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Orbifolds, orbispaces and global homotopy theory

BRANKO JURAN

Given an orbifold, we construct a global spectrum representing its stable global homotopy type. Global spectra now represent orbifold cohomology theories which automatically satisfy certain properties such as additivity and the existence of Mayer–Vietoris sequences. Moreover, the value at a global quotient orbifold $M//G$ can be identified with the G -equivariant cohomology of the manifold M . Examples of orbifold cohomology theories which are represented by global spectra include Borel and Bredon cohomology theories and orbifold K -theory. This also implies that these cohomology groups are independent of the presentation of an orbifold as a global quotient orbifold.

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1 Introduction

The purpose of this paper is to use global homotopy theory for studying the homotopy theory of orbifolds as suggested by Schwede in [33, Preface].

Orbifolds are objects from geometric topology which model spaces that locally arise as quotients of smooth manifolds under smooth group actions. Special cases are *global quotient orbifolds* $M//G$ of a smooth manifold M under a smooth action of a compact Lie group G .

Orbifolds were first defined by Satake [31] who called them V -manifolds. The definition generalizes the definition of a manifold by endowing the euclidean charts with a finite group action. This notion was later used and popularized by Thurston [37], who started calling them orbifolds and used them to study three-manifolds. In a more general approach, due to Moerdijk and Pronk [25], one defines an orbifold as a special kind of Lie groupoid. Connections between groupoids and orbifolds were first studied by Haefliger [12].

It is essential that an orbifold carries more information than just the underlying topological quotient space: It also keeps track of the group actions involved. The homotopy type of a global quotient orbifold $M//G$ should relate to the G -equivariant homotopy type of the underlying manifold M . For example, the cohomology of $M//G$ is usually defined as the G -equivariant cohomology of M . However, the group G may vary for different orbifolds and two quotients can represent the same orbifold, eg for H a subgroup of G , the quotients $M//H$ and $(G \times_H M)//G$ are equivalent as orbifolds. The presentation of an orbifold X as a global quotient orbifold $M//G$ hence is nonintrinsic data of the orbifold X . However, an orbifold cohomology theory should capture the G -equivariant behavior of the orbifold X . A conceptual treatment of *orbifold cohomology theories* must necessarily pay attention to the interplay between different equivariant cohomology theories. This leads us to global homotopy theory.

Global homotopy theory studies homotopy types with simultaneous, compatible group actions of all compact Lie groups. A particular framework for stable global homotopy theory was developed by Schwede in [33]. He uses the category of orthogonal spectra together with *global equivalences*. The notion of global equivalences is much more restrictive than the usual one of stable equivalences between orthogonal spectra. Two orthogonal spectra are globally equivalent if they represent the same *global homotopy type* and not just the same nonequivariant homotopy type. Localizing the category of orthogonal spectra at the global equivalences yields the *stable global homotopy category* $\mathcal{GH}$. There is a functor $\mathcal{GH} \rightarrow G\text{-}\mathcal{SH}$ from the stable global homotopy category to the G -equivariant stable homotopy category for any Lie group G which assigns the *underlying G -equivariant homotopy type* to a global homotopy type.

Similarly, the category of orthogonal spaces together with global equivalences of orthogonal spaces models unstable global homotopy types. Orthogonal spectra, ie stable global homotopy types, represent cohomology theories on these orthogonal spaces.

The goal of this paper is to construct a well-behaved functor from orbifolds to orthogonal spaces, so that we can use orthogonal spectra to represent cohomology theories on orbifolds.

Firstly, we use a different model for unstable global homotopy theory, the category $\mathbf{T}_{\mathbf{Orb}}$ of **Orb**-spaces, introduced by Gepner and Henriques in [13]. This category is defined to be the category of topological presheaves on the category **Orb**, which is the full subcategory of the topological category of topological groupoids **TGrpd** spanned by the groupoids with a single object and a compact Lie group as automorphisms. Hence, the restricted Yoneda embedding gives rise to a functor $Y : \mathbf{TGrpd} \rightarrow \mathbf{T}_{\mathbf{Orb}}$.

We verify that this functor is homotopical, ie it sends the so-called *essential equivalences* of topological groupoids to weak equivalences of **Orb**-spaces.

Theorem (Corollary 2.22) *The functor*

$$Y : \mathbf{TGrpd} \rightarrow \mathbf{T}_{\mathbf{Orb}}$$

*sends essential equivalences of topological groupoids to weak equivalences of **Orb**-spaces.*

Precomposing the above functor with the forgetful functor $\mathbf{Orbfld} \rightarrow \mathbf{TGrpd}$ hence yields a functor sending essential equivalences of orbifolds to weak equivalences of **Orb**-spaces and therefore induces a functor

$$|-|_{\mathbf{gl}} : \mathbf{Orbfld}_{\infty} \rightarrow (\mathbf{TOrb})_{\infty}$$

on ∞ -categorical localizations which we call the *global realization* functor.

Using a zigzag of Quillen equivalences established by K\"orschgen [17] and Schwede [34] between the category of **Orb**-spaces and the category of orthogonal spaces with global equivalences, there is a suspension functor $\Sigma_+^{\infty} : (\mathbf{TOrb})_{\infty} \rightarrow \mathcal{GH}$ to the ∞ -category of global spectra. We define the *global motivic* of an orbifold as

$$\mathbf{M}(\mathbb{G}) = \Sigma_+^{\infty} |\mathbb{G}|_{\mathbf{gl}}.$$

Any other global spectrum E now represents a cohomology theory on orbifolds, by defining the k^{th} *orbifold cohomology group* represented by E of $\mathbb{G}$ as

$$E^k(\mathbb{G}) = \mathcal{GH}(\mathbf{M}(\mathbb{G}), E[k]).$$

It fulfills the following properties:

Theorem (Theorem 4.6) *Let $E \in \mathcal{GH}$ be a global spectrum.*

(i) *The orbifold cohomology functor $E^k : \mathbf{Orbfld}^{\text{op}} \rightarrow \mathbf{Ab}$ takes essential equivalences of orbifolds to isomorphisms.*

(ii) *For every family $\{\mathbb{G}^i\}_{i \in I}$ of orbifolds, the natural map*

$$E^k\left(\coprod_{i \in I} \mathbb{G}^i\right) \rightarrow \prod_{i \in I} E^k(\mathbb{G}^i)$$

is an isomorphism.

(iii) *Let $\mathbb{G}$ be an orbifold covered by two open suborbifolds $\mathbb{U}$ and $\mathbb{V}$. There is a long exact sequence of orbifold cohomology groups*

$$\dots \rightarrow E^k(\mathbb{G}) \xrightarrow{(i_{\mathbb{U}}^*, i_{\mathbb{V}}^*)} E^k(\mathbb{U}) \oplus E^k(\mathbb{V}) \xrightarrow{j_{\mathbb{U}}^* - j_{\mathbb{V}}^*} E^k(\mathbb{U} \cap \mathbb{V}) \rightarrow E^{k+1}(\mathbb{G}) \rightarrow \dots$$

for every orthogonal spectrum E . Here, $i_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{G}$, $i_{\mathbb{V}} : \mathbb{V} \rightarrow \mathbb{G}$, $j_{\mathbb{U}} : \mathbb{U} \cap \mathbb{V} \rightarrow \mathbb{U}$ and $j_{\mathbb{V}} : \mathbb{U} \cap \mathbb{V} \rightarrow \mathbb{V}$ denote the inclusion morphisms.

(iv) *Let G be a compact Lie group acting almost freely on a manifold M . The orbifold cohomology of $G \ltimes M$ is isomorphic to the G -equivariant cohomology of M , ie*

$$E^k(G \ltimes M) \cong E_G^k(M).$$

Parts (i) and (iv) imply that orbifold cohomology groups are independent of the presentation of the orbifold as a global quotient orbifold.

Examples of equivariant cohomology theories which are represented by orthogonal spectra in the global sense include equivariant K -theory, Borel cohomology theories and Bredon cohomology theories. These cohomology theories hence are examples of *orbifold cohomology theories* in our sense. Particularly, we show that Bredon cohomology extends to a well-behaved cohomology theory on all orbifolds. These examples are discussed in greater detail in Section 4.

Relations to other work

Since this article was first posted, other papers have studied the relation between orbifolds and global homotopy theory, eg [6; 32]. The main difference is that those papers model orbifolds as differentiable stacks, whereas we work with the topological category of Lie groupoids and invert essential equivalences. There are several comparisons between these approaches, eg in [13; 18; 28], but the author is not aware of a precise comparison to the model used in this paper.

Organization of the paper

In Section 2.1 we recall the concept of an orbifold as a Lie groupoid. In Section 3, we discuss the zigzag of Quillen equivalences between models for orbispaces. The most important technical result is Corollary 3.13, saying that these equivalences preserve models for global quotient spaces. Finally, we associate an orthogonal space to an orbifold and define what an *orbifold cohomology theory* is in Section 4.

Conventions

Throughout this paper, $\mathbf{T}$ denotes the category of compactly generated, weak Hausdorff spaces together with continuous maps, as defined in [36]. This definition was first used by McCord in [22]. Other sources for details on this topic are [8, Section 7.9; 19, Appendix A; 33, Appendix A]. A *space* is always assumed to be an object of $\mathbf{T}$. Moreover, $\mathbf{T}_*$ denotes the category of based, compactly generated, weak Hausdorff spaces together with based continuous maps.

Let G be a topological group, ie a group object in $\mathbf{T}$. We write $G\text{-}\mathbf{T}$ for the category of (left) G -spaces together with G -equivariant continuous maps. Moreover, $G\text{-}\mathbf{T}_*$ denotes the category of based G -spaces together with based G -equivariant continuous maps.

A *manifold* is always assumed to be a smooth manifold which is second countable and Hausdorff. The category of smooth manifolds together with smooth maps is denoted by $\mathbf{Mfld}$. We require all occurring group actions of Lie groups on manifolds to be smooth.

We will use the theory of ∞ -categories as developed by Lurie in [20]. For $\mathcal{C}$ a topological category with a class of weak equivalences W , we write $\mathcal{C}_\infty = \mathbf{N}(\mathcal{C})[W^{-1}]$ for the ∞ -categorical localization of the homotopy coherent nerve of $\mathcal{C}$ at W in the sense of [20, Definitions 1.1.5.5 and 1.3.4.1]. In the case where $\mathcal{C}$ admits the additional structure of a bicomplete compactly generated topological model category, this agrees with the ∞ -categorical localization of the underlying 1-category, as can be shown using [21, Theorem 1.3.4.20; 30, Corollary 13.2.4].

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2 The global homotopy type of orbifolds

2.1 Orbifolds

We recall the approach to orbifolds via Lie groupoids. An excellent overview of this approach can be found in Moerdijk's paper [23]. A closely related approach using differentiable stacks can be found in [18].

Definition 2.1 (topological groupoid) A *topological groupoid* is a groupoid object in $\mathbf{T}$, that is a tuple $\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, s, t, e, i, \circ)$ where $\mathbb{G}_0$ and $\mathbb{G}_1$ are spaces and $s: \mathbb{G}_1 \rightarrow \mathbb{G}_0$, $t: \mathbb{G}_1 \rightarrow \mathbb{G}_0$, $e: \mathbb{G}_0 \rightarrow \mathbb{G}_1$, $i: \mathbb{G}_1 \rightarrow \mathbb{G}_1$ and $\circ: \mathbb{G}_1 \times_{s,t} \mathbb{G}_1 \rightarrow \mathbb{G}_1$ are continuous maps such that the underlying sets of $\mathbb{G}_0$ (as objects) and $\mathbb{G}_1$ (as arrows) together with s as the source map, t as the target map, e as the identity map, i as the inverse map and $\circ$ as the composition map form a (small) groupoid.

A *morphism* of topological groupoids $f: \mathbb{G} \rightarrow \mathbb{H}$ is a pair of continuous maps $f_0: \mathbb{G}_0 \rightarrow \mathbb{H}_0$ and $f_1: \mathbb{G}_1 \rightarrow \mathbb{H}_1$ which is a functor between the underlying groupoids. A *2-morphism* between two morphisms of topological groupoids $f, g: \mathbb{G} \rightarrow \mathbb{H}$ is a continuous map $\mathbb{G}_0 \rightarrow \mathbb{H}_1$ which is a natural transformation between f and g .

The category of topological groupoids is enriched over itself. We topologize the set $\text{Map}(\mathbb{G}, \mathbb{H})_0$ of morphisms between $\mathbb{G}$ and $\mathbb{H}$ as a subspace of $\text{map}(\mathbb{G}_1, \mathbb{H}_1)$. Moreover, $\text{Map}(\mathbb{G}, \mathbb{H})_1$ is the space of 2-morphisms between $\mathbb{G}$ and $\mathbb{H}$ topologized as a subspace of $\text{map}(\mathbb{G}_1, \mathbb{H}_1) \times \text{map}(\mathbb{G}_1, \mathbb{H}_1) \times \text{map}(\mathbb{G}_0, \mathbb{H}_1)$ where the inclusion into the first factor is given by the source functor, the one into the second factor by the target functor and the one into the third factor by the underlying map of the natural transformation itself.

The category of topological groupoids is denoted by **TGrpd**.

Definition 2.2 (Lie groupoid) Similarly, a *Lie groupoid* is a tuple $\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, s, t, e, i, \circ)$ where $\mathbb{G}_0$ and $\mathbb{G}_1$ are manifolds, $s: \mathbb{G}_1 \rightarrow \mathbb{G}_0$, $t: \mathbb{G}_1 \rightarrow \mathbb{G}_0$ are smooth submersions and $e: \mathbb{G}_0 \rightarrow \mathbb{G}_1$, $i: \mathbb{G}_1 \rightarrow \mathbb{G}_1$ and $\circ: \mathbb{G}_1 \times_{s,t} \mathbb{G}_1 \rightarrow \mathbb{G}_1$ are smooth maps such that the underlying sets form a groupoid as above.

Morphisms and 2-morphisms are defined just as for topological groupoids with smooth maps instead of continuous ones. The category of Lie groupoids is denoted by **LieGrpd**.

The category of Lie groupoids is not enriched over itself, but it is enriched over the category of topological groupoids by taking the C^∞ -topology on mapping spaces. This topology is finer than the compact-open topology.

If s is a submersion, then t automatically is a submersion because $t = s \circ i$ and i is a self-inverse diffeomorphism. The requirement that $s, t: \mathbb{G}_1 \rightarrow \mathbb{G}_0$ are submersions ensures that the fibered product $\mathbb{G}_1 \times_{s,t} \mathbb{G}_1$ exists in **Mfld** and that the underlying space can be computed in **T**.

The forgetful functor **Mfld** $\rightarrow$ **T** induces a forgetful functor **LieGrpd** $\rightarrow$ **TGrpd**. In particular, every Lie groupoid can be regarded as a topological groupoid. We will not mention this when using this fact.

Example 2.3 We will give some examples without explicitly mentioning or defining all maps.

- Let G be a topological group (or Lie group, respectively). Let $\mathbb{B}G$ denote the topological groupoid (or Lie groupoid, respectively) with objects space $\mathbb{B}G_0 = *$ and arrow space $\mathbb{B}G_1 = G$. The identity map of $\mathbb{B}G$ sends $*$ to the identity of G . Composition and inverse maps are given by the corresponding maps of G .
- More generally, let G be a topological group and A be a G -space. Define the *action groupoid* $G \ltimes A$ with $(G \ltimes A)_0 = A$ and $(G \ltimes A)_1 = G \times A$. The map $s: G \times A \rightarrow A$ is the projection map and the map $t: G \times A \rightarrow A$ is the action map of A .

An arrow $g: x \rightarrow y$ in $G \ltimes A$ from x to y in A may be described as an element g of G such that $g.x = y$. The composition is defined by using the composition of G .

The topological groupoid $G \ltimes A$ carries the structure of a Lie groupoid if G is a Lie group acting smoothly on a manifold $M = A$.

Definition 2.4 (essential equivalence) An *essential equivalence* of topological groupoids (or Lie groupoids, respectively) is a morphism $f: \mathbb{G} \rightarrow \mathbb{H}$ of groupoids such that the following holds true:

- **(essentially surjective)** The map $\mathbb{H}_1 \times_{s,f_0} \mathbb{G}_0 \rightarrow \mathbb{H}_0$ induced by $t: \mathbb{H}_1 \rightarrow \mathbb{H}_0$ on the first factor is a surjection admitting local sections (or surjective submersion, respectively).
- **(fully faithful)** The square

$$\begin{array}{ccc} \mathbb{G}_1 & \xrightarrow{f_1} & \mathbb{H}_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ \mathbb{G}_0 \times \mathbb{G}_0 & \xrightarrow{(f_0, f_0)} & \mathbb{H}_0 \times \mathbb{H}_0 \end{array}$$

is a pullback in **T** (or in **Mfld**, respectively).

We say that two topological groupoids (or Lie groupoids, respectively) are *essentially equivalent* if there is a zigzag of essential equivalences between them. This is often also called *Morita equivalent*.

Definition 2.5 (stabilizer group) Let $\mathbb{G}$ be a topological groupoid and let x be a point in $\mathbb{G}_0$. We write

$$\mathbb{G}_x = (s, t)^{-1}(x, x)$$

for the *stabilizer group* of x . This becomes a topological group with the subspace topology.

Definition 2.6 (proper, foliation groupoid) Let $\mathbb{G}$ be a Lie groupoid. We say that $\mathbb{G}$ is *proper* if the map $(s, t): \mathbb{G}_1 \rightarrow \mathbb{G}_0 \times \mathbb{G}_0$ is proper. We say that $\mathbb{G}$ is a *foliation groupoid* if $\mathbb{G}_x$ is discrete for all $x \in \mathbb{G}_0$.

Lemma 2.7 *Properness and being a foliation groupoid are invariant under essential equivalences.*

Proof It follows immediately from the fully faithfulness that essential equivalences induce isomorphism on stabilizer groups. This proves that being a foliation groupoid is invariant under essential equivalence. The statement for properness is proven in [24, Proposition 5.26]. $\square$

Definition 2.8 (orbifold) An *orbifold* is a proper foliation groupoid. We write **Orbld** for the full subcategory of orbifolds in **LieGrpd**.

Remark 2.9 Any essential equivalence between proper Lie groupoids is an essential equivalence of underlying topological groupoids. The essentially surjectiveness condition holds true because any submersions admits local sections. Let P denote the topological pullback of the diagram in the fully faithfulness condition. The induced map $\mathbb{G}_1 \rightarrow P$ is bijective and continuous, so we need to check that its inverse is continuous, too. We may check this locally at a point x in the codomain P . Let K be a compact neighborhood of the image of x under the structure map $P \rightarrow \mathbb{G}_0 \times \mathbb{G}_0$. The restriction of the map $\mathbb{G}_1 \rightarrow P$ to the preimage of K under $(s, t): \mathbb{G}_1 \rightarrow \mathbb{G}_0 \times \mathbb{G}_0$ and the preimage of K under the structure map $P \rightarrow \mathbb{G}_0 \times \mathbb{G}_0$ is a bijective map from a compact space to a Hausdorff space and hence a homeomorphism.

Example 2.10 (global quotient orbifold) Let M be a manifold together with an almost free action of a compact Lie group G . The action groupoid $G \ltimes M$ is an orbifold and is called a *global quotient orbifold*.

In fact, every orbifold arises as such a global quotient orbifold, up to essential equivalence.

Theorem 2.11 [27, Corollary 1.3] *Every orbifold $\mathbb{G}$ is essentially equivalent to $G \ltimes M$ for some manifold M together with an almost free action of a compact Lie group G on M .*

For $\mathbb{G}$ a topological groupoid, we write $\|\mathbb{G}\|$ for the fat geometric realization of the nerve of $\mathbb{G}$ as defined in [35, Appendix A]. The fat geometric realization has a better homotopical behavior than the usual geometric realization, ie we have the following:

Proposition 2.12 *Let $f: \mathbb{G} \rightarrow \mathbb{H}$ be an essential equivalence of topological groupoids. Then the induced map $\|f\|: \|\mathbb{G}\| \rightarrow \|\mathbb{H}\|$ is a weak equivalence.*

Proof This is mentioned as a corollary of [26, Theorem 1.1]. $\square$

However, in contrast to the usual geometric realization, the fat geometric realization does not commute with products on the nose, but it does up to weak equivalence:

Proposition 2.13 *Let $\mathbb{G}$ and $\mathbb{H}$ be two topological groupoids. The natural map*

$$\|\mathbb{G} \times \mathbb{H}\| \rightarrow \|\mathbb{G}\| \times \|\mathbb{H}\|$$

is a weak equivalence and admits a retract. These retracts together with the inclusion of the 0-simplex $* \rightarrow \|\ast\|$ assemble into a lax symmetric monoidal structure on the functor

$$\|-\|: \mathbf{TGrpd} \rightarrow \mathbf{T}$$

with respect to the cartesian monoidal structure on both sides.

Proof The above map is a weak equivalence by [35, Proposition A.1(iii)]. The nerve functor commutes with products and therefore is strong symmetric monoidal. The claim about the lax symmetric monoidality of the fat geometric realization is proven in [11, Appendix B]. $\square$

Construction 2.14 Recall that the category of topological groupoids is enriched over itself. Applying the above lax symmetric monoidal functor to the topological groupoids $\mathbb{Map}(\mathbb{H}, \mathbb{G})$, we obtain a topological enrichment on the category $\mathbf{TGrpd}$ with mapping spaces

$$\mathbf{TGrpd}(\mathbb{H}, \mathbb{G}) = \|\mathbb{Map}(\mathbb{H}, \mathbb{G})\|.$$

Similarly, we can change enrichment of the category of orbifolds from topological groupoids to topological spaces.

The category of orbifolds is usually defined as the categorical localization of the category of proper foliation groupoids at the essential equivalences. There are several explicit construction for this. A brief explanation using a calculus of fractions can be found in [23, Section 2]. An explanation how to invert those morphisms in a 2-categorical way can be found in [28]; see [18] for a comparison of different approaches. We will eventually consider the higher categorical version of this construction and define the ∞ -category of orbifolds $\mathbf{Orbld}_\infty$ as the ∞ -categorical localization of the homotopy coherent nerve of the category of orbifolds at the essential equivalences.

2.2 From orbifolds to Orb-spaces

We recall how an orbifold gives rise to an **Orb**-space, a particular model for the homotopy theory of orbispaces due to Gepner and Henriques [13, Section 4]. The main result of this section is Corollary 2.23, saying that the resulting functor $F: \mathbf{Orbld} \rightarrow \mathbf{TOrb}$ sends essential equivalences of orbifolds to weak equivalences of **Orb**-spaces. An immediate corollary is that the functor F extends to the ∞ -categorical localizations and yields a functor

$$|-|_{\text{gl}}: \mathbf{Orbld}_\infty \rightarrow (\mathbf{TOrb})_\infty$$

from the ∞ -category of orbifolds to the ∞ -category of **Orb**-spaces.

Definition 2.15 The topological category **Orb** is the full topological subcategory of **TGrpd** spanned by the topological groupoid $\mathbb{B}G$ for all compact Lie groups G . An **Orb-space** is a continuous functor $X : \mathbf{Orb}^{\text{op}} \rightarrow \mathbf{T}$. A *morphism* of **Orb**-spaces is a continuous natural transformation. A map between **Orb**-spaces is called a weak equivalence if it is levelwise a weak equivalence. We write $\mathbf{TOrb}$ for the category of **Orb**-spaces.

Remark 2.16 If G, H are compact Lie groups, then the morphism space $\mathbf{Orb}(H, G) = \|\text{Map}(\mathbb{B}H, \mathbb{B}G)\|$ is homeomorphic to $\text{hom}(H, G) \times_G EG$, the homotopy orbit space of the conjugation action of the group G on the space $\text{hom}(H, G)$ of continuous homomorphisms; compare [13, Remark 4.5].

Now we recall how to associate an **Orb**-space to an orbifold. This definition is taken from [13, Section 4.2].

Construction 2.17 We define the topological functor

$$Y : \mathbf{TGrpd} \rightarrow \mathbf{TOrb}, \quad \mathbb{G} \mapsto (K \mapsto \|\text{map}(\mathbb{B}K, \mathbb{G})\|),$$

as the restricted topological Yoneda embedding and

$$F : \mathbf{Orbfld} \rightarrow \mathbf{TOrb}$$

for the composite of the forgetful functor $\mathbf{Orbfld} \rightarrow \mathbf{TGrpd}$ and Y .

A key result for our purposes is that the functor $Y : \mathbf{TGrpd} \rightarrow \mathbf{TOrb}$ sends essential equivalences of topological groupoids to weak equivalences of **Orb**-spaces. This follows from the following:

Proposition 2.18 *Let K be a topological group. Let $f : \mathbb{G} \rightarrow \mathbb{H}$ be an essential equivalence of topological groupoids. Then the induced map of groupoids*

$$f_* : \text{Map}(\mathbb{B}K, \mathbb{G}) \rightarrow \text{Map}(\mathbb{B}K, \mathbb{H})$$

is again an essential equivalence.

The proof splits up into several lemmas. Let

$$p_{\mathbb{G}} : \text{Map}(\mathbb{B}K, \mathbb{G}) \rightarrow \text{Map}(\mathbb{B}e, \mathbb{G}) \cong \mathbb{G}$$

be induced by the unique morphism $\mathbb{B}e \rightarrow \mathbb{B}K$ where e denotes the trivial subgroup of $\mathcal{L}$.

Lemma 2.19 *Let K be a topological group and let $f : \mathbb{G} \rightarrow \mathbb{H}$ be an essential equivalence of topological groupoids. Then the diagram*

$$\begin{array}{ccc} \text{Map}(\mathbb{B}K, \mathbb{G})_0 & \xrightarrow{(p_{\mathbb{G}})_0} & \mathbb{G}_0 \\ (f_*)_0 \downarrow & & \downarrow f_0 \\ \text{Map}(\mathbb{B}K, \mathbb{H})_0 & \xrightarrow{(p_{\mathbb{H}})_0} & \mathbb{H}_0 \end{array}$$

is a pullback.

Proof Recall that $\text{Map}(\mathbb{B}K, \mathbb{H})_0$ is topologized as a subset of $\text{map}(K, \mathbb{H}_1)$. Evaluation at K hence induces a map $\text{Map}(\mathbb{B}K, \mathbb{H})_0 \times K \rightarrow \mathbb{H}_1$. One can check on elements that the following diagram commutes:

$$\begin{array}{ccccc} \text{Map}(\mathbb{B}K, \mathbb{H})_0 \times K & \xrightarrow{(p_{\mathbb{G}})_0 \circ \text{pr}_1} & \mathbb{H}_0 & \xleftarrow{f_0} & \mathbb{G}_0 \\ \downarrow & & \downarrow \Delta_{\mathbb{H}_0} & & \downarrow \Delta_{\mathbb{G}_0} \\ \mathbb{H}_1 & \xrightarrow{(s_{\mathbb{H}}, t_{\mathbb{H}})} & \mathbb{H}_0 \times \mathbb{H}_0 & \xleftarrow{(f_0, f_0)} & \mathbb{G}_0 \times \mathbb{G}_0 \end{array}$$

Using that

$$(\text{Map}(\mathbb{B}K, \mathbb{H})_0 \times K) \times_{\mathbb{H}_0} \mathbb{G}_0 \cong (\text{Map}(\mathbb{B}K, \mathbb{H})_0 \times_{\mathbb{H}_0} \mathbb{G}_0) \times K,$$

we obtain an induced map on pullbacks

$$(\text{Map}(\mathbb{B}K, \mathbb{H})_0 \times_{\mathbb{H}_0} \mathbb{G}_0) \times K \rightarrow \mathbb{H}_1 \times_{\mathbb{H}_0 \times \mathbb{H}_0} (\mathbb{G}_0 \times \mathbb{G}_0) \cong \mathbb{G}_1,$$

where the last isomorphism uses that f is an essential equivalence. One can now check on elements that the adjoint map

$$\text{Map}(\mathbb{B}K, \mathbb{H})_0 \times_{\mathbb{H}_0} \mathbb{G}_0 \rightarrow \text{map}(K, \mathbb{G}_1)$$

takes values in $\text{Map}(\mathbb{B}K, \mathbb{G})_0$ and that this map is inverse to the map

$$\text{Map}(\mathbb{B}K, \mathbb{G})_0 \rightarrow \text{Map}(\mathbb{B}K, \mathbb{H})_0 \times_{\mathbb{H}_0} \mathbb{G}_0$$

induced by $(f_*)_0$ and $(p_{\mathbb{G}})_0$. □

We write $s_{\mathbb{B}K, \mathbb{G}}$ and $t_{\mathbb{B}K, \mathbb{G}}$ for the source and target map of the topological groupoid $\text{Map}(\mathbb{B}K, \mathbb{G})$.

Lemma 2.20 *Let K be a topological group and let $\mathbb{G}$ be a topological groupoid. The diagram*

$$\begin{array}{ccc} \text{Map}(\mathbb{B}K, \mathbb{G})_1 & \xrightarrow{s_{\mathbb{B}K, \mathbb{G}}} & \text{Map}(\mathbb{B}K, \mathbb{G})_0 \\ (p_{\mathbb{G}})_1 \downarrow & & \downarrow (p_{\mathbb{G}})_0 \\ \mathbb{G}_1 & \xrightarrow{s_{\mathbb{G}}} & \mathbb{G}_0 \end{array}$$

is a pullback.

Proof Similarly as in the proof of Lemma 2.19, evaluation at an element of K and diagonal maps induce a map

$$\text{Map}(\mathbb{B}K, \mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1 \times K \rightarrow \mathbb{G}_1 \times_{\mathbb{G}_0 \times \mathbb{G}_0} (\mathbb{G}_1 \times \mathbb{G}_1).$$

There is a homeomorphism

$$\mathbb{G}_1 \times_{\mathbb{G}_0 \times \mathbb{G}_0} (\mathbb{G}_1 \times \mathbb{G}_1) \cong \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1 \times_{\mathbb{G}_0} \mathbb{G}_1$$

sending $(g_1, (g_2, g_3))$ to (g_3, g_1, g_2^{-1}) . Composition in $\mathbb{G}$ hence induces a map

$$\text{Map}(\mathbb{B}K, \mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1 \times K \rightarrow \mathbb{G}_1.$$

Consider the map

$$\mathrm{Map}(\mathbb{B}K, \mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1 \rightarrow \mathrm{map}(K, \mathbb{G}_1) \times \mathrm{map}(K, \mathbb{G}_1) \times \mathrm{map}(*, \mathbb{G}_1),$$

which is induced by the inclusion $\mathrm{Map}(\mathbb{B}K, \mathbb{G})_0 \hookrightarrow \mathrm{map}(K, \mathbb{G}_1)$, the adjoint of the above map and the homeomorphism $\mathbb{G}_1 \cong \mathrm{map}(*, \mathbb{G}_1)$. One can check on elements that this map takes values in $\mathrm{Map}(\mathbb{B}K, \mathbb{G})_1$ and that it is inverse to the map from $\mathrm{Map}(\mathbb{B}K, \mathbb{G})_1$ to the pullback $\mathrm{Map}(\mathbb{B}K, \mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1$ induced by $s_{\mathbb{B}K, \mathbb{G}}$ and $(p_{\mathbb{G}})_1$. $\square$

Remark 2.21 Similarly one shows that the diagram in Lemma 2.20 is a pullback when replacing the source maps with target maps.

Proof of Proposition 2.18 We first prove that f_* is essentially surjective as in Definition 2.4 of essential equivalences. Let

$$P = \mathrm{Map}(\mathbb{B}K, \mathbb{H})_1 \times_{\mathrm{Map}(\mathbb{B}K, \mathbb{H})_0} \mathrm{Map}(\mathbb{B}K, \mathbb{G})_0$$

denote the pullback along $s_{\mathbb{B}K, \mathbb{H}}$ and $(f_*)_0$. Consider the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Map}(\mathbb{B}K, \mathbb{G})_0 & \xrightarrow{(p_{\mathbb{G}})_0} & \mathbb{G}_0 \\ & \nearrow & \downarrow & & \nearrow \\ P & \xrightarrow{\varphi} & \mathbb{H}_1 \times_{\mathbb{H}_0} \mathbb{G}_0 & & \mathbb{G}_0 \\ & \searrow & \downarrow (f_*)_0 & & \downarrow f_0 \\ & & \mathrm{Map}(\mathbb{B}K, \mathbb{H})_0 & \xrightarrow{(p_{\mathbb{H}})_0} & \mathbb{H}_0 \\ & \nearrow s_{\mathbb{B}K, \mathbb{H}} & \downarrow & & \nearrow s_{\mathbb{H}} \\ \mathrm{Map}(\mathbb{B}K, \mathbb{H})_1 & \xrightarrow{(p_{\mathbb{H}})_1} & \mathbb{H}_1 & & \mathbb{H}_0 \end{array}$$

where the unlabeled maps out of the pullbacks are the structure maps and φ is the map induced by the functoriality of pullbacks.

The squares on the left and on the right are pullbacks by definition. The square in the back also is a pullback by Lemma 2.19. Hence, the square in the front is a pullback. Consider the following commutative diagram:

$$\begin{array}{ccccc} P & \xrightarrow{\quad} & \mathrm{Map}(\mathbb{B}K, \mathbb{H})_1 & \xrightarrow{t_{\mathbb{B}K, \mathbb{H}}} & \mathrm{Map}(\mathbb{B}K, \mathbb{H})_0 \\ \downarrow \varphi & & \downarrow (p_{\mathbb{H}})_1 & & \downarrow (p_{\mathbb{H}})_0 \\ \mathbb{H}_1 \times_{\mathbb{H}_0} \mathbb{G}_0 & \xrightarrow{\quad} & \mathbb{H}_1 & \xrightarrow{t_{\mathbb{H}}} & \mathbb{H}_0 \end{array}$$

We have just shown that the left square is a pullback and the right square is a pullback by Remark 2.21. Therefore, the outer square also is a pullback. The composition of the lower maps is a surjection admitting

local sections by assumption on f . Now we can conclude that the composition of the maps in the top row also is a surjection that admits local sections.

To see that f_* is fully faithful as in Definition 2.4, consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \text{Map}(\mathbb{B}K, \mathbb{G})_0 & \xrightarrow{(f_*)_0} & \text{Map}(\mathbb{B}K, \mathbb{H})_0 \\
 & \nearrow s_{\mathbb{B}K, \mathbb{G}} & \downarrow & & \nearrow s_{\mathbb{B}K, \mathbb{H}} \\
 \text{Map}(\mathbb{B}K, \mathbb{G})_1 & \xrightarrow{(f_*)_1} & \text{Map}(\mathbb{B}K, \mathbb{H})_1 & & \\
 \downarrow (p_{\mathbb{G}})_1 & & \downarrow (p_{\mathbb{H}})_1 & & \downarrow (p_{\mathbb{H}})_0 \\
 \mathbb{G}_1 & \xrightarrow{f_1} & \mathbb{H}_1 & & \mathbb{H}_0 \\
 & \nearrow s_{\mathbb{G}} & \downarrow & & \nearrow s_{\mathbb{H}} \\
 & & \mathbb{G}_0 & \xrightarrow{f_0} & \mathbb{H}_0
 \end{array}$$

The square in the back is a pullback by Lemma 2.19. The squares on the left and on the right are pullbacks by Lemma 2.20. Therefore, the square in the front is also a pullback. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbb{G}_1 & \xrightarrow{f_1} & \mathbb{H}_1 \\
 & \nearrow (p_{\mathbb{G}})_1 & \downarrow & & \nearrow (p_{\mathbb{H}})_1 \\
 \text{Map}(\mathbb{B}K, \mathbb{G})_1 & \xrightarrow{(f_*)_1} & \text{Map}(\mathbb{B}K, \mathbb{H})_1 & & \\
 \downarrow (s_{\mathbb{B}K, \mathbb{G}}, t_{\mathbb{B}K, \mathbb{G}}) & & \downarrow (s_{\mathbb{B}K, \mathbb{H}}, t_{\mathbb{B}K, \mathbb{H}}) & & \downarrow (s_{\mathbb{H}}, t_{\mathbb{H}}) \\
 & \nearrow ((p_{\mathbb{G}})_0)^2 & (\mathbb{G}_0)^2 & \xrightarrow{(f_0)^2} & (\mathbb{H}_0)^2 \\
 & & \downarrow & & \nearrow ((p_{\mathbb{H}})_0)^2 \\
 (\text{Map}(\mathbb{B}K, \mathbb{G})_0)^2 & \xrightarrow{((f_*)_0)^2} & (\text{Map}(\mathbb{B}K, \mathbb{H})_0)^2 & &
 \end{array}$$

The square in the back is a pullback by assumption on f . The lower square is a pullback by Lemma 2.19. We have just shown that the square on the top is also a pullback. All in all, we can conclude that the square in the front also is a pullback. $\square$

Propositions 2.18 and 2.12 imply that the induced map

$$\|\text{Map}(\mathbb{B}K, \mathbb{G})\| \rightarrow \|\text{Map}(\mathbb{B}K, \mathbb{H})\|$$

is a weak equivalence for every essential equivalence $f: \mathbb{G} \rightarrow \mathbb{H}$. We hence proved the following:

Corollary 2.22 *The functor*

$$Y : \mathbf{TGrpd} \rightarrow \mathbf{TOrb}$$

*from Construction 2.17 sends essential equivalences of topological groupoids to weak equivalences of **Orb**-spaces.*

Corollary 2.23 *The functor*

$$F : \mathbf{Orb fld} \rightarrow \mathbf{TOrb}$$

*from Construction 2.17 sends essential equivalences of orbifolds to weak equivalences of **Orb**-spaces.*

Definition 2.24 Because the functor $F : \mathbf{Orb fld} \rightarrow \mathbf{TOrb}$ takes essential equivalences of orbifolds to weak equivalences of **Orb**-spaces, it extends in an essentially unique way to a functor of the ∞ -categorical realizations and yields a functor

$$(2-1) \quad |-\|_{\text{gl}} : \mathbf{Orb fld}_{\infty} \rightarrow (\mathbf{TOrb})_{\infty},$$

we refer to $|M|_{\text{gl}}$ as the *global realization* of the orbifold M .

The following theorem summarizes the main properties of the global realization functor. For $\widetilde{U}$ an open subset of the object manifold $\mathbb{G}_0$ of a Lie groupoid $\mathbb{G}$, we define the *restricted groupoid* $\mathbb{G}|_{\widetilde{U}}$ of $\mathbb{G}$ to $\widetilde{U}$ as the Lie groupoid with object manifold $\widetilde{U}$ and arrow manifold $(s, t)^{-1}(\widetilde{U})$. The structure maps are given by restricting the corresponding maps of $\mathbb{G}$. We moreover write

$$\mathbb{G}_0/\mathbb{G}_1 = \text{coeq}(s, t : \mathbb{G}_1 \rightrightarrows \mathbb{G}_0)$$

for the *orbit space* of $\mathbb{G}$.

Theorem 2.25 (i) *The global realization functor (2-1) takes countable disjoint unions of orbifolds to coproducts of **Orb**-spaces.*

(ii) *Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\widetilde{U}$ and $\widetilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \rightarrow \mathbb{G}_0/\mathbb{G}_1$. Then the square*

$$(2-2) \quad \begin{array}{ccc} |\mathbb{G}|_{\widetilde{U} \cap \widetilde{V}}|_{\text{gl}} & \longrightarrow & |\mathbb{G}|_{\widetilde{U}}|_{\text{gl}} \\ \downarrow & & \downarrow \\ |\mathbb{G}|_{\widetilde{V}}|_{\text{gl}} & \longrightarrow & |\mathbb{G}|_{\text{gl}} \end{array}$$

*is a pushout in the ∞ -category of **Orb**-spaces, where all morphisms are induced by the respective inclusions.*

(iii) *For every orbifold $\mathbb{G}$, the underlying space of the global realization $|\mathbb{G}|_{\text{gl}}$ is weakly equivalent to $\|\mathbb{G}\|$.*

Proof (i) The functor $F : \mathbf{Orbfld} \rightarrow \mathbf{T}_{\mathbf{Orb}}$ preserves coproducts because they can be computed levelwise and $\|\mathbb{M}\text{ap}(\mathbb{B}K, -)\| : \mathbf{Orbfld} \rightarrow \mathbf{T}$ preserves coproducts for any universal subgroup K of $\mathcal{L}$ by a similar reasoning as for pushouts in the next part of the proof. Disjoint union of $\mathbf{Orb}$ -spaces preserves weak equivalences, so coproducts in the 1-category $\mathbf{T}_{\mathbf{Orb}}$ represent coproducts in the ∞ -category $\mathbf{Orb}$ -spaces.

(ii) We first check that for every compact Lie group K , the commutative diagram of orbifolds

$$(2-3) \quad \begin{array}{ccc} \mathbb{G}|_{\tilde{U} \cap \tilde{V}} & \xrightarrow{j_U} & \mathbb{G}|_{\tilde{U}} \\ j_V \downarrow & & \downarrow i_U \\ \mathbb{G}|_{\tilde{V}} & \xrightarrow{i_V} & \mathbb{G} \end{array}$$

is sent to a pushout square in $\mathbf{T}$ under the functor $\|\mathbb{M}\text{ap}(\mathbb{B}K, -)\| : \mathbf{Orbfld} \rightarrow \mathbf{T}$. Fat geometric realization is a left adjoint and therefore commutes with pushouts. We can hence also check that the functor which assigns the n^{th} space of the nerve of $\mathbb{M}\text{ap}(\mathbb{B}K, \mathbb{G})$ to a topological groupoid $\mathbb{G}$ sends (2-3) to a pushout. It can be checked on elements that this is true for the underlying sets. It is also a pushout in $\mathbf{T}$ because all maps are sent to inclusions of open subsets.

The natural map from the homotopy pushout to a pushout along open inclusions is a weak equivalence; see [9, Corollary 1.6]. This applies to the above situation. The functor F from Construction 2.17 hence takes (2-3) to a levelwise homotopy pushout in $\mathbf{T}_{\mathbf{Orb}}$.

We are now going to prove that a levelwise homotopy pushout in $\mathbf{T}_{\mathbf{Orb}}$ is a homotopy pushout. Let P denote the diagram $(\cdot \leftarrow \cdot \rightarrow \cdot)$. Pushouts can be computed levelwise. The homotopy pushout functor in $\mathbf{T}_{\mathbf{Orb}}$ therefore is the left derived functor of

$$(\text{colim}_P)^{\mathbf{Orb}} : [P, \mathbf{T}_{\mathbf{Orb}}] \cong [\mathbf{Orb}, [P, \mathbf{T}]] \rightarrow [\mathbf{Orb}, \mathbf{T}] = \mathbf{T}_{\mathbf{Orb}}.$$

It follows from general model category theory that the homotopy pushout can be computed as the pushout of a cofibrant replacement in the projective model structure on $[P, \mathbf{T}_{\mathbf{Orb}}]$; compare [14, Corollary 5.1.6]. We claim that a cofibrant diagram in $[P, \mathbf{T}_{\mathbf{Orb}}] \cong [\mathbf{Orb}, [P, \mathbf{T}]]$ levelwise consists of diagrams in $[P, \mathbf{T}]$ for which the natural map from the homotopy pushout to the pushout is a weak equivalence. This is the case for diagrams in $[P, \mathbf{T}]$ where both arrows of P are mapped to Hurewicz cofibrations; see eg [9, Theorem A.7]. As can be seen from the characterization given in [14, Theorem 5.1.3], a cofibrant object in $[P, \mathbf{T}_{\mathbf{Orb}}]$ maps the two arrows of P to cofibrations between $\mathbf{Orb}$ -spaces and these cofibrations are levelwise Hurewicz cofibrations by [16, Proposition 2.2(iv)].

(iii) Let e denote the trivial subgroup of $\mathcal{L}$. Then the groupoid $\mathbb{M}\text{ap}(\mathbb{B}e, \mathbb{G})$ is isomorphic to $\mathbb{G}$, and so the value of the $\mathbf{Orb}$ -space $|\mathbb{G}|_{\mathbb{G}}$ at the trivial group is isomorphic to $\|\mathbb{G}\|$. $\square$

3 A comparison of global quotient constructions

There are different models for the ∞ -category of orbispaces. In this section we recall a result by Körschgen [17] establishing a Quillen equivalence between the **Orb**-space model of Gepner and Henriques [13] and the $\mathcal{L}$ -space model of Schwede [34] as well as a result by Schwede [34] establishing a Quillen equivalence between the latter and the orthogonal space model from [33].

Each of these models comes with a specific *global quotient functor* with source the category of G -spaces. The goal of this section is to prove that the above Quillen equivalences are compatible with these global quotient constructions.

3.1 Orb-spaces versus $\mathbf{O}_{\text{gl}}$ -spaces

We recall the definition of the topological monoid $\mathcal{L}$, which contains every compact Lie group as a subgroup. A space with an action of $\mathcal{L}$ then gives rise to a G -space for an arbitrary compact Lie group G by restricting the action of $\mathcal{L}$ to G .

The following is mainly taken from [34]. We require all representations to be orthogonal representations on inner product spaces.

Definition 3.1 ($\mathcal{L}$ -space) Let $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ be the space of linear isometric self-embeddings of $\mathbb{R}^\infty$ together with the standard inner product that carries the compactly generated function space topology as described in [34, Section 1]. The space $\mathcal{L}$ together with the composition of self-embeddings is a topological monoid.

An $\mathcal{L}$ -space A is a space together with a continuous action of the topological monoid $\mathcal{L}$. A *morphism* of $\mathcal{L}$ -spaces is an $\mathcal{L}$ -equivariant continuous map of the underlying spaces. We write $\mathcal{L}\text{-}\mathbf{T}$ for the category of $\mathcal{L}$ -spaces.

We let G be a compact Lie group. We recall that a *complete G -universe* is a G -representation of countably infinite dimension into which every finite-dimensional G -representation embeds by an equivariant linear isometry.

Definition 3.2 (universal subgroup) A subgroup G of $\mathcal{L}$ is *universal* if it is compact, admits the structure of a Lie group, and the tautological G -action through the inclusion $G \hookrightarrow \mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ makes $\mathbb{R}^\infty$ a complete G -universe.

By [34, Proposition 1.5], every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$. Moreover, every isomorphism between universal subgroups of $\mathcal{L}$ is given by conjugation with an invertible element of $\mathcal{L}$.

Definition 3.3 ($\mathbf{O}_{\text{gl}}$ -spaces and $\mathbf{Orb}'$ -spaces) We recall two topological categories $\mathbf{O}_{\text{gl}}$ and $\mathbf{Orb}'$ with the same objects, namely the universal subgroups of the linear isometries monoid $\mathcal{L}$. In $\mathbf{Orb}$, the space of morphisms between two universal subgroups G and H of $\mathcal{L}$ is given by

$$\mathbf{O}_{\text{gl}}(H, G) = \text{map}^{\mathcal{L}}(\mathcal{L}/H, \mathcal{L}/G).$$

The composition in $\mathbf{O}_{\text{gl}}$ is composition of $\mathcal{L}$ -equivariant maps.

To define the morphism spaces in $\mathbf{Orb}'$ we consider the space

$$\tilde{E}(H, G) = \{(\alpha, \varphi) \in \text{hom}(H, G) \times \mathcal{L} \mid h\varphi = \varphi\alpha(h) \text{ for all } h \in H\}.$$

The group G acts on $\tilde{E}(H, G)$ via $g.(\alpha, \varphi) = (\alpha^g, \varphi g^{-1})$ where $(\alpha^g)(h) = g\alpha(h)g^{-1}$ for $h \in H$. The morphism space is then given by the homotopy orbit space

$$\mathbf{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG.$$

The composition is defined in [17, Section 3.1].

In much the same way as for the indexing category $\mathbf{Orb}$, and $\mathbf{O}_{\text{gl}}$ -space is a contravariant continuous functor from $\mathbf{O}_{\text{gl}}$ to the category $\mathbf{T}$ of spaces; and an $\mathbf{Orb}'$ -space is a contravariant continuous functor from $\mathbf{Orb}'$ to $\mathbf{T}$. Morphisms of $\mathbf{O}_{\text{gl}}$ -spaces and $\mathbf{Orb}'$ -spaces are natural transformations. We write $\mathbf{T}_{\mathbf{O}_{\text{gl}}}$ and $\mathbf{T}_{\mathbf{Orb}'}$ for the categories of $\mathbf{O}_{\text{gl}}$ -spaces and $\mathbf{Orb}'$ -spaces, respectively.

Construction 3.4 K\"orschen [17] provided a zigzag of Quillen equivalences relating the category $\mathbf{T}_{\mathbf{O}_{\text{gl}}}$ and $\mathbf{T}_{\mathbf{Orb}}$ through the category $\mathbf{T}_{\mathbf{Orb}'}$, each equipped with the projective model structures. The right Quillen functors are restriction functors along continuous functors $f_1: \mathbf{Orb}' \rightarrow \mathbf{Orb}$ and $f_2: \mathbf{Orb}' \rightarrow \mathbf{O}_{\text{gl}}$ between the indexing categories. Both functors are the identity on objects. On morphism space, the first functor

$$f_1(H, G): \mathbf{Orb}'(H, G) = \tilde{E}(H, G) \times_G EG \rightarrow \text{map}(H, G) \times_G EG \cong \mathbf{Orb}(H, G)$$

arises from the projection $\tilde{E}(H, G) \rightarrow \text{map}(H, G)$ by applying the homotopy orbit construction. The behavior of the second functor on morphism spaces is

$$f_2(H, G): \tilde{E}(H, G) \times_G EG \rightarrow \text{map}^{\mathcal{L}}(\mathcal{L}/H, \mathcal{L}/G) = \mathbf{O}_{\text{gl}}(H, G), \quad [\alpha, \varphi] \mapsto \varphi^b,$$

where $\varphi^b: \mathcal{L}/H \rightarrow \mathcal{L}/G$ is defined by $\varphi^b(fH) = f\varphi G$.

K\"orschen proved in [17, Propositions 2.18 and 2.22] that all the maps $f_1(H, G)$ and $f_2(H, G)$ are weak equivalences. So in particular, both functors $f_1: \mathbf{Orb}' \rightarrow \mathbf{Orb}$ and $f_2: \mathbf{Orb}' \rightarrow \mathbf{O}_{\text{gl}}$ are Dwyer–Kan equivalences of topologically enriched categories; see [17, Corollary 3.13]. Consequently, the restriction functors

$$f_1^*: \mathbf{T}_{\mathbf{Orb}} \rightarrow \mathbf{T}_{\mathbf{Orb}'} \quad \text{and} \quad f_2^*: \mathbf{T}_{\mathbf{O}_{\text{gl}}} \rightarrow \mathbf{T}_{\mathbf{Orb}'}$$

are fully homotopical and right Quillen functors for the projective model structure; see [13, Lemma A.6] or [16, Theorem 3.5]. So the induced functors of localizations are equivalences of ∞ -categories.

3.2 $\mathbf{O}_{\text{gl}}$ -spaces versus orthogonal spaces

In this section we recall the ∞ -categorical essence of Schwede's paper [34], in the form of an equivalence between the ∞ -categories of $\mathbf{O}_{\text{gl}}$ -spaces and global spaces.

Global spaces are the ∞ -category underlying the global model structure on the category of orthogonal spaces [33].

Definition 3.5 (orthogonal spaces) The topological category $\mathbf{L}$ has all finite-dimensional real inner product spaces as its object. The morphism space $\mathbf{L}(V, W)$ between two inner product spaces is the space of linear isometric maps between V and W topologized as a Stiefel manifold. An *orthogonal space* is a continuous functor from $\mathbf{L}$ to the category $\mathbf{T}$ of spaces; *morphisms* of orthogonal spaces are the natural transformations. We write $\mathbf{Spc}$ for the category of orthogonal spaces and morphisms between them.

In [33, Definition 1.1.2], Schwede introduced the notion of *global equivalence* for morphisms between orthogonal spaces. This notion is much finer than the nonequivariant notion of equivalence; orthogonal spaces encode compatible G -equivariant homotopy types for all compact Lie groups that are invariant under global equivalences. The global equivalences are part of a model structure on the category of orthogonal spaces, the *global model structure* of [33, Proposition 1.2.21].

Now we recall that and how the ∞ -category of global spaces is equivalent to the ∞ -category of $\mathbf{O}_{\text{gl}}$ -spaces. We define the value of an orthogonal space X at $\mathbb{R}^\infty$ by

$$X(\mathbb{R}^\infty) = \operatorname{colim}_{V \in s(\mathbb{R}^\infty)} X(V),$$

where the colimit is taken over the poset $s(\mathbb{R}^\infty)$ of finite-dimensional subspaces of $\mathbb{R}^\infty$. The functoriality of X extends by colimits to a continuous action of $\mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) = \mathcal{L}$ on $X(\mathbb{R}^\infty)$; see [34, Conjecture 3.2].

Construction 3.6 We introduce a functor $\Psi: \mathbf{Spc} \rightarrow \mathbf{T}_{\mathbf{O}_{\text{gl}}}$ from the category of orthogonal spaces to the category of $\mathbf{O}_{\text{gl}}$ -spaces. We let X be an orthogonal space. The value of the $\mathbf{O}_{\text{gl}}$ -space $\Psi(X)$ at a universal subgroup G of the linear isometries monoid $\mathcal{L}$ is

$$\Psi(X)(G) = X(\mathbb{R}^\infty)^G.$$

An orthogonal space is *closed* if all of its structure maps are closed embeddings. It is shown in [33, Proposition 1.1.17] that the functor $\Psi: \mathbf{Spc} \rightarrow \mathbf{T}_{\mathbf{O}_{\text{gl}}}$ takes global equivalences between closed orthogonal spaces to weak equivalences of $\mathbf{O}_{\text{gl}}$ -spaces. The following result is the ∞ -categorical essence of [34].

Theorem 3.7 The functor $\Psi: \mathbf{Spc} \rightarrow \mathbf{T}_{\mathbf{O}_{\text{gl}}}$ admits a left derived functor

$$\Psi_\infty: \mathbf{Spc}_\infty \rightarrow (\mathbf{T}_{\mathbf{O}_{\text{gl}}})_\infty$$

that is an equivalence of ∞ -categories.

Proof In [34, Theorem 3.9] Schwede constructed a left Quillen equivalence $L: \mathbf{Spc} \rightarrow \mathcal{L}\text{-}\mathbf{T}$ between the global model structure on $\mathcal{L}\text{-}\mathbf{T}$ and the positive global model structure on $\mathcal{L}\text{-}\mathbf{T}$. This functor hence derives to an equivalence of ∞ -categories

$$\mathbf{Spc}_\infty \simeq (\mathcal{L}\text{-}\mathbf{T})_\infty.$$

The result [34, Theorem 2.5] exhibits another Quillen equivalence between the global model structure on $\mathcal{L}\text{-}\mathbf{T}$ and the projective model structure on the category of $\mathbf{O}_{\text{gl}}$ -spaces. The right adjoint $\Phi: \mathcal{L}\text{-}\mathbf{T} \rightarrow \mathbf{T}_{\mathbf{O}_{\text{gl}}}$ of this Quillen equivalence is “of Elmendorf type” in that it records the fixed points for all universal subgroups of $\mathcal{L}$ and the natural maps between these. The right adjoint preserves all equivalences, so it extends to an equivalence of ∞ -categories

$$(\mathcal{L}\text{-}\mathbf{T})_\infty \simeq (\mathbf{T}_{\mathbf{O}_{\text{gl}}})_\infty.$$

In combination we obtain an equivalence of ∞ -categories from $\mathbf{Spc}_\infty$ to $(\mathbf{T}_{\mathbf{O}_{\text{gl}}})_\infty$ that is left derived from the functor $\Phi \circ L: \mathbf{Spc} \rightarrow \mathbf{T}_{\mathbf{O}_{\text{gl}}}$. Finally, [34, Proposition 3.7] provides a natural morphism of $\mathcal{L}$ -spaces

$$\xi(X): L(X) \rightarrow X(\mathbb{R}^\infty),$$

which is a global equivalence if X is cofibrant in the global model structure on orthogonal spaces. So

$$\Phi(\xi(X)): \Phi(L(X)) \rightarrow \Phi(X(\mathbb{R}^\infty)) = \Psi(X)$$

is natural morphism of $\mathbf{O}_{\text{gl}}$ -spaces that is a weak equivalence whenever X is cofibrant. Since the left derived functor of $\Phi \circ L$ is an equivalence of ∞ -categories, so is the left derived functor of Ψ . $\square$

Summarizing the above discussion, there is a chain of equivalences of ∞ -categories:

$$(\mathbf{T}_{\text{Orb}})_\infty \xrightarrow{f_1^*} (\mathbf{T}_{\text{Orb}'})_\infty \xleftarrow{f_2^*} (\mathbf{T}_{\mathbf{O}_{\text{gl}}})_\infty \xleftarrow{\Psi} \mathbf{Spc}_\infty.$$

We denote the resulting composite of those equivalences or their inverses, respectively, by

$$(3-1) \quad \omega: (\mathbf{T}_{\text{Orb}})_\infty \rightarrow \mathbf{Spc}_\infty.$$

3.3 Comparing orbispace quotients

The **Orb**-space model and the orthogonal space model for global spaces both come with specific constructions that implement the “orbispace quotient” $A//G$ of a compact Lie group G acting on a space A . We need to know that these two kinds of orbispace quotients match up under the equivalence (3-1). For orthogonal spaces, the orbispace quotient $A//G$ is modeled by the following semifree orthogonal space. The construction depends on a choice of faithful G -representation, but it is of course independent up to global equivalence of this choice; see [33, Proposition 1.1.26].

Definition 3.8 (semifree orthogonal spaces) Let V be a finite-dimensional faithful representation of a compact Lie group G . We define the *semifree orthogonal space* functor

$$\mathbf{L}_{G,V}: G\text{-}\mathbf{T} \rightarrow \mathbf{Spc}$$

as follows: Let A be a G -space. For an inner product space W , we set

$$(\mathbf{L}_{G,V}A)(W) = \mathbf{L}(V, W) \times_G A,$$

where G acts on $\mathbf{L}(V, W)$ by precomposing the action on V . The structure maps act by postcomposition.

We write $\bar{F}: \mathbf{TGrpd} \rightarrow \mathbf{TOrb}$ for the composite of the Yoneda embedding and restriction along the full embedding $\mathbf{TOrb}^{\text{op}} \rightarrow \mathbf{TGrpd}^{\text{op}}$; this functor is one ingredient in the functor Construction 2.17 that induces the global realization (2-1). So the $\mathbf{Orb}$ -space $\bar{F}(\mathbb{G})$ associated to a topological groupoid $\mathbb{G}$ is given on objects by $\bar{F}(\mathbb{G})(K) = \|\mathbb{M}\text{ap}(\mathbb{B}K, \mathbb{G})\|$.

For the rest of this section, we let G be a universal subgroup of the linear isometries monoid $\mathcal{L}$, we let V be a finite-dimensional faithful G -subrepresentation of the complete G -universe $\mathbb{R}^\infty$, and we let A be a G -space. By adapting constructions from K\"orschgen's paper [17], we will now construct a functor $Q: G\text{-}\mathbf{T} \rightarrow \mathbf{TOrb}'$ and two natural transformations of $\mathbf{Orb}'$ -spaces

$$f_1^*(\bar{F}(G \ltimes A)) \xleftarrow{l} Q(A) \xrightarrow{v} f_2^*(\Psi(\mathbf{L}_{G,V}A)).$$

We will then show in Theorem 3.12 that the two morphisms are weak equivalences of $\mathbf{Orb}'$ -spaces whenever A is Hausdorff. So in this case the equivalences witness that the two orbispace quotients $A//G$ in the worlds of $\mathbf{Orb}$ -spaces and orthogonal spaces correspond under the equivalence ω from (3-1).

Construction 3.9 Let H be another universal subgroup of $\mathcal{L}$. We define

$$\tilde{E}(H, G, A) = \{(\alpha, \varphi, a) \in \text{hom}(H, G) \times \mathcal{L} \times A \mid h\varphi = \varphi\alpha(h) \text{ and } \alpha(h)a = a \text{ for all } h \in H\}.$$

The group G acts on $\tilde{E}(H, G, A)$ by

$$g.(\alpha, \varphi, a) = (\alpha^g, \varphi g^{-1}, g.a).$$

We define the $\mathbf{Orb}'$ -space $Q(A)$ at the group H by

$$Q(A)(H) = \tilde{E}(H, G, A) \times_G EG \cong \|G \ltimes \tilde{E}(H, G, A)\|.$$

Functoriality is defined similarly as is [17, Definition 3.5]: We define a composition law of topological groupoids

$$(H \ltimes \tilde{E}(K, H)) \times (G \ltimes \tilde{E}(H, G, A)) \rightarrow G \ltimes \tilde{E}(K, G, A),$$

which is given by

$$H \ltimes \tilde{E}(K, H) \times G \ltimes \tilde{E}(H, G, A) \rightarrow G \ltimes \tilde{E}(K, G, A), \quad (h, \alpha, \varphi), (g, \beta, \psi, x) \mapsto (g\beta(h), \beta \circ \alpha, \varphi\psi, x),$$

on arrows. The verification that this indeed gives rise to a functor between topological groupoids is analogous to the one in [17, Proposition 3.7]. Using Proposition 2.13, we obtain a map

$$\mathbf{Orb}'(K, H) \times Q(A)(H) \rightarrow Q(A)(K)$$

by applying fat geometric realizations on both sides. In the case where $A = *$, then $Q(*) (H) = \mathbf{Orb}'(H, G)$, and this construction reduces to the composition maps in the category $\mathbf{Orb}'$ as defined in [17, Proposition 3.9]. The verification of the associativity and unitality with respect to the composition in $\mathbf{Orb}'$ are a computation on the level of groupoids similar to the one for $A = *$ in [17, Proposition 3.8]. This concludes the definition of the $\mathbf{Orb}'$ -space $Q(A)$.

Next we define the weak equivalence of $\mathbf{Orb}'$ -spaces $l: Q(A) \rightarrow f_1^*(\bar{F}(G \ltimes A))$.

Construction 3.10 We let H be another universal subgroup of $\mathcal{L}$. The projection

$$\rho_A^H: \tilde{E}(H, G, A) \rightarrow \mathbb{M}\text{ap}(\mathbb{B}H, \mathbb{G} \ltimes A)_0 = \{(\alpha, x) \in \text{hom}(H, G) \times A \mid x \in A^{\alpha(H)}\}, \quad \rho_A^H(\alpha, \varphi, x) = (\alpha, x),$$

participates in a pullback square

$$\begin{array}{ccc} \tilde{E}(H, G, A) & \longrightarrow & \tilde{E}(H, G) \\ \rho_A^H \downarrow & & \downarrow \rho^H \\ \mathbb{M}\text{ap}(\mathbb{B}H, G \ltimes A)_0 & \longrightarrow & \text{hom}(H, G) \end{array}$$

where the horizontal maps are also projections (forgetting the factor A) and the map $\rho^H = \rho_*^H$ is also defined in [17, Definition 2.12]. The map ρ_A^H gives rise to the map

$$\rho_A^H \times_G EG: \tilde{E}(H, G, A) \times_G EG \rightarrow \mathbb{M}\text{ap}(\mathbb{B}H, G \ltimes A)_0 \times_G EG.$$

Note that

$$\begin{aligned} ((f_1)^* \circ F)(G \ltimes A)(H) &= \|\mathbb{M}\text{ap}(\mathbb{B}H, G \ltimes A)\| \\ &\cong \|G \ltimes \mathbb{M}\text{ap}(\mathbb{B}H, G \ltimes A)_0\| \cong \mathbb{M}\text{ap}(\mathbb{B}H, G \ltimes A)_0 \times_G EG. \end{aligned}$$

Using this homeomorphism, the map $\rho_A^H \times_G EG$ rewrites as a map

$$l(H): Q(A)(H) \rightarrow ((f_1)^* \circ F)(G \ltimes A)(H)$$

and a computation on the level of groupoids similar to [17, Proposition 3.8] shows that these maps are compatible with the action maps. We obtain a morphism of $\mathbf{Orb}'$ -spaces $l: Q(A) \rightarrow ((f_1)^* \circ F)(G \ltimes A)$.

Finally, we define the weak equivalence of $\mathbf{Orb}'$ -spaces $v: Q(A) \rightarrow ((f_2)^* \circ \Psi)(\mathbf{L}_{G, V} A)$.

Construction 3.11 We let V be a finite-dimensional faithful G -subrepresentation of the complete G -universe $\mathbb{R}^\infty$. Then restriction of linear isometric embeddings from $\mathbb{R}^\infty$ to V is map of $\mathcal{L}$ -spaces $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \rightarrow \mathbf{L}(V, \mathbb{R}^\infty)$ that is also G -equivariant for precomposition. If H is another universal subgroup of $\mathcal{L}$, we define a continuous map

$$\nu(H): \mathcal{Q}(A)(H) = \tilde{E}(H, G, A) \times_G EG \rightarrow (\mathbf{L}(V, \mathbb{R}^\infty) \times_G A)^H = (f_2^*(\Psi(\mathbf{L}_{G,V}A)))(H)$$

by

$$\nu(H)[\alpha, \varphi, a; x] = [\varphi|_V, a].$$

For varying universal subgroups H , these maps form a morphism of $\mathbf{Orb}'$ -spaces

$$\nu: \mathcal{Q}(A) \rightarrow ((f_2)^* \circ \Psi)(\mathbf{L}_{G,V}A).$$

This can be proven similarly as in [17, Proposition 3.12] because functoriality on both sides does not interact with the elements of A .

The following is the main result of this section.

Theorem 3.12 *Let G be a universal subgroup of the linear isometries monoid $\mathcal{L}$, let V be a finite-dimensional faithful G -subrepresentation of the complete G -universe $\mathbb{R}^\infty$, and let A be a Hausdorff G -space. Then the morphisms of $\mathbf{Orb}'$ -spaces*

$$f_1^*(\bar{F}(G \ltimes A)) \xleftarrow{l} \mathcal{Q}(A) \xrightarrow{\nu} f_2^*(\Psi(\mathbf{L}_{G,V}A))$$

are weak equivalences.

Proof Let H be a universal subgroup of $\mathcal{L}$. The projection $\rho^H: \tilde{E}(H, G) \rightarrow \text{hom}(H, G)$ is a fiber bundle with contractible fiber by [17, Proposition 2.15] and so is the pullback

$$\rho_A^H: \tilde{E}(H, G, A) \rightarrow \mathbb{M}\text{ap}(\mathbb{B}H, \mathbb{G} \ltimes A)_0.$$

The proof that $l(H) = \rho_A^H \times_G EG$ is a weak equivalence is completely analogous to the one in [17, Proposition 2.18], exploiting that the homotopy quotient construction $- \times_G EG$ is homotopical.

We now turn to the proof that the second map is an equivalence. The space A is Hausdorff by assumption, the space $\text{hom}(H, G)$ is Hausdorff because it is metrizable, the space $\mathcal{L}$ is Hausdorff by [34, Proposition A.1]. The product of Hausdorff spaces is Hausdorff and we conclude that the space $\tilde{E}(H, G, A)$ is Hausdorff because it is a subspace of a Hausdorff space. The natural map

$$\tilde{E}(H, G, A) \times_G EG \rightarrow \tilde{E}(H, G, A)/G$$

from the homotopy quotient to the orbit space is a weak equivalence for all universal subgroups H of $\mathcal{L}$ by [17, Theorem A.7] because $\tilde{E}(H, G, A)$ is a free G -space and Hausdorff.

Now we exploit that the map

$$\tilde{E}(H, G, A)/G \rightarrow (\mathcal{L} \times_G A)^H, \quad [\alpha, \varphi, x] \mapsto [\varphi, x],$$

is well defined and a homeomorphism. We can adapt the proof of [17, Proposition 2.20]. More in detail, the proof does not use any properties of $\mathcal{L}$ other than being a $(H \times G)$ -space such that the restricted G -action is free. These assumption also hold for the space $\mathcal{L} \times A$ where $H \times G$ acts via

$$(k, g).(\varphi, x) = (k\varphi g^{-1}, g.x)$$

for $k \in H$, $g \in G$, $\varphi \in \mathcal{L}$ and $x \in A$. We may replace $\mathcal{L}$ with $\mathcal{L} \times A$. The space $\tilde{E}(H, G)$ then becomes $\tilde{E}(H, G, A)$ and $(\mathcal{L}/G)^H$ becomes $(\mathcal{L} \times_G A)^H$.

The final ingredient is [34, Proposition A.10], which implies that the restriction map

$$\rho_V: \mathcal{L} \times_G A = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \times_G A \rightarrow \mathbf{L}(V, \mathbb{R}^\infty) \times_G A = (\mathbf{L}_{G,V} A)(\mathbb{R}^\infty)$$

induces a weak equivalence on H -fixed point spaces. $\square$

We now apply the previous theorem to the main case of interest for us, namely global quotient orbifolds.

Corollary 3.13 *Let M be a manifold together with an almost free action of a universal subgroup G of $\mathcal{L}$ on M . Then the image of the global realization $|G \ltimes M|_{\text{gl}}$ under the equivalence $\omega: (\mathbf{TOrb})_\infty \rightarrow \mathbf{Spc}_\infty$ defined in (3-1) is equivalent to the semifree global space $\mathbf{L}_{G,V} M$, where V is any faithful G -representation.*

Proof This follows from Theorem 3.12, also using that f_1^* and f_2^* are fully homotopical as all objects are fibrant in the projective model structure and that the semifree global space $\mathbf{L}_{G,V}$ is closed by [33, Example 1.1.14], so that we can directly apply the functor Ψ from Construction 3.6 without replacing the object cofibrantly. $\square$

4 Orbifold cohomology theories from global spectra

In this section we use global spectra in the sense of Schwede [33] to define orbifold cohomology theories. We then discuss various properties and examples of our construction.

In [33, Chapter 4], Schwede developed a setting for global stable homotopy theory based on orthogonal spectra; see [33, Definition 3.1.1]. For a compact Lie group G , every orthogonal spectrum X can be made an orthogonal G -spectrum via the trivial G -action; thus, the G -equivariant homotopy groups $\pi_*^G(X)$ are defined, compare [33, (3.1.11)]. A morphism $f: X \rightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced homomorphism $\pi_k^G(f): \pi_k^G(X) \rightarrow \pi_k^G(Y)$ is an isomorphism for every compact Lie group G and all integers k .

Definition 4.1 The ∞ -category of global spectra $\mathcal{GH}$ is the ∞ -categorical localization of the homotopy coherent nerve of the topological category $\mathbf{Sp}$ of orthogonal spectra at the class of global equivalences.

Remark 4.2 Usually, one would define $\mathcal{GH}$ as the ∞ -categorical localization of the 1-category of orthogonal spectra at the global equivalences. However, one can first regard orthogonal spectra as a simplicial model category and then use [30, Theorem 13.2.1] to deduce that there exists simplicial (co)fibrant replacement functors. Combining this with [21, Theorem 1.3.4.20] implies that the two definitions agree.

By [33, Theorem 4.3.18], the global equivalences of orthogonal spectra are part of a stable, cofibrantly generated, proper, topological model structure. The ∞ -category of global spectra is thus complete and cocomplete; compare [3, Theorem 2.5.9]. Since the model structure is stable, the suspension functor on its homotopy category is invertible, so the ∞ -category $\mathcal{GH}$ is stable; compare [21, Corollary 1.4.2.27].

The *suspension spectrum* functor

$$\Sigma_+^\infty : \mathbf{Spc} \rightarrow \mathbf{Sp}$$

is defined in [33, Conjecture 4.1.7]. Given an orthogonal space X , the value of $\Sigma_+^\infty X$ at an inner product space V is $(\Sigma_+^\infty X)(V) = S^V \wedge X(V)_+$. The functor Σ_+^∞ takes global equivalences of orthogonal spaces to global equivalences of orthogonal spectra [33, Corollary 4.1.9], and it is a left Quillen functor with respect to the two global model structures, see [33, Theorem 4.3.17(v)]. So it descends to a colimit preserving functor

$$\Sigma_+^\infty : \mathbf{Spc}_\infty \rightarrow \mathcal{GH}$$

of underlying ∞ -categories, for which we use the same name.

Definition 4.3 (global motiv) We write $\mathbf{M} : \mathbf{Orbfld}_\infty \rightarrow \mathcal{GH}$ for the composite

$$\mathbf{Orbfld}_\infty \xrightarrow{|\cdot|_{\text{gl}}} (\mathbf{T}_{\text{Orb}})_\infty \xrightarrow[\sim]{\omega} \mathbf{Spc}_\infty \xrightarrow{\Sigma_+^\infty} \mathcal{GH}$$

of the global homotopy type, the equivalence ω from (3-1), and the suspension spectrum functor. We call $\mathbf{M}(\mathbb{G})$ the *global motiv* of the orbifold $\mathbb{G}$.

The following theorem summarizes the main properties of the global motiv functor. All four parts are consequences of the corresponding properties of the global realization, see Theorem 2.25, and the fact that the functor Σ_+^∞ preserves colimits. The last item also uses Corollary 3.13.

Theorem 4.4 (i) *The global motiv functor takes essential equivalences of orbifolds to global equivalences of global spectra.*

(ii) *The global motiv functor takes disjoint unions of orbifolds to coproducts of global spectra.*

(iii) *Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\tilde{U}$ and $\tilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \rightarrow \mathbb{G}_0/\mathbb{G}_1$. Then the square*

$$(4-1) \quad \begin{array}{ccc} \mathbf{M}(\mathbb{G}|_{\tilde{U} \cap \tilde{V}}) & \longrightarrow & \mathbf{M}(\mathbb{G}|_{\tilde{U}}) \\ \downarrow & & \downarrow \\ \mathbf{M}(\mathbb{G}|_{\tilde{V}}) & \longrightarrow & \mathbf{M}(\mathbb{G}) \end{array}$$

is a bicartesian square in the ∞ -category of global spectra, where all morphisms are induced by the respective inclusions.

(iv) *Let M be a manifold together with an almost free action of a universal subgroup G of $\mathcal{L}$ on M . Then the global motiv $\mathbf{M}(G \ltimes M)$ is equivalent to the suspension spectrum $\Sigma_+^\infty \mathbf{L}_{G,V} M$.*

Definition 4.5 (orbifold cohomology) Let $\mathbb{G}$ be an orbifold and let E be an orthogonal spectrum and let k be an integer. We define the k^{th} orbifold cohomology group represented by E of $\mathbb{G}$ by

$$E^k(\mathbb{G}) = \mathcal{GH}(\mathbf{M}(\mathbb{G}), E[k]).$$

This extends to a functor $E^k: \mathbf{Orbld}^{\text{op}} \rightarrow \mathbf{Ab}$ and morphisms between orthogonal spectra induce natural transformations between the associated orbifold cohomology theories.

The first three items of Theorem 4.4 immediate imply the following properties of the orbifold cohomology groups. Part (iv) follows from the equivalence between the global motiv $\mathbf{M}(G \ltimes M)$ and the suspension spectrum $\Sigma_+^\infty \mathbf{L}_{G,V} M$, and the adjunction isomorphism

$$E_G^k(M) = G\text{-}\mathcal{SH}(\Sigma_{G,+}^\infty M, E[k]) \cong \mathcal{GH}(\Sigma_+^\infty \mathbf{L}_{G,V} M, E[k])$$

from [33, Remark 4.5.25].

Theorem 4.6 Let E be an orthogonal spectrum.

(i) The orbifold cohomology functor $E^k: \mathbf{Orbld}^{\text{op}} \rightarrow \mathbf{Ab}$ takes essential equivalences of orbifolds to isomorphisms.

(ii) For every family $\{\mathbb{G}^i\}_{i \in I}$ of orbifolds, the natural map

$$E^k\left(\coprod_{i \in I} \mathbb{G}^i\right) \rightarrow \prod_{i \in I} E^k(\mathbb{G}^i)$$

is an isomorphism.

(iii) Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\tilde{U}$ and $\tilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \rightarrow \mathbb{G}_0/\mathbb{G}_1$. There is a long exact sequence of orbifold cohomology groups

$$\cdots \rightarrow E^k(\mathbb{G}) \xrightarrow{(i_U^*, i_V^*)} E^k(\mathbb{G}|_{\tilde{U}}) \oplus E^k(\mathbb{G}|_{\tilde{V}}) \xrightarrow{j_U^* - j_V^*} E^k(\mathbb{G}|_{\tilde{U} \cap \tilde{V}}) \rightarrow E^{k+1}(\mathbb{G}) \rightarrow \cdots$$

for every orthogonal spectrum E . Here, $i_U: \mathbb{G}|_{\tilde{U}} \rightarrow \mathbb{G}$, $i_V: \mathbb{G}|_{\tilde{V}} \rightarrow \mathbb{G}$, $j_U: \mathbb{G}|_{\tilde{U} \cap \tilde{V}} \rightarrow \mathbb{G}|_{\tilde{U}}$ and $j_V: \mathbb{G}|_{\tilde{U} \cap \tilde{V}} \rightarrow \mathbb{G}|_{\tilde{V}}$ denote the inclusion morphisms.

(iv) Let G be a compact Lie group acting almost freely on a manifold M . The orbifold cohomology of $G \ltimes M$ is isomorphic to the G -equivariant cohomology of M , ie

$$E^k(G \ltimes M) \cong E_G^k(M).$$

Example 4.7 (cohomology of the classifying space, Borel cohomology) Let E be a nonequivariant generalized cohomology theory. By Brown's representability theorem, there is an orthogonal spectrum representing E which we also denote by E . The forgetful functor $U: \mathcal{GH} \rightarrow \mathcal{SH}$ admits a right adjoint $b: \mathcal{SH} \rightarrow \mathcal{GH}$; see [33, Theorem 4.5.1]. This right adjoint admits an explicit lift to the category of orthogonal spectra described in [33, Conjecture 4.5.21].

Let $\mathbb{G}$ be an orbifold. Since the underlying space of the global realization $|\mathbb{G}|_{\text{gl}}$ is the nonequivariant realization $|\mathbb{G}|$, the underlying nonequivariant homotopy type of the global motivic $\mathbf{M}(\mathbb{G})$ is the unreduced suspension spectrum of $|\mathbb{G}|$. So we obtain a chain of natural isomorphisms

$$\begin{aligned}(bE)^k(\mathbb{G}) &= \mathcal{GH}(\mathbf{M}(\mathbb{G}), bE[k]) \cong \mathcal{SH}(U(\mathbf{M}(\mathbb{G})), E[k]) \\ &\cong \mathcal{SH}(\Sigma_+^\infty |\mathbb{G}|, E[k]) = E^k(|\mathbb{G}|).\end{aligned}$$

We used that the functor b commutes with the respective shift functors in $\mathcal{SH}$ and $\mathcal{GH}$ because its adjoint U does so. We just showed that the orbifold cohomology theory represented by bE can be identified with the cohomology represented by E of the classifying space. For $G \ltimes M$ a global quotient orbifold, we have $|G \ltimes M| \cong M \times_G EG$. Therefore, bE represents the equivariant Borel cohomology represented by E for all global quotient orbifolds.

Example 4.8 (Bredon cohomology) Let F be a global Mackey functor as defined in [33, Section 4.2]. By [33, Theorem 4.4.9], there is an *Eilenberg–MacLane spectrum* HF , ie an object of the global stable homotopy category with equivariant homotopy groups concentrated in degree 0, and such that $\pi_0^G(HF) \cong F(G)$ for all compact Lie groups G with corresponding transfer and restriction maps. The underlying orthogonal G -spectrum $(HF)_G$ represents Bredon cohomology for the underlying G -coefficient system of $F|_F$, the underlying contravariant functor of the restriction of F to a G -Mackey functor; see [7, Example 3.2.16]. For $G \ltimes M$ a global quotient orbifold, we obtain

$$HF^k(G \ltimes M) \cong HF_G^k(M) \cong H_G^k(M, F|_G)$$

by Theorem 4.6(iv). We conclude that the spectrum HF represents Bredon cohomology for global quotient orbifolds. This in particular proves that the Bredon cohomology of an orbifold is independent of the choice of the presentation as a global quotient orbifold. Adem and Ruan [2, Section 3, Remark 5.11] proved this for the special case where $F = \mathbf{R}\mathbf{U}_{\mathbb{Q}}$ is the rationalized representation functor, Pronk and Scull [29, Proposition 5.11] proved this for what they call an *orbifold coefficient system*. Moreover, Bredon cohomology extends to an actual cohomology theory on all orbifolds.

Construction 4.9 (Atiyah–Hirzebruch spectral sequence) The full subcategories of globally connective spectra and globally coconnective spectra form a t -structure on $\mathcal{GH}$; see [33, Theorem 4.4.9]. The heart of this t -structure consists of those spectra whose homotopy groups are concentrated in degree 0, ie Eilenberg–MacLane spectra. The functor $\underline{\pi}_0$ induces an equivalence between this heart and the category of global Mackey functors. This t -structure is induced by a t -model structure on $\mathbf{Sp}$ in the sense of Fausk and Isaksen [10, Definition 4.1]. They give a short argument why this is true for model categories of spectra which also applies in our case.

Their paper [10, Section 10] explains how to construct an Atiyah–Hirzebruch spectral sequence from a t -model structure on a stable model category. We briefly recall the most important ideas and adjust them to our situation: There is a $(q-1)$ -connective cover $Y_{\geq q}$ for any spectrum Y and any integer q and natural

transformations $Y_{\geq q+1} \rightarrow Y_{\geq q}$ in $\mathcal{GH}$. The homotopy cofiber of this map is a shifted Eilenberg–MacLane spectrum of type $\underline{\pi}_q(Y)$.

For X another orthogonal spectrum, we apply $\mathcal{GH}(X, -[-p-q])$ to the distinguished triangle

$$Y_{\geq q+1} \rightarrow Y_{\geq q} \rightarrow (H\underline{\pi}_q(Y))[q] \rightarrow Y_{\geq q+1}[1]$$

to get an exact couple with $D_{p,q}^2 = \mathcal{GH}(X, Y_{\geq q}[-p-q])$ and $E_{p,q}^2 = \mathcal{GH}(X, (H\underline{\pi}_q(Y))[-p])$.

For $X = (\Sigma_+^\infty \circ R \circ \mathbf{R}_{\text{fib}} \circ \mathcal{L}_{\text{Lie}})(\mathbb{G})$, we obtain the following:

Proposition 4.10 *Let Y be an orthogonal spectrum and let $\mathbb{G}$ be an orbifold. There is a spectral sequence with $E_{p,q}^2 = (H\underline{\pi}_q(Y))^{-p}(\mathbb{G})$ — Bredon cohomology of $\mathbb{G}$ with coefficient system $\underline{\pi}_q(Y)$ — which conditionally converges to $Y^{p+q}(\mathbb{G})$.*

Here, conditional convergences of spectral sequences is defined in Boardman’s paper [5, Definition 5.10]. It also provides several criteria for deducing strong convergence from conditional convergence.

Proof The statement follows from Fausk and Isaksen’s theorem [10, Theorem 10.1]. We need to check that $(\Sigma_+^\infty \circ R \circ \mathbf{R}_{\text{fib}} \circ \mathcal{L}_{\text{Lie}})(\mathbb{G})$ is bounded below and that $\text{holim}_{n \rightarrow \infty} Y_{\geq n}$ is contractible.

The first condition holds because suspension spectra are globally connective by [33, Proposition 4.1.11].

Let G be a compact Lie group and k be an integer. The functor $\pi_0^G: \mathcal{GH} \rightarrow \mathbf{Ab}$ is corepresentable by [33, Theorem 4.4.3(i)]. We hence obtain a Milnor short exact sequence; see eg [4, Lemma 0919]:

$$0 \rightarrow \lim_{n \rightarrow \infty}^1 \pi_{k+1}^G(Y_{\geq n}) \rightarrow \pi_k^G(\text{holim}_{n \rightarrow \infty} Y_{\geq n}) \rightarrow \lim_{n \rightarrow \infty} \pi_k^G(Y_{\geq n}) \rightarrow 0$$

The homotopy groups $\pi_k^G(Y_{\geq n})$ eventually vanish. The left and the right term of the sequence therefore vanish and so does the middle term. $\square$

Example 4.11 (orbifold K -theory) Firstly, we briefly discuss the geometric definition of orbifold K -theory via orbifold vector bundles. A full definition and proofs of the following statements can be found in [1, Section 3.3]. Let $\mathbb{G}$ be a compact orbifold, that is a proper foliation groupoid where $\mathbb{G}_0$ is compact. A $\mathbb{G}$ -space is a smooth manifold M together with a smooth *anchor map* $\alpha: M \rightarrow \mathbb{G}_0$ and a smooth *action map* $\mu: \mathbb{G}_1 \times_{\mathbb{G}_0} M \rightarrow M$ which satisfy the usual unitality and associativity conditions. A $\mathbb{G}$ -vector bundle is a $\mathbb{G}$ -space M where the anchor map $\alpha: M \rightarrow \mathbb{G}_0$ is a complex vector bundle such that the action of $\mathbb{G}$ is fiberwise linear. Similarly as for manifolds, one can define Whitney sums and tensor products of $\mathbb{G}$ -vector bundles.

The *orbifold K -theory* $\mathbf{K}_{\text{Orbifd}}(\mathbb{G})$ of $\mathbb{G}$ is defined as the Grothendieck ring of isomorphism classes of $\mathbb{G}$ -vector bundles. This is a functorial construction (using pullbacks of vector bundles) and one can prove that every essential equivalence between orbifolds induces an isomorphism on these rings; compare [2, Proposition 4.2].

Moreover, the K -theory of a global quotient orbifold $G \ltimes M$ may be identified with the G -equivariant K -theory of M , ie there is an isomorphism

$$\mathbf{K}_{\text{Orbld}}(G \ltimes M) \cong \mathbf{K}_G(M),$$

which is natural in M ; see [1, Proposition 3.6].

On the other hand, there is the *periodic global K -theory spectrum* $\mathbf{KU}$ which represents K -theory in the following sense: for every finite G -CW complex A , there is a natural isomorphism

$$\mathbf{K}_G(A) \cong \mathbf{KU}_G^0(A).$$

This applies when $A = M$ is a compact manifold together with a G -action on M . The definition of $\mathbf{KU}$ is due to Joachim [15]. See [33, Section 6.4] for a self-contained exposition from a global perspective.

By also using Theorem 4.6(iv), we finally obtain an isomorphism

$$\mathbf{K}_{\text{Orbld}}(G \ltimes M) \cong \mathbf{KU}^0(G \ltimes M).$$

We conclude that the ring spectrum $\mathbf{KU}$ represents orbifold K -theory for compact global quotient orbifolds. Moreover, both sides are invariant under essential equivalences of orbifolds. We conclude that $\mathbf{KU}$ represents orbifold cohomology for all compact orbifolds by Theorem 2.11.

It is however not clear that this isomorphism is natural with respect to all maps of orbifolds.

Example 4.12 (rationalized K -theory) The rationalized K -theory is represented by the rationalized periodic global K -theory spectrum $\mathbf{KU}_{\mathbb{Q}}$. By [38, Theorem 1.6], this spectrum decomposes up to $\mathcal{F}$ in-global equivalence into a sum of shifted Eilenberg–MacLane spectra

$$\mathbf{KU}_{\mathbb{Q}} \simeq \bigvee_{n \in \mathbb{Z}} H(\mathbf{RU}_{\mathbb{Q}})[2n],$$

which is enough for our purposes. This implies that there is an isomorphism

$$\mathbf{KU}^0(\mathbb{G}) \otimes \mathbb{Q} \cong \bigoplus_{n \in \mathbb{Z}} (H(\mathbf{RU}_{\mathbb{Q}}))^{2n}(\mathbb{G})$$

between the rationalized K -theory and Bredon cohomology with coefficients system $\mathbf{RU}_{\mathbb{Q}}$ of an orbifold $\mathbb{G}$. For $\mathbb{G} = G \ltimes M$ a global quotient orbifold of a compact manifold M under the action of a compact Lie group G , this rewrites as

$$\mathbf{K}_{\text{Orbld}}(G \ltimes M) \otimes \mathbb{Q} \cong \bigoplus_{n \in \mathbb{Z}} H_G^{2n}(M, \mathbf{RU}_{\mathbb{Q}})$$

by Examples 4.11 and 4.8. This has been noted before by Adem and Ruan [2, Remark 5.11].

