A}$ then it does so once, and C links distinct knot circles. For convenience let $\mathcal{L}^c = \mathcal{A} \setminus \mathcal{L}_h$ denote the components of $\mathcal{A}$ not in $\mathcal{L}_h$. If C is a crossing circle of $\mathcal{L}^c$, then, it either links the fixed component K_f or only knot circles of $\mathcal{L}^c$.

7.2 The standard ball

Our next objective is to associate a two-sphere S_α^+ with every arc α of $K_f \setminus \mathcal{D}$, with the property that $S_\alpha^+ \cap \mathcal{A}$ consists of two points on K_f. This is formally stated and proved at the end of this subsection in Proposition 7.9. To build this two-sphere, we first need to introduce a number of properties and terminology related to separating quadruples associated with a signature sublink $\mathcal{L}_h$ of $\mathcal{A}$. As noted in the introduction to Section 7, this two-sphere will play an essential role in classifying the flat FALs under consideration in this section.

Choose an orientation on K_f, say counterclockwise, and note that the crossing disks of $\mathcal{D}$ partition K_f into n oriented arcs. Let α be an *open arc* of $K_f \setminus \mathcal{D}$ and use the orientation on K_f to order the disks of $\mathcal{D}$ (and associated components of $\mathcal{L}_h$) so that α goes from D_n to D_1. Index the components of $\mathcal{K}$ and $\mathcal{C}$ so that $C_i \in \mathcal{C}$ bounds D_i and links $K_i \in \mathcal{K}$, and use this ordering to index crossing circles $\mathcal{C}_{\mathcal{K}}$ and separating quadruples $\mathcal{Q}$. Also number the arcs of $K_f \setminus \mathcal{D}$ so that α_i goes from D_{i-1} to D_i, for $2 \le i \le n$ (α could be considered α_1 in this ordering, but we continue to refer to it as α because of the special role it plays). Thus α induces an ordering on components and disks of the signature sublink $\mathcal{L}_h$, other than K_f, which we call the *α-ordering* of $\mathcal{L}_h$.

Let S^2_{ij} denote the separating quadruple $Q_{ij} \in \mathcal{Q}$ thought of as a two-sphere embedded in $\mathbb{S}^3$. The crossing circles $\{C_i, C_j, C_{ij}\}$ and disks $\{D_i, D_j, D_{ij}, D^{\ell}_{ij}\}$ of Q_{ij} are subsets of S^2_{ij}; whereas, the knot circle components $\{K_i, K_j, K_f\}$ each puncture S^2_{ij} twice. Here, D^{ℓ}_{ij} is a longitudinal disk, with longitudinal punctures along the crossing circles C_i, C_j, and C_{ij}, as discussed at the beginning of Section 6. Moreover, $\mathbb{S}^3 \setminus S^2_{ij}$ is two open three-balls, one of which contains α. Define the *inside* $\mathbb{I}_{ij}$ of Q_{ij} (or of S^2_{ij}) to be the open three-ball component of $\mathbb{S}^3 \setminus S^2_{ij}$ that does not contain α (thinking of α as *outside* each Q_{ij}). Observe that S^2_{ij} is the boundary of $\mathbb{I}_{ij}$, so the closure is $\overline{\mathbb{I}_{ij}} = \mathbb{I}_{ij} \cup S^2_{ij}$.

We highlight the consequence of Lemma 7.3(iv) that insides are either disjoint or nested.

Lemma 7.4 *Let $\mathcal{A}$ be a flat FAL whose complement admits a unique reflection surface and a type-changing homeomorphism h to another flat FAL complement. Let $\mathcal{L}_h$ be a signature sublink of $\mathcal{A}$ and $\mathcal{Q}$ be the set of all separating quadruples determined by $\mathcal{C}_{\mathcal{K}}$. Finally, let α be an open arc of $K_f \setminus \mathcal{D}$ inducing insides for each separating quadruple of $\mathcal{Q}$.*

If $Q_{ij}, Q_{kl} \in \mathcal{Q}$ are distinct separating quadruples, then their insides $\mathbb{I}_{ij}$ and $\mathbb{I}_{kl}$ are either disjoint or properly nested (so $\mathbb{I}_{ij}$ is a proper subset of $\mathbb{I}_{kl}$, or vice versa).

Proof We let α_{ij} denote the arc of K_f that is inside Q_{ij}, so that α_{ij} runs from D_i to D_j and is disjoint from α. If α_{ij} and α_{kl} overlap, but are not nested, then the disks D_i, D_j alternate with D_k, D_l around K_f. This implies the separating quadruples Q_{ij} and Q_{kl} intersect but not along a crossing disk in $\mathcal{D}$, contradicting Lemma 7.3(iv). Thus if the arcs α_{ij} and α_{kl} overlap then they are nested, and we conclude the insides $\mathbb{I}_{ij}$ and $\mathbb{I}_{kl}$ are either nested or disjoint. □

We now use the insides of separating quadruples and set inclusion to define a partial ordering on the set $\mathcal{Q}$.

Definition 7.5 Let α be an open arc of $K_f \setminus \mathcal{D}$ inducing insides on elements of $\mathcal{Q}$. The separating quadruple Q_{ij} is *inside* Q_{kl}, denoted by $Q_{ij} \prec_\alpha Q_{ij}$, if $\mathbb{I}_{ij}$ is a proper subset of $\mathbb{I}_{kl}$.

Observe that the inside relation, being defined using set inclusion, is indeed a strict partial ordering on $\mathcal{Q}$. First, the subset relation is transitive so the inside relation is as well. Second, the *proper* subset restriction implies the inside relation is neither reflexive nor symmetric, so it is a strict partial ordering.

The inside relation is most easily seen by choosing the midpoint of α to be infinity in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ and viewing the link from infinity as in Figure 21. The separating quadruple Q_{12} is not related to any other separating quadruple of $\mathcal{Q}$, while both relations $Q_{35} \prec_\alpha Q_{25}$ and $Q_{56} \prec_\alpha Q_{57}$ hold.

We now introduce more terminology that will be helpful in the ensuing discussion.

