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Involutive Khovanov homology and equivariant knots

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For strongly invertible knots, we define an involutive version of Khovanov homology, and from it derive a pair of integer-valued invariants $(\underline{s}, \overline{s})$, which is an equivariant version of Rasmussen's s-invariant. Using these invariants, we reprove that the infinite family of knots J_n introduced by Hayden each admits exotic pairs of slice disks. Our construction is intended to give a Khovanov-theoretic analogue of the formalism given by Dai, Mallick and Stoffregen in involutive knot Floer theory.

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1 Introduction

A *strongly invertible knot* is a knot K in S^3 equipped with an involution τ of S^3 that reverses the orientation of K. While strongly invertible knots have been studied for decades (see Sakuma [26]), recent developments in knot theory and low-dimensional topology gave rise to new directions in its research and its applications, as in Watson [33], Snape [30], Hayden [12], Hayden and Sundberg [13], Alfieri and Boyle [2], Boyle and Issa [7], Lipshitz and Sarkar [21] and Dai, Mallick and Stoffregen [10].

Dai, Mallick and Stoffregen [10] use *involutive knot Floer homology* to define integer-valued invariants $\underline{V}^{\tau}_0, \overline{V}^{\tau}_0, \underline{V}^{\iota\tau}_0, \overline{V}^{\iota\tau}_0$ of strongly invertible knots, and show that there is an infinite family of knots J_n, each admitting exotic pairs of slice disks. Previously, the result for the special case $J_0 = 17nh_{73}$ (also known as the *positron knot*) was proved by Hayden and Sundberg [13] using Khovanov homology, by distinguishing the cobordism maps induced from the two slice disks. In this paper, we adapt the formalism of involutive knot Floer homology to the Khovanov side, and recover the general result systematically.

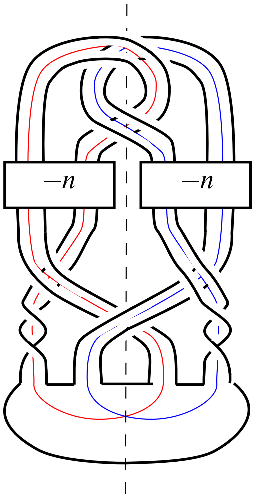

Figure 1: The knot J_n and the two slice disks D_n, D'_n obtained by compressing along the two colored circles.

For strongly invertible knots, we define an involutive version of Khovanov homology, called the *involutive Khovanov homology*, and from it derive a pair of integer-valued invariants $(\underline{s}, \overline{s})$, which is an equivariant version of Rasmussen's *s-invariant* [25] for ordinary knots.

Theorem 1 *For each $n \geq 0$, the strongly invertible knot J_n of Figure 1 has*

$$\underline{s}(J_n) = 0 < 2 \leq \overline{s}(J_n).$$

As we shall see later, Theorem 1 implies that each J_n never admits a *simple isotopy-equivariant* slice disk. In particular, the two slice symmetric disks D_n, D'_n of J_n depicted in Figure 1 are not smoothly isotopic rel J_n. Combining Theorem 1 with the fact that D_n and D'_n are topologically isotopic, which is proved by Hayden [12] using the result of Conway and Powell [9], we may conclude that these disks form an exotic pair of slice disks of J_n. This argument is completely analogous to the proof of [10, Theorem 7.11].

The definition of involutive Khovanov homology follows the formalism of involutive knot Floer homology; see Hendricks and Manolescu [14], Zemke [36], Alfieri, Kang and Stipsicz [3] and Dai, Mallick and Stoffregen [10]. Given a strongly invertible knot diagram (D, τ), there is an induced involution τ on the Khovanov complex $\mathrm{CKh}(D)$ over $\mathbb{F}_2$. Using this we define:

Definition 1.1 The *involutive Khovanov complex* of (D, τ) is defined by

$$\mathrm{CKhI}(D, \tau) = \mathrm{Cone}\big(\mathrm{CKh}(D) \xrightarrow{Q(1+\tau)} Q\mathrm{CKh}(D)\big),$$

where Q is a formal variable of $Q^2 = 0$. The homology of $\mathrm{CKhI}(D, \tau)$ is denoted by $\mathrm{KhI}(D, \tau)$ and is called the *involutive Khovanov homology* of (D, τ).

Theorem 2 *The isomorphism class of* $\mathrm{KhI}(D,\tau)$ *is an invariant of the strongly invertible knot* (K,τ).

Hereafter, we make τ implicit and omit it from CKhI and KhI. We note that Definition 1.1 is also valid for the deformed versions of Khovanov homology obtained by replacing the defining Frobenius algebra, and Theorem 2 still holds. There are also reduced versions, denoted by $\mathrm{CKhI}_r(D)$ and $\mathrm{KhI}_r(D)$, which are also invariants of strongly invertible knots.

Our equivariant invariant $(\underline{s},\overline{s})$ is defined using the Bar-Natan's deformation, given by the Frobenius algebra $A = R[X]/(X(X+H))$ over the ring $R = \mathbb{F}_2[H]$ with $\deg(H) = -2$. Let us first recall the definition of the $\mathbb{F}_2$-*Rasmussen invariant* $s = s^{\mathbb{F}_2}$, as characterized by Kotelskiy, Watson and Zibrowius in [18, Proposition 3.8]. For any knot K, it is known that the reduced Bar-Natan homology $\mathrm{BN}_r(K)$ has a single $\mathbb{F}_2[H]$-tower in homological grading 0. Then $s(K)$ is defined as the quantum grading of its generator:

$$\mathrm{BN}_r(K) \cong h^0 q^{s(K)} \mathbb{F}_2[H] \oplus (\mathrm{Tor}).$$

For a strongly invertible knot K, it is proved that the *reduced involutive Bar-Natan homology* $\mathrm{BNI}_r(K)$ has two $\mathbb{F}_2[H]$-towers, one in homological grading 0 and another in homological grading 1. Thus we may define $\underline{s}(K), \overline{s}(K)$ as the quantum gradings of their generators:

$$\mathrm{BNI}_r(K) \cong h^0 q^{\underline{s}(K)} \mathbb{F}_2[H] \oplus h^1 q^{\overline{s}(K)} \mathbb{F}_2[H] \oplus (\mathrm{Tor}).$$

The pair $(\underline{s}(K), \overline{s}(K))$ is called the *equivariant Rasmussen invariant* of K.

Theorem 3 *The equivariant Rasmussen invariant* $(\underline{s}(K), \overline{s}(K))$ *is an invariant of the strongly invertible knot* K, *satisfying*

$$\underline{s}(K) \le s(K) \le \overline{s}(K).$$

As is true for the ordinary s-invariant, our equivariant s-invariant is directly computable using computers. In particular, we have computed the invariants for the three strongly invertible knots given by Hayden and Sundberg in [13], which are proved therein to admit pairs of nonsmoothly isotopic slice disks.

Proposition 1.2 *The three strongly invertible slice knots* $K = m(9_{46}), 15n_{103488}, 17nh_{73}$ *have*

$$\underline{s}(K) = 0 < 2 = \overline{s}(K).$$

Table 1 shows the computed BNI_r for $J_0 = 17nh_{73}$, from which we can see that there are indeed two $\mathbb{F}_2[H]$ summands, one in bigrading $(0,0)$ and another in $(1,2)$, and the remaining summands are copies of $\mathbb{F}_2[H]/(H) = \mathbb{F}_2$. The computed BNI_r for the other two knots will be given in Section 5. Combined with Corollary 1.11, we recover that these knots admit pairs of nonsmoothly isotopic slice disks.

We further study properties of the equivariant invariant $(\underline{s},\overline{s})$, that are analogous to that of the ordinary s.

Proposition 1.3 *For the mirror* K^* *of* K, *we have*

$$\underline{s}(K^*) = -\overline{s}(K).$$

18	.	.	.	.	.	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	.	.	.	.	.	.	.	.
14	.	.	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$	.	.
12	.	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$	.	.	.
10	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
8	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}^3$	$\mathbb{F}^3$	$\mathbb{F}$	.	.	.	.	.
6	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
4	.	.	.	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$	.	.	.	.	.	.	.	.
2	.	.	$\mathbb{F}$	$\mathbb{F}[H] \oplus \mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.	.	.
0	.	.	$\mathbb{F}[H]$	.	.	.	.	.	.	.	.	.	.	.
-2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.	.	.	.	.	.
	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11

Table 1: $\mathrm{BNI}_r(J_0)$.

Proposition 1.4 *For another strongly invertible knot K',*

$$\underline{s}(K) + \underline{s}(K') \le \underline{s}(K \,\#\, K') \le \underline{s}(K) + \overline{s}(K') \le \overline{s}(K \,\#\, K') \le \overline{s}(K) + \overline{s}(K').$$

Here # denotes the equivariant connected sum.

Proposition 1.5 *Let K^+, K^- be strongly invertible knots such that K^- is obtained by applying an "equivariant negative crossing change" to K^+* (see Definition 4.15). *Then*

$$\underline{s}(K^-) \le \underline{s}(K^+) \le \underline{s}(K^-) + 2a,$$

where $a = 1$ if the move is performed on-axis, and $a = 2$ if performed off-axis. The same holds for $\overline{s}$.

Proposition 1.6 *Let K^+, K^- be strongly invertible knots such that K^- is obtained by applying a "4-strand equivariant generalized negative crossing change" to K^+* (see Definition 4.24). *Then*

$$\underline{s}(K^-) \le \underline{s}(K^+) \le \underline{s}(K^-) + 4a,$$

where $a = 1$ if the move is performed on-axis, and $a = 2$ if performed off-axis. The same holds for $\overline{s}$.

Proposition 1.7 *The positive (p,q)-torus knot $T_{p,q}$ has*

$$\underline{s}(T_{p,q}) = \overline{s}(T_{p,q}) = (p-1)(q-1)$$

with respect to the unique inverting involution.

The lower bound for the 4-genus

$$|s(K)| \le 2g_4(K)$$

is one of the significant properties of s, that led to the reproof of the *Milnor conjecture* [24]. This is an implication of its behavior under cobordisms. In order to prove an analogous result for the equivariant invariant, we impose the following condition on cobordisms.

Definition 1.8 Let L, L' be two strongly invertible links sharing the same involution τ of S^3. A *simple equivariant cobordism* between L and L' is an oriented cobordism S in $S^3 \times I$ between L, L' satisfying $(\tau \times \mathrm{id})(S) = S$. A *simple isotopy-equivariant cobordism* between L, L' is defined similarly, except that $(\tau \times \mathrm{id})(S)$ is only required to be isotopic to S rel boundary.

Proposition 1.9 *Suppose there is a connected, simple isotopy-equivariant cobordism S between two strongly invertible knots K, K'. Then,*

$$|\underline{s}(K) - \underline{s}(K')| \leq -\chi(S) \quad \text{and} \quad |\overline{s}(K) - \overline{s}(K')| \leq -\chi(S).$$

In particular, $\underline{s}$ and $\overline{s}$ are invariant under simple isotopy-equivariant concordances.

We may naturally define the *simple equivariant genus* $\widetilde{\mathrm{sg}}_4(K)$ and the *simple isotopy-equivariant genus* $\widetilde{\mathrm{sig}}_4(K)$ of a strongly invertible knot K. Obviously, we have inequalities

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \widetilde{g}_4(K) \arrow[dr, phantom, sloped, "\leq"] & \\
g_4(K) \arrow[ur, phantom, sloped, "\leq"] \arrow[dr, phantom, sloped, "\leq"] & & \widetilde{\mathrm{sg}}_4(K) \;\leq\; \widetilde{g}_3(K) \\
 & \widetilde{\mathrm{sig}}_4(K) \arrow[ur, phantom, sloped, "\leq"] &
\end{tikzcd}$$

where g_4 is the ordinary 4-genus, while $\widetilde{g}_3$ and $\widetilde{g}_4$ are the equivariant 3- and 4-genus, respectively. Proposition 1.9 implies:

Corollary 1.10 *Both $|\underline{s}(K)|$ and $|\overline{s}(K)|$ bound $2\,\widetilde{\mathrm{sig}}_4(K)$ from below.*

Corollary 1.11 *If either $|\underline{s}(K)|$ or $|\overline{s}(K)|$ is greater than $2g_4(K)$, then no slice surfaces S of K realizing $g(S) = g_4(K)$ are simple isotopy-equivariant. In particular, S and $(\tau \times \mathrm{id})(S)$ are not smoothly isotopic rel K.*

Now Theorem 1 and the implication that the two slice disks D_n, D'_n of Figure 1 are not smoothly isotopic rel J_n follows from Theorem 3, Propositions 1.2 and 1.6 and Corollary 1.11, for we have

$$0 = \underline{s}(J_0) \leq \underline{s}(J_1) \leq \cdots \leq \underline{s}(J_n) \leq s(J_n) = 0 \quad \text{and} \quad 2 = \overline{s}(J_0) \leq \overline{s}(J_1) \leq \cdots \leq \overline{s}(J_n).$$

We finally remark that the restriction to *simple* equivariant cobordisms is due to the way the cobordism maps on Khovanov homology are defined. However, this restriction may lead to a deeper understanding of the *equivariant concordance group* $\widetilde{\mathscr{C}}$, defined by Sakuma in [26, Section 4]. It is known that the Smith conjecture fails in higher dimensions, in particular, there is an involution $\overline{\tau}$ on S^4 whose fixed-point set is a knotted sphere; see Gordon [11]. This implies that nonsimple equivariant cobordisms do exist. We may define the *simple equivariant concordance group* $\widetilde{\mathrm{s}\mathscr{C}}$, whose elements are simple equivariant concordance classes of directed strongly invertible knots, with the operation given by the equivariant connected sum. We question whether the surjective homomorphism $\widetilde{\mathrm{s}\mathscr{C}} \to \widetilde{\mathscr{C}}$ is injective or not, and also the existence of knots such that $\widetilde{g}_4(K) < \widetilde{\mathrm{sg}}_4(K)$.

Organization This paper is organized as follows. In Section 2, we define the *involutive Khovanov complex* CKhI for *involutive link diagrams*, and prove its invariance up to chain homotopy under the *involutive Reidemeister moves*. The reduced version is also introduced therein. In Section 3, we define the *equivariant Rasmussen invariant* $(\underline{s}, \overline{s})$ for *strongly invertible links*, not in the way presented above, but instead using the divisibility of the *equivariant Lee classes*. In Section 4, the above-stated properties of $(\underline{s}, \overline{s})$ are proved, including the proof that the two definitions of $(\underline{s}, \overline{s})$ coincide. In Section 5, the proof of Theorem 1 will be restated, followed by an observation of the result. In Section 6, we briefly state that an analogous construction is possible for 2-periodic links by considering a modified involution. In the appendix, we give a list of KhI and BNI for prime knots with up to 7 crossings, obtained by direct computations.

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2 Involutive Khovanov homology

Throughout this paper, we work in the smooth category and assume all objects and maps to be smooth. We assume that the reader is familiar with the construction of Khovanov homology [17], in particular Bar-Natan's reformulation given in [5]. Let R be a commutative ring with unity of char $R = 2$, and A a Frobenius algebra of the form $A = R[X]/(X(X+h))$ with $h \in R$, determined by $\varepsilon(1) = 0$, $\varepsilon(X) = 1$. For a link diagram D, let $\mathrm{CKh}(D; R, h)$ denote the Khovanov chain complex of D obtained from the above Frobenius algebra A, and $\mathrm{Kh}(D; R, h)$ its homology. Typically we consider the following cases:

$$(R, h) = (\mathbb{F}_2, 0),\ (\mathbb{F}_2, 1),\ (\mathbb{F}_2[H], H),$$

each corresponding to the original Khovanov homology [17], the filtered and the bigraded Bar-Natan homology [5] over $\mathbb{F}_2$. For the third case, we assume that H has $\deg(H) = -2$, so that the chain complex admits a bigrading. We usually make (R, h) implicit and omit it from the notations.

2.1 Definition

Hereafter, we assume that any involution τ on S^3 has fixed-point set S^1. (From the resolution of the Smith conjecture [32; 6], the fixed-point set of an involution of S^3 with nonempty fixed-point set is necessarily an unknotted circle.) The fixed-point set $\mathrm{Fix}(\tau)$ is called the *axis* of τ. We follow [22] for the definitions of involutive links and their equivalence.

Definition 2.1 An *involutive link* (L, τ) in S^3 is an oriented link L equipped with an involution τ on S^3 such that $\tau(L) = L$, possibly altering the orientations on some of its components. Two involutive links are *equivalent* if they are isotopic through involutive links. An involutive link (L, τ) is *strongly invertible* if τ reverses the orientation of L, *2-periodic* if τ preserves the orientation of L. Strongly invertible links and 2-periodic links are together called *equivariant links*.

We will draw involutive link diagrams so that the axis is projected as a straight vertical line, and the diagram is symmetric with respect to the axis, as in Figure 1. For an involutive link diagram (D,τ), the induced involution τ on $\mathrm{CKh}(D)$ is defined as follows. For each state s for D, there is a unique state s' such that $D(s)$ and $D(s')$ are symmetric with respect to the axis $\mathrm{Fix}(\tau)$. For each standard generator x in state s, its image $\tau(x)$ is defined to be the generator in the state s' that corresponds to x under the symmetry. It is obvious that τ commutes with d and is an involution on $\mathrm{CKh}(D)$ with bidegree $(0,0)$.

Definition 2.2 The *involutive Khovanov complex* of (D,τ) is defined by

$$\mathrm{CKhI}(D,\tau) = \mathrm{Cone}\big(\mathrm{CKh}(D) \xrightarrow{Q(1+\tau)} Q\mathrm{CKh}(D)\big),$$

where Q is a formal variable of $Q^2=0$. The homology of $\mathrm{CKhI}(D,\tau)$ is denoted by $\mathrm{KhI}(D,\tau)$ and is called the *involutive Khovanov homology* of (D,τ).

Remark 2.3 Our $\mathrm{CKhI}(D,\tau)$ is different from the triply graded complex $\mathrm{CKh}_\tau(D)$ given in [22], where $\mathrm{CKh}_\tau(D) = \mathrm{CKh}(D;\mathbb{F}_2,0)$ as $\mathbb{F}_2$-modules, but the differential is defined as $\partial = d+1+\tau$.