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Higher symplectic capacities

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We construct new families of symplectic capacities indexed by certain symmetric polynomials, defined using rational symplectic field theory. In particular, we introduce a sequence of capacities based on an $\mathcal{L}_\infty$ structure on linearized contact homology and rational curve counts with local tangency constraints. We prove various structural properties of these capacities and give some preliminary computations which show that they give state of the art symplectic embedding obstructions in basic examples.

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1 Introduction

1.1 Context

1.1.1 Symplectic embeddings Given a symplectic manifold (M^{2n}, ω) and a function $H: M \rightarrow \mathbb{R}$, flowing along the symplectic gradient of H defines a family of symplectomorphisms from M to itself. Such Hamiltonian dynamical systems provide the geometric basis of classical mechanics and appear in various guises throughout mathematics and physics. It has been known since the nineteenth century that symplectomorphisms preserve the canonical volume form $\frac{1}{n!}\omega^{\wedge n}$, a result known as Liouville’s theorem in statistical mechanics. However, in his seminal 1985 paper [30] on pseudoholomorphic curves, Gromov showed that symplectomorphisms are much more subtle and mysterious than volume-preserving maps. The first proof of concept for this underlying richness is Gromov’s celebrated “nonsqueezing theorem”,

which states that a large ball cannot be symplectically squeezed into a narrow infinite cylinder. More precisely, for $0 < a' < a$ there is no symplectic embedding

$$B^{2n}(a) \hookrightarrow B^{2n}(a') \times \mathbb{C}^{n-1}.$$

Here both sides are equipped with their canonical symplectic structures coming from the restriction of the standard symplectic form $\sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{C}^n$, and $B^{2n}(a)$ denotes the ball of area a (ie radius $\sqrt{a/\pi}$). Such an embedding is perfectly allowed by volume considerations alone, so this shows that symplectic geometry imposes fundamental and previously unanticipated constraints.

Understanding the precise nature of symplectic transformations is still a largely open field of research. After the nonsqueezing theorem, a natural next toy question is when can one symplectic ellipsoid be embedded into another. We define the symplectic ellipsoid $E(a_1, \dots, a_n)$ by

$$E(a_1, \dots, a_n) := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{k=1}^n \frac{\pi |z_k|^2}{a_k} \leq 1 \right\}.$$

For convenience, we will typically assume $a_1 \leq \dots \leq a_n$. We have the special cases $E(a, \dots, a) = B^{2n}(a)$ and $E(a, \infty, \dots, \infty) = B^2(a) \times \mathbb{C}^{n-1}$.

Question 1.1 For which $a_1, \dots, a_n$ and $a'_1, \dots, a'_n$ is there a symplectic embedding of $E(a_1, \dots, a_n)$ into $E(a'_1, \dots, a'_n)$?

Since the volume of $E(a_1, \dots, a_n)$ is given by $\frac{1}{n!} a_1 \cdots a_n$, the first basic obstruction is

$$a_1 \cdots a_n \leq a'_1 \cdots a'_n.$$

Also, Gromov's nonsqueezing theorem can be rephrased as saying that there is no symplectic embedding

$$E(a, \dots, a) \hookrightarrow E(a', \infty, \dots, \infty)$$

unless $a \leq a'$, and this implies that in general we must have

$$a_1 \leq a'_1.$$

1.1.2 Symplectic capacities For $n = 1$, an embedding $B^2(a_1) \hookrightarrow B^2(a'_1)$ evidently exists if and only if $a_1 \leq a'_1$. For $n = 2$, the answer to Question 1.1 is now completely understood, or at least reduced to pure combinatorics. Before recalling the result, it is useful to formalize symplectic embedding obstructions using the notion of a symplectic capacity. Roughly, a symplectic capacity c is any invariant of symplectic manifolds which takes values in positive real numbers and is monotone with respect to symplectic embeddings. More precisely, the symplectic capacities considered in this paper will be defined for symplectic manifolds with nonempty convex boundary (although one could formally extend them to all symplectic manifolds if desired). We will also distinguish between “exact symplectic capacities”, which are monotone only for exact symplectic embeddings, and “nonexact symplectic capacities”, which are monotone for all symplectic embeddings.¹ By default, “capacity” will mean exact, but we note that

¹See Definition 3.7 for the precise definition of exact symplectic embedding.

this distinction is immaterial eg for Liouville domains with $H^1(X; \mathbb{R}) = 0$ (see Remark 3.10). One often also requires a symplectic capacity c to scale like area, ie $c(M, a\omega) = ac(M, \omega)$, which rules out symplectic volume in higher dimensions. This property will indeed hold for all the capacities discussed in this paper. Finally, some authors include the normalization condition $c(B^{2n}) = c(B^2 \times \mathbb{C}^{n-1}) = 1$, but this property will *not* hold for most of the capacities considered in this paper.²

An important sequence of symplectic capacities $c_1^{\text{EH}}, c_2^{\text{EH}}, c_3^{\text{EH}}, \dots$ was defined by Ekeland and Hofer in [16; 17] using variational methods. The first Ekeland–Hofer capacity satisfies

$$c_1^{\text{EH}}(B^{2n}(r)) = c_1^{\text{EH}}(B^2(r) \times \mathbb{C}^{n-1}) = r,$$

and so in particular recovers Gromov’s nonsqueezing theorem. More generally, for $k \in \mathbb{Z}_{>0}$, the k^{th} Ekeland–Hofer capacity of the ellipsoid $E(a_1, \dots, a_n)$ has been computed to be the k^{th} smallest element in the infinite matrix $(ia_j : i \in \mathbb{Z}_{>0}, 1 \leq j \leq n)$. For example, the first few values for $E(1, 2)$ are 1, 2, 2, 3, 4, 5, $\dots$ and the first few values for $E(1.5, 1.5)$ are 1.5, 1.5, 3, 3, 4.5, 4.5, 6, $\dots$, and hence c_2^{EH} rules out the existence of a symplectic embedding $E(1, 2) \hookrightarrow E(1.5, 1.5)$.

Although the Ekeland–Hofer capacities give various obstructions beyond Gromov’s nonsqueezing theorem, the four-dimensional ellipsoid embedding problem is controlled by another sequence of symplectic capacities $c_1^{\text{ECH}}, c_2^{\text{ECH}}, c_3^{\text{ECH}}, \dots$ defined by Hutchings in [38] using embedded contact homology (ECH). Unlike the Ekeland–Hofer capacities, these capacities are gauge-theoretic in nature and are only defined in dimension four, but in that dimension they tend to give very strong obstructions. The k^{th} ECH capacity of the ellipsoid $E(a, b)$ has been computed to be the $(k+1)^{\text{st}}$ smallest element in the infinite matrix $(ia + jb : i, j \in \mathbb{Z}_{\geq 0})$. For example, the first few values for $E(1, 2)$ are 1, 2, 2, 3, 3, 4, 4, 4, $\dots$ and the first few values for $E(1.5, 1.5)$ are 1.5, 1.5, 3, 3, 3, 4.5, 4.5, 4.5, 4.5, $\dots$, etc. McDuff proved the following result in [50]:

Theorem 1.2 *There is a symplectic embedding of $E(a, b)$ into $E(c, d)$ if and only if $c_k^{\text{ECH}}(E(a, b)) \leq c_k^{\text{ECH}}(E(c, d))$ for all $k \in \mathbb{Z}_{>0}$.*

Up to a scaling factor we can write $E(a, b)$ as $E(1, x)$ for some $x \geq 1$. In the special case $c = d$ (ie the target is a ball), the answer can be summarized in terms of a function $f : [1, \infty) \rightarrow [1, \infty)$, with $f(x)$ defined to be the infimal c such that there exists a symplectic embedding $E(1, x) \hookrightarrow E(c, c)$. McDuff and Schlenk [52] gave an explicit characterization of the function $f(x)$, discovering the following rather subtle features:

- For $x < \tau^4$, $f(x)$ is piecewise linear, with values determined by ratios of Fibonacci numbers (here $\tau = \frac{1+\sqrt{5}}{2}$, the golden ratio).
- For $x > (\frac{17}{6})^2$, $f(x) = \sqrt{x}$ (ie embeddings in this regime are obstructed only by volume).
- For $\tau^4 < x < (\frac{17}{6})^2$, $f(x)$ alternates between the above two behaviors.

²In fact, the “strong Viterbo conjecture” states that there should be a *unique* capacity satisfying this normalization condition, when restricted to convex domains in $\mathbb{R}^{2n}$.

1.1.3 Stabilized symplectic embeddings In higher dimensions, Question 1.1 is still largely open. One surprising development was Guth's construction [31] of symplectic embeddings

$$E(1, \underbrace{S, \dots, S}_{n-1}) \hookrightarrow E(R, R, \underbrace{\infty, \dots, \infty}_{n-2})$$

for some finite R and S arbitrarily large. This shows that an infinite amount of squeezing is possible in the second factor, in significant contrast to the first factor. Hind and Kerman managed to refine Guth's embedding and obtained the following optimal result:

Theorem 1.3 [34; 59] *There exists a symplectic embedding*

$$E(1, \underbrace{\infty, \dots, \infty}_{n-1}) \hookrightarrow E(R, R, \underbrace{\infty, \dots, \infty}_{n-2})$$

if and only if $R \geq 3$.

More precisely, it was shown in [59] that the existence of a symplectic embedding

$$E(1, \underbrace{S, \dots, S}_{n-1}) \hookrightarrow E(R, R, \underbrace{\infty, \dots, \infty}_{n-2})$$

for all sufficiently large S implies the existence of a symplectic embedding

$$E(1, \underbrace{\infty, \dots, \infty}_{n-1}) \hookrightarrow E(R, R, \underbrace{\infty, \dots, \infty}_{n-2}).$$

In the sequel we will often not distinguish between these two. More recent subsequent progress has focused on the following special case of Question 1.1:

Question 1.4 (“stabilized ellipsoid embedding problem”) *For which a, b, c, d is there a symplectic embedding of $E(a, b, \underbrace{\infty, \dots, \infty}_{n-2})$ into $E(c, d, \underbrace{\infty, \dots, \infty}_{n-2})$?*

Any symplectic embedding $\underbrace{E(a, b)}_{n-2} \hookrightarrow \underbrace{E(c, d)}_{n-2}$ can be stabilized to an embedding $E(a, b, \infty, \dots, \infty) \hookrightarrow E(c, d, \infty, \dots, \infty)$, but the converse need not hold. Indeed, the volume obstruction disappears after stabilizing, as do many of the ECH obstructions. For the special case $c = d$, ie embeddings of the form $E(1, x, \infty, \dots, \infty) \hookrightarrow E(c, c, \infty, \dots, \infty)$, notable progress has been made, for instance we have:

Theorem 1.5 [11; 12; 34; 51] *For $1 \leq x \leq \tau^4$, the optimal c is given by $f(x)$, ie the optimal embedding comes from the four-dimensional case. For $x = b_n = \frac{g_{4n+6}}{g_{4n+2}}$ with g_i the i^{th} Fibonacci number (eg $b_0 = 8$, $b_1 = \frac{55}{8}$, etc), or for $x = 3d - 1$ with $d \in \mathbb{Z}_{>0}$, the optimal c is given by $\frac{3x}{x+1}$.*

Although for $x > \tau^4$ the optimal embedding is not known, Hind [33] has produced embeddings

$$(1-1) \quad E(1, x) \times \mathbb{C}^{n-2} \hookrightarrow E\left(\frac{3x}{x+1}, \frac{3x}{x+1}\right) \times \mathbb{C}^{n-2},$$

which are conjectured in [51] to be optimal. We mention that in all known cases the answer to Question 1.4 is independent of n provided $n \geq 3$.

In this paper we primarily focus on the problem of obstructing high-dimensional symplectic embeddings. The results comprising Theorem 1.5 have relied on the following basic strategy. First, given a four-dimensional symplectic embedding $E(1, x) \hookrightarrow E(c, c)$, one finds a rigid punctured pseudoholomorphic curve C in the symplectic completion of the complementary cobordism $E(c, c) \setminus E(1, x)$ (see Section 3.1 for more on the geometric setup). The existence of such a curve is proved using embedded contact homology in [11], using obstruction bundle gluing in [51], and by neck stretching closed curves in $\mathbb{CP}^2$ in [12]. One then argues that this curve meaningfully extends to the stabilization as a product curve $C \times \{\text{pt}\}$. Assuming this, a deformation argument shows that a similar curve $\tilde{C}$ must also exist in the complementary cobordism of any hypothetical symplectic embedding $E(1, x, \infty, \dots, \infty) \hookrightarrow E(c, c, \infty, \dots, \infty)$. Finally, positivity of energy for $\tilde{C}$ together with an application of Stokes' theorem gives a lower bound for c in terms of x .

The process of stabilizing the curve C cannot always be achieved. Indeed, for an SFT-type curve C with positive ends asymptotic to Reeb orbits $\gamma_1^+, \dots, \gamma_k^+$ and negative ends asymptotic to Reeb orbits $\gamma_1^-, \dots, \gamma_l^-$, the Fredholm index is given by

$$(1-2) \quad \text{ind}(C) = (n-3)(2-2g-k-l) + \sum_{i=1}^k \text{CZ}(\gamma_i^+) - \sum_{j=1}^l \text{CZ}(\gamma_j^-) + 2c_1 \cdot [C]$$

(see Section 3.2). Since stabilizing by $\mathbb{C}$ has the effect of increasing n by 1 and also increasing each Conley–Zehnder index $\text{CZ}(\gamma)$ by 1, the index after stabilizing is

$$(1-3) \quad \text{ind}(C \times \{\text{pt}\}) = 2 - 2g - 2l + \text{ind}(C).$$

This means that $C \times \{\text{pt}\}$ will typically not be rigid, and might potentially even have negative index, in which case it ought to disappear after a small perturbation. However, a basic observation from [34] is that for genus-zero curves with exactly one negative end, the index is unchanged by stabilization, and in fact such curves do generally persist to give higher-dimensional obstructions. One of the main motivations of this paper is to understand whether these obstructions can be understood more systematically in terms of symplectic capacities which are defined in any dimension.

1.2 Floer theory perspective

Before discussing our main results, we first motivate and illustrate the concept of “higher symplectic capacities” from the perspective of Floer theory, showing how to construct new capacities based on algebraic structures on symplectic cohomology. The eager reader may skip ahead to the next subsection, where we turn our attention to rational symplectic field theory, which will be our main setting for the bulk of this paper.

Let X^{2n} be a connected Liouville domain, ie a compact symplectic manifold with boundary, whose symplectic form ω admits a global primitive λ restricting to a positive contact form on ∂X . We associate

to X its symplectic cohomology $\mathrm{SH}(X)$, defined over say $\mathbb{K} = \mathbb{Q}$. We can also consider the underlying cochain complex $\mathrm{SC}(X)$, with its natural increasing action filtration

$$\mathcal{F}_{\leq a} \mathrm{SC}(X) \subset \mathcal{F}_{\leq a'} \mathrm{SC}(X), \quad 0 \leq a < a',$$

with $\bigcup_{a \in \mathbb{R}_{\geq 0}} \mathcal{F}_{\leq a} \mathrm{SC}(X) = \mathrm{SC}(X)$. Roughly, as a $\mathbb{K}$ -module $\mathrm{SC}(X)$ has one generator for each critical point of a chosen Morse function on X , and two generators $\hat{\gamma}, \check{\gamma}$ for each Reeb orbit γ of ∂X , with differential ∂ counting Floer cylinders. The Morse cochain complex of X (defined over $\mathbb{K}$) sits inside $\mathrm{SC}(X)$ as the subcomplex of action-zero elements:

$$C(M) = \mathcal{F}_{\leq 0} \mathrm{SC}(X) \subset \mathrm{SC}(X).$$

In particular, there is a canonical action-zero element $e \in \mathcal{F}_{\leq 0} \mathrm{SC}(X)$ represented by the minimum of the Morse function, which we will refer to as the unit (even when not discussing product structures). Given any exact symplectic embedding $X \hookrightarrow X'$, we have a filtration-preserving chain map $\Phi: \mathrm{SC}(X') \rightarrow \mathrm{SC}(X)$ sending unit to unit.

We define a symplectic capacity from this data as follows:

$$\mathfrak{c}(X) := \inf\{a : \partial(x) = e \text{ for some } x \in \mathcal{F}_{\leq a} \mathrm{SC}(X)\}.$$

Moreover, given an exact symplectic embedding $X \hookrightarrow X'$, we have the monotonicity property $\mathfrak{c}(X) \leq \mathfrak{c}(X')$. Indeed, this follows from a simple diagram chase, since $x \in \mathcal{F}_{\leq a} \mathrm{SC}(X')$ with $\partial x = e$ implies

$$\partial(\Phi x) = \Phi(\partial x) = \Phi(e) = e,$$

with $\Phi(x) \in \mathcal{F}_{\leq a} \mathrm{SC}(X)$.

One can compute $\mathfrak{c}(B^{2n}(r)) = r$ and $\mathfrak{c}(B^2(R) \times \mathbb{C}^{n-1}) = R$,³ and hence the capacity $\mathfrak{c}$ recovers Gromov's nonsqueezing theorem. However, in order to capture more refined obstructions we need more capacities. Recently, Gutt and Hutchings [32] showed that by replacing symplectic cohomology with its S^1 -equivariant analogue, we can extend $\mathfrak{c}(X)$ to a sequence of capacities $\mathfrak{c}_1^{\mathrm{GH}}(X), \mathfrak{c}_2^{\mathrm{GH}}(X), \mathfrak{c}_3^{\mathrm{GH}}, \dots$ which conjecturally agree with the Ekeland–Hofer capacities. To explain the construction, recall that $\mathrm{SC}(X)$ naturally has the structure of an S^1 -complex, which means we have a sequence of operations $\Delta_i: \mathrm{SC}(X) \rightarrow \mathrm{SC}(X)$ of degree $1 - 2i$,⁴ with $\Delta_0 = \partial$, satisfying the relations

$$\sum_{i+j=k} \Delta_i \circ \Delta_j = 0$$

for all $k \in \mathbb{Z}_{\geq 0}$ (see eg [6; 25; 63]). Here Δ_1 descends to the BV operator on $\mathrm{SH}(X)$, while $\Delta_{k \geq 2}$ is a “higher BV operator” compensating for the failure of Δ_1 to satisfy $\Delta_1^2 = 0$ on the nose. Let

³Technically since $B^2(R) \times \mathbb{C}^n$ is not compact, this statement actually holds for a smoothing of $B^2(R) \times B^{2n-2}(S)$ with S sufficiently large.

⁴In the sequel we will discuss $\mathbb{Z}$ gradings assuming they are available, ie $2c_1(X) = 0$, or else interpret them in the appropriate quotient of $\mathbb{Z}$. See Section 3.4 for more on our grading conventions.

$\mathbb{K}[u^{-1}]$ be a shorthand for the $\mathbb{K}[u]$ -module $\mathbb{K}[u, u^{-1}]/(u\mathbb{K}[u])$. We define S^1 -equivariant symplectic cochains SC_{S^1} by

$$\mathrm{SC}_{S^1}(X) := \mathrm{SC}(X) \otimes_{\mathbb{K}[u]} \mathbb{K}[u^{-1}],$$

with differential ∂_{S^1} given by

$$\partial_{S^1}(x) = \sum_{i=0}^{\infty} \Delta_i(x) u^i$$

for $x \in \mathrm{SC}_{S^1}(X)$.⁵

There is a natural increasing action filtration on $\mathrm{SC}_{S^1}(X)$, and we have canonical elements $u^{-k}e \in \mathrm{SC}_{S^1}(X)$ for $k \in \mathbb{Z}_{\geq 0}$. We define $\mathfrak{c}_k^{\mathrm{GH}}(X) \in \mathbb{R}_{>0}$ by

$$\mathfrak{c}_k^{\mathrm{GH}}(X) := \inf\{a : \partial_{S^1}(x) = u^{-k+1}e \text{ for some } x \in \mathcal{F}_{\leq a} \mathrm{SC}_{S^1}(X)\}.$$

Similar to before, exact symplectic embeddings $X \hookrightarrow X'$ induce transfer maps $\mathrm{SC}_{S^1}(X') \rightarrow \mathrm{SC}_{S^1}(X)$, from which it follows that $\mathfrak{c}_k^{\mathrm{GH}}(X) \leq \mathfrak{c}_k^{\mathrm{GH}}(X')$.

To understand why the equivariant theory offers more quantitative data, we can rephrase the construction as follows. By quotienting out the Morse complex of X , we obtain a homology-level long exact sequence

$$(1-4) \quad \cdots \rightarrow H(X) \rightarrow \mathrm{SH}(X) \rightarrow \mathrm{SH}_+(X) \xrightarrow{\delta} H(X) \rightarrow \cdots,$$

where the map δ is the connecting homomorphism. We have the equivalent homology-level description

$$\mathfrak{c}(X) = \inf\{a : \delta(x) = [e] \text{ for some } x \in \mathcal{F}_{\leq a} \mathrm{SH}_+(X)\}.$$

In fact, we can define more general “spectral capacities” by replacing $[e] = 1 \in H^0(X)$ by any class in $H^*(X)$. However, if X is say a star-shaped domain⁶ in $\mathbb{C}^n$, we have $\mathrm{SH}_+(X) \cong H(X) \cong \mathbb{K}$, so there is essentially only one spectral invariant. In contrast, in the S^1 -equivariant version we replace $H(X)$ by $H_{S^1}(X) \cong H(X) \otimes H(\mathbb{C}\mathbb{P}^\infty)$, giving rise to countably many independent classes even when X is contractible.

Let us now observe that symplectic cohomology has various additional algebraic structures which we can try to exploit to define further capacities. Indeed, there is a Lie bracket on S^1 -equivariant symplectic cohomology $\mathrm{SH}_{S^1}(X)$, which corresponds to the string bracket when X is a cotangent bundle. On the chain level this is expected to come from an underlying $\mathcal{L}_\infty$ structure, ie we have a sequence of operations $\ell_{S^1}^1, \ell_{S^1}^2, \ell_{S^1}^3, \dots$, where $\ell_{S^1}^k$ is a k -to-one operation of degree $4 - 3k$ (see Section 2 for more details on $\mathcal{L}_\infty$ structures and our conventions). Here $\ell_{S^1}^1$ coincides with the equivariant differential ∂_{S^1} , and $\ell_{S^1}^2$ descends to the Lie bracket on homology, while $\ell_{S^1}^{k \geq 3}$ does not descend to homology.

⁵There are two common variations on this definition of the S^1 -equivariant complex: the *negative* version, in which one works over $\mathbb{K}[[u]]$, and the *periodic* version, in which one works over $\mathbb{K}[[u]][u^{-1}]$. The version we use above is akin to taking homotopy orbits of an S^1 -action.

⁶By *star-shaped domain* in $\mathbb{C}^n$ we mean any subdomain $M^{2n} \subset \mathbb{C}^n$ whose boundary is smooth and transverse to the radial vector field. The restriction of the standard Liouville form $\sum_{i=1}^n \frac{1}{2}(x_i dy_i - y_i dx_i)$ makes M into a Liouville domain. All star-shaped domains are diffeomorphic and in fact Liouville deformation equivalent, but typically not symplectomorphic.

It is useful to observe that the data of this $\mathcal{L}_\infty$ algebra can be encoded in terms of the bar complex $\mathcal{BSC}_{\mathcal{S}^1}(X)$, which is a single chain complex with bar differential $\widehat{\ell}_{\mathcal{S}^1} : \mathcal{BSC}_{\mathcal{S}^1}(X) \rightarrow \mathcal{BSC}_{\mathcal{S}^1}(X)$ (see Definition 2.4). Here as a $\mathbb{K}$ -module $\mathcal{BSC}_{\mathcal{S}^1}(X)$ is the (reduced, graded) symmetric tensor algebra $\overline{\mathcal{SSC}}_{\mathcal{S}^1}(X)$ on $\mathcal{SC}_{\mathcal{S}^1}(X)$. In particular, we have a distinguished subcomplex with trivial differential, which is spanned by words in the elements $u^{-k}e$, $k \in \mathbb{Z}_{\geq 0}$, and which we identify with $\mathcal{BK}[u^{-1}] \cong \overline{\mathcal{S}}\mathbb{K}[u^{-1}]$. We now cook up a family of symplectic capacities as follows. Given an element $\mathfrak{b} \in \overline{\mathcal{S}}\mathbb{K}[u^{-1}]$, define $\mathfrak{d}_{\mathfrak{b}}$ by

$$\mathfrak{d}_{\mathfrak{b}}(X) := \inf\{a : \mathfrak{b} = \widehat{\ell}_{\mathcal{S}^1}(x) \text{ for some } x \in \mathcal{F}_{\leq a}\mathcal{BSC}_{\mathcal{S}^1}(X)\}.$$

As with $\mathfrak{c}_k^{\text{GH}}(X)$ above, it follows from the invariance properties of symplectic cochains that $\mathfrak{d}_{\mathfrak{b}}(X)$ is independent of all choices involved in its construction and is monotone with respect to exact symplectic embeddings.

Similar to (1-4), quotienting out the subcomplex $\mathcal{BC}(X) \subset \mathcal{BSC}(X)$ gives a long exact sequence on homology

$$(1-5) \quad \cdots \rightarrow H(\mathcal{BC}(X)) \rightarrow H(\mathcal{BSC}_{\mathcal{S}^1}(X)) \rightarrow H(\mathcal{B}_+\mathcal{SC}_{\mathcal{S}^1}(X)) \xrightarrow{\delta} H(\mathcal{BC}(X)) \rightarrow \cdots.$$

If $\text{SH}_{\mathcal{S}^1}(X) = 0$ (or equivalently $\text{SH}(X) = 0$), then the homology of $\mathcal{BSC}_{\mathcal{S}^1}(X)$ also necessarily vanishes. This occurs eg if X is a star-shaped domain in $\mathbb{C}^n$, in which case the connecting map in (1-5) gives an isomorphism

$$(1-6) \quad H(\mathcal{B}_+\mathcal{SC}_{\mathcal{S}^1}(X)) \cong \overline{\mathcal{S}}\mathbb{K}[u^{-1}],$$

and we see that $\mathfrak{d}_{\mathfrak{b}}(X)$ is precisely the spectral invariant of the homology class mapping to $\mathfrak{b} \in \overline{\mathcal{S}}\mathbb{K}[u^{-1}]$.

Remark 1.6 The bar complex $\mathcal{BSC}_{\mathcal{S}^1}(X)$ has another increasing filtration by word length, that is, $\mathcal{B}_{\leq l}\mathcal{SC}_{\mathcal{S}^1}(X) \subset \mathcal{B}_{\leq l'}\mathcal{SC}_{\mathcal{S}^1}(X)$ for $1 \leq l < l'$, and we can refine the construction and define capacities $\mathfrak{d}_{\mathfrak{b}}^{\leq l}$ by

$$\mathfrak{d}_{\mathfrak{b}}^{\leq l}(X) := \inf\{a : \mathfrak{b} = \widehat{\ell}_{\mathcal{S}^1}(x) \text{ for some } x \in \mathcal{F}_{\leq a}\mathcal{B}_{\leq l}\mathcal{SC}_{\mathcal{S}^1}(X)\}.$$

In the special case with $l = 1$ and $\mathfrak{b} = u^{-k+1} \in \overline{\mathcal{S}}\mathbb{K}[u^{-1}]$ for some $k \in \mathbb{Z}_{\geq 1}$ (ie $\mathfrak{b}$ is a word of length one), $\mathfrak{d}_{u^{-k+1}}^{\leq 1}$ reduces to $\mathfrak{c}_k^{\text{GH}}$, so the family of capacities $\{\mathfrak{d}_{\mathfrak{b}}^{\leq l}\}$ naturally extends the sequence of Gutt–Hutchings capacities.

Remark 1.7 (other structures on symplectic cohomology) In characteristic zero, symplectic cohomology SH is known to have the structure of a BV-algebra (see, for example, [1]), ie the BV operator is compatible with a product and a bracket, both given by counting pairs of pants but with asymptotic markers treated differently. On the chain level the product has an underlying $\mathcal{A}_\infty$ structure $\mu^1, \mu^2, \mu^3, \dots$, where μ^k has degree $2 - k$, and the bracket has an underlying $\mathcal{L}_\infty$ structure $\ell^1, \ell^2, \ell^3, \dots$, where ℓ^k has degree $3 - 2k$. It is tempting to also use these nonequivariant structures to define symplectic capacities, but unfortunately the subcomplex of $\mathcal{BSC}$ spanned by words consisting only of e (ie the analogue of $\mathcal{BK}[u^{-1}]$) is acyclic in the case of the $\mathcal{A}_\infty$ structure (due to a nontrivial product), and it is just one dimensional in the case of the $\mathcal{L}_\infty$ structure (for grading reasons).

1.3 Main results

Finally, we arrive at the main thrust of this paper.

1.3.1 Stable symplectic capacities As before, let X^{2n} be a connected Liouville domain. In Section 3 we discuss rational symplectic field theory as a filtered $\mathcal{L}_\infty$ algebra whose underlying chain complex is the contact homology algebra $\text{CHA}(\partial X)$, which is generated as a $\mathbb{K}$ -algebra by (good) Reeb orbits in ∂X . This linearizes to a filtered $\mathcal{L}_\infty$ structure whose underlying chain complex is linearized contact homology $\text{CH}_{\text{lin}}(X)$,⁷ which is generated as a $\mathbb{K}$ -module by (good) Reeb orbits in ∂X .

In Section 6 we construct a family of dimensionally stable symplectic capacities as follows. Let $\mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$ denote the graded $\mathbb{K}$ -module spanned by formal variables $t_1, t_2, t_3, \dots$ with $|t_i| = -2 - 2i$, and let $\bar{S}\mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$ denote its (reduced) symmetric tensor algebra, that is, the $\mathbb{K}$ -module spanned by nonempty words in the generators $t_1, t_2, t_3, \dots$ (we can identify this with the polynomial algebra $\mathbb{K}[t_1, t_2, t_3, \dots]$ modulo the constants). We view $\mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$ as an $\mathcal{L}_\infty$ algebra over $\mathbb{K}$ with all $\mathcal{L}_\infty$ operations trivial, and we view $\bar{S}\mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$ as its bar complex (which also has trivial differential).

Theorem 1.8 (see Sections 6.1 and 6.2) *For each $b \in \bar{S}\mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$, there is a symplectic capacity $g_b(X)$ defined for Liouville domains X which satisfies the following properties:*

- **Scaling** *If X' is given by scaling the symplectic form of X by $c \in \mathbb{R}_{>0}$, then $g_b(X') = c g_b(X)$.*
- **Monotonicity** *Given a symplectic embedding $X \xhookrightarrow{s} X'$ between Liouville domains of the same dimension we have $g_b(X) \leq g_b(X')$.*
- **Stabilization** *We have $g_b(X \times B^2(S)) = g_b(X)$ whenever $S > g_b(X)$, provided that X satisfies the index positivity Assumption 5.6 (this holds, for example, if $X \subset \mathbb{R}^{2n}$ is a convex domain).⁸*

It follows from monotonicity that g_b is a symplectomorphism invariant. The stabilization property can be viewed as a quantitative refinement of the property $g_k(X \times \mathbb{C}) = g_k(X)$.

The definition of g_b is parallel to that of d_b discussed in Section 1.2, but with using the $\mathcal{L}_\infty$ structure underlying linearized contact homology $\text{CH}_{\text{lin}}(X)$ in place of $\text{SC}_{S^1,+}(X)$, and replacing the connecting homomorphism δ in (1-5) with a geometrically defined $\mathcal{L}_\infty$ homomorphism $\epsilon_{\text{lin}} \ll \mathcal{T}^\bullet p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \mathbb{K}\langle t_1, t_2, t_3, \dots \rangle$ which counts punctured curves satisfying a local tangency constraint in the symplectic completion $\widehat{X}$ of X (local tangency constraints and more general geometric constraints are discussed in Section 5). The fact that g_b is monotone under general (that is, not necessarily exact) symplectic embeddings is nonobvious and is based on the discussion of nonexact functoriality and deformations by Maurer–Cartan elements which appears in Section 4.

⁷In this paper by default all SFT structures are taken at chain level, and we will be explicit whenever passing to homology (similar conventions are used eg in [3]).

⁸Strictly speaking $X \times B^2(S)$ is a Liouville domain with corners, but this holds for any sufficiently close smoothing of the corners. In fact, in Section 6.2.2 we formulate a more general setting in which stabilization holds.

Disclaimer 1.9 The invariants defined in this paper are based on rational symplectic field theory, and we take as a black box its existence and expected properties as outlined eg in [20] — see Section 3 for more details. It is well known that the general construction requires working with a virtual perturbation framework, for which several proposals have been put forward in varying degrees of completeness and generality. Apart from the general structural properties outlined in Section 3 we will not otherwise discuss particularities of any virtual framework. See Remark 3.1 below for a more detailed discussion.

The indexing set for the capacities $\{\mathfrak{g}_b\}$ is rather large, but a specialization of this family yields a more user-friendly sequence of capacities $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \dots$ as follows.⁹

Theorem 1.10 (see Sections 6.1 and 6.2) *For each $k \in \mathbb{Z}_{\geq 1}$ there is a symplectic capacity $\mathfrak{g}_k(X)$ defined for Liouville domains X which satisfies the scaling, monotonicity, and stabilization properties from Theorem 1.8, as well as:*

- **Upper bounds from closed curves** *If (M, ω_M) is a closed symplectic manifold of the same dimension as X and there is a symplectic embedding $X \hookrightarrow M$, then we have $\mathfrak{g}_k(X) \leq [\omega_M] \cdot A$ provided that $\text{GW}_{M,A} \langle \mathcal{T}^{k-1} p \rangle \neq 0$ for some $A \in H_2(M)$.*
- **Reeb orbits** *If ∂X has nondegenerate Reeb dynamics, there is a collection of Reeb orbits $\gamma_1, \dots, \gamma_l$ in ∂X satisfying $\sum_{i=1}^l [\gamma_i] = 0 \in H_1(X)$ such that $\mathfrak{g}_k(X) = \sum_{i=1}^l A(\gamma_i)$.*

In the first bullet above, $\text{GW}_{M,A} \langle \mathcal{T}^{k-1} p \rangle$ denotes the Gromov–Witten-type count of rational curves satisfying the local tangency constraint $\langle \mathcal{T}^{k-1} p \rangle$ (see Section 5.2). For the second bullet, if $X \subset \mathbb{R}^{2n}$ is a $2n$ -dimensional star-shaped domain, or more generally if $c_1(X) = 0$, then we moreover have $\sum_{i=1}^l \text{CZ}(\gamma_i) = l(n-3) + 2k + 2$, where CZ denotes the Conley–Zehnder index with respect to any global symplectic trivialization of TX .

Intuitively, $\mathfrak{g}(X)$ can be interpreted as the minimal energy¹⁰ of a rigid genus-zero punctured pseudoholomorphic curve in $\widehat{X}$ which passes through a specified point p and is tangent to order $k-1$ to a specified smooth local divisor D near p . Of course, this interpretation comes with various caveats:

- The asymptotic orbits must form a cycle with respect to the differential on the bar complex (this is automatic for an irrational ellipsoid for degree parity reasons).
- The cycle may in fact be a $\mathbb{K}$ -linear combination of (good) Reeb orbits.
- Curves are counted algebraically rather than geometrically, and in general these counts could be virtual.

⁹These are also denoted by $\mathfrak{g} \langle \mathcal{T}^{k-1} p \rangle$ in the body of the paper.

¹⁰The notion of energy used for SFT curves is a modification of the naive symplectic area, which is typically infinite — see Section 3.1.

Analogous to Remark 1.6, using the word length filtration on bar complexes (ie restricting the number of positive ends our curves can have), we can define extended exact symplectic capacities $\mathfrak{g}_b^{\leq l}$ for $l \in \mathbb{Z}_{\geq 1}$ and $b \in \mathbb{K}[t_1, t_2, t_3, \dots]$, and similarly we have $\mathfrak{g}_k^{\leq l}$ for $k, l \in \mathbb{Z}_{\geq 1}$. The special case $l = 1$ corresponds to curves with one positive end, and at least for “nice” domains X we expect

$$(1-7) \quad \mathfrak{g}_k^{\leq 1}(X) = c_k^{\text{GH}}(X),$$

that is, the family $\{\mathfrak{g}_k^{\leq l}\}$ naturally extends the sequence of Gutt–Hutchings capacities.

Addendum 1.11 Indeed, (1-7) is established in [60] for any Liouville domain X^{2n} satisfying $\pi_1(X) = 0$ and $2c_1(X) = 0$.

Remark 1.12 In the body of the paper we work over the Novikov ring in lieu of using action filtrations. This allows us to define capacities for a larger class of domains which are not necessarily exact, and it also allows us to discuss Maurer–Cartan theory (see Section 4).

Remark 1.13 In contrast to the capacities c_k^{GH} , the capacities $\mathfrak{g}_k$ and more generally $\mathfrak{g}_b$ are actually monotone even for nonexact symplectic embeddings. This follows from the Cieliebak–Latschev formalism (see Section 4), which describes cobordism maps for nonexact symplectic embeddings. We note, however, that the word length refinements $\mathfrak{g}_k^{\leq l}$ and $\mathfrak{g}_b^{\leq l}$ are only monotone for exact symplectic embeddings.

1.3.2 First computations and applications In Section 6.3, we give the following first computations:

Theorem 1.14 (1) $\mathfrak{g}_k(B^4) = \lceil \frac{1}{3}(k+1) \rceil$ for $k \geq 1$ congruent to 2 mod 3.

(2) $\mathfrak{g}_k(E(1, x)) = k$ for $1 \leq k \leq x$.

(3) $\mathfrak{g}_k(P(1, x)) = \min(k, x + \lceil \frac{1}{2}(k-1) \rceil)$ for $k \geq 1$ odd.

Here $P(a_1, \dots, a_n)$ denotes the polydisk $B^2(a_1) \times \dots \times B^2(a_n)$. Items (1), (2), (3) correspond to Propositions 6.6, 6.8, 6.9, respectively.

Comparing the obstructions from the first two items in Theorem 1.14 with the embedding (1-1), it follows that the capacities $\{\mathfrak{g}_k\}$ give optimal obstructions for the problem

$$E(1, 3d-1) \times \mathbb{C}^{n-2} \hookrightarrow E(c, c) \times \mathbb{C}^{n-2}$$

for any $d \geq 1, n \geq 3$, recovering a recent result of McDuff [51] (see Theorem 1.5). Similarly, they give optimal obstructions for the problem

$$E(1, 2d-1) \times \mathbb{C}^{n-2} \hookrightarrow P(c, c) \times \mathbb{C}^{n-2}$$

for any $d \geq 1, n \geq 3$.

The computations in Theorem 1.14 also imply various new obstructions for symplectic embeddings, which we illustrate with some examples.

k	1	3	5	7	9
$\mathfrak{g}_k(P(1, 2))$	1	3	4	5	6
$\mathfrak{g}_k(P(c, c))$	c	$2c$	$3c$	$4c$	$5c$
$c_k^{\text{EH}}(P(1, 2))$	1	3	5	7	9
$c_k^{\text{EH}}(P(c, c))$	c	$3c$	$5c$	$7c$	$9c$

Table 1: Sample computations using Theorem 1.14.

Example 1.15 Consider the symplectic embedding problem

$$P(1, 2) \times \mathbb{C}^{n-2} \hookrightarrow P(c, c) \times \mathbb{C}^{n-2}.$$

The capacities $\{\mathfrak{g}_k\}$ give the lower bound $c \geq \frac{3}{2}$ (for convenience see Table 1). The Ekeland–Hofer capacities c_k^{EH} give only $c \geq 1$, and the ECH capacities are not applicable here.

Addendum 1.16 The recent preprint [43] gives a nearly complete description of the stabilized four-dimensional polydisk into polydisk problem; see also [65, Section 4.4].

Example 1.17 Consider the symplectic embedding problem

$$P(1, 3) \times \mathbb{C}^{n-2} \hookrightarrow B^4(c) \times \mathbb{C}^{n-2}.$$

The capacities $\{\mathfrak{g}_k\}$ give the lower bound $c \geq \frac{5}{2}$, while the Ekeland–Hofer capacities give only $c \geq 2$ (for convenience see Table 2). According to Schlenk [61, Proposition 4.3.9], symplectic folding produces a symplectic embedding of $P(1, a)$ into $B^4(c)$ for any $a > 2$ and $c > 2 + \frac{a}{2}$.

Actually, using the stabilization part of Theorem 1.10, we find eg that for any $S \geq 5$ there is no symplectic embedding $P(1, 3) \times B^2(S) \hookrightarrow B^4(c) \times \mathbb{C}^{n-2}$ unless $c \geq \frac{5}{2}$.

Example 1.18 In the followup work [13], we use the capacities $\{\mathfrak{g}_k\}$ and the computations in Theorem 1.14 to give optimal obstructions for certain stabilized embeddings of four-dimensional ellipsoids into polydisks, in particular proving Conjecture 1.4 from [10].

k	1	3	5	7	9
$\mathfrak{g}_k(P(1, 3))$	1	3	5	6	7
$\mathfrak{g}_k(B^4(c))$	c	$2c$	$2c$	$3c$	$4c$
$c_k^{\text{EH}}(P(1, 3))$	1	3	5	7	9
$c_k^{\text{EH}}(B^4(c))$	c	$2c$	$3c$	$4c$	$5c$

Table 2: More sample computations using Theorem 1.14.

For the problem

$$E(1, x) \times \mathbb{C}^{n-2} \xrightarrow{\mathcal{S}} B^4(c) \times \mathbb{C}^{n-2},$$

it turns out that the simplified capacities $\{\mathfrak{g}_k\}$ do *not* always give optimal obstructions. Indeed, we give a general formula for $\mathfrak{g}_k(E(a, b))$ (in fact for all four-dimensional convex toric domains) in the follow up [54], and we observe that these capacities do not always recover the known optimal results in Theorem 1.5. In hindsight this is not too surprising, since the simplification from $\{\mathfrak{g}_b\}$ to $\{\mathfrak{g}_k\}$ involves a rather coarse minimization procedure.

On the other hand, for abstract reasons which we explain at the end of Section 6.3, the set of capacities $\mathfrak{g}_b$ necessarily give optimal obstructions for all known cases covered in Theorem 1.5, suggesting:

Conjecture 1.19 *The capacities $\{\mathfrak{g}_b\}$ provide a complete set of obstructions for Question 1.4.*

Addendum 1.20 In the followup work [65] we give an explicit algorithmic computation of $\mathfrak{g}_b(X)$ for X a convex toric domain. In particular, this gives many new obstructions for stabilized ellipsoid embeddings. However, since the combinatorics are rather involved, the full case of Conjecture 1.19 is at present still open.

Remark 1.21 The indexing set $\bar{S}\mathbb{K}[t]$ for the above capacities is quite large, and is also somewhat redundant, since at the very least the capacities only depend on $\mathfrak{b} \in \bar{S}\mathbb{K}[t]$ up to multiplication by an invertible element of $\mathbb{K}$. We are also not using in any way the algebra structure of $\mathbb{K}[t]$. One can use persistent homology theory to find, for two given Liouville domains X, X' , a countable set of $\{\mathfrak{g}_b\}$ which carries all of the obstructions of the full family — see [65, Section 4.2].

1.3.3 Further capacities from rational symplectic field theory The local tangency constraints appearing in the definition of $\mathfrak{g}_b$ are key to the dimensional stability property, but we can also consider various other geometric constraints and define corresponding (typically unstable) capacities. Indeed, the family $\{\mathfrak{g}_b\}$ is part of larger family given by replacing local tangency constraints with more general multibranch local tangency constraints of the form $\langle \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p \rangle$. Roughly, this requires curves to pass through the specified point p in b branches, with the i^{th} branch tangent to order m_i to the local divisor D (see Section 5.6). We denote this family of capacities by $\{\mathfrak{r}_b\}$, where now $\mathfrak{b} \in \bar{S}\mathbb{K}[t_1, t_2, t_3, \dots]$. In lieu of the stabilization property we have a disjoint union property (see Section 6.2.4), the ECH analogue of which has many important ramifications (see, for example, [14; 15; 42]).

Whereas the capacities $\mathfrak{g}_b$ are defined using the $\mathcal{L}_\infty$ structure on linearized contact homology $\text{CH}_{\text{lin}}(X)$, the capacities $\{\mathfrak{r}_b\}$ require the $\mathcal{L}_\infty$ on the full contact homology algebra $\text{CHA}(X)$. This algebraic structure, which we describe in Section 3.4.2, gives a perspective on rational symplectic field theory which is particularly useful for defining symplectic capacities.¹¹

¹¹The recent preprint [46] explains in detail the relationship between our constructions and the original RSFT formalism of [20].

At first glance still more generally, we can also impose multibranch local tangency constraints at several distinct points. Namely, we specify distinct points $p_1, \dots, p_r \in \widehat{X}$ and corresponding smooth local divisors at those points, and consider constraints of the form

$$\llbracket \underbrace{\mathcal{T}^\bullet p_1, \dots, \mathcal{T}^\bullet p_1}_{b_1}, \dots, \underbrace{\mathcal{T}^\bullet p_r, \dots, \mathcal{T}^\bullet p_r}_{b_r} \rrbracket$$

for $b_1, \dots, b_r \in \mathbb{Z}_{\geq 1}$. Notably, in the special case without any tangency conditions this specializes to a family of “blowup capacities”, denoted by $\tau_{\llbracket p_1^{\times b_1}, \dots, p_r^{\times b_r} \rrbracket}$, which can be also be interpreted in terms of blowing up X at r distinct points and then restricting to certain homology classes of curves. It follows from known results and basic properties of these capacities that the full family of blowup capacities gives (together with the volume constraint) complete obstructions to the four-dimensional ellipsoid embedding problem (as well as the more general embedding problems considered in [9]). The further special case $\tau_k := \tau_{\llbracket p_1, \dots, p_r \rrbracket}$, which roughly corresponds to the minimum energy of a (not necessarily connected) rational curve passing through r distinct points, was first sketched by Hutchings in [39] (this was an important inspiration for this paper).

As we explain in Section 5.7, in dimension four the above capacities involving constraints at multiple points can actually be reduced to the capacities τ_b . Indeed, we present an explicit formula describing how to combine constraints at different $p_1, \dots, p_r$ into constraints at a single point p (a special case involving two points appears in Gathmann’s work [26]). This can be used to give recursive formulas for curve counts with tangency constraints in terms of curve counts without tangency constraints, eg in [53] we apply this technique to count closed curves in $\mathbb{CP}^2$ with multibranch tangency constraints. In particular, the capacities $\{\tau_b\}$ recover the obstructions coming from both the blowup capacities $\tau_{\llbracket p_1^{\times b_1}, \dots, p_r^{\times b_r} \rrbracket}$ and the stable capacities $\{g_b\}$, and we expect the full family to play an important role in the general case of Question 1.1.

Remark 1.22 It is also possible to define higher-genus analogues of τ_b using the full symplectic field theory [20], although we do not pursue this here. It is an interesting question whether higher-genus curves play any role in higher-dimensional symplectic embedding obstructions.

1.4 Outline

In Section 2 we review the necessary background on filtered $\mathcal{L}_\infty$ algebras, which forms the algebraic backbone of the paper. In Section 3 we discuss rational symplectic field theory, formulated using the language of filtered $\mathcal{L}_\infty$ algebras. In Section 4 we describe the Cieliebak–Latschev formalism in this framework, which among other things is useful for studying the role of compactifications. In Section 5 we construct the $\mathcal{L}_\infty$ augmentations, which are interpreted as counting curves with geometric constraints and defined formally using RSFT cobordism maps. Finally, in Section 6 we give the general construction of the capacities, prove some key properties, and present the aforementioned applications.

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2 Filtered $\mathcal{L}_\infty$ algebras

In this section we go over the necessary background on filtered $\mathcal{L}_\infty$ algebras, meanwhile setting the notation and conventions for the rest of the paper. Roughly speaking, an $\mathcal{L}_\infty$ algebra is like a graded Lie algebra which is equipped with a differential, and where the graded Jacobi identity holds only up to a coherent sequence of higher homotopies. Thus we have a graded module V and multilinear k -to-one operations ℓ^k for $k = 1, 2, 3, \dots$, such that ℓ^1 is a differential, ℓ^2 descends to a Lie bracket on the ℓ^1 cohomology, ℓ^3 gives a chain nullhomotopy of the Jacobi relation for ℓ^2 , and so on. The homology inherits the structure of a graded Lie algebra, and the special case where $\ell^k \equiv 0$ for all $k \geq 3$ is equivalent to the notion of a differential graded Lie algebra (DGLA). Thus $\mathcal{L}_\infty$ algebras and DGLAs are Lie analogues of the slightly more familiar notions of $\mathcal{A}_\infty$ algebras and differential graded algebras (DGAs).

As it turns out, DGLAs and $\mathcal{L}_\infty$ naturally occur all over geometry and topology, typically modeling “chain-level objects” for which the underlying homology is a graded Lie algebra. Even in cases when the natural model is a DGLA (and in fact one can always abstractly replace an $\mathcal{L}_\infty$ algebra by an equivalent DGLA), one is more or less forced to face $\mathcal{L}_\infty$ notions when considering the homotopy category. For example, a key property of $\mathcal{L}_\infty$ homomorphisms is that quasi-isomorphisms are automatically homotopy equivalences. This statement fails for the naive homotopy category of DGLAs, for which the naive notion of homomorphism turns out to be too “strict”.

In symplectic geometry, $\mathcal{L}_\infty$ algebras arise naturally when considering rational pseudoholomorphic curves with multiple positive (ie input) punctures and one negative (output) puncture. For the quantitative applications we have in mind, it will be essential for us to consider $\mathcal{L}_\infty$ algebras which are additionally equipped with filtrations. In terms of geometry these filtrations will be related to the symplectic action functional. One way to encode this extra data is by working over the Novikov ring and recording energies in pseudoholomorphic curve counts. The resulting category of filtered $\mathcal{L}_\infty$ algebras is very rich and is central to our constructions of quantitative symplectic invariants.

In what follows, we adopt some of the notation and conventions for $\mathcal{L}_\infty$ algebras from [23; 28; 68] and other sources. Although the basic definitions are essentially standard, there is some small room for discrepancy arising from sign conventions, grading conventions, etc.

2.1 $\mathcal{L}_\infty$ algebra basics

2.1.1 Review of $\mathcal{A}_\infty$ algebras To set the stage for $\mathcal{L}_\infty$ notions, we first discuss the slightly more familiar case of $\mathcal{A}_\infty$ algebras. Let $\mathbb{K}$ be a fixed commutative ring containing $\mathbb{Q}$ (eg $\mathbb{Q}$ itself). Let V be a graded module over $\mathbb{K}$. In general we will only assume $\mathbb{Z}/2$ gradings, although the following discussion applies equally if V has a $\mathbb{Z}$ grading (which occurs in favorable cases). Let $s\mathbb{K}$ denote the graded $\mathbb{K}$ -module consisting of a single copy of $\mathbb{K}$ in degree -1 , and let $sV := s\mathbb{K} \otimes V$ be the “suspension” of V . Thus $(sV)^k$ is identified with V^{k+1} , and there is a natural degree -1 map $s: V \rightarrow sV$ sending v to $s(v) := 1 \otimes v$, where $1 \in \mathbb{K}$ is the unit. For $x \in V^k$, we put $|x| = k$ and $|sx| = k - 1$. In what follows, the ubiquitous use of suspensions gives a convenient way of describing the gradings and signs in $\mathcal{L}_\infty$ equations; they can be ignored on a first pass if one is not too concerned with these.

A succinct way to define an $\mathcal{A}_\infty$ algebra structure on V is as a degree $+1$ coderivation $\hat{\mu}$ on the reduced tensor coalgebra $\bar{T}sV$ which satisfies $\hat{\mu}^2 = 0$. In more detail, let $\otimes^k V$ denote the tensor product (over $\mathbb{K}$) of k copies of V , and set $\bar{T}V = \bigoplus_{k \geq 1} \otimes^k V$. Although $\bar{T}V$ is an algebra, at present it is more relevant that it is a coalgebra. Here the line over T refers to the fact that we start at $k = 1$ rather than $k = 0$, so $\bar{T}V$ is the *reduced* tensor coalgebra on V . In general, a graded coassociative coalgebra is the dual notion of a graded associative algebra, consisting of a graded $\mathbb{K}$ -module C and a degree-zero comultiplication map $\Delta: C \rightarrow C \otimes C$ satisfying the coassociativity condition $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$. In the case of $\bar{T}V$, the comultiplication Δ is given by

$$\Delta(v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_k),$$

which is indeed coassociative.

Given a coassociative coalgebra (C, Δ) , a coderivation is a map $\delta: C \rightarrow C$ satisfying the co-Leibniz condition $\Delta \circ \delta = (1 \otimes \delta) \circ \Delta + (\delta \otimes 1) \circ \Delta$. Here and throughout we implement the Koszul–Quillen sign convention that $f \otimes g$ pairs with $x \otimes y$ to give $(-1)^{|x||g|} f(x) \otimes g(y)$, for $f, g: C \rightarrow C$ and $x, y \in C$. For example, for δ of degree 1 we have $(1 \otimes \delta)(x \otimes y) = (-1)^{|x|} x \otimes \delta(y)$.

Definition 2.1 An $\mathcal{A}_\infty$ algebra is a graded $\mathbb{K}$ -module V and a degree $+1$ coderivation¹² $\hat{\mu}: \bar{T}(sV) \rightarrow \bar{T}(sV)$ satisfying $\hat{\mu}^2 = 0$.

A basic fact is that any coderivation $\hat{\mu}: \bar{T}V \rightarrow \bar{T}V$ is uniquely determined by the sequence of maps $\mu^k: \otimes^k V \rightarrow V$ given by the compositions

$$\otimes^k V \hookrightarrow \bar{T}V \xrightarrow{\hat{\mu}} \bar{T}V \rightarrow V,$$

¹²The degree of $+1$ for $\hat{\mu}$ reflects the fact that we choose cohomological conventions throughout. Some references instead adapt homological conventions, in which case $\hat{\mu}$ has degree -1 .

ie inclusion followed by $\hat{\mu}$ followed by projection. Indeed, any sequence of $\mathbb{K}$ -linear maps $\mu^k: \otimes^k V \rightarrow V$ uniquely extends to a coderivation $\hat{\mu}: \bar{T}V \rightarrow \bar{T}V$ such the induced maps $\otimes^k V \rightarrow V$ recover the μ^k . Explicitly, the extension is given by

$$(2-1) \quad \hat{\mu}(v_1 \otimes \cdots \otimes v_n) := \sum_{\substack{1 \leq i \leq n-k+1 \\ 1 \leq k \leq n}} (1^{\otimes(i-1)} \otimes \mu^k \otimes 1^{\otimes(n-i-k+1)})(v_1 \otimes \cdots \otimes v_n).$$

We can therefore equivalently define an $\mathcal{A}_\infty$ algebra structure as a sequence of degree-one maps $\mu^k: \otimes^k(sV) \rightarrow sV$ satisfying the quadratic relations

$$\sum_{\substack{1 \leq i \leq n-k+1 \\ 1 \leq k \leq n}} (-1)^{|sv_1| + \cdots + |sv_{i-1}|} \mu^{n-k+1}(sv_1 \otimes \cdots \otimes sv_{i-1} \otimes \mu^k(sv_i \otimes \cdots \otimes sv_{i+k-1}) \otimes sv_{i+k} \otimes \cdots \otimes sv_n) = 0.$$

Or, we could work instead with the desuspended operations $s^{-1} \circ \mu^k \circ s^{\otimes k}: \otimes^k V \rightarrow V$, which have degree $2 - k$ and satisfy the same quadratic relations except for some additional Koszul signs. In any case, we will often refer to an $\mathcal{A}_\infty$ algebra by its underlying $\mathbb{K}$ -module V when the rest of the structure is implicit.

Remark 2.2 Recall that a DGA is an associative $\mathbb{K}$ -algebra equipped with a degree $+1$ differential ∂ and a degree-zero associative product satisfying the graded Leibniz rule $\partial(a \cdot b) = \partial(a) \cdot b + (-1)^{|a|} a \cdot \partial(b)$. In the special case of an $\mathcal{A}_\infty$ algebra with $\mu^k = 0$ for all $k \geq 3$, we recover the notion of a DGA on V by setting $x \cdot y := s^{-1} \mu^2 \circ (s \otimes s)(x \otimes y) = (-1)^{|x|} s^{-1} \mu^2(sx, sy)$.

2.1.2 Symmetric tensor coalgebras and $\mathcal{L}_\infty$ algebras Let Σ_k denote the symmetric group on k elements. For V a graded $\mathbb{K}$ -module, we consider the signed action of Σ_k on $\otimes^k V$, given by

$$\sigma(v_1 \otimes \cdots \otimes v_k) = \diamond(V, \sigma; v_1, \dots, v_n) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $\diamond(V, \sigma; v_1, \dots, v_n) \in \{1, -1\}$ is the associated Koszul sign

$$\diamond(V, \sigma; v_1, \dots, v_n) = (-1)^{\{ |v_i| | v_j| : 1 \leq i < j \leq n, \sigma(i) > \sigma(j) \}}$$

(that is, swapping v_i and v_{i+1} “costs” $(-1)^{|v_i| | v_{i+1}|}$). Let $\odot^k V := (\otimes^k V) / \Sigma_k$ denote the quotient space. Then $\bar{S}V := \bigoplus_{k \geq 1} \odot^k V$ is the reduced symmetric tensor coalgebra on V . For $v_1 \otimes \cdots \otimes v_k \in \otimes^k V$, we denote its image in $\odot^k V$ by $v_1 \odot \cdots \odot v_k$. Let $\text{Sh}(i, k-i)$ denote the subset of permutations $\sigma \in \Sigma_k$ satisfying $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$. The comultiplication $\Delta: \bar{S}V \rightarrow \bar{S}V \otimes \bar{S}V$ is given by

$$\Delta(v_1 \odot \cdots \odot v_k) := \sum_{i=1}^{k-1} \sum_{\sigma \in \text{Sh}(i, k-i)} \diamond(\sigma, V; v_1, \dots, v_k) (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \odot \cdots \odot v_{\sigma(k)}).$$

This comultiplication is cocommutative in the sense that $\mathfrak{R} \circ \Delta = \Delta$, where $\mathfrak{R}: \bar{S}V \otimes \bar{S}V \rightarrow \bar{S}V \otimes \bar{S}V$ is given by $\mathfrak{R}(x \otimes y) = (-1)^{|x| | y|} y \otimes x$.

Definition 2.3 An $\mathcal{L}_\infty$ algebra is a graded $\mathbb{K}$ -module V and a degree $+1$ coderivation $\hat{\ell}: \bar{S}(sV) \rightarrow \bar{S}(sV)$ satisfying $\hat{\ell}^2 = 0$.

As in the $\mathcal{A}_\infty$ case, any coderivation $\hat{\ell}: \bar{S}V \rightarrow \bar{S}V$ is uniquely determined by the maps $\ell^k: \odot^k V \rightarrow V$ given by the composition $\odot^k V \hookrightarrow \bar{S}V \xrightarrow{\hat{\ell}} \bar{S}V \rightarrow V$ of inclusion followed by $\hat{\ell}$ followed by projection. Indeed, any sequence of $\mathbb{K}$ -linear maps $\ell^k: \odot^k V \rightarrow V$ extends uniquely to a coderivation $\hat{\ell}: \bar{S}V \rightarrow \bar{S}V$ via the formula

$$\begin{aligned} \hat{\ell}(v_1 \odot \cdots \odot v_n) &:= \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(k, n-k)} \diamond(\sigma, V; v_1, \dots, v_n) \ell^k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)} \\ &= \sum_{k=1}^n \sum_{\sigma \in \Sigma_k} \frac{\diamond(\sigma, V; v_1, \dots, v_n)}{k!(n-k)!} \ell^k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}. \end{aligned}$$

An $\mathbb{K}$ -linear map $\ell^k: \odot^k(sV) \rightarrow sV$ is the same as a symmetric $\mathbb{K}$ -linear map $\otimes^k(sV) \rightarrow sV$, ie one for which interchanging inputs sv_i, sv_{i+1} introduces a sign $(-1)^{|sv_i||sv_{i+1}|}$. This is in turn equivalent to the desuspension $s^{-1} \circ \ell^k \circ s^{\otimes k}: \otimes^k V \rightarrow V$ being skew-symmetric, that is, interchanging inputs v_i, v_{i+1} introduces a sign $-(-1)^{|v_i||v_{i+1}|}$. We can thus equivalently define an $\mathcal{L}_\infty$ algebra as a sequence of maps $\ell^k: \odot^k sV \rightarrow sV$ of degree 1 satisfying the quadratic relations

$$(2-2) \quad \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(k, n-k)} \diamond(\sigma, V; sv_1, \dots, sv_n) \ell^{n-k+1}(\ell^k(sv_{\sigma(1)} \odot \cdots \odot sv_{\sigma(k)}) \odot sv_{\sigma(k+1)} \odot \cdots \odot sv_{\sigma(n)}) = 0,$$

or in terms of the degree $2-k$ desuspended maps $s^{-1} \circ \ell^k \circ s^{\otimes k}: \otimes^k V \rightarrow V$, which are skew-symmetric and satisfy the same relations with a few extra signs.

Definition 2.4 Given an $\mathcal{L}_\infty$ algebra as above, the chain complex $(\bar{S}(sV), \hat{\ell})$ is called its *bar complex*, denoted by $\mathcal{B}V$.

Remark 2.5 The bar complex is generally much larger than the chain complex (V, ℓ^1) . There is also an $\mathcal{A}_\infty$ analogue. Technically it is the *reduced* bar complex, whereas the unreduced version has underlying $\mathbb{K}$ -module $S(sV)$ instead of $\bar{S}(sV)$. There is also a related but somewhat different complex, the Chevalley–Eilenberg complex of V , which plays an important role in Lie algebra cohomology. It is defined by dualizing the k -to-one operations ℓ^k to produce one-to- k operations on the dual space $(sV)^\vee$, and then assembling these into a differential on the symmetric tensor algebra of $(sV)^\vee$ via the cobar construction.

Remark 2.6 Recall that a DGLA is a graded $\mathbb{K}$ -module with a degree-one differential ∂ and a degree-zero bracket $[-, -]$ which is graded skew-symmetric, $[x, y] = -(-1)^{|x||y|}[y, x]$, satisfying the Jacobi identity

$$[[x, y], z] + (-1)^{|z|(|x|+|y|)}[[z, x], y] + (-1)^{|x|(|y|+|z|)}[[y, z], x] = 0$$

and the Leibniz rule

$$\partial[x, y] = [\partial x, y] + (-1)^{|x|}[x, \partial y].$$

This corresponds to the special case of an $\mathcal{L}_\infty$ algebra with $\ell^k \equiv 0$ for all $k \geq 3$. More precisely, given such an $\mathcal{L}_\infty$ algebra structure on V , we recover the structure of a DGLA by setting $[x, y] := s^{-1}[-, -] \circ (s \otimes s)(x \otimes y) = (-1)^{|x|} s^{-1} \ell^2(sx, sy)$.

Remark 2.7 Instead of working with the quotient spaces $\otimes^k V / \Sigma_k$, we could alternatively work with the fixed point spaces $(\otimes^k V)^{\Sigma_k}$ and define the reduced symmetric tensor coalgebra by $\bar{S}_{\text{fix}} V := \bigoplus_{k \geq 1} (\otimes^k V)^{\Sigma_k}$. Here the coproduct on $\bar{S}_{\text{fix}} V$ is given by the restriction of the coproduct on $\bar{T} V$. Indeed, the obvious projection map from $\bar{S}_{\text{fix}} V$ to $\bar{S} V$ is an isomorphism of coalgebras, with inverse $N: \bar{S} V \rightarrow \bar{S}_{\text{fix}} V$ given by

$$N(v_1 \odot \cdots \odot v_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \sigma \cdot (v_1 \otimes \cdots \otimes v_k).$$

2.1.3 $\mathcal{L}_\infty$ homomorphisms, quasi-isomorphisms, homotopies, and homotopy equivalences We now define more key players in the homotopy category of $\mathcal{L}_\infty$ algebras. For coalgebras (C, Δ) and (C', Δ') , an $\mathbb{K}$ -module map $F: C \rightarrow C'$ is a coalgebra homomorphism if

$$\Delta' \circ F = (F \otimes F) \circ \Delta.$$

Consider two $\mathcal{L}_\infty$ algebras V and V' with corresponding coderivations $\hat{\ell}: \bar{S}(sV) \rightarrow \bar{S}(sV)$ and $\hat{\ell}': \bar{S}(sV') \rightarrow \bar{S}(sV')$.

Definition 2.8 An $\mathcal{L}_\infty$ homomorphism from V to V' is a degree-zero coalgebra homomorphism $\hat{\Phi}: \bar{S}(sV) \rightarrow \bar{S}(sV')$ satisfying $\hat{\Phi} \circ \hat{\ell} = \hat{\ell}' \circ \hat{\Phi}$.

A coalgebra homomorphism $\hat{\Phi}: \bar{S}(sV) \rightarrow \bar{S}(sV')$ is uniquely determined by the maps $\Phi^k: \odot^k(sV) \rightarrow sV'$ given by the composition

$$\odot^k(sV) \hookrightarrow \bar{S}(sV) \xrightarrow{\hat{\Phi}} \bar{S}(sV') \rightarrow sV',$$

and any sequence of maps $\Phi^k: \odot^k(sV) \rightarrow sV'$ can be uniquely extended to a coalgebra homomorphism $\hat{\Phi}: \bar{S}(sV) \rightarrow \bar{S}(sV')$ by the formula

$$\hat{\Phi}(v_1 \odot \cdots \odot v_n) := \sum_{\substack{k \geq 1 \\ i_1 + \cdots + i_k = n}} \sum_{\sigma \in \Sigma_n} \frac{\diamond(\sigma, V; v_1, \dots, v_n)}{k! i_1! \cdots i_k!} (\Phi^{i_1} \odot \cdots \odot \Phi^{i_k})(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}).$$

We will typically write an $\mathcal{L}_\infty$ homomorphism as $\Phi: V \rightarrow V'$ and think of it either as a sequence of maps $s^{-1} \circ \Phi^k \circ s^{\odot k}: \odot^k V \rightarrow V$, $k = 1, 2, 3, \dots$, where Φ^k has degree $1 - k$, or equivalently as a single map of bar complexes $\hat{\Phi}: \bar{S}(sV) \rightarrow \bar{S}(sV')$ of degree 0.

Now consider an $\mathcal{L}_\infty$ algebra V with operations $\ell_V^k: \odot^k sV \rightarrow sV$, and suppose that C is a commutative differential graded algebra (CDGA) with differential $\partial: C \rightarrow C$. In this case one can define a new $\mathcal{L}_\infty$

algebra with underlying $\mathbb{K}$ -module $V \otimes_{\mathbb{K}} C$. The $\mathcal{L}_{\infty}$ operations $\ell_{V \otimes C}^k: \odot^k(sV \otimes C) \rightarrow sV \otimes C$ are given by

$$\ell_{V \otimes C}^1(sv \otimes c) = \ell_V^1(sv) \otimes c + (-1)^{|sv|} sv \otimes \partial(c)$$

for $k = 1$, and

$$\ell_{V \otimes C}^k(sv_1 \otimes c_1, \dots, sv_k \otimes c_k) = (-1)^{\sum_{j < i} |sv_i| |c_j|} \ell_V^k(sv_1, \dots, sv_k) \otimes c_1 \cdots c_k$$

for $k \geq 2$.

Let $\mathbb{K}[t, dt]$ denote the CDGA which is freely generated as a graded algebra by symbols t and dt with degrees $|t| = 0$ and $|dt| = 1$, with differential $\partial(t) = dt$ and $\partial(dt) = 0$. We view $\mathbb{K}[t, dt]$ as the algebraic differential forms on the interval $[0, 1]$. As a special case of the above construction, we can consider $V \otimes \mathbb{K}[t, dt]$. Following [23], we can write a typical element of $sV \otimes \mathbb{K}[t, dt]$ as $P(t) + Q(t) dt$, where $P(t)$ and $Q(t)$ are polynomials with coefficients in sV . The $\mathcal{L}_{\infty}$ operations can then be written as

$$\ell^1(P(t) + Q(t) dt) = \ell^1(P(t)) + (-1)^{|P(t)|} \frac{dP}{dt} dt + \ell^1(Q(t)) dt$$

and

$$\begin{aligned} \ell^k(P_1(t) + Q_1(t) dt, \dots, P_k(t) + Q_k(t) dt) = \\ \ell^k(P_1(t), \dots, P_k(t)) + \sum_{i=1}^k (-1)^{|P_{i+1}| + \dots + |P_k|} \ell^k(P_1(t), \dots, Q_i(t), \dots, P_k(t)) dt \end{aligned}$$

for $k \geq 2$.

For $\mathcal{L}_{\infty}$ homomorphisms $\Phi: V \rightarrow V'$ and $\Psi: V' \rightarrow V''$, the composite $\mathcal{L}_{\infty}$ homomorphism $V \rightarrow V''$ is defined on the level of bar complexes by simply $\widehat{\Psi} \circ \widehat{\Phi}: \overline{S}(sV) \rightarrow \overline{S}(sV'')$. For any $t_0 \in [0, 1]$, there is an $\mathcal{L}_{\infty}$ homomorphism $\text{Eval}_{t=t_0}: V \otimes \mathbb{K}[t, dt] \rightarrow V$, whose component with one input and one output sends $P(t) + Q(t) dt$ to $P(t_0)$, and which has no higher components.

Definition 2.9 Two $\mathcal{L}_{\infty}$ homomorphisms $\Phi, \Psi: V \rightarrow V'$ are $\mathcal{L}_{\infty}$ homotopic if there exists an $\mathcal{L}_{\infty}$ homomorphism $H: V \rightarrow V' \otimes \mathbb{K}[t, dt]$ such that $\text{Eval}_{t=0} \circ H = \Phi$ and $\text{Eval}_{t=1} \circ H = \Psi$.

A basic fact is that $\mathcal{L}_{\infty}$ homotopy is an equivalence relation (see [23, Section 2]).

Definition 2.10 Two $\mathcal{L}_{\infty}$ algebras V and V' are homotopy equivalent if there exist $\mathcal{L}_{\infty}$ homomorphisms $\Phi: V \rightarrow V'$ and $\Psi: V' \rightarrow V$ such that the compositions $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are homotopic to the identity (that is, the $\mathcal{L}_{\infty}$ endomorphism whose component with one input and one output is the identity and whose higher components are trivial).

2.2 Filtered $\mathcal{L}_{\infty}$ algebras

An $\mathcal{L}_{\infty}$ homomorphism Φ from V to V' is a *quasi-isomorphism* if the homology-level map $H(V, \ell_V^1) \rightarrow H(V', \ell_{V'}^1)$ induced by Φ^1 is an isomorphism. The following is a standard consequence of the homological perturbation lemma:

Theorem 2.11 *If $\mathbb{K}$ is a field, an $\mathcal{L}_\infty$ quasi-isomorphism of $\mathcal{L}_\infty$ algebras over $\mathbb{K}$ is automatically an $\mathcal{L}_\infty$ homotopy equivalence.*

In particular, any $\mathcal{L}_\infty$ algebra which is acyclic (with respect to ℓ^1) is homotopy equivalent to the trivial $\mathcal{L}_\infty$ algebra. This means that $\mathcal{L}_\infty$ algebras up to homotopy equivalence, at least over a field, are not powerful enough to model the geometric situations we have in mind. For example, for any star-shaped domain in $\mathbb{C}^n$, the $\mathcal{L}_\infty$ structures on both symplectic cochains and S^1 -equivariant symplectic cochains are acyclic, even though there are plenty of nonsymplectomorphic pairs. To remedy this and incorporate more quantitative symplectic geometric input, we now consider $\mathcal{L}_\infty$ algebras with filtrations.

Let R be a commutative ring. We say that R has a *decreasing filtration* if there are R -linear subspaces $\mathcal{F}_{\geq a} R \subset R$ for $a \in \mathbb{R}$, with

- $\mathcal{F}_{\geq a} R \subset \mathcal{F}_{\geq a'} R$ for $a > a'$,
- $(\mathcal{F}_{\geq a} R) \cdot (\mathcal{F}_{\geq a'} R) \subset \mathcal{F}_{\geq a+a'} R$,
- $\bigcup_{a \in \mathbb{R}} \mathcal{F}_{\geq a} R = R$.

The filtration makes R into a topological ring, wherein a basis of neighborhoods at $x \in R$ is given by $x + \mathcal{F}_{\geq a} R$, $a \in \mathbb{R}$. A sequence $x_i \in R$ is said to be *Cauchy* if for any a there exists N such that $x_i - x_j \in \mathcal{F}_{\geq a} R$ for all $i, j \geq N$, and we say that R is *complete* if every Cauchy sequence converges.

To give our main example, suppose that $\mathbb{K}$ is a commutative ring containing $\mathbb{Q}$ as before. The (universal) Novikov ring $\Lambda_{\geq 0}$ over $\mathbb{K}$ is the ring of formal series

$$\Lambda_{\geq 0} := \left\{ \sum_{i=1}^{\infty} c_i T^{a_i} : c_i \in \mathbb{K}, a_i \in \mathbb{R}_{\geq 0}, \lim_{i \rightarrow \infty} a_i = +\infty \right\},$$

where T is a formal variable. The filtration is given by

$$\mathcal{F}_{\geq a} \Lambda_{\geq 0} := T^a \cdot \Lambda_{\geq 0} = \left\{ \sum_{i=1}^{\infty} c_i T^{a_i} : a_i \geq a \right\}$$

for $a \geq 0$ and $\mathcal{F}_{\geq a} V := V$ for $a < 0$. There is also the (universal) Novikov field¹³ Λ over $\mathbb{K}$, defined by

$$\Lambda := \left\{ \sum_{i=1}^{\infty} c_i T^{a_i} : c_i \in \mathbb{K}, a_i \in \mathbb{R}, \lim_{i \rightarrow \infty} a_i = +\infty \right\},$$

which is also complete and contains $\Lambda_{\geq 0}$ as a subring.

If V is an R -module, a decreasing filtration consists of R -linear subspaces $\mathcal{F}_{\geq a} V \subset V$, $a \in \mathbb{R}$, satisfying

- $\mathcal{F}_{\geq a} V \subset \mathcal{F}_{\geq a'} V$ for $a > a'$,
- $(\mathcal{F}_{\geq a} R) \cdot (\mathcal{F}_{\geq a'} V) \subset \mathcal{F}_{\geq a+a'} V$,
- $\bigcup_{a \in \mathbb{R}} \mathcal{F}_{\geq a} V = V$.

¹³Strictly speaking Λ is not a field unless $\mathbb{K}$ is a field, in which case Λ is the field of fractions of $\Lambda_{\geq 0}$.

The filtration similarly induces a topology on V , and we define completeness for V in essentially the same way as for R . A filtration on V induces one on $\bar{S}(sV)$, namely

$$\mathcal{F}_{\geq a}(\odot^k sV) := \bigcup_{a_1 + \dots + a_k \geq a} \mathcal{F}_{\geq a_1} sV \odot \dots \odot \mathcal{F}_{\geq a_k} sV.$$

Definition 2.12 A *filtered $\mathcal{L}_\infty$ algebra* over R is an $\mathcal{L}_\infty$ algebra over R whose underlying R -module V is equipped with a decreasing filtration $\{\mathcal{F}_{\geq a} V\}$ such that the coderivation $\hat{\ell} : \bar{S}(sV) \rightarrow \bar{S}(sV)$ preserves the induced filtration on $\bar{S}(sV)$.

In terms of the operations $\ell^k : \odot^k sV \rightarrow sV$, this corresponds to having

$$\ell^k(\mathcal{F}_{\geq a_1} sV \odot \dots \odot \mathcal{F}_{\geq a_k} sV) \subset \mathcal{F}_{\geq (a_1 + \dots + a_k)} sV.$$

We similarly define filtered $\mathcal{L}_\infty$ homomorphisms, homotopies, and homotopy equivalences by requiring all structure maps to respect filtrations; see the aforementioned references for more details.

Any filtered R -module can be canonically completed, and we will always implicitly assume this has been done. A filtered $\mathcal{L}_\infty$ algebra V is *complete* if the underlying R -module is complete. Filtered $\mathcal{L}_\infty$ algebras have better functoriality properties; for example, Maurer–Cartan elements can be pushed forward (see Section 2.3 below). Any filtered $\mathcal{L}_\infty$ algebra can be canonically completed by completing the R -module V and extending the $\mathcal{L}_\infty$ operations, and we will also assume this has been done. Moreover, from now on in filtered settings we take all tensor products and direct sums to be completed, although we suppress this from the notation. For example, for V a filtered $\mathcal{L}_\infty$ algebra, $\bar{S}(sV)$ will denote the naive reduced symmetric tensor algebra of sV , but completed with respect to the natural filtration-induced topology.

In this paper, all of the filtered $\mathcal{L}_\infty$ algebras we encounter from symplectic geometry will be defined over $\Lambda_{\geq 0}$, such that the underlying $\Lambda_{\geq 0}$ -module V is torsion-free. In this case, the filtration on V is actually redundant, as it is uniquely determined by $\mathcal{F}_{\geq a} V = T^a \cdot V$ for $a \geq 0$ and $\mathcal{F}_{\geq a} V = V$ for $a < 0$. From now on we will always assume the filtration is of this form and will therefore typically omit the qualifier “filtered”, unless we wish to emphasize the presence of filtrations.

Remark 2.13 One defines $\mathcal{L}_\infty$ algebras with *increasing* filtrations in an analogous way. In the context of exact symplectic manifolds, the invariants we consider can also be defined as $\mathcal{L}_\infty$ algebras over $\mathbb{K}$ with increasing filtrations, where $\mathcal{F}_{\leq a} V$ roughly corresponds to the span of (words of) Reeb orbits with (total) action at most a (see Section 1). By contrast, we will see in Section 3 that working with decreasing filtrations over the Novikov ring as above remembers only *differences* between actions. Despite this apparent loss of information, the latter will suffice to define symplectic capacities, and is more effective when discussing eg nonexact symplectic manifolds or Maurer–Cartan theory.

2.3 Maurer–Cartan theory

Given a filtered $\mathcal{L}_\infty$ algebra V over $\Lambda_{\geq 0}$, an element $\mathfrak{m} \in \mathcal{F}_{>0} V := \bigcup_{\varepsilon > 0} \mathcal{F}_{\geq \varepsilon} V$ with $|\mathfrak{m}| = 0$ is *Maurer–Cartan* if it satisfies

$$\sum_{k=1}^{\infty} \frac{1}{k!} \ell^k(\underbrace{s\mathfrak{m}, \dots, s\mathfrak{m}}_k) = 0.$$

All of the summands land in the degree-one part of sV ; equivalently, we have $|\mathfrak{m}| = 1$ and all of the summands land in the degree-two part of V . In the unfiltered setting this equation involves an infinite sum and hence is actually ill defined. However, since V is filtered we can make sense of the Maurer–Cartan equation by asking the infinite sum to converge to zero in the filtration-induced topology. Equivalently, if

$$\exp(s\mathfrak{m}) := \sum_{k=1}^{\infty} \frac{1}{k!} (\underbrace{s\mathfrak{m} \odot \dots \odot s\mathfrak{m}}_k) \in \mathcal{B}V,$$

then $\mathfrak{m}$ satisfies the Maurer–Cartan equation if and only if $\widehat{\ell}(\exp(s\mathfrak{m})) = 0$.

Two Maurer–Cartan elements $\mathfrak{m}, \mathfrak{m}' \in V$ are said to be *gauge equivalent* if there exists a Maurer–Cartan element $g \in V \otimes \Lambda_{\geq 0}[t, dt]$ with $\text{Eval}_{t=0}(g) = \mathfrak{m}$ and $\text{Eval}_{t=1}(g) = \mathfrak{m}'$. Also, given an $\mathcal{L}_\infty$ homomorphism $\Phi: V \rightarrow V'$, we can push forward a Maurer–Cartan element $\mathfrak{m} \in V$ to one in V' via

$$\Phi_*(\mathfrak{m}) := \sum_{k=1}^{\infty} \frac{1}{k!} s^{-1} \Phi^k(\underbrace{s\mathfrak{m}, \dots, s\mathfrak{m}}_k).$$

A basic fact (see, for example, [23, Section 2]) is that the set of Maurer–Cartan elements modulo gauge equivalence is a filtered $\mathcal{L}_\infty$ homotopy invariant of V .

Given a Maurer–Cartan element $\mathfrak{m} \in V$, we can deform ℓ^1 to a new differential $\ell_{\mathfrak{m}}^1$, defined by

$$\ell_{\mathfrak{m}}^1(sx) := \sum_{i=1}^{\infty} \frac{1}{i!} \ell^{1+i}(sx, \underbrace{s\mathfrak{m}, \dots, s\mathfrak{m}}_i).$$

In fact, the entire $\mathcal{L}_\infty$ structure gets deformed via

$$\ell_{\mathfrak{m}}^k(sx_1, \dots, sx_k) := \sum_{i=1}^{\infty} \frac{1}{i!} \ell^{k+i}(sx_1, \dots, sx_k, \underbrace{s\mathfrak{m}, \dots, s\mathfrak{m}}_i).$$

Since $\mathfrak{m}$ lives in the positive part $\mathcal{F}_{>0} V$ of the filtration, these deformed operations preserve the same decreasing filtration,¹⁴ and one can check that they still satisfy the $\mathcal{L}_\infty$ structure equations. Also, since $|\mathfrak{m}| = 0$, all of these terms have the same degree, although it is sometimes useful to relax this condition by asking only that $|\mathfrak{m}|$ be even, in which case we still get a deformed $\mathcal{L}_\infty$ structure but with only a $\mathbb{Z}/2$ grading. We will denote the $\mathcal{L}_\infty$ algebra deformed by the Maurer–Cartan element $\mathfrak{m}$ by $V_{\mathfrak{m}}$.

¹⁴We point out that if we instead try to work with *increasing* filtrations as in Remark 2.13 then the analogous deformed $\mathcal{L}_\infty$ operations do not a priori preserve the filtration.

Remark 2.14 Maurer–Cartan elements play an important role in deformation theory, and their most standard usage is in *formal* deformation theory. For V an unfiltered $\mathcal{L}_\infty$ algebra over $\mathbb{K}$, we can tensor V with say the formal power series ring $R[[q]]$ and then use the q -adic filtration to make sense of the Maurer–Cartan equation in $V \otimes R[[q]]$. We call such Maurer–Cartan elements *formal* to distinguish them from the *filtered* Maurer–Cartan elements considered above. In fact, the $\mathcal{L}_\infty$ algebra V defines a *formal deformation functor* which inputs an Artinian local $\mathbb{K}$ -algebra A and outputs the set of Maurer–Cartan elements in $V \otimes A$ modulo gauge equivalence. Deformation problems in geometry typically give rise to such formal deformation functors, and the Deligne philosophy states that, at least in characteristic zero, every nice deformation functor should be represented by some natural $\mathcal{L}_\infty$ algebra V (see [47] for a modern treatment).

3 Rational symplectic field theory

In this section we describe the basic geometry and algebra of rational symplectic field theory, which will serve as the backbone for the subsequent sections of this paper. After explaining the general geometric setup, defining punctured pseudoholomorphic curves, and recalling their compactness properties, we finally present the relevant RSFT formalism. Our approach is somewhat nonstandard in that we phrase everything in terms of filtered $\mathcal{L}_\infty$ algebras, giving a language which is particularly useful for quantitative applications, and also for identifying parallels in Floer theory. We will subsequently build on this formalism in Section 5 by introducing additional $\mathcal{L}_\infty$ augmentations, which we use in Section 6 to construct symplectic capacities.

Symplectic field theory is an elaborate algebraic formalism which packages together counts of punctured pseudoholomorphic curves in symplectic manifolds with certain types of cylindrical ends. After Gromov’s seminal compactness theorem and the subsequent development of Gromov–Witten theory, Floer homology, and so on, it was discovered that punctured pseudoholomorphic curves in symplectic cobordisms also admit a good compactness theory [4; 35]. In [20], Eliashberg, Givental, and Hofer outlined the general formalism of symplectic field theory, which incorporates pseudoholomorphic curves of all genera with arbitrary numbers of positive and negative punctures. In a spirit akin to topological field theories, one associates data to both contact manifolds and to symplectic cobordisms between contact manifolds. A key tool in SFT is the “stretching the neck” procedure, which allows one to cut symplectic manifolds along contact-type hypersurfaces and thereby decompose symplectic manifolds into potentially simpler pieces. For example, this procedure makes it possible to decompose the Gromov–Witten invariants of a closed symplectic manifold into the SFT invariants of a symplectic filling and a symplectic cap. Eliashberg, Givental, and Hofer also described simplified invariants such as rational symplectic field theory, which involves only genus-zero curves, and the contact homology algebra, which further restricts to rational curves with only one positive end. One can also consider relative analogues, replacing contact manifolds by Legendrian submanifolds and symplectic cobordisms by Lagrangian cobordisms between Legendrians.

Since its introduction, SFT has produced a large number of impressive geometric applications, with many more still expected. Most but not all applications so far have centered around the contact homology algebra and its linearized version, and especially their relative cousins. In general, computing the full SFT in examples and making fruitful use of its elaborate structure, especially in the presence of higher-genus curves, rotating asymptotic markers, chain-level structures, etc is still a highly active area of research.

Remark 3.1 (on transversality and virtual techniques in SFT) As already mentioned in Disclaimer 1.9, it is well known that the compactified moduli spaces of curves involved in SFT are not generally transversely cut out within a classical perturbation framework. The typical situation is that, for a generic choice of almost complex structure J , all relevant somewhere injective curves will be regular, but their multiple covers might appear with higher than expected dimension. For example, one often encounters negative index branched covers of trivial cylinders in symplectizations which necessarily persist for any choice of almost complex structure. Even in the seemingly innocuous case of cobordisms between four-dimensional ellipsoids, some of the counts one wishes to perform might be represented by multilevel buildings. Such configurations have expected codimension at least one and hence ought not to exist for a generic J , if transversality were to hold.

This means that the structures outlined in [20] can only be achieved by interpreting curve counts in a suitable virtual sense. The situation is similar in Gromov–Witten theory or Floer theory, although in those cases one can often get away with classical perturbation techniques under additional topological assumptions such as semipositivity. There are several different virtual frameworks being developed to achieve the goals of SFT. See, for example, [22; 37; 44; 58]. Arguments in this paper are based on the general structural properties of rational symplectic field theory and not on the particulars of any virtual perturbation scheme. For some of the computations given in Section 6.3, we implicitly invoke the standard assumption that, for generic J , any curve which is already regular will persist in the presence of virtual perturbations.¹⁵

Addendum 3.2 In the followup paper [54], we construct another sequence of capacities, denoted by $\{\tilde{g}_k\}$, which circumvents SFT transversality and agrees with the sequence $\{g_k\}$ in many situations. We expect a similar approach to extend to many of the other capacities introduced in this paper.

3.1 Geometric setup

In order to make sense of symplectic invariants defined in terms of pseudoholomorphic curves, we will need to restrict the class of symplectic manifolds under consideration and choose various pieces of auxiliary data (eg an almost complex structure J). Adapting the general setup of [20], we focus our attention on certain symplectic cobordisms between contact manifolds. The formalism is simplest when these cobordisms are exact, although we also explain in Section 4 how this can be extended to the case of nonexact cobordisms.

¹⁵This is sometimes referred to as the “Obamacare axiom”, the slogan being “if you like your curves you can keep them”.

Recall that a *contact form* α on an odd-dimensional oriented manifold Y^{2n-1} is a one-form such that $\alpha \wedge (d\alpha)^{\wedge(n-1)} > 0$. A (cooriented) contact structure on Y is a hyperplane distribution $\xi \subset TY$ of the form $\xi = \ker \alpha$ for some contact form α . By “strict contact manifold” we will mean a pair (Y, α) consisting of a manifold Y and a contact one-form α . The contact form α is preserved by a canonical vector field R_α called the Reeb vector field, defined by the properties $d\alpha(R_\alpha, -) = 0$ and $\alpha(R_\alpha) = 1$. For $T > 0$, a T -periodic orbit is by definition a T -periodic trajectory of R_α , ie a path $\gamma: [0, T] \rightarrow Y$ such that $\dot{\gamma}(t) = R_\alpha(\gamma(t))$ and $\gamma(T) = \gamma(0)$. The period of γ is equivalently given by $\int \gamma^* \alpha$, and we also say that γ has *action* $\mathcal{A}_\alpha(\gamma) = T$. In the sequel we will frequently omit decorative subscripts and superscripts unless needed to clarify ambiguities, for example denoting $\mathcal{A}_\alpha$ by simply $\mathcal{A}$. We will also often suppress symplectic forms, contact forms, etc from the notation when no confusion should arise.