The term *inside* will frequently be used in the context of link components or thrice-punctured spheres to imply containment within a separating quadruple. For example, the disk D_3 of Figure 21 is inside the separating quadruple Q_{25} since $D_3 \subset \mathbb{I}_{25}$, as are the open arcs α_3, α_4, α_5 of K_f. More generally, recall that two separating quadruples Q_{ij} and Q_{kl} intersect in at most one disk of $\mathcal{D}$ (together with its punctures). This implies that if $Q_{ij} \prec_\alpha Q_{kl}$, then the components C_{ij}, D_{ij}, and D^{ℓ}_{ij} of Q_{ij} are inside Q_{kl} as well.

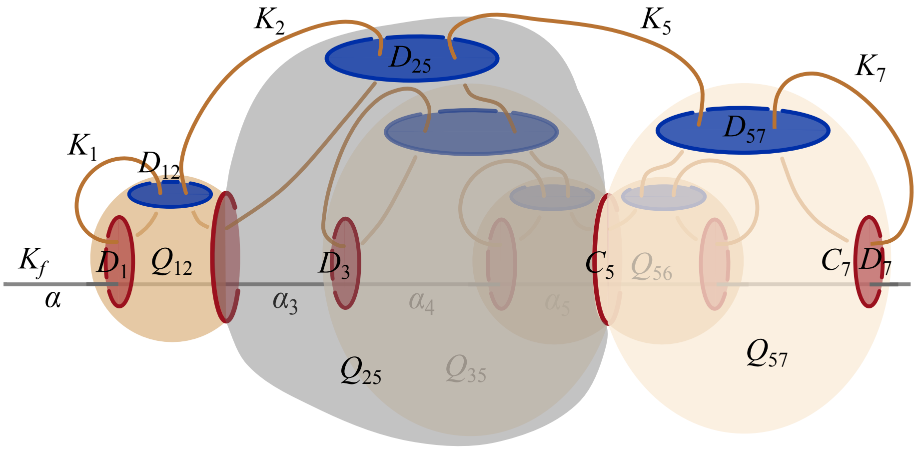


Figure 21: A signature sublink with three outermost quadruples.

A component, disk, or arc is *outside* the separating quadruple Q_{ij} if it is in the open three-ball of $\mathbb{S}^3 \setminus Q_{ij}$ containing α. Thus the disks D_1 and D_{12} of Figure 21 are outside Q_{25}. More generally, if disjoint separating quadruples are not comparable in the inside partial ordering, then they are outside each other.

We will also have occasion to describe separating quadruples as being on the same or opposite sides of a common disk in $\mathcal{D}$. Suppose two separating quadruples $Q, Q' \in \mathcal{Q}$ share a common disk $D \in \mathcal{D}$. Then the closed 3-balls $\overline{\mathbb{I}}$ and $\overline{\mathbb{I}'}$ share a common boundary disk D and have interiors that are either nested or disjoint. Define Q and Q' to be on the *same side* of the crossing disk $D \in \mathcal{D}$ if their insides are nested, and on *opposite sides* if they are disjoint. For example, the separating quadruples Q_{56} and Q_{57} in Figure 21 are on the same side of D_5 while Q_{25} and Q_{57} are on opposite sides.

Recall that an element $Q \in \mathcal{Q}$ is *maximal* if $Q_{ij} \prec_\alpha Q$ whenever Q_{ij} and Q are comparable. Thus the separating quadruples Q_{12}, Q_{25}, and Q_{57} of Figure 21 are maximal elements. The term *outermost quadruple* will be used for maximal elements in the inside partial ordering on $\mathcal{Q}$, as it is more intuitive.

We now highlight some important properties of this partial ordering.

Lemma 7.6 *Let $\mathcal{A}$ be a flat FAL whose complement admits a unique reflection surface and a type-changing homeomorphism h to another flat FAL complement. Let $\mathcal{L}_h$ be a signature sublink, α an open arc of $K_f \setminus \mathcal{D}$, and let $\prec_\alpha$ denote the induced inside relation on $\mathcal{Q}$. The inside relation is a strict partial ordering on $\mathcal{Q}$ with the following properties*:

(i) *Distinct separating quadruples of $\mathcal{Q}$ are not comparable if and only if their insides are disjoint.*

(ii) *Two separating quadruples that share a crossing disk in $D \in \mathcal{D}$ are comparable if and only if they are on the same side of D.*

(iii) *Each $Q_{ij} \in \mathcal{Q}$ is either outermost or contained in a unique outermost element of $\mathcal{Q}$.*

(iv) *The disk $D_1 \in \mathcal{D}$ is part of a unique outermost quadruple, and is not inside any element of $\mathcal{Q}$. The same result is true of the disk $D_n \in \mathcal{D}$.*

Proof Let Q and Q' be distinct separating quadruples in $\mathcal{Q}$. When their insides are properly nested, Q and Q' are comparable; however, if Q and Q' have disjoint insides they are not comparable. Lemma 7.4 shows these are the only two cases, proving statement (i). To see that statement (ii) holds, suppose $Q, Q' \in \mathcal{Q}$ are distinct separating quadruples that share the crossing disk $D \in \mathcal{D}$. By statement (i) they are not comparable if and only if their insides are disjoint which, by definition, is equivalent to saying they are on opposite sides of D.

Now, we consider statement (iii). Suppose $Q \in \mathcal{Q}$ is not outermost, so that there is a $Q' \in \mathcal{Q}$ with $Q \prec_\alpha Q'$. If Q'' is another separating quadruple with $Q \prec_\alpha Q''$, then the insides of Q' and Q'' intersect nontrivially, and statement (i) implies they are comparable. Thus the set of all separating quadruples larger than Q is a finite linearly ordered subset, and so contains a unique maximal element.