Hereafter we make τ implicit and omit it from CKhI and KhI. When explicitly describing elements and maps, we often regard $\mathrm{CKhI}(D)$ as the direct sum of two copies of $\mathrm{CKh}(D)$,

$$\mathrm{CKhI}(D) = \mathrm{CKh}(D) \oplus \mathrm{CKh}(D)[1],$$

with the differential given by

$$\begin{pmatrix} d & \\ 1+\tau & d \end{pmatrix}.$$

As in the noninvolutive case, the differential preserves the quantum grading if $h=0$ or $\deg(h)=-2$; or is quantum-grading nondecreasing if $h\neq 0$ and $\deg(h)=0$. In the former case, we regard $\mathrm{CKhI}(D)$ as a bigraded complex, and in the latter as a filtered complex.

Formally, a chain complex over $\mathbb{F}_2$ equipped with an involution τ is called a *τ-complex*. For τ-complex C, let C_τ denote the complex

$$C_\tau = \mathrm{Cone}(1+\tau).$$

This correspondence is functorial in the following sense. For homotopic chain maps f,g (of any degree) with homotopy h, we write

$$f \simeq_h g$$

to mean

$$f+g = dh+hd.$$

A chain map f between τ-complexes C,C' is *homotopy τ-equivariant* if $\tau f \simeq f\tau$. A *τ-conjugate* of f is defined by

$$f^\tau = \tau f \tau.$$

Obviously f is homotopy τ-equivariant if and only if $f \simeq f^\tau$. Two homotopy τ-equivariant chain maps f, g with homotopies

$$\tau f \simeq_{h_f} f\tau \quad \text{and} \quad \tau g \simeq_{h_g} g\tau$$

are *coherently homotopic* with homotopy h if $f \simeq_h g$ and

$$\tau h + h\tau \simeq h_g + h_f.$$

The following lemmas are easy to verify and will be used throughout this paper.

Lemma 2.4 *A homotopy τ-equivariant chain map $f\colon C \to C'$ with homotopy $\tau f \simeq_{h_f} f\tau$ induces a chain map*

$$f_\tau\colon C_\tau \to C'_\tau \quad \text{given by} \quad f_\tau = \begin{pmatrix} f & \\ h_f & f \end{pmatrix}.$$

In particular, we have

$$1_\tau = 1.$$

For another homotopy τ-equivariant chain map $g\colon C' \to C''$ with homotopy $\tau g \simeq_{h_g} g\tau$, the composition gf is homotopy τ-equivariant with homotopy

$$h_{gf} = h_g f + g h_f,$$

and the above correspondence gives

$$(gf)_\tau = g_\tau f_\tau.$$

Lemma 2.5 *Suppose that $f, g\colon C \to C'$ are homotopy τ-equivariant chain maps with homotopies $\tau f \simeq_{h_f} f\tau$ and $\tau g \simeq_{h_g} g\tau$, and are coherently homotopic with homotopies h and k satisfying $f \simeq_h g$ and $\tau h + h\tau \simeq_k h_g + h_f$. Then the induced maps*

$$f_\tau, g_\tau\colon C_\tau \to C'_\tau$$

are homotopic with homotopy

$$\begin{pmatrix} h & \\ k & h \end{pmatrix}.$$

Lemma 2.6 *Let f be a homotopy τ-equivariant homotopy equivalence with a homotopy τ-equivariant homotopy inverse g and coherent homotopies*

$$gf \simeq 1 \quad \text{and} \quad fg \simeq 1.$$

Then the induced map

$$f_\tau\colon C_\tau \to C'_\tau$$

is a homotopy equivalence with homotopy inverse g_τ.

2.2 Invariance

Next we prove the invariance of KhI. In [22], Lobb and Watson introduced the eight *involutive Reidemeister moves* (see Figure 2) and proved that two involutive links are equivalent if and only if those diagrams are related by a sequence of involutive Reidemeister moves and equivariant planar isotopies. Theorem 2 follows from the following stronger proposition.

Proposition 2.7 *Let D, D' be two involutive link diagrams related by one of the involutive Reidemeister moves. Then there is a chain homotopy equivalence*

$$\rho\colon \mathrm{CKhI}(D) \to \mathrm{CKhI}(D')$$

such that the following diagram commutes:

$$\begin{tikzcd} Q\mathrm{CKh}(D)[1] \arrow[r, hook] \arrow[d, "\rho"] & \mathrm{CKhI}(D) \arrow[r, two heads] \arrow[d, "\rho"] & \mathrm{CKh}(D) \arrow[d, "\rho"] \\ Q\mathrm{CKh}(D')[1] \arrow[r, hook] & \mathrm{CKhI}(D') \arrow[r, two heads] & \mathrm{CKh}(D') \end{tikzcd}$$

Here the two vertical arrows on the left and the right are given by the composition of the standard chain homotopy equivalences given in [5] that corresponds to some decomposition of the involutive move into a sequence of ordinary Reidemeister moves.

In order to prove Proposition 2.7, instead of explicitly constructing chain homotopy equivalences and chain homotopies for each of the moves, we prove the existences of the desired maps in a uniform way.

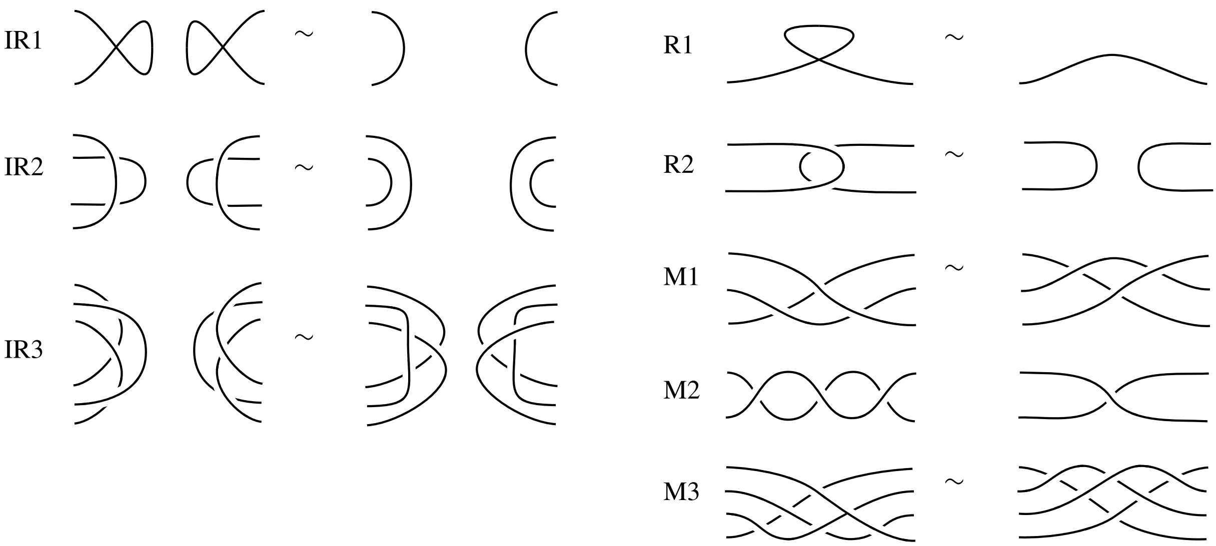


Figure 2: Involutive Reidemeister moves.

2.2.1 General strategy Any involutive Reidemeister move can be decomposed into a sequence of ordinary Reidemeister moves. By composing the corresponding maps given in [5] we get explicit homotopy equivalences F, G between $\mathrm{CKh}(D)$ and $\mathrm{CKh}(D')$, together with homotopies $GF \simeq_H I$ and $FG \simeq_{H'} I$. In general, these maps are not strictly τ-equivariant. Nevertheless, we can prove the following.

Claim 2.8 *F and G are homotopy τ-equivariant, ie there exists homotopies $\tau F \simeq_{h_F} F\tau$ and $\tau G \simeq_{h_G} G\tau$.*

Claim 2.9 *H and H' are coherent homotopies, ie there exists homotopies $\tau H + H\tau \simeq h_G F + G h_F$ and $\tau H' + H'\tau \simeq h_F G + F h_G$.*

Then the invariance follows from Lemma 2.6. Thus the proof is reduced to proving Claims 2.8 and 2.9 for each of the moves. Recall from [5, Definition 8.5] that a tangle diagram T is Kh-*simple* if any degree 0 automorphism (up to homotopy) of $C(T)$ is homotopic to $\pm I$. Here $C(T)$ is the *formal Khovanov complex* introduced in [5], which is a complex in the additive closure of the category $\mathrm{Cob}^3(\partial T)$. Here we strengthen the condition as follows.

Definition 2.10 A chain complex C (in any additive category) is *simple* if any degree 0 automorphism (up to homotopy) of C is homotopic to $\pm I$, and any degree $n \neq 0$ self-chain map $C \to C$ is null-homotopic. A tangle diagram T is Kh-*simple* if the complex $C(T)$ is simple.

A complex C being simple is equivalent to the condition that the homology of the End-complex $\mathrm{End}(C) = \mathrm{Hom}(C, C)$ has only $\pm I$ as the degree 0 multiplicative units (with respect to the composition) and its homology is supported only on degree 0. One can easily prove that [5, Lemmas 8.6–8.9] also hold for our stronger definition, namely:

Lemma 2.11 *Simplicity is preserved under chain-homotopy equivalences.*

Lemma 2.12 *Parings are* Kh*-simple. Here, a pairing is a tangle diagram that has no crossings and no closed components.*

Lemma 2.13 *A tangle diagram T is* Kh*-simple if and only if TX is* Kh*-simple. Here TX is a tangle diagram obtained by adding one extra crossing X somewhere along the boundary of T.*

The following lemmas will also be useful. Here C and C' are complexes in any additive category.

Lemma 2.14 *Suppose there are two degree 0 chain homotopy equivalences $F, F' \colon C \to C'$ with homotopy inverses G, G' respectively. If C is simple, then F and F', as well as G and G', are homotopic up to sign.*

Proof The map GF' is an automorphism on C with a homotopy inverse $G'F$. Since C is simple, we have $GF' \simeq \pm I$. Assuming that $GF' \simeq I$, we get $F \simeq FGF' \simeq F'$ and $G \simeq GF'G' \simeq G'$. □

Lemma 2.15 *Suppose there are degree* 0 *chain maps* $F, F' \colon C \to C'$ *and* $G, G' \colon C' \to C$, *together with homotopies* $F \simeq_{h_F} F'$, $G \simeq_{h_G} G'$, $GF \simeq_H I$ *and* $G'F' \simeq_{H'} I$. *If* C *is simple, then* $H - H'$ *and* $h_G F + G' h_F$ *are homotopic.*

Proof On one hand, we have

$$GF - G'F' = d(H - H') + (H - H')d.$$

On the other hand, we have

$$\begin{aligned} GF - G'F' &= (G - G')F + G'(F - F') \\ &= (dh_G + h_G d)F + G'(dh_F + h_F d) \\ &= d(h_G F + G'h_F) + (h_G F + G'h_F)d. \end{aligned}$$

Thus $(H - H') - (h_G F + G'h_F)$ is a degree -1 chain map, which is necessarily null-homotopic since C is simple. Therefore $H - H' \simeq h_G F + G'h_F$. □

2.2.2 IR1–IR3

Proof of Proposition 2.7, cases IR1–IR3 We define maps F, G by the compositions

$$F = F_2 F_1 \quad \text{and} \quad G = G_1 G_2,$$

where F_1, F_2 are the standard maps corresponding to the Reidemeister moves performed on the left and the right part of the diagram respectively (see Figure 3), and G_1, G_2 are those homotopy inverses. The conjugate maps are given by

$$F^\tau = F_2^\tau F_1^\tau \quad \text{and} \quad G^\tau = G_1^\tau G_2^\tau.$$

Since F_1 and F_2 (resp. G_1 and G_2) are commutative, we may write

$$F^\tau = F_1^\tau F_2^\tau \quad \text{and} \quad G^\tau = G_2^\tau G_1^\tau.$$

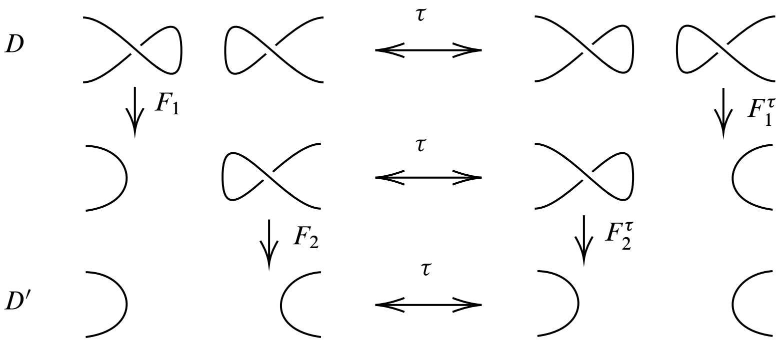


Figure 3: F and F^τ.

Since the tangle parts appearing in the move are Kh-simple, using Lemmas 2.14 and 2.15 we obtain homotopies with the desired properties such as $F_1 \simeq F_2^\tau$ and $F_2 \simeq F_1^\tau$. Thus we obtain the desired properties stated in Claims 2.8 and 2.9, such as $F \simeq F^\tau$. Here we are implicitly extending the maps defined locally for tangle diagrams to maps defined globally for the link diagrams, using the planar algebra structures explained in [5, Section 5]. □

2.2.3 R1, R2 and M1–M3

Proof of Proposition 2.7, cases R1, R2 and M1–M3 Observe that each of these moves occurs locally on a disk that intersects the axis, and the tangles appearing in the move are Kh-simple. Thus Claims 2.8 and 2.9 immediately follow from Lemmas 2.14 and 2.15. □

Remark 2.16 By using the explicit description of the maps, for some of the moves we may take the desired homotopies in a much simpler form. For R1 and R2, the homotopy equivalences F, G and the homotopies H, H' are strictly τ-equivariant. For R3, we may take $\tau F \simeq_{h_F} F\tau$ and $\tau G \simeq_{h_G} G\tau$ so that $Gh_F = 0$, $Fh_G = 0$, and $\tau H + H\tau = h_G F$, $\tau H' + H'\tau = h_F G$.

2.3 Reduced version

Next, we define a reduced version of the involutive Khovanov homology. Let us first recall the definition of the reduced complex in the noninvolutive setting. For a pointed link diagram D, the *reduced Khovanov complex*[1] $\mathrm{CKh}_r(D)$ is defined as the subcomplex of $\mathrm{CKh}(D)$ generated by the standard generators each labeled X on the pointed circle. The *coreduced complex* $\mathrm{CKh}'_r(D)$ is defined as the quotient $\mathrm{CKh}(D)/\mathrm{CKh}_r(D)$.

Definition 2.17 A *pointed involutive link* (L, τ) is an involutive link equipped with a basepoint on $\mathrm{Fix}(\tau) \cap L$.

Note that 2-periodic links cannot be pointed, since $\mathrm{Fix}(\tau) \cap L = \varnothing$. For a pointed involutive link diagram D, it is obvious that the (co)reduced complexes are invariant under the involution τ. Thus we may define:

Definition 2.18 Let (D, τ) be a pointed involutive link diagram. The *reduced involutive Khovanov complex* is defined as

$$\mathrm{CKhI}_r(D, \tau) = \mathrm{Cone}(\mathrm{CKh}_r(D) \xrightarrow{Q(1+\tau)} Q\mathrm{CKh}_r(D)).$$

Similarly, the *coreduced involutive Khovanov complex* is defined as

$$\mathrm{CKhI}'_r(D, \tau) = \mathrm{Cone}(\mathrm{CKh}'_r(D) \xrightarrow{Q(1+\tau)} Q\mathrm{CKh}'_r(D)).$$

[1] The conventional notation for the reduced Khovanov complex is $\widetilde{\mathrm{CKh}}$. Here we changed the notation to avoid putting too much decorations on the letters.

Again, we usually omit τ from the notations of CKhI_r. In the noninvolutive setting, the reduced and the coreduced complexes can be interchanged by the following automorphism σ.

Definition 2.19 The Frobenius algebra automorphism σ on A is defined by

$$1 \mapsto 1, \quad X \mapsto X + h.$$

Its induced automorphism on $\mathrm{CKh}(D)$ is also denoted by σ.