Every Reeb orbit γ has an underlying *simple* (ie embedded) Reeb orbit $\bar{\gamma}$. We denote by κ_γ the covering multiplicity of γ , and sometimes write $\gamma = \bar{\gamma}^{\kappa_\gamma}$. We will assume throughout that every simple Reeb orbit $\bar{\gamma}$ has a chosen basepoint $m_{\bar{\gamma}}$ in its image. This also induces a basepoint on the image of any multiple cover $\bar{\gamma}^{\kappa_\gamma}$. Parametrized Reeb orbits come in S^1 -families, but from now we will consider all orbits in the same S^1 -family to be equivalent and work only with the representative having $\gamma(0)$ as its basepoint.

Remark 3.3 Since we do not distinguish between Reeb orbits in the same S^1 -family, we are effectively working with unparametrized Reeb orbits everywhere, and indeed SFT is “by default” an S^1 -equivariant theory. In fact it is also possible to use SFT to construct nonequivariant invariants by a more refined use of asymptotic markers; see Bourgeois and Oancea’s model for symplectic cohomology [5].

Observe that $d\alpha$ makes $\ker \alpha$ into a $(2n-2)$ -dimensional symplectic vector bundle over Y . Given a trivialization Ξ of this symplectic vector bundle over a Reeb orbit γ , the linearized Reeb flow along γ can be identified with a path of $(2n-2) \times (2n-2)$ symplectic matrices starting at the identity. Unless otherwise stated, we will always assume the contact form α is nondegenerate, which means this path ends on a matrix without 1 as an eigenvalue. Associated to any such path of symplectic matrices is its *Conley–Zehnder index* $CZ_\Xi(\gamma) \in \mathbb{Z}$, which depends on the homotopy class of the trivialization Ξ . In fact, the parity of $CZ_\Xi(\gamma)$ is determined just by the sign of $\mathbb{I} - P_\gamma$, where P_γ is the linearized Poincaré return map of γ , and hence it does not depend on any choice of trivialization.

To any contact manifold (Y, α) we can associate the symplectization, that is, the symplectic manifold $\mathbb{R}_s \times Y$ with exact symplectic form $d(e^s \alpha)$. We will also consider the half-symplectizations given by $(\mathbb{R}_\pm \times Y, d(e^s \alpha))$, where $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ (with s still used to denote the coordinate on the first factor). Due to the presence of a negative end, symplectizations do not satisfy the original “geometrically bounded” criterion of Gromov’s compactness theorem. Nevertheless, it turns out there is a good compactness theory for pseudoholomorphic curves in symplectizations, and more generally in symplectic manifolds with ends modeled on positive or negative half-symplectizations.

Definition 3.4 A *compact symplectic cobordism* is a compact symplectic manifold (X, ω) equipped with a contact form α on ∂X such that $\alpha = \lambda|_{\partial X}$ for some primitive one-form λ for ω defined near ∂X . A boundary component of X is called *positive* if $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ is positive with respect to the boundary orientation, otherwise it is called *negative*. We denote by $\partial^+ X$ the union of all positive boundary components of X and $\alpha^+ := \alpha|_{\partial^+ X}$ the induced contact form, and similarly for $\partial^- X$ and α^- .

Remark 3.5 We will often suppress the contact form α from the notation, or we will write (X, ω, α) when we wish to make its role explicit. Observe that the value of a symplectic capacity c on a symplectic filling (X, ω, α) is manifestly independent of the choice of α (due to monotonicity under symplectic embeddings), and the same is true for an exact symplectic capacity provided that $H^1(\partial X; \mathbb{R}) = 0$ (see Remark 3.10 below).

Definition 3.6 A compact symplectic cobordism with $\partial^- X = \emptyset$ is called a *symplectic filling* of $(\partial^+ X, \alpha^+)$, and a compact symplectic cobordism with $\partial^+ X = \emptyset$ is called a *symplectic cap* of $(\partial^- X, \alpha^-)$.

Given a compact symplectic cobordism (X, ω) with $\omega|_{\mathcal{O}_P(\partial X)} = d\lambda$, we can form a *completed symplectic cobordism* (also called a *symplectic manifold with cylindrical ends* in [20]) by attaching a positive half-symplectization $(\mathbb{R}_+ \times \partial^+ X, d(e^s \alpha^+))$ to the positive boundary of X and attaching a negative half-symplectization $(\mathbb{R}_- \times \partial^- X, d(e^s \alpha^-))$ to the negative boundary of X . Indeed, a standard observation is that the *Liouville vector field* Z_λ , defined near ∂X by the condition $Z_\lambda \lrcorner d\lambda = \lambda$, gives a natural collar coordinate s near ∂X , with respect to which λ looks like $e^s \alpha$, and hence can be smoothly extended to the cylindrical ends. Note that Z is outwardly transverse to each positive boundary component of X and inwardly transverse to each negative boundary component of X . We denote the completion by $(\widehat{X}, \widehat{\omega})$.

A compact symplectic cobordism is called *exact* if the one-form λ extends to a primitive for ω on all of X . This is also called a *Liouville cobordism* (with λ typically viewed as part of the data), and the case with no negative end is known as a *Liouville domain*.

Definition 3.7 Given symplectic fillings (X, ω, α) and (X', ω', α') , an *exact symplectic embedding* is a symplectic embedding $\iota: X \hookrightarrow X'$ such that $Q := X' \setminus \iota(\text{Int } X)$ is an exact symplectic cobordism with respect to the primitives α' on $\partial X'$ and $\iota_* \alpha$ on $\iota(\partial X)$.

This is equivalent to the existence of a one-form $\theta \in \Omega^1(Q)$ such that $d\theta = \omega'$, $\theta|_{\partial X'} = \alpha'$, and $\theta|_{\iota(\partial X)} = \iota_* \alpha$.

Lemma 3.8 Let (X, ω) be a symplectic filling of $(\partial X, \alpha)$ and let (X', λ') be a Liouville domain. Given a symplectic embedding $\iota: X \hookrightarrow X'$, put $Q := X' \setminus \iota(\text{Int } X)$, $\partial^+ Q := \partial X'$, and $\partial^- Q := \iota(\partial X)$. Then ι is exact if and only if $[\lambda'|_{\partial^- Q} - \iota_* \alpha] \in H^1(\partial^- Q; \mathbb{R})$ lies in the image of restriction map $H^1(Q, \partial^+ Q; \mathbb{R}) \rightarrow H^1(\partial^- Q; \mathbb{R})$.

Proof If ι is exact, then by definition there exists $\theta \in \Omega^1(Q)$ such that $d\theta = \omega'$, $\theta|_{\partial^+ Q} = \alpha' := \lambda'|_{\partial^+ Q}$, and $\theta|_{\partial^- Q} = \iota_*\alpha$. Then $\eta := \lambda' - \theta \in \Omega^1(Q)$ is closed and satisfies $\eta|_{\partial^+ Q} \equiv 0$ and $\eta|_{\partial^- Q} = \lambda'|_{\partial^- Q} - \iota_*\alpha$.

Conversely, let $\eta \in \Omega^1(Q)$ be a closed one-form such that $\eta|_{\partial^+ Q} \equiv 0$ and $[\eta|_{\partial^- Q}] = [\lambda'|_{\partial^- Q} - \iota_*\alpha] \in H^1(\partial^- Q; \mathbb{R})$. After a small modification (by adding an exact one-form supported near $\partial^- Q$) we can assume that $\eta|_{\partial^- Q} = \iota_*\alpha$, whence $\theta := \lambda' - \eta$ satisfies $d\theta = \omega'$, $\theta|_{\partial^+ Q} = \alpha'$, and $\theta|_{\partial^- Q} = \iota_*\alpha$. $\square$

Corollary 3.9 *A symplectic embedding of a symplectic filling (X, ω, α) into a Liouville domain (X, λ') is automatically exact provided that $H^1(\partial X; \mathbb{R}) = 0$.*

Remark 3.10 Given Liouville domains (X, λ) , (X', λ') , a symplectic embedding $\iota: X \hookrightarrow X'$ is automatically exact provided that $H^1(X; \mathbb{R}) = 0$, or more generally that the restriction map $H^1(X; \mathbb{R}) \rightarrow H^1(\partial X; \mathbb{R})$ vanishes. Indeed, in this case $[\lambda'|_{\partial^- Q} - \iota_*\alpha] \in H^1(\partial^- Q; \mathbb{R})$ vanishes if and only if $[\iota^*\lambda'|_{\partial X} - \alpha] = [(\iota^*\lambda' - \lambda)|_{\partial X}] \in H_1(\partial X; \mathbb{R})$ vanishes, and the latter clearly lies in the image of $H^1(X; \mathbb{R}) \rightarrow H^1(\partial X; \mathbb{R})$. Also, any generalized Liouville embedding in the sense of [32, Definition 1.23] (ie $[(\iota^*\lambda' - \lambda)|_{\partial X}] \in H^1(\partial X; \mathbb{R})$ vanishes) is exact.

Remark 3.11 Our definition of compact symplectic cobordism is sometimes called a *strong symplectic cobordism*. There are several other important types of symplectic cobordisms commonly appearing in the literature, for example *weak symplectic cobordisms*, *Weinstein cobordisms*, and *Stein cobordisms*. See [48] for a more comprehensive discussion.

Remark 3.12 (on gradings of orbits in symplectic fillings) Suppose that (X, ω) is a symplectic filling of (Y, α) . As mentioned above, each Reeb orbit of (Y, α) has a canonical $\mathbb{Z}/2$ grading. In some cases, this can naturally be upgraded. For example, if we assume that $2c_1(X, \omega) = 0$ and $H_1(X; \mathbb{Z}) = 0$, then each Reeb orbit γ of (Y, α) bounds a spanning surface in X , and there is a unique (up to homotopy) trivialization Ξ of the symplectic vector bundle (TX, ω) over γ which extends over this spanning surface. Two different choices of spanning surfaces define an element $A \in H_2(X; \mathbb{Z})$, and the two resulting Conley–Zehnder indices of γ differ by $2c_1(X, \omega) \cdot A$. Therefore, in this case each Reeb orbit has a canonical Conley–Zehnder index $\text{CZ}(\gamma) \in \mathbb{Z}$.

More generally, if $2c_1(X, \omega) = 0$ and $H_1(X, \omega)$ is torsion-free, we can still assign an integer Conley–Zehnder index to each Reeb orbit, although it is no longer canonical. Rather, the space of choices is a torsor over $H_1(X; \mathbb{Z})$, corresponding to a choice of trivialization of the symplectic vector bundle (TX, ω) over a set of basis elements for $H_1(X; \mathbb{Z})$.

Now suppose we have a symplectic cobordism (X_1, ω_1) with positive end $(\partial^+ X_1, \alpha_1^+)$ and another symplectic cobordism (X_2, ω_2) with negative end $(\partial^- X_2, \alpha_2^-)$. Assuming there is a strict contactomorphism $(\partial^+ X_1, \alpha_1^+) \cong (\partial^- X_2, \alpha_2^-)$ (ie a diffeomorphism $\partial^+ X_1 \cong \partial^- X_2$ pulling back α_2^- to α_1^+), we can use the collar coordinates to glue the positive boundary of X_1 with the negative boundary of X_2 , forming a new

symplectic cobordism $(X_1, \omega_1) \odot (X_2, \omega_2)$. In particular, if (X_1, ω_1) is a filling and (X_2, ω_2) is a cap, then $(X_1, \omega_1) \odot (X_2, \omega_2)$ is a closed symplectic manifold. One of the key features of SFT is that curves in such a glued cobordism can be understood in terms of curves in either of the two pieces.

Suppose that (Y, α) is a strict contact manifold, and let $\Gamma^+ = (\gamma_1^+, \dots, \gamma_{s^+}^+)$ and $\Gamma^- = (\gamma_1^-, \dots, \gamma_{s^-}^-)$ be tuples of Reeb orbits. We denote by $H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ the homology group of two-chains Σ in Y with $\partial\Sigma = \sum_{i=1}^{s^+} \gamma_i^+ - \sum_{j=1}^{s^-} \gamma_j^-$, modulo boundaries of three-chains in Y . This group is a torsor over $H_2(Y; \mathbb{Z})$. Integrating over the closed two-form $d\alpha$ defines a symplectic energy homomorphism

$$[d\alpha] \cdot - : H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \rightarrow \mathbb{R}.$$

In fact, by Stokes' theorem we have simply

$$[d\alpha] \cdot A = \sum_{i=1}^{s^+} \mathcal{A}_\alpha(\gamma_i^+) - \sum_{j=1}^{s^-} \mathcal{A}_\alpha(\gamma_j^-).$$

Remark 3.13 The notion of symplectic energy which we consider is purely homological and will mostly only be considered in the context of J -holomorphic curves or buildings. It is not the same as symplectic area, which would be given by integrating $d(e^s \alpha)$ and is typically infinite for the curves we consider.

Also, after choosing a trivialization Ξ of the symplectic vector bundle $(\ker \alpha, d\alpha)$ over the Reeb orbits of Γ^+ and Γ^- , we have a relative first Chern class homomorphism

$$c_1^\Xi(Y, \alpha) \cdot - : H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

For $[\Sigma] \in H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$, $c_1^\Xi(Y, \alpha) \cdot [\Sigma]$ is the obstruction to finding a nonvanishing section of $(\ker \alpha, d\alpha)$ over Σ which is constant along the ends with respect to the trivialization Ξ .

Similarly, suppose that (X, ω) is a symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) , and let $\Gamma^\pm = (\gamma_1^\pm, \dots, \gamma_{s^\pm}^\pm)$ be a tuple of Reeb orbits of $(Y^\pm, \alpha^\pm)$. Let $H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ denote the homology group of two-chains Σ in X with

$$\partial\Sigma = \sum_{i=1}^{s^+} \gamma_i^+ - \sum_{j=1}^{s^-} \gamma_j^-,$$

modulo boundaries of three-chains in X . Integrating over the symplectic form ω defines a symplectic energy homomorphism

$$[\omega] \cdot - : H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \rightarrow \mathbb{R}.$$

If (X, ω) is an *exact* symplectic cobordism, by Stokes' theorem we have

$$[\omega] \cdot A = \sum_{i=1}^{s^+} \mathcal{A}_{\alpha^+}(\gamma_i^+) - \sum_{j=1}^{s^-} \mathcal{A}_{\alpha^-}(\gamma_j^-).$$

Given a trivialization Ξ of the symplectic vector bundle (TX, ω) , we also have a relative first Chern class homomorphism

$$c_1^\Xi(X, \omega) \cdot - : H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

In the situation of a glued symplectic cobordism $(X_1, \omega_1) \odot (X_2, \omega_2)$ with $\Gamma_1^+ = \Gamma_2^-$, there is a natural concatenation map

$$\begin{aligned} H_2(X_1, \Gamma_1^+ \cup \Gamma_1^-; \mathbb{Z}) \otimes H_2(X_2, \Gamma_2^+ \cup \Gamma_2^-; \mathbb{Z}) &\rightarrow H_2(X_1 \odot X_2, \Gamma_2^+ \cup \Gamma_1^-; \mathbb{Z}), \\ A_1 \otimes A_2 &\mapsto A_1 \odot A_2. \end{aligned}$$

There are also similar concatenation maps for gluing a symplectization to a symplectic cobordism or gluing two symplectizations, of the form

$$\begin{aligned} H_2(X, \Gamma_1^+ \cup \Gamma_1^-; \mathbb{Z}) \otimes H_2(\partial^+ X, \Gamma_2^+ \cup \Gamma_2^-; \mathbb{Z}) &\rightarrow H_2(X, \Gamma_2^+ \cup \Gamma_1^-; \mathbb{Z}), \\ H_2(\partial^- X, \Gamma_1^+ \cup \Gamma_1^-; \mathbb{Z}) \otimes H_2(X, \Gamma_2^+ \cup \Gamma_2^-; \mathbb{Z}) &\rightarrow H_2(X, \Gamma_2^+ \cup \Gamma_1^-; \mathbb{Z}), \\ H_2(Y, \Gamma_1^+ \cup \Gamma_1^-; \mathbb{Z}) \otimes H_2(Y, \Gamma_2^+ \cup \Gamma_2^-; \mathbb{Z}) &\rightarrow H_2(Y, \Gamma_2^+ \cup \Gamma_1^-; \mathbb{Z}). \end{aligned}$$

The symplectic energy homomorphisms are additive with respect to these concatenation maps, as are the relative first Chern class maps (provided we pick appropriately compatible trivializations).

When discussing pseudoholomorphic curves, it will occasionally be convenient to let $\widehat{\widehat{X}}$ denote the compactification of $\widehat{X}$ given by replacing $\mathbb{R}_+ = [0, \infty)$ with $\overline{\mathbb{R}}_+ := [0, +\infty]$ and $\mathbb{R}_- = (-\infty, 0]$ with $\overline{\mathbb{R}}_- := [-\infty, 0]$. We of course have a natural identification $H_2(\widehat{\widehat{X}}, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \cong H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$. We will be considering punctured curves in $\widehat{X}$ which extend by continuity to curves $u: \Sigma \rightarrow \widehat{\widehat{X}}$, and hence naturally define classes in $A_u \in H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$.

Remark 3.14 (on symplectizations as cobordisms) For (Y, α) a strict contact manifold, the symplectization $(\mathbb{R}_s \times Y, d(e^s \alpha))$ looks roughly like the completion of the compact symplectic cobordism $(X_M, \omega_M) := ([-M, M] \times Y, d(e^s \alpha))$ as $M \rightarrow 0$. However, the existence of a global $\mathbb{R}$ -action by s -translations means that pseudoholomorphic curves in symplectizations behave somewhat differently, and we also define symplectic energy differently in symplectizations.

3.2 Punctured pseudoholomorphic curves

Given a strict contact manifold (Y, α) , an almost complex structure J on the symplectization $(\mathbb{R}_s \times Y, d(e^s \alpha))$ is *admissible* if

- J is invariant under s -translations and restricts to a $d\alpha$ -compatible almost complex structure on $\ker \alpha$,
- J sends ∂_s to R_α .

For a Riemann surface Σ and a point $p \in \Sigma$, an *asymptotic marker at p* is a real half-line in $T_p \Sigma$. There is a circle's worth of choices of asymptotic markers at p . Given an asymptotic marker at p , there is a contractible space of holomorphic identifications of $\mathcal{O}p(p)$ with the open unit disk $\mathbb{D}^2 \subset \mathbb{C}$ in such a way that the half-line at p is identified with the positive real direction at the origin. Endowing $\mathbb{D}^2$ with standard polar coordinates (r, θ) , any such identification induces local polar coordinates on $\mathcal{O}p(p)$.

Now fix an admissible J , and consider a collection of Reeb orbits $\gamma_1^+, \dots, \gamma_{s^+}^+$ and $\gamma_1^-, \dots, \gamma_{s^-}^-$ (possibly with repeats) with respect to α . A *pseudoholomorphic curve in the symplectization $\mathbb{R} \times Y$ with positive ends $\gamma_1^+, \dots, \gamma_{s^+}^+$ and negative ends $\gamma_1^-, \dots, \gamma_{s^-}^-$* consists of

- a closed Riemann surface Σ , with almost complex structure denoted by j ,
- a collection of pairwise distinct points $z_1^+, \dots, z_{s^+}^+, z_1^-, \dots, z_{s^-}^- \in \Sigma$, each equipped with an asymptotic marker,
- a map $u: \dot{\Sigma} \rightarrow \mathbb{R} \times Y$ satisfying $du \circ j = J \circ du$, where $\dot{\Sigma}$ denotes the punctured Riemann surface $\Sigma \setminus \{z_1^+, \dots, z_{s^+}^+, z_1^-, \dots, z_{s^-}^-\}$,
- for each z_i^+ , with corresponding coordinates (r, θ) compatible with the asymptotic marker, we have $\lim_{r \rightarrow 0} (\pi_{\mathbb{R}} \circ u)(r e^{i\theta}) = +\infty$ and $\lim_{r \rightarrow 0} (\pi_Y \circ u)(r e^{i\theta}) = \gamma_i(\frac{1}{2\pi} T_i^+ \theta)$,
- for each z_j^- , with corresponding coordinates (r, θ) compatible with the asymptotic marker, we have $\lim_{r \rightarrow 0} (\pi_{\mathbb{R}} \circ u)(r e^{i\theta}) = -\infty$ and $\lim_{r \rightarrow 0} (\pi_Y \circ u)(r e^{-i\theta}) = \gamma_j(\frac{1}{2\pi} T_j^- \theta)$.

Here $\pi_{\mathbb{R}}: \mathbb{R} \times Y \rightarrow \mathbb{R}$ and $\pi_Y: \mathbb{R} \times Y \rightarrow Y$ denote the two projection maps defined on the ends, and $T_k^\pm$ denotes the period of $\gamma_k^\pm$. We will freely identify the marked Riemann surface Σ with the punctured Riemann surface $\dot{\Sigma}$, which carry essentially the same data, and correspondingly we also refer to the marked points $z_k^\pm$ as punctures. For brevity, we sometimes refer to the above data as simply a “curve” and denote it by slight abuse of notation by $u: \Sigma \rightarrow \mathbb{R} \times Y$.

For shorthand, put $\Gamma^\pm = (\gamma_1^\pm, \dots, \gamma_{s^\pm}^\pm)$. Let $\mathcal{M}_{Y,g}^J(\Gamma^+; \Gamma^-)$ denote the moduli space of J -holomorphic curves in $\mathbb{R} \times Y$ with positive ends $\gamma_1^+, \dots, \gamma_{s^+}^+$ and negative ends $\gamma_1^-, \dots, \gamma_{s^-}^-$, with domain varying over all connected Riemann surfaces Σ of genus g with varying punctures $z_1^+, \dots, z_{s^+}^+, z_1^-, \dots, z_{s^-}^- \in \Sigma$ and varying asymptotic markers. In this moduli space two such curves are equivalent if the corresponding holomorphic maps differ by a biholomorphic reparametrization of their domains sending one ordered set of punctures with asymptotic markers to the other. A curve $u: \Sigma \rightarrow \mathbb{R} \times Y$ is called *simple* if it does not factor as $u' \circ f$ for a holomorphic map $f: \Sigma \rightarrow \Sigma'$ with $\deg(f) > 1$ and another curve $u': \Sigma' \rightarrow \mathbb{R} \times Y$. A curve u which does admit such a covering is called a *multiple cover* of u' .

For generic admissible J , the expected dimension of the moduli space $\Gamma^\pm = (\gamma_1^\pm, \dots, \gamma_{s^\pm}^\pm)$ is given by the index formula

$$\text{ind}(u) = (n-3)(2-2g-s^+-s^-) + \sum_{i=1}^{s^+} \text{CZ}(\gamma_i^+) - \sum_{j=1}^{s^-} \text{CZ}(\gamma_j^-) + 2c_1(Y, \alpha) \cdot A_u.$$

Here $A_u \in H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ is the homology class naturally associated to the curve u . Although the Conley–Zehnder index and relative first Chern class terms both depend on choices of trivializations, the full expression for $\text{ind}(u)$ does not.

An important feature of the moduli space $\mathcal{M}_{Y,g}^J(\Gamma^+; \Gamma^-)$ is that it admits a free $\mathbb{R}$ -action given by translations in the s coordinate, since J itself is globally s -translation invariant. By a Sard–Smale-type argument, for generic admissible J the quotient space $\mathcal{M}_{Y,g}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ near a simple curve is a smooth orbifold of dimension

$$(n-3)(2-2g-s^+-s^-) + \sum_{i=1}^{s^+} \text{CZ}(\gamma_i^+) - \sum_{j=1}^{s^-} \text{CZ}(\gamma_j^-) + 2c_1(Y, \alpha) \cdot A_u - 1.$$

For u regular, the existence of a free $\mathbb{R}$ -action implies that $\text{ind}(u) \geq 1$ unless u is a union of trivial cylinders. Here by *trivial cylinder* we mean a pseudoholomorphic curve $u: \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times Y$ such that $\pi_Y \circ u$ is constant in the $\mathbb{R}$ direction of the domain.

Similarly, let (X, ω) be a compact symplectic cobordism, with λ a primitive for ω on $\mathcal{O}p(\partial X)$ and $\alpha_{\pm} := \lambda|_{\partial_{\pm} X}$ the induced contact forms. Let $(\widehat{X}, \widehat{\omega})$ denote the completion. An almost complex structure J on $\widehat{X}$ is *admissible* if

- J is compatible with $\widehat{\omega}$,
- on $\widehat{X} \setminus X$, J is invariant under s -translations and restricts to a $d\alpha^{\pm}$ -compatible almost complex structure on $\ker \alpha^{\pm}$,
- on the end $\mathbb{R}_{\pm} \times \partial_{\pm} X$, J sends ∂_s to $R\alpha_{\pm}$.

Given Reeb orbits $\Gamma^+ = (\gamma_1^+, \dots, \gamma_{s^+}^+)$ with respect to α^+ and $\Gamma^- = (\gamma_1^-, \dots, \gamma_{s^-}^-)$ with respect to α^- , we define the moduli space $\mathcal{M}_{X,g}^J(\Gamma^+; \Gamma^-)$ in the same way as above, except that now the target space is the completed symplectic cobordism $\widehat{X}$ rather than the symplectization $\mathbb{R} \times Y$. For generic admissible J , near a simple curve u the moduli space $\mathcal{M}_{X,g}^J(\Gamma^+; \Gamma^-)$ is a smooth orbifold of dimension

$$\text{ind}(u) = (n-3)(2-2g-s^+-s^-) + \sum_{i=1}^{s^+} \text{CZ}(\gamma_i^+) - \sum_{j=1}^{s^-} \text{CZ}(\gamma_j^-) + 2c_1(X, \omega) \cdot A_u,$$

and there is no longer any $\mathbb{R}$ -action due to the lack of s -translation symmetry. As before, the element $A_u \in H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ is the homology class associated to u .

We define $\mathcal{M}_{Y,g,A}^J(\Gamma^+; \Gamma^-) \subset \mathcal{M}_{Y,g}^J(\Gamma^+; \Gamma^-)$ as the subspace of curves with associated homology class $A_u = A \in H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$. For $r \geq 1$ we can also define $\mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)$ in the same way except that the domain of any curve u is equipped with r additional pairwise distinct marked points $z_1, \dots, z_r \in \dot{\Sigma}$, and biholomorphic reparametrizations must respect these marked points (with their

ordering). The expected dimension of $\mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ is

$$(n-3)(2-2g-s^+-s^-) + \sum_{i=1}^{s^+} \text{CZ}(\gamma_i^+) - \sum_{j=1}^{s^-} \text{CZ}(\gamma_j^-) + c_1(X, \omega) \cdot A_u + 2r - 1.$$

The moduli spaces $\mathcal{M}_{X,g,A}^J(\Gamma^+; \Gamma^-)$ and $\mathcal{M}_{X,g,A,r}^J(\Gamma^+; \Gamma^-)$ are defined similarly.

Given a contact manifold (Y, α) , an admissible almost complex structure J , and a curve $u: \Sigma \rightarrow \mathbb{R} \times Y$ in $\mathcal{M}_{Y,g}^J(\Gamma^+; \Gamma^-)$, we have

$$0 \leq \int u^*(d\alpha) = [d\alpha] \cdot A_u = \sum_{i=1}^{s^+} \mathcal{A}(\gamma_i^+) - \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^-),$$

where the first inequality follows from compatibility of J with $d\alpha$ along $\ker \alpha$. In particular, this implies

$$(3-1) \quad \sum_{i=1}^{s^+} \mathcal{A}(\gamma_i^+) = \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^-) + [d\alpha] \cdot A_u \geq \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^-),$$

that is, the action is nondecreasing from the top to the bottom of a curve. Moreover, this inequality is strict unless the curve is a union of trivial cylinders or a multiple cover thereof.

Similarly, for (X, ω) a symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) , J an admissible almost complex structure, and $u: \Sigma \rightarrow \widehat{X}$ a curve in $\mathcal{M}_{X,g}^J(\Gamma^+; \Gamma^-)$, the symplectic energy of u is nonnegative and is given by the integral of the piecewise smooth two-form

$$(d\alpha^+)|_{\mathbb{R}_+ \times \partial^+ X} + \omega|_X + (d\alpha^-)|_{\mathbb{R}_- \times \partial^- X}.$$

This agrees with $[\omega] \cdot A_u$, where $A_u \in H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \cong H_2(\widehat{X}, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ is the homology class associated to u . In the case that the symplectic cobordism (X, ω) is *exact*, Stokes' theorem applied thrice gives

$$(3-2) \quad \sum_{i=1}^{s^+} \mathcal{A}(\gamma_i^+) = \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^-) + [\omega] \cdot A_u \geq \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^-).$$

3.3 SFT compactness and neck stretching

The SFT compactness theorem extends Gromov's compactness theorem to punctured curves in symplectic cobordisms. As in the case of closed curves in closed symplectic manifolds, we must add nodal configurations à la Kontsevich's stable map compactification. However, we must additionally allow multilevel pseudoholomorphic "buildings", consisting of various curves in different symplectic cobordisms whose asymptotic Reeb orbits agree end-to-end to form a chain.

More precisely, let (Y, α) be a strict contact manifold, and let J be an admissible almost complex structure. Let $\overline{\mathcal{M}}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ denote the partially compactified version of $\mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ in which

curves are no longer required to be connected and they are allowed to develop nodes, provided that no irreducible component is a constant sphere with less than three special points, a constant torus with no special points, or a disjoint union of trivial cylinders without marked points. The SFT compactness theorem [4] in this situation states that any sequence of curves $u_1, u_2, u_3, \dots \in \mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ converges in a certain suitable Gromov–Hofer topology to a *pseudoholomorphic building* consisting of curves $v_i \in \bar{\mathcal{M}}_{Y,g_i,A_i,r_i}^J(\Gamma_i^+; \Gamma_i^-)/\mathbb{R}$, $i = 1, \dots, a$, for some $a \geq 1$, such that

- (1) $\Gamma_i^+ = \Gamma_{i+1}^-$ for $i = 1, \dots, a-1$,
- (2) the total arithmetic genus is g ,
- (3) $A_1 \odot \dots \odot A_a = A$,
- (4) $r_1 + \dots + r_a = r$.

Each v_i corresponds to a “level” of the building, and we have listed them from bottom to top, ie v_1 is the “ground level”, etc. Note that symplectization curves are considered only up to the $\mathbb{R}$ -action translating in the $\mathbb{R}$ direction in the target. There is actually another ambiguity in the pseudoholomorphic building which is related to the asymptotic markers. Namely, suppose γ is a Reeb orbit appearing as a positive end of v_i and as a negative end of v_{i+1} , and as usual let κ_γ denote its covering multiplicity. The underlying simple Reeb orbit has a basepoint m_γ and there are κ_γ possibilities for the location of the corresponding asymptotic marker in the domain of v_i , and similarly for v_{i+1} . This gives κ_γ^2 possibilities, but in fact we do not distinguish between orbits of the $\mathbb{Z}_{\kappa_\gamma}$ -action which increments both choices according to the Reeb orientation.

Next, suppose that (X, ω) is a symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) . Let J be any admissible almost complex structure on $\widehat{X}$, and let $J^\pm$ denote the induced translation invariant almost complex structures on the symplectizations of $(Y^\pm, \alpha^\pm)$. Consider a sequence of curves $u_1, u_2, u_3, \dots \in \mathcal{M}_{X,g,A,r}^J(\Gamma^+; \Gamma^-)$. In this case the SFT compactness theorem takes a similar form, with the limiting pseudoholomorphic building consisting of curves $v_1, \dots, v_{a+b+1}$ with $a, b \geq 0$ and

- $v_i \in \bar{\mathcal{M}}_{Y^-,g_i,A_i,r_i}^{J^-}(\Gamma_i^+; \Gamma_i^-)/\mathbb{R}$ for $i = 1, \dots, a$,
- $v_i \in \bar{\mathcal{M}}_{X,g_i,A_i,r_i}^J(\Gamma_i^+; \Gamma_i^-)$ for $i = a+1$,
- $v_i \in \bar{\mathcal{M}}_{Y^+,g_i,A_i,r_i}^{J^+}(\Gamma_i^+; \Gamma_i^-)/\mathbb{R}$ for $i = a+2, \dots, a+b+1$,

again satisfying the conditions

- $\Gamma_i^+ = \Gamma_{i+1}^-$ for $i = 1, \dots, a+b$,
- the total arithmetic genus is g ,
- $A_1 \odot \dots \odot A_{a+b+1} = A$,
- $r_1 + \dots + r_{a+b+1} = r$,

and with the same ambiguities regarding symplectization curves and asymptotic markers.

Next, we recall the neck stretching procedure and the corresponding compactness result. Consider a symplectic cobordism $(X, \omega) = (X_1, \omega_1) \odot (X_2, \omega_2)$ obtained by gluing together two symplectic cobordisms along a common contact boundary (Y, α) . We assume that $\omega|_{\mathcal{O}p(Y)} = d\lambda$, with $\lambda|_Y = \alpha$, and hence the flow of the Liouville vector field Z_λ defines a collar coordinate s on $\mathcal{O}p(Y)$ with respect to which $(\mathcal{O}p(Y), \omega|_{\mathcal{O}p(Y)})$ looks like $((-\epsilon, \epsilon) \times Y, d(e^s \alpha))$. We will describe a family J_t of admissible almost complex structures on (X, ω) which limits to a singular one. We find it convenient to describe J_t on a varying family of symplectic manifolds (X_t, ω_t) which are all diffeomorphic to (X, ω) .¹⁶ Namely, for $t > 0$ set $X_t = X_1 \cup ([-t, 0] \times Y) \cup X_2$, with the symplectic form ω_t given by

- $e^{-t} \omega_1$ on X_1 ,
- $e^t \alpha$ on $[-t, 0] \times Y$,
- ω_2 on X_2 .

Let J be an admissible almost complex structure on X which on $(-\epsilon, \epsilon) \times Y$ is the restriction of an admissible almost complex structure J_Y on the symplectization of Y . Let J_i denote the restriction of J to X_i for $i = 1, 2$. We endow X_t with the almost complex structure J_t given by

- J_1 on X_1 ,
- the restriction of J_Y on $[-t, 0] \times Y$,
- J_2 on X_2 .

In this situation the SFT compactness theorem states that any sequence of curves $u_1, u_2, u_3, \dots \in \mathcal{M}_{X, g, r, A}^{J_{t_i}}(\Gamma^+; \Gamma^-)$ with $t_i \rightarrow +\infty$ converges to a pseudoholomorphic building consisting of curves $v_1, \dots, v_{a+b+c+2}$ with $a, b, c \geq 0$ and

- $v_i \in \overline{\mathcal{M}}_{\partial^- X_1, g_i, A_i, r_i}^{J_1^-}(\Gamma_i^+, \Gamma_i^-)/\mathbb{R}$ for $i = 1, \dots, a$,
- $v_i \in \overline{\mathcal{M}}_{X_1, g_i, A_i, r_i}^{J_1}(\Gamma_i^+, \Gamma_i^-)$ for $i = a + 1$,
- $v_i \in \overline{\mathcal{M}}_{Y, g_i, A_i, r_i}^{J_Y}(\Gamma_i^+, \Gamma_i^-)/\mathbb{R}$ for $i = a + 2, \dots, a + b + 1$,
- $v_i \in \overline{\mathcal{M}}_{X_2, g_i, A_i, r_i}^{J_2}(\Gamma_i^+, \Gamma_i^-)$ for $i = a + b + 2$,
- $v_i \in \overline{\mathcal{M}}_{\partial^+ X_2, g_i, A_i, r_i}^{J_2^+}(\Gamma_i^+, \Gamma_i^-)/\mathbb{R}$ for $i = a + b + 3, \dots, a + b + c + 2$,

satisfying the same conditions as before regarding matching ends and so on.

Given a moduli space $\mathcal{M}_{Y, g, A, r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ of expected dimension zero, it makes sense at least in principle to assign a rational number $\#\mathcal{M}_{Y, g, A, r}^J(\Gamma^+; \Gamma^-)/\mathbb{R} \in \mathbb{Q}$ by counting the number of elements, according to coherently chosen signs as in [20, Section 1.8] and weighted by $\frac{1}{|\text{Aut}|}$, where Aut is the automorphism group of a given curve. Typically, in order to get relations between these curve counts one needs not just compactness but also gluing results describing the local structure of compactified

¹⁶Each (X_t, ω_t) is a fortiori symplectomorphic to (X, ω) in the case that (X_1, ω_1) is a Liouville domain.

moduli spaces near their boundary strata. Ideally we would like to say that for example each moduli space $\mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ admits a compactification $\overline{\mathcal{M}}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ as a smooth stratified space, with combinatorially indexed boundary strata which correspond precisely to the limiting pseudoholomorphic buildings described in the SFT compactness theorem. Each additional node should increase the codimension by two and each additional level should increase the codimension by one. In particular, we would like to argue that for generic J all moduli spaces $\mathcal{M}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ of negative expected dimension are empty, while all moduli spaces of expected dimension zero or one are regular, with the one-dimensional moduli spaces admitting compactifications $\overline{\mathcal{M}}_{Y,g,A,r}^J(\Gamma^+; \Gamma^-)/\mathbb{R}$ as one-dimensional manifolds, with boundary points consisting of two-level pseudoholomorphic buildings. Although this can generally only be achieved in a virtual sense, we can still read off relations between curve counts by looking at the boundary strata of expected codimension one in compactified moduli spaces of expected dimension one.

In the sequel we will often omit J from the notation when it is implicit from the context. From now on we will only consider genus-zero curves and thus we omit g from the notation, ie we put $\mathcal{M}_{Y,A,r}(\Gamma^+; \Gamma^-)$ instead of $\mathcal{M}_{Y,g=0,A,r}(\Gamma^+; \Gamma^-)$ and so on.

3.4 Rational symplectic field theory formalism

In this subsection we describe the algebraic formalism underpinning rational symplectic field theory. Firstly, to a strict contact manifold (Y, α) we associate the contact homology algebra $\text{CHA}(Y)$. This is a CDGA¹⁷ over $\Lambda_{\geq 0}$, freely generated as an algebra by a variable for each good¹⁸ Reeb orbit of (Y, α) , with differential counting curves in the symplectization of Y with one positive end and any nonnegative number of negative ends. An exact symplectic cobordism (X, ω) between (Y^+, α^+) and (Y^-, α^-) induces a CDGA homomorphism $\Phi_X: \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$. In the special case that (X, ω) is a filling of (Y, α) , we get a CDGA augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$. Given such an augmentation, we can “linearize” $\text{CHA}(Y)$ to obtain a much smaller chain complex $\text{CH}_{\text{lin}}(X)$ called the linearized contact homology. This can be interpreted geometrically as the chain complex over $\Lambda_{\geq 0}$ generated by the good Reeb orbits of (Y, α) , with differential counting “anchored” cylinders in the symplectization of (Y, α) , that is, cylinders with some number of extra negative punctures bounding pseudoholomorphic planes in the filling (X, ω) .

Next, we incorporate rational curves with more than one positive end. In the original formulation of [20], one associates to (Y, α) a differential Poisson algebra with variables q_γ, p_γ for each good Reeb orbit, with differential defined in terms of rational curves in the symplectization of (Y, α) , and with symplectic cobordisms inducing correspondences. Instead, roughly following the “ q -variable only” approach from [39], we will associate to (Y, α) an $\mathcal{L}_\infty$ structure over $\Lambda_{\geq 0}$ whose underlying chain

¹⁷Recall that by default we work at chain level, with passage to homology explicitly notated.

¹⁸A Reeb orbit is “bad” if it is an even cover of another Reeb orbit whose Conley–Zehnder index has a different parity; otherwise it is “good”. Bad orbits are excluded for reasons having to do with signs and coherent orientations of moduli spaces (see [20, Section 1.8]).

complex is $\text{CHA}(Y)$. This approach seems more natural for extracting quantitative invariants. We can also “linearize” this $\mathcal{L}_\infty$ structure to arrive at a much smaller $\mathcal{L}_\infty$ structure with underlying chain complex $\text{CH}_{\text{lin}}(X)$.

Addendum 3.15 After the first version of this paper appeared, the recent work [46] discussed how our quantitative RSFT formalism can be recast in terms of the original formulation of [20]; we refer the reader to [46] for more details.

Remark 3.16 (on the relationship between SFT and Floer theory) Rational symplectic field theory is related to Floer theory as follows. Over $\mathbb{Q}$, linearized contact homology $\text{CH}_{\text{lin}}(X)$ was shown in [6] to be chain homotopy equivalent to $\text{SC}_{S^1,+}(X)$, the positive version of S^1 -equivariant symplectic cochains. The same authors showed that there is also a nonequivariant version of linearized contact homology, $\text{CH}_{\text{lin,non-eq}}(X)$, with two variables $\hat{\gamma}, \check{\gamma}$ for each good Reeb orbit γ , which is equivalent to positive symplectic cochains $\text{SC}_+(X)$. There is a natural chain map from $\text{CH}_{\text{lin,non-eq}}(X)$ to the Morse cochain complex $C(X)$, and the mapping cone gives “filled nonequivariant linearized contact homology”, $\text{CH}_{\text{lin,non-eq,filled}}(X)$, which is equivalent to symplectic cochains $\text{SC}(X)$.

Next, [18] constructs an $\mathcal{L}_\infty$ coalgebra structure on $\text{SC}_{S^1}(X, \omega)$, defined by counting Floer curves with one positive end and several negative ends, with negative ends carrying certain weights which vary over a simplex. Via the cobar construction, this can also be packaged as a CDGA whose underlying algebra is the symmetric tensor algebra on $\text{SC}_{S^1}(X)$. This latter CDGA is shown in [18] to be equivalent to the contact homology algebra $\text{CHA}(Y)$, and there is also a nonequivariant version which is equivalent to a nonequivariant version of the contact homology algebra. Moreover, there are also $\mathcal{L}_\infty$ structures extending the differentials on $\text{SC}(X)$ and $\text{SC}_{S^1}(X)$, which can be defined by using the mapping telescope framework of [2].

More speculatively, it seems reasonable to expect that the $\mathcal{L}_\infty$ structure on $\text{CHA}(X)$ (described in Section 3.4.2), and perhaps also a nonequivariant analogue thereof, should have equivalent models in Floer theory. Roughly, one ought to count rational Floer curves with several positive and negative punctures, combining the mapping telescope approach of [2] with the simplex approach of [18]. This would give a Floer-theoretic analogue of rational symplectic field theory. However, to the author’s knowledge such a construction has not been attempted in the literature.

3.4.1 The contact homology algebra and its linearization We first recall the definition of the contact homology algebra. Given a strict contact manifold (Y, α) , let $\mathcal{X}$ denote the T -adic completion of the $\Lambda_{\geq 0}$ -module generated by a formal variable x_γ for each good Reeb orbit γ of (Y, α) . We endow $\mathcal{X}$ with a grading such that $|x_\gamma| = n - 3 - \text{CZ}(\gamma)$. Along the lines of Remark 3.12, this is only a $\mathbb{Z}/2$ grading, although in many cases it can be upgraded to say a $\mathbb{Z}$ grading (see Remark 3.17 below for the relevant discussion). Let $\mathcal{A}$ denote the T -adic completion of $S\mathcal{X}$. That is, $\mathcal{A}$ is the free supercommutative unital

$\Lambda_{\geq 0}$ -algebra generated by the good Reeb orbits of (Y, α) .¹⁹ Elements of $\mathcal{A}$ are $\Lambda_{\geq 0}$ -linear combinations of “words” $x_{\gamma_1} \odot \cdots \odot x_{\gamma_k}$ for $k \geq 0$, which we will write more briefly as $x_{\gamma_1} \cdots x_{\gamma_k}$. In particular, there is the “empty word” $e = 1 \in \Lambda_{\geq 0} = \odot^0 \mathcal{X}$, which serves as the multiplicative unit.

The contact homology algebra $\text{CHA}(Y)$ is by definition the CDGA over $\Lambda_{\geq 0}$ with underlying graded algebra $\mathcal{A}$, with degree-one differential ∂ defined as follows. Pick an arbitrary total ordering $\preceq$ of the good Reeb orbits of (Y, α) . For $l \geq 0$, let $\mathfrak{S}_l(Y, \alpha)$ denote the set of l -tuples $\Gamma = (\gamma_1, \dots, \gamma_l)$ of good Reeb orbits such that $\gamma_1 \preceq \cdots \preceq \gamma_l$. Equivalently, a tuple $(\gamma_1, \dots, \gamma_l) \in \mathfrak{S}_l(Y, \alpha)$ can be written as

$$(\underbrace{\eta_1, \dots, \eta_1}_{i_1}, \dots, \underbrace{\eta_m, \dots, \eta_m}_{i_m})$$

with $\eta_1 < \cdots < \eta_m$ for some $i_1, \dots, i_m \geq 1$ such that $i_1 + \cdots + i_m = l$.²⁰ Set $\mathfrak{S}(Y, \alpha) = \bigcup_{l \geq 0} \mathfrak{S}_l(Y, \alpha)$. Note that there is an empty tuple $\emptyset \in \mathfrak{S}_0(Y, \alpha) \subset \mathfrak{S}(Y, \alpha)$. For $\Gamma \in \mathfrak{S}(Y, \alpha)$, put $\mu_\Gamma := i_1! \cdots i_m!$, $\kappa_\Gamma := \kappa_{\eta_1}^{i_1} \cdots \kappa_{\eta_m}^{i_m}$, $\text{CZ}(\Gamma) := \sum_{j=1}^l \text{CZ}(\gamma_j)$, and $x_\Gamma = x_{\gamma_1} \cdots x_{\gamma_l}$.

For $l \geq 0$, we define a map $\partial_l: \mathcal{X} \rightarrow \odot^l \mathcal{X}$ by

$$(3-3) \quad \partial_l(x_\gamma) = \sum_{\substack{\Gamma^- \in \mathfrak{S}_l(Y, \alpha) \\ A \in H_2(Y, (\gamma) \cup \Gamma^-; \mathbb{Z})}} \# \overline{\mathcal{M}}_{Y, A}((\gamma); \Gamma^-) / \mathbb{R} \frac{T^{[d\alpha] \cdot A}}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} x_{\Gamma^-}.$$

Here we define $\# \overline{\mathcal{M}}_{Y, A}((\gamma); \Gamma^-) / \mathbb{R} \in \mathbb{Q}$ to be zero unless it is expected to be a count of points, ie unless

$$\text{ind}_{Y, A}((\gamma); \Gamma^-) = (n-3)(1-l) + \text{CZ}(\gamma) - \text{CZ}(\Gamma^-) + 2c_1(Y, \alpha) \cdot A = 1.$$

By nonnegativity of energy we have $[d\alpha] \cdot A \geq 0$, that is, $T^{[d\alpha] \cdot A} \in \Lambda_{\geq 0}$. The differential for the contact homology algebra is then defined on generators by

$$\partial(x_\gamma) = \sum_{l=0}^{\infty} \partial_l(x_\gamma),$$

extended to more general words by the Leibniz rule

$$\partial(x_{\gamma_1} \cdots x_{\gamma_l}) = \sum_{i=1}^l (-1)^{|x_{\gamma_1}| + \cdots + |x_{\gamma_{i-1}}|} x_{\gamma_1} \cdots x_{\gamma_{i-1}} \partial(x_{\gamma_i}) x_{\gamma_{i+1}} \cdots x_{\gamma_l}.$$

Informally, the sum is simply over all rigid (up to target translations) curves in the symplectization of Y with one positive puncture γ and several (possibly none) unordered negative punctures. Each term $1/i_j!$ in μ_{Γ^-} accounts for the superfluous ordering of the punctures asymptotic to γ_j , and each factor $1/\kappa_{\gamma_i}$ in κ_{Γ^-} is related to the fact that there is a κ_γ -fold redundancy when gluing at a pair of positive and negative punctures both asymptotic to γ . By exactness the sums involved are actually finite and hence would be well defined even without using the Novikov ring.

¹⁹Recall from Section 2.2 that our conventions are such that both $\mathcal{X}$ and $S\mathcal{X}$ are always completed with respect to the T -adic topology.

²⁰We could alternatively forego the total ordering at the cost of later dividing out by $m!$ in (3-3) below — see [20, Section 2.1].

The relation $\partial^2 = 0$ comes from the structure of the boundary strata of the compactified moduli space $\overline{\mathcal{M}}_{Y,A}((\gamma); \Gamma^-)/\mathbb{R}$ for $\text{ind}_{Y,A}((\gamma); \Gamma^-) = 2$. Namely, each boundary stratum corresponds to a two-level pseudoholomorphic building in the symplectization of (Y, α) . Each level must have index one and hence consists of a single connected curve of index one together with some number of trivial cylinders. The structure coefficient $\langle \partial^2 x_\gamma, x_{\gamma_1} \cdots x_{\gamma_l} \rangle$ corresponds to the count of points in $\partial \overline{\mathcal{M}}_Y((\gamma); \Gamma^-)/\mathbb{R}$, and hence is zero.

Remark 3.17 (on gradings) In many cases it is possible and desirable to upgrade the $\mathbb{Z}/2$ grading on $\text{CHA}(Y)$ to say a $\mathbb{Z}$ grading. Here we focus on the case that (Y, α) has a filling (X, ω) with $2c_1(X, \omega) = 0$ and $H_1(X; \mathbb{Z}) = 0$ (eg star-shaped domains). In this situation, every Reeb orbit γ of (Y, α) has a canonical integral Conley–Zehnder index $\text{CZ}(\gamma) \in \mathbb{Z}$. If we were to follow the conventions of [20], we would replace $\mathcal{X}$ by $\mathcal{Q}$, the $\Lambda_{\geq 0}$ -module generated by a formal variable q_γ for each good Reeb orbit, with the somewhat different grading $|q_\gamma| = \text{CZ}(\gamma) + n - 3$. Defining $\text{CHA}(Y)$ in otherwise the same way, the resulting CDGA is $\mathbb{Z}$ -graded, with differential of degree -1 (homological convention), product of degree 0, and unit in degree 0.

However, if we use $\mathcal{Q}$, the $\mathcal{L}_\infty$ structures on $\text{CHA}(Y)$ and $\text{CH}_{\text{lin}}(X)$ do not easily conform to the grading framework of Section 2. Instead, we will implement a different grading convention which is more naturally geared for discussing $\mathcal{L}_\infty$ structures and also matches up with standard grading conventions for symplectic cohomology. Our conventions for symplectic cochains $\text{SC}^*(X)$ are that orbits γ are graded by $n - \text{CZ}(\gamma)$, so that the differential has degree $+1$ (cohomological convention), the pair of pants product has degree 0 with unit in degree 0, and for M a closed spin manifold the Viterbo isomorphism reads $SH^*(T^*M) \cong H_{n-*}(\mathcal{L}M)$. With this grading, the k^{th} $\mathcal{L}_\infty$ operation on SC^* has degree $3 - 2k$, ie it corresponds to an $\mathcal{L}_\infty$ structure on $s\text{SC}^*$ within the framework of Section 2. With the same conventions, the k^{th} $\mathcal{L}_\infty$ operation on $\text{SC}_{S^1}^*$ has degree $4 - 3k$, corresponding to an $\mathcal{L}_\infty$ structure on $s^2\text{SC}_{S^1}^*$.

We can modify our definition of $\text{CHA}(Y)$ as follows. We define $\mathcal{X}$ as before, except that we now work over $\Lambda_{\geq 0}[\hbar, \hbar^{-1}]$, where $\hbar$ is a formal variable of degree $6 - 2n$ (whose only role is to correct gradings), and the differential is given by

$$\partial(x_\gamma) = \sum_{l=0}^{\infty} \partial_l(x_\gamma) \hbar^{l-1}.$$

This makes $\text{CHA}(Y)$ into a $\mathbb{Z}$ -graded chain complex with differential of degree $+1$. Similarly, when discussing the $\mathcal{L}_\infty$ structure on $\text{CHA}(Y)$, we can modify ℓ_l^k by including an additional factor $\hbar^{l-1}$. The result is a $\mathbb{Z}$ -graded $\mathcal{L}_\infty$ structure on $s^{-1}\mathcal{A}$. As explained in Section 3.4.3, this also “linearizes” to an $\mathcal{L}_\infty$ structure on $s^{-1}\text{CH}_{\text{lin}}(X)$, that is, with underlying $\Lambda_{\geq 0}$ -module $s^{-1}\mathcal{X}$. Since the generator $s^{-1}x_\gamma$ of $s^{-1}\mathcal{X}$ has degree $|s^{-1}x_\gamma| = n - 2 - \text{CZ}(\gamma)$, this matches the grading on $s^2\text{SC}_{S^1,+}^*$.

Having a $\mathbb{Z}$ grading can be very useful for computations. At any rate, from now on we will mostly suppress $\hbar$ and other $\mathbb{Z}$ grading considerations from the notation for simplicity. We also note that the formal variable $\hbar$ is not needed when discussing the $\mathcal{L}_\infty$ structure on $\text{CH}_{\text{lin}}(X)$, which involves curves with only one negative end.

Next, suppose that (X, ω) is an *exact* symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) . We get an associated unital CDGA homomorphism $\Phi: \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$ defined as follows. Define a map $\Phi_l: \mathcal{X}_{Y^+} \rightarrow \odot^l \mathcal{X}_{Y^-}$ by

$$\Phi_l(x_\gamma) = \sum_{\substack{\Gamma^- \in \mathfrak{S}_l(Y^-, \alpha^-) \\ A \in H_2(X, (\gamma) \cup \Gamma^-; \mathbb{Z})}} \# \overline{\mathcal{M}}_{X,A}((\gamma); \Gamma^-) \frac{T^{[\omega] \cdot A}}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} x_{\Gamma^-}.$$

In this case, a summand can only be nonzero if $\text{ind}_{X,A}((\gamma); \Gamma^-) = 0$, and again by nonnegativity of energy we have $[\omega] \cdot A \geq 0$. We then define

$$\Phi(x_\gamma) = \sum_{l=0}^{\infty} \Phi_l(x_\gamma),$$

extended to more general words multiplicatively by

$$\Phi(x_{\gamma_1} \cdots x_{\gamma_k}) = \Phi(x_{\gamma_1}) \cdots \Phi(x_{\gamma_k}).$$

The fact that Φ is a chain map follows by looking at the boundary strata of $\overline{\mathcal{M}}_{X,A}((\gamma); \Gamma^-)$ for $\text{ind}_{X,A}((\gamma); \Gamma^-) = 1$. Namely, each point of $\partial \overline{\mathcal{M}}_{X,A}((\gamma); \Gamma^-)$ is a two-level pseudoholomorphic building with an index-one level in the symplectization of Y and an index-zero level in X , in either order, and these translate to $\partial_{Y^-} \circ \Phi$ and $\Phi \circ \partial_{Y^+}$, respectively. Here the Novikov coefficients also decompose consistently along such boundary degenerations (see the discussion of energy in Section 3.1). As a special case, if (X, ω) is a symplectic filling of (Y, α) , we get an augmentation of $\text{CHA}(Y)$, ie a CDGA homomorphism $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ sending e to 1. Here $\Lambda_{\geq 0}$ is the $\mathcal{L}_\infty$ algebra equipped with the trivial differential. Strictly speaking, we should also shift the degree of $\Lambda_{\geq 0}$ in order for ϵ to have degree 0, but we will mostly suppress this from the notation to avoid clutter (and similarly for the more general augmentations appearing in Section 5).

The algebra $\mathcal{A} = S\mathcal{X}$ has an increasing “word length” filtration given by $\mathcal{F}_{\leq m} \mathcal{A} = \bigoplus_{k=0}^m \odot^k \mathcal{X}$. The differential ∂ typically does not respect this filtration due to the presence of the term ∂_0 . However, given an augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ we can perform an algebraic operation called “linearization” which replaces $\text{CHA}(Y)$ with an isomorphic CDGA whose differential does preserve the word length filtration. Namely, let $F^\epsilon: \mathcal{A} \rightarrow \mathcal{A}$ be the CDGA homomorphism defined on generators by $F^\epsilon(x_\gamma) = x_\gamma + \epsilon(x_\gamma)$ and extended multiplicatively, ie

$$\begin{aligned} F^\epsilon(x_1 \cdots x_k) &= (x_1 + \epsilon(x_1)) \cdots (x_k + \epsilon(x_k)) \\ &= \sum_{i=0}^k \sum_{\sigma \in \text{Sh}(i, k-i)} \diamond(\sigma) \epsilon(x_{\sigma(1)}) \cdots \epsilon(x_{\sigma(i)}) x_{\sigma(i+1)} \cdots x_{\sigma(k)}. \end{aligned}$$

Note that F^ϵ is invertible, with inverse $F^{\epsilon^{-1}} = F^{-\epsilon}$. We define a new differential on $\mathcal{A}$ by $\partial^\epsilon := F^\epsilon \circ \partial \circ F^{\epsilon^{-1}}$. Then F^ϵ defines a CDGA isomorphism $(\mathcal{A}, \partial) \xrightarrow{\cong} (\mathcal{A}, \partial^\epsilon)$, and it trivializes the augmentation in the sense that $\epsilon \circ F^{\epsilon^{-1}} = 0$. Moreover, letting $\pi_i: \mathcal{A} \rightarrow \odot^i \mathcal{X}$ denote the projection map, we have

$\pi_0 \circ \partial_\epsilon = 0$. In particular, this means that $\partial^{\text{lin}} := \pi_1 \circ \partial^\epsilon : \mathcal{X} \rightarrow \mathcal{X}$ squares to zero. Given a symplectic filling (X, ω) , the linearized contact homology $\text{CH}_{\text{lin}}(X)$ is by definition the chain complex $(\mathcal{X}, \partial^{\text{lin}})$, with the linearization taken with respect to the filling-induced augmentation $\epsilon : \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$.

In fact, $\text{CH}_{\text{lin}}(X)$ has the following more geometric interpretation. The structure coefficient $\langle \partial^{\text{lin}}(x_\gamma), x_\eta \rangle$ counts rational curves in the symplectization of Y with one positive end γ , one negative end η , and $k \geq 0$ extra negative ends at Reeb orbits $\gamma_1^-, \dots, \gamma_k^-$, each weighted by $\epsilon(\gamma_i^-)$, that is, by the count of holomorphic planes in X with positive end γ_i^- . We refer to such a configuration as a cylinder in the symplectization of Y with “anchors” in the filling X . More generally, one can speak of anchored curves of a given topological type, wherein one allows the component in $\mathbb{R} \times Y$ to be possibly disconnected and to have any number of extra negative ends, provided that each extra negative end is matched with the positive end of a component in X and the total topological type is as specified.

Remark 3.18 In some cases, one can ignore the anchors altogether and simply interpret $\text{CH}_{\text{lin}}(X)$ as a count of cylinders in the symplectization of Y . For example, if $2c_1(X^{2n}, \omega) = 0$ and (Y, α) has no Reeb orbits with $\text{CZ}(\gamma) + n - 3 = 0$, then anchors can be ignored since there are no index-zero holomorphic planes in X . In this case linearized contact homology is sometimes also called “cylindrical contact homology”. A similar remark applies to the $\mathcal{L}_\infty$ structure on $\text{CH}_{\text{lin}}(X)$ described below, provided that there are no index-zero k -punctured holomorphic spheres in X for $k \in \mathbb{Z}_{\geq 1}$.

3.4.2 The $\mathcal{L}_\infty$ structure on the contact homology algebra The differential for the contact homology algebra is actually the first term $\ell^1 = \partial$ in a sequence of operations $\ell^k : \odot^k \mathcal{A} \rightarrow \mathcal{A}$, $k \geq 1$, which fit together to form an $\mathcal{L}_\infty$ algebra structure on $s^{-1}\mathcal{A}$. Here the map $\ell^k : \odot^k \mathcal{A} \rightarrow \mathcal{A}$ is defined by counting rational curves with k positive ends in the symplectization of (Y, α) . Firstly, we define a map $\odot^k \mathcal{X} \rightarrow \odot^l \mathcal{X}$ by

$$\ell_l^k(x_{\gamma_1}, \dots, x_{\gamma_k}) := \sum_{\substack{\Gamma^- \in \mathfrak{S}_l(Y, \alpha) \\ A \in H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z})}} \# \overline{\mathcal{M}}_{Y, A}(\Gamma^+; \Gamma^-) / \mathbb{R} \frac{T[d\alpha] \cdot A}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} x_{\Gamma^-},$$

where we have set $\Gamma^+ = (\gamma_1, \dots, \gamma_k)$. We then define

$$\ell^k(x_{\gamma_1}, \dots, x_{\gamma_k}) := \sum_{l=0}^{\infty} \ell_l^k(x_{\gamma_1}, \dots, x_{\gamma_k}).$$

We extend the input to more general words by requiring ℓ^k to satisfy the Leibniz rule in each input, ie given $\Gamma_1^+, \dots, \Gamma_k^+ \in \mathfrak{S}(Y, \alpha)$ with $\Gamma_i = \gamma_1, \dots, \gamma_m$ for some i , we have

$$\begin{aligned} & \ell^k(x_{\Gamma_1^+}, \dots, x_{\Gamma_{i-1}^+}, x_{\gamma_1} \cdots x_{\gamma_m}, x_{\Gamma_{i+1}^+}, \dots, x_{\Gamma_k^+}) \\ &:= \sum_{j=1}^m (-1)^{|x_{\gamma_1}| \cdots |x_{\gamma_{i-1}}|} x_{\gamma_1} \cdots x_{\gamma_{j-1}} \ell^k(x_{\Gamma_1^+}, \dots, x_{\Gamma_{i-1}^+}, x_{\gamma_j}, x_{\Gamma_{i+1}^+}, \dots, x_{\Gamma_k^+}) x_{\gamma_{j+1}} \cdots x_{\gamma_m}. \end{aligned}$$

To see that the maps $\ell^1, \ell^2, \ell^3, \dots$ satisfy the $\mathcal{L}_\infty$ relations, consider the compactified moduli space $\overline{\mathcal{M}}_{Y, \mathcal{A}}(\Gamma^+; \Gamma^-)/\mathbb{R}$ for $\text{ind}_{Y, \mathcal{A}}(\Gamma^+; \Gamma^-) = 2$. Each boundary stratum corresponds to a two-level pseudo-holomorphic building. Each level has index one and hence consists of an index-one component together with some number of trivial cylinders, such that the total glued curve is connected. Then the count of points in $\partial \overline{\mathcal{M}}_Y(\Gamma^+; \Gamma^-)/\mathbb{R}$ precisely corresponds to the structure coefficient $\langle (\widehat{\ell})^2(x_{\gamma_1} \odot \cdots \odot x_{\gamma_k}), x_{\Gamma^-} \rangle$, and hence is zero. Since the operations $\ell^1, \ell^2, \ell^3, \dots$ are derivations in each input, this suffices to establish the $\mathcal{L}_\infty$ relations for all inputs.

Remark 3.19 In fact, the $\mathcal{L}_\infty$ algebra $(\overline{S}\mathcal{A}, \widehat{\ell})$ has a much richer algebraic structure, since $\mathcal{A}$ additionally has a commutative product which is compatible with the $\mathcal{L}_\infty$ operations, making it into a special case of a $\mathcal{G}_\infty$ (homotopy Gerstenhaber) algebra. However, in this paper we do not attempt to make this precise or utilize this finer structure in any systematic way.

Next, suppose that (X, ω) is an exact symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) . We define an $\mathcal{L}_\infty$ homomorphism from $\text{CHA}(Y^+)$ to $\text{CHA}(Y^-)$, consisting of maps $\Phi^k: \odot^k \mathcal{A}_{Y^+} \rightarrow \mathcal{A}_{Y^-}$, $k \geq 1$, as follows. In general, given any two partitions

$$I_1 \sqcup \cdots \sqcup I_a = J_1 \sqcup \cdots \sqcup J_b = \{1, \dots, N\},$$

let $G(I_1, \dots, I_a; J_1, \dots, J_b)$ denote the graph with $a + b + N$ vertices, labeled $A_1, \dots, A_a, B_1, \dots, B_b, v_1, \dots, v_N$, with

- an edge joining A_i and v_j if $j \in I_i$,
- an edge joining B_i and v_j if $j \in J_i$.

Now consider words $w_1, \dots, w_k \in \mathcal{A}_{Y^+}$ of the form

$$(w_1, \dots, w_k) = (\underbrace{x_1 \cdots x_{i_1}}_{i_1}, \underbrace{x_{i_1+1} \cdots x_{i_2}}_{i_2}, \dots, \underbrace{x_{i_1+\cdots+i_{k-1}+1} \cdots x_{i_1+\cdots+i_k}}_{i_k}),$$

and set $N := i_1 + \cdots + i_k$. Put $I_1 = \{1, \dots, i_1\}, \dots, I_k = \{i_1 + \cdots + i_{k-1} + 1, \dots, i_1 + \cdots + i_k\}$, so that the subsets $I_1, \dots, I_k \subset \{1, \dots, N\}$ define a partition $I_1 \sqcup \cdots \sqcup I_k = \{1, \dots, N\}$. We define

$$(3-4) \quad \Phi^k(w_1, \dots, w_k)$$

$$:= \sum_{\substack{J_1 \sqcup \cdots \sqcup J_m = \{1, \dots, N\} \\ \Gamma_1^-, \dots, \Gamma_m^- \in \mathfrak{S}(Y^-, \alpha^-) \\ A_1, \dots, A_m \in H_2(X, \Gamma_i^+ \cup \Gamma_i^-; \mathbb{Z})}} \# \overline{\mathcal{M}}_{X, A_1}(\Gamma_1^+; \Gamma_1^-) \cdots \# \overline{\mathcal{M}}_{X, A_m}(\Gamma_m^+; \Gamma_m^-) \frac{T[\omega] \cdot (A_1 + \cdots + A_m)}{\mu_{\Gamma_1^-} \cdots \mu_{\Gamma_m^-} \kappa_{\Gamma_1^-} \cdots \kappa_{\Gamma_m^-}} x_{\Gamma_1^-} \cdots x_{\Gamma_m^-},$$

where the word $\Gamma_i^+ \in \mathfrak{S}_{|J_i|}(Y^+, \alpha^+)$ correspond to J_i for $i = 1, \dots, m$, and the sum is over all partitions $J_1 \sqcup \cdots \sqcup J_m = \{1, \dots, N\}$ with the property that the graph $G(I_1, \dots, I_k; J_1, \dots, J_m)$ is simply connected. Informally, we can view each word w_i as a cluster of Reeb orbits in Y^+ with a punctured sphere abstractly bounding their union from above, and $\Phi^k(w_1, \dots, w_k)$ is the sum of all ways of attaching a collection of rational curves in X such that the total curve is connected and genus zero. See Figure 1 for an illustration.

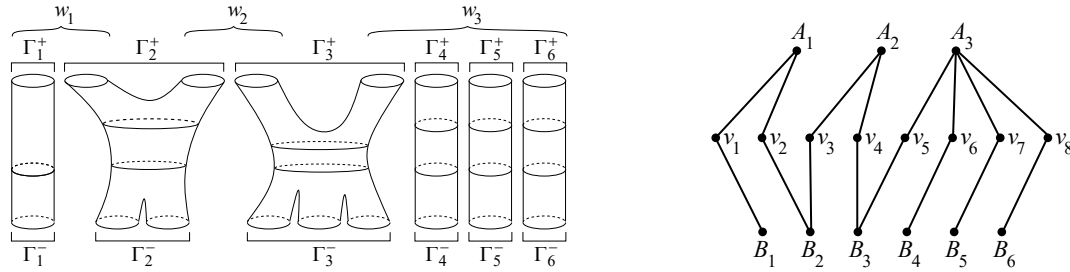


Figure 1: Left: a collection of curves in $\widehat{X}$ contributing to $\langle \Phi^3(w_1, w_2, w_3), x_{\Gamma_1^-} \cdots x_{\Gamma_6^-} \rangle$. Right: the corresponding graph $G(I_1, I_2, I_3; J_1, \dots, J_6)$, where $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $I_3 = \{5, 6, 7, 8\}$, $J_1 = \{1\}$, $J_2 = \{2, 3\}$, $J_3 = \{4, 5\}$, $J_4 = \{6\}$, $J_5 = \{7\}$, $J_6 = \{8\}$ (in general the elements of $J_1, \dots, J_m$ need not appear in consecutive order).

To see that the maps $\Phi^1, \Phi^2, \Phi^3, \dots$ define an $\mathcal{L}_\infty$ homomorphism, we look at the boundary of the compactified moduli space $\overline{\mathcal{M}}_X(\Gamma^+; \Gamma^-)$ for $\text{ind}_{X, \omega}(\Gamma^+; \Gamma^-) = 1$, with $\Gamma^+ = (\gamma_1, \dots, \gamma_k)$. It suffices to check the relation $\widehat{\ell}_{Y^-} \circ \widehat{\Phi} - \widehat{\Phi} \circ \widehat{\ell}_{Y^+} = 0$ on elements $x_{\gamma_1} \odot \cdots \odot x_{\gamma_k} \in \overline{\mathcal{S}}\mathcal{A}_{Y^+}$, with $\widehat{\Phi}: \overline{\mathcal{S}}\mathcal{A}_{Y^+} \rightarrow \overline{\mathcal{S}}\mathcal{A}_{Y^-}$ the induced map on bar complexes. This follows because the coefficient of x_{Γ^-} in this relation precisely corresponds to the count of points in $\partial \overline{\mathcal{M}}_X(\Gamma^+; \Gamma^-)$.

Given two different versions Φ, Φ' of the cobordism map $\text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$ defined by two different almost complex structures J_0 and J_1 (and other perturbation data) for the cobordism, there is a filtered $\mathcal{L}_\infty$ homotopy relating Φ and Φ' . Indeed, we can pick a 1-parameter family of almost complex structures J_t joining J_0 and J_1 , and consider a t -parametrized analogue of the map Φ defined by counting solutions to an appropriate t -parametrized moduli space of curves in the symplectic completion of X . According to [20, Theorem 2.4.5], the resulting algebraic output is a homotopy between the rational SFT potentials associated to J_0 and J_1 , formulated in terms of a certain Hamilton–Jacobi differential equation, and in our setup this gives rise to a filtered $\mathcal{L}_\infty$ homotopy between filtered $\mathcal{L}_\infty$ homomorphisms as in Section 2.

Suppose now that we have a symplectic cobordism $X = X_1 \odot X_2$ which is the concatenation of exact symplectic cobordisms X_1, X_2 with common contact boundary $Y = \partial^+ X_1 = \partial^- X_2$. In this situation we have induced filtered $\mathcal{L}_\infty$ homomorphisms

- $\Phi_{X_2}: \text{CHA}(\partial^+ X_2) \rightarrow \text{CHA}(Y)$,
- $\Phi_{X_1}: \text{CHA}(Y) \rightarrow \text{CHA}(\partial^- X_1)$,
- $\Phi_X: \text{CHA}(\partial^+ X_2) \rightarrow \text{CHA}(\partial^- X_1)$.

By a neck stretching analogue of the above homotopy (using the compactification described in Section 3.3), we get a filtered $\mathcal{L}_\infty$ homotopy

$$(3-5) \quad \Phi_X \sim \Phi_{X_1} \circ \Phi_{X_2}$$

as filtered $\mathcal{L}_\infty$ homomorphisms $\text{CHA}(\partial^+ X_2) \rightarrow \text{CHA}(\partial^- X_1)$.

Observe that a priori $\text{CHA}(Y)$ depends on various choices, namely a contact one-form α , an admissible almost complex structure J on $\mathbb{R} \times Y$, and any additional perturbation data needed to achieve transversality. We will sometimes write, for example, $\text{CHA}(Y, \alpha)$ if we wish to emphasize the role of the contact form. However, it is possible to show that, up to filtered $\mathcal{L}_\infty$ homotopy equivalence, $\text{CHA}(Y)$ is actually independent of the choice of almost complex structure J on $\mathbb{R} \times Y$ and any other perturbation data, and also under replacing the contact one-form α with another of the type $\alpha' = \alpha + df$ for a smooth function $f: Y \rightarrow \mathbb{R}$. Here it is instructive to observe that, for any small $\varepsilon > 0$, we can find an exact compact symplectic cobordism $[0, \varepsilon] \times Y$ with negative contact boundary (Y, α) and positive contact boundary $(Y, e^\varepsilon \alpha')$ (if $\alpha' = \alpha$ we take the Liouville one-form to be simply $e^\varepsilon \alpha$). This induces an $\mathcal{L}_\infty$ homomorphism $\text{CHA}(Y, \alpha') \rightarrow \text{CHA}(Y, \alpha)$, where we may choose different almost complex structures and perturbation data for the domain and target, and which preserves the filtrations up to small distortions (coming from identifying $\text{CHA}(Y, e^\varepsilon \alpha')$ with $\text{CHA}(Y, \alpha')$). We can similarly define a map in the other direction, $\text{CHA}(Y, \alpha) \rightarrow \text{CHA}(Y, \alpha')$, and by the discussion in the previous paragraph the compositions in either order are, up to small distortions, filtered $\mathcal{L}_\infty$ homotopic to the identity maps. By taking more care with energy considerations one can remove these small distortions to give genuine filtered $\mathcal{L}_\infty$ homotopy equivalences.

Incidentally we will not need this in order to establish invariance properties of the capacities defined in the sequel, since as in Remark 3.5 it suffices to show monotonicity under (exact) symplectic embeddings, which will follow directly from versions of the neck stretching composition (3-5) (see Section 6.2). Indeed, suppose that $\mathfrak{c}$ is a real-valued invariant of symplectic fillings which is monotone under exact symplectic embeddings and scales like area, but which a priori depends on various auxiliary choices. By the above we must have $\mathfrak{c}(X, \omega, \alpha, J) < e^\varepsilon \mathfrak{c}(X, \omega, \alpha', J') < e^{2\varepsilon} \mathfrak{c}(X, \omega, \alpha, J)$, and hence $\mathfrak{c}(X, \omega, \alpha, J) = \mathfrak{c}(X, \omega, \alpha', J')$ since ε is arbitrary. If $\mathfrak{c}$ is moreover monotone under arbitrary symplectic embeddings, then $\mathfrak{c}(X, \omega, \alpha)$ is entirely independent of the choice of the contact one-form α , ie it is a symplectomorphism invariant.

3.4.3 The $\mathcal{L}_\infty$ structure on linearized contact homology Given a symplectic filling (X, ω) , we can also linearize the $\mathcal{L}_\infty$ structure on $\text{CHA}(X)$ to get a much smaller $\mathcal{L}_\infty$ structure whose underlying chain complex is $\text{CH}_{\text{lin}}(X)$. More precisely, we define operations $\ell_{\text{lin}}^k: \odot^k \mathcal{X} \rightarrow \mathcal{X}$ which together form an $\mathcal{L}_\infty$ algebra structure on $s^{-1} \mathcal{X}$, and such that $\ell_{\text{lin}}^1 = \partial_{\text{lin}}$ is the linearized contact homology differential.

As a special case of the cobordism map Φ from Section 3.4.2, the filling X induces an $\mathcal{L}_\infty$ augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$. Let $w_1, \dots, w_k \in \mathcal{A}_Y$ be words as above. We define an $\mathcal{L}_\infty$ homomorphism $F^\epsilon: \mathcal{A} \rightarrow \mathcal{A}$ by

$$(3-6) \quad F^{\epsilon; k}(w_1, \dots, w_k) = \sum_{\substack{J_1 \sqcup \dots \sqcup J_m \sqcup K = \{1, \dots, N\} \\ A_1, \dots, A_m \in H_2(X, \Gamma_i^+ \cup \emptyset; \mathbb{Z})}} \# \overline{\mathcal{M}}_{X, A_1}(\Gamma_1^+; \emptyset) \cdots \# \overline{\mathcal{M}}_{X, A_m}(\Gamma_m^+; \emptyset) T^{[\omega] \cdot (A_1 + \dots + A_m)} x_{\Gamma_K},$$

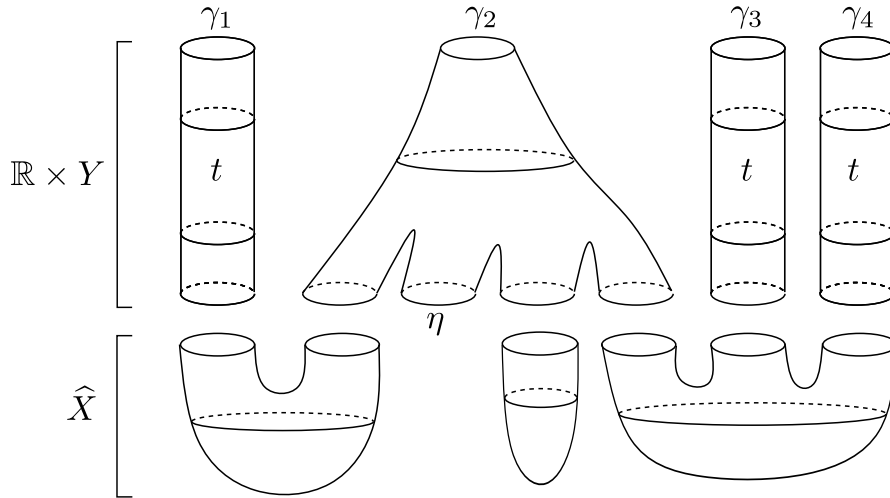


Figure 2: A typical configuration contributing to the coefficient $\langle \ell_{\text{lin}}^4(x_{\gamma_1}, x_{\gamma_2}, x_{\gamma_3}, x_{\gamma_4}), x_\eta \rangle$. Here the cylinders labeled by t are trivial cylinders.

where $\Gamma_i^+ \in \mathfrak{S}_{|J_i|}(Y, \alpha)$ corresponds to J_i for $i = 1, \dots, m$, $\Gamma_K \in \mathfrak{S}_{|K|}(Y, \alpha)$ corresponds to K , and the sum is over all partitions $J_1 \sqcup \dots \sqcup J_m \sqcup K = \{1, \dots, N\}$ with the property that the graph $G(I_1, \dots, I_k; J_1, \dots, J_m)$ is simply connected. Note that the map $F^{\epsilon;1}$ agrees with the automorphism of $\mathcal{A}$ defined in Section 3.4.1 (and denoted by F^ϵ there). Moreover, the $\mathcal{L}_\infty$ homomorphism F^ϵ is invertible, with inverse $F^{-\epsilon}$. We can therefore define a new $\mathcal{L}_\infty$ structure on $s^{-1}\mathcal{A}$ via

$$(3-7) \quad \widehat{\ell}^\epsilon := \widehat{F}^\epsilon \circ \widehat{\ell} \circ \widehat{F}^{-\epsilon}.$$

This corresponds to defining new $\mathcal{L}_\infty$ operations $\ell^{\epsilon;1}, \ell^{\epsilon;2}, \ell^{\epsilon;3}, \dots$, which have the property that $\pi_0 \circ \ell_\epsilon^k = 0$ for each k . Therefore, we can restrict the operations $\ell^{\epsilon;k}$ to inputs from $\mathcal{X}$ and postcompose with the projection π_1 to get $\mathcal{L}_\infty$ operations $\ell_{\text{lin}}^k: \odot^k \mathcal{X} \rightarrow \mathcal{X}$ for $k = 1, 2, 3, \dots$. In particular, the differential ℓ_{lin}^1 agrees with the linearized contact homology differential ∂_{lin} from before.

As in the case of the linearized contact homology differential, we can interpret the $\mathcal{L}_\infty$ operations on CH_{lin} more geometrically as follows. The structure coefficient $\langle \ell_{\text{lin}}^k(x_{\gamma_1}, \dots, x_{\gamma_k}), x_\eta \rangle$ counts rational curves in the symplectization of Y with k positive ends $\gamma_1, \dots, \gamma_k$ and one negative end η , plus some number of additional anchors in X . Unlike the case of ∂_{lin} , the symplectization level can actually be disconnected, with one nontrivial component and some number of trivial cylinders, and the anchors in X can have multiple positive ends. See Figure 2 for an illustration.

Suppose that (X_0, ω_0) and (X_1, ω_1) are symplectic fillings, with a symplectic embedding $X_0 \hookrightarrow X_1$ such that $X_1 \setminus X_0$ is an exact symplectic cobordism. By linearizing the $\mathcal{L}_\infty$ cobordism map $\Phi: \text{CHA}(\partial^+ X_1) \rightarrow \text{CHA}(\partial^+ X_0)$, we get an $\mathcal{L}_\infty$ homomorphism

$$\Phi_{\text{lin}}: \text{CH}_{\text{lin}}(X_1) \rightarrow \text{CH}_{\text{lin}}(X_0),$$

well defined up to filtered $\mathcal{L}_\infty$ homotopy. Also, analogous to the neck stretching discussion at the end of Section 3.4.2, given a symplectic filling W and exact symplectic cobordisms X_1, X_2 with $\partial^+ W = \partial^- X_1$ and $\partial^+ X_1 = \partial^- X_2$, the fillings given by concatenating W, X_1, X_2 induce filtered $\mathcal{L}_\infty$ homomorphisms

- $\Phi_{X_2, \text{lin}}: \text{CH}_{\text{lin}}(W \odot X_1 \odot X_2) \rightarrow \text{CH}_{\text{lin}}(W \odot X_1),$
- $\Phi_{X_1, \text{lin}}: \text{CH}_{\text{lin}}(W \odot X_1) \rightarrow \text{CH}_{\text{lin}}(W),$
- $\Phi_{X_1 \odot X_2, \text{lin}}: \text{CH}_{\text{lin}}(W \odot X_1 \odot X_2) \rightarrow \text{CH}_{\text{lin}}(W),$

and there is a filtered $\mathcal{L}_\infty$ homotopy

$$\Phi_{X_1 \odot X_2, \text{lin}} \sim \Phi_{X_1, \text{lin}} \circ \Phi_{X_2, \text{lin}}: \text{CH}_{\text{lin}}(W \odot X_1 \odot X_2) \rightarrow \text{CH}_{\text{lin}}(W).$$

See also [46, Section 2.7] for another perspective on the structure of linearized RSFT.

4 The Cieliebak–Latschev element

A *nonexact* symplectic cobordism (X, ω) between (Y^+, α^+) and (Y^-, α^-) does *not* generally induce chain maps

$$\text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-), \quad \text{CH}_{\text{lin}}(Y^+) \rightarrow \text{CH}_{\text{lin}}(Y^-),$$

let alone $\mathcal{L}_\infty$ homomorphisms. The basic issue is that an index-one curve in X can break into a configuration involving “reverse anchors”, ie rational curves in X with only negative ends. These reverse anchors, which pose an obstruction to defining a chain map, are ruled out for exact symplectic cobordisms by Stokes’ theorem. Nevertheless, inspired by [24], Cieliebak and Latschev observed that we can still define transfer maps, provided we appropriately deform the target. In fact, the reverse anchors assemble to define a Maurer–Cartan element cl in $\text{CHA}(Y^-)$, and there is an $\mathcal{L}_\infty$ homomorphism

$$\Phi: \text{CHA}(Y^+) \rightarrow \text{CHA}_{\text{cl}}(Y^-),$$

where the latter denotes the $\mathcal{L}_\infty$ structure with deformed operations $\ell_{\text{cl}}^1, \ell_{\text{cl}}^2, \ell_{\text{cl}}^3, \dots$ as in Section 2.3.

More precisely, we define $\text{cl} \in \text{CHA}(Y^-)$ by

$$\text{cl} := \sum_{\substack{\Gamma^- \in \mathfrak{S}(Y^-, \alpha^-) \\ A \in H_2(X, \emptyset \cup \Gamma^-; \mathbb{Z})}} \# \overline{\mathcal{M}}_{X, A}(\emptyset; \Gamma^-) \frac{T^{[\omega] \cdot A}}{\mu_{\Gamma^-} \kappa_{\Gamma^-}} x_{\Gamma^-}.$$

Recall that by definition $\text{CHA}(Y^-)$ is T -adically complete, and hence the above infinite sum is well defined since by the SFT compactness theorem there can only be finitely many nonzero terms with bounded T exponent.

To see that this is a Maurer–Cartan element, one analyzes the boundary strata of the spaces $\overline{\mathcal{M}}_{X, A}(\emptyset; \Gamma^-)$ in the case that $\text{ind}_{X, A}(\emptyset; \Gamma^-) = 1$. Namely, each boundary stratum is a two-level building, with an index-zero level in X and an index-one level in the symplectization of Y , and this contributes to a term of

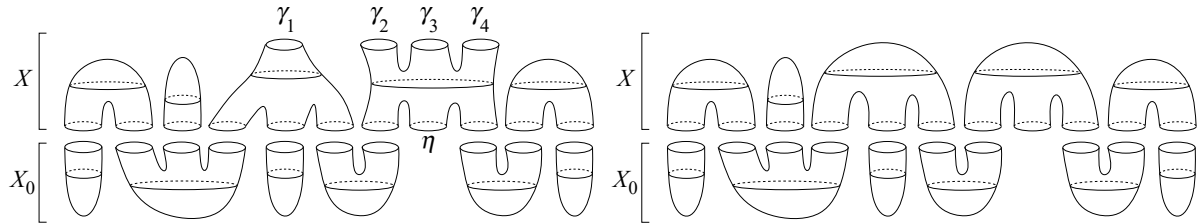


Figure 3: Left: a typical configuration contributing to the coefficient $\langle \Phi_{\text{lin}}^4(x_{\gamma_1}, x_{\gamma_2}, x_{\gamma_3}, x_{\gamma_4}), x_\eta \rangle$, in the case of a nonexact symplectic cobordism X . Right: a typical configuration contributing to the Maurer–Cartan element $\text{cl}_{\text{lin}} \in \text{CH}_{\text{lin}}(X_0)$.

the form $\ell^k(\text{cl}, \dots, \text{cl})$. A similar analysis, looking at moduli spaces of index-one curves in X , confirms that Φ is an $\mathcal{L}_\infty$ homomorphism from $\text{CHA}(Y^+)$ to $\text{CHA}_{\text{cl}}(Y^-)$. Arguing as in Section 3.4.1, one can also show that cl is independent of any choices up to gauge equivalence.

Remark 4.1 We can view a symplectic filling (X, ω) of (Y, α) as a cobordism with negative end the empty set and trivial Cieliebak–Latschev element. In particular, in this case the $\mathcal{L}_\infty$ augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ is defined without any deformation.

Now suppose that (X_0, ω_0) is a symplectic filling of (Y^-, α^-) , and let (X, ω) be a (possibly nonexact) symplectic cobordism with positive end (Y^+, α^+) and negative end (Y^-, α^-) . In this situation we have a linearized version of $\text{cl} \in \text{CHA}(Y^-)$, which is a Maurer–Cartan element cl_{lin} for the $\mathcal{L}_\infty$ structure on $\text{CH}_{\text{lin}}(X_0)$. Namely, we push forward cl under the $\mathcal{L}_\infty$ homomorphism $F^\epsilon: \text{CHA}(Y^-) \rightarrow \text{CHA}(Y^-)$, and then set

$$\text{cl}_{\text{lin}} := \pi_1(F_*^\epsilon(\text{cl})) \in \text{CH}_{\text{lin}}(X_0).$$

Let $\text{CH}_{\text{lin}, \text{cl}_{\text{lin}}}(X_0)$ denote the $\mathcal{L}_\infty$ structure on $\text{CH}_{\text{lin}}(X_0)$ deformed by cl_{lin} . We can also linearize Φ to define an $\mathcal{L}_\infty$ homomorphism $\Phi_{\text{lin}}: \text{CH}_{\text{lin}}(X_0 \odot X) \rightarrow \text{CH}_{\text{lin}, \text{cl}_{\text{lin}}}(X_0)$. See Figure 3.