The argument for statement (iv) follows from the facts that every disk in $\mathcal{D}$ is either inside or on an outermost element of $\mathcal{Q}$, and that disks adjacent to α are not inside any element of $\mathcal{Q}$. To see the first fact, note that the knot circle K_i punctures D_i, and Lemma 7.1(iii) proves that K_i is linked by some crossing circle $C_{il} \in \mathcal{C}_\mathcal{K}$. The crossing circle C_{il} generates a separating quadruple $Q_{il} \in \mathcal{Q}$ which contains the disk D_i, by Lemma 7.3. We just verified that Q_{il} is either outermost or inside an outermost $Q \in \mathcal{Q}$. If Q_{il} is outermost then D_i is part of an outermost quadruple; otherwise, D_i is either on or inside Q. Thus every $D_i \in \mathcal{D}$ is either inside or on an outermost element of $\mathcal{Q}$. Now suppose $D_i \subset \mathbb{I}_{jk}$ for some disk $D_i \in \mathcal{D}$ and where $\mathbb{I}_{jk}$ is the inside of an outermost quadruple $Q_{jk} \in \mathcal{Q}$. Then both arcs of K_f adjacent to D_i are inside Q_{jk} as well. By definition of inside, however, the arc $\alpha \subset K_f$ is outside every element of $\mathcal{Q}$ so disks adjacent to α are not inside any element of $\mathcal{Q}$. Since α is adjacent to D_1 and D_n, they cannot be inside any separating quadruple and must be part of an outermost separating quadruple. To see uniqueness, note that all separating quadruples containing D_1 must be on the side of D_1 opposite α. Hence every pair of separating quadruples containing D_1 are comparable, making all such separating quadruples a linearly ordered subset, which must have a unique maximal element. The same observations hold for the disk D_n. □

Consider the set of all outermost separating quadruples, which we denote by $\{Q_1, Q_2, \dots, Q_l\}$. No two outermost separating quadruples are comparable, so their insides $\{\mathbb{I}_i\}$ are disjoint by Lemma 7.6(i). The open arcs $\beta_i = \mathbb{I}_i \cap K_f$, therefore, are disjoint as well. By definition of inside, the arc α is disjoint from all $\{\mathbb{I}_i\}$, so the arcs $\{\beta_i\}$ can be ordered as they are encountered starting at α and traversing K_f according to its orientation. We assume the sequence $\{Q_1, Q_2, \dots, Q_l\}$ is listed using this order on the $\{\beta_i\}$.

As an example, for a given α, there is a unique outermost quadruple $\{Q_1\}$ if and only if $Q_1 = Q_{1n}$. An alternative characterization is that there is a crossing circle $C_{1n} \in \mathcal{C}_\mathcal{K}$ linking the first and last knot circle in the α-ordering of $\mathcal{K}$.

If Q_i, Q_j are not consecutive in the ordered sequence $\{Q_1, Q_2, \dots, Q_l\}$, there is some Q_k between them along K_f and they cannot share a disk of $\mathcal{D}$. Hence, nonconsecutive separating quadruples in the sequence $\{Q_1, Q_2, \dots, Q_l\}$ are disjoint. Consecutive outermost quadruples Q_i, Q_{i+1}, on the other hand, can either share a disk or be disjoint. They are disjoint if there is an arc of K_f between β_i and β_{i+1}. We say that consecutive quadruples Q_i, Q_{i+1} are *adjacent* if $Q_i \cap Q_{i+1} = D_{j_i}$ for some $D_{j_i} \in \mathcal{D}$. Since the insides of Q_i and Q_{i+1} are disjoint, if they are adjacent they are on opposite sides of D_{j_i} (Lemma 7.6(ii)). The subsequence $\{Q_i, \dots, Q_j\}$ of $\{Q_1, Q_2, \dots, Q_l\}$, is a *maximally adjacent subsequence* if consecutive quadruples in the subsequence are adjacent while the pairs $\{Q_{i-1}, Q_i\}$ and $\{Q_j, Q_{j+1}\}$ are disjoint. The ordered sequence of all outermost quadruples $\{Q_1, Q_2, \dots, Q_l\}$ partitions into maximally adjacent subsequences. For example, the maximally adjacent subsequence of Figure 21 is $\{Q_{12}, Q_{25}, Q_{57}\}$.

Let $\{Q_1, \dots, Q_m\}$ be the initial maximally adjacent subsequence, so that consecutive quadruples of the subsequence are adjacent, while Q_m and Q_{m+1} are not. We will use the subsequence $\{Q_1, \dots, Q_m\}$ to define the standard ball associated with α.

Some elementary observations are in order before we define the standard ball. Let Q_j be a outermost separating quadruple with inside $\mathbb{I}_j$, and note that its closure $\overline{\mathbb{I}_j}$ is a closed three-ball with boundary sphere Q_j (we abuse notation and use Q_j to refer to the two-sphere embedded in $\mathbb{S}^3$ corresponding to this separating quadruple). Now suppose Q_{j-1} and Q_j are adjacent, outermost separating quadruples which share the crossing disk $D_{l_j} \in \mathcal{D}$. Since Q_{j-1} and Q_j are outermost, Lemma 7.6(i) implies their insides are disjoint. The union $\overline{\mathbb{I}_{j-1}} \cup \overline{\mathbb{I}_j}$ is a closed 3-ball, since it is two closed 3-balls (with disjoint interiors) glued along a common disk in their boundary spheres. The open disk $D^\circ_{l_j}$ is interior to $\overline{\mathbb{I}_{j-1}} \cup \overline{\mathbb{I}_j}$, so the boundary is $\partial(\overline{\mathbb{I}_{j-1}} \cup \overline{\mathbb{I}_j}) = (Q_{j-1} \cup Q_j) \setminus D^\circ_{l_j}$. These observations extend to subsequences $\{Q_i, \dots, Q_k\}$ of adjacent, outermost separating quadruples. For each outermost quadruple Q_j in the subsequence, form the closed three-ball $\overline{\mathbb{I}_j} = \mathbb{I}_j \cup Q_j$. Then $\bigcup_{j=i}^k \overline{\mathbb{I}_j}$ is a closed three-ball, because it is a sequence of closed three-balls with disjoint interiors in which only consecutive balls are glued together disk on their boundary spheres. Moreover, the boundary sphere of $\bigcup_{j=i}^k \overline{\mathbb{I}_j}$ is given explicitly by $\bigl(\bigcup_{j=i}^k Q_j\bigr) \setminus \bigl(\bigcup_{j=i+1}^k D^\circ_{l_j}\bigr)$, where $D_{l_j} = Q_{j-1} \cap Q_j$. Applying this to the initial maximally adjacent subsequence $\{Q_1, \dots, Q_m\}$ yields the standard ball associated with α.