Proposition 2.20 *The automorphism σ is an involution that commutes with τ.*

Thus σ induces an involution on $\mathrm{CKhI}(D)$, which is again denoted σ. Analogous to the noninvolutive case, we have:

Proposition 2.21 *There are isomorphisms*

$$\mathrm{CKhI}'_r(D) \cong \sigma(\mathrm{CKhI}_r(D)) \quad \textit{and} \quad \mathrm{CKhI}_r(D) \cong \mathrm{CKhI}(D)/\sigma(\mathrm{CKhI}_r(D)).$$

Proof Immediate from [28, Propositions 3.12 and 3.14]. $\square$

Proposition 2.22 *There is a short exact sequence*

$$\mathrm{CKhI}_r(D) \hookrightarrow \mathrm{CKhI}(D) \twoheadrightarrow \mathrm{CKhI}'_r(D).$$

Proof The short exact sequence

$$\mathrm{CKh}_r(D) \hookrightarrow \mathrm{CKh}(D) \twoheadrightarrow \mathrm{CKh}'_r(D)$$

is τ-equivariant, and hence we obtain maps between exact sequences

$$\begin{tikzcd}
\mathrm{CKhI}_r \arrow[r, hook] \arrow[d, equal] & \mathrm{CKhI} \arrow[r, two heads] \arrow[d, equal] & \mathrm{CKhI}/\mathrm{CKhI}_r \arrow[d, dashed] \\
\mathrm{Cone}(\mathrm{CKh}_r \to Q\mathrm{CKh}_r) \arrow[r, hook] & \mathrm{Cone}(\mathrm{CKh} \to Q\mathrm{CKh}) \arrow[r, two heads] & \mathrm{Cone}(\mathrm{CKh}'_r \to Q\mathrm{CKh}'_r)
\end{tikzcd}$$

The right dashed arrow is an isomorphism from the five lemma. $\square$

Proposition 2.23 *The short exact sequence of Proposition 2.22 splits.*

Before proceeding to the proof, let us review the corresponding results in the noninvolutive setting. Shumakovitch [29] proved that the $\mathbb{F}_2$-Khovanov homology splits (ie when $(R,h) = (\mathbb{F}_2, 0)$), and lately Wigderson [35] extended this result to the $\mathbb{F}_2$-bigraded Bar-Natan homology (ie $(R,h) = (\mathbb{F}_2[H], H)$). Here we briefly review Wigderson's construction, which works whenever $\operatorname{char} R = 2$. First, as an R-module, the coreduced complex $\mathrm{CKh}'_r(D)$ can be identified with the submodule of $\mathrm{CKh}(D)$ generated by the standard generators each labeled 1 on the pointed circle. Then $\mathrm{CKh}(D) = \mathrm{CKh}'_r(D) \oplus \mathrm{CKh}_r(D)$ as R-modules, and the differential d of $\mathrm{CKh}(D)$ can be described as

$$d = \begin{pmatrix} d_1 & \\ f & d_X \end{pmatrix},$$

where d_X and d_1 are differentials of $\mathrm{CKh}_r(D)$ and $\mathrm{CKh}'_r(D)$, respectively, and

$$f\colon \mathrm{CKh}'_r(D) \to \mathrm{CKh}_r(D)$$

is the map given by restricting d to $\mathrm{CKh}'_r(D)$ and then projecting onto $\mathrm{CKh}_r(D)$. It follows from $d^2 = 0$ that f is a chain map, and thus $\mathrm{CKh}(D)$ may be regarded as the cone of f. Next, a null-homotopy κ of f is constructed as follows. For each $i \geq 0$, define

$$\kappa_i\colon \mathrm{CKh}'_r(D) \to \mathrm{CKh}_r(D), \quad x = \underline{1} \otimes \cdots \mapsto \sum_{ⓞ^X, \dots, ⓘ^X} \underline{X} \otimes \cdots.$$

Here, the underline indicates the label for the pointed circle, and the sum runs over all choices of $i+1$ circles C_j labeled X in x. Inside the summation, the label on the pointed circle is changed from 1 to X while the labels of C_j are changed from X to 1. Then define

$$\kappa = \sum_{i \geq 0} h^i \kappa_i \colon \mathrm{CKh}'_r(D) \to \mathrm{CKh}_r(D).$$

It is proved in [35] that κ gives a null-homotopy of f, and hence $1 + \kappa$ gives a section of the quotient map $\mathrm{CKh}(D) \to \mathrm{CKh}'_r(D)$.

Proof of Proposition 2.23 Suppose D is a pointed involutive link diagram. It is obvious from the construction of κ that it commutes with τ, and hence

$$\begin{pmatrix} 1+\kappa & \\ & 1+\kappa \end{pmatrix}$$

gives a section of the quotient map $\mathrm{CKhI}(D) \to \mathrm{CKhI}'_r(D)$. □

Finally, we prove the invariance of the (co)reduced involutive homologies. In order to prove an analogue of Proposition 2.7, we need extra consideration for the moves that involve the basepoint, which are R1 and M1 with the basepoint placed on the horizontal strand. In fact, one can check that the corresponding maps do not restrict to the reduced complexes

$$\begin{tikzcd} \mathrm{CKh}_r(D) \arrow[r, hook] \arrow[d, dotted, "\times"] & \mathrm{CKh}(D) \arrow[d, "\rho"] \\ \mathrm{CKh}_r(D') \arrow[r, hook] & \mathrm{CKh}(D) \end{tikzcd}$$

Thus we restrict the diagrams and the moves that are allowed for the reduced complexes.

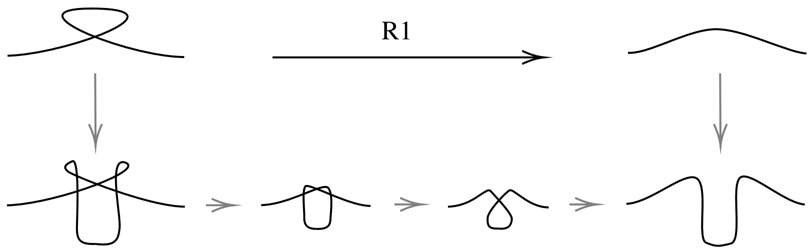


Figure 4: Modifying the R1-move.

Definition 2.24 A diagram of a pointed involutive link diagram is *normal* if the basepoint of the link is placed at the bottommost on the axis, and the horizontal strand containing the basepoint is directed rightwards.

Obviously, any pointed involutive link possesses a normal diagram. Moreover,

Proposition 2.25 *Suppose D, D' are normal pointed involutive link diagrams that represent the same pointed involutive link. Then there is a sequence of involutive Reidemeister moves whose intermediate diagrams are also normal.*

Proof Take any sequence of involutive Reidemeister moves between D and D'.

$$D = D_0 \to D_1 \to \cdots \to D_N = D'.$$

We modify this sequence, by increasing the number of moves if necessary, so that the intermediate diagrams are all normal.

Step 1 For each move $D_i \to D_{i+1}$, we may transform the diagrams by pulling down the horizontal strands that contain the basepoints so that they are placed at the bottommost on the axes (which can be realized by sequences of involutive Reidemeister moves). Moreover, we can show that the two transformed diagrams D_i', D_{i+1}' can be related by a sequence of involutive Reidemeister moves that fix the basepoints:

$$\begin{tikzcd}
D_i \arrow[rrr, "\text{move}"] \arrow[d, "\text{pull}"'] & & & D_{i+1} \arrow[d, "\text{pull}"] \\
D_i' \arrow[r, "\text{move}"] & \cdots \arrow[rr, "\text{move}"] & & D_{i+1}'
\end{tikzcd}$$

The claim is clear when the original move occurs above the pointed strand, or when the moves are off-axis which are IR1–IR3. We must consider the case where the move occurs below the pointed strand, or contains the pointed strand itself. Figure 4 depicts the modification for the R1 move that contains the pointed strand. The claim for the remaining moves R2 and M1–M3 can be checked similarly.

Step 2 The remaining work is to undo the changes in the direction of the pointed strand due to the R1 moves. Note that applying an R1 move to the bottom strand is equivalent to half-twisting the upper parts and then applying an overall half-rotation with respect to the axis. Since the diagram is involutive, a half-rotation has the effect of only changing the orientations on the components of the link. We modify each R1 move by only twisting the upper parts (which can be represented by a sequence of involutive moves) while keeping the bottom strand fixed. Since the bottom strands of D and D' are both pointed rightwards, the R1 moves in total must be applied an even number of times. Thus the effects of skipping the overall half-rotations cancel, and we obtain a desired sequence of involutive Reidemeister moves between D and D'. □

Proposition 2.26 *Let D and D' be normal diagrams related by an involutive Reidemeister move. The chain homotopy equivalence ρ and the corresponding chain homotopies given in Proposition 2.7 restrict to*

$$\rho\colon \mathrm{CKhI}_r(D) \to \mathrm{CKhI}_r(D')$$

and the following diagram commutes:

$$\begin{tikzcd}
Q\mathrm{CKh}_r(D)[1] \arrow[r, hook] \arrow[d, "\rho"] & \mathrm{CKhI}_r(D) \arrow[r, two heads] \arrow[d, "\rho"] & \mathrm{CKh}_r(D) \arrow[d, "\rho"] \\
Q\mathrm{CKh}_r(D')[1] \arrow[r, hook] & \mathrm{CKhI}_r(D') \arrow[r, two heads] & \mathrm{CKh}_r(D')
\end{tikzcd}$$

The same statement holds for the coreduced counterparts.

Proof Since the basepoints are fixed by the move, the chain maps and chain homotopies of Proposition 2.7 restrict to the (co)reduced complexes. □

We conclude that for a pointed involutive link L with normal diagram D, the chain homotopy equivalence classes of the (co)reduced complexes $\mathrm{CKhI}_r(D)$, $\mathrm{CKhI}'_r(D)$ are invariants of L. Those homologies are denoted by $\mathrm{KhI}_r(L)$ and $\mathrm{KhI}'_r(L)$ and called the *(co)reduced involutive Khovanov homologies* of L.

2.4 Mirrors

Next, we study the behavior of the involutive complexes under mirrors. The arguments are straightforward extensions of [28, Section 3.5.2] to the involutive setting. Consider the standard perfect pairing on A

$$\langle\cdot,\cdot\rangle\colon A\otimes A\to R$$

given by $\langle x,y\rangle=\varepsilon(xy)$. The associated duality isomorphism $\mathsf{D}\colon A\to A^*$ such that $\langle x,y\rangle=\mathsf{D}(x)(y)$ is given by

$$\mathsf{D}(1)=X^*,\quad \mathsf{D}(X)=1^*+hX^*,$$

where $\{1^*,X^*\}$ is the dual basis for A^* to the basis $\{1,X\}$ for A. For convenience, put $Y=X+h$, and consider another basis $\{1,Y\}$ for A and its dual basis $\{1^\dagger,Y^\dagger\}$ for A^*. Then we have

$$\mathsf{D}(1)=Y^\dagger,\quad \mathsf{D}(X)=1^\dagger,$$

and

$$\langle X, X\rangle = h, \quad \langle X, Y\rangle = \langle Y, X\rangle = 0, \quad \langle Y, Y\rangle = h.$$

Note that when $h = 0$, the duality isomorphism D coincides with the ordinary self-dual isomorphism.

For a link diagram D, the duality isomorphism D induces a chain isomorphism

$$\mathsf{D}\colon \mathrm{CKh}(D^*) \xrightarrow{\cong} \mathrm{CKh}(D)^*,$$

where D^* denotes the mirror of D, and $\mathrm{CKh}(D)^*$ denotes the dual complex of $\mathrm{CKh}(D)$ with bigrading $(\mathrm{CKh}(D)^*)^{i,j} = \mathrm{CKh}^{-i,-j}(D)^*$. This gives a perfect pairing

$$\langle \cdot, \cdot\rangle\colon \mathrm{CKh}(D) \otimes \mathrm{CKh}(D^*) \to R.$$

Now suppose (D, τ) is an involutive link diagram.

Lemma 2.27 $$\mathsf{D}\tau = \tau^*\mathsf{D}\colon \mathrm{CKh}(D^*) \to \mathrm{CKh}(D)^*.$$

Proof By considering an $1X$-labeled generator x for D^* and an $1Y$-labeled generator y for D, we easily see that $\mathsf{D}(\tau x)(y) = \mathsf{D}(x)(\tau y)$ holds. □

Lemma 2.28 $$\mathrm{CKhI}(D)^*[1] \cong \mathrm{Cone}\big(\mathrm{CKh}(D)^* \xrightarrow{1+\tau^*} \mathrm{CKh}(D)^*\big).$$

Proof Obvious. □

Proposition 2.29 *There is an isomorphism*

$$\mathsf{D}\colon \mathrm{CKhI}(D^*) \xrightarrow{\cong} \mathrm{CKhI}(D)^*[1].$$

Proof The above results give isomorphisms,

$$\begin{aligned}\mathrm{CKhI}(D^*) &= \mathrm{Cone}\big(\mathrm{CKh}(D^*) \xrightarrow{1+\tau} \mathrm{CKh}(D^*)\big)\\ &\cong \mathrm{Cone}\big(\mathrm{CKh}(D)^* \xrightarrow{1+\tau^*} \mathrm{CKh}(D)^*\big)\\ &\cong \mathrm{CKhI}(D)^*[1].\end{aligned}$$ □

Next we see that the duality isomorphism also respects the (co)reduced complexes. By Proposition 2.21, the coreduced complex $\mathrm{CKhI}'_r(D)$ may be identified with the subcomplex $\sigma(\mathrm{CKhI}_r(D))$, and the reduced complex $\mathrm{CKhI}_r(D)$ with the quotient $\mathrm{CKhI}(D)/\sigma(\mathrm{CKhI}_r(D))$. Thus there is a short exact sequence in the reversed direction,

$$\mathrm{CKhI}'_r(D) \xrightarrow{i'} \mathrm{CKhI}(D) \xrightarrow{p'} \mathrm{CKhI}_r(D).$$

Proposition 2.30 *The isomorphism* D *of Proposition 2.29 induces isomorphisms on the* (co)*reduced complexes*, *such that the following diagram commutes*

$$\begin{tikzcd}
\mathrm{CKhI}_r(D^*) \arrow[r, "i"] \arrow[d, dashed, "\mathsf{D}"] & \mathrm{CKhI}(D^*) \arrow[r, "p"] \arrow[d, "\mathsf{D}"] & \mathrm{CKhI}'_r(D^*) \arrow[d, dashed, "\mathsf{D}"] \\
\mathrm{CKhI}_r(D)^* \arrow[r, "(p')^*"] & \mathrm{CKhI}(D)^* \arrow[r, "(i')^*"] & \mathrm{CKhI}'_r(D)^*
\end{tikzcd}$$

Proof Combine Proposition 2.29 with [28, Proposition 3.36]. □

Proposition 2.31 *There are perfect pairings*

$$\langle\cdot,\cdot\rangle\colon \mathrm{CKhI}(D)\otimes\mathrm{CKhI}(D^*)\to R \quad \textit{and} \quad \langle\cdot,\cdot\rangle_r\colon \mathrm{CKhI}_r(D)\otimes\mathrm{CKhI}_r(D^*)\to R$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{CKhI}_r(D)\otimes\mathrm{CKhI}_r(D^*) \arrow[r, "\langle\cdot,\cdot\rangle_r"] \arrow[d, "i\otimes i"] & R \arrow[d, "h\cdot"] \\
\mathrm{CKhI}(D)\otimes\mathrm{CKhI}(D^*) \arrow[r, "\langle\cdot,\cdot\rangle"] & R
\end{tikzcd}$$

Proof Combine Proposition 2.29 with [28, Propositions 3.33, 3.37]. □

3 Equivariant Rasmussen invariants

The focus of this section is strongly invertible knots and links. We mainly consider the case where $h \neq 0$, typically

$$(R,h) = (\mathbb{F}_2, 1),\ (\mathbb{F}_2[H], H),\ (\mathbb{F}_2[H^{\pm}], H).$$

3.1 Equivariant Lee classes

Recall that in the noninvolutive setting, if $h \in R$ is invertible, then for a link diagram D the homology $\mathrm{Kh}(D)$ is generated by the *Lee classes* $\alpha(D,o)$ of D, each corresponding to an orientation o on D; see [19; 31]. Here we recall the construction.

Algorithm 3.1 Given a link diagram D, the *ab-coloring* on its Seifert circles is defined as follows: separate $\mathbb{R}^2$ into regions by the Seifert circles of D, and color the regions in the checkerboard fashion, with the unbounded region colored white. For each Seifert circle, let it inherit the orientation from D, and assign to it a if it sees a black region to the left with respect to the orientation, or b otherwise; see Figure 5.

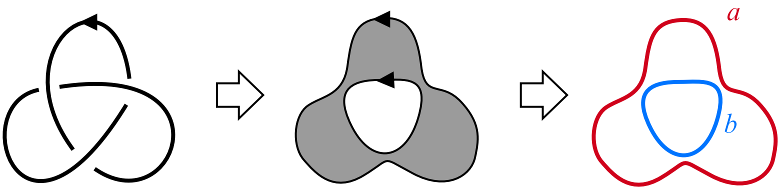


Figure 5: The ab-coloring on the Seifert circles of K.

Definition 3.2 Let D be a link diagram. There is a unique state s of D where the resolved diagram $D(s)$ gives the Seifert circles of D. With the ab-coloring on the Seifert circles, define an element $\alpha(D)$ in $\mathrm{CKh}(D)$ for that state by labeling each circle by X if it is colored a, and $Y = X + h$ if it is colored b. Similarly, for any orientation o on D, we define an element $\alpha(D,o)$ by the same procedure after reorienting D by o. These elements $\alpha(D,o)$ are in fact cycles, and are called the *Lee cycles* of D. The homology classes are called the *Lee classes* of D.

That $\alpha(D,o)$ is indeed a cycle can be seen from the fact that each crossing of D connects two differently colored strands in the resolved diagram, and merging the strands results in $XY = 0$. Note that the automorphism σ on $\mathrm{CKh}(D)$ interchanges $\alpha(D,o)$ and $\alpha(D,\overline{o})$, where $\overline{o}$ is the reversed orientation of o. We will frequently consider such pairs, so we write $\beta(D,o)$ for $\alpha(D,\overline{o})$. In particular when o is the given orientation of D, we write $\alpha(D)$ for $\alpha(D,o)$ and $\beta(D)$ for $\beta(D,o) = \alpha(D,\overline{o})$. The homological gradings of $\alpha(D,o)$ are all even, and in particular $\alpha(D)$, $\beta(D)$ has homological grading 0.

Proposition 3.3 *For a strongly invertible link diagram D, the Lee cycles are invariant under τ.*

Proof We only prove the case when o is the given orientation of D, since the other cases can be proved by reorienting D by o. From the definition of τ, it is obvious that the orientation-preserving state s is preserved by τ. Since $D(s)$, the diagram obtained by resolving D according to s, is symmetric with respect to the axis, we see that each circle C in $D(s)$ is either disjoint from the axis or intersects the axis exactly twice. For the first case, it is obvious that τ preserves the label on C. For the second case, there is a unique circle C' that is symmetric with C. Since D is strongly invertible, C and C' are oriented oppositely with respect to the reflection about the axis, and hence oriented equally as circles in $\mathbb{R}^2$. Thus from Algorithm 3.1 the two circles are colored the same. This shows that $\alpha(D)$ is invariant under τ. □

Hereafter we assume that D is a strongly invertible link diagram. For each orientation o on D, there are two elements $\alpha(D,o)$, $Q\alpha(D,o)$ in $\mathrm{CKhI}(D)$, which are denoted hereafter by $\underline{\alpha}(D,o)$ and $\overline{\alpha}(D,o)$. Proposition 3.3 implies that these are cycles in $\mathrm{CKhI}(D)$. By abuse of notation, the homology classes of $\underline{\alpha}(D,o)$, $\overline{\alpha}(D,o)$ are also denoted by the same symbols.