Now consider the case that $(M, \beta) = (X^-, \omega^-) \odot (X^+, \omega^+)$ is a closed symplectic manifold which decomposes into a symplectic filling (X^-, ω^-) and a symplectic cap (X^+, ω^+) with common contact boundary (Y, α) . For simplicity, assume there are no rigid closed curves in (X^-, ω^-) or (X^+, ω^+) individually. We have the Cieliebak–Latschev element $\text{cl} \in \text{CHA}(Y)$, and the augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ induced by the filling X^- . We can also consider the genus-zero Gromov–Witten invariant

$$\text{GW}_M = \sum_{A \in H_2(M; \mathbb{Z})} T^{[\omega] \cdot A} \text{GW}_{M, A} \in \Lambda_{\geq 0},$$

where for a homology class $A \in H_2(M; \mathbb{Z})$ we put $\text{GW}_{M, A} = \#\overline{\mathcal{M}}_{M, A} \in \mathbb{Q}$. In this situation, by analyzing what happens when we neck stretch closed curves in M along the contact-type hypersurface Y , we find

$$\epsilon_*(\text{cl}) = \text{GW}_M,$$

where $\epsilon_*(\text{cl}) \in \Lambda_{\geq 0}$ is defined as in Section 2.3 by

$$\epsilon_*(\text{cl}) = \sum_{k=1}^{\infty} \frac{1}{k!} \epsilon^k(\underbrace{\text{cl}, \dots, \text{cl}}_k) = \pi_1 \hat{\epsilon}(\exp(\text{cl})).$$

Similarly, we have

$$(4-1) \quad (\epsilon_{\text{lin}})_*(\text{cl}_{\text{lin}}) = \text{GW}_M.$$

In the next section we define various generalizations of the $\mathcal{L}_{\infty}$ augmentation ϵ which count curves with additional geometric constraints, and the corresponding analogues of (4-1) will be the key ingredient used in Section 6.2.3 to give upper bounds on capacities.

In some cases we can make a more refined statement at the level of individual homology classes. For example, suppose that X^- is a symplectic filling of Y such that $H_1(Y; \mathbb{Z}) = H_2(Y; \mathbb{Z}) = H_2(X^-) = 0$ (eg X^- is a star-shaped domain). Let X^+ be a cap of Y , and set $M = X^- \odot X^+$. In this situation, each homology group $H_2(X^+; \emptyset \cup \Gamma^-; \mathbb{Z})$ is naturally identified with $H_2(X^+; \mathbb{Z})$, and the inclusion $H_2(X^+; \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z})$ is an isomorphism. Under these identifications, curves in X^+ define classes in $H_2(M; \mathbb{Z})$, and we have

$$\hat{\ell}(\exp(\text{cl})_A) = 0,$$

where $\exp(\text{cl})_A$ denotes the component of $\exp(\text{cl})$ in the class $A \in H_2(M; \mathbb{Z})$. Moreover,

$$\epsilon_*(\exp(\text{cl})_A) = T^{[\omega] \cdot A} \text{GW}_{M,A},$$

and similarly for the other relations discussed above.

5 Geometric constraints and augmentations

Let (Y, α) be a strict contact manifold, with symplectic filling (X, ω) . As discussed in Section 3.4, the filling induces an $\mathcal{L}_{\infty}$ augmentation $\epsilon: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$, which is used to define the linearized $\mathcal{L}_{\infty}$ algebra $\text{CH}_{\text{lin}}(X)$. Recall that the augmentation ϵ counts index-zero, (possibly disconnected) rational curves in X . In this section, we define various other $\mathcal{L}_{\infty}$ augmentations $\text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$. These can be interpreted as counting rational curves in X which become rigid after imposing additional geometric constraints:

- requiring curves to pass through r generic points $p_1, \dots, p_r \in X$,
- requiring curves to pass through a chosen point in $p \in X$ with an order- m local tangency constraint at that point,
- requiring curves to have an order- b self-contact singularity at a chosen point $p \in X$ (for example, $b = 2$ corresponds to having a node at p),

as well as hybrids of these conditions. By counting curves in X which satisfy these extra constraints, we get corresponding $\mathcal{L}_\infty$ augmentations $\text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$, denoted by

$$\epsilon \ll p_1, \dots, p_r \gg, \quad \epsilon \ll \mathcal{T}^m p \gg, \quad \epsilon \ll \underbrace{p, \dots, p}_b \gg,$$

respectively. The first type of constraint is a standard part of symplectic field theory and is described (in slightly different language) in [20]. The second type of constraint is based on the setup of Cieliebak and Mohnke [8]. The local tangency constraints we consider are closely related to but subtly different from a gravitational descendant condition at a point (see Remark 5.5 for more details). The third type of constraint is most naturally formulated in terms of curves in a multiple blowup of X in a specified homology class. In Section 5.5, we combine all of these constraints into a unified framework by considering curves in a symplectic cobordism with extra negative ends. The basic idea is to count curves with specified ends after removing small neighborhoods of points $p_1, \dots, p_r$ from X . This allows us to combine constraints into general augmentations of the form $\epsilon \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg$, which can all be defined within a standard SFT framework. By plugging these augmentations in the formalism of Section 6.1, these give rise to symplectic capacities. In Section 5.7, we discuss a relation between the three types of constraints, based on the fact that constraints at different points in the target space can be “pushed together” into constraints at a single point.

Remark 5.1 The above constraints can also be posed for closed curves in say a closed symplectic manifold (M, β) , giving rise to Gromov–Witten-type invariants rather than $\mathcal{L}_\infty$ augmentations. We denote the corresponding curve counts in homology class $A \in H_2(M; \mathbb{Z})$ by

$$\text{GW}_{M,A} \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg \in \mathbb{Q}.$$

Most of the above $\mathcal{L}_\infty$ augmentations are *not* stable under products with $\mathbb{C}$. Indeed, each constraint $\mathcal{T}^{m_j^i} p_k$ prescribes a negative end of a curve, whereas we only expect curves with one negative end to stabilize (see (1-3)). The exception is the $\mathcal{L}_\infty$ augmentation $\epsilon \ll \mathcal{T}^m p \gg$, which is formulated in terms of a single negative end. As such, it linearizes to an augmentation

$$\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0},$$

which is dimensionally stable. The details are discussed in Section 5.4.

5.1 Point constraints

Let (X^{2n}, ω) be a symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) . The induced $\mathcal{L}_\infty$ cobordism map $\Phi : \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$ is defined in terms of index-zero moduli spaces $\overline{\mathcal{M}}_{X,A}(\Gamma^+; \Gamma^-)$. We can also consider the versions $\overline{\mathcal{M}}_{X,A,r}(\Gamma^+; \Gamma^-)$ with an additional r (ordered) marked points freely varying

in the domain. These moduli spaces come with evaluation maps $\text{ev}: \overline{\mathcal{M}}_{X,A,r}(\Gamma^+; \Gamma^-) \rightarrow X^{\times r}$. Fixing pairwise distinct points $p_1, \dots, p_r \in X$, we define the moduli space of curves with r point constraints by

$$\overline{\mathcal{M}}_{X,A}(\Gamma^+; \Gamma^-)_{\leq p_1, \dots, p_r} := \text{ev}^{-1}(\{p_1\} \times \dots \times \{p_r\}).$$

Note that each point constraint cuts down the expected dimension by $2n - 2$, so the moduli space $\overline{\mathcal{M}}_{X,A}(\Gamma^+; \Gamma^-)_{\leq p_1, \dots, p_r}$ having index zero corresponds to $\overline{\mathcal{M}}_{X,A,r}(\Gamma^+; \Gamma^-)$ having index $r(2n - 2)$. The combinatorial structure of the boundary strata of $\overline{\mathcal{M}}_{X,A}(\Gamma^+; \Gamma^-)_{\leq p_1, \dots, p_r}$ is such that a typical element of $\partial \overline{\mathcal{M}}_{X,A}(\Gamma^+; \Gamma^-)_{\leq p_1, \dots, p_r}$ consists of a pseudoholomorphic building with one “main” level in X and some number of levels in the symplectization $\mathbb{R} \times Y^\pm$. Note that the curves in the main level may be disconnected, and the point constraints $p_1, \dots, p_r$ are distributed arbitrarily between the different components in X .

We define the point-constrained $\mathcal{L}_\infty$ cobordism map $\Phi_{\leq p_1, \dots, p_r}: \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$ in the same way that we defined the $\mathcal{L}_\infty$ cobordism map $\Phi: \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$, but now imposing the point constraints $p_1, \dots, p_r$. This can also be seen as a special case of the blowup $\mathcal{L}_\infty$ homomorphism discussed in Section 5.3 below, where the extension of (3-4) is given by (5-1). In particular, if (X, ω) is a symplectic filling, we get an $\mathcal{L}_\infty$ augmentation

$$\epsilon_{\leq p_1, \dots, p_r}: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}.$$

As in Section 3.4, it follows from the general invariance properties of rational symplectic field theory that $\epsilon_{\leq p_1, \dots, p_r}$ is independent of all choices, including the precise locations of the points $p_1, \dots, p_r \in X$ (assuming that X is connected), up to filtered $\mathcal{L}_\infty$ homotopy.

Remark 5.2 More generally, instead of requiring curves to pass through points $p_1, \dots, p_r$, we could require them to pass through r chosen cycles in X . However, in this paper we will only make use of the point class, since this is the only homology class relevant to all symplectic fillings regardless of their topological type.

Remark 5.3 There is another $\mathcal{L}_\infty$ augmentation $\text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ which is very similar to $\epsilon_{\leq p_1, \dots, p_r}$ but does not reference the filling (X, ω) . It is defined in the same way except that we pick the points $p_1, \dots, p_r$ to lie in $\mathbb{R} \times Y$ rather than X , and we count curves in $\mathbb{R} \times Y$ rather than in X . These curves still have no negative ends, and here we think of $\mathbb{R} \times Y$ as a symplectic cobordism rather than a symplectization (ie we do not quotient by translations in the target). In some cases these two $\mathcal{L}_\infty$ augmentations coincide (up to filtered $\mathcal{L}_\infty$ homotopy). Namely, if for every $\Gamma^+ \in \mathfrak{S}(Y, \alpha)$ we have $\# \overline{\mathcal{M}}_X(\Gamma^+; \emptyset) = 0$, then by a neck stretching argument the relevant curves in $\widehat{X}$ are in correspondence with curves in $\mathbb{R} \times Y$.

For example, this holds for ellipsoids $E(a_1, \dots, a_n)$ for index reasons. More generally, if we have $c_1(X, \omega) = 0$ and every Reeb orbit γ of Y satisfies $\text{CZ}(\gamma) > |n - 3|$, then for all $\Gamma^+ = (\gamma_1^+, \dots, \gamma_k^+) \in \mathfrak{S}(Y, \alpha)$ we have $\text{ind } \overline{\mathcal{M}}_X(\Gamma^+; \emptyset) > 0$, so there are no rigid genus-zero anchors in X . Indeed, we have

$$\text{ind } \overline{\mathcal{M}}_X(\Gamma^+; \emptyset) = (n - 3)(2 - k) + \sum_{i=1}^k \text{CZ}(\gamma_i^+) > (n - 3)(2 - k) + k|n - 3| \geq 0.$$

For $r = 1$, the $\mathcal{L}_\infty$ augmentation $\epsilon \ll p \gg: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ induces a linearized $\mathcal{L}_\infty$ augmentation $\epsilon_{\text{lin}} \ll p \gg: \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$. As mentioned in the introduction, its first component $\epsilon_{\text{lin}}^1 \ll p \gg: \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$ is part of the SFT model for symplectic cohomology considered in [5]. For $r \geq 2$ the $\mathcal{L}_\infty$ augmentation $\epsilon \ll p_1, \dots, p_r \gg: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ does *not* induce an $\mathcal{L}_\infty$ augmentation $\text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$. Indeed, even the chain map $\epsilon^1 \ll p_1, \dots, p_r \gg: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ does not induce a chain map $\text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$. The essential reason is that an index-one curve in X with two point constraints can degenerate into a configuration for which the marked points lie on different curve components in X .

5.2 Local tangency constraints

Following Cieliebak and Mohnke [7; 8], we can consider curves in a symplectic cobordism (X, ω) satisfying certain local tangency constraints. Let J be an admissible almost complex structure on $\widehat{X}$ which is integrable near a chosen point $p \in X$, and let D be a germ of complex divisor in $\mathcal{O}p(p)$. Suppose we have a J -holomorphic map $u: \Sigma \rightarrow \widehat{X}$ and a marked point $z \in \Sigma$ such that $u(z) = p$. In this situation there is a well-defined notion of u being tangent to D of order m at z , which one can formulate in local coordinates and then show to be coordinate-independent (see [8] for details). Let $\mathcal{M}_{X,A}(\Gamma^+; \Gamma^-) \ll \mathcal{T}^m p \gg$ be defined in the same way as $\mathcal{M}_{X,A}(\Gamma^+, \Gamma^-)$, but now with the constituent curves having one marked point z which maps to p and is tangent to D of order m at z . It is shown in [8, Proposition 3.1] that for generic J (which can be assumed to be fixed and integrable near p), the moduli space $\mathcal{M}_{X,A}(\Gamma^+; \Gamma^-) \ll \mathcal{T}^m p \gg$ is, at least near a somewhere injective curve u , a smooth manifold of dimension

$$(n-3)(2-s^+-s^-) + \sum_{i=1}^{s^+} \text{CZ}(\gamma_i^+) - \sum_{j=1}^{s^-} \text{CZ}(\gamma_j^-) + 2c_1(X, \omega) \cdot A_u - (2n-2) - 2m.$$

Note that the $\mathcal{T}^m p$ constraint is a real codimension $2n-2+2m$ condition. If we fix distinct points $p_1, \dots, p_r \in X$ and a divisor germ D_i at each p_i , for $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$ we can similarly define the moduli space $\mathcal{M}_{X,A}(\Gamma^+, \Gamma^-) \ll \mathcal{T}^{m_1} p_1, \dots, \mathcal{T}^{m_r} p_r \gg$ by considering curves with marked points $z_1, \dots, z_r$, with z_i mapping to p_i and tangent to D_i of order m_i .

Now suppose that (X, ω) is a symplectic filling of (Y, α) . By counting index-zero curves in compactified moduli spaces $\overline{\mathcal{M}}_{X,A}(\Gamma^+, \emptyset) \ll \mathcal{T}^{m_1} p_1, \dots, \mathcal{T}^{m_r} p_r \gg$, we get an $\mathcal{L}_\infty$ augmentation

$$\epsilon \ll \mathcal{T}^{m_1} p_1, \dots, \mathcal{T}^{m_r} p_r \gg: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}.$$

In the case $r = 1$, we also have an induced linearized $\mathcal{L}_\infty$ augmentation $\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg: \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$.

Remark 5.4 In a closed symplectic manifold, the analogous counts of closed curve with tangency constraints are considered in [53]. It is shown in that setting that these give enumerative invariants which can be defined for all semipositive symplectic manifolds using only classical transversality techniques.

Remark 5.5 (on the relationship between tangency constraints and gravitational descendants) Here we compare curve counts with the tangency constraint $\langle\!\langle \mathcal{T}^m p \rangle\!\rangle$ to the gravitational descendant condition $\langle\!\langle \psi^m p \rangle\!\rangle$. For simplicity, assume that (M^{2n}, β) is a closed symplectic manifold, and let $A \in H_2(M; \mathbb{Z})$ be a homology class such that

$$\text{ind}_{M,A} = 2n - 6 + 2c_1(M, \beta) \cdot A - (2n - 2) - 2m = 0.$$

In the context of algebraic Gromov–Witten theory, recall that one defines $\text{GW}_{M,A} \langle\!\langle \psi^m p \rangle\!\rangle \in \mathbb{Q}$ as follows. Consider the space $\overline{\mathcal{M}}_{M,A,1}$ of genus-zero stable maps to M in the homology class A with one marked point. This space comes with an evaluation map $\text{ev}: \overline{\mathcal{M}}_{M,A,1} \rightarrow \mathbb{CP}^2$ and a complex line bundle $\mathbb{L} \rightarrow \overline{\mathcal{M}}_{M,A,1}$ whose fiber over a curve is the cotangent space to the curve at the marked point. We then set

$$\text{GW}_{M,A} \langle\!\langle \psi^m p \rangle\!\rangle := \int_{[\overline{\mathcal{M}}_{M,A}]^{\text{vir}}} \text{ev}^*([p]^\vee) \cup \underbrace{c_1(\mathbb{L}) \cup \cdots \cup c_1(\mathbb{L})}_m,$$

where $[p]^\vee \in H^{2n}(M; \mathbb{Q})$ denotes the point class and $[\overline{\mathcal{M}}_{M,A}]^{\text{vir}} \in H_{2n-4+2c_1(M,\beta) \cdot A}(M; \mathbb{Q})$ is the virtual fundamental class. The case of open curves is more subtle, since then $\mathbb{L}$ is a line bundle over a space with codimension-one boundary, but it does also appear in the literature (see, for example, [19; 21; 57]). Now suppose that D is a smooth divisor passing through $p \in M$. In [27] (see also [66]) it is observed that the line bundle $\mathbb{L}^{\otimes k}$ for $k \in \mathbb{Z}_{>0}$ has a partially defined section σ_k , given by the k -jet of a curve at the marked point in the direction normal to D . The intersection of the locus $\text{ev}^{-1}(p) \cap \sigma_1^{-1}(0) \cap \cdots \cap \sigma_m^{-1}(0)$ with the main stratum $\mathcal{M}_{M,A,1}$ consists of curves with smooth domain which are tangent to D to order m at the marked point. The count of such curves would give precisely $m! \text{GW}_{M,A} \langle\!\langle \psi^m p \rangle\!\rangle$, provided we knew that the sections $\sigma_1, \dots, \sigma_m$ were sufficiently transverse and nonvanishing along the boundary strata $\overline{\mathcal{M}}_{M,A,1} \setminus \mathcal{M}_{M,A,1}$. However, it turns out that these sections always vanish on the boundary strata, and this typically gives additional contributions to the descendant invariant.

As a simple example, we have $2! \text{GW}_{\mathbb{CP}^1, \deg=2} \langle\!\langle \psi^2 p \rangle\!\rangle = \frac{1}{2}$, which follows easily from the genus-zero topological recursion formula (see, for example, [45]). By contrast, we have $\text{GW}_{\mathbb{CP}^1, \deg=2} \langle\!\langle \mathcal{T}^2 p \rangle\!\rangle = 0$. In fact, the compactified moduli space $\overline{\mathcal{M}}_{\mathbb{CP}^1, \deg=2} \langle\!\langle \mathcal{T}^2 p \rangle\!\rangle$ is empty, essentially because there are no degree-two Hurwitz maps $\mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ with an order-three branch point over p . As a slightly more interesting example, we have $4! \text{GW}_{\mathbb{CP}^2, 2[L]} \langle\!\langle \psi^4 p \rangle\!\rangle = 3$, whereas $\text{GW}_{\mathbb{CP}^2, 2[L]} \langle\!\langle \mathcal{T}^4 p \rangle\!\rangle = 1^{21}$ (see Section 5.7 below). One can show that discrepancies in both cases come from a configuration with two degree-one curves joined by a central ghost component containing one marked point.

Nevertheless, at least in the case of a global divisor, Gathmann [27] gives explicit descriptions of the extra boundary contributions occurring in the descendant invariants, described in terms of “comb configurations” related to the ghost configurations above. This is related to the general fact that relative Gromov–Witten

²¹As pointed out to the author by Chiu-Chu Melissa Liu, one can verify this computation by elementary projective geometry. However, if one wishes to use the standard integrable almost complex structure, one must take the local divisor D to be sufficiently generic to avoid degenerate solutions.

invariants can be reduced to absolute Gromov–Witten invariants, possibly with gravitational descendants (see [49; 56]). It might also be possible to extend this approach to the local divisor case in order to give explicit descriptions of the descendant invariants $\text{GW}_{M,A} \llbracket \psi^m p \rrbracket$ in terms of the tangency invariants $\text{GW}_{M,A} \llbracket \mathcal{T}^m p \rrbracket$, plus extra discrepancy configurations.

5.3 Blowup constraints

We also count curves with nodal singularities or higher-order self-contact singularities. These conditions are most naturally posed by considering curves in a suitable symplectic blowup at finitely many points. First suppose that (M^{2n}, β) is a closed symplectic manifold with $2n \geq 4$. Fix distinct points $p_1, \dots, p_r \in X$ and let $\tilde{X}$ denote a symplectic blowup of X at the points $p_1, \dots, p_r$. Recall that the symplectic blowup construction starts with a collection of disjoint Darboux balls $B_1, \dots, B_r$, with B_i centered at p_i and having area a_i (ie radius $\sqrt{a_i/\pi}$). Then $\tilde{X}$ is given by removing each B_i from X and collapsing its boundary along the Hopf fibration. Note that $\tilde{X}$ depends on the points $p_1, \dots, p_r$ and balls $B_1, \dots, B_r$ up to symplectomorphism, but not up to symplectic deformation equivalence. Each collapsed boundary ∂B_i gives rise to an exceptional divisor $\mathcal{E}_i \subset \tilde{X}$, which is a symplectically embedded copy of $\mathbb{C}\mathbb{P}^{n-1}$ whose line $L_i \subset \mathcal{E}_i$ has area a_i . The homology group $H_2(\tilde{X}; \mathbb{Z})$ naturally splits as $H_2(X; \mathbb{Z}) \oplus \mathbb{Z}\langle [L_1], \dots, [L_r] \rangle$, and we have $[\mathcal{E}_i] \cdot [L_i] = (-1)^{n-1} [\text{pt}]$, where $[\text{pt}] \in H_0(\tilde{X}; \mathbb{Z})$ is the point class. Therefore we can uniquely write any class $\tilde{A}$ in $H_2(\tilde{X}; \mathbb{Z})$ as $\tilde{A} = A + C_1[L_1] + \dots + C_r[L_r]$ for $C_1, \dots, C_r \in \mathbb{Z}$, where $\sum_{i=1}^r C_i[L_i]$ is the “exceptional part” of $\tilde{A}$.

For $b_1, \dots, b_r \geq 1$, we introduce the notation

$$\text{GW}_X \llbracket \underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r} \rrbracket := \sum_{A \in H_2(X; \mathbb{Z})} T^{[\omega] \cdot A} \text{GW}_{X,A} \llbracket \underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r} \rrbracket,$$

where

$$\text{GW}_{X,A} \llbracket \underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r} \rrbracket$$

is defined to be $b_1! \cdots b_r! \text{GW}_{\tilde{X}, A - b_1[L_1] - \dots - b_r[L_r]} \sim$. Here the extra combinatorial factor $b_1! \cdots b_r!$ has the effect of ordering the points mapping to each exceptional divisor $\mathcal{E}_i$, which is convenient since the marked points in the definition of $\text{GW}_{X,A} \llbracket p_1, \dots, p_r \rrbracket$ are also ordered. The Novikov exponent $T^{[\omega] \cdot A}$ ignores the exceptional part $-b_1[L_1] - \dots - b_r[L_r]$, and hence is independent of the sizes a_i of the blowup, which we think of as being arbitrarily small.

Now suppose that (X, ω) is a symplectic cobordism between (Y^+, α^+) and (Y^-, α^-) . We can define an $\mathcal{L}_\infty$ homomorphism

$$\Phi \llbracket \underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r} \rrbracket : \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$$

as follows. The homology group $H_2(\tilde{X}, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ naturally splits as a direct sum of $H_2(X, \Gamma^+ \cup \Gamma^-; \mathbb{Z})$ and $\mathbb{Z}[[L_1], \dots, [L_r]]$. The definition of

$$\Phi \ll (\underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r}) \gg$$

is similar to that of the $\mathcal{L}_\infty$ homomorphism $\Phi: \text{CHA}(Y^+) \rightarrow \text{CHA}(Y^-)$ from Section 3.4.2, with

$$\Phi^k \ll (\underbrace{(p_1, \dots, p_1)}_{b_1}, \dots, \underbrace{(p_r, \dots, p_r)}_{b_r}) \gg (w_1, \dots, w_k)$$

given by

$$(5-1) \quad \sum_{\substack{J_1 \sqcup \dots \sqcup J_m = \{1, \dots, N\} \\ \Gamma_1^-, \dots, \Gamma_m^- \in \mathfrak{S}(Y^-, \alpha^-)}} \# \bar{\mathcal{M}}_{\tilde{X}, \tilde{A}_1}(\Gamma_1^+; \Gamma_1^-) \cdots \# \bar{\mathcal{M}}_{\tilde{X}, \tilde{A}_m}(\Gamma_m^+; \Gamma_m^-) \frac{T^{[\omega] \cdot (A_1 + \dots + A_m)} b_1! \cdots b_r!}{\mu_{\Gamma_1^-} \cdots \mu_{\Gamma_m^-} \kappa_{\Gamma_1^-} \cdots \kappa_{\Gamma_m^-}} x_{\Gamma_1^-} \cdots x_{\Gamma_m^-},$$

but where the sum is now over all homology classes $\tilde{A}_1, \dots, \tilde{A}_m \in H_2(\tilde{X}, \Gamma_i^+ \cup \Gamma_i^-; \mathbb{Z})$ such that the total exceptional part of $\tilde{A}_1 + \dots + \tilde{A}_m$ is $-b_1[L_1] - \dots - b_r[L_r]$. In the case that (X, ω) is a filling of (Y, α) , we get an $\mathcal{L}_\infty$ augmentation.

This does not generally induce an $\mathcal{L}_\infty$ augmentation of $\text{CH}_{\text{lin}}(X)$, unless $r = 1$ and $b_1 = 1$. Also, the case $b_1 = \dots = b_r = 1$ agrees with $\Phi \ll p_1, \dots, p_r \gg$ from the previous subsection.

5.4 Behavior under dimensional stabilization

Here we discuss the extent to which curve counts persist to higher dimensions. We will show that the linearized contact homology $\mathcal{L}_\infty$ algebra $\text{CH}_{\text{lin}}(X)$ is invariant under taking a product with $\mathbb{C}$, in a suitable sense. Similar statements hold for the $\mathcal{L}_\infty$ augmentation $\epsilon \ll \mathcal{T}^k p \gg: \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$ and the $\mathcal{L}_\infty$ cobordism map $\text{CH}_{\text{lin}}(X') \rightarrow \text{CH}_{\text{lin}}(X)$ given an exact symplectic embedding $X \hookrightarrow X'$. Together, these will directly translate into a stabilization result for symplectic capacities in Section 6.2.2.

For simplicity we will assume that X is a Liouville domain, with Liouville one-form λ . Then Reeb orbits in ∂X have well-defined action, and for technical reasons in this subsection it will be more convenient to work over $\mathbb{K} = \mathbb{Q}$ rather than $\Lambda_{\geq 0}$, with $\text{CH}_{\text{lin}}(X)$ endowed with its *increasing* action filtration $\mathcal{F}_{<a} \text{CH}_{\text{lin}}(X)$, $a \in \mathbb{R}_{>0}$ (this is possible by our exactness assumption). Since our capacities will be defined in terms of certain action minimizers subject to suitable constraints, we can safely ignore words of Reeb orbits of sufficiently high total action.

We will also make a simplifying assumption which allows us to ignore anchor curves in $\hat{X}$ for index reasons. As in Remark 3.12, given Reeb orbits $\gamma_1, \dots, \gamma_k \in \partial X$ with vanishing total homology class in $H_1(X)$, we can use a symplectic trivialization of TX defined over a spanning surface to define the Conley–Zehnder indices $\text{CZ}(\gamma_1), \dots, \text{CZ}(\gamma_k)$, and the sum $\sum_{i=1}^k \text{CZ}(\gamma_i)$ is independent of the choice of spanning surface provided that $2c_1(X) = 0$. We will make the following assumption, which directly implies that any rational punctured curve without negative ends in $\mathbb{R} \times \partial X$ or $\hat{X}$ has strictly positive index.

Assumption 5.6 (“index positivity”) We have $2c_1(X) = 0$, and for any Reeb orbits $\gamma_1, \dots, \gamma_k$ in ∂X satisfying $\sum_{i=1}^k [\gamma_i] = 0 \in H_1(X)$ we have $\sum_{i=1}^k \text{CZ}(\gamma_i) > (n-3)(k-2)$.

For instance, this holds if there is a global trivialization of TX such that $\text{CZ}(\gamma) > |n-3|$ for all Reeb orbits γ in ∂X . Recall that a convex domain with smooth boundary in $\mathbb{R}^{2n}$ is a fortiori “dynamically convex” in the sense that all Reeb orbits γ satisfy $\text{CZ}(\gamma) \geq n+1$ [36].

Remark 5.7 The papers [11; 12; 34; 51] also prove stabilization theorems in specific examples which are similar in spirit to and provide inspiration for the one in this paper.

Naively, the main point for stabilization is that, for a split almost complex structure, curves in the product $X \times \mathbb{C}$ are given by a product of curves in each factor, and curves in $\mathbb{C}$ effectively must be constant. However, there are a few subtleties to address before making this precise. For one, the product $X \times \mathbb{C}$ is not strictly speaking a symplectic filling, due to its noncompactness. If we replace $\mathbb{C}$ with the two-ball $B^2(a)$ of area a , then $X \times B^2(a)$ is compact but has corners. We can smooth the corners to obtain an honest symplectic filling, but the smoothing process is not entirely canonical. In particular, different choices of smoothings lead to different Reeb dynamics. Moreover, an admissible almost complex structure for the smoothed product is no longer split, so we cannot quite argue via the projection to the $\mathbb{C}$ factor. Another issue is that the index of a punctured curve typically changes under stabilization and can potentially become negative, meaning that stabilized curves might be irregular and disappear after a small perturbation. Since CH_{lin} effectively involves curves with one negative end, invariance under dimensional stabilization is at least plausible based on index considerations as in Section 1.1.3 (in contrast to CHA, which involves multiple negative ends). However, the $\mathcal{L}_\infty$ structure maps of CH_{lin} technically count *anchored* curves with one negative end, hence the symplectization components could still have multiple negative ends and some care is needed.

To address these issues, we will take some care in constructing the smoothing of $X \times B^2(S)$, and we then use intersection theory for punctured pseudoholomorphic curves to rule out any “unexpected curves” after stabilizing. In more detail, assume that ∂X has nondegenerate Reeb orbits. By [54, Lemma 3.6.2], for any $S \in \mathbb{R}_{>0}$ we can construct a smoothing $X \rtimes B^2(S)$ of $X \times B^2(S)$ such that

- $X \times \{0\} \subset X \rtimes B^2(S) \subset X \times B^2(S)$,
- each Reeb orbit γ in ∂X with action less than S is also a Reeb orbit γ^{stab} in $\partial(X \rtimes B^2(S))$ lying in $\partial X \times \{0\}$, and it is nondegenerate with normal Conley–Zehnder $\text{CZ}^\perp(\gamma^{\text{stab}}) = 1$,
- any other Reeb orbit of $\partial(X \rtimes B^2(S))$ has action greater than or equal to S .

Here the *normal Conley–Zehnder index* measures winding in the direction normal to the divisor $X \times \{0\} \subset X \rtimes B^2(S)$ and is taken with respect to the obvious choice trivialization of its symplectic normal bundle coming from the product splitting of $X \times B^2(S)$ (see eg [55] or [54, Section 3.6] for more details).

As a shorthand put $X^{\text{stab}} := X \rtimes B^2(S)$. Any (partial) trivialization of TX naturally extends to a split trivialization of TX^{stab} along $X \times \{0\}$, and for a Reeb orbit γ in ∂X with $\text{CZ}^\perp(\gamma^{\text{stab}}) = 1$ we have

$\text{CZ}(\gamma^{\text{stab}}) = \text{CZ}(\gamma) + 1$. Using Assumption 5.6, any rational punctured curve C in $\mathbb{R} \times \partial X^{\text{stab}}$ or $\widehat{X}^{\text{stab}}$ with all positive ends lying in ∂X and no negative ends satisfies

$$(5-2) \quad \text{ind}(\widetilde{C}) > 2.$$

On the level of $\mathbb{K}$ -modules we have a natural inclusion $J : \text{CH}_{\text{lin}}(X) \hookrightarrow \text{CH}_{\text{lin}}(X^{\text{stab}})$ sending γ to γ^{stab} . We seek to show that for good Reeb orbits $\gamma_1, \dots, \gamma_k$ in ∂X with total action less than S we have

$$(5-3) \quad J(\ell_{\text{lin}}^k(\gamma_1, \dots, \gamma_k)) = \ell_{\text{lin}}^k(\gamma_1^{\text{stab}}, \dots, \gamma_k^{\text{stab}}).$$

We take an admissible almost complex structure $J_{X^{\text{stab}}}$ on $\widehat{X}^{\text{stab}}$ which restricts to an admissible almost complex structure J_X on $\widehat{X}$, and similarly let $J_{\partial X^{\text{stab}}}$ be an admissible almost complex structure on $\mathbb{R} \times \partial X^{\text{stab}}$ which restricts to an admissible almost complex structure $J_{\partial X}$ on $\mathbb{R} \times \partial X$. We assume that $J_{X^{\text{stab}}}$ and J_X restrict to $J_{\partial X^{\text{stab}}}$ and $J_{\partial X}$, respectively, on their cylindrical ends.

Proposition 5.8 *For any $S > 0$ there is a natural filtration-preserving injective chain map*

$$(5-4) \quad \mathcal{F}_{<S} \mathcal{BCH}_{\text{lin}}(X) \hookrightarrow \mathcal{F}_{<S} \mathcal{BCH}_{\text{lin}}(X \rtimes B^2(S)).$$

Remark 5.9 The appearance of bar complexes here reflects the fact that $\mathcal{F}_{<S} \text{CH}_{\text{lin}}(X)$ is not itself an $\mathcal{L}_{\infty}$ subalgebra of $\text{CH}_{\text{lin}}(X)$, since $\gamma_1, \dots, \gamma_k \in \mathcal{F}_{<S} \text{CH}_{\text{lin}}(X)$ only implies $\ell^k(\gamma_1, \dots, \gamma_k) \in \mathcal{F}_{<kS} \text{CH}_{\text{lin}}(X)$, whereas $\mathcal{F}_{<S} \mathcal{BCH}_{\text{lin}}(X)$ effectively encodes all $\mathcal{L}_{\infty}$ operations in $\text{CH}_{\text{lin}}(X)$ such that the inputs have total action less than S .

Proof of Proposition 5.8 For a Reeb orbit η in ∂X^{stab} , the compactified moduli space $\overline{\mathcal{M}}^{\text{stab}}$ computing the term $\langle \ell_{\text{lin}}^k(\gamma_1^{\text{stab}}, \dots, \gamma_k^{\text{stab}}), \eta \rangle$ consists of anchored pseudoholomorphic buildings C involving one or more symplectization levels in $\mathbb{R} \times \partial X^{\text{stab}}$ and possibly a level in $\widehat{X}^{\text{stab}}$ (these are the anchors). By our action assumption, every asymptotic Reeb orbit which appears in a component of C has action less than S and hence must lie in $\partial X \times \{0\}$. In particular, we have $\eta = \gamma_0^{\text{stab}}$ for some Reeb orbit γ_0 in ∂X . Let $\overline{\mathcal{M}}$ denote the compactified moduli space computing the corresponding unstabilized term $\langle \ell_{\text{lin}}^k(\gamma_1, \dots, \gamma_k), \gamma_0 \rangle$. By our choice of almost complex structures there is a natural inclusion $\overline{\mathcal{M}} \hookrightarrow \overline{\mathcal{M}}^{\text{stab}}$. Given $C \in \overline{\mathcal{M}}^{\text{stab}}$, we seek to show that every constituent curve component of C lies in $\mathbb{R} \times \partial X$ or $\widehat{X}$, ie C corresponds to an element of $\overline{\mathcal{M}}$.

Assume for the moment that the unstabilized moduli space $\overline{\mathcal{M}}$ is regular, and in particular for a configuration in $\overline{\mathcal{M}}$ every constituent curve component has nonnegative index. Then, since the total index is 1, every configuration in $\overline{\mathcal{M}}$ consists of just a single curve component, which necessarily lies in $\mathbb{R} \times \partial X$ (anchors in $\widehat{X}$ are ruled out thanks to Assumption 5.6). By Lemma 5.10 below, every component C_0 of C with at least one negative end must lie in $\mathbb{R} \times (\partial X \times \{0\})$, so it has nonnegative index as a curve in $\mathbb{R} \times \partial X$, and hence also as a curve in $\mathbb{R} \times \partial X^{\text{stab}}$ (recall the index stabilization formula (1-3) from Section 1). One can also show that C_0 is still regular as a curve in the stabilization and counts with the same sign (see [54, Section A]). Since the total index of C is 1 and every component of C without any negative ends has index at least 3 by (5-2), we conclude that C has just a single component and it is contained in $\mathbb{R} \times (\partial X \times \{0\})$, and hence indeed corresponds to an element of $\overline{\mathcal{M}}$.

Lemma 5.10 Any $J_{\partial X^{\text{stab}}}$ -holomorphic curve component in $\mathbb{R} \times \partial X^{\text{stab}}$, all of whose asymptotic Reeb orbits have action less than S and having at least one negative end, is entirely contained in $\mathbb{R} \times (\partial X \times \{0\})$.

Proof This is a minor variation on the proof of Lemma 3.6.3 in [54]. Using a version of Siefring's intersection theory [64], for such a curve component C_0 which does *not* lie in Q , we have

$$(5-5) \quad \text{push}(C_0) \cdot Q = C_0 \cdot Q - \sum_{z \text{ positive puncture}} \text{wind}_z + \sum_{z \text{ negative puncture}} \text{wind}_z,$$

where Q denotes the (noncompact) divisor $\mathbb{R} \times (\partial X \times \{0\})$. Here $C \cdot Q$ is the count of intersection points between C_0 and Q before perturbing, which is necessarily nonnegative by positivity of intersections. Meanwhile, $\text{push}(C) \cdot Q$ is the homological intersection after perturbing C_0 at infinity in the direction normal to Q via the aforementioned trivialization, whence $\text{push}(C) \cdot Q = 0$. We also have the winding number bounds $\text{wind}_z \leq \lfloor \text{CZ}^\perp(\gamma_z)/2 \rfloor = 0$ for z a positive puncture and $\text{wind}_z \geq \lceil \text{CZ}^\perp(\gamma_z)/2 \rceil = 1$ for z a negative puncture (see [67, Section 3.5]). Since this violates (5-5), we conclude that C_0 must lie in Q . $\square$

Lastly, recall that in the above we assumed regularity for $\overline{\mathcal{M}}$, which in general can only hold in a virtual sense. More precisely, we compute $\#\overline{\mathcal{M}}$ using virtual perturbations in $\mathbb{R} \times \partial X$ and $\widehat{X}$, which we can then extend trivially in the normal direction to obtain virtual perturbations in $\mathbb{R} \times \partial X^{\text{stab}}$ and $\widehat{X}^{\text{stab}}$ for the computation of $\#\overline{\mathcal{M}}^{\text{stab}}$. With these choices (5-5) remains valid, and the rest of the above proof carries over. $\square$

Similarly, for the local tangency augmentation we have:

Proposition 5.11 Under the inclusion of (5-4), we can further arrange that the chain map

$$\mathcal{F}_{<S} \mathcal{BCH}_{\text{lin}}(X \rtimes B^2(S)) \rightarrow \mathcal{BK}$$

induced by the $\mathcal{L}_\infty$ augmentation $\epsilon_{X \rtimes B^2(S)} \ll \mathcal{T}^m p \gg : \mathcal{CH}_{\text{lin}}(X \rtimes B^2(S)) \rightarrow \mathbb{K}$ restricts to the corresponding chain map $\mathcal{F}_{<S} \mathcal{BCH}_{\text{lin}}(X) \rightarrow \mathcal{BK}$ induced by $\epsilon_X \ll \mathcal{T}^m p \gg : \mathcal{F}_{<S} \mathcal{CH}_{\text{lin}}(X) \rightarrow \mathbb{K}$.

The proof is closely analogous to that of Proposition 5.8. Here we take the local divisor in $X \rtimes B^2(S)$ to be of the form $D \times B^2(\varepsilon)$ for small $\varepsilon > 0$, where $D \subset X$ is the local divisor at $p \in X$. We then appeal to the following analogue of Lemma 5.10, noting that a curve in $\widehat{X}^{\text{stab}}$ satisfying the constraint $\ll \mathcal{T}^m p \gg$ cannot be disjoint from $\widehat{X}^{\text{stab}}$.

Lemma 5.12 Any $J_{X^{\text{stab}}}$ -holomorphic curve in $\widehat{X}^{\text{stab}}$ all of whose asymptotic Reeb orbits have action less than C is either disjoint from $\widehat{X} \times \{0\}$ or entirely contained in it.

Remark 5.13 The formulation and proof of the analogous statement for the $\mathcal{L}_\infty$ cobordism map induced by the stabilization $X \rtimes B^2(S) \hookrightarrow X' \rtimes B^2(S)$ induced by an exact symplectic embedding $X \hookrightarrow X'$ follow along very similar lines, and we leave the details to the reader.

5.5 Constraints via extra negative ends

We now give an alternative formulation of the local tangency constraint $\llcorner \mathcal{T}^m p \gg$, defined by excising a skinny ellipsoid and counting curves with an extra negative end. This has the advantage of putting the $\mathcal{L}_\infty$ augmentations $\epsilon_{\llcorner \mathcal{T}^m p \gg} : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$ into a unified SFT framework and sheds light on the nature of the local tangency constraint.

Let (X^{2n}, ω) be a symplectic filling of (Y, α) , and suppose we are interested in understanding the $\mathcal{L}_\infty$ augmentation $\epsilon_{X, \text{lin}} \llcorner \mathcal{T}^m p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$. To see how the constraint $\llcorner \mathcal{T}^m p \gg$ can be reformulated in terms of extra negative ends, let $U \subset X$ be a small neighborhood of p which is symplectomorphic to the ball $B^{2n}(\varepsilon)$ of area ε . By stretching the neck along the boundary of U , we find that $\epsilon_{X, \text{lin}} \llcorner \mathcal{T}^k p \gg$ is $\mathcal{L}_\infty$ homotopic to the composition $\epsilon_{U, \text{lin}} \llcorner \mathcal{T}^k p \gg \circ \Phi_{\text{lin}}$, where Φ_{lin} denotes the $\mathcal{L}_\infty$ cobordism map $\Phi_{\text{lin}} : \text{CH}_{\text{lin}}(X) \rightarrow \text{CH}_{\text{lin}}(U)$. Here $\epsilon_{U, \text{lin}} \llcorner \mathcal{T}^m p \gg : \text{CH}_{\text{lin}}(U) \rightarrow \Lambda_{\geq 0}$ is identified with the $\mathcal{L}_\infty$ augmentation $\epsilon_{B^{2n}(\varepsilon), \text{lin}} \llcorner \mathcal{T}^m p \gg : \text{CHA}(B^{2n}(\varepsilon)) \rightarrow \Lambda_{\geq 0}$, and can be defined either using Morse–Bott techniques as in the previous section or by slightly perturbing $B^{2n}(\varepsilon)$ to $E(\varepsilon + \delta_1, \dots, \varepsilon + \delta_n)$, where $0 < \delta_1 < \dots < \delta_n$ are small compared to ε and rationally independent. The data associated to $B^{2n}(\varepsilon)$ is essentially the same as that of the unit ball B^{2n} , up to rescaling action. Recall that, after listing the Reeb orbits of $E(\varepsilon + \delta_1, \dots, \varepsilon + \delta_n)$ in order of increasing action, the k^{th} orbit has Conley–Zehnder index $n - 1 + 2k$, and in particular the $\mathcal{L}_\infty$ operations on $\text{CH}_{\text{lin}}(B^{2n})$ are all trivial for degree parity reasons. We will not endeavor to compute the $\mathcal{L}_\infty$ augmentation $\epsilon_{B^{2n}, \text{lin}} \llcorner \mathcal{T}^m p \gg$ here, although the computations for $\mathbb{CP}^2$ in [53] can be viewed as the special case where all inputs are the long simple Reeb orbit of the (slightly perturbed) four-ball. Presently, the main point is that we can use this augmentation to recover the augmentation $\epsilon_{X, \text{lin}} \llcorner \mathcal{T}^m p \gg$ from the cobordism map Φ_{lin} in a “universal” way.

However, to obtain a cleaner reformulation of the constraint $\llcorner \mathcal{T}^m p \gg$, it is more convenient to replace the round neighborhood U by a neighborhood V which is symplectomorphic to a small skinny ellipsoid $E_{\text{sk}} := E(\varepsilon, L, \dots, L)$, where ε and L are both small and we have $L \gg \varepsilon$. The reason is that the $\mathcal{L}_\infty$ augmentation $\epsilon_{E_{\text{sk}}, \text{lin}} : \text{CH}_{\text{lin}}(E_{\text{sk}}) \rightarrow \Lambda_{\geq 0}$ is particularly simple. Indeed, let α_1 denote the simple Reeb orbit of ∂E_{sk} of action ε , and let α_k denote its k -fold iterate for $k \in \mathbb{Z}_{>0}$. All other Reeb orbits of ∂E_{sk} have action at least L , and for L sufficiently large compared to ε they also have arbitrarily large Conley–Zehnder indices.

Proposition 5.14 *For L sufficiently large relative to ε , $\epsilon_{\text{lin}}^k \llcorner \mathcal{T}^m p \gg(\alpha_{m_1}, \dots, \alpha_{m_k})$ is equal to $T^{(m+1)\varepsilon}$ if $k = 1$ and $m_1 = m + 1$, and 0 otherwise.*

Proof We find as in the proof of Proposition 5.11 that the compactified moduli space

$$\overline{\mathcal{M}}_{E_{\text{sk}}}((\alpha_{m_1}, \dots, \alpha_{m_k}); \emptyset) \llcorner \mathcal{T}^m p \gg$$

reduces to

$$\overline{\mathcal{M}}_{B^2(\varepsilon)}((\alpha_{m_1}, \dots, \alpha_{m_k}); \emptyset) \llcorner \mathcal{T}^m p \gg.$$

We can easily compute $\#\overline{\mathcal{M}}_{B^2(\varepsilon)}((\alpha_{m_1}, \dots, \alpha_{m_k}); \emptyset) \llcorner \mathcal{T}^m p \gg$ as a count of Hurwitz covers of $\mathbb{CP}^1$ with a branch point of order $m+1$ over $p \in B^2$ and branch points of orders $m_1, \dots, m_k$ over ∞ . In particular we must have $m_1 + \dots + m_k = m+1$. As an element of $\overline{\mathcal{M}}_{E_{\text{sk}}}((\alpha_{m_1}, \dots, \alpha_{m_k}); \emptyset) \llcorner \mathcal{T}^m p \gg$, the index of such a curve is computed to be $2k-2$, which means that $\epsilon_{\text{lin}}^k \llcorner \mathcal{T}^m p \gg (\alpha_{m_1}, \dots, \alpha_{m_k})$ vanishes for degree reasons unless $k=1$. $\square$

The upshot is that the $\mathcal{L}_\infty$ augmentation $\epsilon_{X, \text{lin}} \llcorner \mathcal{T}^m p \gg$ can be described (up to $\mathcal{L}_\infty$ homotopy) in terms of the $\mathcal{L}_\infty$ cobordism map $\Phi_{\text{lin}}: \text{CH}_{\text{lin}}(X) \rightarrow \text{CH}_{\text{lin}}(V)$:

Corollary 5.15 *In the above setting we have*

$$(5-6) \quad \epsilon_{X, \text{lin}}^k \llcorner \mathcal{T}^m p \gg (-, \dots, -) = T^{(m+1)\varepsilon} \cdot \langle \Phi_{\text{lin}}^k(-, \dots, -), \alpha_{m+1} \rangle.$$

Here for $\gamma_1, \dots, \gamma_k \in \text{CH}_{\text{lin}}(X)$, $\langle \Phi_{\text{lin}}^k(\gamma_1, \dots, \gamma_k), \alpha_{m+1} \rangle$ is the coefficient of the generator α_{m+1} in $\Phi_{\text{lin}}^k(\gamma_1, \dots, \gamma_k) \in \text{CH}_{\text{lin}}(V)$. The extra factor $T^{(m+1)\varepsilon}$ is present to compensate for the fact that the orbits of V have not-quite-zero action.

Remark 5.16 In principle we can similarly apply neck stretching to reformulate the blowup augmentation

$$\epsilon_X \llcorner \underbrace{p, \dots, p}_b \gg: \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$$

and its generalizations in terms of curves in $X \setminus E_{\text{sk}}$. Indeed, we find that the $\mathcal{L}_\infty$ augmentation

$$\epsilon_X \llcorner \underbrace{p, \dots, p}_b \gg$$

is $\mathcal{L}_\infty$ homotopic to the composition

$$\epsilon_{E_{\text{sk}}} \llcorner \underbrace{p, \dots, p}_b \gg \circ \Phi,$$

where $\Phi: \text{CHA}(Y) \rightarrow \text{CHA}(E_{\text{sk}})$ denotes the $\mathcal{L}_\infty$ homomorphism associated to the cobordism $X \setminus E_{\text{sk}}$. However, there is no simple analogue of Proposition 5.14 in this setting. For example, consider the term $\epsilon_{E_{\text{sk}}}^1 \llcorner p, p \gg (\alpha_2)$, where α_2 denotes the two-fold cover of the simple Reeb orbit of ∂E_{sk} of minimal action. One can show using the relative adjunction formula and ECH writhe bounds [41] that there are no somewhere injective planes in the one-point blowup of E_{sk} with top end asymptotic to α_2 . However, there is a two-level building representing such a term, with top level in $\mathbb{R} \times \partial E_{\text{sk}}$ consisting of a double cover of a trivial cylinder with top end asymptotic to α_2 and two bottom ends asymptotic to α_1 , and with bottom level in E_{sk} consisting of two planes each asymptotic to α_1 . The count of such buildings and hence the determination of the coefficient $\epsilon_{E_{\text{sk}}} \llcorner p, p \gg (\alpha_1, \alpha_1)$ depends on the particularities of the chosen virtual perturbations used, ie it is not completely canonical.

5.6 Multibranch local tangency constraints

Now suppose we have a symplectic filling (X, ω) of (Y, α) , and fix integers $b_1, \dots, b_r \geq 1$ and $m_j^i \geq 1$ for $1 \leq i \leq r$, $1 \leq j \leq b_i$. We can also define an $\mathcal{L}_\infty$ augmentation

$$\epsilon \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg : \text{CHA}(Y) \rightarrow \Lambda_{\geq 0},$$

which counts curves “multibranch” tangency constraints at the points $p_1, \dots, p_r$. For example, the constraint $\ll \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p \gg$ corresponds to curves with b local branches passing through p , such that the i^{th} local branch has tangency order m_i for the local divisor D for $i = 1, \dots, b$. These augmentations have not yet appeared in the literature, although analogous counts for closed curves in closed symplectic four-manifolds are discussed in detail in [53, Section 2.3].

We can also put these counts into an SFT framework as follows. Let $X' = X \setminus (V_1 \cup \dots \cup V_r)$ be the resulting symplectic cobordism after removing r pairwise disjoint neighborhoods $V_1, \dots, V_r \subset X$ each symplectomorphic to a small skinny ellipsoids as in E_{sk} above. Let $\Phi_{X'} : \text{CHA}(Y) \rightarrow \text{CHA}(\partial V_1 \cup \dots \cup \partial V_r)$ denote the $\mathcal{L}_\infty$ cobordism map associated to X' . We have $\text{CHA}(\partial V_1 \cup \dots \cup \partial V_r) = \text{CHA}(\partial V_1) \otimes \dots \otimes \text{CHA}(\partial V_r)$ as $\Lambda_{\geq 0}$ -modules, with both $\mathcal{L}_\infty$ structures trivial. We define

$$\epsilon \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg (\gamma_1, \dots, \gamma_k)$$

by

$$T^{A(w_1) + \dots + A(w_r)} \mu_{\Gamma_1} \cdots \mu_{\Gamma_r} \cdot \langle \Phi(\gamma_1, \dots, \gamma_k), \Gamma_1 \otimes \dots \otimes \Gamma_r \rangle,$$

where $w\Gamma_i = \alpha_{m_1^i+1} \cdots \alpha_{m_{b_i}^i+1} \in \text{CHA}(\partial V_i)$ for $i = 1, \dots, r$. Here the combinatorial factor $\mu_{\Gamma_1} \cdots \mu_{\Gamma_r}$ plays the role of $b_1! \cdots b_r!$ from Section 5.3 and has the effect of ordering the constrained marked points. That is, we count rational curves in X' with positive ends $\gamma_1, \dots, \gamma_k$ and b_i specified negative ends on each ∂V_i , and each negative end is an m_j^i -fold iterate of the simple short orbit α_1 . As explained in [53], in the case of a closed symplectic manifold (M, β) , by similarly counting curves in $M \setminus (V_1 \cup \dots \cup V_r)$ with specified negative ends (and no positive ends), we define invariants

$$\text{GW}_{M,A} \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg \in \mathbb{Q}.$$

Suppose that $M = X^- \odot X^+$ is a closed symplectic manifold given by concatenating a symplectic filling X^- and a symplectic cap X^+ with common contact boundary Y . Extending the discussion in Section 4, using geometric constraints contained in X^- , neck stretching along Y decomposes closed curve invariants of M into punctured curves in X^+ (which are encoded by the Cieliebak–Latschev element) and punctured curves in X^- carrying the geometric constraints. Namely, we have eg

$$(5-7) \quad (\epsilon_{\text{lin}})_* \ll \mathcal{T}^m p \gg (\text{cl}_{\text{lin}}) = \text{GW}_M \ll \mathcal{T}^m p \gg,$$

and more generally

$$(5-8) \quad \epsilon_* \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg (\text{cl})$$

$$(5-9) \quad = \text{GW}_M \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg,$$

provided that there are no rigid closed curves in X^- satisfying the same constraints (so that closed curves are forced to split nontrivially under neck stretching).

5.7 Pushing points together

As observed in [53], we can combine multibranch tangency constraints at several different points to produce various constraints at a single point. For simplicity, we restrict to the case of dimension four. The starting point is the following heuristic, which appears in the work [26] of Gathmann. Given an index-zero curve satisfying point constraints at q and p , consider what happens as we vary the locations of the point constraints such that q approaches p along some tangent vector v to p . If the marked points in the domain stay separated during the deformation, the limit should be a curve with two marked points which both map to p_1 , and we can interpret this as a curve with a node at p . On the other hand, if the marked points collide in the domain, the result is a curve with a single marked point mapping to p , where it is tangent to the complexification of v . Gathmann makes this precise with the following formula for curves in $\mathbb{CP}^2$:

$$(5-10) \quad \mathrm{GW}_{\mathbb{CP}^2, \mathcal{A}} \langle q, p, - \rangle = \mathrm{GW}_{\mathbb{CP}^2, \mathcal{A}} \langle \mathcal{T} p, - \rangle + \mathrm{GW}_{\mathbb{CP}^2, \mathcal{A}} \langle p, p, - \rangle,$$

where the symbol $-$ is a shorthand for any additional constraints not involving q or p . The last term can also be written as $2\mathrm{GW}_{\widetilde{\mathbb{CP}^2}, \mathcal{A}-\mathcal{E}}$.

Using the formulation from the previous subsection, we can make the process of pushing points together precise by surrounding two small skinny ellipsoidal neighborhoods V_1, V_2 by another such neighborhood V , and then stretching the neck along the boundary of V . In the example above with two point constraints $q \in V_1$ and $p \in V_2$, we start with a curve with an α_1 negative end on each of $\partial V_1, \partial V_2$. After neck stretching, in the cobordism $V \setminus (V_1 \cup V_2)$ there are two possibilities:

- a union of two cylinders, one with an α_1 positive end on ∂V and a negative end on ∂V_1 and one with an α_1 positive end on ∂V and a negative end on ∂V_2 ,
- a pair of pants with one α_2 positive end on ∂V and an α_1 negative end on each of $\partial V_1, \partial V_2$.

Looking at the remaining part of the curve in $\mathbb{CP}^2 \setminus V$, the first situation corresponds precisely to $\mathrm{GW}_{\mathbb{CP}^2, \mathcal{A}} \langle \mathcal{T} p \rangle$, while the second situation corresponds precisely to $\mathrm{GW}_{\mathbb{CP}^2, \mathcal{A}} \langle p, p \rangle$.²²

In general, given constraints at two distinct points $q, p \in M$, we can push them together as above by surrounding $V_1 \cup V_2$ by V and stretching the neck. By studying curves in the symplectic cobordism $V \setminus (V_1 \cup V_2)$, we get the following outcome (the details appear in [53]):

²²Strictly speaking this decomposition only holds virtually, as on the level of J -holomorphic moduli spaces the pair of pants is actually represented by a two-level building. Nevertheless, one can show using obstruction bundle gluing techniques that it behaves as a unique pair of pants for the purposes of (5-10), and similarly for (5-11) below; see [53, Section 4.2] for details.

Theorem 5.17 (“pushing points together”) *If (M^4, β) is a closed four-dimensional symplectic manifold, we have*

$$(5-11) \quad \text{GW}_{M,A} \ll (\mathcal{T}^m q), (\mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p), - \gg \\ = \text{GW}_{M,A} \ll (\mathcal{T}^m p, \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p), - \gg + \sum_{i=1}^b \text{GW}_{M,A} \ll (\mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_i+m} p, \dots, \mathcal{T}^{m_b} p), - \gg.$$

Similarly, if (X^4, ω) is a four-dimensional symplectic filling, we have the $\mathcal{L}_\infty$ homotopy

$$(5-12) \quad \epsilon \ll (\mathcal{T}^m q), (\mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p), - \gg \\ \sim \epsilon \ll (\mathcal{T}^m p, \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p), - \gg + \sum_{i=1}^b \epsilon \ll (\mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_i+m} p, \dots, \mathcal{T}^{m_b} p), - \gg.$$

In particular we can reduce all constraints at different points to a positive linear combination of constraints at a single point. Moreover, by applying this formula recursively in a somewhat clever way, we can reduce invariants with tangency constraints to linear combinations of invariants without tangency constraints (but possibly with constraints at multiple points). A recursive algorithm for doing this is given in [53]. For example, this technique can be applied to recursively compute the invariants $\text{GW}_{\mathbb{CP}^2, d[L]} \ll \mathcal{T}^{3d-2} p \gg \in \mathbb{Q}$ in terms of Gromov–Witten invariants of blowups of $\mathbb{CP}^2$ at finitely many points, which in turn are readily computable (see [29]). Putting $T_d := \text{GW}_{\mathbb{CP}^2, d[L]} \ll \mathcal{T}^{3d-2} p \gg$, the first few values are

$$T_1 = 1, \quad T_2 = 1, \quad T_3 = 4, \quad T_4 = 26, \quad T_5 = 217, \quad T_6 = 2110,$$

etc. Strictly speaking, in order to apply the upper bound in Section 6.2.3 we only need to know that these numbers are nonzero, which follows from [53, Corollary 4.1.3].

6 Construction of capacities

6.1 The general construction

We now construct symplectic capacities using the $\mathcal{L}_\infty$ augmentations from the previous section. For notational simplicity we will assume that all domains under discussion are connected.²³ Firstly, we have the ones based on the $\mathcal{L}_\infty$ structure on linearized contact homology. Given (X, ω) a symplectic filling of (Y, α) , we have $\mathcal{L}_\infty$ augmentations

$$\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$$

for $m \geq 0$. We assemble these into a single $\mathcal{L}_\infty$ homomorphism

$$\epsilon_{\text{lin}} \ll \mathcal{T}^\bullet p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}[t],$$

²³Note that if X is say a disconnected symplectic filling, then there are for example several different $\mathcal{L}_\infty$ augmentations $\epsilon \ll \mathcal{T}^m p \gg : \text{CH}_{\text{lin}}(X) \rightarrow \Lambda_{\geq 0}$, depending on which component of X the point p lies in. The freedom to choose points in different components plays a role in the disjoint union lower bound discussed in Section 6.2.4 below.

with the t^m coefficient given by $\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg$. For $\mathfrak{b} \in \bar{S}\mathbb{K}[t] \subset \bar{S}\Lambda_{\geq 0}[t]$, put

$$\mathfrak{g}_b^{\leq l}(X) := \inf\{a : T^a \mathfrak{b} = \widehat{\epsilon_{\text{lin}} \ll \mathcal{T}^\bullet p \gg}(x) \text{ for some } x \in \mathcal{B}_{\leq l} \text{CH}_{\text{lin}}(X) \text{ with } \widehat{\ell}(x) = 0\}.$$

Recall here that $\widehat{\ell}$ is the differential on the bar complex $\mathcal{BCH}_{\text{lin}}(X)$. If X is a Liouville domain, it is easy to check that this definition is equivalent to the one given in the introduction. We also define the simplified capacities

$$\begin{aligned} \mathfrak{g}^{\leq l} \ll \mathcal{T}^m p \gg(X) &:= \inf\{a : T^a = \pi_1 \circ \widehat{\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg}(x) \text{ for some } x \in \mathcal{B}_{\leq l} \text{CH}_{\text{lin}}(X) \text{ with } \widehat{\ell}(x) = 0\} \\ &= \inf_{\substack{\mathfrak{b} \in \bar{S}\mathbb{K}[t] \\ \pi_1(\mathfrak{b}) = t^m}} \mathfrak{g}_b^{\leq l}(X), \end{aligned}$$

where π_1 denotes the projection $\bar{S}\mathbb{K}[t] \rightarrow \mathbb{K}[t]$ to the word length one subspace.

Remark 6.1 If we are interested in obstructing symplectic embeddings between two symplectic ellipsoids, it turns out that word length refined capacities $\mathfrak{g}_b^{\leq l}$ provide no extra data beyond the capacities $\mathfrak{g}_b$. Indeed, for X an ellipsoid we have $\mathfrak{g}_b^{\leq l}(X) = \mathfrak{g}_b(X)$ if $\mathfrak{b} \in \bar{S}_{\leq l}\mathbb{K}[t]$, and $\mathfrak{g}_b^{\leq l}(X) = 0$ otherwise. This follows from the fact that (working over $\mathbb{K}$) the map $\mathcal{BCH}_{\text{lin}}(X) \rightarrow \mathcal{B}\mathbb{K}$ induced by $\epsilon_{\text{lin}} \ll \mathcal{T}^m p \gg$ is an isomorphism at chain level, and hence there is a unique element in $\mathcal{BCH}_{\text{lin}}(X)$ mapping to any given $\mathfrak{b} \in \bar{S}\mathbb{K}[t]$.

On the other hand, in some cases the capacities $\mathfrak{g}_b^{\leq l}$ do provide stronger obstructions. As a simple example, if we consider exact symplectic embeddings $D^*S^1 \hookrightarrow B^2$, the capacities $\mathfrak{g}_b$ give the volume obstruction, whereas the capacities $\mathfrak{g}_b^{\leq 1}$ show that no such embedding exists (note that there $\mathfrak{g}_b^{\leq 1}(D^*S^1) = \infty$ since there are no contractible Reeb orbits).

We also define the capacities based on the $\mathcal{L}_\infty$ structure on the full contact homology algebra as follows. We assemble the $\mathcal{L}_\infty$ augmentations

$$\epsilon \ll \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p \gg : \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$$

into a single $\mathcal{L}_\infty$ homomorphism

$$\epsilon \ll \mathcal{T}^\bullet p, \dots, \mathcal{T}^\bullet p \gg : \text{CHA}(Y) \rightarrow S\Lambda_{\geq 0}[t],$$

where the coefficient of $t^{m_1} \odot \dots \odot t^{m_b}$ is given by $\epsilon \ll \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p \gg$. The corresponding map on bar complexes is

$$\widehat{\epsilon} \ll \mathcal{T}^\bullet p, \dots, \mathcal{T}^\bullet p \gg : \mathcal{BCHA}(Y) \rightarrow \bar{S}S\Lambda_{\geq 0}[t].$$

For $\mathfrak{b} \in \bar{S}S\mathbb{K}[t] \subset \bar{S}S\Lambda_{\geq 0}[t]$, we define the capacity

$$\mathfrak{r}_b^{\leq l} \ll \mathcal{T}^\bullet p, \dots, \mathcal{T}^\bullet p \gg(X) \in \mathbb{R}_{>0}$$

by

$$\inf\{a : T^a \mathfrak{b} = \widehat{\epsilon} \ll \mathcal{T}^\bullet p, \dots, \mathcal{T}^\bullet p \gg(x) \text{ for some } x \in \mathcal{B}_{\leq l} \text{CHA}(Y) \text{ with } \widehat{\ell}(x) = 0\}.$$

We also define the simplified capacities

$$(6-1) \quad \mathfrak{r}^{\leq l} \ll \mathcal{T}^{m_1} p, \dots, \mathcal{T}^{m_b} p \gg := \inf_{\substack{\mathfrak{b} \in \overline{SS}\mathbb{K}[t] \\ \pi_1(\mathfrak{b}) = t^{m_1} \odot \dots \odot t^{m_b}}} \mathfrak{r}_{\mathfrak{b}}^{\leq l}(X),$$

where now π_1 denotes the projection $\overline{SS}\mathbb{K}[t] \rightarrow S\mathbb{K}[t]$.

Remark 6.2 We can also define capacities using the $\mathcal{L}_\infty$ augmentations involving several points in the target space

$$\epsilon \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg : \text{CHA}(Y) \rightarrow S\Lambda_{\geq 0}[t].$$

By the discussion in Section 5.7, in dimension four these capacities with $r > 1$ can all be related to ones with $r = 1$.

6.2 Properties

We now deduce some fundamental properties of the capacities constructed in the previous subsection, and in particular justify Theorems 1.8 and 1.10. The scaling and Reeb orbit properties are more or less immediate from the definitions.

6.2.1 Monotonicity It follows from the basic functoriality properties discussed in Sections 3.4 and 5 that all of the above capacities are monotone with respect to exact symplectic embeddings. Indeed, by working on the level of bar complexes this follows from simple diagram chases as explained in the introduction. Less obviously, these capacities are all monotone with respect to nonexact symplectic embeddings, provided we omit the word length refinement “ $\leq l$ ”. Indeed, let X, X' be symplectic fillings of Y, Y' , respectively, and suppose we have a symplectic embedding $X \hookrightarrow X'$. According to Section 4, there is a Maurer–Cartan element $\mathfrak{cl} \in \text{CHA}(Y)$, and an $\mathcal{L}_\infty$ homomorphism $\Phi : \text{CHA}(Y') \rightarrow \text{CHA}_{\mathfrak{cl}}(Y)$. Let $\epsilon_Y \ll - \gg : \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ and $\epsilon_{Y'} \ll - \gg : \text{CHA}(Y') \rightarrow \Lambda_{\geq 0}$ denote any one of the augmentations from Section 5, and let $\mathfrak{r} \ll - \gg$ denote the corresponding simplified capacity as in (6-1) (with $l = \infty$). We can deform $\epsilon_Y \ll - \gg$ by $\mathfrak{cl}$ to define an $\mathcal{L}_\infty$ augmentation

$$\epsilon_{Y, \mathfrak{cl}} \ll - \gg : \text{CHA}_{\mathfrak{cl}}(Y) \rightarrow \Lambda_{\geq 0}$$

such that the composition of Φ and $\epsilon_{Y, \mathfrak{cl}} \ll - \gg$ is $\mathcal{L}_\infty$ homotopic to $\epsilon_{Y'} \ll - \gg$. Suppose that we have $\mathfrak{r} \ll - \gg(X') = a$, so that we have $x \in \mathcal{BCHA}(Y')$ satisfying $\ell(x) = 0$ and $T^a = \pi_1 \widehat{\epsilon_{Y'} \ll - \gg}(x)$. Then

$$0 = \widehat{\Phi} \ell(x) = \widehat{\ell}_{\mathfrak{cl}} \widehat{\Phi}(x) = \widehat{\ell}(\widehat{\Phi}(x) \odot \exp(\mathfrak{cl}))$$

and

$$T^a = \pi_1 \widehat{\epsilon_{Y'} \ll - \gg}(x) = \pi_1 \widehat{\epsilon_{Y, \mathfrak{cl}} \ll - \gg} \widehat{\Phi}(x) = \pi_1 \widehat{\epsilon_Y \ll - \gg}(\widehat{\Phi}(x) \odot \exp(\mathfrak{cl})).$$

In other words, the element $\widehat{\Phi}(x) \odot \exp(\mathfrak{cl})$ exhibits the upper bound $\mathfrak{r} \ll - \gg(X) \leq a$. From this, nonexact monotonicity for the corresponding capacity $\mathfrak{r} \ll - \gg$ follows, but observe that $\widehat{\Phi}(x) \odot \exp(\mathfrak{cl}) \in \mathcal{BCHA}(Y)$ need not lie in the same word-filtration level as $x \in \mathcal{BCHA}(Y')$. Similar reasoning applies to the capacities $\mathfrak{r}_{\mathfrak{b}}$ and the linearized versions.

6.2.2 Behavior under stabilization

Proposition 6.3 For any Liouville domain (X, ω) satisfying the index positivity Assumption 5.6, we have

$$\mathfrak{g}_b^{\leq l}(X \times B^2(S)) = \mathfrak{g}_b^{\leq l}(X)$$

for any $S \geq \mathfrak{g}_b^{\leq l}(X)$.

Here $\mathfrak{g}_b^{\leq l}(X \times B^2(S))$ is by definition the supremum of $\mathfrak{g}_b^{\leq l}(X')$ over all Liouville domains X' admitting an exact symplectic embedding into $X \times B^2(S)$.

Proof It suffices to establish

$$(6-2) \quad \mathfrak{g}_b^{\leq l}(X \times B^2(S)) = \mathfrak{g}_b^{\leq l}(X)$$

for $X \times B^2(S)$ any smoothing of $X \times B^2(S)$ as in Section 5.4, since there are smoothings of this form which are arbitrarily C^0 close to $X \times B^2(S)$. But (6-2) follows immediately from Propositions 5.8 and 5.11, which together show that the data used to define $\mathfrak{g}_b^{\leq l}(X)$ and $\mathfrak{g}_b^{\leq l}(X \times B^2(S))$ coincide for inputs of total action less than S . By the definition of $\mathfrak{g}_b$ as an infimum, it is easy to see that this discrepancy is irrelevant if we have $S \geq \mathfrak{g}_b^{\leq l}(X)$. $\square$

Proposition 6.3 also implies the corresponding stabilization property for the simplified capacities $\mathfrak{g}^{\leq l} \ll \mathcal{T}^m p \gg$.

Remark 6.4 We do not know whether there is a natural analogue of Proposition 6.3 for the capacities $\mathfrak{r} \ll p_1, \dots, p_r \gg$ constructed by Hutchings. If X has dimension $2n$, adding a point constraint is a codimension $2n - 2$ condition, which is dimension dependent. For example, for $n = 2$ the $\mathcal{L}_\infty$ augmentations $\epsilon_{\text{lin}} \ll \mathcal{T}^4 p \gg$ and $\epsilon_{\text{lin}} \ll p_1, \dots, p_5 \gg$ have the same degree, whereas for $n = 3$ the augmentations $\epsilon_{\text{lin}} \ll \mathcal{T}^4 p \gg$ and $\epsilon_{\text{lin}} \ll p_1, p_2, p_3 \gg$ have the same degree.

6.2.3 Upper bounds from closed curves Let (X, ω) be a symplectic filling of (Y, α) , and assume that $H_1(Y; \mathbb{Z}) = H_2(Y; \mathbb{Z}) = H_2(X) = 0$. Let (M, β) a closed symplectic manifold, and suppose there is a symplectic embedding $X \hookrightarrow M$. By the discussion in Section 4, for each $A \in H_2(M; \mathbb{Z})$ we have an element $\exp(\text{cl})_A \in \mathcal{BCHA}(Y)$ satisfying $\hat{\ell}(\exp(\text{cl})_A) = 0$ and its image

$$\pi_1 \hat{\epsilon} \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_r^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg (\exp(\text{cl})_A)$$

agrees with the corresponding closed curve count

$$T^{[\omega] \cdot A} \text{GW}_{M,A} \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_r^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg \in \mathbb{Q}.$$

Provided that this latter count is nonzero, this immediately implies the upper bound

$$\mathfrak{r} \ll (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_r^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \gg (X) \leq [\omega] \cdot A.$$

Similarly, we have $\exp(\text{cl}_{\text{lin}})_A \in \mathcal{BCH}_{\text{lin}}(X)$ satisfying $\widehat{\ell}(\exp(\text{cl}_{\text{lin}})_A) = 0$ and

$$\pi_1 \widehat{\epsilon}_{\text{lin}} \langle \mathcal{T}^m p \rangle (\exp(\text{cl}_{\text{lin}})_A) = T^{[\omega] \cdot A} \text{GW}_{M,A},$$

which gives the upper bound

$$\mathfrak{g} \langle \mathcal{T}^m p \rangle (X) \leq [\omega] \cdot A$$

provided that $\text{GW}_{M,A}$ is nonzero.

For example, in the case that $M = \mathbb{CP}^2(a)$, ie $\mathbb{CP}^2$ with its standard Fubini–Study symplectic form normalized so that a line has area a , we have

$$\mathfrak{r} \langle p_1, \dots, p_{3d-1} \rangle (X) \leq da \quad \text{and} \quad \mathfrak{g} \langle \mathcal{T}^{3d-2} p \rangle (X) \leq da$$

for any symplectic filling X admitting a symplectic embedding $X \hookrightarrow \mathbb{CP}^2(a)$. Similarly, in the case $M = \mathbb{CP}^1(a) \times \mathbb{CP}^1(a')$, ie $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the symplectic form chosen such that the $\mathbb{CP}^1 \times \{\text{pt}\}$ has area a and $\{\text{pt}\} \times \mathbb{CP}^1$ has area a' , we have

$$\mathfrak{r} \langle p_1, \dots, p_{2d+1} \rangle (X) \leq a' + da \quad \text{and} \quad \mathfrak{g} \langle \mathcal{T}^{2d} p \rangle (X) \leq a' + da$$

whenever there is a symplectic embedding $X \hookrightarrow \mathbb{CP}^1(a) \times \mathbb{CP}^1(a')$. Here we are using the fact that the local tangency Gromov–Witten invariants $\text{GW}_{\mathbb{CP}^2, d[L]} \langle \mathcal{T}^{3d-2} p \rangle$ and $\text{GW}_{\mathbb{CP}^1 \times \mathbb{CP}^1, d[L_1] + [L_2]} \langle \mathcal{T}^{2d} p \rangle$ are nontrivial for all $d \in \mathbb{Z}_{\geq 1}$ (see Corollary 4.1.3 and Example 5.1.4 in [53]), while nonvanishing of the standard Gromov–Witten invariants

$$\text{GW}_{\mathbb{CP}^2, d[L]} \langle p_1, \dots, p_{3d-1} \rangle \quad \text{and} \quad \text{GW}_{\mathbb{CP}^1 \times \mathbb{CP}^1, d[L_1] + [L_2]} \langle p_1, \dots, p_{2d+1} \rangle$$

is well known.

6.2.4 Disjoint union lower bound As observed by Hutchings, the capacities $\mathfrak{r} \langle p_1, \dots, p_r \rangle$ satisfy a disjoint union lower bound, similar to the case of ECH capacities. Consider points $p_1, \dots, p_r \in X$ which are pairwise distinct. Suppose that $X_1^{2n}, \dots, X_l^{2n}$, $l \geq 1$, are symplectic fillings, with a symplectic embedding of the disjoint union

$$X_1 \sqcup \dots \sqcup X_l \hookrightarrow X.$$

Put $Y = \partial X$ and $Y_i = \partial X_i$ for $i = 1, \dots, l$. Since the $\mathcal{L}_\infty$ augmentation $\epsilon \langle p_1, \dots, p_r \rangle : \text{CHA}(Y) \rightarrow \Lambda_{\geq 0}$ does not depend on the locations of the points $p_1, \dots, p_r \in X$ up to $\mathcal{L}_\infty$ homotopy, we can distribute them as we like amongst the subdomains $X_1, \dots, X_l$. Given a partition $r = j_1 + \dots + j_l$ with $j_1, \dots, j_l \geq 1$, let us assume that exactly j_i of the points lie in X_i . More specifically, let us suppose that we have a partition

$$\{p_1, \dots, p_r\} = \{p_1^1, \dots, p_{j_1}^1\} \sqcup \dots \sqcup \{p_1^l, \dots, p_{j_l}^l\},$$

where $p_1^i, \dots, p_{j_i}^i$ denote the points lying in X_i . By stretching the neck along $Y_1 \sqcup \dots \sqcup Y_l$, we find that the following relation holds up to $\mathcal{L}_\infty$ homotopy:

$$\epsilon_X \langle p_1, \dots, p_r \rangle = \bigotimes_{i=1}^l (\epsilon_{X_i} \langle p_1^i, \dots, p_{j_i}^i \rangle) \circ \Phi,$$

where $\Phi: \text{CHA}(Y) \rightarrow \bigotimes_{i=1}^l \text{CHA}(Y_i)$ denotes the $\mathcal{L}_\infty$ cobordism map associated to $X \setminus (X_1 \sqcup \cdots \sqcup X_l)$. This relation implies

$$\mathfrak{r}_{\leq p_1, \dots, p_r \geq}(X) \geq \sum_{i=1}^l \mathfrak{r}_{\leq p_1^i, \dots, p_{j_i}^i \geq}(X_i).$$

Since the points $p_1, \dots, p_r$ are all distinct (assuming that X is connected), their precise locations are irrelevant, and we can introduce the shorthand

$$\mathfrak{r}_r(X) := \mathfrak{r}_{\leq p_1, \dots, p_r \geq}.$$

Maximizing the lower bound over all partitions, we have

$$\mathfrak{r}_r(X) \geq \max_{j_1 + \dots + j_l = r} \sum_{i=1}^l \mathfrak{r}_{j_i}(X_i).$$

More generally, consider the capacity

$$\mathfrak{r}_{\leq (\mathcal{T}^{m_1^1} p_1, \dots, \mathcal{T}^{m_{b_1}^1} p_1), \dots, (\mathcal{T}^{m_1^r} p_r, \dots, \mathcal{T}^{m_{b_r}^r} p_r) \geq}(X).$$

By distributing the points $p_1, \dots, p_r$ amongst the subdomains $X_1, \dots, X_l$ as above, we get a lower bound of the form

$$\sum_{i=1}^l \mathfrak{r}_{\leq (\mathcal{T}^- p_1^i, \dots, \mathcal{T}^- p_1^i), \dots, (\mathcal{T}^- p_{j_i}^i, \dots, \mathcal{T}^- p_{j_i}^i) \geq}(X_i),$$

where $(\mathcal{T}^- p_j^i, \dots, \mathcal{T}^- p_j^i)$ corresponds to $(\mathcal{T}^{m_1^c} p_c, \dots, \mathcal{T}^{m_{b_c}^c} p_c)$ if p_j^i is the c^{th} point.

Observe that in particular any ball packing of X^{2n} induces a lower bound for the capacities of X . For example, in the case $2n = 4$, this lower bound can be made explicit using the computations for B^4 given in Section 6.3 below. Namely, suppose that X^4 admits a packing by balls $B^4(a_1), \dots, B^4(a_l)$ with areas $a_1 \geq \dots \geq a_l$. Then we get the lower bounds

$$\mathfrak{g}_{\leq \mathcal{T}^{k-1} p \geq}(X) \geq a_1 \left\lceil \frac{1}{3}(k+1) \right\rceil, \quad \mathfrak{r}_k(X) \geq \max_{j_1 + \dots + j_a = k} \sum_{i=1}^l a_i \left\lceil \frac{1}{3}(j_i+1) \right\rceil, \quad \mathfrak{r}_{\leq \underbrace{p, \dots, p}_k \geq}(X) \geq k a_1.$$

These three capacities all count curves satisfying constraints of the same codimension, but their lower bounds get progressively stronger.

As an important special case, the ellipsoid $E(1, x)$ for x rational admits a canonical ball packing $\bigsqcup_{i=1}^l B^4(a_i) \hookrightarrow E(1, x)$ with $a_1 \geq \dots \geq a_l$ related to the continued fraction expansion of x (see [62] for details). This decomposition plays a central role in the solution of the four-dimensional ellipsoid embedding problem. For example, for $x = \frac{55}{8}$ we have a packing of $E(1, \frac{55}{8})$ by 6 balls of area 1, 1 ball of area $\frac{7}{8}$, and 7 balls of area $\frac{1}{8}$.

6.3 First computations and applications

We end this paper by giving the computations stated in the introduction.

Proposition 6.5 *For $r \geq 1$, we have $\tau_{\leq p_1, \dots, p_r}(B^4) = \lceil \frac{1}{3}(r+1) \rceil$.*

Note that B^4 actually has Morse–Bott Reeb dynamics, with simple Reeb orbits in ∂B^4 corresponding to fibers of the Hopf fibration $S^3 \rightarrow \mathbb{CP}^1$. Alternatively, we make the Reeb orbits nondegenerate by slightly perturbing to an irrational ellipsoid $E(1, 1 + \delta)$ for $\delta > 0$ arbitrarily small. After such a perturbation, there are precisely two simple Reeb orbits in $\partial E(1, 1 + \delta)$, with actions 1 and $1 + \delta$, respectively. Let $\alpha_1, \alpha_2, \alpha_3, \dots$ denote the iterates of the shorter simple Reeb orbit, and let $\beta_1, \beta_2, \beta_3, \dots$ denote the iterates of the longer simple Reeb orbit. The orbits $\alpha_1, \alpha_2, \alpha_3, \dots$ have actions 1, 2, 3, $\dots$ and Conley–Zehnder indices 3, 7, 11, $\dots$, while the orbits $\beta_1, \beta_2, \beta_3, \dots$ have actions $1 + \delta, 2 + \delta, 3 + \delta, \dots$ and Conley–Zehnder indices 5, 9, 13, $\dots$. Recall that we are computing Conley–Zehnder indices with respect to a global trivialization τ_{ex} of the contact vector bundle (see [53, Section 3.2]).

Proof The lower bound $\tau_{\leq p_1, \dots, p_r}(B^4) \geq \lceil \frac{1}{3}(r+1) \rceil$ follows easily by considering the actions and indices of Reeb orbits. For any Reeb orbit γ we have $\text{CZ}(\gamma) \leq 1 + 4\mathcal{A}(\gamma)$. The capacity $\tau_{\leq p_1, \dots, p_r}(B^4)$ is represented by a (possibly disconnected) rigid rational curve in B^4 with r point constraints. Consider a connected such curve with positive ends $\gamma_1, \dots, \gamma_k$. Note that k is at most $\mathcal{A} = \sum_{i=1}^k \mathcal{A}(\gamma_i)$. Index considerations then give

$$0 = (2-3)(2-k) + \sum_{i=1}^k \text{CZ}(\gamma_i) - 2r \leq 6\mathcal{A} - 2 - 2r,$$

ie $\mathcal{A} \geq \frac{1}{3}(r+1)$. Since $\mathcal{A}$ is an integer (up to a factor of δ), this gives $\mathcal{A} \geq \lceil \frac{1}{3}(r+1) \rceil$. The analysis for a disconnected curve is similar.

As for the upper bound $\tau_{\leq p_1, \dots, p_r}(B^4) \leq \lceil \frac{1}{3}(r+1) \rceil$, since $\text{GW}_{\mathbb{CP}^2, d[L]}(\leq p_1, \dots, p_{3d-1}) \neq 0$ for all $d \in \mathbb{Z}_{>0}$, the upper bound from Section 6.2.3 reads

$$\tau_{\leq p_1, \dots, p_{3d-1}}(B^4) \leq d.$$

The upper bound for all other r follows from the observation that $\tau_{\leq p_1, \dots, p_r}$ is nondecreasing with r (see Remark 5.3). $\square$

Proposition 6.6 *We have $\mathfrak{g}_{\leq \mathcal{T}^{m-1}p}(B^4) = \lceil \frac{1}{3}(m+1) \rceil$ for all $m \geq 1$ congruent to 2 mod 3.*

Proof The lower bound $\mathfrak{g}_{\leq \mathcal{T}^{m-1}p}(B^4) \geq \lceil \frac{1}{3}(m+1) \rceil$ follows as in the proof of Proposition 6.5. Similarly, if $m = 3d-1$ for some $d \in \mathbb{Z}_{>0}$, $\mathfrak{g}_{\leq \mathcal{T}^{m-1}p}(B^4)$ follows from the upper bound from Section 6.2.3, using the fact that B^4 symplectically embeds in $\mathbb{CP}^2$ and we have $\text{GW}_{\mathbb{CP}^2, d[L]}(\leq \mathcal{T}^{3d-2}p) \neq 0$ for all $d \geq 1$ by [53, Corollary 4.1.3] $\square$

Remark 6.7 We expect the above computation to be valid for all $m \in \mathbb{Z}_{>0}$, but this does not follow directly from the proof given, since it is not clear from our definition whether $\mathfrak{g}_{\leq \mathcal{T}^m p} \geq$ is nondecreasing with m .

Proposition 6.8 For $1 \leq m \leq x$, we have $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (E(1, x)) = m$.

Proof We can assume without loss of generality that x is irrational. As before let $\alpha_1, \alpha_2, \alpha_3, \dots$ denote the iterates of the short simple Reeb orbit and $\beta_1, \beta_2, \beta_3, \dots$ denote the iterates of the long simple Reeb orbit. The Reeb orbit α_1 in $\partial E(1, x)$ of minimal action has $\text{CZ}(\alpha_1) = 3$, and it bounds a unique holomorphic plane in $E(1, x)$ passing through p . There is a unique m -fold cover of this plane with a branch point at the preimage of p and one positive end α_m . It then satisfies a $\mathcal{T}^{m-1} p$ condition, and it gives the upper bound $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (E(1, x)) \leq m$.

As for the lower bound $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (E(1, x)) \geq m$, note that any representative curve with energy less than m would necessarily have at least two positive ends by index considerations. After replacing the tangency constraint with the skinny ellipsoidal constraint as in Section 5.5, it is easy to rule out such curves using the relative adjunction formula and ECH writhe bounds [40]. See [53, Lemma 4.2.3] for details. $\square$

Proposition 6.9 We have $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (P(1, x)) = \min(m, x + \lceil \frac{1}{2}(m-1) \rceil)$ for all odd $m \geq 1$.

Proof The upper bound $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (P(1, x)) \leq m$ follows from Proposition 6.8, using the fact that $P(1, x)$ symplectically embeds into $E(1 + \delta, x')$ for all $\delta > 0$ and x' sufficiently large. For $m = 2d + 1$ with $ed \geq 0$, the upper bound $\mathfrak{g}_{\leq \mathcal{T}^{2d} p} \geq (P(1, x)) \leq x + d$ follows from the symplectic embedding of $P(1, x)$ into $\mathbb{CP}^1(1) \times \mathbb{CP}^1(x)$, together with the fact that the count of curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ in bidegree $(d, 1)$ and full codimension tangency constraint at a point is nontrivial, ie $\text{GW}_{\mathbb{CP}^1 \times \mathbb{CP}^1, A=d[D_1]+[D_2]} \leq \mathcal{T}^{2d} p \neq 0$ (see [53, Example 5.1.4]).

As for the necessary lower bounds, consider a curve or building u as in the definition of the capacity $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p} \geq (P(1, x)) \geq$. As in [41, Section 5.3], we can find a C^0 -small perturbation $\tilde{P}(1, x)$ of $P(1, x)$ such that in an arbitrarily large action window the Reeb orbits are of the form

- (1) $\alpha_{i,j}$ for $(i, j) \in \mathbb{Z}_{\geq 1}^2$ with $\text{CZ}(\alpha_{i,j}) = 2i + 2j$ and $\mathcal{A}(\alpha_{i,j}) \approx i + jx$,
- (2) $\beta_{i,j}$ for $(i, j) \in \mathbb{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$ with $\text{CZ}(\beta_{i,j}) = 1 + 2i + 2j$ and $\mathcal{A}(\beta_{i,j}) \approx i + jx$.

Here the pair of orbits $\alpha_{i,j}, \beta_{i,j}$ is roughly related to the fact that the strict product of an orbit in $\partial B^2(1)$ and an orbit in $\partial B^2(x)$ gives rise to a Morse–Bott circle of Reeb orbits, which then perturbs to two Reeb orbits with Conley–Zehnder indices differing by 1. Meanwhile, $\beta_{i,0}$ and $\beta_{0,i}$ roughly correspond to orbits of the form $\partial B^2(1) \times \{0\}$ and $\{0\} \times \partial B^2(x)$, respectively, in $\partial \tilde{P}(1, x)$. See also [65, Section 5.3] for more details.