Definition 7.7 Let α be an arc of $K_f \setminus \mathcal{D}$ with initial maximally adjacent subsequence $\{Q_1, \dots, Q_m\}$ of outermost separating quadruples. Let $\mathbb{I}_j$ be the inside of Q_j, and $\overline{\mathbb{I}_j}$ its closure. The *standard ball* of α, denoted by $\mathbb{B}^3_\alpha$, is the union

$$\mathbb{B}^3_\alpha = \bigcup_{j=1}^m \overline{\mathbb{I}_j},$$

and let $S^2_\alpha = \bigl(\bigcup_{j=1}^m Q_j\bigr) \setminus \bigl(\bigcup_{j=2}^m D^\circ_{l_j}\bigr)$ denote the boundary sphere of $\mathbb{B}^3_\alpha$, where $D_{l_j} = Q_{j-1} \cap Q_j$.

In what follows, components of $\mathcal{A}$ that intersect S^2_α will be important so, for convenience, we introduce some notation. Each $Q_j \in \{Q_1, \dots, Q_m\}$ is some $Q_{l_j l_{j+1}} \in \mathcal{Q}$, and it is with this notation we defined $D_{l_j} = Q_{j-1} \cap Q_j$, for $2 \le j \le m$. Rather than using double-subscripts, we adopt lowercase letters to represent components of the Q_j. For example, denote the knot circles $K_{l_j}, K_{l_{j+1}}$ of $Q_j = Q_{l_j l_{j+1}}$ by k_j and k_{j+1}, respectively, let $c_{j,j+1} = C_{l_j l_{j+1}}$, while $d^\ell_{j,j+1}$ denotes the longitudinal disk $D^\ell_{l_j l_{j+1}}$. We continue to use K_f for the fixed knot circle puncturing Q_j. Please refer to Figure 22 for an illustration of this notation.

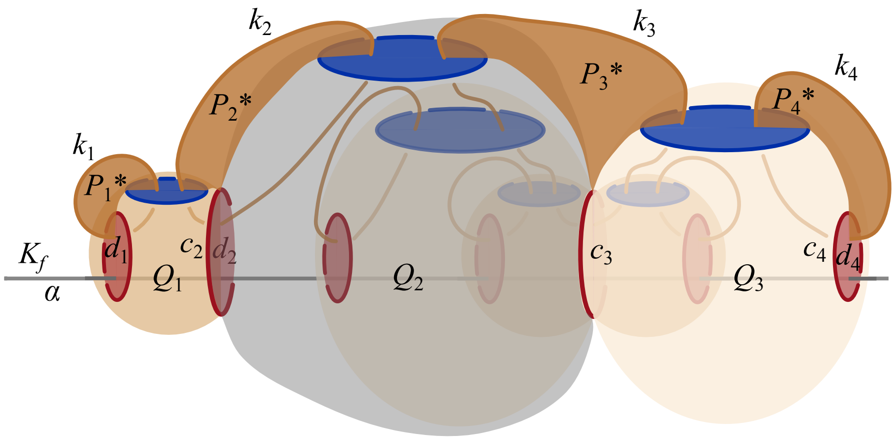


Figure 22: The standard ball with disks $\mathcal{P}_i^*$.

Now consider the boundary S^2_α of the standard ball with this notation. The crossing disk shared by the outermost quadruples Q_{j-1} and Q_j is $d_j = D_{l_j}$, and k_j is the knot circle puncturing them. Further, by Lemma 7.6(iv), the disk $D_1 = d_1$ lies on the separating quadruple Q_1, so that $k_1 = K_1$. This implies $S^2_\alpha = \big(\bigcup_{j=1}^m Q_j\big) \setminus \big(\bigcup_{j=2}^m d_j^\circ\big)$, and S^2_α is punctured twice by each of the knot circles $K_f, k_1, k_2, \dots, k_m, k_{m+1}$.

The boundary S^2_α of the standard ball, then, is a two-sphere embedded in $\mathbb{S}^3$ that is intersected by the link $\mathcal{A}$ in many components of $\mathcal{L}_h$. Our immediate goal is to extend S^2_α to an embedded two-sphere S^+_α that intersects $\mathcal{A}$ in exactly two points of K_f. Since $\mathcal{A}$ is hyperbolic, the desired S^+_α cannot define a connect-sum decomposition, and one component of $\mathbb{S}^3 \setminus S^+_\alpha$ must be a standard ball-arc pair. This significantly restricts the link $\mathcal{A}$ and allows us to prove that $\mathcal{A} = \mathcal{L}_h$ (Theorem 7.11), which ultimately leads to our main result.

Some preliminary definitions, and a technical lemma, are necessary before constructing the two-sphere S^+_α. Lemma 7.3 shows that the structure of a signature link outlined in Lemma 5.3 (a longitudinal disk and separating quadruple) persists in signature sublinks, even in the (potential) presence of additional components. In the proof of Lemma 5.3 the inside $\mathcal{P}_i$ of the knot circle K_i can be described as the component of the reflection surface bounded by K_i and not containing K_f.

Now let k_i be a knot circle in $\mathcal{K}$ puncturing the standard ball. Analogously define the inside $\mathcal{P}_i$ of k_i to be that component of the reflection surface not containing K_f, including the boundary curve k_i. Each k_i punctures the boundary S^2_α twice, so half of the closed disk $\mathcal{P}_i$ is inside and half outside of $\mathbb{B}^3_\alpha$. We let $\mathcal{P}_i^*$ denote the portion of $\mathcal{P}_i$ outside of $\mathbb{B}^3_\alpha$. Precisely we have $\mathcal{P}_i^* = \mathcal{P}_i \setminus \mathbb{B}^3_\alpha$. Note $\mathcal{P}_i^*$ is a half-open disk with the arc of k_i outside $\mathbb{B}^3_\alpha$ part of its boundary. For $2 \le i \le m$, the interior of $\mathcal{P}_i^*$, then, is an open disk whose boundary consists of an arc of k_i, geodesics on each of Q_{i-1} and Q_i, and single point of the crossing circle c_i on $Q_{i-1} \cap Q_i$ (see Figure 22). Note that, for $i = 1, m+1$, $\mathcal{P}_i^*$ is also an open disk, but whose boundary only consists of an arc of k_i and a geodesic on Q_i.