Definition 3.4 The cycles $\underline{\alpha}(D,o)$, $\overline{\alpha}(D,o)$ in $\mathrm{CKhI}(D)$ are called the *equivariant Lee cycles* of D, and those homology classes the *equivariant Lee classes* of D.

Note that $\underline{\alpha}(D,o)$ has even homological grading whereas $\overline{\alpha}(D,o)$ has odd. In particular, $\underline{\alpha}(D), \underline{\beta}(D)$ have homological grading 0, and $\overline{\alpha}(D), \overline{\beta}(D)$ have 1.

Example 3.5 For the left-handed trefoil diagram of Figure 5, we have

$$\underline{\alpha}(D) = X \otimes Y \in \mathrm{CKhI}^0(D) \quad \text{and} \quad \overline{\alpha}(D) = Q(X \otimes Y) \in \mathrm{CKhI}^1(D),$$

where X corresponds to the outer circle, and Y to the inner circle. The other cycles $\underline{\beta}(D), \overline{\beta}(D)$ are obtained by swapping X and Y.

The following proposition states the relation between the ordinary Lee cycles $\{\alpha(D,o)\}$ and the equivariant Lee cycles $\{\underline{\alpha}(D,o), \overline{\alpha}(D,o)\}$.

Proposition 3.6 *In the short exact sequence*

$$\mathrm{CKh}(D)[1] \xhookrightarrow{Q} \mathrm{CKhI}(D) \xrightarrow{q} \mathrm{CKh}(D)$$

the ordinary and the equivariant Lee cycles correspond as

$$\alpha(D,o) \xmapsto{Q} \overline{\alpha}(D,o) \quad \text{and} \quad \underline{\alpha}(D,o) \xmapsto{q} \alpha(D,o).$$

It is well known in the noninvolutive setting that the $\mathbb{Q}$*-Lee homology* is freely generated by the Lee classes [19, Theorem 4.2]. More generally, whenever $h \in R$ is invertible, then the corresponding homology is freely generated by the Lee classes [31, Theorem 4.2], [27, Proposition 2.9]. Analogous statement also hold in the involutive setting.

Proposition 3.7 *If $h \in R$ is invertible, the involutive Khovanov homology* $\mathrm{KhI}(D)$ *is freely generated by the $2^{|D|+1}$ equivariant Lee classes.*

Proof The proof is completely similar to the proof for the noninvolutive case given in [34], also explained in detail in [20]. To explain briefly, first note that when h is invertible, we may take $\{X, Y\}$ as a basis for A. An *admissible coloring* of D is a coloring with a or b on the edges of D such that each crossing admits a resolution that determines the colors of the two arc segments accordingly; see Figure 6. Now $\mathrm{CKh}(D)$ can be decomposed into subcomplexes, each corresponding to an admissible coloring of D, generated by the XY-labeled generators that match the coloring. If there is a crossing such that the four incident edges are colored the same (as in the first two pictures of Figure 6), then the generators can be canceled in pairs, resulting in a trivial complex. By contracting all such subcomplexes, we will be left with subcomplexes each corresponding to an admissible coloring such that the four incident edges at each crossing have two different colors (as in the third picture of Figure 6), which in turn corresponds one-to-one to an orientation o of D. Such subcomplex is generated by the single Lee cycle $\alpha(D,o)$, and thus $\mathrm{CKh}(D)$ is chain homotopy equivalent to a complex generated by the Lee cycles with trivial differential.

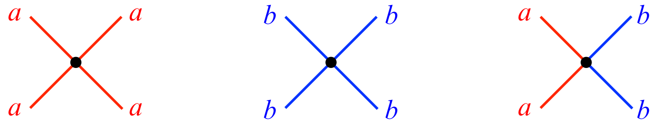


Figure 6: Local picture of an admissibly colored diagram.

This method also works in our case by performing the same cancellations in both $\mathrm{CKh}(D)$ and $Q\mathrm{CKh}(D)$. The only concern is that there are arrows

$$Q(1+\tau)\colon \mathrm{CKh}(D) \to Q\mathrm{CKh}(D)$$

in the differential of $\mathrm{CKhI}(D)$, but it will not affect the cancellation process, since the arrows are only running from the $\mathrm{CKh}(D)$ to $Q\mathrm{CKh}(D)$ and hence will not produce new arrows between the remaining canceling pairs. □

In [27, Proposition 2.13] we showed in the noninvolutive setting that the behaviors of the Lee classes under the Reidemeister moves can be described explicitly. The same formula also holds in the involutive setting. Hereafter, $w(D)$ denotes the writhe of D, and $r(D)$ denotes the number of Seifert circles of D. The difference function δf of a unary function f is defined as $\delta f(x, y) = f(y) - f(x)$.

Proposition 3.8 *Suppose $h \in R$ is invertible. Let D, D' be strongly invertible link diagrams related by an involutive Reidemeister move. Under the isomorphism*

$$\rho\colon \mathrm{KhI}(D) \to \mathrm{KhI}(D')$$

given in Proposition 2.7, the equivariant Lee classes modulo torsions correspond as

$$\underline{\alpha}(D) \xmapsto{\rho} h^j \underline{\alpha}(D'), \quad \overline{\alpha}(D) \xmapsto{\rho} h^j \overline{\alpha}(D'),$$
$$\underline{\beta}(D) \xmapsto{\rho} h^j \underline{\beta}(D'), \quad \overline{\beta}(D) \xmapsto{\rho} h^j \overline{\beta}(D'),$$

where

$$j = \frac{\delta w(D, D') - \delta r(D, D')}{2}.$$

Similar statements also hold for the other Lee classes, after appropriately reorienting D and D' with respect to the choice of the orientation o on D.

Proof From Proposition 3.7, the long exact sequence induced from the short exact sequence of Proposition 3.6 splits, and from Proposition 2.7 the isomorphism ρ fits into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Kh}(D)[1] \arrow[r] \arrow[d, "\rho"] & \mathrm{KhI}(D) \arrow[r] \arrow[d, dashed, "\rho"] & \mathrm{Kh}(D) \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & \mathrm{Kh}(D')[1] \arrow[r] & \mathrm{KhI}(D') \arrow[r] & \mathrm{Kh}(D') \arrow[r] & 0
\end{tikzcd}$$

From [27, Proposition 2.13] and the fact that ρ preserves the homological grading, it follows that

$$\begin{tikzcd} \alpha(D) \arrow[r, mapsto] \arrow[d, mapsto, "\rho"] & \overline{\alpha}(D) \arrow[d, mapsto, dashed, "\rho"] \\ h^j\alpha(D') \arrow[r, mapsto] & h^j\overline{\alpha}(D') \end{tikzcd} \qquad \begin{tikzcd} \underline{\alpha}(D) \arrow[r, mapsto] \arrow[d, mapsto, dashed, "\rho"] & \alpha(D) \arrow[d, mapsto, "\rho"] \\ h^j\underline{\alpha}(D') \arrow[r, mapsto] & h^j\alpha(D') \end{tikzcd}$$

The proof for the β-classes is similar. $\square$

Next, we consider the reduced setting. Hereafter, whenever the (co)reduced complexes are considered, it is implicitly assumed that the diagram is pointed and normal. Let $O(D)$ be the set of all orientations on D, and $O^+(D)$ be the subset of $O(D)$ consisting of orientations o whose orientation on the based component coincides with that of D.

First, we review the noninvolutive setting, which is extensively studied in [28]. For each $o \in O^+(D)$, the cycle $\alpha(D,o)$ lies in the reduced complex $\mathrm{CKh}_r(D) \subset \mathrm{CKh}(D)$. This will be denoted by $\alpha_r(D,o)$ for the sake of distinction. The counterpart $\beta(D,o)$ lies in $\sigma\mathrm{CKh}'_r(D) \subset \mathrm{CKh}(D)$, which is isomorphic to the coreduced complex $\mathrm{CKh}'_r(D)$. Let $\beta_r(D,o)$ denote the corresponding cycle in $\mathrm{CKh}'_r(D)$. $\beta_r(D,o)$ can be described by simply replacing the label of $\beta(D,o)$ on the pointed circle from $Y = \sigma(X)$ to 1.

Now we return to the involutive setting. The *(co)reduced equivariant Lee cycles* $\underline{\alpha}_r(D,o)$, $\overline{\alpha}_r(D,o) \in \mathrm{CKhI}_r(D)$ and $\underline{\beta}_r(D,o)$, $\overline{\beta}_r(D,o) \in \mathrm{CKhI}'_r(D)$ are defined in the same way. The following propositions can be easily verified.

Proposition 3.9 *Under the maps in the short exact sequence of Proposition 2.22, for each $o \in O^+(D)$, the Lee cycles in the unreduced, reduced, and coreduced complexes correspond as*

$$\underline{\alpha}(D,o) \overset{i}{\longmapsto} \underline{\alpha}_r(D,o), \quad \underline{\beta}(D,o) \overset{p}{\longmapsto} h\underline{\beta}_r(D,o),$$
$$\overline{\alpha}(D,o) \overset{i}{\longmapsto} \overline{\alpha}_r(D,o), \quad \overline{\beta}(D,o) \overset{p}{\longmapsto} h\overline{\beta}_r(D,o).$$

Proposition 3.10 *If $h \in R$ is invertible, then the reduced homology $\mathrm{KhI}_r(D)$ is freely generated by the $2^{|D|}$ reduced equivariant Lee classes $\{\underline{\alpha}_r(D,o), \overline{\alpha}_r(D,o)\}_{o \in O^+(D)}$. Similarly, the coreduced homology $\mathrm{KhI}'_r(D)$ is freely generated by the $2^{|D|}$ coreduced equivariant Lee classes $\{\underline{\beta}_r(D,o), \overline{\beta}_r(D,o)\}_{o \in O^+(D)}$.*

Proposition 3.11 *Suppose $h \in R$ is invertible. Let D, D' be two strongly invertible link diagrams (pointed and normal) related by an involutive Reidemeister move. Under the isomorphism given in Proposition 2.26*

$$\rho\colon \mathrm{KhI}_r(D) \to \mathrm{KhI}_r(D'),$$

the reduced equivariant Lee classes correspond as

$$\underline{\alpha}_r(D) \overset{\rho}{\longmapsto} h^j\underline{\alpha}_r(D'), \quad \overline{\alpha}_r(D) \overset{\rho}{\longmapsto} h^j\overline{\alpha}_r(D'), \qquad \text{where } j = \frac{\delta w(D,D') - \delta r(D,D')}{2}.$$

Similar statements hold for the coreduced counterparts.

Finally, we state the correspondence of the unreduced and the (co)reduced Lee cycles under the splitting of Proposition 3.12. First, we consider the noninvolutive setting.

Proposition 3.12 *Under the splitting of* [35]

$$\mathrm{CKh}(D) \cong \mathrm{CKh}'_r(D) \oplus \mathrm{CKh}_r(D),$$

for each $o \in O^+(D)$, the unreduced and the (co)reduced Lee cycles correspond as

$$\alpha(D,o) \mapsto (0, \alpha_r(D,o)), \quad \beta(D,o) \mapsto (h\beta_r(D,o), \alpha_r(D,o)).$$

Proof Here we only give a sketch. The isomorphism

$$\mathrm{CKh}(D) = \mathrm{Cone}(f) \xrightarrow{\cong} \mathrm{CKh}'_r(D) \oplus \mathrm{CKh}_r(D)$$

is given by

$$\begin{pmatrix} 1 & \\ \kappa & 1 \end{pmatrix}$$

using the null-homotopy κ described in Section 2.3. From this description, it is obvious that α maps to $(0,\alpha)^T$. To see that β maps to $(h\beta_r, \alpha)^T$, put

$$\beta = \underline{Y} \otimes x.$$

Here the underline indicates the label corresponding to the pointed circle. With the vector notation β is represented as $(h\underline{1} \otimes x, \underline{X} \otimes x)^T$. Note that $\underline{1} \otimes x$ is exactly β_r, so it remains to prove that

$$h\kappa(\underline{1} \otimes x) + \underline{X} \otimes x = \alpha_r.$$

If we define $\kappa_{-1}\colon \mathrm{CKh}'_r(D) \to \mathrm{CKh}_r(D)$ by

$$\kappa_{-1}(\underline{1} \otimes \cdots) = \underline{X} \otimes \cdots \quad \text{and} \quad \overline{\kappa} = \sum_{i \geq -1} h^{i+1}\kappa_i\colon \mathrm{CKh}'_r(D) \to \mathrm{CKh}_r(D),$$

then the aimed equation can be written as

$$\overline{\kappa}(\beta_r) = \alpha_r.$$

This is a purely algebraic problem and can be proved by the induction on the number of the Seifert circles. □

Proposition 3.13 *Under the splitting given in Proposition 2.23*

$$\mathrm{CKhI}(D) \cong \mathrm{CKhI}'_r(D) \oplus \mathrm{CKhI}_r(D),$$

the equivariant Lee cycles correspond as

$$\begin{aligned} \underline{\alpha}(D,o) &\mapsto (0, \underline{\alpha}_r(D,o)), & \underline{\beta}(D,o) &\mapsto (h\underline{\beta}_r(D,o), \underline{\alpha}_r(D,o)), \\ \overline{\alpha}(D,o) &\mapsto (0, \overline{\alpha}_r(D,o)), & \overline{\beta}(D,o) &\mapsto (h\overline{\beta}_r(D,o), \overline{\alpha}_r(D,o)), \end{aligned}$$

for each $o \in O^+(D)$.

Proof Immediate from Proposition 3.12. □

3.2 Divisibility of equivariant Lee classes

Hereafter we assume that R is a PID and h is prime (hence nonzero and noninvertible), typically $(R,h) = (\mathbb{F}_2[H], H)$.

Definition 3.14 Let M be a finitely generated free R-module. The *h-divisibility* of an element z in M is defined by

$$d_h(z) = \max\{k \geq 0 \mid z \in h^k M\}.$$

Divisibilities can be compared by homomorphisms. Suppose M, N are finitely generated free R-modules, $f\colon M \to N$ is a homomorphism, and $z \in M$, $w \in N$ are elements such that $f(z) = h^j w$ for some $j \in \mathbb{Z}$. Then we have

$$d_h(z) \leq j + d_h(w).$$

In particular when f is an isomorphism, the equality holds. Divisibilities can also be compared after inverting h. Namely, if z, w are elements in M such that $z \otimes 1 = h^j(w \otimes 1)$ in $h^{-1}M = M \otimes_R (h^{-1}R)$, then we have

$$d_h(z) = j + d_h(w).$$

See [28, Lemmas 4.2–4.7] for details.

Definition 3.15 For each orientation o on D, define nonnegative integers $\underline{d}_h(D,o)$ and $\overline{d}_h(D,o)$ by the *h-divisibility* (modulo torsion) of the equivariant Lee classes $\underline{\alpha}(D,o), \overline{\alpha}(D,o) \in \mathrm{KhI}(D)/\operatorname{Tor}$ respectively, ie

$$\begin{aligned}
\underline{d}_h(D,o) &= \max\{k \geq 0 \mid \underline{\alpha}(D,o) \in (h^k)(\mathrm{KhI}(D)/\operatorname{Tor})\},\\
\overline{d}_h(D,o) &= \max\{k \geq 0 \mid \overline{\alpha}(D,o) \in (h^k)(\mathrm{KhI}(D)/\operatorname{Tor})\}.
\end{aligned}$$

Example 3.16 Consider the left-handed trefoil diagram of Figure 5. In Example 3.5 we had

$$\underline{\alpha}(D) = X \otimes Y \in \mathrm{CKhI}^0(D).$$

Now, consider the element x of homological grading -1 depicted in the left side of Figure 7. Note that x is τ-invariant and hence $(d + 1 + \tau)x = dx = X \otimes X$. Now

$$\underline{\alpha}(D) \sim X \otimes Y + X \otimes X = h(X \otimes 1),$$

and we have $\underline{d}_h(D) \geq 1$. In fact in this case the equality holds. Similarly we have $\overline{d}_h(D) = 1$.

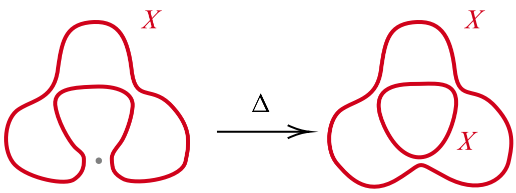


Figure 7: Elements x and dx.

In [27; 28] we defined a similar quantity in the noninvertible setting, namely for each orientation o on D,

$$d_h(D,o) = \max\{k \geq 0 \mid \alpha(D,o) \in (h^k)(\mathrm{Kh}(D)/\,\mathrm{Tor}\,)\}.$$

Proposition 3.17 $$\underline{d}_h(D,o) \leq d_h(D,o) \leq \overline{d}_h(D,o).$$

Proof Immediate from Proposition 3.6. □

Proposition 3.18 $$\underline{d}_h(\bigcirc) = \overline{d}_h(\bigcirc) = 0.$$

Proposition 3.19 *If D is a positive diagram, then $\underline{d}_h(D) = \overline{d}_h(D) = 0$.*

Proof Take a sequence of involutive diagrams

$$D \to D_1 \to \cdots \to D_N$$

by symmetrically resolving positive crossings, so that the final diagram D_N is a disjoint union of symmetric circles. This induces a sequence of quotient maps

$$\mathrm{CKhI}(D) \to \mathrm{CKhI}(D_1) \to \cdots \to \mathrm{CKhI}(D_N)$$

and gives

$$0 \leq \underline{d}_h(D) \leq \cdots \leq \underline{d}_h(D_N) = 0,$$

and similarly $\overline{d}_h(D) = 0$. □

The following lemmas will be used in the coming sections, each stating that $\underline{d}_h(D,o), \overline{d}_h(D,o)$ can be described in several ways. Here, the h-divisibilities are considered modulo torsions.