Suppose first that at least one of the positive ends of u is not of the form $\beta_{i,0}$ for some i . It then follows from straightforward action and index considerations that the energy of u is at least $x + \lceil \frac{1}{2}(m-1) \rceil$. For

example, suppose that u has positive ends $\beta_{0,s}, \beta_{s_1,0}, \dots, \beta_{s_k,0}$ for $k, s_1, \dots, s_k \geq 1$. We then have

$$0 = \text{ind}(u) = 2s + 2k + 2 \sum_{i=1}^k s_i - 2m,$$

that is, $\sum_{i=1}^k s_i = m - s - k$. Since $s_i \geq 1$ for all i , we have

$$\sum_{i=1}^k s_i \geq m - s - \sum_{i=1}^k s_i,$$

and hence $\sum_{i=1}^k s_i \geq \frac{1}{2}(m - s)$. This gives

$$\mathcal{A}(u) = sx + \sum_{i=1}^k s_i \geq x + s - 1 + \frac{1}{2}(m - s) \geq x + \left\lceil \frac{1}{2}(m - 1) \right\rceil$$

as desired. The other cases can be handled similarly.

Otherwise, we can assume that the positive ends of u are of the form $\Gamma^+ = (\beta_{s_1,0}, \dots, \beta_{s_k,0})$ for some $k, s_1, \dots, s_k \geq 1$, and it suffices to show that the energy $\sum_{i=1}^k s_i$ of u is at least m . Arguing as in Addendum 1.20, we can arrange that any such curve must be contained in the two-dimensional slice corresponding to the symplectic completion of $B^2(1) \times \{0\}$. But then u must be a Hurwitz cover of this slice, and in particular we must have $m = \sum_{i=1}^k s_i$. $\square$

As mentioned in the introduction, the above computations of $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p}$ give various new obstructions for stabilized embedding problems. However, these capacities do not recover many of the optimal obstructions underlying Theorem 1.5, and hence they do not generally give optimal obstructions to Question 1.4. On the other hand, we now explain why the more general capacities $\mathfrak{g}_b(E(1, x))$ for $b \in \bar{S}\mathbb{K}[t]$ do recover the obstructions of Theorem 1.5, and why we expect Conjecture 1.19 to hold (see also Addendum 1.20). To keep the exposition concrete, we consider the symplectic embedding problem

$$E\left(1, \frac{13}{2}\right) \times \mathbb{C}^{n-2} \hookrightarrow B^4(c) \times \mathbb{C}^{n-2},$$

and for convenience we work with the increasing action filtrations over $\mathbb{K}$ as in the introduction.

According to [11], the optimal obstruction is $c \geq \frac{13}{5}$. By contrast, the Ekeland–Hofer capacities give the weaker obstruction $c \geq 2$, and it can be shown that the capacities $\mathfrak{g}_{\leq \mathcal{T}^{m-1} p}$ also only give $c \geq 2$ (see [54]). After slightly perturbing $E(1, \frac{13}{2})$ to $E(1, \frac{13}{2} + \delta)$ for $\delta > 0$ sufficiently small, let α_1 and β_1 denote the simple Reeb orbits of actions 1 and $\frac{13}{2} + \delta$, respectively. Including iterates, in order of increasing action the orbits are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \alpha_7, \dots$, with Conley–Zehnder indices 3, 5, 7, 9, 11, 13, 15, 17, $\dots$, etc. Similarly, after perturbing $B^4(c)$ to $E(c, c + \delta')$, let α'_1 and β'_1 denote the primitive Reeb orbits of actions 1 and $1 + \delta'$, respectively, and including iterates we have Reeb orbits $\alpha'_1, \beta'_1, \alpha'_2, \beta'_2, \alpha'_3, \beta'_3, \dots$, etc.

Recall that the differentials on $\mathcal{BCH}_{\text{lin}}(E(c, c + \delta'))$ and $\mathcal{BCH}_{\text{lin}}(E(1, \frac{13}{2} + \delta))$ are both trivial for degree parity reasons, and hence passing to homology has no effect. We also have that the cobordism map $\mathcal{BCH}_{\text{lin}}(E(c, c + \delta')) \rightarrow \mathcal{BCH}_{\text{lin}}(E(1, \frac{13}{2} + \delta))$ is an isomorphism (with inverse given by the cobordism in the reverse direction up to scaling). The maps $\hat{e} \ll \mathcal{T}^\bullet p \gg : \mathcal{BCH}_{\text{lin}}(E(c, c + \delta')) \rightarrow \bar{S}\mathbb{K}[t]$ and $\hat{e} \ll \mathcal{T}^\bullet p \gg : \mathcal{BCH}_{\text{lin}}(E(1, \frac{13}{2} + \delta)) \rightarrow \bar{S}\mathbb{K}[t]$ are also isomorphisms, using the identification of $\bar{S}\mathbb{K}[t]$ with $\mathcal{BCH}_{\text{lin}}(E_{\text{sk}})$. In particular, for any given $\mathfrak{b} \in \bar{S}\mathbb{K}[t]$, there is a unique element $w \in \mathcal{BCH}_{\text{lin}}(E(1, \frac{13}{2} + \delta))$ such that $\hat{e} \ll \mathcal{T}^\bullet p \gg(w) = \mathfrak{b}$, and the capacity $\mathfrak{g}_{\mathfrak{b}}(E(1, \frac{13}{2} + \delta))$ is simply the action of w .

The Cristofaro-Gardiner–Hind obstruction comes from the existence of a curve C in the complementary cobordism $E(c, c + \delta') \setminus E(1, \frac{13}{2} + \delta)$ with five positive ends each asymptotic to β'_1 , and one negative end asymptotic to α_{13} (see [11]). Such a curve has Fredholm index zero, and Stokes’ theorem gives $5c \geq 13$. The word $\beta_l \odot \beta_l \odot \beta_l \odot \beta_l \odot \beta_l$ defines an element in $\mathcal{BCH}_{\text{lin}}(E(c, c + \delta'))$, and its image $y \in \bar{S}\mathcal{CH}_{\text{lin}}(E(1, \frac{13}{2} + \delta))$ is α_s^{13} plus a sum of elements of word length at least two. Let $\mathfrak{b}$ be the image of $\beta_l \odot \beta_l \odot \beta_l \odot \beta_l \odot \beta_l$ under the map $\hat{e} \ll \mathcal{T}^\bullet p \gg : \mathcal{BCH}_{\text{lin}}(E(c, c + \delta')) \rightarrow \bar{S}\mathbb{K}[t]$. Note that, by the above observation about vanishing differentials, $y \in \bar{S}\mathcal{CH}_{\text{lin}}(E(1, \frac{13}{2} + \delta))$ is the unique element mapping to $\mathfrak{b} \in \bar{S}\mathbb{K}[t]$, so we have $\mathfrak{g}_{\mathfrak{b}}(E(1, \frac{13}{2})) = \mathcal{A}(y) \geq 13$ (this relies on the existence of C). We then have $\mathfrak{g}_{\mathfrak{b}}(B^4(c)) = 5c$ and $\mathfrak{g}_{\mathfrak{b}}(E(1, \frac{13}{2})) \geq 13$, and these together recover the optimal obstruction $c \geq \frac{13}{5}$. A similar argument shows that any obstruction coming from the existence of a somewhere injective rational curve with one negative end should be visible to $\mathfrak{g}_{\mathfrak{b}}$ for some $\mathfrak{b} \in \bar{S}\mathbb{K}[t]$.

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The cohomology of biquotients via a product on the two-sided bar construction

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APPENDIX WRITTEN JOINTLY WITH MATTHIAS FRANZ

We compute the Borel equivariant cohomology ring of the left K -action on a homogeneous space G/H , where G is a connected Lie group, H and K are closed connected subgroups, and 2 as well as the torsion primes of the Lie groups are units of the coefficient ring. As a special case, this gives the singular cohomology rings of biquotients $H \backslash G/K$. This depends on a version of the Eilenberg–Moore theorem developed in the appendix, where a novel multiplicative structure on the two-sided bar construction $\mathbf{B}(A'', A, A')$ is defined, valid when $A'' \leftarrow A \rightarrow A'$ is a pair of maps of homotopy Gerstenhaber algebras.

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Homogeneous spaces, which can be realized as coset spaces G/H for G a transitively acting Lie group and H the stabilizer of a point, are arguably the most highly symmetric, most canonical, and most thoroughly investigated objects of study in differential geometry after Lie groups. A generalization of perennial interest, which offers many interesting examples in positive-curvature geometry, is the class of *biquotients*, the orbit spaces $K \backslash G/H$ of G under free left–right actions by products $K \times H$ of two closed subgroups.

The cohomology rings of a Lie group G over $\mathbb{Q}$ and those finite fields $\mathbb{F}_p$ for which $H^*(G; \mathbb{Z})$ lacks p -torsion have been known to be exterior algebras since the fundamental 1941 work of Hopf [17], and that of a homogeneous space G/H with connected stabilizer H has been known over $\mathbb{R}$ since work of Henri Cartan from 1950 [9]:

$$H^*(G/H) \cong \mathrm{Tor}_{H^*(BG)}(\mathbb{R}, H^*(BH)).$$

In his 1952 dissertation [6, Section 30], Borel used the Serre spectral sequence of the Borel fibration $G \rightarrow G/H \rightarrow BH$ to show the same ring isomorphism also holds when H is of maximal rank, with $\mathbb{F}_p$ coefficients if $H^*(G; \mathbb{Z})$ and $H^*(H; \mathbb{Z})$ lack p -torsion and with $\mathbb{Z}$ coefficients if they are torsion free. Beyond this, however, ring isomorphisms proved hard to find. Starting with Baum’s 1962 dissertation, a program began to obtain a more general result, additively at least, using the then-new Eilenberg–Moore spectral sequence of the fibration $G/H \rightarrow BH \rightarrow BG$. Cartan’s result implies this spectral sequence’s collapse over $\mathbb{R}$, and Baum proved collapse under hypotheses covering the best-studied homogeneous spaces. Subsequent work of authors including Baum [2], Larry Smith [30], Gugenheim and May [15],

Munkholm [26], and Joel Wolf [34] in the 1960s–70s proved collapse under substantially more general hypotheses and with accordingly more involved proof techniques. Again, however, almost all of these proofs provided only the additive structure of $H^*(G/H)$, and those which in special cases gave the ring structure could be factored through Borel’s theory. A collapse result guaranteeing a multiplicative isomorphism was provided by Franz only in 2019 [11].

Arguably the most general of the Eilenberg–Moore collapse results actually applies to a more general case than homogeneous spaces: Munkholm proves the collapse of the spectral sequence corresponding to the total space of a pullback bundle

$$\begin{array}{ccc} X \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

when X , B , and E have polynomial cohomology on countably many generators and the fundamental group $\pi_1(B)$ is trivial [26].¹ A biquotient—and hence, in particular, a homogeneous space—fits into this setting when $X = BK$, $B = BG$, and $E = BH$ are models of classifying spaces chosen in such a way that $BH \rightarrow BG$ is a fiber bundle, and Singhof [29] used Munkholm’s result in this case to determine that $H^*(K \backslash G/H)$ is additively isomorphic to $\mathrm{Tor}_{H^*(BG)}^*(H^*(BK), H^*(BH))$, and simply to Tor^0 in the special case where the rank of G is the sum of the ranks of K and H . Because the Eilenberg–Moore spectral sequence is concentrated in the 0th column in this case, there is no extension problem, so this is in fact a ring isomorphism. One might hope Singhof’s and Franz’s theorems were instances of a more general result, and our initial motivation was to show this hope is justified.

Theorem 0.1 *Let G be a connected Lie group, H and K closed connected subgroups, and k a principal ideal domain in which 2 and the torsion primes of G , H , and K are units. Then the Borel equivariant cohomology ring of the left translation action of K on G/H , or, equivalently, of the two-sided action of $K \times H$ on G by $(x, h) \cdot g := xgh^{-1}$, is*

$$H_K^*(G/H; k) \cong H_{K \times H}^*(G; k) \cong \mathrm{Tor}_{H^*(BG; k)}^*(H^*(BK; k), H^*(BH; k)).$$

In particular, if the two-sided action of $K \times H$ on G is free, the cohomology ring of the biquotient $K \backslash G/H$ is given by

$$H^*(K \backslash G/H; k) \cong \mathrm{Tor}_{H^*(BG; k)}^*(H^*(BK; k), H^*(BH; k)).$$

See Example 6.1 for a sample computation. This result in fact extends to what could be called *generalized biquotients* by analogy with generalized homogeneous spaces; see Remark 4.3, where we recover a known result on the cohomology of certain free loop spaces.

In broadest outline, our proof of Theorem 0.1 uses the Eilenberg–Moore theorem to show the cohomology of the homotopy pullback of the diagram $BK \rightarrow BG \leftarrow BH$ is $\mathrm{Tor}_{C^*(BG)}^*(C^*(BK), C^*(BH))$ and constructs maps between $H^*(B\Gamma)$ and $C^*(B\Gamma)$ for $\Gamma \in \{G, K, H\}$ which then induce an isomorphism with

¹ And additionally, in characteristic 2, the cup-1 squares of polynomial generators of $H^*(X)$ and $H^*(E)$ vanish.

$\mathrm{Tor}_{H^*(BG)}(H^*(BK), H^*(BH))$. This much it has in common with many collapse results, but our result also shows that this map takes the classical product on the Tor of cochain algebras to that on the Tor of cohomology rings. The way this multiplicativity is established closely follows Franz's proof, but among the technical underpinnings necessary to extend his approach from a fibration to a general pullback is one substantial innovation.

A Tor of DGAs in full generality is not endowed with a product, and the product that is defined on $\mathrm{Tor}_{C^*(BG)}(C^*(BK), C^*(BH))$ is synthetic, arising from the homological external product rather than a multiplicative structure on the resolution itself. As a consequence, in previous collapse results, one could not say whether the isomorphisms shown were multiplicative. But there is a cochain complex $\mathbf{B}(A', A, A'')$, the two-sided bar construction, functorial in spans $A' \leftarrow A \rightarrow A''$ of DGA maps, whose cohomology is $\mathrm{Tor}_A(A', A'')$ under mild conditions. In Theorem A.1, assuming the maps in the span are actually maps of so-called *homotopy Gerstenhaber algebras*, a type of DGA with extra structure (of which cochain algebras are the main examples), we are able to define a product on $\mathbf{B}(A', A, A'')$ inducing a product on $\mathrm{Tor}_A(A', A'')$ which specializes to the known products when A', A , and A'' are cochain algebras or cohomology rings. This product may be the point of greatest interest in this paper; it is certainly the most difficult. Then the maps between $H^*(B\Gamma)$ and $C^*(B\Gamma)$ alluded to in the previous paragraph, chosen with sufficient care and assuming 2 is a unit of the coefficient ring k , will preserve this novel product in the transition from $\mathbf{B}(H^*(BK), H^*(BG), H^*(BH))$ to $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$.

Outline The layout of the paper is as follows:

- (1) In Section 1 we recall algebraic conventions and the two-sided bar construction.
- (2) In Section 2 we discuss extended homotopy Gerstenhaber algebras and recall results showing normalized cochain algebras and cohomology rings are examples.
- (3) In Section 3 we recall the quasi-isomorphisms we need: A_∞ -algebra maps λ from $H^*(B\Gamma)$ to $C^*(B\Gamma)$ for $\Gamma \in \{K, G, H\}$ and DGA maps $f: C^*(BT) \rightarrow H^*(BT)$ for $T \in \{T_K, T_H\}$ maximal tori in K and H . These f will be defined so as to annihilate the error terms distinguishing λ from a genuine DGA map, so that the composites $H^*(BK) \rightarrow C^*(BK) \rightarrow C^*(BT_K) \rightarrow H^*(BT_K)$ and $H^*(BH) \rightarrow C^*(BH) \rightarrow C^*(BT_H) \rightarrow H^*(BT_H)$ are just the functorially induced $H^*(B(T_K \hookrightarrow K))$ and $H^*(B(T_H \hookrightarrow H))$. The DGA quasi-isomorphisms f do not necessarily exist if T is replaced by a more general Lie group, and to construct them we need a simplicial model for BT , which necessitates the replacements discussed in Section 4.

It is an important technical point in showing the three A_∞ -maps λ from $H^*(B\Gamma)$ to $C^*(B\Gamma)$ induce a map of Tors that they are essentially functorial up to homotopy. This fact, and our control over the error term annihilated by f , come from the existence of a certain auxiliary structure on extended homotopy Gerstenhaber algebras, a so-called *strongly homotopy commutative* algebra structure Φ whose existence was proven by Franz. The explicit formula for Φ plays a key role in the appendix. The fact that Φ , so defined, is a structure of the type sought follows from an extraordinarily involved cochain-level computation [10].

(4) In Section 4 we apply the enhanced version of the Eilenberg–Moore theorem proven in the appendix to the span $BK \rightarrow BG \leftarrow BH$ whose homotopy pullback $K \backslash G / H$ we are interested in, showing that $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$ carries a weak multiplicative structure inducing the cup product on $H^*(K \backslash G / H)$. We then show this product is preserved when we replace $BK \rightarrow BG \leftarrow BH$ with a simplicial model making available the DGA formality maps discussed in the previous paragraph.

(5) In Section 5 we weave these threads together to prove Theorem 0.1. We first construct a quasi-isomorphism

$$\Theta: \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \rightarrow \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)),$$

roughly to be thought of as $\mathbf{B}(\lambda_K, \lambda_G, \lambda_H)$, which is not necessarily multiplicative. Letting $T_K \leq K$ and $T_H \leq H$ be maximal tori, we then map the codomain into $\mathbf{B}(C^*(BT_K), C^*(BG), C^*(BT_H))$, using the maps $\rho: C^*(B\Gamma) \rightarrow C^*(BT_\Gamma)$ for $\Gamma \in \{K, H\}$ to define a strictly multiplicative map

$$\Psi = \mathbf{B}(f\rho, \text{id}, f\rho): \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \rightarrow \mathbf{B}(H^*(BT_K), C^*(BG), H^*(BT_H)),$$

inducing an injection in cohomology. Because we have chosen f and λ to compose nicely, the composite $\Psi\Theta$ is multiplicative up to homotopy, and because the multiplicative map $H^*(\Psi)$ is injective, this shows $H^*(\Theta)$ is multiplicative as well, concluding the proof.

(6) Finally in the appendix, the technical core of the work, we construct a product on the two-sided bar construction and use it to prove an Eilenberg–Moore theorem (Theorem A.27) in which the product on the cohomology of the pullback of $X \rightarrow B \leftarrow E$ arises from the nonassociative product on $\mathbf{B}(C^*(X), C^*(B), C^*(E))$.

Remark 0.2 (added in proof) Since the submission of the present article, several new results have been obtained. Franz [12] has shown that the product on the two-sided bar construction mentioned above can indeed be extended to an A_∞ -structure; this had been anticipated in Remark A.26. Moreover, when the domain is equipped with this A_∞ -structure, the quasi-isomorphism ξ of Theorem A.27 is enhanced to a map of A_∞ -algebras. Munkholm [26, Section 9] had already introduced a product on $\text{Tor}_A(A', A'')$, for DGA maps $A \leftarrow A' \rightarrow A''$ meeting certain hypotheses. Carlson [8] has resurrected and studied the properties of this product and determined [7] that under mild flatness hypotheses, the product introduced in the present article induces Munkholm’s product in cohomology.

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1 Algebras, twisting cochains, and bar constructions

In this background section we establish notational conventions and recount some foundational lemmas. Nothing is original here, save possibly some of the lemmas on two-sided twisted tensor products and bar constructions, which if not published are still likely known.

Fix forever a commutative base ring k with unity over which we will consider cochain complexes, and with respect to which all tensor products and Hom-modules will be taken. We will take as understood the Koszul sign convention and the notions of derivation, coderivation, differential graded k -algebra (henceforth DGA), and differential graded k -coalgebra (DGC). A commutative DGA is a CDGA. All *algebras* we consider are *nonnegatively graded, associative, and connected* unless otherwise noted, and all *coalgebras* *nonnegatively graded, coassociative, and cocomplete*. All DGA maps preserve unit and augmentation, and all DGC maps preserve counit and coaugmentation. All ideals of DGAs will be *two-sided differential* ideals.

We also assume understood suspensions, desuspensions, tensor products, and Hom-modules of cochain complexes, homomorphisms of DGAs and DGCs, and the classical (reduced) bar construction of a DGA. We nevertheless need to establish notation:

Notation 1.1 (1) Given a cochain complex $B = \bigoplus B_n$, its differential is canonically written d_B . For the tensor differential on a tensor product $\bigotimes B^{(i)}$ of complexes $B^{(i)}$ we write $d_\otimes$. We write $|x|$ for the degree n of a homogeneous element $x \in B_n$. The *desuspension* $s^{-1}B = \{s^{-1}b : b \in B\}$ is the (re)graded k -module given by $(s^{-1}B)_n := B_{n+1}$, equipped with the differential $d_{s^{-1}B} : s^{-1}b \mapsto -s^{-1}db$. The desuspension map $s^{-1} : b \mapsto s^{-1}b$ is a degree- (-1) isomorphism of cochain complexes with inverse $s : s^{-1}B \xrightarrow{\sim} B$, the *suspension*.

(2) A DGA A is a list comprising, besides the underlying cochain complex (A, d_A) , a canonically named multiplication $\mu_A : A \otimes A \rightarrow A$, unit $\eta_A : k \rightarrow A$, and augmentation $\epsilon_A : A \rightarrow k$, the clarifying decorations suppressed when practicable. The augmentation ideal is $\bar{A}$ and we write $\mu^{[n]} : A^{\otimes n} \rightarrow A$ for iterated multiplication.

(3) A DGC comprises a cochain complex (C, d_C) , comultiplication $\Delta_C : C \rightarrow C \otimes C$, counit $\epsilon_C : C \rightarrow k$, and coaugmentation $\eta_C : k \rightarrow C$. *Cocompleteness* of C means that every element is annihilated by some iterate of the reduced comultiplication $c \mapsto \Delta c - 1 \otimes c - c \otimes 1$.

(4) We use $\bullet$ to abbreviate indices or exponents representing an indefinite number of tensor factors to be determined from context; for instance, if A is a graded algebra, $a_\bullet$ denotes a pure tensor $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$, where n is to be gleaned contextually. A repeated $\bullet$ implies summation unless explicitly stated otherwise, so that for example $d_\otimes = \text{id}^\bullet \otimes d_A \otimes \text{id}^\bullet : A^{\otimes n} \rightarrow A^{\otimes n}$ represents the sum of the n maps $\text{id}^{\otimes \ell} \otimes d_A \otimes \text{id}^{\otimes n-\ell-1}$ applying d_A to one tensor factor. Given a pure tensor $a_\bullet \in A^{\otimes n}$, an expression involving the string $a_\bullet$ multiple times represents a sum over order-preserving subdivisions of $a_\bullet$ into tensor factors.

(5) Our notation for tensor-factor permutations is typically cyclic. For example, $(12): A \otimes B \rightarrow B \otimes A$ denotes the k -linear map given by $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$. We may at times also abusively use arguments to indicate permutations; for example we may instead of (123) write $\tau_{a \otimes b; c}: A \otimes B \otimes C \rightarrow C \otimes A \otimes B$ for the map taking $a \otimes b \otimes c$ to $(-1)^{|c||a|+|c||b|} c \otimes a \otimes b$. Amongst all permutations, *shuffles* will be of particular interest for us: a (p, q) -*shuffle* is a permutation π of $\{1, \dots, p+q\}$ such that $\pi(i) < \pi(j)$ whenever $i < j$ both lie in $\{1, \dots, p\}$ or both lie in $\{p+1, \dots, p+q\}$. Thus shuffles interleave two blocks, leaving the order within each block unaffected. To avoid writing Koszul signs, we generally manipulate such maps directly, without evaluating on elements.

Definition 1.2 The graded module underlying the *bar construction* $\mathbf{BA}$ of a DGA A is the direct sum of $\mathbf{B}_n A := (s^{-1} \bar{A})^{\otimes n}$ for $n \geq 0$. The projection $\mathbf{BA} \rightarrow \mathbf{B}_n A$ is denoted by pr_n . Pure tensors are written as bar-words $[a_1 | \dots | a_n] \in \mathbf{B}_n A$, or $[a_\bullet]$ when the *length* $\ell([a_1 | \dots | a_n]) = n$ is immaterial. The empty word $[] \in \mathbf{B}_0 A$ is the image of $1 \in k$ under the coaugmentation; the counit can be identified with pr_0 . The comultiplication is the *deconcatenation* $\Delta_{\mathbf{BA}}: [a_\bullet] \mapsto \sum_{p=0}^{\ell(a_\bullet)} [a_1 | \dots | a_p] \otimes [a_{p+1} | \dots | a_{\ell(a_\bullet)}]$. We will employ Einstein–Sweedler notation for the values of iterated deconcatenations: $\Delta_{\mathbf{BA}}^{[n]}[a_\bullet] =: [a_{(1)}] \otimes \dots \otimes [a_{(n)}]$, implicitly summing over partitions of $[a_\bullet]$ into n bar-subwords.

The sum of the tensor differentials on the $\mathbf{B}_n A = (s^{-1} \bar{A})^{\otimes n}$ is the *internal differential* d_{int} ; the *external differential* $d_{\text{ext}}: \mathbf{B}_n A \rightarrow \mathbf{B}_{n-1} A$ is the bar-deletion operation $\text{id}^\bullet \otimes s^{-1} \mu_A s^{\otimes 2} \otimes \text{id}^\bullet$ for $n \geq 2$ and is 0 on $\mathbf{B}_{\leq 1} A$. The total differential $d_{\mathbf{BA}} := d_{\text{int}} + d_{\text{ext}}$ makes $\mathbf{BA}$ a DGC. This DGC structure is functorial in DGA maps $f: A \rightarrow B$, with the DGC map $\mathbf{B}f: \mathbf{BA} \rightarrow \mathbf{BB}$ given by $(s^{-1} f s)^{\otimes n}: [a_\bullet] \mapsto [f a_1 | \dots | f a_n] =: [f a_\bullet]$ on $\mathbf{B}_n A$.

Neglecting the DGC structure, there is a structurally irrelevant but sometimes notationally convenient *concatenation* map $\Delta_{\leftarrow}: [a_1 | \dots | a_p] \otimes [b_1 | \dots | b_q] \mapsto [a_1 | \dots | a_p | b_1 | \dots | b_q]$ which comes from viewing $\mathbf{BA}$ as the tensor algebra on $s^{-1} \bar{A}$.

Definition 1.3 Given a DGC C and DGA A , we write $\text{Hom}_n(C, A)$ for the k -module of k -linear maps f sending each C_j to A_{j+n} , and set the degree $|f|$ to n for such a map. We write $\text{Hom}(C, A)$ for the graded module $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_n(C, A)$. This module becomes a cochain complex under the differential $D = d_{\text{Hom}(C, A)}$ given by $Df := d_A f - (-1)^{|f|} f d_C$ and a DGA with respect to the *cup product* $f \smile g := \mu_A(f \otimes g) \Delta_C$, unity $\eta_A \in C: C \rightarrow k \rightarrow A$, and augmentation $f \mapsto \eta_A f \in C$. If $h \in \text{Hom}_0(C, A)$ satisfies $h \eta_C = \eta_A$, then a cup inverse $h^{\smile -1} := \sum_{\ell=0}^{\infty} (\eta_A \in C - h)^{\smile \ell}$ is well defined by cocompleteness of C . An element $t \in \text{Hom}_1(C, A)$ satisfying the three conditions

$$\epsilon_A t = 0 = t \eta_C, \quad D t = t \smile t$$

is called a *twisting cochain*.

Example 1.4 For each connected DGA A , the composition

$$t_A: \mathbf{BA} \xrightarrow{\text{pr}_1} s^{-1} \bar{A} \xrightarrow{s} \bar{A} \hookrightarrow A$$

of degree 1 is a twisting cochain called the *tautological twisting cochain*. The assignment $t_{(-)}$ is a natural transformation between the functors assigning a DGA A the underlying graded modules of $\mathbf{B}A$ and of A .

Any twisting cochain $C \rightarrow A$ extends uniquely to a DGC map $C \rightarrow \mathbf{B}A$, and inversely, given a DGC map $F: C \rightarrow \mathbf{B}A$, postcomposing $t_A: \mathbf{B}A \rightarrow A$ gives a twisting cochain $t_A F: C \rightarrow A$, thus prescribing a bijection between twisting cochains $C \rightarrow A$ and DGC maps $C \rightarrow \mathbf{B}A$.

Example 1.5 Given a DGA A , one can check that the map

$$t_{\nabla} = \mu_A(t_A \otimes \epsilon_A + \epsilon_A \otimes t_A): \mathbf{B}A \otimes \mathbf{B}A \rightarrow A,$$

which takes $[a] \otimes 1$ and $1 \otimes [a]$ to a and annihilates all other pure tensors, is a twisting cochain if and only if A is a CDGA. In this case, the uniquely induced DGC map

$$\mu_{\nabla}: \mathbf{B}A \otimes \mathbf{B}A \rightarrow \mathbf{B}A,$$

called the *shuffle product*, is a product making $\mathbf{B}A$ a DG Hopf algebra. The shuffle product takes $[a_{\bullet}] \otimes [b_{\bullet}] \in \mathbf{B}_p A \otimes \mathbf{B}_q A$ to the sum of all (p, q) -shuffles (with Koszul sign) of $[a_{\bullet}|b_{\bullet}]$.

More generally, let B be another DGA. Then the *shuffle map*

$$\nabla: \mathbf{B}A \otimes \mathbf{B}B \rightarrow \mathbf{B}(A \otimes B)$$

is the direct sum of the maps $\mathbf{B}_p A \otimes \mathbf{B}_q B \rightarrow \mathbf{B}_{p+q}(A \otimes B)$ sending $[a_{\bullet}] \otimes [b_{\bullet}]$ to the sum of all tensor (p, q) -shuffles of $[a_1 \otimes 1 | \cdots | a_p \otimes 1 | 1 \otimes b_1 | \cdots | 1 \otimes b_q]$. This is a DGC map, and if $A = B$ is a CDGA, then the composition $\mathbf{B}\mu \circ \nabla$ is the product μ_{∇} of the previous paragraph.

The bijection between DGC maps and twisting cochains preserves suitable homotopy notions.

Definition 1.6 A DGC *homotopy* from one DGC map $F: C \rightarrow B$ to another such map G is a degree- (-1) map $H: C \rightarrow B$ such that

$$\epsilon_B H = 0, \quad H \eta_C = 0, \quad DH = G - F, \quad \text{and} \quad \Delta_B H = F \otimes H + H \otimes G.$$

We write $H: F \simeq G$ in this situation. Taking $B = \mathbf{B}A$ for an augmented DGA A , we may translate to an appropriate notion of homotopy for twisting cochains. Given two twisting cochains $t, u \in \text{Hom}_1(C, A)$, a *twisting cochain homotopy* from the former to the latter is a map $h \in \text{Hom}_0(C, A)$ satisfying the three conditions

$$\epsilon_C h = \epsilon_A, \quad h \eta_A = \eta_C, \quad \text{and} \quad Dh = t \smile h - h \smile u.$$

We again write $h: t \simeq u$.

Proposition 1.7 Given a DGC homotopy $H: C \rightarrow \mathbf{B}A$, the map $h = \eta_A \epsilon_C + t_A H: C \rightarrow A$ is a twisting cochain homotopy, and the assignment $H \mapsto h$ is a bijection from the set of DGC homotopies $F \simeq G$ to the set of twisting cochain homotopies $t_A F \simeq t_A G$.

Aside from providing a tractable encoding of DGC maps $C \rightarrow \mathbf{BA}$ and homotopies therebetween, twisting cochains $C \rightarrow A$ can also be harnessed to produce new differentials on $C \otimes A$ which we will use to define the two-sided bar construction.

Definition 1.8 Let C be a DGC, A' and A'' DGAs, M a differential right C -comodule, and N a differential left A'' -module. One defines the *cap product* with an element $\phi \in \text{Hom}(C, A'')$ by

$$\delta_\phi^R := (\text{id}_M \otimes \mu_N)(\text{id}_M \otimes \phi \otimes \text{id}_N)(\Delta_M \otimes \text{id}_N): M \otimes N \rightarrow M \otimes N, \quad x \otimes y \mapsto \pm x_{(1)} \otimes \phi(x_{(2)})y.$$

It is standard that if $\phi = t''$ is a twisting cochain, then $d_\otimes - \delta_{t''}^R$ is a differential on $M \otimes N$, making it a cochain complex we denote by $M \otimes_{t''} N$ and call a *twisted tensor product*. When $M = C$ and $N = A''$, this prescription makes $C \otimes_{t''} A''$ a differential left C -module and a differential right A'' -module.

A symmetric construction produces a cap product endomorphism δ_ϕ^L on $P \otimes Q$ from a graded linear $\phi: C \rightarrow A'$, a differential left C -comodule Q , and a differential right A' -module P , and if $\phi = t'$ is a twisting cochain then $\delta_{t'}^L + d_\otimes$ is a differential on $P \otimes Q$; we again write $P \otimes_{t'} Q$ for the twisted tensor product. Applying this construction to $P = A'$ and $Q = C \otimes_{t''} A''$, we obtain a *two-sided twisted tensor product* [19, Remarks II.5.4]:

$$A' \otimes_{t'} C \otimes_{t''} A'' := A' \otimes_{t'} (C \otimes_{t''} A'').$$

This is an (A', A'') -bimodule, and given ideals $\mathfrak{a}' \trianglelefteq A'$ and $\mathfrak{a}'' \trianglelefteq A''$, we write $(\mathfrak{a}', \mathfrak{a}'')$ for the (A', A'') -subbimodule $\mathfrak{a}' \otimes C \otimes A'' + A' \otimes C \otimes \mathfrak{a}''$.

We will need some lemmas describing when (homotopy-)commutative squares of maps

$$(1.9) \quad \begin{array}{ccccc} A'_0 & \xleftarrow{t'_0} & C_0 & \xrightarrow{t''_0} & A''_0 \\ \downarrow & & \downarrow & & \downarrow \\ A'_1 & \xleftarrow{t'_1} & C_1 & \xrightarrow{t''_1} & A''_1 \end{array}$$

induce a map $A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \rightarrow A'_1 \otimes_{t'_1} C_1 \otimes_{t''_1} A''_1$ of twisted tensor products. These properties seem to be well known in the one-sided case (see e.g. Huebschmann [18, page 360] and Franz [11, Section 7]), so we can safely suppress the proofs of their easily guessed two-sided generalizations.

Lemma 1.10 Let $G: C_0 \rightarrow C_1$ be a DGC map, $f': A'_0 \rightarrow A'_1$ and $f'': A''_0 \rightarrow A''_1$ DGA maps, and $t'_j: C_j \rightarrow A'_j$ and $t''_j: C_j \rightarrow A''_j$ twisting cochains for $j \in \{0, 1\}$. Then $f' \otimes G \otimes f''$ is a cochain map

$$A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \rightarrow A'_1 \otimes_{t'_1} C_1 \otimes_{t''_1} A''_1$$

if $t'_1 G = f' t'_0$ and $t''_1 G = f'' t''_0$. If G is one such DGC map and $\tilde{G}$ another, and $H: C_0 \rightarrow C_1$ is a DGC homotopy $G \simeq \tilde{G}$ such that $t'_1 H$ and $t''_1 H$ are 0, then $f' \otimes H \otimes f''$ is a cochain homotopy $f' \otimes G \otimes f'' \simeq f' \otimes \tilde{G} \otimes f''$.

Corollary 1.11 Let C_0 and C_1 be DGCs, A' and A'' DGAs, $t': C_1 \rightarrow A'$ and $t'': C_1 \rightarrow A''$ twisting cochains, and $G: C_0 \rightarrow C_1$ a DGC homomorphism. Then $t' \circ G: C_0 \rightarrow A'$ and $t'' \circ G: C_0 \rightarrow A''$ are twisting cochains and

$$\text{id}_{A'} \otimes G \otimes \text{id}_{A''}: A' \otimes_{t'G} C_0 \otimes_{t''G} A'' \rightarrow A' \otimes_{t'} C_1 \otimes_{t''} A''$$

a cochain map.

Lemma 1.12 Let C be a DGC, A' and A'' DGAs, $t'_j: C \rightarrow A'$ and $t''_j: C \rightarrow A''$ twisting cochains for $j \in \{0, 1\}$, and $\alpha' \leq A'$ and $\alpha'' \leq A''$ ideals, and suppose there exist homotopies $h': t'_0 \simeq t'_1$ and $h'': t''_1 \simeq t''_0$ with $h'\bar{C} \leq \alpha'$ and $h''\bar{C} \leq \alpha''$.² Then the composition

$$(\delta_{h'}^L \otimes \text{id}_{A''})(\text{id}_{A'} \otimes \delta_{h''}^R): A' \otimes C \otimes A'' \rightarrow A' \otimes C \otimes A''$$

is a cochain isomorphism $A' \otimes_{t'_0} C \otimes_{t''_0} A'' \xrightarrow{\sim} A' \otimes_{t'_1} C \otimes_{t''_1} A''$ congruent to the identity modulo (α', α'') .

We now define the twisted tensor products of greatest interest to us:

Definition 1.13 Let A, A' , and A'' be augmented DGAs and $F': \mathbf{B}A \rightarrow \mathbf{B}A'$ and $F'': \mathbf{B}A \rightarrow \mathbf{B}A''$ DGC maps.³ Associated to F' and F'' are twisting cochains $t' = t_{A'}F'$ and $t'' = t_{A''}F''$, and we define the *two-sided bar construction* as the two-sided twisted tensor product

$$\mathbf{B}(A', A, A'') := A' \otimes_{t_{A'}F'} \mathbf{B}A \otimes_{t_{A''}F''} A''.$$

We will write a pure tensor $a' \otimes [a_\bullet] \otimes a'' \in \mathbf{B}(A', A, A'')$ as $a'[a_\bullet]a''$. The differential of $\mathbf{B}(A', A, A'')$ preserves the length filtration $\mathbf{B}_{\leq \ell}(A', A, A'') := A' \otimes \mathbf{B}_{\leq \ell}A \otimes A''$.

Example 1.14 This two-sided bar construction, when we take $F' = \mathbf{B}f'$ and $F'' = \mathbf{B}f''$ for DGA maps $f': A \rightarrow A'$ and $f'': A \rightarrow A''$, coincides with the familiar classical two-sided bar construction of differential graded algebras. In this case, the twisting cochains t' and t'' defining the twisted tensor product $\mathbf{B}(A', A, A'') = A' \otimes_{t'} \mathbf{B}A \otimes_{t''} A''$ are simply $t_{A'}\mathbf{B}f' = f't_A$ and $f''t_A$, respectively.

We see already from the preceding lemmas that DGA maps from A' and A'' and DGC maps from $\mathbf{B}A$ induce maps of two-sided bar constructions, and want to extend this technique to induce maps from commutative diagrams of DGC maps of the form

$$(1.15) \quad \begin{array}{ccccc} \mathbf{B}A'_0 & \xleftarrow{F'_0} & \mathbf{B}A_0 & \xrightarrow{F''_0} & \mathbf{B}A''_0 \\ G' \downarrow & & \downarrow G & & \downarrow G'' \\ \mathbf{B}A'_1 & \xleftarrow{F'_1} & \mathbf{B}A_1 & \xrightarrow{F''_1} & \mathbf{B}A''_1 \end{array}$$

² Note this second homotopy goes the opposite direction to the one one might expect.

³ These are A_∞ -algebra maps from A to A' and A'' in the language to be introduced in Section 3.

We write $G'_{(1)}: \overline{A}'_0 \rightarrow \overline{A}'_1$ for the composite $t_{A'_1} \circ G' \circ s_{A'_0}^{-1}$ and similarly for $G''_{(1)}$; later, in Definition 3.1, we will see $\eta\epsilon + G'_{(1)}: A'_0 \rightarrow A'_1$ and $\eta\epsilon + G''_{(1)}: A''_0 \rightarrow A''_1$ can be understood as a sort of approximate DGA map.

Proposition 1.16 (Wolf⁴ [34, Theorem 7]; cf Gugenheim and Munkholm [16, Theorem 3.5*]) *Suppose given a strictly commuting diagram of DGC maps as in (1.15). Then, in the notation of Definition 1.2, we may define a cochain map $\mathbf{B}(A'_0, A_0, A''_0) \rightarrow \mathbf{B}(A'_1, A_1, A''_1)$ by*

$$(1.17) \quad \mathbf{B}(G', G, G'') = (\Upsilon' \otimes G \otimes \Upsilon'')(\text{id}_{A'_0} \otimes \Delta_{\mathbf{B}A_0}^{[3]} \otimes \text{id}_{A''_0}),$$

where

$$\begin{aligned} \Upsilon'|_{k \otimes \mathbf{B}A_0} &= \text{id}_k \otimes \epsilon_{\mathbf{B}A_0}: & c'[a_{(1)}] &\mapsto c'\epsilon[a_{(1)}], \\ \Upsilon'|_{\overline{A}'_0 \otimes \mathbf{B}A_0} &= t_{A'_1} G' \Delta_{\leftarrow} (s_{A'_0}^{-1} \otimes F'_0): & a'[a_{(1)}] &\mapsto \pm t_{A'_1} G'([a'] \otimes F'_0[a_{(1)}]), \\ \Upsilon''|_{\mathbf{B}A_0 \otimes k} &= \epsilon_{\mathbf{B}A_0} \otimes \text{id}_k: & [a_{(3)}]c'' &\mapsto \epsilon[a_{(3)}]c'', \\ \Upsilon''|_{\mathbf{B}A_0 \otimes \overline{A}''_0} &= t_{A''_1} G'' \Delta_{\leftarrow} (F''_0 \otimes s_{A''_0}^{-1}): & [a_{(3)}]a'' &\mapsto \pm t_{A''_1} G''(F''_0[a_{(3)}] \otimes [a'']). \end{aligned}$$

If $t_{A'_1} G'$ takes $\mathbf{B}_{\geq 2} A'_0$ into an ideal $\mathfrak{a}' \trianglelefteq A'_1$ and $t_{A''_1} G''$ takes $\mathbf{B}_{\geq 2} A''_0$ into $\mathfrak{a}'' \trianglelefteq A''_1$, then

$$\mathbf{B}(G', g, G'') \equiv (\eta\epsilon + G'_{(1)}) \otimes G \otimes (G''_{(1)} + \eta\epsilon) \pmod{(\mathfrak{a}', \mathfrak{a}'')}.$$

Notation 1.18 In the event that the maps G , G' , and G'' in (1.15), are respectively $\mathbf{B}g$, $\mathbf{B}g'$, and $\mathbf{B}g''$ for $g: A_0 \rightarrow A_1$, $g': A'_0 \rightarrow A'_1$, and $g'': A''_0 \rightarrow A''_1$, we abuse notation by writing the map of Proposition 1.16 as $\mathbf{B}(g', g, g'') := \mathbf{B}(G', G, G'')$. Such a map is also a special case of that of Lemma 1.10.

We are interested in two-sided bar constructions because they provide functorial resolutions:

Proposition 1.19 (see Barthel, May, and Riehl [1, after Proposition 10.19]) *Suppose k is a principal ideal domain and $A \rightarrow A'$ a map of DGAs flat over k . Then given another DGA map $A \rightarrow A''$, the cohomology of the classical two-sided bar construction $\mathbf{B}(A', A, A'') = \mathbf{B}(A', A, A) \otimes_A A''$ of Example 1.14 is $\text{Tor}_A(A', A'')$. If $H^*(A)$ and $H^*(A'')$ are flat over k , then the E_2 page of the associated algebraic Eilenberg–Moore spectral sequence is $\text{Tor}_{H^*(A)}(H^*(A'), H^*(A''))$.*

Discussion 1.20 For the proof of Theorem A.13, we will need a more explicit expression for the differential on the classical two-sided bar construction $\mathbf{B}(A', A, A'')$ of Example 1.14. Note that the total differential is the sum of the tensor differential $d_{\otimes}$ on $A' \otimes \mathbf{B}A \otimes A''$ and two cap products. Separating the summand $\text{id}_{A'} \otimes d_{\mathbf{B}A} \otimes \text{id}_{A''}$ into its internal and external components, $d_{\mathbf{B}(A', A, A'')}$ decomposes as

⁴ Wolf [34] cites his unpublished dissertation [33] for a proof of this result; however, this work was not publicly accessible at time of writing. (It is now available at the author's webpage.) For our proof, which turns out to be nontrivial, see the first version of this paper (arXiv 2106.02986v1). Proofs of the lemmas and many of the standard claims in this section can be found there as well.

the sum of tensor differentials and an “external” differential which is the sum of the two cap products and the bar-deletion differential. Explicitly, appealing to Definition 1.8, this external differential $d_{\mathbf{B}(A', A, A'')}^{\text{ext}}$ is given on the summand $\mathbf{B}_\ell(A', A, A'')$ by

$$(1.21) \quad \begin{aligned} & \mu_{A'}(\text{id}_{A'} \otimes f's) \otimes \text{id}_{s^{-1}A}^{\otimes \ell-1} \otimes \text{id}_{A''} \\ & + \text{id}_{A'} \otimes d_{\text{ext}} \otimes \text{id}_{A''} \\ & - \text{id}_{A'} \otimes \text{id}_{s^{-1}A}^{\otimes \ell-1} \otimes \mu_{A''}(f''s \otimes \text{id}_{A''). \end{aligned}$$

The minus sign before the third term of (1.21) is important and not consistently noted in the literature.

2 Homotopy Gerstenhaber algebras

We ultimately want to describe the multiplicative structure on $\text{Tor}_{C^*(B)}(C^*(X), C^*(E))$ in terms of a product on the two-sided bar construction. The special property of a normalized singular cochain algebra A enabling us to do so will turn out to be the existence of a DG Hopf algebra structure on $\mathbf{B}A$, or in other words, a DGC map $\mathbf{B}A \otimes \mathbf{B}A \rightarrow \mathbf{B}A$ making it a DGA.⁵

Definition 2.1 Let A be a DGA such that $\mathbf{B}A$ admits a multiplication making it a DG Hopf algebra. If the twisting cochain $\mathbf{E} := t_A \mu_{\mathbf{B}A} : \mathbf{B}A \otimes \mathbf{B}A \rightarrow A$ satisfies $\mathbf{E}_{j,\ell} := \mathbf{E}|_{\mathbf{B}_j A \otimes \mathbf{B}_\ell A} = 0$ for $j \geq 2$, we call A , equipped with $\mu_{\mathbf{B}A}$ (equivalently, with $\mathbf{E}$), a *homotopy Gerstenhaber algebra* (HGA).

Notation 2.2 We will on occasion write the values of the product $\mu_{\mathbf{B}A}$ in infix notation as $[a_\bullet] * [b_\bullet]$. It will be useful later to translate $\mathbf{E}_{1,\bullet}$ into a degree-zero map on $A \otimes \mathbf{B}A$ by taking

$$\mathfrak{E} := \mathbf{E}_{1,\bullet}(s^{-1} \otimes \text{id}_{\mathbf{B}A}) : A \otimes \mathbf{B}A \rightarrow A, \quad a[b_\bullet] \mapsto \mathbf{E}([a] \otimes [b_\bullet]).$$

We will also have occasion to use the operations $E_\ell = \mathbf{E}_{1,\ell} \circ (s^{-1})^{\otimes 1+\ell} : A \otimes A^{\otimes \ell} \rightarrow A$, whose values we write as $E_\ell(a; b_\bullet) = E_\ell(a \otimes b_1 \otimes \cdots \otimes b_\ell)$. An HGA structure on A is more commonly phrased as a list of conditions [11, (6.2)–(6.4)] on the E_ℓ .

Remark 2.3 Unitality of $\mu_{\mathbf{B}A}$ implies $\mathbf{E}_{0,1}$ and $\mathbf{E}_{1,0}$ must both be $s : s^{-1} \bar{A} \xrightarrow{\sim} \bar{A}$ and $\mathbf{E}_{j,0}$ and $\mathbf{E}_{0,\ell}$ must be 0 for $j, \ell \geq 2$. The “ $t\eta = 0$ ” clause in the definition of a twisting cochain in Definition 1.3 implies $\mathbf{E}_{0,0} = \mathbf{E}\eta_{\mathbf{B}A} = 0$. For notational convenience, we will extend $\mathbf{E}_{0,1}$ and $\mathbf{E}_{1,0}$ to both be $s : s^{-1} A \rightarrow A$ (thus respectively sending $[1] \otimes []$ and $[] \otimes [1]$ to 1) and extend the operations $\mathbf{E}_{1,\ell}$ to $(s^{-1} A)^{\otimes 1+\ell}$ for $\ell \geq 1$ by setting them to 0 on pure tensors any of whose factors lies in $s^{-1} \text{im } \eta_A$. That is, we will sometimes consider bar-words containing a letter c in the coefficient ring k , but such words are to be annihilated by $\mathbf{E}_{j,\ell}$ unless $(j, \ell) = (1, 0)$ or $(0, 1)$.

⁵ Under our blanket assumption that A is graded and connected, a bialgebra structure on $\mathbf{B}A$ admits a unique antipode defined by a formula analogous to the one defining the cup inverse in Definition 1.3 [14, Propositions 1.4.14 and 1.4.24].

Example 2.4 If A is a commutative DGA, then the shuffle product μ_{∇} of Example 1.5 makes the bar construction $\mathbf{B}A$ a DG Hopf algebra, so that A becomes an HGA. The corresponding twisting cochain $t_{\nabla} = \mathbf{E}$ satisfies $\mathbf{E} = \mathbf{E}_{0,1} + \mathbf{E}_{1,0}$, so the operations E_{ℓ} vanish for $\ell \geq 1$.

The Maurer–Cartan identity $D\mathbf{E} = \mathbf{E} \smile \mathbf{E}$ for the twisting cochain $\mathbf{E}: \mathbf{B}A \otimes \mathbf{B}A \rightarrow A$ defining an HGA structure on A yields, upon mild rearrangement, the useful formulae

$$(2.5) \quad \mathfrak{E}(\mu_A \otimes \text{id}_{\mathbf{B}A}) = \sum_{\pi} \mu_A(\mathfrak{E} \otimes \mathfrak{E})\pi: \bar{A} \otimes \bar{A} \otimes \mathbf{B}A \rightarrow A, \quad \mathfrak{E}(a_1 a_2 [b_{\bullet}]) = \sum \pm \mathfrak{E}(a_1 [b_{\bullet}]) \mathfrak{E}(a_2 [b_{\bullet}]),$$

where the sum is over shuffles $\pi: [a_1 | a_2] \otimes [b_{\bullet}] \mapsto \pm [a_1 | b_{\bullet}] \otimes [a_2 | b_{\bullet}]$, and

$$(2.6) \quad \begin{aligned} D_{\otimes} \mathfrak{E} &= d_A \mathfrak{E} - \mathfrak{E} d_{\otimes} \mu_A(s \otimes \mathfrak{E})(12) + \mathfrak{E}(\text{id}_A \otimes d_{\mathbf{B}(A,A,A)}^{\text{ext}}) - \mu_A(\mathfrak{E} \otimes s), \\ (D_{\otimes} \mathfrak{E})(a[b_{\bullet}]) &= \pm b_1 \mathfrak{E}(a[b_2 | b_{\bullet} | b_{\ell}]) \pm \mathfrak{E}(a[b_1 | b_{\bullet} | b_p b_{p+1} | b_{\bullet} | b_{\ell}]) \pm \mathfrak{E}(a[b_1 | b_{\bullet} | b_{\ell-1}]) b_{\ell} \end{aligned}$$

for $\ell \geq 2$, with the middle term omitted if $\ell = 1$.⁶ Here $d_{\otimes}$ refers to the “internal” differential $d_{\bar{A}} \otimes \text{id}_{\mathbf{B}A} + \text{id}_{\bar{A}} \otimes d_{\otimes}$ on $\bar{A} \otimes \mathbf{B}A$, omitting the external differential on the $\mathbf{B}A$ factor, and $D_{\otimes}$ is the derivation on $\text{Hom}(\bar{A} \otimes \mathbf{B}A, A)$ defined with respect to this internal differential and d_A .

Critically, there is a natural way to make a variant of the cochain algebra into an HGA:

Definition 2.7 The *normalized cochain algebra* $C^*(X_{\bullet}; k)$ on a simplicial set $X_{\bullet}$ is the DG subalgebra containing all and only cochains vanishing on each degenerate simplex. It is augmented with respect to the map $C^*(X_{\bullet}; k) \twoheadrightarrow C^*(x_0; k) \simeq k$ induced by the inclusion of the simplicial subset associated to any chosen basepoint $x_0 \in X_0$.

The so-called *interval-cut operations*, among which the classical cup product and Steenrod cup- i products are the most famous examples, are known [24, Theorem 2.15] to define on $C^*(X_{\bullet}; k)$ the action of a symmetric DG-operad $\mathcal{X}$ called the *sequence operad* [24] or *surjection operad* [3]. This operad is known to be a quotient of the DG-operad $\mathcal{E}$ associated to the classical Barratt–Eccles simplicial operad [3, Theorem 1.3.2], an E_{∞} -operad filtered by an increasing sequence of suboperads $F_n \mathcal{E}$ whose geometric realizations are equivalent to the little n -cubes operads, and the sequence operad is accordingly filtered by quotients $F_n \mathcal{X}$ [3, Lemma 1.6.1]. As it turns out, an HGA structure on a DGA A is precisely an algebra structure over $F_2 \mathcal{X}$ [23; 24, Theorem 4.1; 3, Section 1.6.6] (cf Franz [13, (3.13)] for the signs of the operations). Franz [10, Section 3.2] defines an *extended homotopy Gerstenhaber algebra* structure on a DGA A to be an algebra structure over a certain suboperad $F'_3 \mathcal{X}$ of $F_3 \mathcal{X}$ containing $F_2 \mathcal{X}$. This structure comes in the form of operations $F_{p,q}: A^{\otimes p} \otimes A^{\otimes q} \rightarrow A$ for $p, q \geq 1$, in addition to the E_{ℓ} , satisfying several compatibility conditions. An *extended HGA homomorphism* $f: A \rightarrow B$ is a DGA map which is simultaneously an $F'_3 \mathcal{X}$ -algebra homomorphism (which is to say it distributes over the E_{ℓ} and the $F_{p,q}$). Thus we have the following:

⁶ Compare Franz [13, Lemma 3.1] for conditions for a map $C \rightarrow \Omega C \otimes \Omega C$ to be a twisting cochain inducing a DGC structure on the cobar construction ΩC of a DGC C .

Corollary 2.8 For any pointed simplicial set $X_\bullet$, its algebra $C^*(X_\bullet)$ of normalized cochains is naturally an extended HGA. Any CDGA A is naturally an extended HGA.

Convention 2.9 All CDGAs in this work, particularly cohomology rings, come equipped with this trivial extended HGA structure. Consequently, if A is an extended HGA and B is a CDGA, then an extended HGA map $f: A \rightarrow B$ annihilates the values of $F_{p,q}$ and of E_ℓ for $\ell \geq 1$.

3 Strong homotopy commutativity

As adverted to in the introduction, we will require maps between DGAs that are not multiplicative, but still preserve the multiplication up to coherent homotopy.

Definition 3.1 Given two A_∞ -algebras A and B , an A_∞ -map from the former to the latter is defined to be a DGC homomorphism $F: \mathbf{B}A \rightarrow \mathbf{B}B$. Such a map F is determined by the sequence of compositions

$$F_{(n)}: A^{\otimes n} \xrightarrow[\sim]{(s^{-1})^{\otimes n}} \mathbf{B}_n A \hookrightarrow \mathbf{B}A \xrightarrow{F} \mathbf{B}B \xrightarrow{t_B} B.$$

If A and B are DGAs, then we write $F_{(1)}^+ = F_{(1)} + \eta_B \epsilon_A: A \rightarrow B$. This is a cochain map, and if $H^*(F_{(1)}^+): H^*(A) \rightarrow H^*(B)$ is an isomorphism, we call F an A_∞ -quasi-isomorphism.⁷

We will see it is possible to construct an A_∞ -quasi-isomorphism from the CDGA $H^*(A)$ with trivial differential to A itself when A is a DGA with polynomial cohomology. To do so, we need two auxiliary notions:

Proposition 3.2 [26, Proposition 3.3] There exists an **internal tensor product** of A_∞ -maps which, given DGAs A, A', B , and B' and DGC maps $F: \mathbf{B}A \rightarrow \mathbf{B}A'$ and $G: \mathbf{B}B \rightarrow \mathbf{B}B'$, produces a DGC map

$$F \otimes G: \mathbf{B}(A \otimes B) \rightarrow \mathbf{B}(A' \otimes B').$$

The operation $(F, G) \mapsto F \otimes G$

- extends the tensor product of DGA maps in the sense that if $f: A \rightarrow A'$ and $g: B \rightarrow B'$ are DGA maps, then $\mathbf{B}f \otimes \mathbf{B}g = \mathbf{B}(f \otimes g)$,
- is functorial in each variable separately,
- is unital in the sense that $F \otimes \text{id}_k$ and $\text{id}_k \otimes F$ can be identified with F , and
- is associative in the sense that given a third pair C, C' of DGAs and a DGC map $H: \mathbf{B}C \rightarrow \mathbf{B}C'$, the iterated products $(F \otimes G) \otimes H$ and $F \otimes (G \otimes H): \mathbf{B}(A \otimes B \otimes C) \rightarrow \mathbf{B}(A' \otimes B' \otimes C')$ agree.

This notion of internal tensor product is connected to the ordinary external one by the shuffle map ∇ of Example 1.5.

⁷ The map $F_{(1)}^+$ can be seen as a DGA map up to homotopy, and an ideal $\mathfrak{b} \trianglelefteq B$ containing the image of $t_B F(\mathbf{B}_{\geq 2} A)$ can be seen as a measure of its deviation from multiplicativity, for $F_{(1)}^+$ is a DGA map just if $t_B F(\mathbf{B}_{\geq 2} A) = 0$; and if so, then $F = \mathbf{B}F_{(1)}^+$.

Proposition 3.3 [11, Lemma 4.4] *Let DGAs A'_0, A'_1, A''_0 , and A''_1 and DGC maps $G': \mathbf{B}A'_0 \rightarrow \mathbf{B}A'_1$ and $G'': \mathbf{B}A''_0 \rightarrow \mathbf{B}A''_1$ be given. Then the following square commutes:*

$$\begin{array}{ccc} \mathbf{B}A'_0 \otimes \mathbf{B}A''_0 & \xrightarrow{\nabla} & \mathbf{B}(A'_0 \otimes A''_0) \\ G' \otimes G'' \downarrow & & \downarrow G' \otimes G'' \\ \mathbf{B}A'_1 \otimes \mathbf{B}A''_1 & \xrightarrow{\nabla} & \mathbf{B}(A'_1 \otimes A''_1) \end{array}$$

The internal tensor product allows us to define another notion of homotopy commutativity a priori unrelated to HGAs:

Definition 3.4 A *strongly homotopy commutative algebra* (henceforth *SHC-algebra*) is an augmented DGA A equipped with an A_∞ -map from $A \otimes A$ to A (i.e. a DGC map $\Phi: \mathbf{B}(A \otimes A) \rightarrow \mathbf{B}A$), satisfying the following conditions:

- (1) Its 1-component $\Phi_{(1)}: \bar{A} \otimes \bar{A} \rightarrow \bar{A}$ is the restriction $\mu_A|_{\bar{A} \otimes \bar{A}}$ of the given product on A .
- (2) It is strictly unital in the sense that $\Phi \circ \mathbf{B}(\text{id}_A \otimes \eta_A) = \text{id}_{\mathbf{B}A} = \Phi \circ \mathbf{B}(\eta_A \otimes \text{id}_A)$.
- (3) It is homotopy-associative: there is a homotopy between $\Phi(\Phi \otimes \text{id}_A): \mathbf{B}(A \otimes A \otimes A) \rightarrow \mathbf{B}A$ and $\Phi(\text{id}_A \otimes \Phi)$.
- (4) It is homotopy-commutative: there is a homotopy between Φ and $\Phi \circ \mathbf{B}(12)$, where (12) is the tensor-factor transposition $A \otimes A \xrightarrow{\sim} A \otimes A$ of Notation 1.1(5).

We define the iterates $\Phi^{[n]}: \mathbf{B}(A^{\otimes n}) \rightarrow \mathbf{B}A$ of the structure map Φ by $\Phi^{[2]} := \Phi$ and $\Phi^{[n+1]} := \Phi(\Phi^{[n]} \otimes \text{id}_A)$.

An SHC-algebra structure Φ on a DGA A allows one to combine maps in a useful way: given sequences $(A_j)_{j=1}^n$ of DGAs and $(F_j: \mathbf{B}A_j \rightarrow \mathbf{B}A)_{j=1}^n$ of DGC maps, the composite

$$(3.5) \quad \mathbf{B}\left(\bigotimes_{j=1}^n A_j\right) \xrightarrow{\bigotimes F_j} \mathbf{B}(A^{\otimes n}) \xrightarrow{\Phi^{[n]}} \mathbf{B}A$$

is guaranteed to be another DGC map. We will say this map is *compiled* from the F_j .

Associated to each homogeneous element $a \in A$ is a map from the free DGA on one generator of degree $|a|$ taking this generator to a . If a is a cocycle of even degree, this map factors through the map $\lambda_a: k[x] \rightarrow A$ taking x to a , where the differential on $k[x]$ is trivial. Thus, given a list $\vec{a} = (a_j)$ of even-degree cocycles of A and taking $A_j := k[x_j]$ for $|x_j| = |a_j|$ and $F_j := \mathbf{B}\lambda_{a_j}$, the compilation procedure (3.5) yields a DGC map

$$(3.6) \quad \lambda_{\vec{a}}: \mathbf{B}\left(\bigotimes_{j=1}^n k[x_j]\right) \xrightarrow{\mathbf{B}(\bigotimes \lambda_{a_j})} \mathbf{B}(A^{\otimes n}) \xrightarrow{\Phi^{[n]}} \mathbf{B}A.$$

Then $(\lambda_{\vec{a}})_{(1)}^+$ is easily seen to be the optimist's candidate for a ring homomorphism,

$$k[\vec{x}] := k[x_1, \dots, x_n] \rightarrow A, \quad x_1^{p_1} \cdots x_n^{p_n} \mapsto a_1^{p_1} \cdots a_n^{p_n}.$$

Though this map is in fact almost never multiplicative, it is at least a quasi-isomorphism.

Proposition 3.7 (Stasheff–Halperin [31, Theorem 9] and [26, 7.2]) *If A is an SHC-algebra whose cohomology ring $H^*(A)$ is polynomial on classes represented by even-degree elements $a_j \in A$ and $\lambda_{\vec{a}}$ is defined as in (3.6), then $H^*(\lambda_{\vec{a}})_{(1)}^+ : k[\vec{x}] \rightarrow H^*(A)$ is an isomorphism.*

In the target application, A is the normalized cochain algebra on a classifying space BG . Munkholm [26, Proposition 4.7] (and stating a bit less, Gugenheim and Munkholm [16, Proposition 4.2]) showed that the cochain algebra of a simplicial set admits a natural SHC-algebra structure, but to define the product on the two-sided bar construction we will need later for our variant (Theorem A.27) of the Eilenberg–Moore theorem, we will use a result of Franz defining an SHC-algebra structure in terms of extended HGA operations.

Theorem 3.8 (Franz [10, Theorem 1.1 and (4.2)]) *An extended HGA A admits an SHC-algebra structure whose structure map Φ and associativity and commutativity homotopies are defined in terms of the extended HGA operations on A , and hence are natural in extended HGA maps. Moreover, the composite $\Phi \circ \nabla : \mathbf{BA} \otimes \mathbf{BA} \rightarrow \mathbf{B}(A \otimes A) \rightarrow \mathbf{BA}$ of the shuffle map of Example 1.5 and this structure map is the given product $\mu_{\mathbf{BA}}$ making A an HGA.*

We will require an explicit formula for $t_A \Phi_A$ in the proof of Theorem A.5, but hold off on stating it until then. From Corollary 2.8 and Theorem 3.8, we immediately have the following:

Corollary 3.9 *The normalized cochain algebra $C^*(X_\bullet; k)$ of a pointed simplicial set is naturally an SHC-algebra.*

Corollary 3.10 *For a pointed simplicial set $X_\bullet$ with polynomial cohomology and any list $\vec{a}$ in $C^*(X_\bullet; k)$ of representatives for k -algebra generators of $H^*(X_\bullet; k)$, the DGC map $\lambda_{\vec{a}} : \mathbf{BH}^*(X_\bullet; k) \rightarrow \mathbf{BC}^*(X_\bullet; k)$ given in (3.6) is such that $H^*(\lambda_{\vec{a}})_{(1)}^+$ can be identified with the identity map of $H^*(X_\bullet; k)$.*

We have observed that though it is a quasi-isomorphism in the cases that interest us, the extended 1-component $(\lambda_{\vec{a}})_{(1)}^+ : H^*(X_\bullet) \rightarrow C^*(X_\bullet)$ is rarely a DGA homomorphism on the nose. However, the extended HGA structure and the resulting SHC structure on $C^*(X_\bullet)$ guarantee that, loosely speaking, it is a DGA map *up to an error term* contained in an ideal $\mathfrak{k}_{X_\bullet}$ of $C^*(X_\bullet)$ functorial in $X_\bullet$, independently of the choice of representatives $\vec{a}$ in $C^*(X_\bullet)$; thinking of $\mathfrak{k}_{X_\bullet}$ as a neighborhood of 0, we may consider it a sort of uniform bound on failure to be a DGA map. It will be an important point in the proof of our main result that the bounding ideal $\mathfrak{k}_{X_\bullet}$ lies in the kernel of the formality map f to be described in Theorem 3.16, and hence f annihilates the error term.

Definition 3.11 (Franz [11, (10.2)]) *Given a pointed simplicial set $X_\bullet$, viewing its normalized cochain algebra $C^*(X_\bullet)$ as an extended HGA via Corollary 2.8 we denote by $\mathfrak{k} = \mathfrak{k}_{X_\bullet} \triangleleft C^*(X_\bullet)$ the ideal generated by the following elements, where $a, b, b_\bullet$, and $c_\bullet$ range over $C^*(X_\bullet)$:*

- (1) coboundaries,
- (2) elements of odd degree,
- (3) elements of the form $E_\ell(a; b_\bullet)$ for $\ell = \ell(b_\bullet) \geq 1$,
- (4) elements of the form $F_{p,q}(b_\bullet; c_\bullet)$ with $(p, q) \neq (1, 1)$,
- (5) elements of the form $a \smile_2 E_\ell(b; c_\bullet)$ with $\ell \geq 2$,
- (6) elements of the form $a \smile_2 (\cdots ((b_0 \smile_1 b_1) \smile_1 b_2) \smile_1 \cdots)$ for cocycles a and $b_\bullet$.

From the naturality of the extended HGA structure on a cochain algebra, the following functoriality property of $\mathfrak{k}$ is immediate:

Lemma 3.12 (Franz [11, Proposition 10.1]) *If $\phi: Y_\bullet \rightarrow X_\bullet$ is a map of pointed simplicial sets, then the ideals of Definition 3.11 satisfy $\phi^* \mathfrak{k}_{X_\bullet} \leq \mathfrak{k}_{Y_\bullet}$.*

Morally speaking, then, our maps $\lambda_{\vec{a}}$ are functorial and multiplicative modulo $\mathfrak{k}$:

Theorem 3.13 (Franz [10, Proposition 7.2; 11, Proposition 11.5 and Theorem 11.6]) *Suppose 2 is a unit of k . Then the maps $\lambda_{\vec{a}}: k[\vec{x}] \rightarrow C^*(X_\bullet)$ of (3.6) are functorial modulo $\mathfrak{k}$ in the sense that, given a map $\phi: Y_\bullet \rightarrow X_\bullet$ of simplicial sets with polynomial cohomology and sequences $\vec{a}$ in $C^*(X_\bullet)$ and $\vec{b}$ in $C^*(Y_\bullet)$ representing generators of $H^*(X_\bullet)$ and $H^*(Y_\bullet)$, respectively, the left diagram of (3.14) commutes up to a DGC homotopy $H_\phi: \mathbf{B}H^*(X_\bullet) \rightarrow \mathbf{B}C^*(Y_\bullet)$ whose associated twisting cochain homotopy $\eta\epsilon + t_{C^*(Y_\bullet)}H_\phi$ sends $\mathbf{B}_{\geq 1}H^*(X_\bullet)$ into $\mathfrak{k}_{Y_\bullet}$.*

$$(3.14) \quad \begin{array}{ccc} \mathbf{B}H^*(X_\bullet) & \xrightarrow{\mathbf{B}H^*\phi} & \mathbf{B}H^*(Y_\bullet) & \mathbf{B}(k[\vec{x}] \otimes k[\vec{x}]) & \xrightarrow{\mathbf{B}\mu} & \mathbf{B}(k[\vec{x}]) \\ \lambda_{\vec{a}} \downarrow & & \downarrow \lambda_{\vec{b}} & \lambda_{\vec{a}} \otimes \lambda_{\vec{a}} \downarrow & & \downarrow \lambda_{\vec{a}} \\ \mathbf{B}C^*(X_\bullet) & \xrightarrow[\mathbf{B}C^*\phi]{} & \mathbf{B}C^*(Y_\bullet) & \mathbf{B}(C^*(X_\bullet) \otimes C^*(X_\bullet)) & \xrightarrow[\Phi]{} & \mathbf{B}C^*(X_\bullet) \end{array}$$

Moreover $\lambda_{\vec{a}}$ is multiplicative modulo $\mathfrak{k}$ in the sense that $t_{C^*(X_\bullet)}\lambda_{\vec{a}}$ takes $\mathbf{B}_{\geq 2}(k[\vec{x}])$ into $\mathfrak{k}_{X_\bullet}$ and the right diagram of (3.14) commutes up to a DGC homotopy $H_\mu: \mathbf{B}(k[\vec{x}]^{\otimes 2}) \rightarrow \mathbf{B}C^*(X_\bullet)$ whose associated twisting cochain homotopy $\eta\epsilon + t_{C^*(X_\bullet)}H_\mu$ sends $\mathbf{B}_{\geq 1}(k[\vec{x}]^{\otimes 2})$ into $\mathfrak{k}_{X_\bullet}$.

We remark that the mere existence of such homotopies, without the bound in terms of $\mathfrak{k}$, lies at the heart of Munkholm's collapse proof [26]. To make precise our claim that there is a map $C^*(BT) \rightarrow H^*(BT)$ in the other direction annihilating $\mathfrak{k}$, we recall the classical construction of the simplicial classifying space:

Proposition 3.15 (see e.g. May [20, Section 21]) *Let $G_\bullet$ be a simplicial group. Then there exists a contractible simplicial $G_\bullet$ -space $WG_\bullet$ functorial in $G_\bullet$. That is, if $\phi: G_\bullet \rightarrow H_\bullet$ is a homomorphism of simplicial groups, then $W\phi: WG_\bullet \rightarrow WH_\bullet$ is ϕ -equivariant in the sense that $(W\phi)(x \cdot g) = (W\phi)(x) \cdot \phi(g)$ for $x \in (WG_\bullet)_n$ and $g \in G_n$. The projection $WG_\bullet \rightarrow \overline{W}G_\bullet := WG_\bullet / G_\bullet$ is a principal $G_\bullet$ -bundle, and the base $\overline{W}G_\bullet$ is the classifying simplicial set for simplicial principal $G_\bullet$ -bundles.*

The promised map is then provided by the following result:

Theorem 3.16 (Franz [11, Theorem 9.6 and Proposition 9.7]) *Let $T_\bullet$ be a simplicial abelian group pointed at $1 \in T_0$ and such that the cohomology $H^*(T_\bullet; \mathbb{Z})$ of the normalized cochain complex $C^*(T_\bullet)$ is an exterior algebra on finitely many degree-1 generators. Then there exists a quasi-isomorphism $f = f_T: C^*(\overline{W}T_\bullet) \rightarrow H^*(\overline{W}T_\bullet)$ of extended HGAs, called the **formality map**, which annihilates all extended HGA operations $F_{p,q}$ except for $F_{1,1} = -\smile_2$. If 2 is a unit of k , the formality map can be chosen so as to annihilate all $\smile_2$ -products of cocycles and such that its kernel contains the ideal $\mathfrak{k}_{\overline{W}T_\bullet}$ of Definition 3.11.*

4 Simplicial substitution

In this section we introduce a simplicial form of the homotopy pullback whose cohomology we will ultimately compute.

Discussion 4.1 We use the functorial Milgram model [25; 32; 28] of the universal principal bundle $EG \rightarrow BG$ associated to a Lie group G . It is well known that $K \backslash G/H$ is the homotopy pullback of the diagram $BK \rightarrow BG \leftarrow BH$ when $K \times H$ acts freely on G [29, Section 2]. More generally, writing $i: K \rightarrow G$ and $j: H \rightarrow G$ for the inclusions, the homotopy pullback can be identified as $EK \otimes_K G/H$,⁸ which is the pullback of the diagram $BK \xrightarrow{Bi} BG \leftarrow EG/H$, the homeomorphism being given by $e \otimes gH \mapsto (eK, (ei)(e)gH)$.

It is even better known that $Ej: EH \rightarrow EG$ is H -equivariant and induces a homotopy equivalence $BH \rightarrow EG/H$. The resulting map of triples induces a map of bar constructions

$$\mathbf{B}(C^*(BK), C^*(BG), C^*(EG/H)) \rightarrow \mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$$

which the algebraic Eilenberg–Moore spectral sequence shows is a quasi-isomorphism by Proposition 1.19. Moreover, by Theorem A.27, this map is multiplicative with respect to the product of Theorem A.1, and hence induces a ring quasi-isomorphism.

If we write $\overline{G}$ for the singular complex $\text{Sing } G$ of a connected Lie group, made a simplicial group by equipping each level $(\text{Sing } G)_n$ with the valuetype multiplication of maps $\Delta^n \rightarrow G$, then the counit $|\overline{G}| \rightarrow G$ of the standard adjunction $|-| \dashv \text{Sing}$ between the geometric realization and singular complex functors is a homomorphism and a homotopy equivalence, and is natural in continuous homomorphisms [20, Theorem 16.6(ii)]. For any simplicial group $G_\bullet$, there is a natural $|G_\bullet|$ -equivariant homeomorphism $|WG_\bullet| \xrightarrow{\cong} E|G_\bullet|$ descending to a natural homeomorphism $|\overline{W}G_\bullet| \xrightarrow{\cong} B|G_\bullet|$ [4], so for $G_\bullet = \overline{G}$ we get a composite weak homotopy equivalence $|\overline{W}\overline{G}| \rightarrow E|\overline{G}| \rightarrow EG$ equivariant with respect to the

⁸ This is the quotient of $EK \times G/H$ under the equivalence relation $(ek, gH) \sim (e, kgH)$ for $k \in K$, more commonly denoted by $EK \times_K G/H$, the same notation as a pullback.

homomorphism $|\bar{G}| \rightarrow G$. Long exact homotopy sequences then show the map $\bar{W}\bar{G} = |W\bar{G}|/|\bar{G}| \rightarrow (EG)/G = BG$ is a natural weak homotopy equivalence. By naturality, these maps yield a multiplicative quasi-isomorphism $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \rightarrow \mathbf{B}(C^*|\bar{W}\bar{K}|, C^*|\bar{W}\bar{G}|, C^*|\bar{W}\bar{H}|)$.

Finally, the unit $X_\bullet \mapsto \text{Sing } |X_\bullet|$ of the standard adjunction is a natural weak equivalence of simplicial sets whenever $X_\bullet$ is a Kan complex [20, Theorem 16.6(i)], which we know $\bar{W}\bar{K}$, $\bar{W}\bar{G}$, and $\bar{W}\bar{H}$ are [20, Lemma 21.3], so we have a multiplicative quasi-isomorphism to a more economical bar construction, $\mathbf{B}(C^*|\bar{W}\bar{K}|, C^*|\bar{W}\bar{G}|, C^*|\bar{W}\bar{H}|) \rightarrow \mathbf{B}(C^*(\bar{W}\bar{K}), C^*(\bar{W}\bar{G}), C^*(\bar{W}\bar{H}))$, although our proof does not strictly speaking require this simplification.

All told, we have the following:

Proposition 4.2 *Let G be a compact connected Lie group and H and K closed subgroups, and suppose the coefficient ring k is a principal ideal domain. Then there is an isomorphism of graded algebras*

$$H_K^*(G/H) \cong H^*\mathbf{B}(C^*(\bar{W}\bar{K}), C^*(\bar{W}\bar{G}), C^*(\bar{W}\bar{H})),$$

natural with respect to the diagram $K \hookrightarrow G \hookleftarrow H$.

Proof It remains only to check naturality, but this follows from the functoriality of W , Sing , $|-|$, and C^* and the naturality of the unit, the counit, and the Eilenberg–Moore isomorphism. $\square$

Remark 4.3 May and Neumann observed [21] that the general result [15, Theorem 4.3] behind the Gugenheim–May computation of the cohomology groups of a homogeneous space G/H applies equally in many cases to a *generalized homogenous space*, meaning the homotopy fiber F of a map $B_H \rightarrow B_G$ of path-connected spaces. Here F is to be thought of as G/H for $G = \Omega B_G$ and $H = \Omega B_H$, and we assume a trivial $\pi_1(B_G)$ -action on $H^*(F)$ and that $H^*(B_G)$ is a polynomial algebra over the principal ideal domain k , finitely generated in each degree. The key additional assumption is that there is a map $BT \rightarrow B_H$, where BT is a $K(\mathbb{Z}^n, 2)$, making $H^*(B_H)$ a free module of finite rank over $H^*(BT)$; then $T = \Omega BT$ is called a *maximal torus* of H .

Our result similarly generalizes. Given three path-connected spaces B_G , B_K , and B_H such that K and H admit maximal tori T_K and T_H and $H^*(B_G)$, $H^*(B_K)$, and $H^*(B_H)$ are polynomial algebras of finite type, then replacing $\text{Sing } BT_K$ and $\text{Sing } BT_H$ by $\bar{W}\bar{T}_K$ and $\bar{W}\bar{T}_H$, respectively, the same proof as in Section 5 shows that the cohomology of the homotopy pullback Y of the span $B_K \rightarrow B_G \leftarrow B_H$ is given as a graded group by $\text{Tor}_{H^*(B_G)}(H^*(B_K), H^*(B_H))$, and as a ring if 2 is a unit of k .

We are not aware of great interest in such spaces Y in general, but some special cases have been studied. For instance, if B_K and B_H are points, we recover that $H^*(\Omega B_G)$ is an exterior algebra, and for the span $B_K \rightarrow B_K \times B_K \leftarrow B_K$, with both maps the diagonal, we find the cohomology ring of the free loop space LB_K is the tensor product of $H^*(B_K)$ and $H^*(K)$. This is an instance of a more general result

of Saneblidze [27], derived using Hochschild homology and a model of the total space of the fibration $\Omega Y \rightarrow LY \rightarrow Y$ equipped with a cochain-level product related to the product described in Theorem A.5 and the HGA structure on $C^*(Y)$. His result, however, requires of $H^*(Y)$ only that the cup-1 squares of a set of polynomial generators be zero, without any maximal torus assumption, and of k only that it be a commutative ring.

5 The quasi-isomorphisms

We have finally assembled the necessary ingredients to prove Theorem 0.1.

Notation 5.1 In the calculation that follows, the base ring k is now a principal ideal domain, still usually suppressed in the notation. We will not require that 2 be a unit initially. Let G be a connected Lie group and H and K closed connected subgroups such that BG , BK , and BH have polynomial cohomology over k (equivalently, such that the torsion primes of G , H , and K are units). We will work with normalized cochains on the simplicial models $\overline{W}\overline{G}$ of Section 4, but in the notation for cohomology identify $H^*(BG)$ with $H^*(\overline{W}\overline{G})$ and so on, suppressing the natural isomorphisms induced by the simplicial weak equivalences $\overline{W}\overline{G} \rightarrow \text{Sing}|\overline{W}\overline{G}| \rightarrow \text{Sing}BG$.

From Proposition 4.2 we have a natural isomorphism

$$(5.2) \quad H_K^*(G/H) \cong \text{Tor}_{C^*(\overline{W}\overline{G})}(C^*(\overline{W}\overline{K}), C^*(\overline{W}\overline{H}))$$

of graded algebras, and our goal is to use the maps λ of (3.6) and f of Theorem 3.16 to induce a CGA isomorphism

$$(5.3) \quad H_K^*(G/H) \cong \text{Tor}_{H^*(BG)}(H^*(BK), H^*(BH)),$$

natural in inclusion diagrams $K \hookrightarrow G \leftarrow H$. On the level of graded modules, this isomorphism particularly means the Eilenberg–Moore spectral sequence of the homotopy pullback square of $BK \rightarrow BG \leftarrow BH$ collapses, a theorem of Munkholm [26] we will reprove as Proposition 5.7. To improve this result to a ring isomorphism, as both Tors arise as the cohomology of a two-sided bar construction, we will connect the argument DGAs of the two via maps preserving enough structure to guarantee our novel product on the two-sided bar construction is preserved up to homotopy. The structure of this section closely follows that of Franz [11, Section 12] and specializes to it in the case $K = 1$.

Discussion 5.4 We begin by constructing an additive cochain map between the two bar constructions. Selecting arbitrarily and fixing a list $\vec{a}$ of cocycles representing irredundant generators of $H^*(BG)$, and similarly for $H^*(BK)$ and $H^*(BH)$, we may use (3.6) to construct A_∞ -quasi-isomorphisms $\lambda_G: \mathbf{B}H^*(BG) \rightarrow \mathbf{B}C^*(\overline{W}\overline{G})$, λ_K , and λ_H . Recall that these maps are selected so that if x_j denotes the cohomology class

of $a_j \in C^*(\overline{W}\overline{G})$ and $\text{rk } G = n$, then the extension $(\lambda_G)_{(1)}^+ = \eta\epsilon + (\lambda_G)_{(1)}$ of the 1-component is the additive quasi-isomorphism

$$k[\vec{x}] \cong H^*(BG) \rightarrow C^*(\overline{W}\overline{G}), \quad x_1^{p_1} \cdots x_n^{p_n} \mapsto a_1^{p_1} \cdots a_n^{p_n},$$

and similarly for λ_K and λ_H . We write the canonical twisting cochains $\mathbf{B}H^*(BG) \rightarrow H^*(BG)$ and $\mathbf{B}C^*(\overline{W}\overline{G}) \rightarrow C^*(\overline{W}\overline{G})$ respectively as t_H and t_C , and for any homomorphism $L' \rightarrow L$ of topological groups will write $\rho = \rho_L^L$, for the functorially induced DGA maps $C^*(\overline{W}\overline{L}) \rightarrow C^*(\overline{W}\overline{L}')$ and $H^*(BL) \rightarrow H^*(BL')$. Then the candidate quasi-isomorphism

$$(5.5) \quad \Theta: \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \rightarrow \mathbf{B}(C^*(\overline{W}\overline{K}), C^*(\overline{W}\overline{G}), C^*(\overline{W}\overline{H}))$$

is defined as the composition of the cochain maps

$$\begin{array}{ccc} H^*(BK) \otimes_{t_H \mathbf{B} \rho} \mathbf{B}H^*(BG) \otimes_{t_H \mathbf{B} \rho} H^*(BH) & \xrightarrow{C^*(\overline{W}\overline{K}) \otimes_{t_H \mathbf{B} \rho \lambda_G} \mathbf{B}H^*(BG) \otimes_{t_H \mathbf{B} \rho \lambda_G} C^*(\overline{W}\overline{H})} & C^*(\overline{W}\overline{K}) \otimes_{t_H \mathbf{B} \rho \lambda_G} \mathbf{B}H^*(BG) \otimes_{t_H \mathbf{B} \rho \lambda_G} C^*(\overline{W}\overline{H}) \\ \downarrow \mathbf{B}(\lambda_K, \text{id}, \lambda_H) & \nearrow \theta & \downarrow \mathbf{B}(\text{id}, \lambda_G, \text{id}) \\ C^*(\overline{W}\overline{K}) \otimes_{t_H \lambda_K \mathbf{B} \rho} \mathbf{B}H^*(BG) \otimes_{t_H \lambda_H \mathbf{B} \rho} C^*(\overline{W}\overline{H}) & & C^*(\overline{W}\overline{K}) \otimes_{t_C \mathbf{B} \rho} \mathbf{B}C^*(\overline{W}\overline{G}) \otimes_{t_C \mathbf{B} \rho} C^*(\overline{W}\overline{H}) \end{array}$$

given from beginning to end by Lemmas 1.16, 1.12, and 1.11, where $\theta = (\delta_{hK}^L \otimes \text{id}) \circ (\text{id} \otimes \delta_{hH}^R)$, the twisting cochain homotopies $h^K: \mathbf{B}H^*(BG) \rightarrow C^*(\overline{W}\overline{K})$ and $h^H: \mathbf{B}H^*(BG) \rightarrow C^*(\overline{W}\overline{H})$ coming from Theorem 3.13.

Lemma 5.6 *The map Θ defined in (5.5) satisfies*

$$\Theta \equiv (\lambda_K)_{(1)}^+ \otimes \lambda_G \otimes (\lambda_H)_{(1)}^+ \pmod{(\mathfrak{k}_{\overline{W}\overline{K}}, \mathfrak{k}_{\overline{W}\overline{H}})}.$$

Proof As $t_{C^*(\overline{W}\overline{K})} \lambda_K$ and $t_{C^*(\overline{W}\overline{H})} \lambda_H$ respectively take $\mathbf{B}_{\geq 2} H^*(BK)$ and $\mathbf{B}_{\geq 2} H^*(BH)$ into $\mathfrak{k}_{\overline{W}\overline{K}}$ and $\mathfrak{k}_{\overline{W}\overline{H}}$ by Theorem 3.13, the first factor $\mathbf{B}(\lambda_K, \text{id}, \lambda_H)$ is congruent to $(\lambda_K)_{(1)}^+ \otimes \text{id} \otimes (\lambda_H)_{(1)}^+$ modulo $(\mathfrak{k}_{\overline{W}\overline{K}}, \mathfrak{k}_{\overline{W}\overline{H}})$ by Proposition 1.16. Next, by Theorem 3.13 again, the twisting cochain homotopies h^K and h^H respectively send $\mathbf{B}_{\geq 1} H^*(BG)$ into $\mathfrak{k}_{\overline{W}\overline{K}}$ and $\mathfrak{k}_{\overline{W}\overline{H}}$, so $(\delta_{hK}^L \otimes \text{id})(\text{id} \otimes \delta_{hH}^R)$ is congruent to the identity modulo $(\mathfrak{k}_{\overline{W}\overline{K}}, \mathfrak{k}_{\overline{W}\overline{H}})$ by Lemma 1.12. Finally, $\mathbf{B}(\text{id}, \lambda_G, \text{id})$ is $\text{id} \otimes \lambda_G \otimes \text{id}$. $\square$

Proposition 5.7 *We retain the notation G, K, H from Notation 5.1 and Θ from (5.5).*

(i) *The induced map*

$$H^*(\Theta): \text{Tor}_{H^*(BG)}^*(H^*(BK), H^*(BH)) \rightarrow \text{Tor}_{C^*(\overline{W}\overline{G})}^*(C^*(\overline{W}\overline{K}), C^*(\overline{W}\overline{H}))$$

of graded k -modules is an isomorphism.

(ii) *The Eilenberg–Moore spectral sequence for the homotopy pullback of $BK \rightarrow BG \leftarrow BH$ collapses at the E_2 page.*

Proof (i) Because $H^*(BG)$ and $H^*(BK)$ are flat over k , under the length filtration of two-sided bar constructions discussed in Definition 1.13, the E_2 page of the target under the map of associated filtration

spectral sequences induced by Θ is again $\mathrm{Tor}_{H^*(BG)}^*(H^*(BK), H^*(BH))$ as noted in Proposition 1.19. Since λ_G , $(\lambda_K)_{(1)}^+$, and $(\lambda_H)_{(1)}^+$ are each quasi-isomorphisms, by Lemma 5.6, the map of E_2 pages is the identity map. These are half-plane spectral sequences with exiting differentials and in the associated exact couples (E_1, A_1) one has $A_1^p = 0$ for $p > 0$, and particularly $\varprojlim_p A_1^p = 0$, so they are strongly convergent [5, Theorem 6.1(a)]; hence $H^*(\Theta)$ is a graded k -linear isomorphism [5, Theorem 5.3].