Before constructing S_α^+, we prove a technical lemma showing that no components of $\mathcal{A}$ intersect $\mathcal{P}_i^*$ (other than the boundary arc of k_i). Components of $\mathcal{L}_h$ and its complement $\mathcal{L}^c = \mathcal{A} \setminus \mathcal{L}_h$ will be considered separately.

Lemma 7.8 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ be a flat FAL complement with a unique reflection surface that admits a type-changing homeomorphism h to another flat FAL complement. Let $\mathcal{L}_h$ denote the signature sublink of $\mathcal{A}$. Let α be an arc of $K_f \setminus \mathcal{D}$ with standard ball $\mathbb{B}_\alpha^3$ generated by the initial maximally adjacent subsequence $\{Q_1, \dots, Q_m\}$ of outermost quadruples.*

If $\mathcal{P}_i^*$ is the portion of the reflection surface inside of $k_i \in \mathcal{K}$ but outside $\mathbb{B}_\alpha^3$, then the components of $\mathcal{A}$ are disjoint from the interior of $\mathcal{P}_i^*$.

Proof Initially focus on components of $\mathcal{L}_h$ and consider $\mathcal{P}_i \cap \mathcal{L}_h$. First observe that k_i is the only knot circle of $\mathcal{L}_h$ that is a boundary curve of $\mathcal{P}_i$. To see this, suppose that K is a knot circle of $\mathcal{A}$ interior to $\mathcal{P}_i$. Then K and K_f are on opposite sides of k_i, and cannot be linked by a crossing circle. Since each $K_j \in \mathcal{K}$ is linked to K_f by C_j, we see $K \notin \mathcal{K}$ so K is not in $\mathcal{L}_h$. Thus $\mathcal{P}_i^* \cap \mathcal{L}_h$ consists only of punctures by crossing circles of $\mathcal{L}_h$.

We turn our attention to crossing circles of $\mathcal{L}_h$ which intersect $\mathcal{P}_i^*$. The crossing circle C_i is the only one of $\mathcal{C}$ that intersects $\mathcal{P}_i$, and it does so in one point on the boundary of, not interior to, $\mathcal{P}_i^*$. Now suppose $C \in \mathcal{C}_\mathcal{K}$ is a crossing circle that links k_i, and let Q be the separating quadruple it generates (guaranteed by Lemma 7.3). In this case, Q contains the crossing disk D_i and is comparable to exactly those separating quadruples of $\mathcal{Q}$ on the same side of D_i as Q (Lemma 7.6((ii))). Consider the cases $i = 1$, $2 \le i \le m$, and $i = m+1$ separately.

In the case $i = 1$, the disk d_1 is $D_1 \in \mathcal{D}$ by Lemma 7.6(iv), and every separating quadruple containing D_1 is opposite to α. Thus Q is comparable to Q_1 and, by maximality of Q_1, we have $Q \prec_\alpha Q_1$. This implies the crossing circle C is inside Q_1 and disjoint from $\mathcal{P}_i^*$.

In the case $2 \le i \le m$, the separating quadruple Q shares the disk d_i with both Q_{i-1} and Q_i. The outermost quadruples Q_{i-1} and Q_i are not comparable so must be on opposite sides of d_i. Thus Q is comparable to one of either Q_{i-1} or Q_i, making C interior to that outermost quadruple and disjoint from $\mathcal{P}_i^*$.

Finally consider $\mathcal{P}_{m+1}^*$ which is bounded by an arc of the knot circle k_{m+1}. In this case Q shares the disk d_{m+1} with Q_m (eg disk $\mathcal{P}_4^*$ of Figure 22). If Q and Q_m are on the opposite sides of d_{m+1}, then Q is contained in a unique outermost quadruple Q_{m+1} on the opposite side of d_{m+1} from Q_m (Lemma 7.6). But then the sequence $\{Q_1, Q_2, \dots, Q_m\}$ can be extended by Q_{m+1} to a longer sequence of adjacent, outermost separating quadruples. This contradicts the definition of $\{Q_1, Q_2, \dots, Q_m\}$, so Q and Q_m are on the same side of d_{m+1}. Maximality of Q_m implies that C is inside Q_m and disjoint from $\mathcal{P}_i^*$.

In all cases, then, the interior of $\mathcal{P}_i^*$ is disjoint from crossing circles of $\mathcal{L}_h$. The preceding argument showed that the same is true of knot circles in $\mathcal{L}_h$, so the interior of $\mathcal{P}_i^*$ is disjoint from the signature sublink $\mathcal{L}_h$.

It remains to show that components of $\mathcal{L}^c = \mathcal{A} \setminus \mathcal{L}_h$ do not intersect the interior of $\mathcal{P}_i^*$. The proof amounts to showing that if the set of components of $\mathcal{L}^c$ intersecting $\mathcal{P}_i^*$ is nontrivial, then $\mathcal{A}$ is a split link, hence $\mathcal{L}^c$ must be disjoint from $\mathcal{P}_i^*$.

First recall that any crossing disk punctured by k_i is bounded by a crossing circle in $\mathcal{L}_h$ by Lemma 7.3, and consider a crossing circle $C \in \mathcal{L}^c$ that punctures the interior of $\mathcal{P}_i^*$. Then, if D a crossing disk bounded by C it is disjoint from k_i as well as from separating quadruples in $\mathcal{Q}$ — in other words, D is disjoint from the boundary of $\mathcal{P}_i^*$. Now C punctures $\mathcal{P}_i^*$, so D intersects the reflection surface entirely within $\mathcal{P}_i^*$. In particular, C punctures the interior of $\mathcal{P}_i^*$ twice and any knot circle(s) linked by C are interior to $\mathcal{P}_i^*$.