Lemma 3.20 $$d_h(\underline{\alpha}(D,o)) = d_h(\underline{\beta}(D,o)), \quad d_h(\overline{\alpha}(D,o)) = d_h(\overline{\beta}(D,o)).$$

Proof Consider the automorphism σ. □

Lemma 3.21
$$\begin{aligned} d_h(\underline{\alpha}_r(D,o)) &= d_h(\underline{\beta}_r(D,o)) = \underline{d}_h(D,o),\\ d_h(\overline{\alpha}_r(D,o)) &= d_h(\overline{\beta}_r(D,o)) = \overline{d}_h(D,o). \end{aligned}$$

Proof Immediate from Proposition 3.13. □

Lemma 3.22 $$d_h(\underline{\alpha}(D,o) + \underline{\beta}(D,o)) = \underline{d}_h(D,o) + 1, \quad d_h(\overline{\alpha}(D,o) + \overline{\beta}(D,o)) = \overline{d}_h(D,o) + 1.$$

Proof Again immediate from Proposition 3.13. □

3.3 Definition of the equivariant invariants

Proposition 3.8 and Proposition 3.6 justifies the following definition.

Definition 3.23 For a strongly invertible link L with diagram D, define

$$\underline{s}_h(L) = 2\underline{d}_h(D) + w(D) - r(D) + 1 \quad \text{and} \quad \overline{s}_h(L) = 2\overline{d}_h(D) + w(D) - r(D) + 1.$$

The pair $(\underline{s}_h(L), \overline{s}_h(L))$ is called the *equivariant Rasmussen* invariant of L.

In [27; 28], for a (noninvolutive) link L with diagram D, the invariant $s_h(L)$ is defined as

$$s_h(L) = 2d_h(D) + w(D) - r(D) + 1.$$

Proposition 3.24
$$\underline{s}_h(L) \le s_h(L) \le \overline{s}_h(L).$$

Proof Immediate from Proposition 3.17. □

Corollary 3.25 *For a strongly invertible knot K, we have*

$$\underline{s}_h(K) \le s(K) \le \overline{s}_h(K),$$

where $s(K)$ is the $\mathbb{F}_2$-Rasmussen invariant of K.

Proof Proof that $s_h(K)$ coincides with $s(K)$ when $\mathrm{char}(R) = 2$ is in [28, Theorem 2, Proposition 4.36]. □

Proposition 3.26
$$\underline{s}_h(\bigcirc) = \overline{s}_h(\bigcirc) = 0.$$

Proposition 3.27 *If K is positive with positive diagram D, then*

$$\underline{s}_h(K) = \overline{s}_h(K) = n(D) - r(D) + 1,$$

where $n(D)$ is the number of crossings of D.

Corollary 3.28 *For the positive (p, q)-torus knot $T_{p,q}$,*

$$\underline{s}_h(T_{p,q}) = \overline{s}_h(T_{p,q}) = (p-1)(q-1)$$

with respect to the unique inverting involution τ of $T_{p,q}$.

Example 3.29 For the left-handed trefoil 3_1, we have $\underline{s}_h(K) = \overline{s}_h(K) = -2$ from the computations in Example 3.16. For the right-handed trefoil $m(3_1)$, we have $\underline{s}_h(m(3_1)) = \overline{s}_h(m(3_1)) = 2$ from Corollary 3.28.

4 Properties of the equivariant invariants

In this section properties of $(\underline{s}_h, \overline{s}_h)$ stated in Section 1 will be proved in more generality. Throughout, we assume that R is a PID and h is prime.

4.1 Mirror formula

Proposition 4.1 *Consider a strongly invertible link diagram D and its mirror D^*. With the perfect pairing of Proposition 2.31, for any $o \in O(D)$ and $o' \in O(D^*)$, we have*

$$\langle \underline{\alpha}(D,o), \overline{\alpha}(D^*,o')\rangle = \langle \overline{\alpha}(D,o), \underline{\alpha}(D^*,o')\rangle = \begin{cases} h^{r(D,o)} & \text{if } o' = o^*, \\ 0 & \text{otherwise}, \end{cases}$$

where $r(D,o)$ denotes the number of Seifert circles of D reoriented by o. Similarly for the reduced versions, we have

$$\langle \underline{\alpha}_r(D,o), \overline{\alpha}_r(D^*,o')\rangle_r = \langle \overline{\alpha}_r(D,o), \underline{\alpha}_r(D^*,o')\rangle_r = \begin{cases} h^{r(D,o)-1} & \text{if } o' = o^*, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Obvious from the observation that the Seifert circles of D and D^* are identical, together with $\langle X, X\rangle = \langle Y, Y\rangle = h$ and $\langle X, Y\rangle = 0$. □

Proposition 4.2 *For a strongly invertible knot diagram D which is also pointed and normal, we have*

$$\underline{d}_h(D) + \overline{d}_h(D^*) = \overline{d}_h(D) + \underline{d}_h(D^*) = r(D) - 1.$$

Proof The formula is proved using the reduced Lee classes. Take a generator z of $\mathrm{KhI}^0_r(D) \cong R$ and put

$$\underline{\alpha}_r(D) = ah^d z,$$

where $d = \underline{d}_h(D)$ and $h \nmid a$. Similarly take a generator w of $\mathrm{KhI}^1_r(D^*) \cong R[1]$ and put

$$\overline{\alpha}_r(D^*) = bh^{d'} w,$$

where $d = \overline{d}_h(D^*)$ and $h \nmid b$. The perfect pairing of Proposition 2.31 induces a perfect pairing

$$\langle \cdot, \cdot \rangle_r \colon (\mathrm{KhI}_r(D)/\operatorname{Tor}) \otimes (\mathrm{KhI}_r(D^*)/\operatorname{Tor}) \to R,$$

and from Proposition 4.1 we have

$$\langle \underline{\alpha}_r(D), \overline{\alpha}_r(D^*)\rangle_r = abh^{d+d'}\langle z, w\rangle_r = h^{r(D)-1}.$$

Now $\langle z, w\rangle_r$ must be a unit of R, and since h is assumed to be prime, we must have that $a, b \in R$ are both units and

$$\underline{d}_h(D) + \overline{d}_h(D^*) = r(D) - 1.$$

The other equation follows from a similar argument. □

Proposition 4.3 *For a strongly invertible knot K,*

$$\underline{s}_h(K^*) = -\overline{s}_h(K).$$

Proof Immediate from Proposition 4.2. □

Example 4.4 Compare Example 3.29.

The proof of Proposition 4.2 also shows that the following two elements

$$\underline{\xi}_r(D) = h^{-\underline{d}_h(D)}\underline{\alpha}_r(D) \quad \text{and} \quad \overline{\xi}_r(D) = h^{-\overline{d}_h(D)}\overline{\alpha}_r(D)$$

form a basis of $\mathrm{KhI}_r(D)/\operatorname{Tor} \cong R \oplus R[1]$. Similarly,

$$\underline{\zeta}_r(D) = h^{-\underline{d}_h(D)}\underline{\beta}_r(D) \quad \text{and} \quad \overline{\zeta}_r(D) = h^{-\overline{d}_h(D)}\overline{\beta}_r(D)$$

form a basis of $\mathrm{KhI}'_r(D)/\operatorname{Tor} \cong R \oplus R[1]$. Under the identification of Proposition 3.13, the elements corresponding to $\underline{\xi}_r(D), \overline{\xi}_r(D)$ are

$$\underline{\xi}(D) = h^{-\underline{d}_h(D)}\underline{\alpha}(D) \quad \text{and} \quad \overline{\xi}(D) = h^{-\overline{d}_h(D)}\overline{\alpha}(D),$$

and the elements corresponding to $\underline{\zeta}_r(D), \overline{\zeta}_r(D)$ are

$$\underline{\zeta}(D) = h^{-\underline{d}_h(D)-1}(\underline{\alpha}(D) + \underline{\beta}(D)) \quad \text{and} \quad \overline{\zeta}(D) = h^{-\overline{d}_h(D)-1}(\overline{\alpha}(D) + \overline{\beta}(D)).$$

Thus the four elements $\underline{\zeta}(D)$, $\underline{\xi}(D)$, $\overline{\zeta}(D)$ and $\overline{\xi}(D)$ form a basis of $\mathrm{KhI}(D)/\operatorname{Tor} \cong R^2 \oplus R[1]^2$. Propositions 3.8 and 3.11 imply that all of these classes are invariant under the Reidemeister moves.

In particular when R is graded and $\deg(h) = -2$, we see that

$$\underline{s}_h(K) = \mathrm{gr}_q(\underline{\xi}_r(K)) \quad \text{and} \quad \overline{s}_h(K) = \mathrm{gr}_q(\overline{\xi}_r(K)).$$

Thus when $(R, h) = (\mathbb{F}_2[H], H)$ the definition of $(\underline{s}, \overline{s})$ for strongly invertible knots given in Section 1 coincides with Definition 3.23. We summarize:

Proposition 4.5

$$\begin{aligned}
\mathrm{KhI}(K) &= R\langle \underline{\zeta}(K), \underline{\xi}(K), \overline{\zeta}(K), \overline{\xi}(K)\rangle \oplus (\mathrm{Tor}),\\
\mathrm{KhI}_r(K) &= R\langle \underline{\xi}_r(K), \overline{\xi}_r(K)\rangle \oplus (\mathrm{Tor}),\\
\mathrm{KhI}'_r(K) &= R\langle \underline{\zeta}_r(K), \overline{\zeta}_r(K)\rangle \oplus (\mathrm{Tor}).
\end{aligned}$$

Example 4.6 For the simplest example $D = \bigcirc$, we have

$$\begin{aligned}
\underline{\alpha}(D) &= X, & \underline{\beta}(D) &= Y,\\
\overline{\alpha}(D) &= QX, & \overline{\beta}(D) &= QY,
\end{aligned}$$

and $\underline{d}_h(D) = \overline{d}_h(D) = 0$, so

$$\underline{\xi}(D) = X, \qquad \underline{\zeta}(D) = 1,$$
$$\overline{\xi}(D) = QX, \quad \overline{\zeta}(D) = Q1,$$

and we obtain

$$\mathrm{KhI}(D) = R\langle 1, X, Q1, QX\rangle = A \oplus QA,$$

as expected.

4.2 Connected sum formula

Arguments in this section are inspired by [15], where the connected sum formula for the $\underline{d}$-, $\overline{d}$-invariants in involutive Heegaard Floer homology is proved.

For strongly invertible links L, L', the (equivariant) *disjoint union* $L \sqcup L'$ and the (equivariant) *connected sum* $L \#_b L'$ along an equivariant band b are defined in the obvious ways so that the resulting links are also strongly invertible. Note that different choices of b will in general give nonequivalent links, but here we make the choice implicit and omit b from the notation.[2] The corresponding operations for strongly invertible link diagrams D, D' are also defined. When we write $D \sqcup D'$, it is assumed that D and D' are disjoint as diagrams. When we write $D \# D'$, it is assumed that the band is untwisted and no crossings are produced by the surgery.

Proposition 4.7 *There is a canonical isomorphism*

$$\mathrm{CKhI}(D \sqcup D') \cong \mathrm{Cone}(\,\mathrm{CKh}(D) \otimes \mathrm{CKh}(D') \xrightarrow{Q(1\otimes 1+\tau\otimes\tau)} Q(\mathrm{CKh}(D) \otimes \mathrm{CKh}(D'))\,).$$

Under this identification, we have

$$\underline{\alpha}(D \sqcup D') = \alpha(D) \otimes \alpha(D'), \quad \overline{\alpha}(D \sqcup D') = Q(\alpha(D) \otimes \alpha(D')).$$

Proof Obvious from the canonical isomorphism

$$\mathrm{CKh}(D \sqcup D') \cong \mathrm{CKh}(D) \otimes \mathrm{CKh}(D'),$$

which holds for any (R, h); see [17, Section 7.4]. □

Proposition 4.8 *There are chain maps*

$$\mathrm{CKhI}(D \sqcup D') \underset{\Delta}{\overset{m}{\rightleftarrows}} \mathrm{CKhI}(D \# D')$$

corresponding to the band surgery from $D \sqcup D'$ to $D \# D'$ and its reverse.

Proof The corresponding maps in the noninvolutive setting are τ-invariant. □

[2] For *directed* strongly invertible knots K, K', there is a canonical choice of the band b from K to K', and the equivariant connected sum is defined without ambiguity. See [26].

Proposition 4.9 *For strongly invertible links L, L', we have*

$$\underline{s}_h(L \# L') - 1 \le \underline{s}_h(L \sqcup L') \le \underline{s}_h(L \# L') + 1,$$

and similarly for $\overline{s}_h$.

Proof Under the maps of Proposition 4.8, we have

$$m(\alpha(D) \otimes \alpha(D')) = h\alpha(D \# D') \quad \text{and} \quad \Delta\alpha(D \# D') = \alpha(D) \otimes \alpha(D'),$$

hence

$$\underline{d}_h(D \# D') \le \underline{d}_h(D \sqcup D') \le \underline{d}_h(D \# D') + 1.$$

This gives the desired inequality. □

Lemma 4.10 *Suppose C, C' are τ-complexes over $\mathbb{F}_2$. Put*

$$C_\tau = \mathrm{Cone}(C \xrightarrow{1+\tau} C), \quad C'_\tau = \mathrm{Cone}(C' \xrightarrow{1+\tau} C') \quad \text{and} \quad C^{\otimes}_\tau = \mathrm{Cone}(C \otimes C' \xrightarrow{1\otimes 1+\tau\otimes\tau} C \otimes C').$$

Let $z \in C$ and $z' \in C'$ be τ-invariant cycles. In the following, $\sim$ denotes homologous.

(1) *Let $x, y \in C$ and $x', y' \in C'$ be elements such that*

$$\begin{pmatrix} z \\ 0 \end{pmatrix} \sim \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} z' \\ 0 \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix}$$

in C_τ and in C'_τ respectively. Then in $C^{\otimes}_\tau$,

$$\begin{pmatrix} z \otimes z' \\ 0 \end{pmatrix} \sim \begin{pmatrix} x \otimes x' \\ x \otimes y' + y \otimes \tau x \end{pmatrix}.$$

(2) *Let $x, y \in C$ and $x', y' \in C'$ be elements such that*

$$\begin{pmatrix} z \\ 0 \end{pmatrix} \sim \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ z' \end{pmatrix} \sim \begin{pmatrix} x' \\ y' \end{pmatrix}$$

in C_τ and in C'_τ respectively. Then in $C^{\otimes}_\tau$,

$$\begin{pmatrix} 0 \\ z \otimes z' \end{pmatrix} \sim \begin{pmatrix} x \otimes x' \\ y \otimes x' + \tau x \otimes y' \end{pmatrix}.$$

Proof (1) Put

$$\begin{pmatrix} z \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & \\ 1+\tau & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z' \\ 0 \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} d & \\ 1+\tau & d \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}.$$

Then the boundary of

$$\begin{pmatrix} x \otimes a' + a \otimes x' + a \otimes da' \\ x \otimes b' + b \otimes \tau x' + (1+\tau) a \otimes \tau a' \end{pmatrix}$$

gives the desired relation.

(2) Put
$$\begin{pmatrix} z \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d & \\ 1+\tau & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ z' \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} d & \\ 1+\tau & d \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}.$$
Then the boundary of
$$\begin{pmatrix} x \otimes a' \\ x \otimes b' + \tau a \otimes y' + (1 \otimes 1 + \tau \otimes \tau)(a \otimes a') + da \otimes b' + b \otimes da' \end{pmatrix}$$
gives the desired relation. □

Proposition 4.11 *For strongly invertible knots K, K',*
$$\underline{s}_h(K) + \underline{s}_h(K') \le \underline{s}_h(K \# K') \le \underline{s}_h(K) + \overline{s}_h(K') \le \overline{s}_h(K \# K') \le \overline{s}_h(K) + \overline{s}_h(K').$$

Proof We will prove the first and the third inequalities, which is sufficient to prove Proposition 4.11 from the mirror formula Proposition 4.3. Take $z = \alpha(D) + \beta(D)$ in $\mathrm{CKh}(D)$ and $z' = \alpha(D') + \beta(D')$ in $\mathrm{CKh}(D')$. Observe that under the chain map m of Proposition 4.8, we have
$$\begin{pmatrix} z \otimes z' \\ 0 \end{pmatrix} \overset{m}{\longmapsto} h \begin{pmatrix} \alpha(D \# D') + \beta(D \# D') \\ 0 \end{pmatrix} = h(\underline{\alpha}(D \# D') + \underline{\beta}(D \# D')),$$
$$\begin{pmatrix} 0 \\ z \otimes z' \end{pmatrix} \overset{m}{\longmapsto} h \begin{pmatrix} 0 \\ \alpha(D \# D') + \beta(D \# D') \end{pmatrix} = h(\overline{\alpha}(D \# D') + \overline{\beta}(D \# D')).$$
From Lemma 3.22, there are elements $x, y \in \mathrm{CKh}(D)$ such that,
$$\begin{pmatrix} z \\ 0 \end{pmatrix} \sim h^{\underline{d}_h(D)+1} \begin{pmatrix} x \\ y \end{pmatrix}$$
in $\mathrm{CKhI}(D)$, modulo torsion in homology. By inverting h, we may assume that they are strictly homologous in $h^{-1}\mathrm{CKhI}(D)$. Similarly, there are elements $x', y' \in \mathrm{CKh}(D')$ such that
$$\begin{pmatrix} z' \\ 0 \end{pmatrix} \sim h^{\underline{d}_h(D')+1} \begin{pmatrix} x' \\ y' \end{pmatrix}$$
in $h^{-1}\mathrm{CKhI}(D')$. From Lemma 4.10 (1), we have
$$\begin{pmatrix} z \otimes z' \\ 0 \end{pmatrix} \sim h^{\underline{d}_h(D)+\underline{d}_h(D')+2} \begin{pmatrix} x \otimes x' \\ x \otimes y' + y \otimes \tau x \end{pmatrix}$$
in $h^{-1}\mathrm{CKhI}(D \sqcup D')$. Under the map m of Proposition 4.8, the left-hand side maps to
$$h(\underline{\alpha}(D \# D') + \underline{\beta}(D \# D'))$$
in $h^{-1}\mathrm{CKhI}(D \# D')$. Its homology class in $h^{-1}\mathrm{KhI}(D \# D')$ is the $h^{\underline{d}_h(D \# D')+2}$ multiple of the class $\underline{\zeta}(D \# D')$ of Proposition 4.5. The homology class of
$$m \begin{pmatrix} x \otimes x' \\ x \otimes y' + y \otimes \tau x \end{pmatrix}$$

is also some h^e multiple of $\underline{\zeta}(D \# D')$ for some $e \geq 0$ (because x, y, x' and y' were taken before inverting h). Thus it follows that

$$\underline{d}_h(D) + \underline{d}_h(D') \leq \underline{d}_h(D \# D').$$

This implies the first inequality. Similarly for the third inequality, there are elements $x'', y'' \in \mathrm{CKh}(D')$ such that

$$\begin{pmatrix} 0 \\ z' \end{pmatrix} \sim h^{\overline{d}_h(D')+1} \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

in $h^{-1}\mathrm{CKhI}(D')$. From Lemma 4.10 (2), we have

$$\begin{pmatrix} 0 \\ z \otimes z' \end{pmatrix} \sim h^{\underline{d}_h(D)+\overline{d}_h(D')+2} \begin{pmatrix} x \otimes x'' \\ y \otimes x'' + \tau x \otimes y'' \end{pmatrix}.$$