(ii) In the map of spectral sequences, the codomain is the Eilenberg–Moore spectral sequence of the homotopy pullback of the diagram $BK \leftarrow BG \rightarrow BH$. We have seen the spectral sequence map is a k -linear graded isomorphism from E_2 on, so it is enough to show the domain spectral sequence collapses. But E_2 of this domain sequence is already isomorphic to the sequence's target $H^*\mathbf{B}(H^*(BK), H^*(BG), H^*(BH))$ as a graded k -module. $\square$

To show $H^*(\Theta)$ is multiplicative and natural will involve the formality map of Section 4, and particularly *from here on out, we will need 2 to be a unit in k* . We will soon specialize to maximal tori, but for now let arbitrary compact tori T_K and T_H and simplicial group homomorphisms $\alpha_K: \bar{T}_K \rightarrow \bar{K}$ and $\alpha_H: \bar{T}_H \rightarrow \bar{H}$ be given, and choose formality maps $f_{T_K}: C^*(\bar{W}\bar{T}_K) \rightarrow H^*(BT_K)$ and $f_{T_H}: C^*(\bar{W}\bar{T}_H) \rightarrow H^*(BT_H)$ as guaranteed by Theorem 3.16, recalling that these maps respectively annihilate the ideals $\mathfrak{k}_{\bar{W}\bar{T}_K}$ and $\mathfrak{k}_{\bar{W}\bar{T}_H}$ defined in Definition 3.11. As f and ρ are HGA and hence DGA maps, Lemma 1.10 provides a cochain map of two-sided bar constructions

$$(5.8) \quad \Psi_{\alpha_K, \alpha_H}: C^*(\bar{W}\bar{K}) \otimes_{\rho_{t_C}} \mathbf{BC}^*(\bar{W}\bar{G}) \otimes_{\rho_{t_C}} C^*(\bar{W}\bar{H}) \xrightarrow{f\rho \otimes \mathrm{id} \otimes f\rho} H^*(BT_K) \otimes_{f\rho_{t_C}} \mathbf{BC}^*(\bar{W}\bar{G}) \otimes_{f\rho_{t_C}} H^*(BT_H).$$

Lemma 5.9 *Let G , K , and H be as in Notation 5.1, Θ as in (5.5), and Ψ as in (5.8).*

(i) *The cochain map Ψ is multiplicative with respect to the product $\tilde{\mu}$ of Theorem A.1.*

(ii) *The composite $\Psi\Theta$ is equal to*

$$\begin{aligned} \rho_{T_K}^K \otimes \lambda_G \otimes \rho_{T_H}^H: H^*(BK) \otimes_{\rho_{t_H}} \mathbf{BH}^*(BG) \otimes_{\rho_{t_H}} H^*(BH) \\ \rightarrow H^*(BT_K) \otimes_{f\rho_{t_C}} \mathbf{BC}^*(\bar{W}\bar{G}) \otimes_{f\rho_{t_C}} H^*(BT_H). \end{aligned}$$

(iii) *If α_K and α_H are inclusions of maximal tori, the induced map $H^*(\Psi)$ in cohomology is injective.*

Proof (i) Multiplicativity of Ψ follows from naturality of $\tilde{\mu}$ since f and ρ are HGA maps.

(ii) The restriction $\rho_{T_K}^K$ sends $\mathfrak{k}_{\bar{W}\bar{K}}$ to $\mathfrak{k}_{\bar{W}\bar{T}_K}$ by Lemma 3.12, while by Theorem 3.16 f_K annihilates $\mathfrak{k}_{\bar{W}\bar{T}_K}$, and similarly $f_H \rho_{T_H}^H$ annihilates $\mathfrak{k}_{\bar{W}\bar{H}}$. As Θ is congruent to $(\lambda_K)_{(1)}^+ \otimes \lambda_G \otimes (\lambda_H)_{(1)}^+$ modulo $(\mathfrak{k}_{\bar{W}\bar{K}}, \mathfrak{k}_{\bar{W}\bar{H}})$ by Lemma 5.6, we then have

$$\Psi\Theta = (f\rho \otimes \mathrm{id} \otimes f\rho)\Theta = f\rho(\lambda_K)_{(1)}^+ \otimes \lambda_G \otimes f\rho(\lambda_H)_{(1)}^+.$$

But taking cohomology of the cochain maps

$$H^*(BK) \xrightarrow{(\lambda_K)_{(1)}^+} C^*(\bar{W}\bar{K}) \xrightarrow{\rho} C^*(\bar{W}\bar{T}_K) \xrightarrow{f} H^*(BT_K)$$

yields

$$H^*(BK) \xrightarrow{\text{id}} H^*(BK) \xrightarrow{\rho} C^*(BT_K) \xrightarrow{\text{id}} H^*(BT_K),$$

and the differentials on $H^*(BK)$ and $H^*(BT_K)$ are zero, implying the cochain map $f_K \circ \rho_{T_K}^K \circ (\lambda_K)_{(1)}^+$ is itself $\rho_{T_K}^K$, and similarly for $\rho_{T_H}^H$.

(iii) We factor Ψ as $(f \otimes \text{id} \otimes f)(\rho \otimes \text{id} \otimes \rho)$ and show both factors induce injections in cohomology. As $f \otimes \text{id} \otimes f: \mathbf{B}(C^*(\overline{WT}_K), C^*(\overline{WG}), C^*(\overline{WT}_H)) \rightarrow \mathbf{B}(H^*(BT_K), C^*(\overline{WG}), H^*(BT_H))$ induces the identity map between the E_2 pages of the associated filtration spectral sequences, it is a quasi-isomorphism. As for $\rho \otimes \text{id} \otimes \rho$, consider the map of Serre spectral sequences induced by the map of (vertical) fibrations

$$\begin{array}{ccc} T_K G_{T_H} & \longrightarrow & BT_K \times BT_H \\ \downarrow & & \downarrow \\ K G_H & \longrightarrow & BK \times BH \end{array}$$

The homotopy fiber of both fibrations is $K/T_K \times H/T_H$. Thus the E_2 page of the right spectral sequence is concentrated in even degree, implying the sequence collapses, and the map of spectral sequences implies the left spectral sequence collapses as well; it follows that the left fibration induces an injection in cohomology. But by the naturality clause of Proposition 4.2 this injection is $H^*(\rho \otimes \text{id} \otimes \rho)$. $\square$

Theorem 5.10 *The isomorphism $H^*(\Theta)$ of Proposition 5.7 is multiplicative.*

Proof Since we know from Lemma 5.9 that $H^*(\Psi)$ is injective and multiplicative it will be enough to show the map $\Psi\Theta = \rho \otimes \lambda_G \otimes \rho$ is multiplicative up to homotopy. As $H^*(BK)$ and $H^*(BH)$ are CDGAs, the HGA operations E_k are zero for $k \geq 0$ by Convention 2.9, so the product $\tilde{\mu}$ of (A.6) reduces to $a'[a_\bullet]a'' \cdot b'[b_\bullet]b'' = \pm a'b' \otimes [a_\bullet] * [b_\bullet] \otimes a''b''$, which is just the tensor permutation Π of (A.4) rearranging the factors in the correct order followed by the coordinatewise product $\mu^{\otimes 3}$. Recalling from the proof of Theorem A.1 that Π gives a natural cochain isomorphism from the tensor-square of the two-sided bar construction to the two-sided twisted tensor product with respect to the twisting cochains t' and t'' given by $\rho^{\otimes 2}(t \otimes \eta\epsilon + \eta\epsilon \otimes t)$, we can transfer the desired multiplicativity of $\Psi\Theta$ up to homotopy to a question about maps

$$H^*(BK)^{\otimes 2} \otimes_{t'} (\mathbf{B}H^*(BG))^{\otimes 2} \otimes_{t''} H^*(BH)^{\otimes 2} \rightarrow H^*(BT_K) \otimes_{f\rho_{t_K}} \mathbf{B}C^*(\overline{WG}) \otimes_{f\rho_{t_H}} H^*(BT_H).$$

We want to find a homotopy between the cochain maps

$$(\rho \otimes \lambda_G \otimes \rho)(\mu \otimes \mu_{\mathbf{B}H^*(BG)} \otimes \mu) = \rho\mu \otimes \lambda_G \mu_{\mathbf{B}H^*(BG)} \otimes \rho\mu$$

and

$$(\mu \otimes \mu_{\mathbf{B}C^*(\overline{WG})} \otimes \mu)(\rho^{\otimes 2} \otimes \lambda_G^{\otimes 2} \otimes \rho^{\otimes 2}) = \mu\rho^{\otimes 2} \otimes \mu_{\mathbf{B}C^*(\overline{WG})} \lambda_G^{\otimes 2} \otimes \mu\rho^{\otimes 2}.$$

On tensor factors we have $\rho\mu = \mu\rho^{\otimes 2}$ because ρ is a ring map, while Theorem 3.13 provides a coalgebra homotopy $H_\mu: \lambda_G \mathbf{B}\mu_{H^*(BG)} \simeq \Phi(\lambda_G \otimes \lambda_G)$ such that $t_C H_\mu$ takes $\mathbf{B}_{\geq 1}(H^*(BG)^{\otimes 2})$ into $\mathfrak{k}_{\overline{WG}}$, so that

the twisting cochains $f\rho t_C$ annihilate the image of H_μ . As the shuffle map ∇ is a natural transformation $\otimes \rightarrow \underline{\otimes}$ of bifunctors, we may append the square of Proposition 3.3:

$$\begin{array}{ccccc} (\mathbf{B}H^*(BG))^{\otimes 2} & \xrightarrow{\nabla} & \mathbf{B}(H^*(BG)^{\otimes 2}) & \xrightarrow{\mathbf{B}\mu} & \mathbf{B}(H^*(BG)) \\ \lambda_G \otimes \lambda_G \downarrow & & \lambda_G \otimes \lambda_G \downarrow & & \downarrow \lambda_G \\ (\mathbf{B}C^*(\overline{W}\overline{G}))^{\otimes 2} & \xrightarrow{\nabla} & \mathbf{B}(C^*(\overline{W}\overline{G})^{\otimes 2}) & \xrightarrow{\Phi} & \mathbf{B}C^*(\overline{W}\overline{G}) \end{array}$$

Recalling $\Phi \circ \nabla = \mu_{\mathbf{B}C^*(\overline{W}\overline{G})}$ (Theorem 3.8) and $\mathbf{B}\mu \circ \nabla = \mu_{\mathbf{B}H^*(BG)}$ (Example 1.5), we then see $H_\mu \nabla$ is a DGC homotopy from $\lambda_G \mu_{\mathbf{B}H^*(BG)}$ to $\mu_{\mathbf{B}C^*(\overline{W}\overline{G})} \lambda_G^{\otimes 2}$. As $t_C H_\mu \nabla$ takes the coaugmentation coideal of $(\mathbf{B}H^*(BG))^{\otimes 2}$ into $\mathfrak{k}_{\overline{W}\overline{G}}$, we may apply Lemma 1.10 to see that $f\rho \otimes H_\mu \nabla \otimes f\rho$ is a homotopy doing what we wanted. $\square$

This establishes the ring isomorphism of Theorem 0.1. It remains to show this isomorphism is natural in inclusion diagrams.

Theorem 5.11 *Given a diagram*

$$(5.12) \quad \begin{array}{ccccc} K_1 & \hookrightarrow & G_1 & \hookleftarrow & H_1 \\ \downarrow & & \downarrow & & \downarrow \\ K_0 & \hookrightarrow & G_0 & \hookleftarrow & H_0 \end{array}$$

of continuous group homomorphisms such that the cohomology of each of the groups' classifying spaces is a polynomial ring over k and given cocycle representatives for an irredundant set of polynomial generators of each cohomology ring, so as to define quasi-isomorphisms Θ_1 and Θ_0 as in (5.5), the following induced square commutes:

$$\begin{array}{ccc} \mathrm{Tor}_{H^*(BG_0)}^*(H^*(BK_0), H^*(BH_0)) & \xrightarrow{\mathrm{Tor}_\rho^*(\rho, \rho)} & \mathrm{Tor}_{H^*(BG)}^*(H^*(BK_1), H^*(BH_1)) \\ \downarrow H^*(\Theta_0) \wr & & \wr \downarrow H^*(\Theta_1) \\ H_{K_0}^*(G_0/H_0) & \xrightarrow{\quad\quad\quad} & H_{K_1}^*(G_1/H_1) \end{array}$$

Proof Let $i_{K_1}: T_{K_1} \hookrightarrow K_1$ and $i_{H_1}: T_{H_1} \hookrightarrow H_1$ be inclusions of maximal tori. We expand the description of $\Psi\Theta$ in Lemma 5.9 to the diagram in Figure 1. Note carefully the asymmetry in the bottom square: on the left we have G_0 rather than G_1 , but the tori T_{K_1} and T_{H_1} are subtori of G_1 rather than G_0 . The composite $f\rho \otimes \mathrm{id} \otimes f\rho$ after Θ in the right column is the map $\Psi_{i_{K_1}, i_{H_1}}$ of (5.8) and the composite after Θ the first column is Ψ_{j_K, j_H} , where j_K and j_H are respectively the compositions $T_{K_1} \hookrightarrow K_1 \rightarrow K_0$ and $T_{H_1} \hookrightarrow H_1 \rightarrow H_0$.

Our goal is to show the top square commutes in cohomology, but as $H^*(\Psi_{i_{K_1}, i_{H_1}})$ is injective by Lemma 5.9(iii), it will suffice to postcompose $\Psi_{i_{K_1}, i_{H_1}}$ to the two paths around the square and show the resulting maps induce the same map in cohomology. One confirms tensor factor by tensor factor that the

$$\begin{array}{ccc}
H^*(BK_0) \otimes_{\rho_{t_H}} \mathbf{B}H^*(BG_0) \otimes_{\rho_{t_H}} H^*(BH_0) & \xrightarrow{\rho \otimes \mathbf{B}\rho \otimes \rho} & H^*(BK_1) \otimes_{\rho_{t_H}} \mathbf{B}H^*(BG_1) \otimes_{\rho_{t_H}} H^*(BH_1) \\
\downarrow \Theta_0 & & \downarrow \Theta_1 \\
C^*(\overline{W}\overline{K}_0) \otimes_{\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_0) \otimes_{\rho_{t_C}} C^*(\overline{W}\overline{H}_0) & \xrightarrow{\rho \otimes \mathbf{B}\rho \otimes \rho} & C^*(\overline{W}\overline{K}_1) \otimes_{\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_1) \otimes_{\rho_{t_C}} C^*(\overline{W}\overline{H}_1) \\
\downarrow \rho \otimes \text{id} \otimes \rho & & \downarrow \rho \otimes \text{id} \otimes \rho \\
C^*(\overline{W}\overline{T}_{K_1}) \otimes_{\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_0) \otimes_{\rho_{t_C}} C^*(\overline{W}\overline{T}_{H_1}) & \xrightarrow{\text{id} \otimes \mathbf{B}\rho \otimes \text{id}} & C^*(\overline{W}\overline{T}_{K_1}) \otimes_{\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_1) \otimes_{\rho_{t_C}} C^*(\overline{W}\overline{T}_{H_1}) \\
\downarrow f \otimes \text{id} \otimes f & & \downarrow f \otimes \text{id} \otimes f \\
H^*(BT_{K_1}) \otimes_{f\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_0) \otimes_{f\rho_{t_C}} H^*(BT_{H_1}) & \xrightarrow{\text{id} \otimes \mathbf{B}\rho \otimes \text{id}} & H^*(BT_{K_1}) \otimes_{f\rho_{t_C}} \mathbf{B}C^*(\overline{W}\overline{G}_1) \otimes_{f\rho_{t_C}} H^*(BT_{H_1})
\end{array}$$

Figure 1: A diagram of two-sided bar constructions.

bottom two squares commute, so it will be enough to find a cochain homotopy between the two paths along the outer rectangle. By Lemma 5.9(ii), the composite along the upper right is

$$(5.13) \quad (\rho_{T_{K_1}}^{K_1} \otimes \lambda_{G_1} \otimes \rho_{T_{H_1}}^{H_1})(\rho_{K_1}^{K_0} \otimes \mathbf{B}\rho_{G_1}^{G_0} \otimes \rho_{H_1}^{H_0}) = \rho_{T_{K_1}}^{K_0} \otimes \lambda_{G_1} \mathbf{B}\rho_{G_1}^{G_0} \otimes \rho_{T_{K_1}}^{K_0}$$

and the composite along the lower left is

$$(5.14) \quad (\text{id} \otimes \mathbf{B}\rho_{G_1}^{G_0} \otimes \text{id})(\rho_{T_{K_1}}^{K_0} \otimes \lambda_{G_0} \otimes \rho_{T_{H_1}}^{H_0}) = \rho_{T_{K_1}}^{K_0} \otimes \mathbf{B}\rho_{G_1}^{G_0} \lambda_{G_0} \otimes \rho_{T_{H_1}}^{H_0}.$$

Now Theorem 3.13 provides a DGC homotopy H_ρ between $\rho_{G_1}^{G_0} \lambda_{G_0}$ and $\lambda_{G_1} \rho_{G_1}^{G_0}$ such that $t_C H_\rho$ takes $\mathbf{B}_{\geq 1} H^*(BG_0)$ into $\mathfrak{k}_{\overline{W}\overline{G}_1}$, so that the two twisting cochains $f\rho_{t_C}$ defining the common codomain of (5.13) and (5.14) annihilate the image of H_ρ . Applying Lemma 1.10, we see $f\rho \otimes H_\rho \otimes f\rho$ is the sought-after homotopy between (5.13) and (5.14). $\square$

Corollary 5.15 *The isomorphism $H^*(\Theta)$ of Theorem 0.1 does not depend on the choice of representatives defining Θ in (5.5).*

Proof One can take the vertical maps to each be the identity in (5.12), defining Θ_0 in terms of one set of representatives and Θ_1 in terms of another. $\square$

6 Example

In this section we give an example where Theorem 0.1 provides information not previously available from the standard techniques. This example is in a sense minimal in that in similar examples where the differentials form a regular sequence, the fact that an exterior algebra is a free CGA allows one to resolve the multiplicative extension problem, and in examples with $\text{Tor}^{\leq -2} = 0$ one can resolve the extension problem by noting that elements of Tor^{-1} annihilate one other and lie in odd degree, whereas Tor^0 is concentrated in even degree.

Example 6.1 Let S be the image of the embedding $z \mapsto \text{diag}(z^{-4}, z, z, z)$ of $U(1)$ in $SU(4)$. Theorem 0.1 computes the ring $H_S^*(SU(4)/S; \mathbb{Z}[\frac{1}{2}])$ as

$$\text{Tor}_{H^*(BSU(4); \mathbb{Z}[\frac{1}{2}])}^*(H^*(BS; \mathbb{Z}[\frac{1}{2}]), H^*(BS; \mathbb{Z}[\frac{1}{2}]))$$

(previously, the Sullivan model noted by Kapovich showed the same with $\mathbb{Z}[\frac{1}{2}]$ replaced by $\mathbb{Q}$). One finds explicitly that the induced map $\mathbb{Z}[\frac{1}{2}][c_2, c_3, c_4] \xrightarrow{\sim} H^*BSU(4) \rightarrow H^*BS \xrightarrow{\sim} \mathbb{Z}[\frac{1}{2}][t_2]$ takes the universal Chern classes c_2, c_3 , and c_4 respectively to $-6t^2, -8t^3$, and $-3t^4$, so one can compute the Tor using the associated Koszul complex, as the cohomology of the DGA $\mathbb{Z}[\frac{1}{2}][s_2, t_2] \otimes \wedge[v_3, w_5, x_7]$ with differential annihilating s and t and with

$$dv_3 = 6(s^2 - t^2), \quad dw_5 = 8(s^3 - t^3), \quad \text{and} \quad dx_7 = 3(s^4 - t^4).$$

Setting $p_{2j} = (s^j - t^j)/(s - t)$ for $2 \leq j \leq 4$ one finds that $\text{Tor}^0 \cong \mathbb{Z}[\frac{1}{2}][s, t]/(s - t)(6p_2, 8p_4, 3p_6)$; that Tor^{-1} is the quotient of the $\mathbb{Z}[\frac{1}{2}][s, t]$ -module generated by the cocycles

$$a = 3p_2w_5 - 4p_4v_3 = \frac{1}{2(s-t)}d(v_3w_5), \quad b = 2p_2x_7 - p_6v_3 = \frac{1}{3(s-t)}d(v_3x_7),$$

$$c = 8p_4x_7 - 3p_6w_5 = \frac{1}{s-t}d(w_5x_7),$$

which satisfy $p_6a - 4p_4b + p_2c = 0$, by the submodule generated by $d(v_3w_5), d(v_3x_7)$, and $d(w_5x_7)$; and that Tor^{-2} is the quotient of the free $\mathbb{Z}[\frac{1}{2}][s, t]$ -module generated by the class of

$$e = 3p_6v_3w_5 - 8p_4v_3x_7 + 6p_2w_5x_7 = \frac{1}{(s-t)}d(v_3w_5x_7)$$

by the submodule generated by $d(v_3w_5x_7)$. All told, one finds that $H_S^*(SU(4)/S; \mathbb{Z}[\frac{1}{2}])$ is the quotient of $\mathbb{Z}[\frac{1}{2}][s_2, t_2, e_{14}] \otimes \wedge[a_7, b_9, c_{11}]$ by the ideal generated by

$$(s^3 + s^2t + st^2 + t)a - 4(s^2 + st + t^2)b + (s + t)c, \quad 2(s - t)a, \quad 3(s - t)b, \quad (s - t)c, \quad (s - t)e,$$

$$ab - (s + t)e, \quad ac - 4(s^2 + st + t^2)e, \quad bc - (s^3 + s^2t + st^2 + t^3)e, \quad ae, \quad be, \quad ce \quad \text{and} \quad e^2.$$

This computation resolves, for example, the multiplicative extension problem for products of the classes corresponding to a, b, c , and e on the E_∞ page of the Serre spectral sequence beginning with $E_2 = H^*(BS \times BS) \otimes H^*SU(4)$ and converging to $H_S^*(SU(4)/S; \mathbb{Z})$.

Appendix A product on the two-sided bar construction joint with Matthias Franz

Given that an HGA structure defines a DGA structure on the bar construction of a DGA, one might hope HGA homomorphisms $A' \leftarrow A \rightarrow A''$ similarly induce a DGA structure on the two-sided bar construction. We cannot assert this, but we can at least define a nonassociative product sufficient to prove the variant Eilenberg–Moore theorem (Theorem A.27) needed for the proof of Theorem 0.1.

Theorem A.1 Let $A' \xleftarrow{f'} A \xrightarrow{f''} A''$ be HGA homomorphisms. Then there exists a cochain map

$$\tilde{\mu}: \mathbf{B}(A', A, A'') \otimes \mathbf{B}(A', A, A'') \rightarrow \mathbf{B}(A', A, A''),$$

which we think of as a product, natural in the sense that given a commutative diagram

$$\begin{array}{ccccc} A' & \longleftarrow & A & \longrightarrow & A'' \\ g' \downarrow & & \downarrow g & & \downarrow g'' \\ B' & \longleftarrow & B & \longrightarrow & B'' \end{array}$$

of HGA homomorphisms, the induced map $\mathbf{B}(g', g, g''): \mathbf{B}(A', A, A'') \rightarrow \mathbf{B}(B', B, B'')$ of Notation 1.18 is multiplicative.

Remark A.2 It requires the extended HGA operations $F_{p,q}$ to define the natural homotopy h^c witnessing the homotopy-commutativity axiom for Franz's natural SHC-algebra structure map Φ_A described in Theorem 3.8, but to define Φ_A itself and show that $\Phi_A \circ \nabla = \mu_{\mathbf{B}A}$, which is all we require here, one needs only that A be an HGA.

Proof of Theorem A.1 By Theorem 3.8, there exists a DGC map $\Phi_A: \mathbf{B}(A \otimes A) \rightarrow \mathbf{B}A$ giving an SHC-algebra structure on A , and similarly for A' and A'' . We set

$$(A.3) \quad \tilde{\mu} := \underbrace{\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''}) \circ (\text{id}_{A' \otimes A'} \otimes \nabla \otimes \text{id}_{A'' \otimes A''})}_{\mathbf{B}(\Phi_{A'}, \mu_{\mathbf{B}A}, \Phi_{A''})} \circ \Pi,$$

where the $\mathbf{B}(-, -, -)$ maps are as defined in Proposition 1.16 and the initial map

$$(A.4) \quad \begin{aligned} \Pi: \mathbf{B}(A', A, A'') \otimes \mathbf{B}(A', A, A'') &\rightarrow (A' \otimes A') \otimes_{t'} (\mathbf{B}A \otimes \mathbf{B}A) \otimes_{t''} (A'' \otimes A''), \\ a'[a_\bullet]a'' \otimes b'[b_\bullet]b'' &\mapsto \pm a' \otimes b' \otimes [a_\bullet] \otimes [b_\bullet] \otimes a'' \otimes b'', \end{aligned}$$

is the tensor permutation (2354). Here the two-sided twisted tensor product is determined by the twisting cochains

$$\begin{aligned} t' &= (f')^{\otimes 2} t_{A \otimes 2} \nabla = (f')^{\otimes 2} (t_A \otimes \eta_{A \in \mathbf{B}A} + \eta_{A \in \mathbf{B}A} \otimes t_A): (\mathbf{B}A)^{\otimes 2} \rightarrow (A')^{\otimes 2}, \\ t'' &= (f'')^{\otimes 2} t_{A \otimes 2} \nabla = (f'')^{\otimes 2} (t_A \otimes \eta_{A \in \mathbf{B}A} + \eta_{A \in \mathbf{B}A} \otimes t_A): (\mathbf{B}A)^{\otimes 2} \rightarrow (A'')^{\otimes 2}, \end{aligned}$$

for ∇ the shuffle map of Example 1.5, and it is not hard to check Π is a cochain map. To see

$$\text{id} \otimes \nabla \otimes \text{id}: (A' \otimes A') \otimes_{t'} (\mathbf{B}A \otimes \mathbf{B}A) \otimes_{t''} (A'' \otimes A'') \rightarrow (A' \otimes A') \otimes_{t'} \mathbf{B}(A \otimes A) \otimes_{t''} (A'' \otimes A'')$$

is a cochain map, it is enough by Lemma 1.10 to observe $t' = (f' \otimes f') t_{A \otimes A} \nabla = t_{A' \otimes A'} \mathbf{B}(f' \otimes f') \nabla$ and similarly $t'' = t_{A'' \otimes A''} \mathbf{B}(f'' \otimes f'') \nabla$. To see

$$\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A'')}: \mathbf{B}(A' \otimes A', A \otimes A, A'' \otimes A'') \rightarrow \mathbf{B}(A', A, A'')$$

is a well-defined cochain map, by Proposition 1.16 it is enough to note the diagram

$$\begin{array}{ccccc}
 \mathbf{B}(A' \otimes A') & \xleftarrow{\mathbf{B}(f')^{\otimes 2}} & \mathbf{B}(A \otimes A) & \xrightarrow{\mathbf{B}(f'')^{\otimes 2}} & \mathbf{B}(A'' \otimes A'') \\
 \Phi_{A'} \downarrow & & \downarrow \Phi_A & & \downarrow \Phi_{A''} \\
 \mathbf{B}A' & \xleftarrow{\mathbf{B}f'} & \mathbf{B}A & \xrightarrow{\mathbf{B}f''} & \mathbf{B}A''
 \end{array}$$

commutes by the naturality of Φ in HGAs.

Naturality follows because Π is natural in sextuples of cochain complexes, ∇ is natural in pairs of cochain complexes, Φ is natural in extended HGA maps, and $\mathbf{B}(-, -, -)$ is functorial in triples of DGA maps. $\square$

To make real use of this product, we will require a more explicit formula:

Theorem A.5 *The cochain map $\tilde{\mu}$ of Theorem A.1 is given in terms of the notation set in Definition 2.1 and Notation 2.2 by*

$$\begin{aligned}
 \text{(A.6)} \quad a'[a_\bullet]a'' \otimes b'[b_\bullet]b'' \mapsto & \quad \pm a'b' \otimes [a_\bullet] * [b_{(2)}] \otimes \mathfrak{E}(a''[f''b_{(3)}])b'' \\
 & \pm a'\mathfrak{E}(f'a_1[b'|f'b_{(1)}]) \otimes [a_{(2)}] * [b_{(2)}] \otimes \mathfrak{E}(a''[f''b_{(3)}])b'',
 \end{aligned}$$

where the first term is the sum over all decompositions $[b_\bullet] = [b_{(2)}] \otimes [b_{(3)}]$ into two tensor factors, and the second is the double sum over decompositions $[a_\bullet] = [a_1] \otimes [a_{(2)}]$ where the first tensor factor is of length 1 and decompositions $[b_\bullet] = [b_{(1)}] \otimes [b_{(2)}] \otimes [b_{(3)}]$ into three tensor factors, and where we recall from Remark 2.3 our convention that $\mathfrak{E}(f'a_1[b'|f'b_{(1)}]) = 0$ when $b' = 1$. The signs are those imposed by the Koszul convention. Explicitly, the first sum is

$$\text{(A.7)} \quad \overbrace{(\mu_{A'} \otimes \mu_{\mathbf{B}A} \otimes \mu_{A''} (\mathfrak{E}(\text{id}_{A''} \otimes \mathbf{B}f'') \otimes \text{id}_{A''}) \tau_{[b_{(3)}]; a''})}^{T_0} \circ (\text{id}_{A' \otimes A'} \otimes \text{id}_{\mathbf{B}A} \otimes \Delta_{\mathbf{B}A} \otimes \text{id}_{A'' \otimes A''}) \Pi$$

for $\tau_{[b_{(3)}]; a''}$ the tensor shuffle taking the factors $[b_{(3)}] \otimes a'$ to $(-1)^{|[b_{(3)}]| \cdot |a'|} a' \otimes [b_{(3)}]$ and Π the permutation (2354) from (A.4), and the second is

$$\begin{aligned}
 \text{(A.8)} \quad & \overbrace{(\mu_{A'} (\text{id}_{A'} \otimes \mathfrak{E}(f's \otimes s^{-1} \otimes \mathbf{B}f')) \tau_{b'; [a_1]} \otimes \mu_{\mathbf{B}A} \otimes \mu_{A''} (\mathfrak{E}(\text{id}_{A''} \otimes \mathbf{B}f'') \otimes \text{id}_{A''}) \tau_{[b_{(3)}]; a''})}^{T_1} \\
 & \circ (\text{id}_{A'} \otimes (\text{id}_{A'} - \epsilon_{A'}) \otimes ((\text{pr}_1 \otimes \text{id}_{\mathbf{B}A}) \otimes (\text{id}_{\mathbf{B}A} \otimes \text{id}_{\mathbf{B}A}) \otimes (\epsilon_{\mathbf{B}A} \otimes \text{id}_{\mathbf{B}A})) \Delta_{\mathbf{B}A \otimes \mathbf{B}A}^{[3]} \otimes \text{id}_{A'' \otimes A''}) \Pi,
 \end{aligned}$$

where $\tau_{b'; [a_1]}$ is again a tensor transposition.

Concatenations $\Delta_{\leftarrow}$ (as defined in Definition 1.2) have been omitted in the statement and proof to aid legibility (so far as possible).

Proof We first establish the formula modulo 2. The composition of the first two maps in (A.3) takes a pure tensor $a'[a_\bullet]a'' \otimes b'[b_\bullet]b''$ to $(a' \otimes b') \otimes \nabla([a_\bullet] \otimes [b_\bullet]) \otimes (a'' \otimes b'')$. Recall $\nabla([a_\bullet] \otimes [b_\bullet])$ is the signed sum of terms $[\alpha \otimes \beta]_\bullet := [\alpha_1 \otimes \beta_1] \cdots [\alpha_\ell \otimes \beta_\ell]$ where each $\alpha_j \otimes \beta_j$ is either $a_m \otimes 1$ or $1 \otimes b_m$ for

some m , and the indices of the a -arguments appear in increasing order as one encounters them from left to right, as do the indices of the b -arguments. The map $\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})$ first breaks each term $[\alpha \otimes \beta]_\bullet$ into three factors $[\alpha \otimes \beta]_{(1)} = [\alpha_1 \otimes \beta_1 | \cdots | \alpha_p \otimes \beta_p]$, $[\alpha \otimes \beta]_{(2)}$, and $[\alpha \otimes \beta]_{(3)}$ via $\Delta_{\mathbf{B}(A \otimes 2)}^{[3]}$, and applies f' to each α and β in the first block and f'' to each in the third block. The verification for each of the three tensor factors will run in parallel.

Now there is a case distinction to be made. Write $\alpha'_j = f'\alpha_j$ and so on. If $a' \otimes b' \in \overline{A' \otimes A'}$ and $a'' \otimes b'' \in \overline{A'' \otimes A''}$, then by (1.17), $\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})$ takes $(a' \otimes b') \otimes \nabla([a_\bullet] \otimes [b_\bullet]) \otimes (a'' \otimes b'')$ to

$$(A.9) \quad \pm t_{A'} \Phi_{A'}[a' \otimes b' | \alpha'_1 \otimes \beta'_1 | \cdots | \alpha'_p \otimes \beta'_p] \otimes [\alpha \otimes \beta]_{(2)} \otimes t_{A''} \Phi_{A''}[\alpha''_1 \otimes \beta''_1 | \cdots | \alpha''_q \otimes \beta''_q | a'' \otimes b''],$$

where $p + \ell([\alpha \otimes \beta]_{(2)}) + q = \ell(a_\bullet) + \ell(b_\bullet)$. If instead $a' \otimes b' \in \text{im } \eta_{A' \otimes A'} = k$, then the associated clause $\Upsilon' = \text{id}_k \otimes \epsilon$ from (1.17) applies instead, so the first tensor factor in (A.9) is replaced with $a'b'$, and $p = 0$; similarly, if $a'' \otimes b'' \in \text{im } \eta_{A'' \otimes A''}$, then instead the third tensor factor in (A.9) is $a''b''$, and $q = 0$. In either case, these simpler factors agree with those given in (A.6), (A.7), and (A.8) by our observations and conventions from Remark 2.3 on the vanishing of $\mathbf{E}$, so for the remainder of the verification we may assume $a' \otimes b' \in \overline{A' \otimes A'}$ and $a'' \otimes b'' \in \overline{A'' \otimes A''}$.

At this point it becomes necessary to recall the unsigned formula [10, (4.2)] for $t_A \Phi_A$

$$(A.10) \quad t_A \Phi_A[\alpha \otimes \beta]_\bullet = \sum_{\sigma} \pm \alpha_1 \mathbf{E}[\alpha_2 | \beta_\bullet] \cdots \mathbf{E}[\alpha_n | \beta_\bullet] \beta_n,$$

where the sum is over those permutations σ separately preserving the orders of the indices of the α -arguments and those of the β -arguments, and such that, moreover, at every point, reading left to right, one has encountered the symbol α more often than β . The key point in simplifying this formula in the present context is that $\mathbf{E}[\alpha_j | \beta_\bullet]$ vanishes whenever one of the β -letters other than β_n is 1, by the convention set in Remark 2.3.

Thus in order for $t_{A'} \Phi_{A'}([a' \otimes b'] \otimes [\alpha' \otimes \beta']_\bullet)$ to be nonzero, one of two things must happen. The first possibility is that $p = \ell[\alpha' \otimes \beta']_\bullet = 0$, so that $t_{A'} \Phi_{A'}[a' \otimes b'] = \pm a'b'$. The other possibility is that $b', \beta'_1, \dots, \beta'_{p-1}$ are all non-1. Particularly, if $b' = 1$, then $\pm a'b'$ is again the only term, agreeing with (A.6), (A.7), and (A.8), so from now on we will assume $b' \in \overline{A'}$. Since for each j one of α'_j and β'_j is 1, it follows that $\alpha'_1 = \cdots = \alpha'_{p-1} = 1$. As $\mathbf{E}_{1,0}([1] \otimes []) = 1$ and otherwise $\mathbf{E}_{1,\bullet}([1] \otimes [\beta_\bullet]) = 0$ by the conventions of Remark 2.3, for a term in (A.10) to be nonzero one needs $\alpha'_p \neq 1$, so that $\beta'_p = 1$. Thus the only relevant terms have

$$[\alpha' \otimes \beta']_\bullet = [1 \otimes f'b_1 | \cdots | 1 \otimes f'b_{p-1} | f'a_1 \otimes 1],$$

$$t_{A'} \Phi_{A'}([a' \otimes b'] \otimes [\alpha' \otimes \beta']_\bullet) = \pm a' \mathbf{E}_{1,p}([f'a_1] \otimes [b' | f'b_{(1)}]),$$

where $[f'b_{(1)}] = [f'b_1 | \cdots | f'b_{p-1}]$.

Similarly, in order that $t_{A''} \Phi_{A''}[\alpha''_1 \otimes \beta''_1 | \cdots | \alpha''_q \otimes \beta''_q | a'' \otimes b'']$ be nonzero, the arguments β''_j must be non-1, so each α''_j is 1. Thus the relevant tensor factor is

$$t_{A''} \Phi_{A''}([1 \otimes f''b]_{(3)} \otimes [a'' \otimes b'']) = \pm \mathbf{E}_{1,q}([a''] \otimes [f''b_{(3)}]) b'',$$

which vanishes by convention if $a'' = 1$. As for the middle tensor factor, we know from Theorem 3.8 that $\Phi \circ \nabla = \mu_{\mathbf{B}A}$. Combining our descriptions of the three tensor factors, we have established (A.6).

It remains to determine the signs. The sign for the permutation Π of (A.4) is by definition the Koszul permutation sign. For the composition of the second two maps, recall from (1.17) that before involving $\Phi_{A'}$ and $\Phi_{A''}$, the map $\mathbf{B}(\Phi_{A'}, \mu_{\mathbf{B}A}, \Phi_{A''})$ first breaks $[\alpha \otimes \beta]_{\bullet} = \nabla([a_{\bullet}] \otimes [b_{\bullet}])$ into three chunks via $\Delta_{\mathbf{B}(A \otimes A)}^{[3]}$. As ∇ is a DGC map $\mathbf{B}A \otimes \mathbf{B}A \rightarrow \mathbf{B}(A \otimes A)$ by Example 1.5, we equally well have $\Delta_{\mathbf{B}(A \otimes A)}^{[3]} \nabla = \nabla^{\otimes 3} \Delta_{\mathbf{B}A \otimes \mathbf{B}A}^{[3]}$. Applying the definition (1.17), suppressing concatenations $\Delta_{\leftarrow}$, gives

$$(A.11) \quad \mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})(\text{id}_{A' \otimes A'} \otimes \nabla \otimes \text{id}_{A'' \otimes A''}) \\ = \underbrace{\left(\overbrace{t_{A'} \Phi_{A'} (s_{A' \otimes A'}^{-1} \otimes \mathbf{B}(f' \otimes f') \nabla)}^{Q_1} \otimes \Phi_A \nabla \otimes \overbrace{t_{A''} \Phi_{A''} (\mathbf{B}(f'' \otimes f'') \nabla \otimes s_{A'' \otimes A''}^{-1})}^{Q_3} \right)}_Q \\ \circ (\text{id}_{A' \otimes A'} \otimes \Delta_{\mathbf{B}A \otimes \mathbf{B}A}^{[3]} \otimes \text{id}_{A'' \otimes A''}).$$

We will evaluate Q on terms

$$(a' \otimes b' \otimes [a_{(1)}] \otimes [b_{(1)}]) \otimes ([a_{(2)}] \otimes [b_{(2)}]) \otimes ([a_{(3)}] \otimes [b_{(3)}] \otimes a'' \otimes b'')$$

and compare with the evaluations of T_0 and T_1 from (A.7) and (A.8), recalling from the discussion of nonzero values of $t_{A'} \Phi_{A'}$ and $t_{A''} \Phi_{A''}$ in the previous paragraph that the only nonzero terms in (A.11) arise when $[a_{(3)}] = []$ is of length 0 and either we have $[a_{(1)}] = [b_{(1)}] = []$ of length 0 as well or else $[a_{(1)}] = [a_1]$ is of length 1.

As each of the three tensor factors of Q , of T_0 , and of T_1 is of degree zero, it will be enough to compare these factors separately. The middle tensor factor $\Phi_A \nabla$ of Q is $\mu_{\mathbf{B}A}$ by Theorem 3.8, as in T_0 and T_1 . For the first and third factors, we will need to be explicit about the sign for Φ . The signed version of formula (A.10) for $(\Phi_A)_{(n)} = t_A \circ (\Phi_A)|_{\mathbf{B}_n A} \circ (s_{A \otimes A}^{-1})^{\otimes n}$ is

$$(A.12) \quad (\Phi_A)_{(n)} := (-1)^{n-1} \sum_{\sigma} \mu_A^{[\bullet]} \left(\text{id}_A \otimes \bigotimes_m E_{\ell_m} \otimes \text{id}_A \right) \sigma,$$

where the sum is over shuffles σ as in (A.10); thus, the sequence (ℓ_m) gives a partition of $n-1$, apportioning subsequences of $\beta_1, \dots, \beta_{n-1}$ as arguments of $\mathbf{E}[\alpha_2|-]$ through $\mathbf{E}[\alpha_n|-]$, and E_{ℓ} is $\mathbf{E}_{1,\ell}(s^{-1})^{\otimes 1+\ell}$ as in Notation 2.2; similar arguments hold for $\Phi_{A'}$ and $\Phi_{A''}$.

To interpret Q_1 , note that $\ell[a_{(1)}]$ is either 0 or 1. In the case $[a_{(1)}] = []$, the only term we need to worry about is $t_{A'} \Phi_{A'} s_{A' \otimes A'}^{-1} = (\Phi_{A'})_{(1)}$, which is $\mu_{A'}$ by Definition 3.4(1). This is the first factor of T_0 from (A.7). When $[a_{(1)}] = [a_1]$, we have seen from the discussion above that the only terms on which $t_{A'} \Phi_{A'}$ will potentially not vanish are those of the form $[a' \otimes b' | 1 \otimes f' b]_{(1)} \otimes [f' a_1 \otimes 1]$. Here the notation $[a' \otimes b' | 1 \otimes f' b]_{(1)}$ means some initial subword $[a' \otimes b' | 1 \otimes f' b_1 | \dots | 1 \otimes f' b_{\ell}]$, which is $[a' \otimes b']$ if $\ell = 0$.

A term of $\nabla([a_1][b_{(1)}])$ leading to a term of this form is created by the operation

$$v: [a_1] \otimes [b_{(1)}] \mapsto [a_1 \otimes 1 | 1 \otimes b]_{(1)}$$

followed by the shuffle $\tau = \tau_{[a_1 \otimes 1]; [1 \otimes b]_{(1)}}$ moving $[a_1 \otimes 1]$ past each $[1 \otimes b_j]$, resulting in

$$x := \pm [a' \otimes b'] \otimes [1 \otimes f'b]_{(1)} \otimes [f'a_1 \otimes 1].$$

To determine the sign explicitly, (A.12) tells us we should rewrite things in terms of the operations E_ℓ . When we apply $\Phi_{A'}$ to x , we have seen above that the only nonzero term is $\pm a' E_p(a_1; b', b_{(1)})$, which comes from the summand in (A.12) corresponding to the shuffle

$$\sigma': (a' \otimes b') \otimes \bigotimes_{j=1}^{p-1} (1 \otimes f'b_j) \otimes (f'a_1 \otimes 1) \mapsto (-1)^{|a_1| \cdot (|b'| + |b_{(1)}|)} a' \otimes 1^{\otimes p-1} \otimes f'a_1 \otimes b' \otimes f'b_{(1)} \otimes 1.$$

If we write κ for the map omitting tensor factors 1, then this nonzero term can be written as

$$\mu'_{A'}(\text{id}_{A'} \otimes E_p) \tau_{b' \otimes b_{(1)}; a_1} \kappa(s_{A' \otimes A'}^{-1})^{\otimes p+1} x,$$

where $\tau_{b' \otimes b_{(1)}; a_1}$ is the tensor shuffle moving a_1 past $b' \otimes b_{(1)}$. Since $(s^{-1})^{\otimes n} s^{\otimes n}$ is multiplication by $(-1)^{\binom{n}{2}}$, we conclude from (A.12) that on $(A' \otimes A') \otimes \mathbf{B}_1 A \otimes \mathbf{B}_p A$, we have

$$\begin{aligned} Q_1 &= t_{A'} \Phi_{A'} \circ (s_{A' \otimes A'}^{-1} \otimes \mathbf{B}(f')^{\otimes 2} \tau \nu) \\ &= (-1)^{\binom{p+1}{2}} (\Phi_{A'})_{(p+1)} s_{A' \otimes A'}^{\otimes 1+p} \circ (s_{A' \otimes A'}^{-1} \otimes \mathbf{B}(f')^{\otimes 2} \tau \nu) \\ &= (-1)^{\binom{p+1}{2}} ((-1)^p \mu_{A'}(\text{id}_{A'} \otimes \mathbf{E}_{1,p}(s_{A'}^{-1})^{\otimes 1+p}) \tau_{b' \otimes b_{(1)}; a_1} \kappa) (-1)^p (\text{id}_{A' \otimes A'} \otimes s_{A' \otimes A'}^{\otimes p} \mathbf{B}(f')^{\otimes 2} \tau \nu) \\ &= (-1)^{\binom{p+1}{2}} \mu_{A'}(\text{id}_{A'} \otimes \mathfrak{E}(\text{id}_{A'} \otimes (s_{A'}^{-1})^{\otimes p})) \tau_{b' \otimes b_{(1)}; a_1} \kappa(\text{id}_{A' \otimes A'} \otimes s_{A' \otimes A'}^{\otimes p} \mathbf{B}(f')^{\otimes 2} \tau \nu), \end{aligned}$$

where the accounting of signs is as follows: the $(-1)^{\binom{p+1}{2}}$ comes from $(\Phi_{A'})_{(1+p)} := t_{A'} \Phi_{A'} s_{A' \otimes A'}^{\otimes 1+p}$ in (A.12), the first $(-1)^p$ is the leftmost term of the right-hand side of (A.12), and the second $(-1)^p$ comes from moving $s_{A' \otimes A'}^{\otimes p}$ past $s_{A' \otimes A'}^{-1}$.

We must compare the sign of the value of Q_1 on $([a' \otimes b'] \otimes [a_1 | b_{(1)}])$ with that of the first factor $T_{1,1} = \mu_{A'}(\text{id}_{A'} \otimes \mathfrak{E}(f's \otimes s^{-1} \otimes \mathbf{B}f')) \tau_{b'; [a_1]}$ from (A.8). This is done symbolically by determining which tensor factors are moved past which others in the computation. For these purposes, the factors $\mu_{A'}$, $\text{id}_{A'}$, $\mathfrak{E}$, κ , f' , $\mathbf{B}(f' \otimes f')$, and ν of degree 0 are invisible, as are the tensor factors 1 of degree 0 produced by ν and deleted by κ . In Figure 2 below, Q_1 is in the left column and $T_{1,1}$ in the right, and in each row of each column, operations are listed on the left and arguments on the right, the result of applying the operations appearing on the right in the following row. As usual, moving two symbols y and z past each other incurs the sign $(-1)^{|y||z|}$, and the rule goes equally for $y = s^{\pm 1}$.

In determining whether the signs agree, we can remove cancelling signs within one diagram or pairs of matching signs in both diagrams, referring to this as “cancellation” either way. Then two crossings cancel if both involve transposing the same symbols y and z , or if one crossing transposes s and z while the other transposes s^{-1} and z . After cancellation, the only signs remaining are $(-1)^{|s^{-1}||s^{-1}|} = -1$, arising from the crossing marked with a dot on the right, and the underlined

$$(-1)^{\binom{p+1}{2}} \cdot |(s^{-1})^{\otimes p-1} s^{\otimes p-1}| = (-1)^{\binom{p+1}{2} + \binom{p-1}{2}} = (-1)^{p^2 - p + 1} = -1$$

at the bottom on the left, so the operations in question are indeed the same.

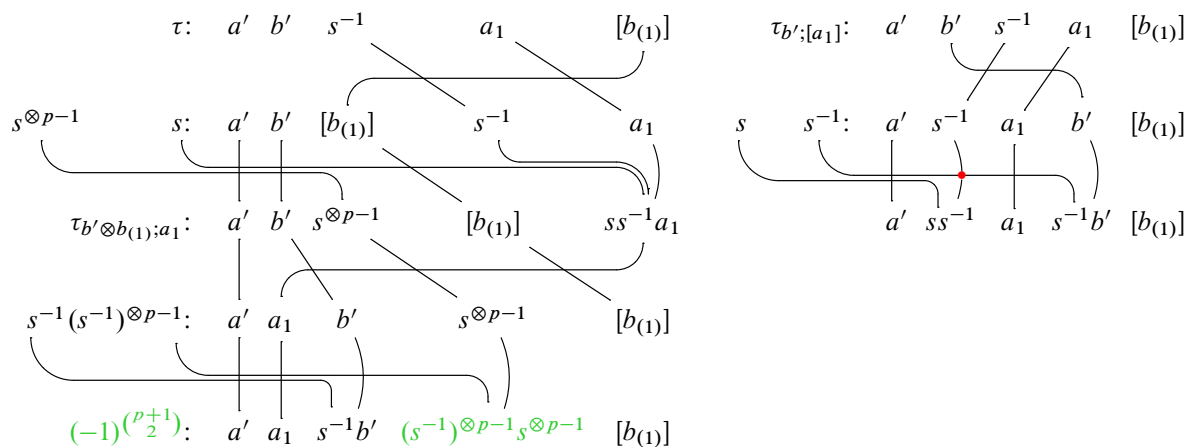


Figure 2

As for Q_3 and $T_{0,3} = T_{1,3}$ (from (A.11), (A.7), and (A.8)), we have seen above that the terms on which the $\Phi_{A''}$ in Q_3 is potentially nonzero are those of the form $[1 \otimes f''b]_{(3)} \otimes [a'' \otimes b'']$. In particular, $[a_{(3)}] = \emptyset$. Recall that we write $q = \ell[b_{(3)}]$. We see the many 1 factors arising because $\nabla(\emptyset \otimes [b_{(3)}]) = [1 \otimes b]_{(3)}$ contribute nothing to the iterated $\mu_{A''}$ -product, so we may omit them with the map κ . When they are omitted, the shuffle involved in the nonvanishing summand of (A.12) simplifies to the shuffle $\tau_{f''b_{(3)};a''}$ switching the A'' tensor factors containing a'' and $f''b_{(3)}$ in $f''b_{(3)} \otimes a'' \otimes b''$. Thus on $\mathbf{B}_q A \otimes A'' \otimes A''$, we can expand Q_3 as

$$\begin{aligned} Q_3 &= t_{A''} \Phi_{A''} \circ (\mathbf{B}(f'')^{\otimes 2} \otimes s_{A'' \otimes A''}^{-1}) \\ &= (-1)^{\binom{q+1}{2}} (\Phi_{A''})_{(1+q)} \circ s_{A'' \otimes A''}^{\otimes 1+q} \circ (\mathbf{B}(f'')^{\otimes 2} \nabla \otimes s_{A'' \otimes A''}^{-1}) \\ &= (-1)^{\binom{q+1}{2}} (-1)^q \mu_{A''} (\mathbf{E}_{1,q} (s_{A''}^{-1})^{\otimes 1+q} \otimes \text{id}_{A''}) \tau_{f''b_{(3)};a''} \kappa \circ (s_{A'' \otimes A''}^{\otimes q} \mathbf{B}(f'')^{\otimes 2} \nabla \otimes \text{id}_{A'' \otimes A''}) \\ &= (-1)^{\binom{q+1}{2} - q} \mu_{A''} (\mathfrak{E}(\text{id}_{A''} \otimes (s_{A''}^{-1})^{\otimes q}) \otimes \text{id}_{A''}) \tau_{f''b_{(3)};a''} \kappa (s_{A'' \otimes A''}^{\otimes q} \mathbf{B}(f'')^{\otimes 2} \nabla \otimes \text{id}_{A'' \otimes A''}), \end{aligned}$$

where, as in the analysis of Q_1 , the $(-1)^{\binom{q+1}{2}}$ comes from $(\Phi_{A''})_{q+1} = t_{A''} \Phi_{A''} s_{A'' \otimes A''}^{\otimes q+1}$ and the $(-1)^q$ is the leftmost factor of the right-hand side of (A.12). We may simplify the exponent of -1 by observing that

$$\binom{q+1}{2} - q = \binom{q+1}{2} - \binom{q}{1} = \binom{q}{2}.$$

In Figure 3 the crossings for Q_3 appear on the left and that for

$$T_{0,3} = T_{1,3} = \mu_{A''} (\mathfrak{E}(\text{id}_{A''} \otimes \mathbf{B}(f'')) \otimes \text{id}_{A''}) \tau_{[b_{(3)}];a''}$$

appears on the right.

When matching crossings are cancelled, the only signs remaining are the underlined $(-1)^{\binom{q}{2}}$ and $|(s^{-1})^{\otimes q} s^{\otimes q}| = (-1)^{\binom{q}{2}}$ on the left, so the operations are equal, concluding the proof. $\square$

With this more explicit formulation of $\tilde{\mu}$, we are able to relate it to the product on an HGA:

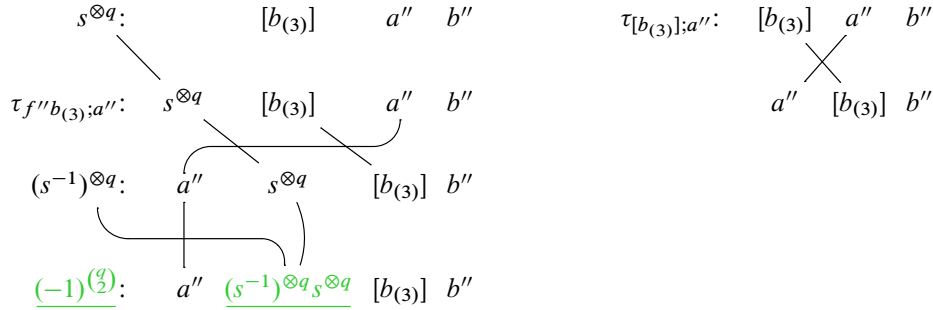
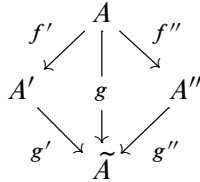


Figure 3

Theorem A.13 Given a commutative diagram



of HGA homomorphisms, the natural cochain map

$$(A.14) \quad \xi := \mu_{\tilde{A}}^{[3]}(g' \otimes \eta_{A \in \mathbf{BA}} \otimes g'') : \mathbf{B}(A', A, A'') \rightarrow \tilde{A}, \quad a'[a_{\bullet}]a'' \mapsto g'(a')\eta_{\tilde{A} \in \mathbf{BA}}[a_{\bullet}]g''(a''),$$

induces a map in cohomology multiplicative with respect to the product on $H^*\mathbf{B}(A', A, A'')$ induced by the product defined by Theorem A.1 and the expected product on $H^*(\tilde{A})$.

Proof By Theorem A.1, $\mathbf{B}(g', g, g'')$ is a multiplicative cochain map $\mathbf{B}(A', A, A'') \rightarrow \mathbf{B}(\tilde{A}, \tilde{A}, \tilde{A})$, so it will be enough to show $\xi : \mathbf{B}(\tilde{A}, \tilde{A}, \tilde{A}) \rightarrow \tilde{A}$ is a cochain map inducing a multiplicative map in cohomology. In other words, we may as well start by assuming $A' = A = A'' = \tilde{A}$ and $f' = f'' = g' = g'' = \text{id}_A$. Let us then agree to write $\mu = \mu_A$ and $\text{id} = \text{id}_A$ and $\epsilon = \eta_{A \in \mathbf{BA}}$, so we can restate (A.14) as

$$(A.15) \quad \xi = \mu^{[3]}(\text{id} \otimes \epsilon \otimes \text{id}) : c'[c_{\bullet}]c'' \mapsto c' \epsilon [c_{\bullet}] c''.$$

It is trivial to check this is a cochain map.

To show the induced map $H^*(\xi)$ in cohomology is multiplicative, we show that

$$(A.16) \quad Dh = \xi \tilde{\mu} - \mu \xi^{\otimes 2} : \mathbf{B}(A, A, A)^{\otimes 2} \rightarrow A$$

for a certain homotopy $h : \mathbf{B}(A, A, A)^{\otimes 2} \rightarrow A$ to be produced momentarily. It will help to adapt (A.6), (A.7), and (A.8) to the present setup:

$$(A.17) \quad \tilde{\mu} = \left(\mu \otimes \mu_{\mathbf{BA}} \otimes \mu(\mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{BA}}) \otimes \text{id}) \right) \circ \Pi' \\ a'b' \otimes [a_{\bullet}] * [b_{(2)}] \otimes \mathfrak{E}(a'' \mid b_{(3)}) \mid b'' \\ + \left(\mu(\text{id} \otimes \mathfrak{E}(s \otimes s^{-1} \otimes \text{id}_{\mathbf{BA}})) \otimes \mu_{\mathbf{BA}} \otimes \mu(\mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{BA}}) \otimes \text{id}) \right) \circ \Pi'' \\ a' \otimes \mathfrak{E}(a_1 \mid b' \mid b_{(1)}) \otimes [a_{(2)}] * [b_{(2)}] \otimes \mathfrak{E}(a'' \mid b_{(3)}) \mid b'',$$

where Π' and Π'' are preparatory tensor permutations. The permutations τ in particular have been absorbed into these without sign change because the operators between τ and Π in (A.7) and (A.8) are of degree zero. Here the value, up to sign (determined by the Koszul convention, but not written out), on the standard pure tensor $a'[a_\bullet]a'' \otimes b'[b_\bullet]b''$ is displayed below the function, to make the argument easier to follow, though in principle these values are optional. We will continue with this notation throughout the proof. Recall that if b' is in the image of the unit map η_A , then the $\mathfrak{E}$ factor in the second term is defined to vanish on bar-words containing it by Remark 2.3.

It is evident from (A.15) that $-\mu\xi^{\otimes 2}$ vanishes unless $\ell = \ell(a_\bullet)$ and $r = \ell(b_\bullet)$ are both zero, in which case it is

$$-\mu^{[6]}(\text{id} \otimes \eta_A \otimes \text{id} \otimes \text{id} \otimes \eta_A \otimes \text{id}) = -\mu^{[4]}_{a'a''b'b''}.$$

As $\mu_{\mathbf{B}A}$ is of degree 0 and $\epsilon_{\mathbf{B}A}$ annihilates $\mathbf{B}_{\geq 1}A$, the first term of $\xi\tilde{\mu}$ vanishes according to (A.15) and (A.17) unless $\ell(a_\bullet) = \ell(b_{(2)}) = 0$, in which case it reduces to

$$\mu^{[3]}(\mu_{a'b'} \otimes \eta_A \otimes \mu(\mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id})) \circ \tau_{a'';b'} \circ \mathfrak{E}(a'' \mid [b_{(3)}]) \mid b''.$$

If additionally $\ell(b_{(3)}) = 0 = r$, this further reduces to

$$\mu^{[3]}(\mu_{a'b'} \otimes \eta_A \otimes \mu(\text{id} \otimes \text{id})) \circ \tau_{a'';b'} = \mu^{[4]}_{a'a''b'b''} \circ \tau_{a'';b'}.$$

The second term of $\xi\tilde{\mu}$ vanishes unless $[a_\bullet] = [a]$ is of length 1 and $[b_{(2)}]$ is of length 0, in which case it contributes

$$\mu^{[4]}(\text{id} \otimes \mathfrak{E}(s \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \circ \pi''_{a' \mid \mathfrak{E}(a_1 \mid [b' \mid b_{(1)}]) \mid \mathfrak{E}(a'' \mid [b_{(3)}]) \mid b''}$$

for π'' running over tensor permutations $a'[a]a'' \otimes b'[b_\bullet]b'' \mapsto a' \otimes [a]b'[b_{(1)}] \otimes a''[b_{(3)}] \otimes b''$. These π'' are the specialization of Π'' from (A.17) to the case $\ell(a_{(2)}) = 0$.

All told, on $\mathbf{B}_\ell(A, A, A) \otimes \mathbf{B}_r(A, A, A)$ one has

$$(A.18) \quad \xi\tilde{\mu} - \mu_A\xi^{\otimes 2} = \begin{cases} \mu^{[4]}_{a'b'a''b''} \tau_{a'';b'} - \mu^{[4]}_{a'a''b'b''} & \text{if } \ell = 0 = r, \\ \mu^{[4]}(\text{id} \otimes \text{id} \otimes \mathfrak{E}(a'' \mid [b_\bullet]) \otimes \text{id}) \tau_{a'';b'} & \text{if } \ell = 0 < r, \\ \mu^{[4]}(\text{id} \otimes \mathfrak{E}(s \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \circ \pi'' & \text{if } \ell = 1, \\ 0 & \text{if } \ell \geq 2. \end{cases}$$

The promised homotopy is given as

$$(A.19) \quad h := \mu^{[4]}_{a' \mid \epsilon[a_\bullet] \mid \mathfrak{E}(a'' \mid [b' \mid b_\bullet]) \mid b''}(\text{id} \otimes \epsilon \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id})$$

To show Dh agrees with $\xi\tilde{\mu} - \mu_A\xi^{\otimes 2}$, we follow the same case distinctions.

If $\ell = \ell(a_\bullet) \geq 2$, then h and hence dh vanish since $\epsilon[a_\bullet] = 0$, and since the differential on $\mathbf{B}(A, A, A)$ takes $\mathbf{B}_{\geq 2}(A, A, A)$ into $\mathbf{B}_{\geq 1}(A, A, A)$, so does $hd_{\mathbf{B}(A, A, A)}^{\otimes 2}$.

If $\ell = 1$, then h and hence dh vanish as before, but hd need not, by the formula for the “external” differential $d_{\mathbf{B}(A, A, A)}^{\text{ext}} = d_{\mathbf{B}(A, A, A)} - d_\otimes$ given in (1.21), since the outer two operators reduce the length of the bar-word in $a'[a]a''$ to zero, taking it respectively to $\pm a'a[a'']$ and $\pm a'[aa'']$. Thus $hd_\otimes$ and $h(\text{id}_{\mathbf{B}(A, A, A)} \otimes d_{\mathbf{B}(A, A, A)})$ vanish, but for $h(d_{\mathbf{B}(A, A, A)} \otimes \text{id}_{\mathbf{B}(A, A, A)})$, plugging (1.21) into (A.19), we get

$$\begin{aligned} \text{(A.20)} \quad & \mu^{[4]}(\text{id} \otimes \epsilon \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \\ & \circ \left((\mu(\text{id} \otimes s) \otimes \eta_{\mathbf{B}A} \otimes \text{id}) \otimes \text{id}_{\mathbf{B}(A, A, A)} - \text{id} \otimes \eta_{\mathbf{B}A} \otimes \mu(s \otimes \text{id}) \otimes \text{id}_{\mathbf{B}(A, A, A)} \right) \\ & \quad \begin{matrix} a' & a & [] & a'' & b'[b_\bullet]b'' & a' & [] & a & a'' & b'[b_\bullet]b'' \end{matrix} \\ & = -\mu^{[4]}(\text{id} \otimes s \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) + \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\mu(s \otimes \text{id}) \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \\ & \quad \begin{matrix} a' & a & \mathfrak{E}(a'' & [b' & | & b_{(1)}]) & b'' & a' & \mathfrak{E}(a & a'' & [b' & | & b_{(1)}]) & b'' \end{matrix}, \end{aligned}$$

the change in signs coming in both terms from the factor s of degree 1 coming from $d_{\mathbf{B}(A, A, A)}^{\text{ext}}$ being moved past the factor s^{-1} of degree -1 in the argument of $\mathfrak{E}$. By the Cartan-esque formula (2.5), the second term on the right-hand side of (A.20) can be replaced by

$$\mu^{[4]}(\text{id} \otimes (\mathfrak{E} \otimes \mathfrak{E})\pi(s \otimes \text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \quad \begin{matrix} a' & (\mathfrak{E} \otimes \mathfrak{E})\pi(a & a'' & [b' & | & b_\bullet]) & b'' \end{matrix},$$

where π runs over shuffles $a \otimes a'' \otimes [b'|b_\bullet] \mapsto a[b'|b_\bullet]_{(1)} \otimes a''[b'|b_\bullet]_{(2)}$ and $[b'|b_\bullet]_{(1)} \otimes [b'|b_\bullet]_{(2)}$ is a deconcatenation of $[b'|b_\bullet]$. There is a case distinction here: The deconcatenation yielding $[] \otimes [b'|b_\bullet]$ recovers

$$\mu^{[4]}(\text{id} \otimes s \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \quad \begin{matrix} a' & a & \mathfrak{E}(a'' & [b' & | & b_\bullet]) & b'' \end{matrix},$$

cancelling the first term of (A.20). Otherwise, b' is assigned to the first word in the deconcatenation, so the value is $[b'|b_{(1)}] \otimes [b_{(3)}]$, and one gets operations

$$(\text{id} \otimes \mathfrak{E}(s \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \mathfrak{E}(\text{id} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \circ \pi'' \quad \begin{matrix} a' & \mathfrak{E}(a_1 & [b' & | & b_{(1)}]) & \mathfrak{E}(a'' & [b_{(3)}]) & b'' \end{matrix}$$

agreeing with the case $\ell = 1$ of (A.18). Because π preserves the relative positions of s and s^{-1} , it does not incur a sign change in moving past them to become π'' .

For the $\ell = 0$ cases, we write the differential on $\mathbf{B}(A, A, A)^{\otimes 2}$ as the sum of the “internal” tensor differential $d_\otimes$ and the “external” differentials $\text{id}_{\mathbf{B}(A, A, A)} \otimes d_{\mathbf{B}(A, A, A)}^{\text{ext}}$ and $d_{\mathbf{B}(A, A, A)}^{\text{ext}} \otimes \text{id}_{\mathbf{B}(A, A, A)}$. We may omit the second external differential in consideration of $hd_{\mathbf{B}(A, A, A)}$ since $d_{\mathbf{B}(A, A, A)}$ vanishes on $\mathbf{B}_0(A, A, A)$, and when $r = 0$ we may omit the first external differential for the same reason. Write $D_\otimes$ for the differential on $\text{Hom}((A \otimes \mathbf{B}A \otimes A)^{\otimes 2}, A)$, where the domain is $\mathbf{B}(A, A, A)^{\otimes 2}$ equipped with the tensor differential $d_\otimes$. Thus $D_\otimes h = d_A h + h d_\otimes$. Because composition is a cochain map, as are μ_A , id_A , $\epsilon_{\mathbf{B}A}$, $\text{id}_{\mathbf{B}A}$, and s^{-1} , we have

$$\text{(A.21)} \quad D_\otimes h = \mu^{[4]}(\text{id} \otimes \eta_{\mathbf{B}A} \otimes (D_\otimes \mathfrak{E})(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}A}) \otimes \text{id}) \quad \begin{matrix} a' & (D_\otimes \mathfrak{E})(a'' & [b' & | & b_\bullet]) & b'' \end{matrix}.$$

Equation (2.6) shows that for fixed r , the function $D_{\otimes} \mathfrak{E}: \bar{A} \otimes \mathbf{B}_{1+r} A \rightarrow A$ is given by

$$(A.22) \quad D_{\otimes} \mathfrak{E} = \mu(s \otimes \mathfrak{E})(12) + \mathfrak{E}(\text{id} \otimes d_{\text{ext}}) - \mu(\mathfrak{E} \otimes s),$$

$$b' \quad \mathfrak{E}(a''[b_{\bullet}]) \quad \mathfrak{E}(a''d_{\text{ext}}[b'|b_{\bullet}]) \quad \begin{cases} \mathfrak{E}(a''[])b' & r = 0, \\ \mathfrak{E}(a''[b'|b_1|b_{\bullet}|b_{r-1}])b_r & r > 0. \end{cases}$$

The second term also vanishes when $r = 0$, and then plugging (A.22) into (A.21) yields

$$(A.23) \quad D_{\otimes} h = \mu^{[4]}(\text{id} \otimes (s \otimes \mathfrak{E}) \tau_{a'';[b']}(\text{id} \otimes s^{-1} \otimes \eta_{\mathbf{B}A}) \otimes \text{id})$$

$$a' \quad b' \quad \mathfrak{E}(a''[]) \quad b''$$

$$= \mu^{[4]} \circ \tau_{a'';b'} - \mu^{[4]}(\text{id} \otimes (\mathfrak{E} \otimes s)(\text{id} \otimes s^{-1} \otimes \eta_{\mathbf{B}A}) \otimes \text{id}) - \mu^{[4]}$$

$$a'b'a''b'' \quad a' \quad \mathfrak{E}(a''[]) \quad b' \quad b'' \quad a'a''b'b''.$$

Here we use that $\mathfrak{E} = \text{id}$ on $A \otimes \mathbf{B}_0 A \cong A$, and the further simplification in the first term comes from the calculation

$$(s \otimes \text{id})\tau_{a'';[b']}(\text{id} \otimes s^{-1}) = \tau_{a'';b'}(\text{id} \otimes s)(\text{id} \otimes s^{-1}) = \tau_{a'';b'}.$$

The $\ell = 0 = r$ case of (A.18) agrees with (A.23), concluding that case.

For $\ell = 0 < r$, substituting (A.22) into (A.21) now gives instead

$$(A.24) \quad D_{\otimes} h = \mu^{[4]}(\text{id} \otimes (s \otimes \mathfrak{E}) \tau_{a'';[b']}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}_r A}) \otimes \text{id})$$

$$a' \quad b' \quad \mathfrak{E}(a''[b_{\bullet}]) \quad b''$$

$$+ \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \mu s^{\otimes 2} \otimes \text{id}_{\mathbf{B}_{r-1} A})(\text{id} \otimes s^{-1} \otimes \text{id}_{s^{-1} \bar{A}} \otimes \text{id}_{\mathbf{B}_{r-1} A}) \otimes \text{id})$$

$$a' \quad \mathfrak{E}(a''[b'b_1|b_2|\dots|b_r]) \quad b''$$

$$+ \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\text{id} \otimes \text{id} \otimes d_{\text{ext}}|_{\mathbf{B}_r A})(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}_r A}) \otimes \text{id})$$

$$a' \quad \mathfrak{E}(a''[b'] \otimes d_{\text{ext}}[b_{\bullet}]) \quad b''$$

$$- \mu^{[4]}(\text{id} \otimes (\mathfrak{E} \otimes s)(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}_{r-1} A} \otimes \text{id}_{s^{-1} \bar{A}}) \otimes \text{id})$$

$$a' \quad \mathfrak{E}(a''[b'|b_1|\dots|b_{r-1}]) \quad b_r \quad b''.$$

As for $h(\text{id}_{\mathbf{B}(A,A,A)} \otimes d_{\mathbf{B}(A,A,A)}^{\text{ext}})$, composing its definition from (1.21) with the expression for h from (A.19) yields

$$(A.25) \quad \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \mu(\text{id} \otimes s) \otimes \text{id}_{\mathbf{B}_{r-1} A}) \otimes \text{id})$$

$$a' \quad \mathfrak{E}(a''[b'b_1|\dots|b_r]) \quad b''$$

$$+ \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes d_{\text{ext}}|_{\mathbf{B}_r A}) \otimes \text{id})$$

$$a' \quad \mathfrak{E}(a''[b'] \otimes d_{\text{ext}}[b_{\bullet}]) \quad b''$$

$$- \mu^{[3]}(\text{id} \otimes \mathfrak{E}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B}_{r-1} A}) \otimes \mu(s \otimes \text{id}))$$

$$a' \quad \mathfrak{E}(a''[b'|b_1|\dots|b_{r-1}]) \quad b_r \quad b''.$$

These three terms are the respective opposites of the second through fourth terms of (A.24), as transitioning from one ordering of symbols to the other involves moving s^{-1} past s in the second and fourth terms and past d_{ext} in the third, incurring each time a sign change of $(-1)^{-1 \cdot 1}$. Thus these three pairs of terms cancel. It remains only to check that the uncanceled first term of (A.24) agrees with the $\ell = 0 < r$ clause

in (A.18). For this, note that id has degree 0, so we have $\tau_{a'';[b]}(\text{id} \otimes s^{-1}) = (s^{-1} \otimes \text{id})\tau_{a'';b'}$, and $\mathfrak{E}$ has degree zero as well, so

$$\begin{aligned} \mu^{[4]}(\text{id} \otimes (s \otimes \mathfrak{E})\tau_{a'';[b]}(\text{id} \otimes s^{-1} \otimes \text{id}_{\mathbf{B},A}) \otimes \text{id}) &= \mu^{[4]}(\text{id} \otimes (s \otimes \mathfrak{E})(s^{-1} \otimes \text{id} \otimes \text{id}_{\mathbf{B},A}) \otimes \text{id})\tau_{a'';b'} \\ &= \mu^{[4]}(\text{id} \otimes \text{id} \otimes \mathfrak{E} \otimes \text{id})\tau_{a'';b'}. \end{aligned} \quad \square$$

Remark A.26 We expect that if A' , A , and A'' are extended HGAs, the product $\tilde{\mu}$ of Theorem A.1 is the 2-component of a differential on $\mathbf{BB}(A', A, A'')$ making $\mathbf{B}(A', A, A'')$ an A_∞ -algebra and making the map ξ of Theorem A.13 the extended 1-component $\Xi_{(1)}^+$ of an A_∞ -map Ξ from $\mathbf{B}(A', A, A'')$ to $\tilde{A}$, but we will leave the exploration of this possibility for another occasion.⁹

We now can use the new product to establish the particular version of the Eilenberg–Moore theorem we will need:

Theorem A.27 (Eilenberg–Moore with induced product) *Suppose the coefficient ring k is a principal ideal domain, and suppose given a pullback square*

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & E \\ \alpha \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

of pointed topological spaces in which $E \rightarrow B$ is a Serre fibration, B and X are path connected, the action of $\pi_1(B)$ on the cohomology of the fiber F is trivial, and

- (1) *each $H^n(F)$ is a finitely generated k -module, or*
- (2) *each $H^n(B)$ and each $H^n(X)$ is a finitely generated k -module.*

Then there is a natural quasi-isomorphism

$$\xi: \mathbf{B}(C^*(X), C^*(B), C^*(E)) \rightarrow C^*(Y), \quad x[b_\bullet]e \mapsto \alpha^*(x) \smile \eta_{C^*(Y) \in \mathbf{B}C^*(E)}[b_\bullet] \smile \beta^*(e),$$

inducing a ring isomorphism

$$\text{Tor}_{C^*(B)}^*(C^*(X), C^*(E)) \xrightarrow{\sim} H^*(Y)$$

with respect to the product on the domain induced by the product given in Theorem A.1.

We rely on the presentation of Gugenheim and May [15, Theorem 3.3] with some modification of hypotheses.

Proof We assume B and X are path connected to guarantee that the homotopy type of the fiber F is well defined and we only have to discuss one group $\pi_1(B)$; one otherwise needs a separate argument for each path component. This assumption then also implies $\pi_1(X)$ acts trivially on $H^*(F)$. Given

⁹ Added in proof: this expectation has now been confirmed; see Franz [12] and Remark 0.2.

any proper projective resolution $P^\bullet$ of $C^*(X)$ as a $C^*(B)$ -module (or resolution in the more general sense of Gugenheim and May [15, Definitions 1.1]), Gugenheim and May show the expected composite filtered map $\vartheta: P^\bullet \otimes_{C^*(B)} C^*(E) \rightarrow C^*(X) \otimes_{C^*(B)} C^*(E) \rightarrow C^*(Y)$ is a quasi-isomorphism [15, Theorem 3.3 and Corollary 3.5] under slightly different finiteness hypotheses, namely that k is Noetherian and the groups $H_n(X; \mathbb{Z})$ and $H_n(B; \mathbb{Z})$ are all finitely generated.

These hypotheses are used only to see the canonical maps $C^*(B) \otimes H^*(F) \rightarrow C^*(B; H^*(F))$ and $C^*(X) \otimes H^*(F) \rightarrow C^*(X; H^*(F))$ are quasi-isomorphisms, using a lemma [15, Lemma 3.2] asserting that if k is a commutative Noetherian ring, G an k -module, and C a chain complex over $\mathbb{Z}$ with each $H_n(C)$ finite, then $\text{Hom}_{\mathbb{Z}}(C, k) \otimes_k G \rightarrow \text{Hom}_{\mathbb{Z}}(C, G)$ is a quasi-isomorphism. The proof of this lemma uses only that $\mathbb{Z}$ is a principal ideal domain, and hence the same argument shows that if k is a principal ideal domain, C now a differential graded k -module, and G a k -module, then $\text{Hom}_k(C, k) \otimes_k G \rightarrow \text{Hom}_k(C, G)$ is a quasi-isomorphism. Thus, assuming k is a principal ideal domain, we can replace Gugenheim and May's hypothesis that each integral homology group $H_n(B; \mathbb{Z})$ and $H_n(X; \mathbb{Z})$ is finitely generated with the weaker hypothesis (2) that the (co)homology groups with coefficients in k are. If, alternatively, we assume (1) that the $H^n(F)$ is finitely generated over k , then again assuming k is a principal ideal domain, the decomposition of each $H^n(F)$ as a finite product of cyclic k -modules shows $C^*(B) \otimes H^n(F) \rightarrow C^*(B; H^n(F))$ and $C^*(X) \otimes H^n(F) \rightarrow C^*(X; H^n(F))$ are isomorphisms of differential graded k -modules.

To see the suppressed details in the proof of the multiplicativity of $H^*(\vartheta)$ with respect to the classical product on Tor [15, page 26], subdivide the vertical cohomological Eilenberg–Zilber map featuring along the upper-right of McCleary's diagram gesturing at such a proof [22, page 255] as

$$H^*(Y) \otimes H^*(Y) \xrightarrow{H^*(\Phi)} H^*(C^*(Y) \otimes C^*(Y)) \xrightarrow{i} H^*((C_*(Y) \otimes C_*(Y))^*) \xleftarrow{EZ^*} H^*(Y \times Y).$$

Then there are evident horizontal maps subdividing the region into three rectangles that can be seen to commute on choosing maps between resolutions [15, Theorem 1.7] and expanding out the definition of the external product.

As k is a principal ideal domain, Proposition 1.19 implies that $\tilde{C} = \mathbf{B}(C^*(X), C^*(B), C^*(B))$ gives a resolution $\tilde{C} \rightarrow C^*(X)$ of $C^*(X)$ as a differential $C^*(B)$ -module and $H^*(\tilde{C})$ is the desired Tor. The map ϑ then specializes to our ξ . As C^* is a functor valued in HGAs, by Theorems A.1 and A.13 there is a natural cochain map $\tilde{\mu}: \tilde{C}^{\otimes 2} \rightarrow \tilde{C}$ such that the quasi-isomorphism ξ is multiplicative up to homotopy in the sense that if μ is the cup product on $C^*(Y)$, then there exists $h: \tilde{C} \rightarrow C^*(Y)$ with $Dh = \xi\tilde{\mu} - \mu\xi^{\otimes 2}$. Thus the k -module isomorphism $H^*(\xi): H^*(\tilde{C}) \rightarrow H^*(Y)$ takes the induced product

$$H^*(\tilde{C}) \otimes H^*(\tilde{C}) \xrightarrow{\sim} H^*(\tilde{C} \otimes \tilde{C}) \xrightarrow{H^*(\tilde{\mu})} H^*(\tilde{C})$$

on Tor to the cup product on $H^*(Y)$. Moreover, since the classical proof shows $H^*(\xi)$ is multiplicative with respect the standard product on Tor and the cup product on $H^*(Y)$, we conclude that $H^*(\tilde{\mu}) \circ \times$ is the standard product. $\square$

Remark A.28 The classical Eilenberg–Moore theorem obtains the ring structure on the domain through the external product and the Eilenberg–Zilber theorem, without reference to a cochain-level product on any complex computing Tor . We went to all this trouble because in Section 5, we need the fact that the product on Tor is induced by such a product in order to show certain maps of Tors are multiplicative and finally obtain Theorem 0.1.

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Envelopes for algebraic patterns

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We generalize Lurie’s construction of the symmetric monoidal envelope of an ∞ -operad to the setting of algebraic patterns. This envelope becomes fully faithful when sliced over the envelope of the terminal object, and we characterize its essential image. Using this, we prove a comparison result that allows us to compare analogues of ∞ -operads over various algebraic patterns. In particular, we show that the G - ∞ -operads of Nardin and Shah are equivalent to “fibrous patterns” over the $(2, 1)$ -category $\mathrm{Span}(\mathbb{F}_G)$ of spans of finite G -sets. When G is trivial this means that Lurie’s ∞ -operads can equivalently be defined over $\mathrm{Span}(\mathbb{F})$ instead of $\mathbb{F}_*$.

18N70; 18N60

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1 Introduction

In Lurie’s seminal work on homotopy-coherent algebra [23], the main objects used to encode algebraic structures are (symmetric) ∞ -operads, which are defined as a certain type of functor of ∞ -categories $\mathcal{O} \rightarrow \mathbb{F}_*$, where $\mathbb{F}_*$ is the category of finite pointed sets. However, as illustrated already in [23], it can sometimes be useful to consider variants of this notion, for instance because they give a combinatorially simpler description of some structure. For example, Lurie also considers *planar* (or nonsymmetric) ∞ -operads, where the category $\mathbb{F}_*$ is replaced by the simplex category Δ^{op} . As a special case of a general comparison theorem [23, Theorem 2.3.3.26] using the theory of *approximations* to ∞ -operads, Lurie proves that there is an equivalence of ∞ -categories between planar ∞ -operads and ∞ -operads over the (symmetric) associative operad Ass , given by pulling back along an explicit map $\Delta^{\mathrm{op}} \rightarrow \mathrm{Ass}$.

Our main goal in this paper is to prove a more general version of such comparisons. Before we explain this result in more detail, let us motivate it by (informally) stating the two main new comparisons we will apply it to:

- In the definition of symmetric ∞ -operads, we can equivalently replace the category $\mathbb{F}_*$ of finite pointed sets by the $(2,1)$ -category $\text{Span}(\mathbb{F})$ of *spans* of finite sets.
- For G a finite group, the G -equivariant ∞ -operads of Nardin and Shah [26] can equivalently be described as ∞ -operads over the $(2,1)$ -category $\text{Span}(\mathbb{F}_G)$ of spans of finite G -sets.

Fibrous patterns The general version of our main result is in the setting of *algebraic patterns* in the sense of Chu and Haugseng [7], which is a general framework for algebraic structures described by “Segal conditions”. More precisely, an algebraic pattern is an ∞ -category $\mathcal{O}$ equipped with a factorization system $(\mathcal{O}^{\text{int}}, \mathcal{O}^{\text{act}})$ of “inert” and “active” morphisms and a full subcategory $\mathcal{O}^{\text{el}} \subset \mathcal{O}^{\text{int}}$ of “elementary” objects. This data lets one define *Segal $\mathcal{O}$ -objects* in a complete ∞ -category $\mathcal{C}$ as functors $F: \mathcal{O} \rightarrow \mathcal{C}$ such that for any object $O \in \mathcal{O}$ the natural map

$$F(O) \rightarrow \lim_{E \in \mathcal{O}^{\text{el}}_{/O}} F(E)$$

is an equivalence, where $\mathcal{O}^{\text{el}}_{/O} := \mathcal{O}^{\text{el}} \times_{\mathcal{O}^{\text{int}}} \mathcal{O}^{\text{int}}_{/O}$ consists of inert morphisms from O to elementary objects. We can then consider a version of ∞ -operads where the category $\mathbb{F}_*$ is replaced by an arbitrary algebraic pattern $\mathcal{O}$; we will refer to them as *fibrous $\mathcal{O}$ -patterns*.¹ Such a fibrous $\mathcal{O}$ -pattern can be defined as a functor $\pi: \mathcal{P} \rightarrow \mathcal{O}$ such that:

- (1) $\mathcal{P}$ has all π -cocartesian lifts of inert morphisms in $\mathcal{O}$.
- (2) For all $O \in \mathcal{O}$, the commutative square of ∞ -categories

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\text{el}}_{/O}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/E} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\text{act}}_{/O} & \longrightarrow & \lim_{E \in \mathcal{O}^{\text{el}}_{/O}} \mathcal{O}^{\text{act}}_{/E} \end{array}$$

is cartesian. This square is constructed in Definition 4.1.2 using the factorization system and the cocartesian lifts from (1).²

The ∞ -category $\text{Fbrs}(\mathcal{O})$ of fibrous $\mathcal{O}$ -patterns is then defined as the subcategory of $\text{Cat}_{\infty/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$ -patterns and whose morphisms are required to preserve cocartesian morphisms over inert maps in $\mathcal{O}$.

¹Under the mild technical assumption that $\mathcal{O}$ is *sound*, our definition of fibrous $\mathcal{O}$ -patterns agrees with the definition of *weak Segal $\mathcal{O}$ -fibrations* studied in [7]; see Proposition 4.1.7. However, we prove some results beyond this case, and here the notion of fibrous $\mathcal{O}$ -pattern we introduce is better behaved for our purposes.

²The bottom horizontal functor is induced by the functors $\alpha_1: \mathcal{O}^{\text{act}}_{/O} \rightarrow \mathcal{O}^{\text{act}}_{/E}$ that are defined for an inert map $\alpha: O \rightarrowtail E$ by sending $\omega: X \rightarrow O$ to the active part of the factorization $\alpha \circ \omega: X \rightarrowtail \alpha_1 X \rightarrowtail E$. The top horizontal functor is defined similarly, by using the cocartesian lifts for inerts.

Let us mention a few examples of algebraic patterns where the corresponding notion of fibrous pattern has already been studied:

- If we take $\mathbb{F}_*$ with the classes of inert and active maps defined as in [23] (see Example 3.1.3) and $\langle 1 \rangle := (\{0, 1\}, 0)$ as the only elementary object, then a fibrous $\mathbb{F}_*$ -pattern is a functor $\pi : \mathcal{P} \rightarrow \mathbb{F}_*$ that has cocartesian lifts for inerts and for which the functor

$$\mathcal{P}^{\text{act}} \times_{\mathbb{F}} \mathbb{F}_{/\langle n \rangle} \simeq \mathcal{P} \times_{\mathbb{F}_*} (\mathbb{F}_*)^{\text{act}}_{/\langle n \rangle} \rightarrow \prod_{\langle n \rangle \twoheadrightarrow \langle 1 \rangle} \mathcal{P} \times_{\mathbb{F}_*} \mathbb{F} \simeq (\mathcal{P}^{\text{act}})^n$$

is an equivalence. We will show in Proposition 4.1.7 that this is precisely equivalent to $\mathcal{P}$ being a (symmetric) ∞ -operad in the sense of Lurie.

- If $\mathcal{O} \rightarrow \mathbb{F}_*$ is an ∞ -operad in the sense of Lurie, then it has a canonical pattern structure for which a fibrous $\mathcal{O}$ -pattern $\pi : \mathcal{P} \rightarrow \mathcal{O}$ is simply an ∞ -operad over $\mathcal{O}$:

$$\text{Fbrs}(\mathcal{O}) \simeq \text{Fbrs}(\mathbb{F}_*)_{/\mathcal{O}} = (\text{Opd}_{\infty})_{/\mathcal{O}}.$$

- Let $\mathbb{F}_*^{\natural}$ denote the algebraic pattern with underlying category $\mathbb{F}_*$ and the same factorization system as before, but with both $\langle 0 \rangle$ and $\langle 1 \rangle$ as elementary objects. Then a fibrous $\mathbb{F}_*^{\natural}$ -pattern is a *generalized ∞ -operad* in the sense of [23].
- If we equip Δ^{op} with the usual inert-active factorization system (see Example 3.1.4) and take $[1]$ as the only elementary object, then a fibrous Δ^{op} -pattern is precisely a planar or nonsymmetric ∞ -operad as in [23]. If we instead take both $[0]$ and $[1]$ as elementary we get *generalized nonsymmetric ∞ -operads* as in [11].
- For a finite group G , the G - ∞ -operads of [26] are precisely fibrous $\mathbb{F}_{G,*}$ -patterns for a certain pattern $\mathbb{F}_{G,*}$ (see Section 5.2).