Now let K be a knot circle of $\mathcal{L}^c$ and recall that every crossing circle in $\mathcal{L}_h$ links only knot circles in $\mathcal{L}_h$. Thus all crossing circles of $\mathcal{A}$ that link K are contained in $\mathcal{L}^c$. Now suppose K intersects $\mathcal{P}_i^*$. Knot circles of $\mathcal{A}$ are disjoint in the reflection surface and K, being in $\mathcal{L}^c$, is disjoint from $\mathcal{Q}$, so K is contained in the interior of $\mathcal{P}_i^*$. Then any crossing circle C that links K punctures $\mathcal{P}_i^*$, and the above argument shows that C links only knot circles interior to $\mathcal{P}_i^*$.

Now suppose that $\mathcal{L}^c$ intersects $\mathcal{P}_i^*$ nontrivially, and let $\mathcal{L}_i^c$ denote the sublink of components of $\mathcal{A}$ that intersect $\mathcal{P}_i^*$. The above argument shows that crossing circles of $\mathcal{L}_i^c$ link only knot circles of $\mathcal{L}_i^c$, and vice versa. Thus the components of flat FAL $\mathcal{A}$ partition into two nonempty subsets, $\mathcal{L}_i^c$ and its complement, in which crossing circles only link knot circles within their respective subset. In an FAL this results in a split link, contradicting the fact that $\mathcal{A}$ is hyperbolic.

Thus $\mathcal{P}_i^*$ is disjoint from $\mathcal{L}^c$, completing the proof that the components of $\mathcal{A}$ are disjoint from the interior of $\mathcal{P}_i^*$. □

Proposition 7.9 *Let $\mathcal{A}$ be a flat FAL whose complement admits a type-changing homeomorphism h, with associated signature sublink $\mathcal{L}_h$. Let K_f be the knot circle component of $\mathcal{L}_h$ whose type is fixed by h, and α be an arc of $K_f \setminus \mathcal{D}$. The standard ball $\mathbb{B}^3_\alpha$ has a neighborhood $N(\mathbb{B}^3_\alpha)$ in $\mathbb{S}^3$ whose boundary is a two-sphere S^+_α such that $S^+_\alpha \cap \mathcal{A}$ is precisely two distinct points of K_f. Moreover, S^+_α can be chosen so that*

(i) *every component, other than K_f, of $\mathcal{A}$ that intersects $\mathbb{B}^3_\alpha$ is inside S^+_α, and*

(ii) *every component of $\mathcal{A}$ that is outside $\mathbb{B}^3_\alpha$ is also outside S^+_α.*

Proof Let K_f be oriented with α an arc of K_f between two consecutive disks of $\mathcal{D}$ and endow components of $\mathcal{L}_h$ with the α-ordering. Further, let $\{Q_1, \dots, Q_m\}$ be the initial maximally adjacent subsequence and let $\mathbb{B}^3_\alpha$ be the standard ball of α with boundary sphere S^2_α. The proof consists of constructing a cell complex X consisting of $\mathbb{B}^3_\alpha$ together with portions of the projection plane. The desired sphere S^+_α will be the boundary of an appropriately chosen regular neighborhood $N(X)$ of this cell complex.

Let $\mathcal{R}$ denote the unique reflection surface of M. As in Lemma 7.8, let $\mathcal{P}_i$ denote the disk of $\mathcal{R} \setminus k_i$ that does not contain K_f, together with its boundary curve k_i. Now define X to be the cell complex

$$X = \mathbb{B}^3_\alpha \cup \Big(\bigcup_{1 \le i \le m+1} \mathcal{P}_i \Big).$$

Again following Lemma 7.8, the standard ball intersects each disk $\mathcal{P}_i$ and we let $\mathcal{P}_i^*$ denote the portion of $\mathcal{P}_i$ outside $\mathbb{B}^3_\alpha$ (ie $\mathcal{P}_i^* = \mathcal{P}_i \setminus \mathbb{B}^3_\alpha$ — see Figure 22).

The first statement of the proposition involves components of $\mathcal{A}$ that intersect $\mathbb{B}^3_\alpha$, and we show that each of these is contained in X. Since the components of the quadruples $\{Q_1, \dots, Q_m\}$ are the only components of $\mathcal{A}$ that intersect the boundary S^2_α of the standard ball, we address those first. All crossing circles in these quadruples lie on the boundary of the standard ball and are, therefore, subsets of X. Recall that the knot circles of $\mathcal{A}$ that puncture S^2_α have been labeled $K_f, k_1, k_2, \dots, k_m, k_{m+1}$, which are not subsets of $\mathbb{B}^3_\alpha$. Since X includes the closed disks $\mathcal{P}_i$, however, each k_i is a subset of X. Thus all components of $\mathcal{A}$ that intersect S^2_α, other than K_f, are subsets of X. The remaining components of $\mathcal{A}$ that intersect $\mathbb{B}^3_\alpha$ are interior to $\mathbb{B}^3_\alpha$. Thus every component of $\mathcal{A}$ that intersects $\mathbb{B}^3_\alpha$ is contained in X and, therefore, interior to every neighborhood of X in $\mathbb{S}^3$.

The goal is to show that an appropriate open neighborhood $N(X)$ of X is an open three-ball whose boundary sphere satisfies the requirements for S^+_α. First observe that each $\mathcal{P}_i^*$ can be retracted into $\mathbb{B}^3_\alpha$ along the disk $\mathcal{P}_i$. Since small enough neighborhoods of $\mathbb{B}^3_\alpha$ are three-balls, the same is true for X. Moreover, since the knot circle K_f punctures S^2_α twice, the same will be true of small enough neighborhoods of X. From now on we assume $N(X)$ is an open three-ball neighborhood of X with boundary sphere S^2_X that is punctured twice by K_f. The previous paragraph demonstrates that S^2_X contains all other components of $\mathcal{A}$ that intersect $\mathbb{B}^3_\alpha$, so S^2_X satisfies the first statement of the proposition as well.

We have shown that components of $\mathcal{A}$ that intersect $\mathbb{B}^3_\alpha$ behave as desired relative to S^2_X. Furthermore, the neighborhood $N(X)$ can be chosen small enough so that its boundary sphere doesn't intersect any components of $\mathcal{A}$ that are disjoint from X. Thus any component of $\mathcal{A}$ that is disjoint from (or outside) X is also outside S^2_X.