By a similar argument, we obtain

$$\underline{d}_h(D) + \overline{d}_h(D') \leq \overline{d}_h(D \# D')$$

which implies the third inequality. □

4.3 Behavior under crossing changes

Proposition 4.12 *Let D^+ be a link diagram with at least one positive crossing, and D^- be a diagram obtained from D^+ by applying a negative crossing change to one of the positive crossings of D^+. There are homological grading preserving chain maps*

$$\mathrm{CKh}(D^+) \underset{\Phi^+}{\overset{\Phi^-}{\rightleftarrows}} \mathrm{CKh}(D^-)$$

such that the Lee cycles correspond as

$$\alpha(D^+) \overset{\Phi^-}{\longmapsto} \alpha(D^-), \quad \alpha(D^-) \overset{\Phi^+}{\longmapsto} h\alpha(D^+).$$

Proof Let x be the positive crossing of D^+ on which the crossing change is performed. Let D_0, D_1 be the 0-, 1-resolved diagram of D^+ at x respectively. Then $\mathrm{CKh}(D^+)$ may be described as a cone of the surgery map

$$\mathrm{CKh}(D_0) \overset{e}{\to} \mathrm{CKh}(D_1).$$

Similarly, by 1-resolving D^- at the corresponding crossing, we see that $\mathrm{CKh}(D^-)$ can be described as a cone of

$$\mathrm{CKh}(D_1) \overset{e'}{\to} \mathrm{CKh}(D_0).$$

The setup is depicted in Figure 8 where the red arcs indicate the corresponding surgery maps. Note that $\alpha(D^+)$ and $\alpha(D^-)$ are identical, and that they both belong to $\mathrm{CKh}(D_0)$.

Now we define chain maps

$$\mathrm{CKh}(D^+) \underset{\Phi^+}{\overset{\Phi^-}{\rightleftarrows}} \mathrm{CKh}(D^-)$$

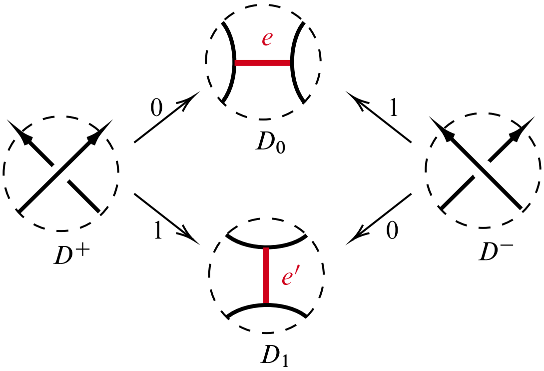


Figure 8: Diagrams $D^{\pm}$ and their resolutions.

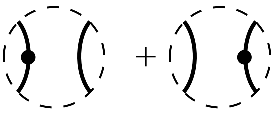
so that they fit into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CKh}(D_0) \arrow[r, "e"] \arrow[d, dashed, shift right, "\Phi^-"'] & \mathrm{CKh}(D_1) \arrow[d, dashed, "0"] \\
\mathrm{CKh}(D_1) \arrow[r, "e'"] \arrow[u, dashed, "0"] & \mathrm{CKh}(D_0) \arrow[r] \arrow[u, dashed, shift right, "\Phi^+"'] & 0
\end{tikzcd}$$

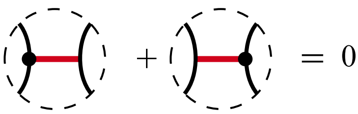
giving the desired chain maps between the complexes. First, define $\Phi^- = \mathrm{id}_{D_0}$, which is obviously a chain map satisfying

$$\alpha(D^+) \overset{\Phi^-}{\longmapsto} \alpha(D^-).$$

Next, we define Φ^+ as



where each dot represents the multiplication by X on the circle it is drawn on. To verify that Φ^+ is a chain map, it suffices to show that $e\Phi^+ = 0 = \Phi^+ e'$. The first equation can be described pictorially as



which obviously holds, since dots can move freely within their connected components. The second equation can be proved similarly. Finally, we see that

$$\alpha(D^-) \overset{\Phi^+}{\longmapsto} h\alpha(D^+)$$

from the local description of $\alpha(D^+)$ together with $X^2 = hX$ and $XY = 0$. $\square$

Remark 4.13 The crossing change maps $\Phi^{\pm}$ of Proposition 4.12 partially appear in [4, Figures 3,4]. Φ^+ also appears in [16, Section 3] in the form of a morphism in the category Cob^3.

Proposition 4.14 *Let L^+, L^- be links such that L^- is obtained by applying a negative crossing change to L^+. Then*

$$s_h(L^-) \le s_h(L^+) \le s_h(L^-) + 2.$$

Proof Let D^+, D^- be diagrams of L^+, L^- respectively, such that D^- is obtained by applying a negative crossing change to a single crossing of D^+. From Proposition 4.12, we have

$$d_h(D^+) \le d_h(D^-) \le d_h(D^+) + 1.$$

Thus,

$$\begin{aligned} s_h(L^-) &= 2d_h(D^-) + w(D^-) - r(D^-) + 1 \\ &\le 2(d_h(D^+) + 1) + (w(D^+) - 2) - r(D^+) + 1 = s_h(L) \\ &\le 2d_h(D^-) + (w(D^-) + 2) - r(D^-) + 1 = s_h(L^-) + 2. \end{aligned}$$ □

Definition 4.15 An *equivariant negative crossing change* on a strongly invertible link L is an operation that is either a single negative crossing change on a crossing lying on the axis of L, or two negative crossing changes on crossings x and $\tau(x)$ lying off the axis.

Proposition 4.16 *Let L^+, L^- be strongly invertible links such that L^- is obtained by applying an equivariant negative crossing change to L^+. Then*

$$\underline{s}_h(L^-) \le \underline{s}_h(L^+) \le \underline{s}_h(L^-) + 2a,$$

where $a = 1$ if the move is performed on-axis, and $a = 2$ if performed off-axis. The same holds for $\overline{s}_h$.

Proof If the move is performed on-axis, the maps $\Phi^\pm$ are strictly τ-equivariant and hence induce maps between the involutive complexes. If the move is performed off-axis, then we may define equivariant crossing change maps

$$\Phi^\pm = \Phi_2^\pm \Phi_1^\pm,$$

where $\Phi_1^\pm$ and $\Phi_2^\pm$ are the nonequivariant crossing change maps corresponding to the off-axis moves. By an argument similar to the proof of Proposition 2.7 for moves IR1–IR3, we see that $\Phi^\pm$ is strictly τ-equivariant and hence induce maps between the involutive complexes. □

4.4 Behavior under generalized crossing changes

Next, we extend the results in the previous section to *generalized crossing changes*, introduced by Cochran and Tweedy in [8].

Definition 4.17 For $n \ge 1$, a $2n$*-strand generalized negative* (*resp. positive*) *crossing change* on a link L is a modification of L by adding a *positive* (resp. *negative*) full twist on $2n$ parallel strands of L, where n strands are oriented one way and the other are oriented the other; see Figure 9

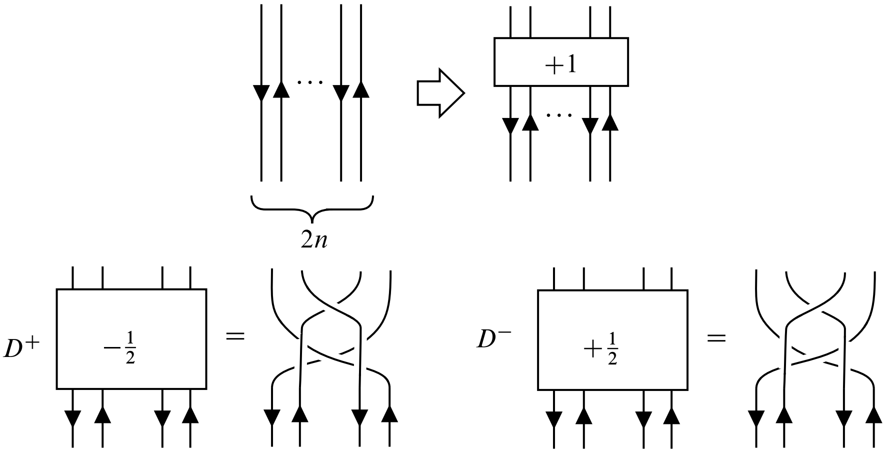


Figure 9: Top: a $2n$-strand generalized negative crossing change. Bottom: diagrams D^+ and D^-.

Note that the case $n = 1$ gives the ordinary negative (resp. positive) crossing change. Hereafter we only consider the case $n = 2$.

Proposition 4.18 *Let D^+, D^- be diagrams that differ locally as in Figure 9. Then there are homological grading preserving chain maps*

$$\mathrm{CKh}(D^+) \underset{\Phi^+}{\overset{\Phi^-}{\rightleftarrows}} \mathrm{CKh}(D^-)$$

such that the Lee classes modulo torsions correspond as

$$\alpha(D^+) \overset{\Phi^-}{\longmapsto} \alpha(D^-), \quad \alpha(D^-) \overset{\Phi^+}{\longmapsto} h^2\alpha(D^+).$$

The idea of the proof is similar to that of Proposition 4.12 but we need some preparations. First recall from [5, Section 4] the definition of *strong deformation retracts*.

Definition 4.19 A chain map $r : C \to C'$ between chain complexes (in any additive category) is a *strong deformation retract*[3] if there is a chain map $i : C' \to C$ (called the *inclusion*) and a homotopy h on C, satisfying

(i) $ri = 1$,

(ii) $ir - 1 = dh + hd$,

(iii) $hi = 0$,

(iv) $rh = 0$, and

(v) $h^2 = 0$.

The following lemma is a generalization of [5, Lemma 4.5].

[3] Conditions (iii)–(v) are called the *side conditions*. In [5, Definition 4.3] only conditions (i)–(iii) are imposed, but we can easily check that the remaining two conditions also hold for the R1 and the R2 maps.

Lemma 4.20 *Suppose X, Y, Z, W are chain complexes (in any additive category), and there are maps f, g, k, l (not necessarily chain maps) such that the square of complexes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "l"] & W \end{tikzcd}$$

forms a chain complex C. Furthermore suppose Y has a strong deformation retract $r\colon Y \to Y'$ with inclusion i and homotopy h. Then the square of complexes

$$\begin{tikzcd} X \arrow[r, "rf"] \arrow[d, "k"'] \arrow[dr, "ghf"] & Y' \arrow[d, "gi"] \\ Z \arrow[r, "l"] & W \end{tikzcd}$$

forms a chain complex C' and is a strong deformation retract of C.

Proof The differentials of C and C' may be expressed by matrices

$$D = \begin{pmatrix} d_X & & & \\ f & d_Y & & \\ k & & d_Z & \\ & g & l & d_W \end{pmatrix}, \quad D' = \begin{pmatrix} d_X & & & \\ rf & d_{Y'} & & \\ k & & d_Z & \\ ghf & gi & l & d_W \end{pmatrix},$$

and one can see that $D^2 = 0$ implies $(D')^2 = 0$. Furthermore, one can check that

$$R = \begin{pmatrix} 1 & & & \\ & r & & \\ & & 1 & \\ & gh & & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & & \\ hf & i & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & & & \\ & h & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

gives a strong deformation retract $R\colon C \to C'$ with inclusion I and homotopy H. □

Lemma 4.21 *Let D^+, D^- be diagrams that differ locally as in Figure 9. The complex* $\mathrm{CKh}(D^+)$ *strongly deformation retracts onto the complex E described in Figure 10. Similarly, the complex* $\mathrm{CKh}(D^-)$ *strongly deformation retracts onto the complex E^- described in Figure 10. (Descriptions for some arrows are omitted, since we will not use them.)*

Proof By considering the 0 and 1 resolutions of the bottom left and the bottom right crossings of D^+, we see that the complex $\mathrm{CKh}(D^+)$ can be expressed as a square of complexes

$$\begin{tikzcd} \mathrm{CKh}(D^+_{00}) \arrow[r, "e"] \arrow[d, "e'"] & \mathrm{CKh}(D^+_{10}) \arrow[d, "e'"] \\ \mathrm{CKh}(D^+_{01}) \arrow[r, "e"] & \mathrm{CKh}(D^+_{11}) \end{tikzcd}$$

where e and e' denote the surgery map corresponding to the bottom left and the bottom right crossings, respectively. Observe that the resolved diagrams D^+_{00}, D^+_{10}, D^+_{01} can be simplified by performing R2

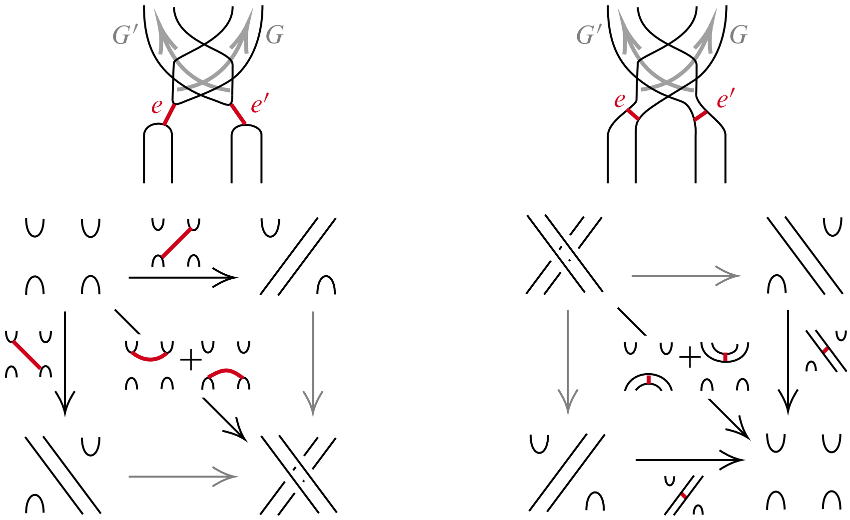


Figure 10: Simplifying $\mathrm{CKh}(D^+)$ and $\mathrm{CKh}(D^-)$. Top: D^+, left, and D^-, right. Bottom: E^+, left, and E^-, right.

moves; see Figure 10. Since the map G for the R2-move defined in [5, Section 4.2] is a strong deformation retract, we may apply Lemma 4.20 repeatedly and obtain a strong deformation retract E^+ of $\mathrm{CKh}(D^+)$ of the form

$$\begin{tikzcd} E^+_{00} \arrow[r, "G'eF"] \arrow[d, "e'"] \arrow[dr, "e'H'eF"] & E^+_{10} \arrow[d, "e'F'"] \\ E^+_{01} \arrow[r, "eF"] & E^+_{11} \end{tikzcd}$$

Here, G and G' are the maps for the R2-moves, F, F' and H, H' are the corresponding inclusions and homotopies. By unraveling the explicit maps, one can check that the maps are given as in Figure 10. Similarly $\mathrm{CKh}(D^-)$ can be expressed as a square of complexes

$$\begin{tikzcd} \mathrm{CKh}(D^-_{00}) \arrow[r, "e"] \arrow[d, "e'"] & \mathrm{CKh}(D^-_{10}) \arrow[d, "e'"] \\ \mathrm{CKh}(D^-_{01}) \arrow[r, "e"] & \mathrm{CKh}(D^-_{11}) \end{tikzcd}$$

and the resolved diagrams D^-_{10}, D^-_{01}, D^-_{11} can be simplified by performing R2 moves. Thus we obtain a strong deformation retract E^- of $\mathrm{CKh}(D^-)$ of the form

$$\begin{tikzcd} E^-_{00} \arrow[r, "Ge"] \arrow[d, "G'e'"] \arrow[dr, "GeH'e'"] & E^-_{10} \arrow[d, "e'"] \\ E^-_{01} \arrow[r, "GeF'"] & E^-_{11} \end{tikzcd}$$

and again one can check that the maps are given as in Figure 10. □

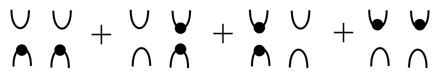
Proof of Proposition 4.18 First we define chain maps

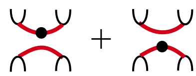
$$E^+ \underset{\phi^+}{\overset{\phi^-}{\rightleftarrows}} E^-$$

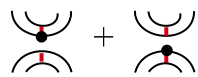
between the two simplified complexes E^+, E^- of Lemma 4.21. For ϕ^-, since E^+_{00} and E^-_{11} are identical, we may define $\phi^- = \mathrm{id}$ on E^+_{00} and 0 elsewhere. Obviously this is a chain map, since we have $\phi^- d^+ = 0 = d^- \phi^-$. Next, ϕ^+ is defined by the sum of the following four maps $\phi_1^+, \phi_2^+, \phi_3^+, \phi_4^+$:

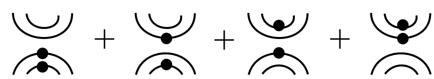
$$\begin{tikzcd}
 & & \underline{E^+_{00}} \arrow[r] \arrow[dr] \arrow[drr] & E^+_{10} \arrow[dr] & \\
 & & & E^+_{01} \arrow[r] & E^+_{11} \\
E^-_{00} \arrow[r] \arrow[dr] \arrow[drr] \arrow[uurr, dashed, "\phi_3^+"] \arrow[urrrr, dashed, "\phi_4^+"] & E^-_{10} \arrow[dr] & & & \\
 & E^-_{01} \arrow[r] & \underline{E^-_{11}} \arrow[uuu, dashed, "\phi_1^+"'] \arrow[uurr, dashed, "\phi_2^+"'] & &
\end{tikzcd}$$

with explicit descriptions:

$\phi_1^+ =$ 

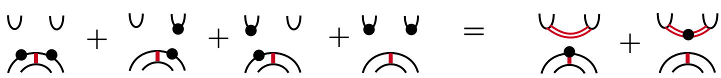
$\phi_2^+ =$ 

$\phi_3^+ =$ 

$\phi_4^+ =$ 

Note that E^+_{00} and E^-_{11} only have homological grading 0 (within the displayed area), whereas E^+_{11} has homological grading in range $[0, 4]$ and E^-_{00} has homological grading in range $[-4, 0]$. One can see that each ϕ_i^+ has domain and codomain in the homological grading 0 parts.