Comparing fibrous patterns Our first main theorem lets us compare fibrous patterns over various bases:

Theorem A *Let $f : \mathcal{O} \rightarrow \mathcal{P}$ be a morphism of algebraic patterns (ie a functor that preserves active and inert morphisms and elementary objects). Suppose furthermore that:*

- The induced functors $\mathcal{O}_{O/}^{\text{el}} \rightarrow \mathcal{P}_{f(O)/}^{\text{el}}$ are coinital for all $O \in \mathcal{O}$.*
- The pattern $\mathcal{P}$ is **sound** in the sense of Definition 3.3.4.*
- The pattern $\mathcal{P}$ is **extendable**: for all $P \in \mathcal{P}$ the canonical functor*

$$\mathcal{P}_{/P}^{\text{act}} \rightarrow \lim_{E \in \mathcal{P}_{P/}^{\text{el}}} \mathcal{P}_{/E}^{\text{act}},$$

is an equivalence.

- The restriction $f^{\text{el}} : \mathcal{O}^{\text{el}} \rightarrow \mathcal{P}^{\text{el}}$ of f is an equivalence of ∞ -categories,*
- The functor $(\mathcal{O}_{/O}^{\text{act}})^{\simeq} \rightarrow (\mathcal{P}_{/f(O)}^{\text{act}})^{\simeq}$ induced by f is an equivalence for all $O \in \mathcal{O}$.*

Then pullback along f gives an equivalence

$$f^* : \text{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \text{Fbrs}(\mathcal{O}).$$

Here the condition of *soundness* is a mild but rather technical assumption, which is satisfied in almost all examples of algebraic patterns we are aware of. We can now state the applications of Theorem A that we mentioned above more precisely:

Corollary B *Let G be a finite group and $\text{Span}(\mathbb{F}_G)$ the $(2, 1)$ -category of spans of finite G -sets; we regard this as an algebraic pattern where the inert and active maps are the backwards and forwards maps, respectively, and the elementary objects are the orbits G/H for H a subgroup of G . There is a functor $\mathbb{F}_{G,*} \rightarrow \text{Span}(\mathbb{F}_G)$ such that pullback along it gives an equivalence*

$$\text{Fbrs}(\text{Span}(\mathbb{F}_G)) \xrightarrow{\cong} \text{Fbrs}(\mathbb{F}_{G,*}) = \text{Opd}_{G,\infty}.$$

If we restrict to those fibrous patterns that are also cocartesian fibrations (these are Segal fibrations, or equivalently Segal objects in Cat_∞) then we recover [26, Theorem 2.3.9] of Nardin and Shah, which says that the ∞ -category of G -symmetric monoidal ∞ -categories is equivalent to the ∞ -category of product-preserving functors $\text{Span}(\mathbb{F}_G) \rightarrow \text{Cat}_\infty$.

In the case of the trivial group $G = \{e\}$, Corollary B yields an equivalence

$$\text{Fbrs}(\text{Span}(\mathbb{F})) \xrightarrow{\sim} \text{Fbrs}(\mathbb{F}_*) = \text{Opd}_\infty$$

between fibrous $\text{Span}(\mathbb{F})$ -patterns and ∞ -operads in the sense of Lurie, given by pulling back along the inclusion of $\mathbb{F}_*$ in $\text{Span}(\mathbb{F})$ as the wide subcategory containing the spans whose backwards map is injective.

Segal envelopes The crux of our strategy for proving Theorem A is a reduction to a comparison between Segal objects in Cat_∞ for the two patterns. For this purpose we need to develop an analogue of Lurie's *symmetric monoidal envelope* for ∞ -operads over a general algebraic pattern.

A symmetric monoidal ∞ -category can be viewed both as a commutative monoid in Cat_∞ (ie a Segal object for $\mathbb{F}_*$) and as an ∞ -operad that is a cocartesian fibration; we thus have a (nonfull) subcategory inclusion $\text{CMon}(\text{Cat}_\infty) \rightarrow \text{Opd}_\infty$. In [23, Section 2.2.4], Lurie shows that this functor has a left adjoint, the symmetric monoidal envelope, which admits a very explicit description as a cocartesian fibration: the envelope of an ∞ -operad $\mathcal{O}$ is simply the fiber product $\mathcal{O} \times_{\mathbb{F}_*} \text{Ar}_{\text{act}}(\mathbb{F}_*)$ where $\text{Ar}_{\text{act}}(\mathbb{F}_*)$ is the full subcategory of the arrow category of $\mathbb{F}_*$ on the active morphisms and the fiber product is over the source functor $\mathbb{F}^{\text{II}} := \text{Ar}_{\text{act}}(\mathbb{F}_*) \rightarrow \mathbb{F}_*$, while the projection to $\mathbb{F}_*$ giving the symmetric monoidal ∞ -category is by the target functor. Moreover, it was observed in [19] that if we instead regard the envelope as a functor to symmetric monoidal ∞ -categories over $(\mathbb{F}, \text{II})$ (that is, finite sets with the disjoint union as symmetric monoidal structure) then it is fully faithful. We want to generalize these results to fibrous $\mathcal{O}$ -patterns for a general algebraic pattern $\mathcal{O}$. To simplify exposition we assume here that $\mathcal{O}$ is both sound and extendable. For such $\mathcal{O}$, unstraightening restricts to give a functor $\text{Seg}_{\mathcal{O}}(\text{Cat}_\infty) \rightarrow \text{Fbrs}(\mathcal{O})$ analogous to the inclusion $\text{CMon}(\text{Cat}_\infty) \rightarrow \text{Opd}_\infty$. Our second main result is a description of the left adjoint of this functor.

Theorem C *Let $\mathcal{O}$ be a sound and extendable pattern. Then:*

- (1) *The unstraightening functor $\text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty}) \rightarrow \text{Fbrs}(\mathcal{O})$ has a left adjoint $\text{Env}_{\mathcal{O}}$ whose value on a fibrous $\mathcal{O}$ -pattern $\mathcal{P}$ is given by the functor $O \mapsto \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}}$.*
- (2) *Slicing $\text{Env}_{\mathcal{O}}$ over $\mathcal{A}_{\mathcal{O}} := \text{Env}_{\mathcal{O}}(\mathcal{O})$ yields a fully faithful embedding*

$$\text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} : \text{Fbrs}(\mathcal{O}) \hookrightarrow \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}},$$

which admits both a left and a right adjoint.

- (3) *An object $\mathcal{C} \rightarrow \mathcal{A}_{\mathcal{O}}$ in $\text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}}$ lies in the essential image of $\text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$ if and only if it is **$\text{Ar}_{\text{act}}(\mathcal{O})$ -equifibered**, ie for every active map $O \rightsquigarrow O'$ in $\mathcal{O}$, the square*

$$\begin{array}{ccc} \mathcal{C}(O) & \xrightarrow{\mathcal{C}(\omega)} & \mathcal{C}(O') \\ \downarrow & & \downarrow \\ \mathcal{O}_{/O}^{\text{act}} & \xrightarrow{\omega_*} & \mathcal{O}_{/O'}^{\text{act}} \end{array}$$

is cartesian.

In Section 4.2 we actually prove more general (but weaker) versions of this statement that do not require $\mathcal{O}$ to be sound or extendable. The comparison of Theorem A can now be shown by recalling a (simpler) comparison theorem for Segal objects from [2], passing to slices and then showing that the equivalence restricts to the essential image of the envelope.

In Section 4.3 we spell out Theorem C in several examples. In particular, for $\mathcal{O} = \mathbb{F}_*$, Theorem C recovers a result of [19], though with an alternative characterization of the image:³

Corollary D *The left adjoint to the forgetful functor $\text{CMon}(\text{Cat}_{\infty}) \rightarrow \text{Opd}_{\infty}$ lifts to a fully faithful functor:*

$$\text{Env} : \text{Opd}_{\infty} \hookrightarrow \text{CMon}(\text{Cat}_{\infty})_{/(\mathbb{F}, \mathbb{I})}.$$

This functor has adjoints on both sides. A symmetric monoidal functor $\pi : (\mathcal{C}, \otimes) \rightarrow (\mathbb{F}, \mathbb{I})$ is in the essential image of Env if and only if the square

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ \pi \times \pi \downarrow & & \downarrow \pi \\ \mathbb{F} \times \mathbb{F} & \xrightarrow{\mathbb{I}} & \mathbb{F} \end{array}$$

is a pullback square in Cat_{∞} .

In Section 5.2 we also give a similar characterization of the essential image of the envelope for G - ∞ -operads, though in that case one has to require additional pullback squares involving the norm maps $\text{Nm}_K^H : \mathcal{C}^K \rightarrow \mathcal{C}^H$.

³See Observation 4.3.2 for a comparison.

Organization In Section 2 we prove a key part of Theorem C, which only depends on the factorization system on an algebraic pattern:

Theorem E *Let $\mathcal{B}$ be an ∞ -category with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$.*

- (1) *The forgetful functor $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ has a left adjoint, which takes $\mathcal{E} \rightarrow \mathcal{B}$ to $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$, where $\mathrm{Ar}_R(\mathcal{B})$ is the full subcategory of $\mathrm{Ar}(\mathcal{B}) := \mathrm{Fun}([1], \mathcal{B})$ spanned by the morphisms in $\mathcal{B}_R$, the fiber product is over evaluation at $0 \in [1]$, and the projection to $\mathcal{B}$ uses evaluation at 1.*
- (2) *The induced functor $\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightarrow (\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})}$ is fully faithful, and a morphism $\mathcal{E} \rightarrow \mathrm{Ar}_R(\mathcal{B})$ in $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}}$ lies in the image of $\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ if and only if it is $\mathrm{Ar}_R(\mathcal{B})$ -**equifibered**, meaning that for every $\varphi: a \rightarrow b$ in $\mathrm{Ar}_R(\mathcal{B})$ the commutative square*

$$\begin{array}{ccc} \mathcal{E}_a & \xrightarrow{\varphi!} & \mathcal{E}_b \\ \downarrow & & \downarrow \\ (\mathcal{B}_R)_{/a} & \xrightarrow{\varphi!} & (\mathcal{B}_R)_{/b} \end{array}$$

is cartesian.

We emphasize that only the second point here is actually new — the first point has already been proved by both Ayala, Mazel-Gee, and Rozenblyum [1] and Shah [31].

We then review algebraic patterns in Section 3, where we also introduce the condition of soundness for patterns. In Section 4 we define fibrous patterns, specialize Theorem E to this context to prove Theorem C, and explore several examples. We are then ready to prove Theorem A in Section 5, where we also discuss the applications and an $(\infty, 2)$ -categorical version of Theorem A.

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2 Envelopes for factorization systems

Our goal in this section is to prove Theorem E. We begin in Section 2.1 by explicitly describing the general procedure of freely adding cocartesian morphisms over a wide subcategory $\mathcal{B}_0$ of $\mathcal{B}$ to a functor $p: \mathcal{E} \rightarrow \mathcal{B}$, and then in Section 2.2 we specialize this to the situation where $\mathcal{B}_0$ is the right class of a factorization system and $\mathcal{E}$ already has p -cocartesian morphisms over the left class. As already mentioned, these results are not new, but we include complete proofs to make the paper more self-contained. In Section 2.3 we

then prove the new part of Theorem E: we observe that for the induced adjunction on slices the left adjoint is fully faithful, and identify its image.

2.1 Adding cocartesian morphisms over a subcategory

Let $\mathcal{B}$ be an ∞ -category equipped with a wide subcategory $\mathcal{B}_0$, and write $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ for the subcategory of $\text{Cat}_{\infty/\mathcal{B}}$ whose objects have all cocartesian lifts of morphisms in $\mathcal{B}_0$ and whose morphisms preserve these. The aim of this subsection is to show that the forgetful functor

$$\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$$

admits an (explicitly defined) left adjoint. Before explaining the construction of the left adjoint, let us first fix some notation: We let $\text{Ar}(\mathcal{B}) := \text{Fun}([1], \mathcal{B})$ denote the arrow ∞ -category of $\mathcal{B}$, and write $\text{Ar}_0(\mathcal{B})$ for the full subcategory of $\text{Ar}(\mathcal{B})$ spanned by morphisms in $\mathcal{B}_0$. The left adjoint of the forgetful functor above is then given by

$$(\mathcal{E} \rightarrow \mathcal{B}) \mapsto (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{B}),$$

where the fiber product is over $\text{ev}_0: \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{B}$, and the map $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{B}$ is given by ev_1 . We will prove this by showing that for any $\mathcal{E} \in \text{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F} \in \text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, restriction yields a natural equivalence

$$\text{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \text{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}),$$

where the left-hand side consists of functors that preserve cocartesian morphisms over $\mathcal{B}_0$. This result is by no means new, and has already appeared in [1; 31], but we include a proof for completeness, as this is the key input needed for our work in this paper.

Notation 2.1.1 Since $\mathcal{B}_0$ is a wide subcategory, the degeneracy map $s_0^*: \mathcal{B} \rightarrow \text{Ar}(\mathcal{B})$ restricts to a functor $i: \mathcal{B} \rightarrow \text{Ar}_0(\mathcal{B})$, taking an object of $\mathcal{B}$ to its identity map. We also have evaluation maps $\text{ev}_0, \text{ev}_1: \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{B}$, and natural transformations $\sigma: i \circ \text{ev}_0 \rightarrow \text{id}$ and $\tau: \text{id} \rightarrow i \circ \text{ev}_1$, given for an object $x \xrightarrow{\varphi} y$ by the squares

$$\begin{array}{ccc} x & \xlongequal{\quad} & x \\ \parallel & & \downarrow \varphi \\ x & \xrightarrow{\varphi} & y \end{array} \quad \text{and} \quad \begin{array}{ccc} x & \xrightarrow{\varphi} & y \\ \downarrow \varphi & & \parallel \\ y & \xlongequal{\quad} & y \end{array}$$

respectively. For any functor $p: \mathcal{E} \rightarrow \mathcal{B}$, the functor i induces a section $i_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$ of the projection $\text{pr}_{\mathcal{E}}: \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{E}$, and σ induces a natural transformation $\sigma_{\mathcal{E}}: i_{\mathcal{E}} \text{pr}_{\mathcal{E}} \rightarrow \text{id}$.

Observation 2.1.2 Suppose $p: \mathcal{E} \rightarrow \mathcal{B}$ is cocartesian over $\mathcal{B}_0$. Then $i_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$ has a left adjoint $\pi_{\mathcal{E}}$, as we will show now (see [29, Theorem 5.2.8.(iii)] for the special case where $\mathcal{B}_0 = \mathcal{B}$): Such an adjoint exists if and only if, given an object $(x, \varphi: px \rightarrow b)$, there is an initial object in the ∞ -category

$$\mathcal{E}_{(x, \varphi)/} := \mathcal{E} \times_{\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})} (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}))_{(x, \varphi)/} \simeq \mathcal{E}_x / \times_{\mathcal{B}_{px/}} \mathcal{B}_b /$$

with the functor $\mathcal{B}_{b/} \rightarrow \mathcal{B}_{px/}$ given by composition with φ . A cocartesian morphism $x \rightarrow \varphi_!x$ is precisely an initial object in the right-hand side that maps to the identity in $\mathcal{B}_{b/}$. Thus $\pi_{\mathcal{E}}$ takes $(x, \varphi: px \rightarrow b)$ to the target $\varphi_!x$ of the cocartesian morphism over φ . Note that we have $\pi_{\mathcal{E}}i_{\mathcal{E}} \simeq \text{id}$, and the unit transformation $\text{id} \rightarrow i_{\mathcal{E}}\pi_{\mathcal{E}}$ is given at (x, φ) by

$$\left(\begin{array}{ccc} x & px & \xrightarrow{\varphi} b \\ \downarrow & \downarrow \varphi & \parallel \\ \varphi_!x & b & \xlongequal{\quad} b \end{array} \right).$$

Moreover, this is an adjunction over $\mathcal{B}$ in the sense that we have commutative squares

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\iota_{\mathcal{E}}} & \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \\ \downarrow p & & \downarrow \text{ev}_1 \\ \mathcal{B} & \xlongequal{\quad} & \mathcal{B} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) & \xrightarrow{\pi_{\mathcal{E}}} & \mathcal{E} \\ \downarrow \text{ev}_1 & & \downarrow p \\ \mathcal{B} & \xlongequal{\quad} & \mathcal{B}. \end{array}$$

For the left square this holds by construction and for the right square we have a transformation $p \circ \pi_{\mathcal{E}} = \text{ev}_1 \circ \iota_{\mathcal{E}} \circ \pi_{\mathcal{E}} \rightarrow \text{ev}_1$ coming from the counit of $\pi_{\mathcal{E}} \dashv \iota_{\mathcal{E}}$. This is an equivalence by the pointwise description of $\pi_{\mathcal{E}}$ given above.

Observation 2.1.3 Given $p: \mathcal{E} \rightarrow \mathcal{B}$, observe that $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$ is cocartesian over $\mathcal{B}_0$, with cocartesian morphisms given by composition in $\text{Ar}_0(\mathcal{B})$. (For instance, we can write $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$ as a pullback $(\mathcal{E} \times \mathcal{B}) \times_{(\mathcal{B} \times \mathcal{B})} \text{Ar}_0(\mathcal{B})$ over $\mathcal{B}$, where all three ∞ -categories appearing are cocartesian over $\mathcal{B}_0$.)

Proposition 2.1.4 If $q: \mathcal{F} \rightarrow \mathcal{B}$ is cocartesian over $\mathcal{B}_0$, composition with $i_{\mathcal{E}}$ gives a functor

$$\text{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}), \mathcal{F}) \rightarrow \text{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}).$$

This is an equivalence, with inverse given by taking $F: \mathcal{E} \rightarrow \mathcal{F}$ to the composite

$$\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \xrightarrow{F \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})} \mathcal{F} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}.$$

Proof Given $G: \mathcal{E} \rightarrow \mathcal{F}$, the definitions of the sections $i_{\mathcal{E}}$ and $i_{\mathcal{F}}$ give

$$(G \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq i_{\mathcal{F}} \circ G,$$

and so we have

$$\pi_{\mathcal{F}} \circ (G \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq \pi_{\mathcal{F}} \circ i_{\mathcal{F}} \circ G \simeq G.$$

In the other direction, given $F: \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_0$, we have to show that F is naturally equivalent to $\pi_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}))$. Here we can write $\text{pr}_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}))$ as the composite

$$\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \xrightarrow{\text{pr}_{\mathcal{E}}} \mathcal{E} \xrightarrow{i_{\mathcal{E}}} \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \xrightarrow{F} \mathcal{F},$$

so that $\sigma_{\mathcal{E}}$ induces a natural transformation

$$\alpha: \text{pr}_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})) \rightarrow F.$$

Note that this is given at $(e, \varphi: p(e) \rightarrow b)$ by the image $F(e, \text{id}_{p(e)}) \rightarrow F(e, \varphi)$ of a cocartesian morphism in $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$, and so is cocartesian in $\mathcal{F}$ since by assumption F preserves cocartesian morphisms over $\mathcal{B}_0$. Projecting to $\mathcal{B}$, we see that $q\alpha$ factors as the projection to $\text{Ar}_0(\mathcal{B})$ followed by the evaluation map $\text{Ar}_0(\mathcal{B}) \times [1] \rightarrow \mathcal{B}$. We can therefore define a natural transformation

$$\beta: \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \times [1] \rightarrow \mathcal{F} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$$

via the commutative diagram

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \times [1] & \xrightarrow{\alpha} & \mathcal{F} \\ \downarrow & & \downarrow q \\ \text{Ar}_0(\mathcal{B}) \times [1] & \searrow \text{ev} & \downarrow \\ \downarrow \tau & & \downarrow \\ \text{Ar}_0(\mathcal{B}) & \xrightarrow{s} & \mathcal{B}. \end{array}$$

Here β is a natural transformation $(Fi_{\mathcal{E}} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})) \rightarrow i_{\mathcal{F}}F$, and takes

$$(e, \varphi: p(e) \rightarrow b) \quad \text{to} \quad (F(e, \text{id}_{p(e)}), \varphi) \rightarrow (F(e, \varphi), \text{id}_c).$$

Composing with $\pi_{\mathcal{F}}$ we get a natural transformation $\pi_{\mathcal{F}}\beta: \pi_{\mathcal{F}} \circ (Fi_{\mathcal{E}} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})) \rightarrow \pi_{\mathcal{F}}i_{\mathcal{F}}F \simeq F$. This is given at (e, φ) by the canonical morphism $\varphi!F(e, \text{id}) \rightarrow F(e, \varphi)$. Since F preserves cocartesian morphisms over $\mathcal{B}_0$, this is an equivalence, and so we have obtained the natural equivalence we required. $\square$

Corollary 2.1.5 *The forgetful functor*

$$\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$$

has a left adjoint given by

$$(\mathcal{E} \rightarrow \mathcal{B}) \mapsto \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) = s^*\mathcal{E} \rightarrow \text{Ar}_0(\mathcal{B}) \xrightarrow{t} \mathcal{B},$$

and unit given by $i_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B})$.

Proof By Proposition 2.1.4, for $\mathcal{E} \in \text{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F} \in \text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ the composite

$$\text{Map}_{\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}), \mathcal{F}) \rightarrow \text{Map}_{\text{Cat}_{\infty/\mathcal{B}}}(\mathcal{E} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}), \mathcal{F}) \xrightarrow{i_{\mathcal{E}}^*} \text{Map}_{\text{Cat}_{\infty/\mathcal{B}}}(\mathcal{E}, \mathcal{F})$$

is an equivalence, hence this natural transformation is indeed the unit of an adjunction. $\square$

Observation 2.1.6 The forgetful functors $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}} \rightarrow \text{Cat}_{\infty}$ detect pullbacks; in particular, the ∞ -category $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ has all pullbacks. Indeed, given morphisms $\mathcal{E}_1 \rightarrow \mathcal{E}_0 \leftarrow \mathcal{E}_2$ in $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, it is easy to see that a morphism in the fiber product $\mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_2$ is cocartesian over $\mathcal{B}_0$ if and only if its images in $\mathcal{E}_1$ and $\mathcal{E}_2$ are cocartesian.

Observation 2.1.7 Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞ -categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f: \mathcal{A} \rightarrow \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Pullback along f clearly gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} & \xrightarrow{f^*} & \mathrm{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}} \\ \downarrow & & \downarrow \\ \mathrm{Cat}_{\infty/\mathcal{B}} & \xrightarrow{f^*} & \mathrm{Cat}_{\infty/\mathcal{A}}. \end{array}$$

We then have an induced Beck–Chevalley transformation between the left adjoints of the vertical maps, given for $p: \mathcal{E} \rightarrow \mathcal{B}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \mathrm{Ar}_0(\mathcal{A}) \rightarrow (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B})) \times_{\mathcal{B}} \mathcal{A},$$

which takes $(e \in \mathcal{E}, a \in \mathcal{A}, p(e) \simeq f(a), \varphi: a \rightarrow a' \in \mathrm{Ar}_0(\mathcal{A}))$ to $(e, f(a), f(\varphi), a)$. Note, however, that this is typically not an equivalence.

2.2 Free fibrations for factorization systems

In this subsection we specialize our previous results to the case of an ∞ -category equipped with a factorization system. We again emphasize that this result already appears in [1; 31].

Notation 2.2.1 In this section we fix an ∞ -category $\mathcal{B}$ with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$; we write $\mathrm{Ar}_L(\mathcal{B})$ and $\mathrm{Ar}_R(\mathcal{B})$ for the full subcategories of $\mathrm{Ar}(\mathcal{B})$ spanned by the morphisms in $\mathcal{B}_L$ and $\mathcal{B}_R$, respectively. We also abbreviate

$$\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} := \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_L\text{-cocart}}$$

Proposition 2.2.2 [7, Proposition 7.3] *Let $(q: \mathcal{C} \rightarrow \mathcal{B}) \in \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$. Then:*

- (1) *The functor $q': \mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$ given by evaluation at the target is a cocartesian fibration.*
- (2) *A morphism $(\alpha, \beta): (c_0, \varphi_0) \rightarrow (c_1, \varphi_1)$ in $\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$ represented by the diagram*

$$\left(\begin{array}{ccccc} c_0 & q(c_0) & \xrightarrow{\varphi_0} & b_0 & \\ \downarrow \alpha & \downarrow q(\alpha) & & \downarrow \beta & \\ c_1 & q(c_1) & \xrightarrow{\varphi_1} & b_1 & \end{array} \right)$$

is a q' -cocartesian lift of $\beta: b_0 \rightarrow b_1$ if and only if $q(\alpha)$ is in $\mathcal{B}_L$ and α is q -cocartesian.

Proof We first show that q' is a locally cocartesian fibration. A locally q' -cocartesian morphism over $\beta: b_0 \rightarrow b_1$ with source $(c_0, \varphi_0: q(c_0) \rightarrow b_0)$ is an initial object in the ∞ -category

$$(\mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_{(c_0, \varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}.$$

We can identify this ∞ -category as the fiber product

$$\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} (\mathcal{B}_{/b_1}^R \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}),$$

where $\mathcal{B}_{/b_1}^R$ denotes the full subcategory of $\mathcal{B}_{/b_1}$ spanned by morphisms in $\mathcal{B}_R$.

We first observe that here $\mathcal{B}_{/b_1}^R \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$ has an initial object, given by

$$\begin{array}{ccc} qc_0 & \xrightarrow{\lambda} & b' \\ & \searrow \beta\varphi_0 & \swarrow \rho \\ & b_1 & \end{array}$$

where (λ, ρ) is the (L, R) -factorization of $\beta\varphi_0$ — this follows from [22, Lemma 5.2.8.19].

The projection $\mathcal{B}_{/b_1}^R \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \rightarrow \mathcal{B}_{/b_1}^R$ is a left fibration, since it is a base change of the left fibration $(\mathcal{B}_{/b_1})_{\beta\varphi_0/} \rightarrow \mathcal{B}_{/b_1}$. The initial object of $\mathcal{B}_{/b_1}^R \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$, which maps to ρ in $\mathcal{B}_{/b_1}^R$, therefore gives an equivalence

$$\mathcal{B}_{/b_1}^R \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \simeq (\mathcal{B}_{/b_1}^R)_{\rho/}$$

by [24, Tag 0199]. We can thus rewrite our expression for the ∞ -category $(\mathcal{C} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))_{(c_0, \varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}$ as

$$(\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}) \times_{\mathcal{B}_{b'/}} (\mathcal{B}_{/b_1}^R)_{\rho/}.$$

A q -cocartesian morphism over λ with source c_0 , which exists by assumption since λ is in $\mathcal{B}_L$, is precisely an initial object of $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ that maps to the initial object in $\mathcal{B}_{b'/}$. We thus have initial objects in $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ and $(\mathcal{B}_{/b_1}^R)_{\rho/}$ that both map to the initial object in $\mathcal{B}_{b'/}$, and these thus give an initial object in the fiber product $(\mathcal{C} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))_{(c_0, \varphi_0)/}$. This shows that if $\alpha: c_0 \rightarrow c_1$ is a q -cocartesian lift of λ , then

$$\left(\begin{array}{ccc} c_0 & q(c_0) \xrightarrow{\varphi_0} & b_0 \\ \downarrow \alpha & \downarrow \lambda & \downarrow \beta \\ c_1 & b' \xrightarrow{\rho} & b_1 \end{array} \right)$$

is a locally q -cocartesian lift of β with source (c_0, φ_0) .

We have thus shown that q' is a locally cocartesian fibration, and the locally q' -cocartesian morphisms are precisely those in (2). To see that q' is a cocartesian fibration it then suffices by [22, Proposition 2.4.2.8] to check that the locally q' -cocartesian morphisms are closed under composition, which in our case is clear. $\square$

Notation 2.2.3 It follows from Proposition 2.2.2 that the construction $\mathcal{E} \mapsto \mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B})$ restricts to a well-defined functor

$$\mathbf{E}: \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}, \quad (\mathcal{E} \rightarrow \mathcal{B}) \mapsto (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}).$$

Proposition 2.2.4 *Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a functor admitting cocartesian lifts for all arrows in $\mathcal{B}_L$ and let $q: \mathcal{F} \rightarrow \mathcal{B}$ be a cocartesian fibration. Then the equivalence of Proposition 2.1.4 restricts to an equivalence*

$$\mathrm{Fun}_{/\mathcal{B}}^{\mathrm{cocart}}(\mathcal{E}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathrm{Fun}_{/\mathcal{B}}^{L\text{-cocart}}(\mathcal{E}, \mathcal{F}).$$

Proof We must show that these full subcategories are identified under the equivalence

$$\mathrm{Fun}_{/\mathcal{B}}^{R\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \mathrm{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})$$

of Proposition 2.1.4. Given a functor $F: \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \rightarrow \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_R$, we must thus check that F preserves all cocartesian morphisms if and only if $F \circ i_{\mathcal{E}}$ preserves cocartesian morphisms over $\mathcal{B}_L$. We write $p': \mathcal{E} \times_{\mathcal{C}} \mathrm{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$ for the map induced by ev_1 .

First, assume that $F: \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \rightarrow \mathcal{F}$ preserves all cocartesian edges. For a p -cocartesian lift $\alpha: c_0 \rightarrow c_1$ of an edge $\beta: b_0 \rightarrow b_1$ in $\mathcal{B}_L$, its image under $i_{\mathcal{E}}$ is the edge

$$\left(\begin{array}{ccc} c_0 & b_0 & \xrightarrow{=} b_0 \\ \downarrow \alpha, & \downarrow \beta & \\ c_1 & b_1 & \xrightarrow{=} b_1 \end{array} \right)$$

in $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$, which is p' -cocartesian by Proposition 2.2.2. In other words, $i_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$ preserves cocartesian lifts over $\mathcal{B}_L$, and hence so does $F \circ i_{\mathcal{E}}$.

For the converse assume that F preserves cocartesian lifts of edges in $\mathcal{B}_R$ and $F \circ i_{\mathcal{E}}$ preserves cocartesian lifts of edges in $\mathcal{B}_L$. We would like to show that a general p' -cocartesian morphism $(\alpha, \beta): (c_0, \varphi_0) \rightarrow (c_1, \varphi_1)$ is sent to a q -cocartesian morphism in $\mathcal{F}$. According to Proposition 2.2.2, the morphism $p(\alpha)$ is in $\mathcal{B}_L$ and α is p -cocartesian. We can fit this morphism into the following diagram by applying the natural transformation $\sigma_{\mathcal{E}}: i_{\mathcal{E}} \mathrm{pr}_{\mathcal{E}} \rightarrow \mathrm{id}$:

$$\begin{array}{ccc} (c_0, \mathrm{id}) & \xrightarrow{(\mathrm{id}, \varphi_0) = (\sigma_{\mathcal{E}})_{(c_0, \varphi_0)}} & (c_0, \varphi_0) \\ (\alpha, q(\alpha)) \downarrow & & \downarrow (\alpha, \beta) \\ (c_1, \mathrm{id}) & \xrightarrow{(\mathrm{id}, \varphi_1) = (\sigma_{\mathcal{E}})_{(c_1, \varphi_1)}} & (c_1, \varphi_1). \end{array}$$

Both horizontal morphisms are cocartesian edges over $\mathcal{B}_R$ (by Proposition 2.2.2) and the left-hand vertical morphism is the image under $i_{\mathcal{E}}$ of a p -cocartesian morphism over $\mathcal{B}_L$. Hence F sends three of the morphisms in the above square to cocartesian edges in $\mathcal{F}$ and it follows by composition and right-cancellation for cocartesian edges that $F(\alpha, \beta)$ is cocartesian too. $\square$

Corollary 2.2.5 *The adjunction of Corollary 2.1.5 restricts to an adjunction*

$$\mathcal{E}: \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} : \mathrm{forget}.$$

Observation 2.2.6 Suppose $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ are ∞ -categories equipped with factorization systems, and that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a functor that preserves both classes of maps in these. Pullback along f then gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} & \xrightarrow{f^*} & \mathrm{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}} \\ \downarrow & & \downarrow \\ \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} & \xrightarrow{f^*} & \mathrm{Cat}_{\infty/\mathcal{A}}^{\mathrm{cocart}}. \end{array}$$

As in Observation 2.1.7, this induces a Beck–Chevalley transformation, but this is typically not an equivalence.

2.3 Full faithfulness on slices

In this subsection we prove the main new result of this section: We observe that the adjunction of Corollary 2.2.5 induces an adjunction

$$\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}})_{/\mathrm{Ar}_R(\mathcal{B})},$$

where the left adjoint is fully faithful, and characterize its image as in Theorem E.

To construct this adjunction, we recall the general construction of adjunctions on slices:

Observation 2.3.1 Given an adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R,$$

where $\mathcal{C}$ admits pullbacks, we have (by [22, Proposition 5.2.5.1]) for any c in $\mathcal{C}$ an induced adjunction

$$L_c : \mathcal{C}_{/c} \rightleftarrows \mathcal{D}_{/Lc} : R_c,$$

where L_c is simply given by applying L , while R_c is defined at $f : d \rightarrow Lc$ by the natural pullback square

$$\begin{array}{ccc} R_c d & \longrightarrow & Rd \\ \downarrow & & \downarrow Rf \\ c & \xrightarrow{\eta_c} & RLc \end{array}$$

over the unit map η_c . The unit for the new adjunction is then given at $c' \rightarrow c$ by the canonical map $c' \rightarrow R_c L_c c'$ obtained by factoring the square

$$\begin{array}{ccc} c' & \xrightarrow{\eta_{c'}} & RLc' \\ \downarrow & & \downarrow \\ c & \xrightarrow{\eta_c} & RLc \end{array}$$

through the pullback, while the counit $L_c R_c d \rightarrow d$ is given by the outer square in the diagram

$$\begin{array}{ccccc} LR_c d & \longrightarrow & LRd & \xrightarrow{\epsilon_d} & d \\ \downarrow & & \downarrow & & \downarrow \\ Lc & \xrightarrow{L\eta_c} & LRLc & \xrightarrow{\epsilon_{Lc}} & Lc, \end{array}$$

where ϵ is the counit of the original adjunction.

Proposition 2.3.2 *By applying the construction of Observation 2.3.1 to the adjunction of Corollary 2.2.5 at the terminal object $(\mathcal{B} \rightrightarrows \mathcal{B}) \in \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ we obtain an adjunction*

$$(1) \quad E: \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} : Q.$$

The left adjoint in this adjunction is fully faithful.

Proof Here E sends $\mathcal{E} \rightarrow \mathcal{B}$ to the cocartesian fibration $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$, equipped with the canonical projection to $\text{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$. The right adjoint Q is given by

$$\mathcal{E} \rightarrow \text{Ar}_R(\mathcal{B}) \mapsto i^* \mathcal{E} = \mathcal{B} \times_{\text{Ar}_R(\mathcal{B})} \mathcal{E} \rightarrow \mathcal{B},$$

where the pullback is taken along the inclusion of the identities $i: \mathcal{B} \rightarrow \text{Ar}_R(\mathcal{B})$. The unit of this adjunction is then the map $\mathcal{E} \rightarrow Q(E(\mathcal{E}))$ obtained from the commutative square of units for the adjunction $E \dashv \text{forget}$ (from Corollary 2.2.5) as the canonical map from $\mathcal{E}$ to the pullback. This square of units is the left-hand square in the commutative diagram

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{i_{\mathcal{E}}} & E(\text{forget}(\mathcal{E})) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{i} & \text{Ar}_R(\mathcal{B}) & \xrightarrow{\text{ev}_0} & \mathcal{B}, \end{array}$$

where the right-hand square is cartesian by construction of $E(\mathcal{E})$ in Notation 2.2.3. Hence the left-hand square is also cartesian and thus the unit $\mathcal{E} \rightarrow Q(E(\mathcal{E}))$ is an equivalence, and so E is indeed fully faithful. $\square$

Now that we have the fully faithful envelope functor all that is left to do to prove Theorem E is to characterize its essential image:

Proposition 2.3.3 *A morphism $\mathcal{D} \rightarrow \text{Ar}_R(\mathcal{B})$ of cocartesian fibrations over $\mathcal{B}$ is in the essential image of the left adjoint E from Proposition 2.3.2 if and only if it is **equifibered**, meaning that for every object*

$\varphi: a \rightarrow b$ in $\text{Ar}_R(\mathcal{B})$, the natural square

$$\begin{array}{ccc} \mathcal{D}_a & \xrightarrow{\varphi!} & \mathcal{D}_b \\ \downarrow & & \downarrow \\ \text{Ar}_R(\mathcal{B})_a & \xrightarrow{\varphi \circ (-)} & \text{Ar}_R(\mathcal{B})_b \end{array}$$

is cartesian.

Proof We begin with the “only if” direction for $(\mathcal{E} \rightarrow \mathcal{B}) \in \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $(\varphi: a \rightarrow b) \in \text{Ar}_R(\mathcal{B})$. We need to show that the left square of the following diagram is cartesian:

$$\begin{array}{ccccc} & & \text{pr}_{\mathcal{E}} & & \\ & \swarrow & & \searrow & \\ (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))_a & \xrightarrow{\varphi!} & (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))_b & \xrightarrow{\text{pr}_{\mathcal{E}}} & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ar}_R(\mathcal{B})_a & \xrightarrow{\varphi \circ (-)} & \text{Ar}_R(\mathcal{B})_b & \xrightarrow{s} & \mathcal{B} \\ & \searrow & & \swarrow & \\ & & s & & \end{array}$$

where the identification of the composite in the top row uses the description of cocartesian morphisms in $(\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))$ from Proposition 2.2.2. This follows since the right-hand square and the outer rectangle are both cartesian.

For the “if” direction we must show that the counit $E(Q(\mathcal{D})) \rightarrow \mathcal{D}$ is an equivalence if $\mathcal{D}$ is equifibered. By Observation 2.3.1 this counit can be factored as the composite of the top horizontal maps in the diagram

$$(\star) \quad \begin{array}{ccccc} E(Q(\mathcal{D})) & \longrightarrow & E(\mathcal{D}) & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Ar}_R(\mathcal{B}) & \longrightarrow & E(\text{Ar}_R(\mathcal{B})) & \longrightarrow & \text{Ar}_R(\mathcal{B}). \end{array}$$

Here the right-hand horizontal maps come from the counit of the adjunction from Corollary 2.2.5. The bottom horizontal composite is an equivalence, so it will suffice to show that the composite rectangle is cartesian. Since the left-hand square is given by E applied to the cartesian square defining Q (as $\text{Ar}_R(\mathcal{B})$ is $E(\mathcal{B})$), and E preserves weakly contractible limits, it suffices to show that the right-hand square is cartesian.

By assumption, the functor $\mathcal{D} \rightarrow \mathcal{B}$ is a cocartesian fibration, and so the projection

$$E(\mathcal{D}) \simeq \mathcal{D} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) \rightarrow \text{Ar}_R(\mathcal{B})$$

is also a cocartesian fibration, with cocartesian morphisms exactly those that project to cocartesian morphisms in $\mathcal{D}$. Consider now the square

$$\begin{array}{ccc} E(\mathcal{D}) & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \pi \\ \text{Ar}_R(\mathcal{B}) & \xrightarrow{t} & \mathcal{B} \end{array}$$

in which the top map is the counit for the adjunction of Corollary 2.2.5. The top map in the square takes cocartesian morphisms over $\text{Ar}_R(\mathcal{B})$ to π -cocartesian morphisms in $\mathcal{D}$. To see this, note that a cocartesian morphism in $E(\mathcal{D})$ over $\text{Ar}_R(\mathcal{B})$ is of the form

$$\left(\begin{array}{ccc} d & \pi(d) & \xrightarrow{\alpha} b \\ \downarrow & \downarrow \varphi & \downarrow \beta \\ \varphi_! d & a & \xrightarrow{\gamma} b' \end{array} \right),$$

and this is by construction sent to the canonical map $\alpha_! d \rightarrow \gamma_! \varphi_! d$, which is indeed cocartesian over β .

Consequently the top right square of $(\star)$ sits as the top face in the cube

$$\begin{array}{ccccc} E(\mathcal{D}) & \xrightarrow{\quad} & \mathcal{D} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & E(\text{Ar}_R(\mathcal{B})) & \xrightarrow{\quad} & \text{Ar}_R(\mathcal{B}) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{Ar}_R(\mathcal{B}) & \xrightarrow{\quad} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \\ & \downarrow & \downarrow & \downarrow & \\ & \text{Ar}_R(\mathcal{B}) & \xrightarrow{\quad} & \mathcal{B} & \end{array}$$

in which the vertical maps are cocartesian fibrations and the maps in the top square preserve cocartesian morphisms. Since the bottom square is obviously cartesian, to show that the top square is cartesian it suffices to check that taking fibers over any $\varphi \in \text{Ar}_R(\mathcal{B})$ yields a cartesian square. We thus want to show that the following square is cartesian:

$$\begin{array}{ccc} E(\mathcal{D})_\varphi & \xrightarrow{\quad} & \mathcal{D}_b \\ \downarrow & & \downarrow \\ E(\text{Ar}_R(\mathcal{B}))_\varphi & \xrightarrow{\quad} & \text{Ar}_R(\mathcal{B})_b. \end{array}$$

Here there is a canonical equivalence $E(\mathcal{D})_\varphi \simeq (\mathcal{D} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}))_\varphi \simeq \mathcal{D}_a$ and similarly $E(\text{Ar}_R(\mathcal{B}))_\varphi \simeq (\text{Ar}_R(\mathcal{B}))_a$. Via these equivalences the horizontal maps are identified with the cocartesian pushforward along φ . The resulting square is then precisely one of the squares that are cartesian by the assumption that $\mathcal{D}$ is equifibered. $\square$

In Section 4.2 it will be notationally convenient to use a “straightened” version of the adjunction (1); to state this we first introduce some notation:

Notation 2.3.4 Let $\mathcal{B}$ be an ∞ -category equipped with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$. Let $\mathcal{R}: \mathcal{B} \rightarrow \text{Cat}_\infty$ be the straightening of the cocartesian fibration $\text{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$. We define the functor

$$\text{St}_{\mathcal{B}}^L: \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightarrow \text{Fun}(\mathcal{B}, \text{Cat}_\infty)_{/\mathcal{R}},$$

which we think of as a form of “straightening relative to the factorization system”, as the composite

$$\text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \xrightarrow{E} (\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} \xrightarrow{\text{St}_{\mathcal{B}}} \text{Fun}(\mathcal{B}, \text{Cat}_\infty)_{/\mathcal{R}},$$

sending $(p: \mathcal{E} \rightarrow \mathcal{B})$ to the straightening of $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) \rightarrow \mathcal{B}$. Dually, we define

$$\text{Un}_{\mathcal{B}}^L: \text{Fun}(\mathcal{B}, \text{Cat}_{\infty})_{/\mathcal{R}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$$

as the composite

$$\text{Fun}(\mathcal{B}, \text{Cat}_{\infty})_{/\mathcal{R}} \xrightarrow{\text{Un}_{\mathcal{B}}} (\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} \xrightarrow{Q} \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}.$$

For a functor $F: \mathcal{B} \rightarrow \text{Cat}$ together with natural transformation $\alpha: F \rightarrow \mathcal{R}$ we then have that $\text{Un}_{\mathcal{B}}^L(\alpha)$ is the pullback

$$\begin{array}{ccc} \text{Un}_{\mathcal{B}}^L(\alpha) & \longrightarrow & \text{Un}_{\mathcal{B}}(F) \\ \downarrow & \lrcorner & \downarrow \text{Un}_{\mathcal{B}}(\alpha) \\ \mathcal{B} & \longrightarrow & \text{Ar}_R(\mathcal{B}). \end{array}$$

This yields the following reformulation of Theorem E:

Theorem 2.3.5 *The functors $\text{St}_{\mathcal{B}}^L$ and $\text{Un}_{\mathcal{B}}^L$ give an adjunction*

$$\text{St}_{\mathcal{B}}^L: \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows \text{Fun}(\mathcal{B}, \text{Cat}_{\infty})_{/\mathcal{R}}: \text{Un}_{\mathcal{B}}^L.$$

The functor $\text{St}_{\mathcal{B}}^L$ is fully faithful and a natural transformation $F \rightarrow \mathcal{R}$ is in the essential image of $\text{St}_{\mathcal{B}}^L$ if and only if it is **equifibered**, meaning that for every object $a \xrightarrow{\varphi} b$ in $\text{Ar}_R(\mathcal{B})$, the natural square

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\varphi)} & F(b) \\ \downarrow & & \downarrow \\ \mathcal{R}(a) & \xrightarrow{\mathcal{R}(\varphi)} & \mathcal{R}(b) \end{array}$$

is cartesian.

A pleasant consequence of Theorem 2.3.5 is that $\text{St}_{\mathcal{B}}^L$ also has a *left* adjoint and that $\text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ is presentable. To see this, we use the following observation:

Observation 2.3.6 Let $\mathcal{C}$ be a presentable ∞ -category, and S a set of morphisms in $\mathcal{C}$. Recall that a morphism $\varphi: X \rightarrow Y$ in $\mathcal{C}$ is *right orthogonal* to S if there exists a unique filler in every commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ f \downarrow & \nearrow & \downarrow \varphi \\ B & \longrightarrow & Y, \end{array}$$

where f is in S . Equivalently, φ is right orthogonal to S if and only if the commutative square

$$\begin{array}{ccc} \text{Map}_{\mathcal{C}}(B, X) & \xrightarrow{f^*} & \text{Map}_{\mathcal{C}}(A, X) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ \text{Map}_{\mathcal{C}}(B, Y) & \xrightarrow{f^*} & \text{Map}_{\mathcal{C}}(A, Y) \end{array}$$

is cartesian for all $f: A \rightarrow B$ in S . This square is in turn cartesian if and only if for all maps $B \rightarrow Y$, the map on fibers

$$\mathrm{Map}_{/Y}(B, X) \rightarrow \mathrm{Map}_{/Y}(A, X)$$

is an equivalence. Thus the map φ is right orthogonal to S if and only if as an object of $\mathcal{C}_{/Y}$ it is local with respect to the set of maps

$$\left\{ \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \swarrow \\ & Y & \end{array} : f \in S \right\}.$$

In particular, the full subcategory of $\mathcal{C}_{/Y}$ spanned by the objects that are right orthogonal to S is an accessible localization of $\mathcal{C}_{/Y}$, and so is also presentable.

Proposition 2.3.7 *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a small ∞ -category equipped with a factorization system. The functor $\mathrm{St}_{\mathcal{B}}^L$ has a left adjoint, which exhibits $\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ as an accessible localization of $\mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_{\infty})_{/\mathcal{R}}$. In particular, $\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ is a presentable ∞ -category.*

Proof The ∞ -category $\mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_{\infty})_{/\mathcal{R}}$ is clearly presentable, and we know that the functor $\mathrm{St}_{\mathcal{B}}^L$ is fully faithful, with its essential image given by functors equifibered over $\mathcal{R}$. It therefore suffices to show that this is the full subcategory of objects in $\mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_{\infty})_{/\mathcal{R}}$ that are local with respect to a set of morphisms.

Let S be the collection of morphisms of the form

$$(y(\varphi) \times \mathrm{id}): y(b) \times [\epsilon] \rightarrow y(a) \times [\epsilon]$$

for $\epsilon \in \{0, 1\}$ and $(\varphi: a \rightarrow b) \in \mathrm{Ar}_{\mathcal{R}}(\mathcal{B})$, where $y(a)(-) := \mathrm{Map}_{\mathcal{B}}(a, -)$ is the Yoneda embedding of $\mathcal{B}^{\mathrm{op}}$; this is a set since $\mathcal{B}$ is by assumption a small ∞ -category. An object $\gamma: F \rightarrow \mathcal{R}$ in $\mathrm{Fun}(\mathcal{B}, \mathrm{Cat}_{\infty})_{/\mathcal{R}}$ is then equifibered if and only if it is right orthogonal to S : The latter means that the commutative squares

$$\begin{array}{ccc} \mathrm{Map}(y(a) \times [\epsilon], F) & \longrightarrow & \mathrm{Map}(y(b) \times [\epsilon], F) \\ \downarrow & & \downarrow \\ \mathrm{Map}(y(a) \times [\epsilon], \mathcal{R}) & \longrightarrow & \mathrm{Map}(y(b) \times [\epsilon], \mathcal{R}) \end{array}$$

are cartesian; by the Yoneda lemma this square can be identified with

$$\begin{array}{ccc} \mathrm{Map}([\epsilon], F(a)) & \longrightarrow & \mathrm{Map}([\epsilon], F(b)) \\ \downarrow & & \downarrow \\ \mathrm{Map}([\epsilon], \mathcal{R}(a)) & \longrightarrow & \mathrm{Map}([\epsilon], \mathcal{R}(b)), \end{array}$$

which is cartesian for $\epsilon = 0, 1$ if and only if the square

$$\begin{array}{ccc} F(a) & \longrightarrow & F(b) \\ \downarrow & & \downarrow \\ \mathcal{R}(a) & \longrightarrow & \mathcal{R}(b) \end{array}$$

is cartesian, since the objects $[0], [1]$ generate Cat_∞ under colimits. The result then follows from Observation 2.3.6. $\square$

Observation 2.3.8 It is easy to see (using the mapping space criterion for cocartesian morphisms) that the forgetful functor $\text{Cat}_{\infty/B}^{L\text{-cocart}} \rightarrow \text{Cat}_{\infty/B}$ preserves limits and filtered colimits. Since both ∞ -categories are presentable by Proposition 2.3.7, it follows by the adjoint functor theorem that this functor has a left adjoint.

Observation 2.3.9 Let $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be ∞ -categories equipped with factorization systems, and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a functor that preserves both classes of maps in these.

The functor f then induces a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i_{\mathcal{A}}} & \text{Ar}_R(\mathcal{A}) & \xrightarrow{\text{ev}_1} & \mathcal{A} \\ \downarrow f & & \downarrow q & & \parallel \\ & & f^* \text{Ar}_R(\mathcal{B}) & \xrightarrow{\text{ev}_1} & \mathcal{A} \\ & & \downarrow \lrcorner & & \downarrow f \\ \mathcal{B} & \xrightarrow{i_{\mathcal{B}}} & \text{Ar}_R(\mathcal{B}) & \xrightarrow{\text{ev}_1} & \mathcal{B}. \end{array}$$

From this we get the following commutative diagram of ∞ -categories:

$$(2) \quad \begin{array}{ccccc} (\text{Cat}_{\infty/B}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} & \xrightarrow{f^*} & (\text{Cat}_{\infty/\mathcal{A}}^{\text{cocart}})_{/f^* \text{Ar}_R(\mathcal{B})} & \xrightarrow{q^*} & (\text{Cat}_{\infty/\mathcal{A}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{A})} \\ \downarrow & & \downarrow & & \downarrow \\ (\text{Cat}_{\infty/B}^{L\text{-cocart}})_{/\text{Ar}_R(\mathcal{B})} & \xrightarrow{f^*} & (\text{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}})_{/f^* \text{Ar}_R(\mathcal{B})} & \xrightarrow{q^*} & (\text{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}})_{/\text{Ar}_R(\mathcal{A})} \\ \downarrow \iota_B^* & & \downarrow \iota_B^* & & \downarrow \iota_A^* \\ \text{Cat}_{\infty/B}^{L\text{-cocart}} & \xrightarrow{f^*} & \text{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}} & & \end{array}$$

(Curved arrows labeled Q_B and Q_A connect the top and middle rows on the left and right respectively.)

Let us write $f^\circledast$ for the composite in the top row, which takes $\mathcal{E} \rightarrow \text{Ar}_R(\mathcal{B})$ to the fiber product $\mathcal{E} \times_{\text{Ar}_R(\mathcal{B})} \text{Ar}_R(\mathcal{A}) \rightarrow \text{Ar}_R(\mathcal{A})$. Passing to vertical left adjoints now yields a Beck–Chevalley transformation

$$E_{\mathcal{A}} f^* \rightarrow f^\circledast E_{\mathcal{B}}.$$

Unwinding the definitions, this is given at $\mathcal{E} \rightarrow \mathcal{B}$ in $\text{Cat}_{\infty/B}^{L\text{-cocart}}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \text{Ar}_R(\mathcal{A}) \rightarrow (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B})) \times_{\text{Ar}_R(\mathcal{B})} \text{Ar}_R(\mathcal{A}).$$

This is an equivalence, so that we also have a commutative square

$$(3) \quad \begin{array}{ccc} \text{Cat}_{\infty/B}^{L\text{-cocart}} & \xrightarrow{f^*} & \text{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}} \\ \downarrow E_{\mathcal{B}} & & \downarrow E_{\mathcal{A}} \\ (\text{Cat}_{\infty/B}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} & \xrightarrow{f^\circledast} & (\text{Cat}_{\infty/\mathcal{A}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{A})}. \end{array}$$

3 Algebraic patterns

In this section we will first review the basic definitions related to algebraic patterns and Segal objects in Section 3.1, and then look at some examples thereof in Section 3.2. We then introduce the condition of *soundness* for algebraic patterns in Section 3.3; this is somewhat technical, but turns out to be the key property needed for some of our results in the next section.

3.1 Algebraic patterns and Segal objects

In this subsection we review the definitions of algebraic patterns and Segal objects, and some related basic notions introduced in [7]. We also introduce a *relative* version of Segal objects, which will show up later.

Definition 3.1.1 An *algebraic pattern* is an ∞ -category $\mathcal{O}$ equipped with a factorization system, whereby every morphism factors (uniquely up to equivalence) as an *inert* morphism followed by an *active* morphism, as well as a collection of *elementary objects*. We write $\mathcal{O}^{\text{int}}$ and $\mathcal{O}^{\text{act}}$ for the subcategories of $\mathcal{O}$ containing only the inert and active morphisms, respectively, and $\mathcal{O}^{\text{el}}$ for the full subcategory of $\mathcal{O}^{\text{int}}$ containing elementary objects and inert morphisms among them. For $X \in \mathcal{O}$, we also write

$$\mathcal{O}_{X/}^{\text{el}} := \mathcal{O}^{\text{el}} \times_{\mathcal{O}^{\text{int}}} \mathcal{O}_{X/}^{\text{int}}$$

for the ∞ -category of inert maps $X \rightarrow E$ with $E \in \mathcal{O}^{\text{el}}$.

Notation 3.1.2 We often indicate inert maps as $X \twoheadrightarrow Y$ and active maps as $X \rightsquigarrow Y$. These arrows are not meant to indicate any particular intuition about inert or active morphisms.

Example 3.1.3 We write $\mathbb{F}_*$ for a skeleton of the category of pointed finite sets, with objects $\langle n \rangle := (\{0, 1, \dots, n\}, 0)$, and we say a morphism $\varphi: \langle n \rangle \rightarrow \langle m \rangle$ is *inert* if φ restricts to an isomorphism $\langle n \rangle \setminus \varphi^{-1}(0) \rightarrow \langle m \rangle \setminus \{0\}$, and *active* if $\varphi^{-1}(0) = \{0\}$. Then the inert and active morphisms form a factorization system on $\mathbb{F}_*$, and we make this an algebraic pattern⁴ by taking $\langle 1 \rangle$ to be the single elementary object.

Example 3.1.4 Another basic example is Δ^{op} , where Δ is the simplex category. Recall that Δ^{op} admits an inert-active factorization system where inert maps are opposite to interval inclusions and active maps are opposite to maps preserving the maximal and minimal elements. To make Δ^{op} an algebraic pattern, we can take the elementary objects to be $[0]$ and $[1]$, in which case we denote the pattern by $\Delta^{\text{op}, [1]}$, or alternatively just $[1]$, in which case the pattern is denoted by $\Delta^{\text{op}, b}$.

The main reason for introducing algebraic patterns is that they describe algebraic structures via Segal conditions:

⁴In [7] this pattern was denoted by $\mathbb{F}_*^b$ to distinguish it from the pattern $\mathbb{F}_*^{[1]}$, where the elementary objects are $\langle 0 \rangle$ and $\langle 1 \rangle$. However, in this paper $\mathbb{F}_* = \mathbb{F}_*^b$ is the key example, so we use a simplified notation for it.

Definition 3.1.5 A functor $F : \mathcal{O} \rightarrow \mathcal{C}$ is a *Segal $\mathcal{O}$ -object* in the ∞ -category $\mathcal{C}$ if for every object $X \in \mathcal{O}$ the induced functor

$$(\mathcal{O}_{X/}^{\text{el}})^{\triangleleft} \rightarrow \mathcal{O} \xrightarrow{F} \mathcal{C}$$

is a limit diagram. If $\mathcal{C}$ has limits for diagrams indexed by $\mathcal{O}_{X/}^{\text{el}}$ for all $X \in \mathcal{O}$, in which case we say that $\mathcal{C}$ is *$\mathcal{O}$ -complete*, then this condition is equivalent to the canonical maps

$$F(X) \rightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} F(E)$$

being equivalences. We refer to Segal $\mathcal{O}$ -objects in the ∞ -category $\mathcal{S}$ of spaces as *Segal $\mathcal{O}$ -spaces* and Segal $\mathcal{O}$ -objects in the ∞ -category Cat_{∞} of ∞ -categories as *Segal $\mathcal{O}$ - ∞ -categories*.

Example 3.1.6 We can identify $(\mathbb{F}_*)_{\langle n \rangle /}^{\text{el}}$ with the set $\{\rho_i : i = 1, \dots, n\}$, where ρ_i is the inert morphism $\langle n \rangle \rightarrow \langle 1 \rangle$ given by

$$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i. \end{cases}$$

A functor $F : \mathbb{F}_* \rightarrow \mathcal{C}$ is then a Segal $\mathbb{F}_*$ -object if for every n the map

$$F(\langle n \rangle) \rightarrow \prod_{i=1}^n F(\langle 1 \rangle),$$

induced by the maps ρ_i , is an equivalence. Thus Segal $\mathbb{F}_*$ -objects are precisely commutative monoids in the sense of [23, Section 2.4.2]. For $\mathcal{C} = \mathcal{S}$, this gives the ∞ -categorical analogue of *special Γ -spaces* in the sense of Segal [30].

Example 3.1.7 Segal $\Delta^{\text{op}, \mathfrak{h}}$ -spaces are precisely *Segal spaces* in the sense of [27], while Segal $\Delta^{\text{op}, \mathfrak{b}}$ -objects in $\mathcal{C}$ are associative monoids (or E_1 -algebras).

Later on, we will also need to consider a *relative* version of Segal objects:

Definition 3.1.8 Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$ -complete ∞ -category. A *relative Segal $\mathcal{O}$ -object* of $\mathcal{C}$ is a morphism $\pi : Y \rightarrow X$ in $\text{Fun}(\mathcal{O}, \mathcal{C})$ such that for every $O \in \mathcal{O}$ the natural commutative square

$$\begin{array}{ccc} Y(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O/}^{\text{el}}} Y(E) \\ \pi(O) \downarrow & & \downarrow \lim_{E \in \mathcal{O}_{O/}^{\text{el}}} \pi(E) \\ X(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O/}^{\text{el}}} X(E) \end{array}$$

is cartesian. We denote by $\text{Seg}_{\mathcal{O}}^{/X}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{O}, \mathcal{C})_{/X}$ the full subcategory whose objects are the X -relative Segal $\mathcal{O}$ -objects.

Observation 3.1.9 If $Y \rightarrow X$ is a relative Segal $\mathcal{O}$ -object of $\mathcal{C}$, then the pasting lemma for cartesian squares implies that a morphism $Z \rightarrow Y$ is a relative Segal $\mathcal{O}$ -object if and only if the composite $Z \rightarrow X$ is one. Moreover, a morphism $X \rightarrow *$ to the terminal object is a relative Segal $\mathcal{O}$ -object if and only if X is a Segal $\mathcal{O}$ -object in $\mathcal{C}$. Combining these two observations, we see that if X is a Segal $\mathcal{O}$ -object of $\mathcal{C}$ then an X -relative Segal $\mathcal{O}$ -object is just a Segal $\mathcal{O}$ -object with a map to X , ie we have

$$\mathrm{Seg}_{\mathcal{O}}^{/X}(\mathcal{C}) = \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})_{/X}$$

as full subcategories of $\mathrm{Fun}(\mathcal{C}, \mathcal{O})_{/X}$.

Lemma 3.1.10 Suppose $X \rightarrow Y$ is a relative Segal $\mathcal{O}$ -object in $\mathcal{C}$. Then for any map $\eta: Y' \rightarrow Y$, the pullback $X' := X \times_Y Y' \rightarrow Y'$ is also a relative Segal $\mathcal{O}$ -object. In other words, pullback along η gives a functor $\eta^*: \mathrm{Seg}_{\mathcal{O}}^{/Y}(\mathcal{C}) \rightarrow \mathrm{Seg}_{\mathcal{O}}^{/Y'}(\mathcal{C})$.

Proof For $O \in \mathcal{O}$, consider the commutative cube

$$\begin{array}{ccccc} X'(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\mathrm{el}}} X'(E) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & X(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\mathrm{el}}} X(E) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ Y'(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\mathrm{el}}} Y'(E) & & \\ & \searrow & \downarrow & \searrow & \\ & Y(O) & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\mathrm{el}}} Y(E) & \end{array}$$

Here the left, right, and front faces are all cartesian, hence so is the back face. □

Lemma 3.1.11 For every presentable ∞ -category $\mathcal{C}$ the full subcategory

$$\mathrm{Seg}_{\mathcal{O}}^{/X}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{O}, \mathcal{C})_{/X}$$

is an accessible localization. In particular, it is a presentable ∞ -category.

Proof Consider the following collection of morphisms in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$:

$$\{\mathrm{colim}_{E \in \mathcal{O}_{X'}^{\mathrm{el}}} y(E) \otimes C \rightarrow y(X) \otimes C\}_{X \in \mathcal{O}, C \in K},$$

where K is a set of κ -compact generators for $\mathcal{C}$, y is the Yoneda embedding for $\mathcal{O}^{\mathrm{op}}$, and $T \otimes C$ for $T \in \mathcal{S}$, $C \in \mathcal{C}$, is the canonical tensoring of $\mathcal{C}$ with $\mathcal{S}$, given by the colimit over T of the constant diagram with value C . A morphism $X \rightarrow Y$ in $\mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is a relative Segal $\mathcal{O}$ -object if and only if it is right orthogonal to this set of morphisms, hence the claim follows from Observation 2.3.6. □

Next, we take a brief look at morphisms between patterns:

Definition 3.1.12 If $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, a *morphism of algebraic patterns* is a functor $f: \mathcal{O} \rightarrow \mathcal{P}$ that preserves inert and active morphisms as well as elementary objects. We say that such a morphism is a *Segal morphism* if for every Segal $\mathcal{P}$ -space F and every $X \in \mathcal{O}$ the functor $f_{X/}^{\text{el}}: \mathcal{O}_{X/}^{\text{el}} \rightarrow \mathcal{P}_{f(X)/}^{\text{el}}$ arising from f induces an equivalence

$$\lim_{\mathcal{P}_{f(X)/}^{\text{el}}} F \xrightarrow{\sim} \lim_{\mathcal{O}_{X/}^{\text{el}}} F \circ f;$$

by [7, Lemma 4.5] this is equivalent to composition with f giving a functor

$$f^*: \text{Seg}_{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Seg}_{\mathcal{O}}(\mathcal{C})$$

for any $\mathcal{O}$ -complete ∞ -category $\mathcal{C}$. The Segal morphisms that occur in practice are those where the functor $f_{X/}^{\text{el}}$ is coinitial for all $X \in \mathcal{O}$; if this is the case we say that f is a *strong Segal morphism*. In the special case where $f_{X/}^{\text{el}}$ is an *equivalence* for every X , we say that f is an *iso-Segal morphism*.

Example 3.1.13 There is a morphism of algebraic patterns $c: \Delta^{\text{op},b} \rightarrow \mathbb{F}_*$, given on objects by $c([n]) = \langle n \rangle$, and with $c(\varphi): \langle n \rangle \rightarrow \langle m \rangle$ for a morphism $\varphi: [m] \rightarrow [n]$ in Δ given by

$$c(\varphi)(i) = \begin{cases} j & \text{if } \varphi(j-1) < i \leq \varphi(j), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is an iso-Segal morphism.

Notation 3.1.14 We write AlgPatt for the ∞ -category of algebraic patterns together with all morphisms of algebraic patterns.

Observation 3.1.15 Composition with a *strong* Segal morphism $f: \mathcal{O} \rightarrow \mathcal{P}$ also preserves *relative* Segal objects: If $X \rightarrow Y$ is a relative Segal $\mathcal{P}$ -object in $\mathcal{C}$, then for $O \in \mathcal{O}$ we have a commutative diagram

$$\begin{array}{ccccc} X(f(O)) & \longrightarrow & \lim_{E \in \mathcal{P}_{f(O)/}^{\text{el}}} X(E) & \xrightarrow{\sim} & \lim_{E' \in \mathcal{O}_{O/}^{\text{el}}} X(f(E')) \\ \downarrow & & \downarrow & & \downarrow \\ Y(f(O)) & \longrightarrow & \lim_{E \in \mathcal{P}_{f(O)/}^{\text{el}}} Y(E) & \xrightarrow{\sim} & \lim_{E' \in \mathcal{O}_{O/}^{\text{el}}} Y(f(E')) \end{array}$$

here both the left and right squares are cartesian, and hence so is the outer composite square. Composition with f thus gives a functor $f^*: \text{Seg}_{\mathcal{P}}^Y(\mathcal{C}) \rightarrow \text{Seg}_{\mathcal{O}}^{f^*Y}(\mathcal{C})$.

We now recall a simple criterion for a Segal morphism to give an equivalence on Segal objects:

Proposition 3.1.16 [2, Corollary 2.64] *Suppose $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, and $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism such that*

- (1) $f^{\text{el}}: \mathcal{O}^{\text{el}} \rightarrow \mathcal{P}^{\text{el}}$ is an equivalence of ∞ -categories,
- (2) for every $O \in \mathcal{O}$, the functor $(\mathcal{O}_{O/}^{\text{act}})^{\simeq} \rightarrow (\mathcal{P}_{f(O)/}^{\text{act}})^{\simeq}$ is an equivalence of ∞ -groupoids.

Then for any complete ∞ -category $\mathcal{C}$ the functor $f^*: \text{Seg}_{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Seg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence, with inverse given by right Kan extension along f .

Proof We refer to [2, Section 2] for a detailed proof, but since this result will play an important role in this paper we recall the key steps for the reader's convenience.

By [7, Proposition 6.3], (2) implies that right Kan extension along f restricts to Segal objects, giving an adjunction

$$f^*: \text{Seg}_{\mathcal{P}}(\mathcal{C}) \rightleftarrows \text{Seg}_{\mathcal{O}}(\mathcal{C}) : f_*.$$

Moreover, the proof of [7, Proposition 6.3] shows that for $F \in \text{Seg}_{\mathcal{O}}(\mathcal{C})$ we have $(f_* F)|_{\mathcal{P}^{\text{el}}} \simeq f_*^{\text{el}}(F|_{\mathcal{O}^{\text{el}}})$.

Condition (1) therefore implies that the counit $f^* f_* F \rightarrow F$ is an equivalence for $F \in \text{Seg}_{\mathcal{P}}(\mathcal{C})$, since it is an equivalence when evaluated on $\mathcal{P}^{\text{el}}$. Moreover, (1) implies that f^* is conservative on Segal objects, again since equivalences are detected on elementary objects. To see that the unit map $G \rightarrow f_* f^* G$ is an equivalence, it then suffices to check this after applying f^* , but then the adjunction $f^* \dashv f_*$ implies that the composite

$$f^* G \rightarrow f^* f_* f^* G \xrightarrow{\sim} f^* G$$

is an equivalence, and here we already saw that the counit is an equivalence. Since both the unit and counit of the adjunction are natural equivalences, it must be an equivalence of ∞ -categories. $\square$

Remark 3.1.17 If $(\mathcal{O}_{/O}^{\text{act}})^{\simeq}$ is a Segal $\mathcal{O}$ -space and $(\mathcal{P}_{/f(O)}^{\text{act}})^{\simeq}$ is a Segal $\mathcal{P}$ -space in Proposition 3.1.16, then it suffices to check (2) for elementary objects in $\mathcal{O}$. This holds, for instance, if $\mathcal{O}$ and $\mathcal{P}$ are extendable (see Definition 3.3.16).

3.2 Examples of algebraic patterns

We now look at some examples of algebraic patterns. Our focus here will be on examples that will be relevant in the next sections; we refer the reader to [7, Section 3] for many other examples.

Example 3.2.1 We have patterns $\Delta^{n, \text{op}, \mathfrak{l}}$ and $\Delta^{n, \text{op}, \mathfrak{b}}$ with underlying category $\Delta^{n, \text{op}} := (\Delta^{\text{op}})^{\times n}$, equipped with the factorization system where the inert and active maps are those that are inert or active in Δ^{op} in each component. Here $(\Delta^{n, \text{op}, \mathfrak{b}})^{\text{el}} = \{([1], \dots, [1])\}$ while $(\Delta^{n, \text{op}, \mathfrak{l}})^{\text{el}}$ consists of all objects whose components are all either $[0]$ or $[1]$. Then Segal $\Delta^{n, \text{op}, \mathfrak{l}}$ -spaces are n -uple Segal spaces, which model n -fold ∞ -categories, while Segal $\Delta^{n, \text{op}, \mathfrak{b}}$ -objects are $\mathbb{E}_n$ -algebras (by the Dunn–Lurie additivity theorem).

Example 3.2.2 Let Θ_n be the inductively defined wreath product $\Delta \wr \Theta_{n-1}$, starting with $\Theta_0 = [0]$; see, for example, [5; 16] for more details. This has a factorization system where the active/inert maps are those whose components in Δ and Θ_{n-1} are both active or inert. There are two interesting pattern structures on Θ_n^{op} : if we define the objects C_i in Θ_n by $C_0 := [0]()$ and $C_i := [1](C_{i-1})$ for $i = 1, \dots, n$, then for $\Theta_n^{\text{op}, \mathfrak{b}}$ we take C_n to be the only elementary object, while for $\Theta_n^{\text{op}, \mathfrak{l}}$ we take all of $C_0, \dots, C_n$. Then Segal $\Theta_n^{\text{op}, \mathfrak{l}}$ -spaces are Rezk's model for (∞, n) -categories [28], while Segal $\Theta_n^{\text{op}, \mathfrak{b}}$ -objects are again $\mathbb{E}_n$ -algebras (see [4]).

Example 3.2.3 Let $\mathbb{F}_*^{\leq k} \subseteq \mathbb{F}_*$ denote the full subcategory containing pointed finite sets of cardinality $\leq k$ (excluding the basepoint). Consider $\mathbb{F}_*^{\leq k}$ as an algebraic pattern by restricting the inert-active factorization system on $\mathbb{F}_*$ and choosing $\langle 1 \rangle$ to be the only elementary object. Segal objects for $\mathbb{F}_*^{\leq k}$ are *arity k -restricted commutative monoids* — a variant of commutative monoids in which the homotopy coherence data is only supplied up to arity k . More generally, if $\mathcal{O}$ is an ∞ -operad then $\mathcal{O}^{\leq k} := \mathbb{F}_*^{\leq k} \times_{\mathbb{F}_*} \mathcal{O}$ has a natural structure of an algebraic pattern whose Segal objects are arity k -restricted $\mathcal{O}$ -monoids. For more details see [2].

The remaining examples we want to discuss are all instances of a general class of algebraic patterns on ∞ -categories of spans. For this purpose we briefly recall the construction of such ∞ -categories — this is originally due to Barwick [3]; see also [18] for a more “model-independent” version.

Construction 3.2.4 Let $\mathfrak{X}$ be an ∞ -category equipped with a pair of wide subcategories $\mathfrak{X}^b$ and $\mathfrak{X}^f$ (where “ b ” stands for *backwards* and “ f ” stands for *forwards*). Following Barwick, we say that the triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ is *adequate* if for every pair of morphisms $\beta: x \rightarrow y$ in $\mathfrak{X}^b$ and $\varphi: y' \rightarrow y$ in $\mathfrak{X}^f$, we have

- (1) the pullback $x' := x \times_y y'$ exists in $\mathfrak{X}$,
- (2) the projection $x' \rightarrow y'$ lies in $\mathfrak{X}^b$,
- (3) the projection $x' \rightarrow x$ lies in $\mathfrak{X}^f$.

Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ Barwick defines an ∞ -category $\text{Span}_{b,f}(\mathfrak{X})$ (denoted in [3] by $\text{A}^{\text{eff}}(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$) such that the objects of $\text{Span}_{b,f}(\mathfrak{X})$ are the objects of $\mathfrak{X}$ and the morphisms from x to y are spans (or correspondences)

$$\begin{array}{ccc} & w & \\ \beta \swarrow & & \searrow \varphi \\ x & & y \end{array}$$

where the arrow β lies in $\mathfrak{X}^b$ and the arrow φ lies in $\mathfrak{X}^f$. The assumption that the triple is adequate allows for a composition law defined by taking pullbacks. If $\mathfrak{X}$ is an ∞ -category with pullbacks, then we can take $\mathfrak{X}^b = \mathfrak{X}^f = \mathfrak{X}$, in which case we just write $\text{Span}(\mathfrak{X})$ for the corresponding ∞ -category of spans.

Observation 3.2.5 By the first part of [18, Proposition 4.9] the ∞ -category $\text{Span}_{b,f}(\mathfrak{X})$ always has a factorization system given by the classes of maps as above with φ or β required to be an equivalence (which we might call the “backwards” and “forwards” maps) and the subcategories of these maps are equivalent to $\mathfrak{X}^{b,\text{op}}$ and $\mathfrak{X}^f$, respectively.

Definition 3.2.6 Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and a full subcategory $\mathfrak{X}_0 \subseteq \mathfrak{X}$, we denote by $\text{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)$ the algebraic pattern given by $\text{Span}_{b,f}(\mathfrak{X})$ with the factorization system whose inert and active maps are the backwards and forward maps, respectively, and with the objects of $\mathfrak{X}_0$ as the elementary objects.

Remark 3.2.7 The Segal condition for $\text{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)$ takes the following form for a functor F :

$$F(x) \simeq \lim_{e \rightarrow x \in (\mathfrak{X}_{0/x}^b)^{\text{op}}} F(e),$$

where $\text{Span}_{b,f}(\mathfrak{X})_{x/}^{\text{el}} \simeq (\mathfrak{X}_{0/x}^b)^{\text{op}}$ with $\mathfrak{X}_{0/x}^b := \mathfrak{X}_0 \times_{\mathfrak{X}^b} \mathfrak{X}_{/x}^b$ and $\mathfrak{X}_0^b$ is the full subcategory of $\mathfrak{X}^b$ containing the objects of $\mathfrak{X}_0$.

Example 3.2.8 Let $\mathbb{F}$ denote the category of finite sets. Since this has pullbacks, Construction 3.2.4 produces an ∞ -category (in fact a (2,1)-category) $\text{Span}(\mathbb{F})$ whose objects are finite sets, and whose morphisms are spans of the form

$$\begin{array}{ccc} & m & \\ \alpha \swarrow & & \searrow \beta \\ n & & n' \end{array}$$

for finite sets n , m , and n' , with composition given by taking pullbacks. We consider $\text{Span}(\mathbb{F}) = \text{Span}(\mathbb{F}; \{\mathbf{1}\})$ as an algebraic pattern by taking the backward maps as inerts, forward maps as actives and $\mathbf{1} \in \text{Span}(\mathbb{F})$ as the only elementary object.

Observation 3.2.9 The category $\mathbb{F}_*$ may be thought of as the wide subcategory $\text{Span}_{\text{inj}, \text{all}}(\mathbb{F})$ of $\text{Span}(\mathbb{F})$ containing only those morphisms where the backwards map is injective. The inert-active factorization system on $\mathbb{F}_*$ then coincides with the one obtained by restriction from $\text{Span}(\mathbb{F})$, and the inclusion $\mathbb{F}_* \rightarrow \text{Span}(\mathbb{F})$ is an iso-Segal morphism.

Example 3.2.10 Let G be a finite group and $\mathbb{F}_G$ the category of finite G -sets. Denote by $\text{Orb}_G \subseteq \mathbb{F}_G$ the collection of G -orbits (ie transitive G -sets). Since $\mathbb{F}_G$ has pullbacks we have an ∞ -category (really a (2,1)-category) $\text{Span}(\mathbb{F}_G)$. Abusing notation slightly, we also denote the span pattern with the orbits as elementary objects by $\text{Span}(\mathbb{F}_G) := \text{Span}(\mathbb{F}_G; \text{Orb}_G)$. Segal objects for this pattern are precisely G -commutative monoids in the sense of [25]; they also appear in [9] where they are called semi-Mackey functors. More generally, for any full subcategory $\mathcal{F} \subseteq \text{Orb}_G$ we have a span pattern $\text{Span}(\mathbb{F}_G; \mathcal{F})$ whose Segal objects may be thought of as G -commutative monoids that are Borel- $\mathcal{F}$ -complete. Segal objects for $\text{Span}(\mathbb{F}_G; \{G/e\})$ appear implicitly in [9], where they are called Borel-equivariant.

Example 3.2.11 As a variant of the previous example, we can consider subcategories $\mathbb{F}_G^f$ of $\mathbb{F}_G$ that are closed under base change; if $\mathbb{F}_G^f$ is moreover closed under finite coproducts, this data is equivalent to an *indexing system* in the sense of [6]. We can then define the span pattern $\text{Span}_{\text{all}, f}(\mathbb{F}_G) := \text{Span}_{\text{all}, f}(\mathbb{F}_G; \text{Orb}_G)$, whose Segal objects we can think of as G -commutative monoids where only transfers that lie in $\mathbb{F}_G^f$ are allowed. As an illustrative example we may consider the extreme case where all forward maps are isomorphisms, ie $\mathbb{F}_G^f := \mathbb{F}_G^{\simeq}$. The corresponding span pattern $\text{Span}_{\text{all}, \simeq}(\mathbb{F}_G; \text{Orb}_G)$ has an underlying ∞ -category equivalent to $\mathbb{F}_G^{\text{op}}$ with all the maps inert and with Orb_G^{op} as the subcategory of elementary objects. Segal objects for this pattern are thus equivalent to presheaves on Orb_G , and by Elemendorf's theorem this ∞ -category is equivalent to that of G -spaces.

Example 3.2.12 A space $X \in \mathcal{S}$ is called m -finite if it is m -truncated and all of its homotopy groups are finite; we let $\mathcal{S}_m \subseteq \mathcal{S}$ denote the full subcategory of m -finite spaces. Since m -finite spaces are closed under finite limits we may consider the span pattern $\text{Span}(\mathcal{S}_m) := \text{Span}(\mathcal{S}_m; *)$. If we write $\mathcal{S}_m^{n\text{-tr}}$ for the wide subcategory of $\mathcal{S}_m$ whose maps are n -truncated, then $(\mathcal{S}_m, \mathcal{S}_m^{n\text{-tr}}, \mathcal{S}_m)$ is also an adequate triple, and we can likewise consider the pattern

$$\text{Span}_{n\text{-tr}, \text{all}}(\mathcal{S}_m) := \text{Span}_{n\text{-tr}, \text{all}}(\mathcal{S}_m; *)$$

for any n . For $n = m - 1$, the Segal objects for $\text{Span}_{(m-1)\text{-tr}, \text{all}}(\mathcal{S}_m)$ are precisely the m -commutative monoids of Harpaz [14]. It also follows from [14, Proposition 5.14] that these are equivalent to Segal objects for $\text{Span}(\mathcal{S}_m)$.

3.3 Sound patterns

In this subsection we define the notion of a *sound* pattern — a technical condition satisfied in almost all the usual examples. This requires first introducing some notation:

Notation 3.3.1 Fix a morphism $\omega: X \rightarrow Y$ in an algebraic pattern $\mathcal{O}$. For every elementary object $(\alpha: Y \rightarrowtail E) \in \mathcal{O}_{Y/}^{\text{el}}$ we denote the inert-active factorization of $\alpha \circ \omega$ as follows:

$$\begin{array}{ccc} X & \xrightarrow{\omega_\alpha} & \omega_\alpha! X \\ \omega \downarrow & & \downarrow (\alpha \circ \omega)^{\text{act}} \\ Y & \xrightarrow{\alpha} & E. \end{array}$$

Factorization defines a functor $\omega(-): \mathcal{O}_{Y/}^{\text{el}} \rightarrow \mathcal{O}_{X/}^{\text{int}}$ by sending α to ω_α .

Definition 3.3.2 For $\omega: X \rightsquigarrow Y$ we define $\mathcal{O}^{\text{el}}(\omega)$ as the pullback

$$\begin{array}{ccc} \mathcal{O}^{\text{el}}(\omega) & \longrightarrow & \text{Ar}(\mathcal{O}_{X/}^{\text{int}}) \\ \downarrow & & \downarrow (s, t) \\ \mathcal{O}_{Y/}^{\text{el}} \times \mathcal{O}_{X/}^{\text{el}} & \xrightarrow{(\omega(-), \text{id})} & \mathcal{O}_{X/}^{\text{int}} \times \mathcal{O}_{X/}^{\text{int}}. \end{array}$$

An object in $\mathcal{O}^{\text{el}}(\omega)$ can thus be represented by a diagram in $\mathcal{O}$ of the following shape:

$$\begin{array}{ccccc} X & \xrightarrow{\omega_\alpha} & \omega_\alpha! X & \rightarrowtail & E' \\ \omega \downarrow & & \downarrow & & \\ Y & \xrightarrow{\alpha} & E, & & \end{array}$$

where the arrows labeled by $\rightarrowtail$ and $\rightsquigarrow$ are required to be inert and active, respectively, E and E' are elementary, and ω is fixed. Morphisms in $\mathcal{O}^{\text{el}}(\omega)$ are natural transformations of such diagrams that are constant at $\omega: X \rightsquigarrow Y$ and inert at all other objects.

Remark 3.3.3 By construction $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{Y/}^{\text{el}} \times \mathcal{O}_{X/}^{\text{el}}$ is the bifibration (see [22, Definition 2.4.7.2]) corresponding to the functor

$$(\mathcal{O}_{Y/}^{\text{el}})^{\text{op}} \times \mathcal{O}_{X/}^{\text{el}} \rightarrow \mathcal{S}, \quad (\alpha: Y \twoheadrightarrow E, \beta: X \twoheadrightarrow E') \mapsto \text{Map}_{\mathcal{O}_{X/}^{\text{int}}}(\omega_{\alpha}, \beta).$$

Definition 3.3.4 We say that a pattern $\mathcal{O}$ is *sound* if for every active morphism $\omega: X \rightsquigarrow Y$ the functor $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{X/}^{\text{el}}$ is coinital.

The point of introducing the condition of soundness is that it allows us to rewrite certain double limits, as described below in Lemma 3.3.8. Before we state this property we look at a first example, namely $\mathbb{F}_*$, where soundness is particularly easy to check; further examples will be given below.

Example 3.3.5 In the pattern $\mathbb{F}_*$ an active morphism $\omega: X_+ \rightsquigarrow Y_+$ is simply a map $\omega: X \rightarrow Y$ in $\mathbb{F}$. The inert undercategory $(\mathbb{F}_*)_{Y_+/}^{\text{int}}$ may be identified with the poset $(\text{Sub}(Y), \supseteq)$ of subsets of Y , by assigning to each $\gamma: Y_+ \twoheadrightarrow Z_+$ the subset $\gamma^{-1}(Z) \subset Y$. The category of elementary objects under Y_+ is given by the one-element subsets, and we may hence identify it with Y itself. For an elementary $\alpha: Y_+ \twoheadrightarrow E$ corresponding to $e \in Y$, the pushforward $\omega_{\alpha!}X_+$ can be identified with $\omega^{-1}(e)_+ \subset X_+$. Hence we have a cartesian square:

$$\begin{array}{ccc} \mathbb{F}_*^{\text{el}}(\omega) & \longrightarrow & \text{Ar}(\text{Sub}(X)) \\ \downarrow & & \downarrow (t,s) \\ X \times Y & \xrightarrow{(\text{id}, \omega^{-1})} & \text{Sub}(X) \times \text{Sub}(X) \end{array}$$

and so $\mathbb{F}_*^{\text{el}}(\omega)$ is the poset of pairs $(x, y) \in X \times Y$ such that $\{x\} \subset \omega^{-1}(y)$. In other words, $y = \omega(x)$ and hence the map $\mathbb{F}_*^{\text{el}}(\omega) \rightarrow (\mathbb{F}_*)_{X_+/}^{\text{el}} \simeq X$ is an equivalence. In particular it is coinital and thus $\mathbb{F}_*$ is sound.

Observation 3.3.6 The composite $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{X/}^{\text{el}} \times \mathcal{O}_{Y/}^{\text{el}} \rightarrow \mathcal{O}_{Y/}^{\text{el}}$ is by construction a cartesian fibration. Its straightening is the functor

$$(\mathcal{O}_{Y/}^{\text{el}})^{\text{op}} \xrightarrow{\omega(-)} (\mathcal{O}_{X/}^{\text{int}})^{\text{op}} \xrightarrow{\mathcal{O}^{\text{el}}} \text{Cat}$$

that sends $\alpha: Y \twoheadrightarrow E$ to the ∞ -category $\mathcal{O}_{\omega_{\alpha!}X/}^{\text{el}}$ of elementaries under $\omega_{\alpha!}X$. Our definition of $\mathcal{O}^{\text{el}}(\omega)$ therefore matches that given in [7, Remark 7.6]. Moreover, a limit over $\mathcal{O}^{\text{el}}(\omega)$ can be rewritten as a double limit, that is for $F: \mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{C}$ we have

$$\lim_{\mathcal{O}^{\text{el}}(\omega)} F \simeq \lim_{\alpha \in \mathcal{O}_{Y/}^{\text{el}}} \lim_{\mathcal{O}_{\omega_{\alpha!}X/}^{\text{el}}} F.$$

If $\mathcal{O}$ is sound, then we can use this to rewrite a limit over $\mathcal{O}_{X/}^{\text{el}}$ as a double limit.

We now show that soundness is inherited along iso-Segal morphisms. Together with Example 3.3.5 this implies that all cartesian patterns of [8, Definition 2.6] are sound, and in particular any ∞ -operad in the sense of Lurie is sound.

Lemma 3.3.7 *Let $f: \mathcal{O} \rightarrow \mathcal{P}$ be an iso-Segal morphism of algebraic patterns. Then $\mathcal{O}$ is sound, if $\mathcal{P}$ is sound. The converse implication holds if we further assume that*

$$\mathrm{Ar}^{\mathrm{act}}(f): \mathrm{Ar}^{\mathrm{act}}(\mathcal{O}) \rightarrow \mathrm{Ar}^{\mathrm{act}}(\mathcal{P})$$

is essentially surjective.

Proof Being a morphism of algebraic patterns, f induces for each active $\omega: X \rightsquigarrow Y$ a morphism of cartesian fibrations

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{el}}(\omega) & \xrightarrow{f} & \mathcal{P}^{\mathrm{el}}(f(\omega)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y/}^{\mathrm{el}} & \xrightarrow[\simeq]{f} & \mathcal{P}_{f(Y)/}^{\mathrm{el}}, \end{array}$$

where the bottom functor is an equivalence because we assumed that f is iso-Segal. On the fibers over some $(\alpha: Y \rightrightarrows E) \in \mathcal{O}_{Y/}^{\mathrm{el}}$ and $f(\alpha) \in \mathcal{P}_{f(Y)/}^{\mathrm{el}}$ we get an induced functor

$$f: \mathcal{O}_{\alpha_! X/}^{\mathrm{el}} \rightarrow \mathcal{P}_{f(\alpha_! X)/}^{\mathrm{el}},$$

which again is an equivalence because f is iso-Segal. Therefore it follows that $f: \mathcal{O}^{\mathrm{el}}(\omega) \rightarrow \mathcal{P}^{\mathrm{el}}(f(\omega))$ is an equivalence. This is also the top map in the square

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{el}}(\omega) & \xrightarrow[\simeq]{f} & \mathcal{P}^{\mathrm{el}}(f(\omega)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{X/}^{\mathrm{el}} & \xrightarrow[\simeq]{f} & \mathcal{P}_{f(X)/}^{\mathrm{el}}. \end{array}$$

Here the right functor is coinitial because $\mathcal{P}$ is sound, and hence the left functor is coinitial, which proves that $\mathcal{O}$ is sound.

For the converse implication, let $\rho: X' \rightsquigarrow Y'$ be some active morphism in $\mathcal{P}$. Because we assume that $\mathrm{Ar}^{\mathrm{act}}(f)$ is essentially surjective, we can write $\rho = f(\omega)$ for $\omega: X \rightsquigarrow Y$ active in $\mathcal{O}$ as before. Then the above argument shows that $\mathcal{P}^{\mathrm{el}}(\rho) \rightarrow \mathcal{P}^{\mathrm{el}}(Y')$ must be coinitial, as it is equivalent to $\mathcal{O}^{\mathrm{el}}(\omega) \rightarrow \mathcal{O}^{\mathrm{el}}(Y)$. $\square$

The crucial application of soundness for us will be through the following lemma: this will be used in the proof of Lemma 4.2.4, which is how the assumption of soundness enters our main theorem.