To finish the proof of statement (ii), then, we must show that every component of $\mathcal{A}$ that is outside $\mathbb{B}^3_\alpha$ is also outside X. Or, contrapositively, show that every component of $\mathcal{A}$ that intersects $X \setminus \mathbb{B}^3_\alpha = \big(\bigcup_{1 \le i \le m+1} \mathcal{P}_i^* \big)$ also intersects $\mathbb{B}^3_\alpha$. Lemma 7.8, however, demonstrates that the only components of $\mathcal{A}$ intersecting the $\mathcal{P}_i^*$ are components of the outermost quadruples $\{Q_1, \dots, Q_m\}$, which also intersect $\mathbb{B}^3_\alpha$. Thus every component of $\mathcal{A}$ that is outside $\mathbb{B}^3_\alpha$ is also outside X, and S^2_X satisfies the second statement of our proposition, making it our desired S^2_+. □

As an example, the sphere S^+_α of Figure 22 contains the entire signature sublink $\mathcal{L}_h$ except the arc α. In fact, the twice-punctured sphere S^+_α of Proposition 7.9 will be used to provide a connect-sum decomposition of $\mathcal{A}$ if $\mathcal{A} \ne \mathcal{L}_h$, contradicting hyperbolicity. Hence it is the key to finishing the proof that $\mathcal{A} = \mathcal{L}_h$.

7.3 Hyperbolicity conditions

We can now use the properties of S_α^+ from Proposition 7.9 along with hyperbolicity conditions, specifically a hyperbolic link can not be a split link and can not be a connect-sum, to prove our main results. Before continuing, let us compare the *signature sublinks* of Definition 7.2 to the *signature links* of Definition 5.1. One glaring difference is that a signature link is assumed to be a flat FAL while a signature sublink, being a sublink of a flat FAL, is not necessarily a flat FAL by itself. The other difference is that all knot components of $\mathcal{K}$ in a signature link are assumed to be on the same side of K_f, and this has yet to be proven for signature sublinks. By *same side of* K_f we mean the same component of its complement in the projection plane. Theorem 7.10 shows that all knot circles in $\mathcal{K}$ for a signature sublink $\mathcal{L}_h$ are indeed on the same side of K_f. Theorem 7.11 proves that $\mathcal{L}_h$ is a flat FAL by proving (the stronger result) that it equals the flat FAL $\mathcal{A}$. With these results in hand it is not hard to finish, in Theorem 7.12, the proof that flat FALs are determined by their complements.

Theorem 7.10 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ and $M' = \mathbb{S}^3 \setminus \mathcal{A}'$ be flat FAL complements, each with unique reflection surfaces. Suppose $h\colon M \to M'$ is a type-changing homeomorphism, and let $\mathcal{L}_h$ be the signature sublink of $\mathcal{A}$. Then all knot circles in $\mathcal{K}$ are on the same side of K_f.*

Proof The result will follow once we show that if knot circles of $\mathcal{K}$ are on opposite sides of K_f, then there is an arc α in $K_f \setminus \mathcal{D}$ such that the two-sphere S_α^+ of Proposition 7.9 provides a connect-sum decomposition of $\mathcal{A}$.

A crossing circle in $\mathcal{C}_\mathcal{K}$ links two knot circles of $\mathcal{K}$ that are on the same side of K_f, since a crossing circle of $\mathcal{A}$ that punctures opposite sides of K_f necessarily links K_f. Therefore, if $C_{ij}, C_{kl} \in \mathcal{C}_\mathcal{K}$ link knot circles on opposite sides of K_f, then the sets of crossing circles $\{C_i, C_j, C_{ij}\}$ and $\{C_k, C_l, C_{kl}\}$ are disjoint. Since separating quadruples can intersect only along common crossing disks (Lemma 7.3(iv)), this implies Q_{ij} and Q_{kl} are disjoint. In particular, outermost separating quadruples that link knot circles of $\mathcal{K}$ on opposite sides of K_f are not adjacent.

Now suppose there are knot circles in $\mathcal{K}$ that are on opposite sides of K_f, and label knot circles of $\mathcal{K}$ so that K_n and K_1 are on opposite sides of K_f. Let α denote the arc of K_f between crossing disks D_n and D_1, and let $\mathbb{B}_\alpha^3$ be the standard ball of α. Since K_f intersects D_1 and D_n at the endpoints of α, they are contained in outermost separating quadruples Q_{1i} and Q_{jn} by Lemma 7.6(iv). The maximally adjacent subsequence $\{Q_1, \dots, Q_m\}$ contains only outermost quadruples on the same side of K_f as Q_1, and create the standard ball $\mathbb{B}_\alpha^3$. Since the outermost quadruple Q_{jn} is on the opposite side from those in $\{Q_1, \dots, Q_m\}$, its components are outside $\mathbb{B}_\alpha^3$. There are components of $\mathcal{A}$, then, on both sides of the two-sphere S_α^+ constructed in Proposition 7.9. But then S_α^+ provides a nontrivial connect-sum decomposition of $\mathcal{A}$, contradicting the hyperbolicity of $\mathcal{A}$. □

Theorem 7.11 *Let $M = \mathbb{S}^3 \setminus \mathcal{A}$ and $M' = \mathbb{S}^3 \setminus \mathcal{A}'$ be flat FAL complements each with unique reflection surfaces. Suppose $h\colon M \to M'$ is a type-changing homeomorphism. Then $\mathcal{A}$ is a signature link.*

Proof To prove this result we show that $\mathcal{A}$ equals its signature sublink $\mathcal{L}_h$. Once we've shown $\mathcal{L}_h = \mathcal{A}$, the signature sublink is a flat FAL and satisfies the final property of Definition 5.1. We assume that $\mathcal{L}^c = \mathcal{A} \setminus \mathcal{L}_h$ is nonempty, and will arrive at a contradiction.

We begin by showing that if C is a crossing circle of $\mathcal{L}^c$, then any crossing disk D bounded by C is punctured only by knot circles in $\mathcal{L}^c$. First, since $C \in \mathcal{A} \setminus \mathcal{L}_h$, Lemma 7.3 implies that D is not punctured by any $K_i \in \mathcal{K}$.