To show that $\phi^+ = \sum_i \phi_i^+$ actually defines a chain map, it suffices to verify that all the diagrams displayed in Figure 11 commute. Here each number in the parenthesis indicates the homological grading. This can be checked by the explicit descriptions of the maps ϕ_i^+ and the differentials of E^+, E^- given in Figure 10. For example, the commutativity of the third diagram is equivalent to



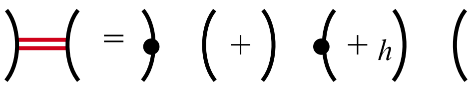
$$\begin{tikzcd} E^-_{10}(-1) \arrow[r, "d"] \arrow[d] & E^-_{11}(0) \arrow[d, "\phi_1^+"] \\ 0 \arrow[r] & E^+_{00}(0) \end{tikzcd} \quad \begin{tikzcd} E^-_{01}(-1) \arrow[r, "d"] \arrow[d] & E^-_{11}(0) \arrow[d, "\phi_1^+"] \\ 0 \arrow[r] & E^+_{00}(0) \end{tikzcd} \quad \begin{tikzcd} E^-_{00}(-1) \arrow[r, "d"] \arrow[d, "d"] & E^-_{11}(0) \arrow[d, "\phi_1^+"] \\ E^-_{00}(0) \arrow[r, "\phi_3^+"] & E^+_{00}(0) \end{tikzcd} \quad \begin{tikzcd} E^-_{00}(-1) \arrow[r, "d"] \arrow[d, "d"] & E^-_{11}(0) \arrow[d, "\phi_2^+"] \\ E^-_{00}(0) \arrow[r, "\phi_4^+"] & E^+_{11}(0) \end{tikzcd}$$

$$\begin{tikzcd} E^-_{00}(0) \arrow[r, "\phi_3^+"] \arrow[d] & E^+_{00}(0) \arrow[d, "d"] \\ 0 \arrow[r] & E^+_{10}(1) \end{tikzcd} \quad \begin{tikzcd} E^-_{00}(0) \arrow[r, "\phi_3^+"] \arrow[d] & E^+_{00}(0) \arrow[d, "d"] \\ 0 \arrow[r] & E^+_{01}(1) \end{tikzcd} \quad \begin{tikzcd} E^-_{00}(0) \arrow[r, "\phi_3^+"] \arrow[d, "\phi_4^+"] & E^+_{00}(0) \arrow[d, "d"] \\ E^+_{11}(0) \arrow[r, "d"] & E^+_{11}(1) \end{tikzcd} \quad \begin{tikzcd} E^-_{11}(0) \arrow[r, "\phi_1^+"] \arrow[d] & E^+_{00}(0) \arrow[d, "d"] \\ 0 \arrow[r] & E^+_{10}(1) \end{tikzcd}$$

$$\begin{tikzcd} E^-_{11}(0) \arrow[r, "\phi_1^+"] \arrow[d] & E^+_{00}(0) \arrow[d, "d"] \\ 0 \arrow[r] & E^+_{01}(1) \end{tikzcd} \qquad \begin{tikzcd} E^-_{11}(0) \arrow[r, "\phi_1^+"] \arrow[d, "\phi_2^+"] & E^+_{00}(0) \arrow[d, "d"] \\ E^+_{11}(0) \arrow[r, "d"] & E^+_{11}(1) \end{tikzcd}$$

Figure 11

Here, a doubled arc in the right hand side represents a handle attachment (or a saddle move applied twice), and the equation can be checked using the *neck cutting relation*:



Verifications are left to the reader. Now the desired maps $\Phi^{\pm}$ between $\mathrm{CKh}(D^+)$ and $\mathrm{CKh}(D^-)$ are defined as

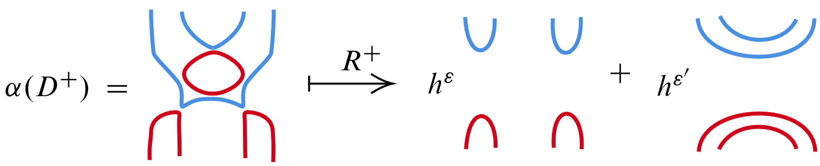
$$\mathrm{CKh}(D^+) \underset{I^+}{\overset{R^+}{\rightleftarrows}} E^+ \underset{\phi^+}{\overset{\phi^-}{\rightleftarrows}} E^- \underset{R^-}{\overset{I^-}{\rightleftarrows}} \mathrm{CKh}(D^-).$$

where $R^{\pm}$ and $I^{\pm}$ are the retractions and inclusions respectively.

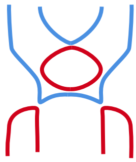
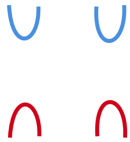
Finally, the correspondence between the Lee classes can be checked by comparing those images in E^+ and E^- under the retractions $R^{\pm}$. The retractions are given by the following matrices

$$R^+ = \begin{pmatrix} G & & & \\ G'eH & G' & & \\ & & G & \\ e'H'eH & e'H' & eH & 1 \end{pmatrix}, \quad R^- = \begin{pmatrix} 1 & & & \\ & G & & \\ & & G' & \\ & & GeH' & G \end{pmatrix}.$$

The image of $\alpha(D^+) \in \mathrm{CKh}(D^+_{00})$ can be directly computed as



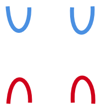
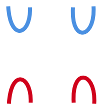
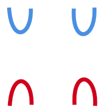
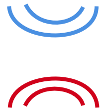
Here $\varepsilon, \varepsilon' \in \{0, 1\}$ are determined by how the arcs are connected outside the displayed area. Similarly, the image of $\alpha(D^-) \in \mathrm{CKh}(D^-_{11})$ is

$\alpha(D^-) =$  $\overset{R^-}{\longmapsto} h^{\varepsilon}$ 

First,

$$\phi^- R^+(\alpha(D^+)) = R^-(\alpha(D^-))$$

is obvious by definition. Next, from the explicit description of ϕ_1^+ and ϕ_2^+,

 $\overset{f_1^+}{\longmapsto} h^2$  and  $\overset{f_2^+}{\longmapsto} h^{(1-\varepsilon)+\varepsilon'+1}$ 

Thus

$$\phi^+ R^-(\alpha(D^-)) = h^2 R^+(\alpha(D^+)),$$

and the proof is complete. □

Proposition 4.22 *Let L^+, L^- be links such that L^- is obtained from L^+ by applying a 4-strand generalized negative crossing change on L^+. Then*

$$s_h(L^-) \le s_h(L^+) \le s_h(L^-) + 4.$$

Proof A generalized negative crossing change can be realized by first applying Reidemeister moves on the four parallel strands and then modifying the negative half-twist part to a positive half-twist. Thus the result follows from Proposition 4.18 by an argument similar to the proof of Proposition 4.14. □

Remark 4.23 In [23, Theorem 1.11], the lower bound

$$s^{\mathbb{Q}}(L^-) \le s^{\mathbb{Q}}(L^+)$$

is given for the $\mathbb{Q}$-Rasmussen invariant (with no restrictions on the number of strands). It is obtained by realizing the move as a connected genus-0 cobordism from L^+ to L^- in $\overline{\mathbb{C}P^2} \setminus (B^4 \sqcup B^4)$. We expect that our map Φ^+ gives (a part of) the combinatorial description of this geometric map.

Now we prove the result for the equivariant case.

Definition 4.24 An *equivariant generalized negative crossing change* on a strongly invertible link L is an operation that is either a generalized negative crossing change on a set of strands lying on the axis, or two generalized negative crossing changes on two sets of strands that correspond by τ, lying off the axis.

Proposition 4.25 *Let L^+, L^- be strongly invertible links such that L^- is obtained by applying an equivariant 4-strand generalized negative crossing change to L^+. Then*

$$\underline{s}_h(L^-) \le \underline{s}_h(L^+) \le \underline{s}_h(L^-) + 4a,$$

where $a = 1$ if the move is performed on-axis, and $a = 2$ if performed off-axis. The same holds for $\overline{s}_h$.

Proof Let D^+, D^- be diagrams of L^+, L^- respectively such that D^+ and D^- are locally related by an equivariant generalized crossing change. First suppose the generalized negative crossing change is performed on-axis. Recall the definition of $\Phi^\pm$

$$\mathrm{CKh}(D^+) \underset{I^+}{\overset{R^+}{\rightleftarrows}} E^+ \underset{\phi^+}{\overset{\phi^-}{\rightleftarrows}} E^- \underset{R^-}{\overset{I^-}{\rightleftarrows}} \mathrm{CKh}(D^-).$$

The middle maps $\phi^\pm$ are strictly τ-equivariant. Moreover, since the tangle diagrams appearing in the moves are Kh-simple, from Lemma 2.14 it follows that the chain homotopy equivalences $R^\pm$ and $I^\pm$ are homotopy τ-equivariant. Thus the composite maps $\Phi^\pm$ are homotopy τ-equivariant and hence induce maps between the involutive complexes.

The proof of the case when the move is performed off-axis is similar to the proof of Proposition 4.16. □

4.5 Behavior under cobordisms

First recall that in the noninvolutive setting, given an oriented cobordism S between links L, L' in $\mathbb{R}^3$, there is a map between the corresponding complexes

$$\phi_S : \mathrm{CKh}(D^+) \to \mathrm{CKh}(D'),$$

where D, D' are diagrams of L, L' under a fixed projection $\mathbb{R}^3 \to \mathbb{R}^2$. The map ϕ_S is obtained by decomposing S into elementary cobordism and composing the corresponding maps between the intermediate complexes. It is proved in [5, Theorem 4] that ϕ_S is invariant up to chain homotopy under isotopies of S rel boundary. We prove an analogous statement in the involutive setting.

Definition 4.26 Let L, L' be two involutive links sharing the same involution τ of S^3. A *simple equivariant cobordism* between L and L' is an oriented cobordism S in $S^3 \times I$ between L, L' satisfying $(\tau \times \mathrm{id})(S) = S$. A *simple isotopy-equivariant cobordism* between L, L' is defined similarly, expect that $(\tau \times \mathrm{id})(S)$ is only required to be isotopic to S rel boundary.

Hereafter we simply write τS for $(\tau \times \mathrm{id})(S)$.

Lemma 4.27 *Let τ be an involution on S^3 and S be a cobordism between links L, L'. Then the following diagram commutes up to homotopy*

$$\begin{tikzcd} \mathrm{CKh}(D) \arrow[r, "\phi_S"] \arrow[d, "\tau"] & \mathrm{CKh}(D') \arrow[d, "\tau"] \\ \mathrm{CKh}(\tau D) \arrow[r, "\phi_{\tau S}"] & \mathrm{CKh}(\tau D') \end{tikzcd}$$

Proof First, assume that S is an elementary cobordism. For the Reidemeister moves R1, R2 and the three Morse moves M1–M3, one can check from the explicit descriptions that the square strictly commutes. For the R3 move, although it does not commute strictly, it follows from Kh-simplicity that $(\phi_S)^\tau = \tau\phi_S\tau$ and $\phi_{\tau S}$ are chain homotopic, so the square commutes up to homotopy.

For a general cobordism S, decompose it into elementary cobordisms as $S = S_1 \cup S_2 \cup \cdots \cup S_N$. Then $\tau S_1 \cup \tau S_2 \cup \cdots \cup \tau S_N$ gives an elementary cobordism decomposition of τS, and we obtain a homotopy commutative diagram

$$\begin{tikzcd}
C(D) \arrow[r, "\phi_{S_1}"] \arrow[d, "\tau"] & C(D_1) \arrow[r] \arrow[d, "\tau"] & \cdots \arrow[r] & C(D_{N-1}) \arrow[r, "\phi_{S_N}"] \arrow[d, "\tau"] & C(D') \arrow[d, "\tau"] \\
C(\tau D) \arrow[r, "\phi_{\tau S_1}"] & C(\tau D_1) \arrow[r] & \cdots \arrow[r] & C(\tau D_{N-1}) \arrow[r, "\phi_{\tau S_N}"] & C(\tau D')
\end{tikzcd}$$

□

Proposition 4.28 *Let S be a simple isotopy-equivariant cobordism between involutive links L and L'. Then there is a cobordism map*

$$\phi_S : \mathrm{CKhI}(D) \to \mathrm{CKhI}(D')$$

such that the following diagram commutes:

$$\begin{tikzcd}
Q\mathrm{CKh}(D)[1] \arrow[r, hook] \arrow[d, "\phi_S"] & \mathrm{CKhI}(D) \arrow[r, two heads] \arrow[d, "\phi_S"] & \mathrm{CKh}(D) \arrow[d, "\phi_S"] \\
Q\mathrm{CKh}(D')[1] \arrow[r, hook] & \mathrm{CKhI}(D') \arrow[r, two heads] & \mathrm{CKh}(D')
\end{tikzcd}$$

Proof By definition S and τS are isotopic rel boundary, so from [5, Theorem 4] there is a homotopy $\phi_S \simeq \phi_{\tau S}$. Together with Lemma 4.27, we have

$$\phi_S \simeq \phi_{\tau S} \simeq (\phi_S)^\tau.$$

Thus the result follows from Lemma 2.4. □

Now we restrict to strongly invertible links, and prove the behavior of the equivariant Lee classes under simple isotopy-equivariant cobordisms.

Proposition 4.29 *Suppose S is a simple isotopy-equivariant cobordism between strongly invertible links L, L', such that every component of S has boundary in L. Then under the map ϕ_S of Proposition 4.28, the equivariant Lee classes modulo torsions for the given orientations correspond as*

$$\underline{\alpha}(D) \xmapsto{\phi_S} h^j \underline{\alpha}(D'), \quad \underline{\beta}(D) \xmapsto{\phi_S} h^j \underline{\beta}(D'),$$
$$\overline{\alpha}(D) \xmapsto{\phi_S} h^j \overline{\alpha}(D'), \quad \overline{\beta}(D) \xmapsto{\phi_S} h^j \overline{\beta}(D'),$$

where

$$j = \frac{\delta w(D, D') - \delta r(D, D') - \chi(S)}{2}.$$

Proof By an argument similar to the proof of Proposition 3.8, the assertions are immediate from [27, Proposition 3.4]. □

Proposition 4.30 *Under the assumption of Proposition 4.29, we have*

$$\underline{s}_h(L) \le \underline{s}_h(L') - \chi(S) \quad \text{and} \quad \overline{s}_h(L) \le \overline{s}_h(L') - \chi(S).$$

Moreover if every component of S has boundary in both L and L', then we have

$$|\underline{s}_h(L) - \underline{s}_h(L')| \le -\chi(S) \quad \text{and} \quad |\overline{s}_h(L) - \overline{s}_h(L')| \le -\chi(S).$$

In particular, both $\underline{s}$ and $\overline{s}$ are invariant under simple isotopy-equivariant link concordances.

Proof Immediate from Proposition 4.29. □

Corollary 4.31 *For a strongly invertible knot K, both $|\underline{s}_h(K)|$ and $|\overline{s}_h(K)|$ bound $2\,\widetilde{\mathrm{sig}}_4(K)$ from below.*

Corollary 4.32 *If either $|\underline{s}_h(K)|$ or $|\overline{s}_h(K)|$ is greater than $2g_4(K)$, then no slice surfaces S of K realizing $g(S) = g_4(K)$ are simple isotopy-equivariant.*

5 The main theorem

In this section specialize to $(R, h) = (\mathbb{F}_2[H], H)$. The invariants $\underline{s}_H, \overline{s}_H$ will be denoted by $\underline{s}, \overline{s}$.

Proposition 1.2 *The three strongly invertible slice knots $K = m(9_{46})$, $15n_{103488}$, $17nh_{73}$ have*

$$\underline{s}(K) = 0 < 2 = \overline{s}(K).$$

Proof Proved by direct computations of BNI_r for the three knots. The result for $J_0 = 17nh_{73}$ is given in Table 1. The results for $m(9_{46})$ and $15n_{103488}$ are given in Table 2. □

	0	1	2	3	4	5	6	7
12	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
10	.	.	.	.	.	.	.	.
8	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
6	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
4	.	.	.	.	.	.	.	.
2	.	$\mathbb{F}[H]$	$\mathbb{F}$	.	.	.	.	.
0	$\mathbb{F}[H]$	.	.	.	.	.	.	.

	0	1	2	3	4	5	6	7	8	9	10	11
18	.	.	.	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	.	.	.	.	.	.
14	.	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
12	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$	.	.	.
10	.	.	.	.	.	.	.	.	.	.	.	.
8	.	.	.	.	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}$	.	.	.	.	.
6	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.	.	.	.
2	.	$\mathbb{F}[H]$	$\mathbb{F}$	.	.	.	.	.	.	.	.	.
0	$\mathbb{F}[H]$	.	.	.	.	.	.	.	.	.	.	.

Table 2: $\mathrm{BNI}_r(m(9_{46}))$, left, and $\mathrm{BNI}_r(15n_{103488})$, right.

Theorem 1 *The strongly invertible knot J_n of Figure 1 has*

$$\underline{s}(J_n) = 0 < 2 \leq \overline{s}(J_n).$$

Proof Immediate from Theorem 3 and Propositions 1.2 and 1.6. □

To give some intuition for the inequality $\underline{s} < \overline{s}$, we give a human readable proof of a weaker inequality

$$\underline{s}(K) = 0 < 2 \leq \overline{s}(K)$$

specifically for $K = m(9_{46})$, inspired by the arguments given in [1]. Figure 12, left, depicts a diagram D of $m(9_{46})$ in the standard form of the pretzel knot $P(-3, 3, -3)$. Given $w(D) = 3$, $r(D) = 8$, it suffices to show that

$$\underline{d}_h(D) = 2 < 3 \leq \overline{d}_h(D).$$

In Figure 12, the central diagram depicts the Lee cycle $\alpha(D)$ of D and the diagram on the right depicts another τ-invariant cycle z, which is obtained from $\alpha(D)$ by altering the labels of the two center circles to 1. By an argument similar to Example 3.16, we may find some τ-invariant element $x \in \mathrm{CKh}^{-1}(D)$ that gives

$$\alpha(D) \sim h^2 z,$$

where $\sim$ denotes homologous. Using $x \in \mathrm{CKhI}^{-1}(D)$ and $Qx \in \mathrm{CKhI}^{0}(D)$, we obtain

$$\underline{\alpha}(D) \sim h^2 z, \quad \overline{\alpha}(D) \sim h^2 Qz,$$

and hence

$$\underline{d}_h(D), \overline{d}_h(D) \geq 2.$$

Here, it is necessary that x is τ-invariant, otherwise $Q(1+\tau)x$ will produce some nonzero term in $Q\mathrm{CKh}(D)$.

Since K is slice, we have $s_h(D) = 0$ and hence $d_h(D) = 2$. Thus from Proposition 3.17 we obtain $\underline{d}_h(D) = 2$. It remains to show that the cycle $Qz \in \mathrm{CKhI}(D)$ is at least once more h-divisible. This is

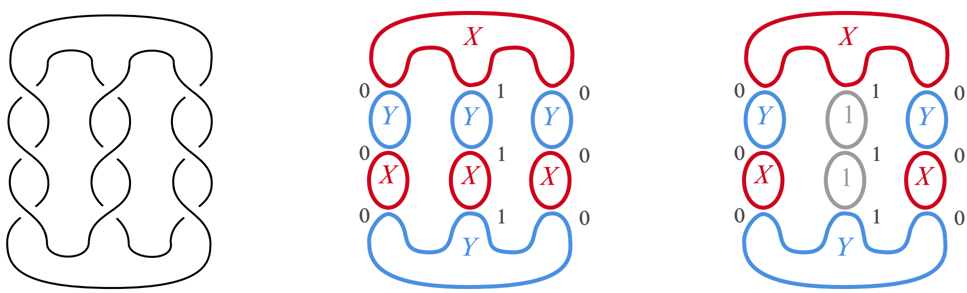


Figure 12: $m(9_{46})$.

equivalent to Qz being null-homologous *modulo* h in $\mathrm{CKhI}(D)$. Thus it suffices to set $h = 0$ and prove that there exists elements $a, b \in \mathrm{CKh}(D)$ such that

$$\begin{tikzcd} & b \arrow[r, mapsto, "d"] \arrow[d, "Q(1+\tau)"] & 0 \\ Qa \arrow[r, mapsto, "d"] & Qz & \end{tikzcd} \qquad d(Qa) + Q(1+\tau)b = Qz.$$

Here it is necessary that $a \in \mathrm{CKh}(D)$ is *not* τ-invariant, otherwise we would also have $z \sim 0$ modulo h in $\mathrm{CKhI}(D)$ and $\underline{d}_h(D) \geq 3$. We leave it to the reader to find explicit elements a, b satisfying the above relation. Hence we conclude that $\overline{d}_h(D) \geq 3$.

This observation demonstrates the following facts:

- A cycle $z \in \mathrm{CKh}(D)$ must be symmetric (ie τ-invariant) to be a cycle in $\mathrm{CKhI}(D)$, whereas $Qz \in \mathrm{CKhI}(D)$ is always a cycle.
- A nontrivial boundary $dx \in \mathrm{CKh}(D)$ must be symmetric to be a boundary in $\mathrm{CKhI}(D)$, whereas $Q(dx) \in \mathrm{CKhI}(D)$ is always a boundary.
- A nonsymmetric cycle $z \in \mathrm{CKh}(D)$ gives a nontrivial boundary $Q(1+\tau)z$ in $Q\mathrm{CKh}(D) \subset \mathrm{CKhI}(D)$.

Therefore there are "less cycles" in $\mathrm{CKh}(D) \subset \mathrm{CKhI}(D)$ and "more boundaries" in $Q\mathrm{CKh}(D) \subset \mathrm{CKhI}(D)$, which allow $\underline{s} < \overline{s}$ to happen. A more simplified proof of Proposition 1.2 using reduction techniques is to be presented in a future paper.

6 On 2-periodic links

Analogous constructions for 2-periodic links are possible by using $\sigma\tau$ instead of τ, where σ is the involution given in Definition 2.19. Since σ is induced from a Frobenius algebra isomorphism, it commutes with any map coming from Bar-Natan's category $\mathrm{Cob}^3_{/l}(B)$, and many of the arguments in Sections 2 and 3 run in parallel for 2-periodic knots and links. The exceptions are (i) equivariant connected sums are not defined and (ii) the reduced complexes cannot be defined for 2-periodic links. Here we only state some of the basic definitions and results.

Definition 6.1 Given an involutive link (D, τ), define

$$\mathrm{CKhI}(D, \sigma\tau) = \mathrm{Cone}\big(\, \mathrm{CKh}(D) \xrightarrow{Q(1+\sigma\tau)} Q\mathrm{CKh}(D) \,\big).$$

Proposition 6.2 *The chain homotopy type of* $\mathrm{CKhI}(D, \sigma\tau)$ *is an invariant of the involutive link.*

Hereafter we assume that (D, τ) is 2-periodic.

Proposition 6.3 *The Lee cycles $\alpha(D, o)$ in $\mathrm{CKh}(D)$ are invariant under $\sigma\tau$.*

Proof Similar to the proof of Proposition 3.3, except that no Seifert circles intersect $\mathrm{Fix}(\tau)$, and that each symmetric pair of Seifert circles are oriented oppositely. □

Definition 6.4 The cycles $\underline{\alpha}(D, o) = \alpha(D, o)$ and $\overline{\alpha}(D, o) = Q\alpha(D, o)$ in $\mathrm{CKhI}(D, \sigma\tau)$ are called the *equivariant Lee cycles* of D.

Proposition 6.5 *If $h \in R$ is invertible, the homology $\mathrm{KhI}(D, \sigma\tau)$ is freely generated by the $2^{|D|+1}$ equivariant Lee classes.*

Hereafter we additionally assume that R is a PID and h is prime.

Proposition 6.6 *Let $\underline{d}(D)$ and $\overline{d}(D)$ be the h-divisibility (modulo torsion) of the equivariant Lee classes $\underline{\alpha}(D), \overline{\alpha}(D)$ in $\mathrm{KhI}(D, \sigma\tau)$. Then the quantities*

$$\underline{s}_h(L) = 2\underline{d}_h(D) + w(D) - r(D) + 1,$$
$$\overline{s}_h(L) = 2\overline{d}_h(D) + w(D) - r(D) + 1,$$

are invariants of the corresponding 2-periodic link L, satisfying

$$\underline{s}_h(L) \le s_h(L) \le \overline{s}_h(L).$$

In particular when K is a 2-periodic knot, then

$$\underline{s}_h(K) \le s(K) \le \overline{s}_h(K)$$

where $s(K)$ is the $\mathbb{F}_2$-Rasmussen invariant.

Proposition 6.7 *Let S be a simple isotopy-equivariant cobordism between 2-periodic links L and L'. Then we have*

$$\underline{s}_h(L) \le \underline{s}_h(L') - \chi(S) \quad \text{and} \quad \overline{s}_h(L) \le \overline{s}_h(L') - \chi(S).$$

Moreover if every component of S has boundary in both L and L', then we have

$$|\underline{s}_h(L) - \underline{s}_h(L')| \le -\chi(S) \quad \text{and} \quad |\overline{s}_h(L) - \overline{s}_h(L')| \le -\chi(S).$$

In particular, both $\underline{s}$ and $\overline{s}$ are invariant under simple isotopy-equivariant link concordances.

Corollary 6.8 *For a 2-periodic knot K, both $|\underline{s}_h(K)|$ and $|\overline{s}_h(K)|$ bound $2\,\widetilde{\mathrm{sig}}_4(K)$ from below.*

Appendix Computations for small prime knots

Here, the reduced involutive Khovanov homologies for prime knots with up to 7 crossings are given, computed by a program[4] developed by the author, with the diagrams given in [22, Section 6.5] as the input data.

Before describing the results, we review some of the basic facts on strongly invertible knots. See [26, Section 3] for the references.

Proposition A.9 (1) *Every torus knot admits exactly one inverting involution.*

(2) *An invertible hyperbolic knot admits exactly one or two inverting involutions*; *two when it has* (*cyclic or free*) *period* 2, *or one otherwise.*

(3) *For a fully amphichiral hyperbolic knot* K,

(a) *if* K *admits a unique inverting involution* τ, *then* $(K, \tau) \cong (K, \tau)^*$,

(b) *if* K *admits two inverting involutions* τ, τ', *then* $(K, \tau) \cong (K, \tau')^*$.

Thus for a torus knot (T) and a fully amphichiral hyperbolic knot (HA), it suffices to perform the computation for only one of its involutions. For a nonamphichiral hyperbolic knot (HN), there are possibly two nonequivalent involutions. In fact, all nonamphichiral hyperbolic knots with up to 7 crossings have exactly two nonequivalent involutions, and are distinguished by Sakuma's η-polynomial; see [26, Appendix].

The following list shows the computation results for strongly invertible prime knots with up to 7 crossings. Each item displays the name of the strongly invertible knot K, together with its type: (T), (HA) or (HN). For a (HN) type knot, the two distinct strongly invertible knots are distinguished by suffixes a, b, such as 5_{2a} and 5_{2b}. Its reduced involutive Khovanov homology $(R, h) = (\mathbb{F}_2, 0)$ is displayed on the left and its reduced involutive Bar-Natan homology $(R, h) = (\mathbb{F}_2[H], H)$ on the right. Note that both theories are bigraded, and the bigrading of each summand can be extracted from its computed generator. For the sake of readability, $\mathbb{F}_2$ is simply written as $\mathbb{F}$ and also $\mathbb{F}_2[H]/(H)$ is replaced with $\mathbb{F}$. From the latter table, one can read off the values of the equivariant Rasmussen invariants $(\underline{s}, \overline{s})$, however all knots in this list have $\underline{s} = \overline{s} = s^{\mathbb{F}_2}$. One can observe that BNI_r is generally not a direct sum of two shifted copies of BN_r, as can be seen in 7_{4b} and 7_{7b}. It is also notable that these two have order two H-torsions in BNI_r.

For the purpose of distinguishing nonequivalent involutions on the same knot, we see that the only successful ones are 7_4 and 7_7, so our invariants are not as strong as Sakuma's η-polynomial and Lobb–Watson's triply graded invariant CKh_τ; see [22]. Some additional structures, such as filtrations, might be given on KhI to further strengthen the invariant.

[4]The program is available at https://github.com/taketo1024/yui.

- 3_1 (T)

	0	1	2	3	4
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$
6	.	.	$\mathbb{F}$	$\mathbb{F}$	.
4	.	.	.	.	.
2	$\mathbb{F}$	$\mathbb{F}$	.	.	.

	0	1	2	3	4
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$
6	.	.	.	.	.
4	.	.	.	.	.
2	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.

- 4_1 (HA)

	-2	-1	0	1	2	3
4	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
2	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
0	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
-2	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
-4	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.

	-2	-1	0	1	2	3
4	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
2	.	.	.	.	.	.
0	.	.	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.
-2	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
-4	.	.	.	.	.	.

- 5_1 (T)

	0	1	2	3	4	5	6
14	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
12	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
10	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
8	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
6	.	.	.	.	.	.	.
4	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.

	0	1	2	3	4	5	6
14	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
12	.	.	.	.	.	.	.
10	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
8	.	.	.	.	.	.	.
6	.	.	.	.	.	.	.
4	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.

- $5_{2a}, 5_{2b}$ (HN)

	0	1	2	3	4	5	6
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
10	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
6	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
4	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.

	0	1	2	3	4	5	6
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
10	.	.	.	.	.	.	.
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
6	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
4	.	.	.	.	.	.	.
2	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.

- $6_{1a}, 6_{1b}$ (HN)

	-2	-1	0	1	2	3	4	5
8	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
6	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
4	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
2	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
0	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
-2	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-4	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.

	-2	-1	0	1	2	3	4	5
8	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
6	.	.	.	.	.	.	.	.
4	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
2	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
0	.	.	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.
-2	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-4	.	.	.	.	.	.	.	.

- $6_{2a}, 6_{2b}$ (HN)

	-2	-1	0	1	2	3	4	5
10	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
8	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
6	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
4	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
2	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
0	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.

	-2	-1	0	1	2	3	4	5
10	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
8	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
6	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
4	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
2	.	.	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.
0	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-2	.	.	.	.	.	.	.	.

- 6_3 (HA)

	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
2	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
0	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
-2	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
-4	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
-6	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.

	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
2	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
0	.	.	.	$\mathbb{F}[H]\oplus\mathbb{F}$	$\mathbb{F}[H]\oplus\mathbb{F}$	.	.	.
-2	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
-4	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-6	.	.	.	.	.	.	.	.

- 7_1 (T)

	0	1	2	3	4	5	6	7	8
20	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
18	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
16	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
14	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
12	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
10	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
8	.	.	.	.	.	.	.	.	.
6	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.

	0	1	2	3	4	5	6	7	8
20	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
18	.	.	.	.	.	.	.	.	.
16	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
14	.	.	.	.	.	.	.	.	.
12	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
10	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	.
6	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.

- $7_{2a}, 7_{2b}$ (HN)

	0	1	2	3	4	5	6	7	8
16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
10	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
8	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
6	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
4	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.

	0	1	2	3	4	5	6	7	8
16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
10	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
6	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.
2	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.

- $7_{3a}, 7_{3b}$ (HN)

18	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
14	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
12	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
10	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
8	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
6	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
4	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

18	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	.	.	.
14	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
12	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
10	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
8	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.
4	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

- 7_{4a} (HN)

16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
12	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
10	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
8	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
6	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.	.	.
4	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.	.
2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
10	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
6	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.
2	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

- 7_{4b} (HN)

16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
10	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
8	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
6	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
4	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
2	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

16	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
14	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
10	.	.	.	.	$\mathbb{F}[H]/(H^2)$	$\mathbb{F}$	.	.	.
8	.	.	.	$\mathbb{F}$	.	.	.	.	.
6	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.
2	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

- $7_{5a}, 7_{5b}$ (HN)

18	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
14	.	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.
12	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
10	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.	.
8	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.	.	.
6	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
4	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

18	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
16	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
14	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
12	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.
10	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
8	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.
4	$\mathbb{F}[H]$	$\mathbb{F}[H]$	.	.	.	.	.	.	.
	0	1	2	3	4	5	6	7	8

- $7_{6a}, 7_{6b}$ (HN)

	-5	-4	-3	-2	-1	0	1	2	3
2	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
0	.	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
-2	.	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.
-4	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
-6	.	.	.	$\mathbb{F}^4$	$\mathbb{F}^4$	.	.	.	.
-8	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.	.	.
-10	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.	.
-12	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.

	-5	-4	-3	-2	-1	0	1	2	3
2	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
0	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
-2	.	.	.	.	.	$\mathbb{F}[H]\oplus\mathbb{F}$	$\mathbb{F}[H]\oplus\mathbb{F}$	.	.
-4	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.
-6	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
-8	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-10	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
-12	.	.	.	.	.	.	.	.	.

- 7_{7a} (HN)

	-4	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.
2	.	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.
0	.	.	.	.	$\mathbb{F}^4$	$\mathbb{F}^4$	.	.	.
-2	.	.	.	$\mathbb{F}^4$	$\mathbb{F}^4$	.	.	.	.
-4	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.	.	.
-6	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.	.
-8	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.

	-4	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
2	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
0	.	.	.	.	$\mathbb{F}[H]\oplus\mathbb{F}^2$	$\mathbb{F}[H]\oplus\mathbb{F}^2$	.	.	.
-2	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
-4	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.
-6	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
-8	.	.	.	.	.	.	.	.	.

- 7_{7b} (HN)

	-4	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.
2	.	.	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.
0	.	.	.	.	$\mathbb{F}^3$	$\mathbb{F}^3$	.	.	.
-2	.	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.
-4	.	.	$\mathbb{F}^2$	$\mathbb{F}^2$	.	.	.	.	.
-6	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.
-8	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.	.	.	.

	-4	-3	-2	-1	0	1	2	3	4
6	.	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$
4	.	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.
2	.	.	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.
0	.	.	.	.	$\mathbb{F}[H]\oplus\mathbb{F}$	$\mathbb{F}[H]\oplus\mathbb{F}$	.	.	.
-2	.	.	.	$\mathbb{F}$	$\mathbb{F}$	.	.	.	.
-4	.	.	$\mathbb{F}[H]/(H^2)$	$\mathbb{F}$	.	.	.	.	.
-6	.	$\mathbb{F}$	.	.	.	.	.	.	.
-8	.	.	.	.	.	.	.	.	.

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