Now suppose that D is punctured by K_f. The crossing disks $\mathcal{D}$ cut K_f into n arcs. Let α be an arc of K_f that punctures the disk D, then construct the standard ball $\mathbb{B}^3_\alpha$ of α and the associated twice-punctured sphere S^+_α of Proposition 7.9. Now the crossing disk D must be outside $\mathbb{B}^3_\alpha$ since it is punctured by α and is disjoint from $S^2_\alpha = \partial \mathbb{B}^3_\alpha$. Then $C = \partial D$ is also outside $\mathbb{B}^3_\alpha$, and C must be outside S^+_α by Proposition 7.9. There are also components of $\mathcal{A}$ inside S^+_α, by Proposition 7.9, since S^+_α contains the components of the maximally adjacent subsequence $\{Q_1, \dots, Q_m\}$, which form a nontrivial subset of $\mathcal{A}$. Thus S^+_α provides a nontrivial connect-sum decomposition of $\mathcal{A}$. Since $\mathcal{A}$ is hyperbolic this is a contradiction, and every crossing circle $C \in \mathcal{L}^c$ links knot circles in $\mathcal{L}^c$.

Further recall that crossing circles of $\mathcal{L}_h$ link only knot circles in $\mathcal{L}_h$. At this stage the components of $\mathcal{A}$ have been partitioned into two nonempty subsets $\mathcal{L}_h$ and $\mathcal{L}^c$ with the property that crossing circles from one subset bound crossing disks punctured only by knot circle(s) from the same set. This is a contradiction, since such an FAL admits a disconnected diagram, indicating it came from a splittable link. Thus the link that generated $\mathcal{A}$ was split, contradicting the hyperbolicity of $\mathcal{A}$.

We conclude that $\mathcal{L}^c$ is empty, and $\mathcal{L}_h = \mathcal{A}$. □

The following theorem highlights our main result: flat FALs are determined by their complements.

Theorem 7.12 *Let $\mathcal{A}$, $\mathcal{A}'$ be flat FALs with homeomorphic complements. Then $\mathcal{A}$ is isotopic to $\mathcal{A}'$.*

Proof First, if $h\colon M \to M'$ is not type changing, then it preserves peripheral systems. Under this assumption, h extends to an isotopy of $\mathbb{S}^3$, making $\mathcal{A}$ and $\mathcal{A}'$ isotopic links. Moving forward, we will assume that h is type changing. Our proof breaks down into just two cases since two flat FAL complements each with a different number of reflection surfaces can not be homeomorphic by Corollary 3.12.

Case I If M contains multiple distinct reflection surfaces, then Corollary 3.12 tells us that $\mathcal{A}$ and $\mathcal{A}'$ are isotopic links.

Case II Suppose M and M' each contain a unique reflection surface. Then by Theorem 7.11 we know that $\mathcal{A}$ has the structure of a signature link $\mathcal{L}$. Thus, there is a unique knot circle K_f whose type is fixed by h, and h changes the type of every crossing circle C_i linking K_f. Each C_i links a knot circle K_i that changes type and h preserves the type of all other crossing circles, each of which link two of the K_i. The fact that $\mathcal{A}$ has all these properties is justified at the beginning of this section in Lemma 7.1.

Since $\mathcal{A}$ is a signature link, we can consider the full-swap homeomorphism $h_f\colon M \to M''$, discussed in Section 5, where $M'' = \mathbb{S}^3 \setminus \mathcal{A}''$ and $\mathcal{A}''$ is also a signature link. This homeomorphism has the same effect on peripheral structures as h and Proposition 5.5 tells us that $\mathcal{A}$ and $\mathcal{A}''$ are isotopic links. Now h_f and h act identically on peripheral systems, so $h_f \circ h^{-1}\colon M' \to M''$ preserves peripheral systems and extends to an isotopy of $\mathbb{S}^3$. This implies that the links $\mathcal{A}'$ and $\mathcal{A}''$ are isotopic. Thus, $\mathcal{A}$ and $\mathcal{A}'$ are isotopic, finishing this case. □

By combining Theorem 3.14 with the work from this section, we can now provide a complete proof of Theorem 1.3, which we first restate.

Theorem 1.3 *Let $\mathcal{A}$ be a flat FAL. Then either*

- *$\mathcal{A}$ is not a signature link and both $\mathcal{A}$ and its complement $M = \mathbb{S}^3 \setminus \mathcal{A}$ have the same symmetry group, or*
- *$\mathcal{A}$ is a signature link and full-swaps on $\mathcal{A}$ generate symmetries of $M = \mathbb{S}^3 \setminus \mathcal{A}$ which are not restrictions of symmetries of $\mathcal{A}$ to M.*

Proof Let $\mathcal{A}$ be a flat FAL with $M = \mathbb{S}^3 \setminus \mathcal{A}$. Note that P_3 is a signature link (see Figure 15) whose complement contains multiple reflection surfaces.

First, suppose A is not a signature link. If M contains multiple reflection surfaces, then Theorem 3.14 tells us that $\mathrm{Sym}(\mathbb{S}^3, \mathcal{A}) = \mathrm{Sym}(\mathbb{S}^3 \setminus \mathcal{A})$. If M contains a unique reflection surface and A is not a signature link, then Theorem 7.11 implies every self-homeomorphism of M is not type changing. Thus, in this case, every self-homeomorphism of M preserves peripheral structures, and so, extends to an isotopy of $\mathbb{S}^3$, as needed.

Now, suppose A is a signature link. Then M admits a full-swap homeomorphism h_f. As discussed in Section 5, full-swaps are compositions of ml-swaps, where these ml-swaps exchange meridional and longitudinal slopes on (distinct) Hopf sublinks of $\mathcal{A}$. Such homeomorphisms do not extend to isotopies of $\mathbb{S}^3$, and so, h_f is a representative for an element of $\mathrm{Sym}(M)$ that does not restrict to an element of $\mathrm{Sym}(\mathbb{S}^3, \mathcal{A})$. □

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Department of Mathematics, Furman University
Greenville, SC, United States

Department of Mathematics, California State University, San Bernardino
San Bernardino, CA, United States

christian.millichap@furman.edu, rtrapp@csusb.edu

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