Lemma 3.3.8 *Let $\mathcal{O}$ be a sound pattern and $\mathcal{C}$ a sufficiently complete ∞ -category. Consider a natural transformation $(\eta: F \Rightarrow G): \mathcal{O}_{X/}^{\mathrm{int}} \rightarrow \mathcal{C}$ such that for all $X \rightrightarrows X' \in \mathcal{O}_{X/}^{\mathrm{int}}$ the square*

$$\begin{array}{ccc} F(X') & \longrightarrow & \lim_{X' \rightrightarrows E \in \mathcal{O}_{X'/}^{\mathrm{el}}} F(E) \\ \eta_{X'} \downarrow & & \downarrow \lim \eta_E \\ G(X') & \longrightarrow & \lim_{X' \rightrightarrows E \in \mathcal{O}_{X'/}^{\mathrm{el}}} G(E) \end{array}$$

is cartesian. Then for every active morphism $\omega: X \rightsquigarrow Y$ the square

$$\begin{array}{ccc} F(X) & \longrightarrow & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} F(\omega_{\alpha!} X) \\ \eta_X \downarrow & & \downarrow \lim \eta_E \\ G(X) & \longrightarrow & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} G(\omega_{\alpha!} X) \end{array}$$

is cartesian.

Proof Consider the commutative cube

$$\begin{array}{ccccc} F(X) & \xrightarrow{\quad} & \lim_{\alpha: Y \twoheadrightarrow E' \in \mathcal{O}_{Y/}^{\text{el}}} F(\omega_{\alpha!} X) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{\beta: X \twoheadrightarrow E \in \mathcal{O}_{X/}^{\text{el}}} F(E) & \xrightarrow{\sim} & \lim_{(\alpha: Y \twoheadrightarrow E', \gamma: \omega_{\alpha!} X \twoheadrightarrow E) \in \mathcal{O}^{\text{el}}(\omega)} F(E) & & \\ \downarrow & & \downarrow & & \\ G(X) & \xrightarrow{\quad} & \lim_{\alpha: Y \twoheadrightarrow E' \in \mathcal{O}_{Y/}^{\text{el}}} G(\omega_{\alpha!} X) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{\beta: X \twoheadrightarrow E \in \mathcal{O}_{X/}^{\text{el}}} G(E) & \xrightarrow{\sim} & \lim_{(\alpha: Y \twoheadrightarrow E', \gamma: \omega_{\alpha!} X \twoheadrightarrow E) \in \mathcal{O}^{\text{el}}(\omega)} G(E). & & \end{array}$$

The front horizontal maps are equivalences because $\mathcal{O}$ is assumed to be sound and hence $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{X/}^{\text{el}}$ is coinital. The left square is cartesian by applying the assumption. We would like to show that the back square is cartesian and by pullback pasting it will suffice to show that the right square is cartesian. We may write the limit over $\mathcal{O}^{\text{el}}(\omega)$ as a double limit, by first right Kan extending along the cartesian fibration $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{Y/}^{\text{el}}$, which is computed by taking limits over the fibers $\mathcal{O}_{\omega_{\alpha!} Y/}^{\text{el}}$, and then taking the limit over $\mathcal{O}_{Y/}^{\text{el}}$. Using this reformulation the right square can be written as an $\mathcal{O}_{Y/}^{\text{el}}$ -limit of diagrams of the form

$$\begin{array}{ccc} F(\omega_{\alpha!} X) & \longrightarrow & \lim_{\omega_{\alpha!} X \twoheadrightarrow E \in \mathcal{O}_{X'/}^{\text{el}}} F(E) \\ \eta_{\omega_{\alpha!} X} \downarrow & & \downarrow \lim \eta_E \\ G(\omega_{\alpha!} X) & \longrightarrow & \lim_{\omega_{\alpha!} X \twoheadrightarrow E \in \mathcal{O}_{X'/}^{\text{el}}} G(E). \end{array}$$

Each of these diagrams is cartesian by assumption, and hence so is their limit. $\square$

We will now check explicitly that the examples of patterns we discussed above are indeed sound. To do so, the following observation will be useful:

Lemma 3.3.9 *For an algebraic pattern $\mathcal{O}$ the following conditions are equivalent:*

- (1) $\mathcal{O}$ is sound.
- (2) For every active morphism $\omega: X \rightsquigarrow Y$ and $\beta: X \twoheadrightarrow E' \in \mathcal{O}_{X/}^{\text{el}}$, the ∞ -category

$$\mathcal{O}_{\beta}^{\text{el}}(\omega) := \mathcal{O}_{Y/}^{\text{el}} \times_{\mathcal{O}_{X/}^{\text{int}}} (\mathcal{O}_{X/}^{\text{int}})_{/\beta}$$

is weakly contractible.

- (3) For every ω and β as in (2) we have $\text{colim}_{\alpha \in (\mathcal{O}_{Y/}^{\text{el}})^{\text{op}}} \text{Map}_{\mathcal{O}_{X/}^{\text{int}}}(\omega_{\alpha}, \beta) \simeq *$.

Proof ((1) $\Leftrightarrow$ (2)) The functor $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{X/}^{\text{el}}$ is a cocartesian fibration. By the dual of [22, Theorem 4.1.3.2.] it is coinital if and only if its fibers are weakly contractible. Unwinding definitions yields the following description of the straightening:

$$\mathcal{O}_{X/}^{\text{el}} \rightarrow \text{Cat}, \quad (\beta: X \rightarrow E') \mapsto \mathcal{O}_{Y/}^{\text{el}} \times_{\mathcal{O}_{X/}^{\text{int}}} (\mathcal{O}_{X/}^{\text{int}})_{/\beta}.$$

((2) $\Leftrightarrow$ (3)) Since $\mathcal{O}^{\text{el}}(\omega) \rightarrow \mathcal{O}_{Y/}^{\text{el}} \times \mathcal{O}_{X/}^{\text{el}}$ is a bifibration, passing to the fiber over $\beta \in \mathcal{O}_{X/}^{\text{el}}$ and taking opposites yields a left fibration $q: \mathcal{O}_{\beta}^{\text{el}}(\omega)^{\text{op}} \rightarrow (\mathcal{O}_{Y/}^{\text{el}})^{\text{op}}$. By [22, Corollary 3.3.4.6], the ∞ -groupoid $|\mathcal{O}_{\beta}^{\text{el}}(\omega)| \simeq |\mathcal{O}_{\beta}^{\text{el}}(\omega)^{\text{op}}|$ can be computed as the colimit of the straightening $\text{St}(q)$, which is given by

$$\text{St}(q): (\mathcal{O}_{Y/}^{\text{el}})^{\text{op}} \rightarrow \mathcal{S}, \quad (\alpha: Y \rightarrow E) \mapsto \text{Map}_{\mathcal{O}_{X/}^{\text{int}}}(\omega_{\alpha}, \beta). \quad \square$$

Observation 3.3.10 Suppose that $\mathcal{O}$ is a pattern such that for all $X \in \mathcal{O}$ the inert under-category $\mathcal{O}_{X/}^{\text{int}}$ is a poset. In this case, spelling out the definition as in Example 3.3.5 we may identify $\mathcal{O}_{\beta}^{\text{el}}(\omega)$ with the following subposet of $\mathcal{O}_{Y/}^{\text{el}}$:

$$\mathcal{O}_{\beta}^{\text{el}}(\omega) \simeq \{(\alpha: Y \rightarrowtail E) \in \mathcal{O}_{Y/}^{\text{el}} \mid \beta = \gamma \circ \omega_{\alpha}\}.$$

Example 3.3.11 For the pattern Δ^{op} the inert under-category $(\Delta^{\text{op}})_{[n]/}^{\text{int}}$ is equivalent to the poset of pairs $(a_0 \leq a_1) \in [n]$. This is elementary in $\Delta^{\text{op}, \flat}$ if and only if $a_1 - a_0 = 1$, and it is elementary in $\Delta^{\text{op}, \natural}$ if and only if $a_1 - a_0 \leq 1$. To check soundness we consider, for an active morphism $\omega: [m] \rightsquigarrow [n]$ in Δ and elementary $(b_0 \leq b_1) \in [n]$, the poset

$$(\Delta^{\text{op}})_{\beta}^{\text{el}}(\omega) \simeq \{(a_0 \leq a_1) \in (\Delta^{\text{op}})_{[m]/}^{\text{el}} \mid \omega(a_0) \leq b_0 \leq b_1 \leq \omega(a_1)\}.$$

In the case of $\Delta^{\text{op}, \flat}$ this poset has a single element, namely that given by $a_0 = \max\{a \in [m] \mid \omega(a) \leq b_0\}$ and $a_1 = a_0 + 1$, which satisfies $\omega(a_1) > b_0$ and hence $\omega(a_1) \geq b_1 = b_0 + 1$. For the pattern $\Delta^{\text{op}, \natural}$ the poset still has a single element if $b_1 = b_0 + 1$ or if $b_1 = b_0$ with $b_i \notin \omega([m])$. But if $b_1 = b_0 = \omega(a)$ for some $a \in [m]$, then the poset is the category

$$(a - 1 \leq a) \rightarrow (a \leq a) \leftarrow (a \leq a + 1),$$

which is not trivial, but still weakly contractible. This shows that $\Delta^{\text{op}, \flat}$ and $\Delta^{\text{op}, \natural}$ are both sound.

Example 3.3.12 The pattern $\mathbb{F}_{*}^{\natural}$ is sound. The inert under-category $(\mathbb{F}_{*}^{\natural})_{A_{+}/}$ is the poset of subsets $U \subset A$. Given an active morphism $\omega: A_{+} \rightarrow B_{+}$ and an elementary $E \subset A$ (ie $|E| \leq 1$), we need to check that the poset of $E' \subset Y$ with $|E'| \leq 1$ and $E \subset \omega^{-1}(E')$ is contractible. If $E = \{a\} \neq \emptyset$, then this poset has exactly one element $E' = \{\omega(a)\}$, and if $E = \emptyset$, then this poset has an initial element $E' = \emptyset$. So the poset is contractible in both cases, which proves that $\mathbb{F}_{*}^{\natural}$ is sound.

Lemma 3.3.13 *Products of sound patterns are sound: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are sound patterns, then $\mathcal{O}_1 \times \mathcal{O}_2$ is also a sound pattern.*

Proof Let $\omega = (\omega_1, \omega_2): (X_1, X_2) \rightsquigarrow (Y_1, Y_2)$ be an active morphism in $\mathcal{O}_1 \times \mathcal{O}_2$. The projection $(\mathcal{O}_1 \times \mathcal{O}_2)^{\text{el}}(\omega) \rightarrow (\mathcal{O}_1 \times \mathcal{O}_2)_{(X_1, X_2)/}^{\text{el}}$ can be identified with the product of the projections

$$\mathcal{O}_1^{\text{el}}(\omega_1) \times \mathcal{O}_2^{\text{el}}(\omega_2) \rightarrow (\mathcal{O}_1)_{X_1/}^{\text{el}} \times (\mathcal{O}_2)_{X_2/}^{\text{el}},$$

which, by assumption, is a product of coinital functors and hence again coinital. $\square$

Example 3.3.14 Applying Lemma 3.3.13 to Example 3.3.11, we see that the patterns $\Delta^{n, \text{op}, \flat}$ and $\Delta^{n, \text{op}, \natural}$ are both sound.

Next, we introduce a further condition for sound patterns; for this we first need some notation:

Notation 3.3.15 By Proposition 2.2.2, evaluation at the target $\text{ev}_1: \text{Ar}_{\text{act}}(\mathcal{O}) \rightarrow \mathcal{O}$ is a cocartesian fibration. Its straightening, denoted by $\mathcal{A}_{\mathcal{O}}: \mathcal{O} \rightarrow \text{Cat}_{\infty}$, takes $X \in \mathcal{O}$ to the ∞ -category $\mathcal{A}_{\mathcal{O}}(X) \simeq \mathcal{O}_{/X}^{\text{act}}$ of active morphisms to X . (Compare with [7, Corollary 7.4 and Remark 7.5].)

Definition 3.3.16 We say an algebraic pattern $\mathcal{O}$ is *soundly extendable* if it is sound and in addition the functor $\mathcal{A}_{\mathcal{O}}$ is a Segal $\mathcal{O}$ - ∞ -category, ie for every $X \in \mathcal{O}$, the functor

$$\mathcal{O}_{/X}^{\text{act}} \rightarrow \lim_{E \in \mathcal{O}_{X/}^{\text{el}}} \mathcal{O}_{/E}^{\text{act}}$$

is an equivalence.

Remark 3.3.17 The notion of a soundly extendable pattern is a mild strengthening of the notion of extendable pattern from [7, Definition 8.5] (which uses a slightly weaker, but more complicated condition than what we are here calling “soundness”). It was shown in [7, Lemma 9.14] that every extendable pattern $\mathcal{O}$ satisfies the condition in Definition 3.3.16, so in particular a sound pattern is extendable if and only if it is soundly extendable. In principle, there could exist extendable patterns that are not sound, but we are not aware of any examples.

Example 3.3.18 The patterns $\mathbb{F}_{*}$, $\Delta^{\text{op}, \natural}$, and $\Delta^{\text{op}, \flat}$ are soundly extendable. Their soundness was verified in Examples 3.3.5 and 3.3.11. For extendability see [7, Examples 8.13 and 8.14]. The pattern $\Theta_n^{\text{op}, \natural}$ is soundly extendable for all n (by [16, Proposition 2.7 and Lemma 3.5]), but note that $\Theta_n^{\text{op}, \flat}$ fails to be extendable for $n > 1$. (See [7, Example 8.15].)

Example 3.3.19 Let $\mathcal{O} \rightarrow \mathbb{F}_{*}$ be an ∞ -operad. Then $\mathcal{O}$ is a soundly extendable pattern. This will follow by Example 4.1.5 and Lemma 4.1.15 in the next section.

Example 3.3.20 The patterns $\mathbb{F}_{*}^{\leq k}$ are sound by Lemma 3.3.7, but not soundly extendable. Indeed, $\mathcal{A}_{\mathbb{F}_{*}^{\leq k}}: \mathbb{F}_{*}^{\leq k} \rightarrow \text{Cat}_{\infty}$ does not satisfy the Segal condition: for any $n \leq k$ the Segal map may be identified with the inclusion

$$(\mathbb{F}^{\times n})^{\leq k} \simeq \mathcal{A}_{\mathbb{F}_{*}^{\leq k}}(n) \rightarrow \mathcal{A}_{\mathbb{F}_{*}^{\leq k}}(1)^{\times n} \simeq (\mathbb{F}^{\leq k})^{\times n},$$

where $(\mathbb{F}^{\leq k})^{\times n}$ is the category of n -tuples of sets such that each set has size $\leq k$, and $(\mathbb{F}^{\times n})^{\leq k}$ denotes the full subcategory on those n -tuples of total size $\leq k$.

Lemma 3.3.21 *Let $\mathcal{O}$ and $\mathcal{P}$ be soundly extendable patterns such that $\mathcal{O}_{O/}^{\text{el}}$ and $\mathcal{P}_{P/}^{\text{el}}$ are weakly contractible for all $O \in \mathcal{O}$ and $P \in \mathcal{P}$. Then $\mathcal{O} \times \mathcal{P}$ is a soundly extendable pattern.*

Proof Soundness follows from Lemma 3.3.13. For extendability we have

$$\begin{aligned} \lim_{(\alpha, \beta): (O, P) \twoheadrightarrow (E, E') \in (\mathcal{O} \times \mathcal{P})_{(O, P)/}^{\text{el}}} (\mathcal{O} \times \mathcal{P})_{/ (E, E')}^{\text{act}} &\simeq \lim_{(\alpha: O \twoheadrightarrow E, \beta: P \twoheadrightarrow E') \in \mathcal{O}_{O/}^{\text{el}} \times \mathcal{P}_{P/}^{\text{el}}} \mathcal{O}_{/ E}^{\text{act}} \times \mathcal{P}_{/ E'}^{\text{act}} \\ &\simeq \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O/}^{\text{el}}} \mathcal{O}_{/ E}^{\text{act}} \times \lim_{\beta: P \twoheadrightarrow E' \in \mathcal{P}_{P/}^{\text{el}}} \mathcal{P}_{/ E'}^{\text{act}} \\ &\simeq \mathcal{O}_{/ O}^{\text{act}} \times \mathcal{P}_{/ P}^{\text{act}}, \end{aligned}$$

where in the second line we used that in any ∞ -category, products distribute over weakly contractible limits. $\square$

Example 3.3.22 The pattern $\Delta^{n, \text{op}, \mathbb{I}}$ is soundly extendable. The case $n = 1$ appears in Example 3.3.18, and for $n > 1$ this follows from Lemma 3.3.21 by observing that $(\Delta^{\text{op}, \mathbb{I}})_{[k]/}^{\text{el}}$ is weakly contractible for all k . (This argument fails for $\Delta^{n, \text{op}, \mathbb{b}}$ since $(\Delta^{\text{op}, \mathbb{b}})_{[0]/}^{\text{el}} = \emptyset$, and indeed this pattern is *not* extendable for $n > 1$.)

Proposition 3.3.23 *The pattern $\text{Span}_{b, f}(\mathfrak{X}; \mathfrak{X}_0)$, as defined in Definition 3.2.6, is*

- (1) *sound if $\mathfrak{X}_{/y}^b \rightarrow \mathfrak{X}_{/y}$ is fully faithful and the inclusion $\mathfrak{X}_{0/y}^b \hookrightarrow \mathfrak{X}_{0/y}$ is cofinal for every $y \in \mathfrak{X}$.*
- (2) *soundly extendable if and only if it is sound and the functor $\mathfrak{X}_{/-}^f: \mathfrak{X}^{b, \text{op}} \rightarrow \text{Cat}_{\infty}$ (defined on morphisms by pullback) is right Kan extended from $\mathfrak{X}_0^{b, \text{op}} \subseteq \mathfrak{X}^{b, \text{op}}$.*

Proof (1) By Lemma 3.3.9(3) the pattern $\text{Span}_{b, f}(\mathfrak{X}; \mathfrak{X}_0)$ is sound if and only if for every $\beta: e' \rightarrow x$ in $\mathfrak{X}^b$ and $\omega: x \rightarrow y$ in $\mathfrak{X}^f$ the following colimit indexed by $\alpha: e \rightarrow y \in \mathfrak{X}_{0/y}^b$ is contractible:

$$\begin{aligned} \text{colim}_{\alpha \in \mathfrak{X}_{0/y}^b} \text{Map}_{\mathfrak{X}_{/x}}(\beta: e' \rightarrow x, \omega^* \alpha: x \times_y e \rightarrow x) \\ \simeq \text{colim}_{\alpha \in \mathfrak{X}_{0/y}^b} \text{Map}_{\mathfrak{X}_{/x}}(\beta: e' \rightarrow x, \omega^* \alpha: x \times_y e \rightarrow x) \quad (\mathfrak{X}_{/x}^b \subset \mathfrak{X}_{/x} \text{ full}) \\ \simeq \text{colim}_{\alpha \in \mathfrak{X}_{0/y}^b} \text{Map}_{\mathfrak{X}_{/y}}(\omega \circ \beta: e' \rightarrow y, \alpha: e \rightarrow x) \quad (\omega! \dashv \omega^*) \\ \simeq \text{colim}_{\alpha \in \mathfrak{X}_{0/y}^b} \text{Map}_{\mathfrak{X}_{0/y}}(\omega \circ \beta: e' \rightarrow y, \alpha: e \rightarrow x) \quad (\mathfrak{X}_{0/y} \subset \mathfrak{X}_{/y} \text{ full}) \\ \simeq |\mathfrak{X}_{0/y}^b \times_{\mathfrak{X}_{0/y}} (\mathfrak{X}_{0/y})_{\omega \circ \beta}|. \end{aligned}$$

By [22, Theorem 4.1.3.1] this ∞ -category is weakly contractible if $\mathfrak{X}_{0/y}^b \rightarrow \mathfrak{X}_{0/y}$ is cofinal, so the claim follows.

(2) Since $\text{Span}_{b, f}(\mathfrak{X}; \mathfrak{X}_0)_{/-}^{\text{act}} \simeq \mathfrak{X}_{/-}^f$, this is a consequence of the fact that a functor is Segal if and only if its restriction to the inert subcategory is right Kan extended from the elementaries by [7, Lemma 2.9]. $\square$

As an important special case, we have:

Corollary 3.3.24 *If $\mathfrak{X}^b = \mathfrak{X}$ then $\text{Span}_{\text{all}, f}(\mathfrak{X}; \mathfrak{X}_0)$ is sound.*

Example 3.3.25 The pattern $\text{Span}(\mathbb{F})$ is soundly extendable.

Example 3.3.26 Let $\mathbb{F}_G^f \subset \mathbb{F}_G$ be closed under base change and coproduct as in Example 3.2.11. The patterns $\text{Span}_{\text{all},f}(\mathbb{F}_G)$ and $\text{Span}_{\text{inj},f}(\mathbb{F}_G)$ are soundly extendable. The slice $(\mathbb{F}_G)_{/A}^f$ decomposes as a product $\prod_{U \in A/G} (\mathbb{F}_G)_{/U}^f$ since the morphisms of $\mathbb{F}_G^f$ are closed under base change. This implies that $(\mathbb{F}_G)_{/-}^f$ is a $\text{Span}_{\text{inj},f}(\mathbb{F}_G)$ -Segal category since the elementary slice category $\text{Span}_{\text{inj},f}(\mathbb{F}_G)_{A/}^{\text{el}} \simeq (\text{Orb}_G)_{/A}^{\text{inj}}$ is equivalent to the discrete set A/G over which we are taking the product. It also follows that $(\mathbb{F}_G)_{/-}^f$ is a Segal $\text{Span}_{\text{all},f}(\mathbb{F}_G)$ -category since

$$(\text{Orb}_G)_{/A}^{\text{inj}} \simeq \text{Span}_{\text{inj},f}(\mathbb{F}_G)_{A/}^{\text{el}} \hookrightarrow \text{Span}_{\text{all},f}(\mathbb{F}_G)_{A/}^{\text{el}}$$

is coinital.

Example 3.3.27 The pattern $\text{Span}(\mathcal{S}_m)$ is soundly extendable. Soundness follows from Corollary 3.3.24. For extendability we need to show that the functor

$$(\mathcal{S}_m)_{/-} : \mathcal{S}_m^{\text{op}} \rightarrow \text{Cat}_{\infty}$$

is right Kan extended from its value at $* \in \mathcal{S}_m^{\text{op}}$. Since being m -truncated can be checked fiberwise over $Y \in \mathcal{S}_m$, this functor is equivalent to $\text{Fun}(-, \mathcal{S}_m)$ by straightening. This is now right Kan extended because $\text{Fun}(X, \mathcal{S}_m) \simeq \lim_X \mathcal{S}_m$. One can show that $\text{Span}_{(m-1)\text{-tr}, \text{all}}(\mathcal{S}_m)$ is also soundly extendable; we will not need this, however.

Finally, we give an example of a pattern that is *not* sound:

Example 3.3.28 We expect that the pattern U^{op} of undirected graphs of Hackney, Robertson, and Yau [13] is sound. However, this pattern does not include the nodeless loop S^1 . In [12], Hackney gives a simpler description of U^{op} and also defines a variant $\widetilde{U}^{\text{op}}$ that does include the nodeless loop. We will now show that this is an example of a nonsound pattern $\mathcal{O} = \widetilde{U}^{\text{op}}$. For the sake of brevity we shall not recall the definition, but rather the following facts:

- The category of elementaries under S^1 is trivial: $\mathcal{O}_{S^1/}^{\text{el}} \simeq *$.
- There is an active morphism $\omega : S^1 \rightsquigarrow S_n^1$ to the n -vertex loop S_n^1 ($n \geq 2$), for which $\mathcal{O}_{S_n^1/}^{\text{el}}$ is the poset of simplices of S_n^1 , which is weakly equivalent to S^1 .

We can now use the characterization of soundness from Lemma 3.3.9(3) in the case of the active morphism $\omega : S^1 \rightsquigarrow S_n^1$ described above. Since $\mathcal{O}_{S^1/}^{\text{el}}$ is trivial (and in this case $\omega_{\alpha!} S^1$ is always elementary), the colimit runs over the constant diagram on the point and hence evaluates to the classifying space of $\mathcal{O}_{S_n^1/}^{\text{el}}$, which is not contractible. Note that this could be resolved by introducing a variant of $\widetilde{U}^{\text{op}}$ where $\text{Map}_{\mathcal{O}}(S^1, S^1) \simeq \text{Map}_{\mathcal{O}}(S^1, e) \simeq O(2)$, in which case $\mathcal{O}_{S^1/}^{\text{el}}$ is equivalent to the ∞ -groupoid S^1 .

4 Fibrous patterns and Segal envelopes

We begin this section by introducing the notion of *fibrous $\mathcal{O}$ -patterns* as a generalization of ∞ -operads over an arbitrary base pattern $\mathcal{O}$ in Section 4.1. We then apply the results of Section 2 to fibrous patterns in Section 4.2, where we prove Theorem C. Finally, in Section 4.3 we give some examples of Segal envelopes.

4.1 Fibrous patterns

In this subsection we introduce the notion of a *fibrous $\mathcal{O}$ -pattern* over a base algebraic pattern $\mathcal{O}$. (We borrow the adjective “fibrous” from [23, Section 2.3.3], where it is used for a somewhat related concept.) Fibrous patterns specialize to give, for example, Lurie’s ∞ -operads and generalized ∞ -operads if we take the base pattern to be $\mathbb{F}_*$ or $\mathbb{F}_*^{\flat}$. The concept is also a variant of the definition of *weak Segal fibrations* given in [7]; as we will see in Proposition 4.1.7 the two notions coincide if the base pattern is sound, ie for almost all interesting examples of patterns, but the definition of fibrous patterns seems to be simpler and better behaved if we do not assume soundness.

Observation 4.1.1 Let $\mathcal{O}$ be an algebraic pattern. If $\pi: \mathcal{P} \rightarrow \mathcal{O}$ has cocartesian lifts of inert morphisms, then applying Proposition 2.2.2 to the inert-active factorization system on $\mathcal{O}$ furnishes a cocartesian fibration $\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \rightarrow \mathcal{O}$ (where this functor is given as $(P, \pi(P) \rightsquigarrow O) \mapsto O$). For a morphism $\omega: O_1 \rightarrow O_2$ in $\mathcal{O}$ the cocartesian transport functor $\omega_!: \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O_1}^{\mathrm{act}} \rightarrow \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O_2}^{\mathrm{act}}$ is given by

$$(P, \varphi: \pi(P) \rightsquigarrow O_1) \mapsto (\alpha_! P, \beta: O' \rightsquigarrow O_2),$$

where

$$\pi(P) \xrightarrow{\alpha} O' \xrightarrow{\beta} O_2$$

is the inert-active factorization of the composite

$$\pi(P) \xrightarrow{\varphi} O_1 \xrightarrow{\omega} O_2$$

and $P \rightarrow \alpha_! P$ is a cocartesian lift of α .

Definition 4.1.2 Let $\mathcal{O}$ be an algebraic pattern. Then a *fibrous $\mathcal{O}$ -pattern* is a functor $\pi: \mathcal{P} \rightarrow \mathcal{O}$ such that:

- (1) $\mathcal{P}$ has all π -cocartesian lifts of inert morphisms in $\mathcal{O}$.
- (2) For all $O \in \mathcal{O}$, the commutative square of ∞ -categories

$$\begin{array}{ccc} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\mathrm{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{/O}^{\mathrm{el}}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\mathrm{act}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{/O}^{\mathrm{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{/O}^{\mathrm{el}}} \mathcal{O}_{/E}^{\mathrm{act}} \end{array}$$

is cartesian. Here the horizontal functors are induced by cocartesian transport along the maps $O \rightarrowtail E$ in $\mathcal{O}_{/O}^{\mathrm{el}}$ for the cocartesian fibrations from Observation 4.1.1, applied to π and $\mathrm{id}_{\mathcal{O}}$.

Observation 4.1.3 Definition 4.1.2(2) says precisely that the straightening of the projection $\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \rightarrow \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$ over $\mathcal{O}$, ie the natural transformation $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathcal{P}): \mathrm{St}_{\mathcal{O}}(\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})) \rightarrow \mathcal{A}_{\mathcal{O}}$, is a relative Segal $\mathcal{O}$ - ∞ -category.

Remark 4.1.4 For many patterns $\mathcal{O}$, the functor $\mathcal{O}_{/-}^{\mathrm{act}}$ is a Segal $\mathcal{O}$ - ∞ -category; this is the case, for instance, if $\mathcal{O}$ is *extendable* in the sense of [7] by [7, Lemma 9.14]. In this case, Observation 3.1.9 implies that (2) is satisfied if and only if the functor $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathcal{P})$ is a Segal $\mathcal{O}$ - ∞ -category, ie the functor

$$\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/\mathcal{O}}^{\mathrm{act}} \rightarrow \lim_{E \in \mathcal{O}_{\mathcal{O}/}^{\mathrm{el}}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\mathrm{act}}$$

is an equivalence for all $\mathcal{O} \in \mathcal{O}$.

Example 4.1.5 Since $\mathbb{F}_{\star}$ is extendable, a fibrous $\mathbb{F}_{\star}$ -pattern is a functor $\pi: \mathcal{P} \rightarrow \mathbb{F}_{\star}$ such that $\mathcal{P}$ has π -cocartesian lifts for inerts, and for all n the functor

$$\mathcal{P}^{\mathrm{act}} \times_{\mathbb{F}} \mathbb{F}_{/\langle n \rangle} \simeq \mathcal{P} \times_{\mathbb{F}_{\star}} (\mathbb{F}_{\star})_{/\langle n \rangle}^{\mathrm{act}} \rightarrow \prod_{\langle n \rangle \twoheadrightarrow \langle 1 \rangle} \mathcal{P} \times_{\mathbb{F}_{\star}} \mathbb{F} \simeq (\mathcal{P}^{\mathrm{act}})^n$$

is an equivalence. This functor takes an object $P \in \mathcal{P}^{\mathrm{act}}$ over $\langle m \rangle$ in $\mathbb{F}_{\star}$ together with an active map $\omega: \langle m \rangle \rightsquigarrow \langle n \rangle$ to the list of objects $(P_1, \dots, P_n)$ where $P \twoheadrightarrow P_j$ is the cocartesian lift of the inert map $\omega_j := (\rho_j \circ \omega)^{\mathrm{int}}: \langle m \rangle \twoheadrightarrow \langle m \rangle_j$ where $\rho_j: \langle n \rangle \twoheadrightarrow \langle 1 \rangle$ is as in Example 3.1.6. We will see later (Proposition 4.1.7) that this condition is equivalent to $\mathcal{P} \rightarrow \mathbb{F}_{\star}$ being an ∞ -operad in the sense of Lurie.

We can rewrite the second condition in Definition 4.1.2 to obtain the following equivalent characterization of fibrous patterns:

Proposition 4.1.6 For any algebraic pattern $\mathcal{O}$, a functor $\pi: \mathcal{P} \rightarrow \mathcal{O}$ is a fibrous $\mathcal{O}$ -pattern if and only if:

- (1) $\mathcal{P}$ has π -cocartesian morphisms over inert morphisms in $\mathcal{O}$.
- (2) For every active morphism $\omega: \mathcal{O}_1 \rightsquigarrow \mathcal{O}_2$ in $\mathcal{O}$, and all objects $X_0 \in \mathcal{P}_{\mathcal{O}_0}$, $X_1 \in \mathcal{P}_{\mathcal{O}_1}$, the commutative square

$$(4) \quad \begin{array}{ccc} \mathrm{Map}_{\mathcal{P}}(X_0, X_1) & \longrightarrow & \lim_{\alpha: \mathcal{O}_2 \twoheadrightarrow E \in \mathcal{O}_{\mathcal{O}_2/}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}(X_0, \omega_{\alpha,!} X_1) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{O}}(\mathcal{O}_0, \mathcal{O}_1) & \longrightarrow & \lim_{\alpha: \mathcal{O}_2 \twoheadrightarrow E \in \mathcal{O}_{\mathcal{O}_2/}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{O}}(\mathcal{O}_0, \omega_{\alpha,!} \mathcal{O}_1) \end{array}$$

is cartesian. Here the horizontal maps are defined using the functor $\omega_{(-)}: \mathcal{O}_{\mathcal{O}_2/}^{\mathrm{el}} \rightarrow \mathcal{O}_{\mathcal{O}_1/}^{\mathrm{int}}$, which was introduced in Notation 3.3.1.

- (3) For every active morphism $\omega: \mathcal{O}_1 \rightsquigarrow \mathcal{O}_2$ in $\mathcal{O}$, the functor

$$\mathcal{P}_{\mathcal{O}_1}^{\simeq} \rightarrow \lim_{\alpha: \mathcal{O}_2 \rightarrow E \in \mathcal{O}_{\mathcal{O}_2/}^{\mathrm{el}}} \mathcal{P}_{\omega_{\alpha,!} \mathcal{O}_1}^{\simeq},$$

induced by cocartesian transport along the inert morphisms $\omega_{\alpha}: \mathcal{O}_1 \twoheadrightarrow \omega_{\alpha,!} \mathcal{O}_1$ in $\mathcal{O}_{\mathcal{O}_1/}^{\mathrm{int}}$, is an equivalence.

Proof A square of ∞ -categories is cartesian if and only if the underlying square of ∞ -groupoids as well as all induced squares of mapping spaces are cartesian. For the square in the definition of a fibrous pattern the underlying square of ∞ -groupoids is

$$\begin{array}{ccc} \mathcal{P}^{\simeq} \times_{\mathcal{O}} (\mathcal{O}/O)^{\simeq} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{P}^{\simeq} \times_{\mathcal{O}} (\mathcal{O}/E)^{\simeq} \\ \downarrow & & \downarrow \\ (\mathcal{O}/O)^{\simeq} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} (\mathcal{O}/E)^{\simeq}. \end{array}$$

This square is cartesian if and only if the map on fibers over each $\omega: O' \rightsquigarrow O$ is an equivalence. This map takes the form

$$(5) \quad \mathcal{P}_{O'}^{\simeq} \rightarrow \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{P}_{\omega_{\alpha,!} O'}^{\simeq}$$

and is induced by the cocartesian transport along the inert morphisms $\omega_{\alpha}: O' \twoheadrightarrow \omega_{\alpha,!} O'$ as in Notation 3.3.1. This is exactly the map from (3), so the square of ∞ -groupoids is cartesian if and only if (3) holds.

Now consider, for two objects $(P, \varphi: \pi(P) \rightsquigarrow O)$ and $(P', \varphi': \pi(P') \rightsquigarrow O) \in \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}}$, the square of mapping spaces

$$(6) \quad \begin{array}{ccc} \text{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}}} ((P, \varphi), (P', \varphi')) & \longrightarrow & \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O'}^{\text{el}}} \text{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}}} ((P, \varphi), (\varphi'_{\alpha,!} P', (\alpha \circ \varphi')^{\text{act}})) \\ \downarrow & & \downarrow \\ \text{Map}_{\mathcal{O}_{/O}^{\text{act}}} (\varphi, \varphi') & \longrightarrow & \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O'}^{\text{el}}} \text{Map}_{\mathcal{O}_{/E}^{\text{act}}} (\varphi, (\alpha \circ \varphi')^{\text{act}}). \end{array}$$

A point in $\text{Map}_{\mathcal{O}_{/O}^{\text{act}}} (\varphi, \varphi')$ is a (necessarily active) morphism $f: \pi(P) \rightsquigarrow \pi(P')$ together with a homotopy $\varphi \simeq \varphi' \circ f$. To compute the fiber of the vertical maps at this point, note that the mapping space in $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}}$ can be computed as

$$\text{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}}} ((P, \varphi), (P', \varphi')) \simeq \text{Map}_{\mathcal{P}} (P, P') \times_{\text{Map}_{\mathcal{O}} (\pi(P), \pi(P'))} \text{Map}_{\mathcal{O}_{/O}^{\text{act}}} (\varphi, \varphi').$$

Hence the map on the vertical fibers of the square is given by

$$(7) \quad \text{Map}_{\mathcal{P}}^f (P, P') \rightarrow \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O'}^{\text{el}}} \text{Map}_{\mathcal{P}}^{\varphi'_{\alpha,!} \circ f} (P, \varphi'_{\alpha,!} P'),$$

where the superscripts indicate fibers over maps in $\mathcal{O}$. This agrees with the map on fibers over f of the square in (2). Therefore (2) implies that the square of mapping spaces is a pullback.

However, we have not shown the converse yet, because we have only considered the fibers in (4) over morphisms $f \in \text{Map}_{\mathcal{O}} (O_0, O_1)$ that are active. Let us now assume that the square of mapping spaces (6) is cartesian. For a general morphism $O_0 \rightarrow O_1$ we can find an inert-active factorization $O_0 \xrightarrow{j} Q \xrightarrow{\xi} O_1$. Since j is inert we can find a cocartesian lift $\tilde{j}: P_0 \rightarrow j_! P_0$ and by virtue of this being cocartesian,

precomposition with $\tilde{j}$ induces the vertical equivalences in the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{P}}^g(j!P, P') & \longrightarrow & \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}^{\varphi'_{\alpha} \circ g}(j!P, \varphi'_{\alpha,!}P') \\ (-) \circ \tilde{j} \Big\downarrow \simeq & & \simeq \Big\downarrow (-) \circ \tilde{j} \\ \mathrm{Map}_{\mathcal{P}}^{g \circ j}(P, P') & \longrightarrow & \lim_{\alpha: O \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}^{\varphi'_{\alpha} \circ g \circ j}(P, \varphi'_{\alpha,!}P'). \end{array}$$

Since g is active, the previous argument shows that the top map is an equivalence. Hence the bottom map is an equivalence and as $f = g \circ j$ was arbitrary this shows that (2) is implied. $\square$

The conditions in Proposition 4.1.6 are reminiscent of Lurie's definition of an ∞ -operad [23]. Note, however, that in conditions (2) and (3) we need to consider *all* active maps in $\mathcal{O}$, while Lurie's definition of ∞ -operads, or the definition of *weak Segal fibrations* in [7], only involve the conditions corresponding to identity maps. If the base pattern is *sound*, however, the conditions for all active maps are implied by this special case:

Proposition 4.1.7 *Suppose $\mathcal{O}$ is a sound pattern. Then a functor $\pi: \mathcal{P} \rightarrow \mathcal{O}$ is a fibrous $\mathcal{O}$ -pattern if and only if it is a weak Segal $\mathcal{O}$ -fibration in the sense of [7, Definition 9.6], ie the conditions of Proposition 4.1.6 hold whenever ω is an identity morphism. Concretely:*

- (1) $\mathcal{P}$ has all π -cocartesian lifts of inert morphisms in $\mathcal{O}$.
- (2) For every $O_1 \in \mathcal{O}$, the functor

$$\mathcal{P}_{O_1}^{\sim} \rightarrow \lim_{\alpha: O_1 \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathcal{P}_E^{\sim},$$

induced by cocartesian transport along $\alpha: O_1 \twoheadrightarrow E$, is an equivalence.

- (3) For all $O_0, O_1 \in \mathcal{O}$, and all objects $X_0 \in \mathcal{P}_{O_0}, X_1 \in \mathcal{P}_{O_1}$, the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{P}}(X_0, X_1) & \longrightarrow & \lim_{\alpha: O_1 \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}(X_0, \alpha!X_1) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{O}}(O_0, O_1) & \longrightarrow & \lim_{\alpha: O_1 \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{O}}(O_0, E) \end{array}$$

is cartesian.

Remark 4.1.8 In [7] (and [23]), the analogue of (2) says that the functor

$$\mathcal{P}_{O_1} \rightarrow \lim_{\alpha: O_1 \twoheadrightarrow E \in \mathcal{O}_{O_1}^{\mathrm{el}}} \mathcal{P}_{\alpha!O_2'}$$

is an equivalence, rather than that the underlying map of ∞ -groupoids is one. However, it follows from (3) that this functor gives an equivalence on mapping spaces, ie it is already fully faithful, and so it is an equivalence if and only if it is an equivalence on underlying ∞ -groupoids. In fact, it would suffice in (2) to assume that the map is merely surjective on π_0 .

Proof of Proposition 4.1.7 Suppose $\pi: \mathcal{P} \rightarrow \mathcal{O}$ is a weak Segal fibration. Consider the functor $F: \mathcal{O}_{1/}^{\text{int}} \rightarrow \mathcal{O}^{\text{int}} \rightarrow \mathcal{S}$ defined by $F(O_1 \twoheadrightarrow O_2) := \mathcal{P}_{O_2}^{\simeq}$ and cocartesian transport along inerts. The natural transformation $\eta: F \Rightarrow *$ to the terminal functor satisfies the conditions of Lemma 3.3.8. The conclusion of the lemma tells us that (2) holds for all $\omega: O_1 \rightsquigarrow O_2$.

For property (3), fix $X_0, X_1 \in \mathcal{P}$ with $\pi(X_0) = O_0$ and $\pi(X_1) = O_1$. Then cocartesian transport along inerts defines a functor

$$F: \mathcal{O}_{1/}^{\text{int}} \rightarrow \mathcal{S}, \quad (\varphi: O_1 \twoheadrightarrow O_2) \mapsto \text{Map}(X_0, \varphi_! X_1),$$

and this admits a canonical natural transformation to the functor $G(\varphi: O_1 \twoheadrightarrow O_2) := \text{Map}(O_0, O_2)$. Applying Lemma 3.3.8 to $\eta: F \Rightarrow G$ shows that (3) holds for all $\omega: O_1 \rightsquigarrow O_2$. $\square$

Example 4.1.9 Fibrous $\mathbb{F}_*$ -patterns are precisely (symmetric) ∞ -operads as defined in [23], while fibrous $\mathbb{F}_*^{\natural}$ -patterns are generalized (symmetric) ∞ -operads. Similarly, fibrous $\Delta^{\text{op},b}$ - and $\Delta^{\text{op},\natural}$ -patterns are nonsymmetric (or planar) ∞ -operads and generalized nonsymmetric ∞ -operads, respectively.

Observation 4.1.10 For a sound pattern $\mathcal{O}$ we can also describe the fibrous $\mathcal{O}$ -patterns that are cocartesian fibrations as the unstraightenings of Segal $\mathcal{O}$ - ∞ -categories, ie as the *Segal $\mathcal{O}$ -fibrations* of [7, Definition 9.1]. This is easy to check directly, but it is also a special case of Lemma 4.2.4 (taking $Y = *$), which we will prove below.

Fibrous $\mathcal{O}$ -patterns admit a canonical pattern structure, which we now introduce:

Definition 4.1.11 Suppose $\pi: \mathcal{P} \rightarrow \mathcal{O}$ is a fibrous $\mathcal{O}$ -pattern. We say a morphism in $\mathcal{P}$ is *inert* if it is π -cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it just lies over an active morphism in $\mathcal{O}$. The inert and active morphisms then form a factorization system on $\mathcal{P}$ by [23, Proposition 2.1.2.5], and we give $\mathcal{P}$ an algebraic pattern structure with this factorization system by taking the elementary objects to be all those that lie over elementary objects in $\mathcal{O}$.

Definition 4.1.12 A *morphism of fibrous $\mathcal{O}$ -patterns* is a commutative triangle

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & \mathcal{P}' \\ & \searrow \pi & \swarrow \pi' \\ & \mathcal{O}, & \end{array}$$

where π and π' are fibrous $\mathcal{O}$ -patterns and f is a morphism of algebraic patterns. It is immediate from the definition of the pattern structures that for this it suffices to require that f preserves inert morphisms. We write $\text{Fbrs}(\mathcal{O})$ for the full subcategory of $\text{AlgPatt}_{/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$ -patterns; this is equivalently a full subcategory of $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$.

Lemma 4.1.13 The inclusion $\text{Fbrs}(\mathcal{O}) \hookrightarrow \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ preserves limits and κ -filtered colimits where κ is a regular cardinal such that $\mathcal{O}_I^{\text{el}}$ is κ -small for all $O \in \mathcal{O}$. Limits and κ -filtered colimits of $\mathcal{O}$ -fibrous patterns can therefore be computed in $\text{Cat}_{\infty/\mathcal{O}}$.

Proof By Observation 2.3.8 the forgetful functor $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{O}}$ preserves limits and κ -filtered colimits, and is also conservative. It therefore suffices to observe that the commutative square that is required to be cartesian for an object of $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ to be a fibrous $\mathcal{O}$ -pattern commutes with limits and κ -filtered colimits of ∞ -categories. Since a limit or filtered colimit of cartesian squares in Cat_{∞} is again cartesian, this implies the result. $\square$

Observation 4.1.14 If $\pi : \mathcal{P} \rightarrow \mathcal{O}$ is a fibrous $\mathcal{O}$ -pattern, then for every object $\bar{X} \in \mathcal{P}$ over X in $\mathcal{O}$, the functor

$$\mathcal{P}_{\bar{X}/}^{\text{el}} \rightarrow \mathcal{O}_{X/}^{\text{el}}$$

is an equivalence. Indeed, since $\mathcal{P}^{\text{int}} \rightarrow \mathcal{O}^{\text{int}}$ is a cocartesian fibration the functor $\mathcal{P}_{\bar{X}/}^{\text{int}} \rightarrow \mathcal{O}_{X/}^{\text{int}}$ is an equivalence, and the above functor is obtained by restricting to the full subcategories of elementary objects. In particular, π is an iso-Segal morphism. More generally, if $f : \mathcal{P} \rightarrow \mathcal{Q}$ is a morphism of fibrous $\mathcal{O}$ -patterns, then f induces an equivalence

$$\mathcal{P}_{\bar{X}/}^{\text{el}} \xrightarrow{\sim} \mathcal{Q}_{f(\bar{X})/}^{\text{el}}$$

for the same reason, so that f is also an iso-Segal morphism.

Lemma 4.1.15 Suppose $\mathcal{O}$ is a sound pattern and $\pi : \mathcal{P} \rightarrow \mathcal{O}$ is $\mathcal{O}$ -fibrous. Then $\mathcal{P}$ is also a sound pattern. Moreover, if $\mathcal{O}$ is soundly extendable, then so is $\mathcal{P}$.

Proof As we just observed that π is iso-Segal, soundness follows from Lemma 3.3.7.

Now assume $\mathcal{O}$ is soundly extendable. Then, by Remark 4.1.4, the functor

$$\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{Y/}^{\text{act}} \rightarrow \lim_{E' \in \mathcal{O}_{Y/}^{\text{el}}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{E'/}^{\text{act}}$$

is an equivalence. Since any morphism in $\mathcal{P}$ that is mapped to an active morphism in $\mathcal{O}$ is active by definition and active morphisms satisfy cancellation, we have that $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{Y/}^{\text{act}} = \mathcal{P}^{\text{act}} \times_{\mathcal{O}^{\text{act}}} \mathcal{O}_{Y/}^{\text{act}}$. Consider the case where $Y = \pi(X)$ for $X \in \mathcal{P}$. Since $\mathcal{P} \rightarrow \mathcal{O}$ is an equivalence on elementary slices, we can rewrite the limit on the right-hand side as a limit over $E \in \mathcal{P}_{X/}^{\text{el}}$ and set $E' := \pi(E)$:

$$\mathcal{P}^{\text{act}} \times_{\mathcal{O}^{\text{act}}} \mathcal{O}_{\pi(X)/}^{\text{act}} \xrightarrow{\sim} \lim_{E \in \mathcal{P}_{X/}^{\text{el}}} \mathcal{P}^{\text{act}} \times_{\mathcal{O}^{\text{act}}} \mathcal{O}_{\pi(E)/}^{\text{act}}.$$

Now, passing to the over-category of $(X, \text{id}_{\pi(X)})$ we obtain an equivalence:

$$\mathcal{P}_{/X}^{\text{act}} \simeq (\mathcal{P}^{\text{act}} \times_{\mathcal{O}^{\text{act}}} \mathcal{O}_{\pi(X)/}^{\text{act}}) / (X, \text{id}_{\pi(X)}) \xrightarrow{\sim} \lim_{E \in \mathcal{P}_{X/}^{\text{el}}} (\mathcal{P}^{\text{act}} \times_{\mathcal{O}^{\text{act}}} \mathcal{O}_{\pi(E)/}^{\text{act}}) / (E, \text{id}_{\pi(E)}) \simeq \lim_{E \in \mathcal{P}_{X/}^{\text{el}}} \mathcal{P}_{/E}^{\text{act}},$$

which shows that $\mathcal{P}$ is soundly extendable. $\square$

Proposition 4.1.16 Suppose we have a commutative triangle of algebraic patterns

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{F} & \mathcal{P} \\ & \searrow q & \swarrow p \\ & \mathcal{O}, & \end{array}$$

where $\mathcal{P}$ is $\mathcal{O}$ -fibrous. Assume further that $\mathcal{O}$ is sound. Then $\mathcal{Q}$ is $\mathcal{O}$ -fibrous if and only if it is $\mathcal{P}$ -fibrous.

Proof By Lemma 4.1.15 $\mathcal{P}$ is also sound, so we may use the characterization from Proposition 4.1.7.

Any inert morphism $\pi: P \rightarrow P'$ in $\mathcal{P}$ is cocartesian over an inert morphism $\omega: O \rightarrow O'$ in $\mathcal{O}$; if $\varphi: Q \rightarrow Q'$ is an inert morphism over ω in $\mathcal{Q}$ such that $F(Q) \simeq P$, then we have $F(\varphi) \simeq \pi$ since F preserves inert morphisms and π is the unique inert morphism over ω with source P . It now follows from [22, Proposition 2.4.1.3] that $\varphi: Q \rightarrow Q'$ is F -cocartesian if and only if it is q -cocartesian. Thus (1) in Proposition 4.1.7 holds for F if and only if it holds for q .

Assuming this holds, then for $Q, Q' \in \mathcal{Q}$, $P = F(Q)$, $P' = F(Q')$ and $O = q(Q)$, $O' = q(Q')$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Q}}(Q, Q') & \longrightarrow & \lim_{\alpha \in \mathcal{O}_{O'}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{Q}}(Q, \alpha_! Q') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{P}}(P, P') & \longrightarrow & \lim_{\alpha \in \mathcal{O}_{O'}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}(P, \alpha_! P') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{O}}(O, O') & \longrightarrow & \lim_{(\alpha: O' \twoheadrightarrow E) \in \mathcal{O}_{O'}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{O}}(O, E). \end{array}$$

Here the bottom square is cartesian since $\mathcal{P}$ is $\mathcal{O}$ -fibrous, so the top square is cartesian if and only if the outer square is cartesian. But since p is an iso-Segal morphism (by Observation 4.1.14) we can rewrite the top square as

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{Q}}(Q, Q') & \longrightarrow & \lim_{\beta \in \mathcal{P}_{P'}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{Q}}(Q, \beta_! Q') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{P}}(P, P') & \longrightarrow & \lim_{(\beta: P' \rightarrow E') \in \mathcal{P}_{P'}^{\mathrm{el}}} \mathrm{Map}_{\mathcal{P}}(P, E'), \end{array}$$

and so (3) in Proposition 4.1.7 holds for F if and only if it holds for q . The proof for (2) is similar. $\square$

Corollary 4.1.17 *If $\mathcal{O}$ is sound and $\pi: \mathcal{P} \rightarrow \mathcal{O}$ exhibits $\mathcal{P}$ as an $\mathcal{O}$ -fibrous pattern, then composition with π gives a functor*

$$\pi_!: \mathrm{Fbrs}(\mathcal{P}) \rightarrow \mathrm{Fbrs}(\mathcal{O}),$$

and this induces an equivalence

$$\mathrm{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathrm{Fbrs}(\mathcal{O})_{/\mathcal{P}}.$$

Example 4.1.18 Let $(\pi: \mathcal{O} \rightarrow \mathbb{F}_*) \in \mathrm{Opd}_{\infty}$ be an ∞ -operad in the sense of Lurie, ie a fibrous $\mathbb{F}_*$ -pattern. Applying Corollary 4.1.17 we obtain an equivalence:

$$\mathrm{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Fbrs}(\mathbb{F}_*)_{/\mathcal{O}} = \mathrm{Opd}_{\infty/\mathcal{O}}.$$

So fibrous $\mathcal{O}$ -patterns are simply ∞ -operads over $\mathcal{O}$.

Lemma 4.1.19 Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism. Then pullback along f restricts to a functor

$$f^*: \text{Fbrs}(\mathcal{P}) \rightarrow \text{Fbrs}(\mathcal{O}), \quad (\pi: \mathcal{F} \rightarrow \mathcal{P}) \mapsto (f^*\pi: \mathcal{F} \times_{\mathcal{P}} \mathcal{O} \rightarrow \mathcal{O}).$$

Proof Suppose $\pi: \mathcal{F} \rightarrow \mathcal{P}$ is a $\mathcal{P}$ -fibrous pattern. Definition 4.1.2(1) for $f^*\mathcal{F}$ follows from the usual description of cocartesian morphisms in a pullback, since f preserves inert morphisms. To prove (2), we observe that $f^*\mathcal{F} \times_{\mathcal{O}} \text{Ar}_{\text{act}}(\mathcal{O}) \simeq \mathcal{F} \times_{\mathcal{P}} \text{Ar}_{\text{act}}(\mathcal{O})$, so that we have a cartesian square

$$\begin{array}{ccc} f^*\mathcal{F} \times_{\mathcal{O}} \text{Ar}_{\text{act}}(\mathcal{O}) & \longrightarrow & \mathcal{F} \times_{\mathcal{P}} \text{Ar}_{\text{act}}(\mathcal{P}) \times_{\mathcal{P}} \mathcal{O} \\ \downarrow & & \downarrow \\ \text{Ar}_{\text{act}}(\mathcal{O}) & \longrightarrow & \text{Ar}_{\text{act}}(\mathcal{P}) \times_{\mathcal{P}} \mathcal{O} \end{array}$$

of cocartesian fibrations over $\mathcal{O}$. Straightening yields the cartesian square

$$\begin{array}{ccc} \text{St}_{\mathcal{O}}^{\text{int}}(f^*\mathcal{F}) & \longrightarrow & \text{St}_{\mathcal{P}}^{\text{int}}(\mathcal{F}) \circ f \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{O}} & \longrightarrow & \mathcal{A}_{\mathcal{P}} \circ f \end{array}$$

of functors $\mathcal{O} \rightarrow \text{Cat}_{\infty}$. By Observation 4.1.3 the natural transformation $\text{St}_{\mathcal{P}}^{\text{int}}(\mathcal{F}) \rightarrow \mathcal{A}_{\mathcal{P}}$ is a relative $\mathcal{P}$ -Segal ∞ -category. This remains true after precomposing with f (by Observation 3.1.15, since f is strong Segal). Hence the right vertical map in the square is a relative $\mathcal{O}$ -Segal ∞ -category and by Lemma 3.1.10 so is the left vertical arrow. Using Observation 4.1.3 again we see that $f^*\mathcal{F}$ is fibrous. $\square$

Example 4.1.20 The morphism $\mathfrak{c}: \Delta^{\text{op},b} \rightarrow \mathbb{F}_*$ from Example 3.1.13 is iso-Segal and hence Lemma 4.1.19 shows that pulling back along it defines a functor:

$$\mathfrak{c}^*: \text{Fbrs}(\mathbb{F}_*) \rightarrow \text{Fbrs}(\Delta^{\text{op},b}).$$

Under the identifications of Example 4.1.9 this is exactly the forgetful functor from (symmetric) ∞ -operads to nonsymmetric ∞ -operads.

Finally, let us note that we can lift the comparison of Proposition 3.1.16 to fibrous patterns:

Proposition 4.1.21 Suppose $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and let $\pi: \mathcal{Q} \rightarrow \mathcal{P}$ be a fibrous pattern. Then $\bar{f}: f^*\mathcal{Q} \rightarrow \mathcal{Q}$ is also a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and thus induces an equivalence

$$\bar{f}^*: \text{Seg}_{\mathcal{Q}}(\mathcal{S}) \xrightarrow{\sim} \text{Seg}_{f^*\mathcal{Q}}(\mathcal{S}).$$

Proof Denote by $\pi': \mathcal{Q}' := f^*\mathcal{Q} \rightarrow \mathcal{O}$ the projection map. Since $\mathcal{Q}$ is fibrous and f is strong Segal, it follows from Lemma 4.1.19 that $\mathcal{Q}'$ is also fibrous. By Observation 4.1.14 we have $\mathcal{Q}_{\mathcal{Q}'}^{\text{el}} \simeq \mathcal{P}_{\pi(\mathcal{Q})}^{\text{el}}/$

and similarly for $\mathcal{Q}'$ and $\mathcal{O}$. The map $(\mathcal{Q}')^{\text{el}}_{\mathcal{Q}'} \rightarrow \mathcal{Q}'^{\text{el}}_{\bar{f}(\mathcal{Q})}$ therefore identifies with $\mathcal{O}_{\pi'(\mathcal{Q})}^{\text{el}} \rightarrow \mathcal{P}_{f(\pi'(\mathcal{Q}))}^{\text{el}}$ which is coinitial by the assumption that f is strong Segal. We conclude that $\bar{f}$ is strong Segal. We proceed by verifying the conditions. Condition (1) of Proposition 3.1.16 is visibly stable under base change so it remains to check (2). Observe that for every object $Q \in \mathcal{Q}'$ that lies over $O \in \mathcal{O}$ we have by [22, Lemma 5.4.5.4] a pullback square of slice ∞ -categories

$$\begin{array}{ccc} \mathcal{Q}'_{/Q} & \longrightarrow & \mathcal{Q}'_{/\bar{f}(Q)} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{O}_{/O} & \longrightarrow & \mathcal{P}_{/f(O)}. \end{array}$$

By assumption the bottom map induces an equivalence on the underlying spaces of active maps and since the square is cartesian the same holds for the top map. $\square$

4.2 Segal envelopes

In this section we will specialize our results from Section 2 to fibrous $\mathcal{O}$ -patterns over an algebraic pattern $\mathcal{O}$. Recall that we have shown that from the inert-active factorization system on $\mathcal{O}$ we obtain an adjunction

$$(-) \times_{\mathcal{O}} \text{Ar}_{\text{act}}(\mathcal{O}) : \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \rightleftarrows (\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})},$$

where the right adjoint is given by pulling back along the map $\mathcal{O} \rightarrow \text{Ar}_{\text{act}}(\mathcal{O})$ given by the degeneracy $[1] \rightarrow [0]$. This can equivalently be interpreted as a “straightening-unstraightening” adjunction

$$\text{St}_{\mathcal{O}}^{\text{int}} : \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \rightleftarrows \text{Fun}(\mathcal{O}, \text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}} : \text{Un}_{\mathcal{O}}^{\text{int}}$$

in which the left adjoint is fully faithful with image the $\mathcal{A}_{\mathcal{O}}$ -equifibered functors.

We can immediately identify the image of the full subcategory $\text{Fbrs}(\mathcal{O})$ under this fully faithful functor:

Proposition 4.2.1 *For any algebraic pattern $\mathcal{O}$, the fully faithful functor $\text{St}_{\mathcal{O}}^{\text{int}}$ identifies $\text{Fbrs}(\mathcal{O})$ with the full subcategory of $\text{Fun}(\mathcal{O}, \text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}}$ spanned by the equifibered maps that are also relative Segal objects. In other words, the functor $\text{St}_{\mathcal{O}}^{\text{int}}$ restricts to a fully faithful functor*

$$\text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} := \text{St}_{\mathcal{O}}^{\text{int}}|_{\text{Fbrs}(\mathcal{O})} : \text{Fbrs}(\mathcal{O}) \hookrightarrow \text{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\text{Cat}_{\infty})$$

with image the equifibered objects. Moreover, for any strong Segal morphism $f : \mathcal{O} \rightarrow \mathcal{P}$, we have a commutative square

$$(8) \quad \begin{array}{ccc} \text{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \text{Fbrs}(\mathcal{O}) \\ \downarrow \text{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}} & & \downarrow \text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \\ \text{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\text{Cat}_{\infty}) & \xrightarrow{f^{\oplus}} & \text{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\text{Cat}_{\infty}), \end{array}$$

where the functor $f^{\otimes}$ is given by the composite

$$\mathrm{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathrm{Cat}_{\infty}) \xrightarrow{f^*} \mathrm{Seg}_{\mathcal{O}}^{/f^*\mathcal{A}_{\mathcal{P}}}(\mathrm{Cat}_{\infty}) \rightarrow \mathrm{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathrm{Cat}_{\infty})$$

of restriction along f and pullback along the natural map $\mathcal{A}_{\mathcal{O}} \rightarrow f^*\mathcal{A}_{\mathcal{P}}$ (see Observation 3.1.15 and Lemma 3.1.10).

Proof From Observation 4.1.3 we know that an object $\mathcal{P}$ of $\mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}}$ is a fibrous $\mathcal{O}$ -pattern if and only if $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathcal{P})$ is a relative Segal $\mathcal{O}$ -object in Cat_{∞} . The commutative square (8) likewise follows by restricting the square (3) in Observation 2.3.9 to full subcategories. $\square$

From this observation we can deduce some pleasant properties of the ∞ -categories of fibrous patterns:

Corollary 4.2.2 *For any algebraic pattern $\mathcal{O}$, the ∞ -category $\mathrm{Fbrs}(\mathcal{O})$ is presentable, and fits in a cartesian square of fully faithful right adjoints*

$$\begin{array}{ccc} \mathrm{Fbrs}(\mathcal{O}) & \xleftarrow{\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}} & \mathrm{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathrm{Cat}_{\infty}) \\ \downarrow & & \downarrow \\ \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}} & \xleftarrow{\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}} & \mathrm{Fun}(\mathcal{O}^{\mathrm{op}}, \mathrm{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}}. \end{array}$$

Proof We know from Proposition 4.2.1 that we have the given cartesian square of fully faithful functors; it remains to show that this is a square in Pr^R . For the bottom horizontal and right vertical functor we have shown this in Proposition 2.3.7 and Lemma 3.1.11, respectively. It now follows that the rest of the diagram also lies in Pr^R , since the diagram is cartesian and by [22, Theorem 5.5.3.18] Pr^R admits pullbacks and the inclusion $\mathrm{Pr}^R \subset \mathrm{Cat}_{\infty}$ preserves them. $\square$

Corollary 4.2.3 (1) *For any algebraic pattern $\mathcal{O}$, the following functors admit left adjoints:*

$$\mathrm{Fbrs}(\mathcal{O}) \hookrightarrow \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{O}}.$$

(2) *For any strong Segal morphism $f: \mathcal{O} \rightarrow \mathcal{P}$, the functor $f^*: \mathrm{Fbrs}(\mathcal{P}) \rightarrow \mathrm{Fbrs}(\mathcal{O})$ admits a left adjoint.*

Proof The first claim was shown in Corollary 4.2.2 and Observation 2.3.8. In particular limits and κ -filtered colimits in $\mathrm{Fbrs}(\mathcal{O})$ for appropriate κ are computed in $\mathrm{Cat}_{\infty/\mathcal{O}}$. This implies that $f^*: \mathrm{Fbrs}(\mathcal{P}) \rightarrow \mathrm{Fbrs}(\mathcal{O})$ preserves limits and κ -filtered colimits, since we know pullback along f preserves limits and filtered colimits as a functor $\mathrm{Cat}_{\infty/\mathcal{P}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{O}}$. Hence the claim follows from the adjoint functor theorem. $\square$

In Proposition 4.2.1 we only showed that the left adjoint $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ restricts to a functor from fibrous patterns to relative Segal objects—in general the right adjoint $\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}$ does not necessarily take relative Segal $\mathcal{O}$ - ∞ -categories over $\mathcal{A}_{\mathcal{O}}$ to fibrous $\mathcal{O}$ -patterns. However, this is the case if $\mathcal{O}$ is sound; to see this, we first need a technical lemma:

Lemma 4.2.4 Let $\mathcal{O}$ be a sound algebraic pattern and let $\gamma: X \rightarrow Y$ be a morphism in $\text{Fun}(\mathcal{O}, \text{Cat}_\infty)$, with $\Gamma: \mathcal{X} \rightarrow \mathcal{Y}$ denoting its unstraightening. Then the following are equivalent:

- (1) $\Gamma: \mathcal{X} \rightarrow \mathcal{Y}$ is a relative Segal object.
- (2) $\text{St}_\mathcal{O}^{\text{int}}(\Gamma): \text{St}_\mathcal{O}^{\text{int}}(\mathcal{X}) \rightarrow \text{St}_\mathcal{O}^{\text{int}}(\mathcal{Y})$ is a relative Segal object, ie the commutative square

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}} \\ \downarrow & & \downarrow \\ \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}} \end{array}$$

is cartesian for all $O \in \mathcal{O}$.

Proof For $O \in \mathcal{O}$, we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}} & & \\ & \searrow & \searrow & \searrow & \\ & \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{/O}^{\text{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}} & \\ & \searrow & \searrow & \searrow & \\ & \mathcal{O}_{/O}^{\text{act}} & \longrightarrow & \lim_{E \in \mathcal{O}_{O'}^{\text{el}}} \mathcal{O}_{/E}^{\text{act}} & \end{array}$$

Here all four functors to the bottom row are cocartesian fibrations, and the morphisms in the top square preserve cocartesian morphisms. We therefore see that condition (2), which asks for the top square to be cartesian, is equivalent to all squares of fibers over $\omega: O' \rightsquigarrow O$ in $\mathcal{O}_{/O}^{\text{act}}$ being cartesian. The relevant square of fibers is

$$\begin{array}{ccc} X(O') & \longrightarrow & \lim_{(\alpha: O \rightarrowtail E) \in \mathcal{O}_{O'}^{\text{el}}} X(\omega_\alpha! O') \\ \downarrow & & \downarrow \\ Y(O') & \longrightarrow & \lim_{(\alpha: O \rightarrowtail E) \in \mathcal{O}_{O'}^{\text{el}}} Y(\omega_\alpha! O'). \end{array}$$

Considering the special case $\omega = \text{id}_O$ we see that (2) implies (1), while to see that the converse holds when $\mathcal{O}$ is sound we apply Lemma 3.3.8 with $F = X$ and $G = Y$. $\square$

Proposition 4.2.5 If the pattern $\mathcal{O}$ is sound, then the adjunction of Notation 2.3.4 restricts to an adjunction

$$\text{Env}_\mathcal{O}^{/A_\mathcal{O}}: \text{Fbrs}(\mathcal{O}) \rightleftarrows \text{Seg}_\mathcal{O}^{/A_\mathcal{O}}(\text{Cat}_\infty): \text{Un}_\mathcal{O}^{\text{int}}.$$

Moreover, if $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism between sound patterns, then in addition to the square (8) we have a commutative square

$$(9) \quad \begin{array}{ccc} \mathrm{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathrm{Cat}_{\infty}) & \xrightarrow{f^{\otimes}} & \mathrm{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathrm{Cat}_{\infty}) \\ \mathrm{Un}_{\mathcal{P}}^{\mathrm{int}} \downarrow & & \downarrow \mathrm{Un}_{\mathcal{O}}^{\mathrm{int}} \\ \mathrm{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \mathrm{Fbrs}(\mathcal{O}). \end{array}$$

Proof We need to show that $\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}: \mathrm{Fun}(\mathcal{O}, \mathrm{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}}$ sends $\mathcal{A}_{\mathcal{O}}$ -relative Segal objects to fibrous $\mathcal{O}$ -patterns. Since we know an object of $\mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}}$ is fibrous if and only if its image under $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ is a relative Segal object, it suffices to show that $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}} \circ \mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}$ preserves relative Segal objects.

Let $X \rightarrow \mathcal{A}_{\mathcal{O}}$ be a relative Segal object; then $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X))$ fits into a cartesian square

$$\begin{array}{ccc} \mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X)) & \longrightarrow & \mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}(X)) \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{O}} & \longrightarrow & \mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}(\mathcal{A}_{\mathcal{O}})) \end{array}$$

obtained from applying $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ to the cartesian square defining $\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X)$. Since relative Segal objects are stable under base change by Lemma 3.1.10, it suffices to show the right vertical map is a relative Segal object, which follows from Lemma 4.2.4. The commutative square (9) follows by restricting the square (2) in Observation 2.3.9 to full subcategories. $\square$

For soundly extendable patterns $\mathcal{O}$ we can furthermore think of this adjunction as being induced by one between fibrous patterns and Segal $\mathcal{O}$ - ∞ -categories:

Theorem 4.2.6 *Let $\mathcal{O}$ be a soundly extendable pattern. Then there is an adjunction*

$$\mathrm{Env}_{\mathcal{O}}: \mathrm{Fbrs}(\mathcal{O}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}_{\infty}),$$

where $\mathrm{Env}_{\mathcal{O}}(\mathcal{P})(X) := \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/X}^{\mathrm{act}}$ and the right adjoint is given by unstraightening. This induces an adjunction

$$\mathrm{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}: \mathrm{Fbrs}(\mathcal{O}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}},$$

where the left adjoint is fully faithful and the image consists of the Segal $\mathcal{O}$ - ∞ -categories that are equifibered over $\mathcal{A}_{\mathcal{O}}$.

Proof It remains to show that the adjunction

$$(-) \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}): \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}} \rightleftarrows \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{cocart}} \simeq \mathrm{Fun}(\mathcal{O}, \mathrm{Cat}_{\infty})$$

from Corollary 2.2.5 restricts to an adjunction between $\mathrm{Fbrs}(\mathcal{O})$ and $\mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}_{\infty})$. Since $\mathcal{A}_{\mathcal{O}}$ is a Segal $\mathcal{O}$ - ∞ -category, we have by Observation 3.1.9 and Proposition 4.2.1 that the left adjoint takes fibrous patterns to Segal $\mathcal{O}$ - ∞ -categories. On the other hand, the right adjoint takes the latter to fibrous patterns by Observation 4.1.10. $\square$

Remark 4.2.7 In the context of Theorem 4.2.6 the right adjoint of $\text{Env}_{\mathcal{O}}$ is faithful and replete. It induces an equivalence between $\text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})$ and the subcategory of $\text{Fbrs}(\mathcal{O})$ whose objects are cocartesian fibrous patterns and whose morphisms preserve all cocartesian edges.

Remark 4.2.8 If $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism between soundly extendable patterns, then pullback/restriction along f gives a commutative square

$$\begin{array}{ccc} \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty}) & \xrightarrow{f^*} & \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty}) \\ \downarrow & & \downarrow \\ \text{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \text{Fbrs}(\mathcal{O}). \end{array}$$

Note, however, that the corresponding Beck–Chevalley transformation is usually not invertible, so we have to slice over $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{O}}$ to get a commutative square of envelopes

$$(10) \quad \begin{array}{ccc} \text{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \text{Fbrs}(\mathcal{O}) \\ \text{Env}_{\mathcal{P}}^{\mathcal{A}_{\mathcal{P}}} \downarrow & & \downarrow \text{Env}_{\mathcal{O}}^{\mathcal{A}_{\mathcal{O}}} \\ \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{P}}} & \xrightarrow{f^{\oplus}} & \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}} \end{array}$$

as a special case of (8).

4.3 Examples of Segal envelopes

Example 4.3.1 For the soundly extendable pattern $\mathbb{F}_{*}$, we know that fibrous patterns are exactly ∞ -operads, while Segal $\mathbb{F}_{*}$ - ∞ -categories are symmetric monoidal ∞ -categories; here $\mathcal{A}_{\mathbb{F}_{*}}$ is the symmetric monoidal category $\mathbb{F}^{\mathbb{I}}$ of finite sets under disjoint union. Hence Theorem 4.2.6 yields an adjunction

$$\text{Env}_{\mathbb{F}_{*}}^{\mathbb{F}^{\mathbb{I}}}: \text{Opd}_{\infty} = \text{Fbrs}(\mathbb{F}_{*}) \rightleftarrows \text{Seg}_{\mathbb{F}_{*}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathbb{F}_{*}}} = \text{CMon}(\text{Cat}_{\infty})_{/(\mathbb{F}, \mathbb{I})}.$$

The left adjoint is fully faithful and a symmetric monoidal functor $\pi: (\mathcal{C}, \otimes) \rightarrow (\mathbb{F}, \mathbb{I})$ lies in the essential image if and only if it is equifibered. This means that the following square is cartesian for all maps $\omega: X \rightarrow Y$ in $\mathbb{F}$:

$$\begin{array}{ccc} \mathcal{C}^X & \xrightarrow{\omega \otimes} & \mathcal{C}^Y \\ \pi^X \downarrow & & \downarrow \pi^Y \\ \mathbb{F}^X & \xrightarrow{\omega \mathbb{I}} & \mathbb{F}^Y. \end{array}$$

Here the horizontal functors tensor over fibers of ω . In fact, it follows by taking products and pasting pullback diagrams⁵ that it suffices to check the case of $\omega: \{1, 2\} \rightarrow \{1\}$.

Observation 4.3.2 The essential image of the sliced envelope functor

$$\text{Env}_{\mathbb{F}_{*}}^{\mathbb{F}^{\mathbb{I}}}: \text{Opd}_{\infty} \hookrightarrow \text{CMon}(\text{Cat}_{\infty})_{/(\mathbb{F}, \mathbb{I})}$$

was first described in [19], but the characterization there looks at first glance quite different from ours. Let us therefore compare these two descriptions:

⁵See Lemma 5.2.16 for an elaboration of this argument.

For a symmetric monoidal functor $\pi: \mathcal{C} \rightarrow \mathbb{F}$, let us write $\mathcal{C}_{(1)} \subset \mathcal{C}$ for the full subcategory of those $x \in \mathcal{C}$ with $|\pi(x)| = 1$. Then the characterization of [19] is that the essential image consists of those π that satisfy the following pair of conditions:

- (1) Every object $x \in \mathcal{C}$ is equivalent to $x_1 \otimes \cdots \otimes x_n$ for some $x_i \in \mathcal{C}_{(1)}$.
- (2) For every $n, m \geq 0$ and any two tuples $x_1, \dots, x_m \in \mathcal{C}_{(1)}$ and $y_1, \dots, y_n \in \mathcal{C}_{(1)}$, the canonical map

$$\coprod_{\varphi: m \rightarrow n} \prod_{i=1}^n \text{Map}_{\mathcal{C}} \left(\bigotimes_{j \in \varphi^{-1}(i)} x_j, y_i \right) \rightarrow \text{Map} \left(\bigotimes_{j=1}^m x_j, \bigotimes_{i=1}^n y_i \right)$$

is an equivalence.

These conditions must be equivalent to our equifiberedness condition since they describe the same full subcategory. To check this more explicitly, we consider the functor

$$D_n: \mathcal{C}^n \rightarrow \mathbb{F}^n \times_{\mathbb{F}} \mathcal{C},$$

which is an equivalence for all n if and only if $p: \mathcal{C} \rightarrow \mathbb{F}$ is equifibered. The functor D_n is essentially surjective if and only if for any $x \in \mathcal{C}$ and a decomposition $\pi(x) = A_1 \amalg \cdots \amalg A_n$ there is a decomposition $x = x_1 \otimes \cdots \otimes x_n$ such that $\pi(x_i) \cong A_i$ compatibly with the decomposition. By choosing the trivial decomposition with $|A_i| = 1$ this recovers condition (1). Conversely, given condition (1) we can decompose x as $\bigotimes_{a \in \pi(x)} y_a$ and then find the desired x_i as $x_i = \bigotimes_{a \in A_i} y_a$.

To see that the full faithfulness of the D_n 's corresponds to condition (2), we first observe that in the presence of condition (1) we can replace condition (2) with the following:

- (2') For every $n \geq 0$ and any two tuples $z_1, \dots, z_n \in \mathcal{C}$ and $y_1, \dots, y_n \in \mathcal{C}$, the canonical map

$$\prod_{i=1}^n \text{Map}_{\mathcal{C}}(z_i, y_i) \rightarrow \coprod_{(\varphi_i: \pi(z_i) \rightarrow \pi(y_i))} \text{Map}^{\amalg_{i=1}^n \varphi_i} \left(\bigotimes_{i=1}^n z_i, \bigotimes_{i=1}^n y_i \right)$$

is an equivalence. Here we write $\text{Map}_{\mathcal{C}}^{\varphi}(a, b)$ for the fiber of $\text{Map}_{\mathcal{C}}(a, b)$ over some $\varphi: \pi(a) \rightarrow \pi(b)$.

To relate this to condition (2), first decompose y_i using condition (1) and use two-out-of-three to reduce to the case where $|\pi(y_i)| = 1$. Then write $z_i = \bigotimes_{j \in \varphi^{-1}(i)} x_j$ and argue as in [19, Remark 2.4.8].

Now we can observe that D_n is fully faithful if and only if condition (2') holds: indeed, the mapping space in $\mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}$ can be described as

$$\begin{aligned} \text{Map}_{\mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}}((x, \pi(x) = A_1 \amalg \cdots \amalg A_n), (y, \pi(y) = B_1 \amalg \cdots \amalg B_n)) \\ \simeq \text{Map}_{\mathbb{F}^n}((A_i), (B_i)) \times_{\text{Map}_{\mathbb{F}}(\pi(x), \pi(y))} \text{Map}_{\mathcal{C}}(x, y) \\ \simeq \coprod_{(\varphi_i: A_i \rightarrow B_i)} \text{Map}^{\amalg \varphi_i}(x, y). \end{aligned}$$

Applying this to the images of $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ under $\mathcal{C}^n \rightarrow \mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}$ yields the desired form.

It is interesting that while in condition (2) we need to quantify over all $n, m \geq 0$, in condition (2') it suffices to consider only the case $n = 2$ as all other cases can be obtained inductively. This works because the objects z_i and y_i in condition (2') are themselves allowed to be composite.

Example 4.3.3 For the soundly extendable pattern $\Delta^{\text{op},b}$ fibrous patterns are nonsymmetric ∞ -operads, while Segal $\Delta^{\text{op},b}$ - ∞ -categories are monoidal ∞ -categories. We therefore define $\text{Opd}_{\infty}^{\text{ns}} := \text{Fbrs}(\Delta^{\text{op},b})$ and $\text{Mon}(\text{Cat}_{\infty}) := \text{Seg}_{\Delta^{\text{op},b}}(\text{Cat}_{\infty})$. The Segal $\Delta^{\text{op},b}$ -category $\mathcal{A}_{\Delta^{\text{op},b}}$ is equivalent to the category Δ_+ of finite (possibly empty) linearly ordered sets, with the monoidal structure given by concatenation. The envelope functor $\text{Env}_{\Delta^{\text{op},b}}^{\Delta_+}$ can then be interpreted as a fully faithful embedding:

$$\text{Env}_{\Delta^{\text{op},b}}^{\Delta_+} : \text{Opd}_{\infty}^{\text{ns}} \hookrightarrow \text{Mon}(\text{Cat}_{\infty})/\Delta_+.$$

Similarly to Example 4.3.1 we can describe the essential image as those monoidal functors $\pi : \mathcal{V} \rightarrow \Delta_+$ for which the following natural square is cartesian:

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes} & \mathcal{V} \\ \pi \downarrow & & \downarrow \pi \\ \Delta_+ \times \Delta_+ & \xrightarrow{\otimes} & \Delta_+. \end{array}$$

Example 4.3.4 For the soundly extendable pattern $\Delta^{\text{op},\natural}$, fibrous patterns are generalized nonsymmetric ∞ -operads as defined in [11], while Segal $\Delta^{\text{op},\natural}$ - ∞ -categories are category objects in Cat_{∞} , ie double ∞ -categories. We thus write $\text{Opd}_{\infty}^{\text{gen,ns}} := \text{Fbrs}(\Delta^{\text{op},\natural})$ and $\text{DbCat}_{\infty} := \text{Seg}_{\Delta^{\text{op},\natural}}(\text{Cat}_{\infty})$. We may regard $(\infty, 2)$ -categories (in the form of complete 2-fold Segal spaces) as those double ∞ -categories $\mathcal{X}_{\bullet}$ such that $\mathcal{X}_0$ is an ∞ -groupoid and which satisfy a completeness condition. In particular, the Segal $\Delta^{\text{op},\natural}$ - ∞ -category $\mathcal{A}_{\Delta^{\text{op},\natural}} \simeq \mathcal{A}_{\Delta^{\text{op},b}}$ may be thought of as the one-object $(\infty, 2)$ -category $\mathfrak{B}\Delta_+$ where the endomorphisms of the single object are Δ_+ , with the monoidal structure corresponding to composition. The envelope functor $\text{Env}_{\Delta^{\text{op},\natural}}^{\mathfrak{B}\Delta_+}$ can then be interpreted as giving a fully faithful embedding:

$$\text{Env}_{\Delta^{\text{op},\natural}}^{\mathfrak{B}\Delta_+} : \text{Opd}_{\infty}^{\text{gen,ns}} \hookrightarrow \text{DbCat}_{\infty}/\mathfrak{B}\Delta_+.$$

The essential image is characterized by a pullback square analogous to the one from Example 4.3.3. The morphisms in $\text{Opd}_{\infty}^{\text{gen,ns}}$ among the cocartesian fibrations that correspond to $(\infty, 2)$ -categories are precisely *lax functors* as defined for instance in [10], so we obtain a description of these in terms of $\text{DbCat}_{\infty}/\mathfrak{B}\Delta_+$. (More generally, we can also consider the envelope for $\Delta^{n,\text{op},\natural}$, which was briefly discussed in [15].)

Example 4.3.5 Let $\mathcal{O} \rightarrow \mathbb{F}_{\ast}$ be an ∞ -operad. Fibrous $\mathcal{O}$ -patterns are, by Example 4.1.18, exactly ∞ -operads over $\mathcal{O}$, while Segal $\mathcal{O}$ - ∞ -categories are precisely $\mathcal{O}$ -monoidal ∞ -categories which we denote by $\text{Mon}_{\mathcal{O}}(\text{Cat}_{\infty}) := \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})$. By Example 3.3.19, $\mathcal{O}$ is soundly extendable and our construction recovers the $\mathcal{O}$ -monoidal envelope of [23, Section 2.2.4]. In particular, this gives a fully faithful embedding

$$\text{Env}_{\mathcal{O}}^{\mathcal{A}_{\mathcal{O}}} : \text{Opd}_{\infty/\mathcal{O}} \rightarrow \text{Mon}_{\mathcal{O}}(\text{Cat}_{\infty})/\mathcal{A}_{\mathcal{O}}.$$

In the case $\mathcal{O} = \mathbb{E}_n$, the ∞ -category $\mathcal{A}_{\mathbb{E}_n}$ admits an alternative description as the $\mathbb{E}_n$ -monoidal ∞ -category of embedded n -disks in $\mathbb{R}^n$.

5 The comparison theorem

In Section 5.1 we use the Segal envelopes to prove the comparison result, Theorem A. We then discuss the application of this to equivariant ∞ -operads, Corollary B, in Section 5.2. Finally, we explain how to upgrade the envelope and comparison equivalences to equivalences of $(\infty, 2)$ -categories in Section 5.3.

5.1 Comparing fibrous patterns

In this subsection we will use Segal envelopes to obtain a criterion for a morphism of patterns $f : \mathcal{O} \rightarrow \mathcal{P}$ to induce via pullback an equivalence

$$f^* : \mathrm{Fbrs}(\mathcal{P}) \xrightarrow{\simeq} \mathrm{Fbrs}(\mathcal{O})$$

between the corresponding ∞ -categories of fibrous patterns. We specialize this to recover some comparison results from [23] without using the technical results on approximations to ∞ -operads from [23, Section 2.3.3]. As new applications, we show that (symmetric) ∞ -operads can also be described as fibrous patterns over $\mathrm{Span}(\mathbb{F})$, and that fibrous patterns over $\mathrm{Span}(\mathcal{S}_m)$ and $\mathrm{Span}_{(m-1)\text{-tr, all}}(\mathcal{S}_m)$ are equivalent.

Theorem 5.1.1 *Suppose $\mathcal{O}$ is a pattern, $\mathcal{P}$ is a soundly extendable pattern, and $f : \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism such that the following conditions hold:*

- (i) $f^{\mathrm{el}} : \mathcal{O}^{\mathrm{el}} \rightarrow \mathcal{P}^{\mathrm{el}}$ is an equivalence of ∞ -categories.
- (ii) $(\mathcal{O}_{/X}^{\mathrm{act}})^{\simeq} \rightarrow (\mathcal{P}_{/f(X)}^{\mathrm{act}})^{\simeq}$ is an equivalence for all $X \in \mathcal{O}$.

Then pullback along f gives an equivalence

$$f^* : \mathrm{Fbrs}(\mathcal{P}) \xrightarrow{\simeq} \mathrm{Fbrs}(\mathcal{O}).$$

Remark 5.1.2 If we also assume that $\mathcal{A}_{\mathcal{O}}^{\simeq} = (\mathcal{O}_{/-}^{\mathrm{act}})^{\simeq}$ is an $\mathcal{O}$ -Segal space, for example if $\mathcal{O}$ is soundly extendable, then it suffices to check condition (ii) when X is elementary.

Example 5.1.3 Let $\mathcal{P}$ be a soundly extendable pattern, and define $\mathcal{O} \subset \mathcal{P}$ as the full subpattern on the “necessary objects” in the sense of [7, Definition 14.7]. This means that $\mathcal{O}$ contains those $X \in \mathcal{P}$ for which there exists an active morphism $X \rightsquigarrow E$ with E elementary. Then Theorem 5.1.1 applies to the full inclusion $\mathcal{O} \subset \mathcal{P}$ and hence restriction yields an equivalence $\mathrm{Fbrs}(\mathcal{P}) \simeq \mathrm{Fbrs}(\mathcal{O})$.

First we show that condition (ii) can always be strengthened as follows.

Lemma 5.1.4 *In the situation of Theorem 5.1.1 the induced natural transformation*

$$\alpha : \mathcal{A}_{\mathcal{O}} \rightarrow f^* \mathcal{A}_{\mathcal{P}}$$

of functors $\mathcal{O} \rightarrow \mathrm{Cat}_{\infty}$ is an equivalence. In particular $\mathcal{A}_{\mathcal{O}}$ is $\mathcal{O}$ -Segal.

Proof By assumption, the functor $\mathcal{A}_{\mathcal{O}}(X) \rightarrow \mathcal{A}_{\mathcal{P}}(f(X))$ is an equivalence on underlying ∞ -groupoids, so it remains to show that it is fully faithful. To see this, observe that given active maps $\varphi: X \rightsquigarrow Y$ and $\varphi': X' \rightsquigarrow Y$, the mapping space $\text{Map}_{\mathcal{O}_{/Y}^{\text{act}}}(\varphi', \varphi)$ is the fiber at φ' of the map $(\varphi \circ -): \mathcal{A}_{\mathcal{O}}^{\sim}(X) \rightarrow \mathcal{A}_{\mathcal{O}}^{\sim}(Y)$. This map fits into a square

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{O}}^{\sim}(X) & \xrightarrow{\sim} & \mathcal{A}_{\mathcal{P}}^{\sim}(fX) \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{O}}^{\sim}(Y) & \xrightarrow{\sim} & \mathcal{A}_{\mathcal{P}}^{\sim}(fY), \end{array}$$

where the horizontal maps are equivalences. Then we also have equivalences on fibers, which gives the desired full faithfulness. Finally we note that $\mathcal{A}_{\mathcal{O}} \simeq f^* \mathcal{A}_{\mathcal{P}}$ implies that $\mathcal{A}_{\mathcal{O}}$ is Segal since $\mathcal{A}_{\mathcal{P}}$ was assumed to be Segal and f^* preserves Segal objects. $\square$

The following lemma tells us that for sound patterns it suffices to check $\text{Ar}_{\text{act}}(\mathcal{O})$ -equifiberedness on active morphisms that end in elementary objects.

Lemma 5.1.5 *Let $\mathcal{O}$ be a sound pattern and let $(\eta: F \Rightarrow G)$ be a relative Segal object over $\mathcal{O}$ in a sufficiently complete ∞ -category $\mathcal{C}$. Suppose that the naturality squares*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\omega)} & F(Y) \\ \downarrow \eta_X & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(\omega)} & G(Y) \end{array}$$

are cartesian for active morphisms $\omega: X \rightsquigarrow Y$ where Y is elementary. Then they are also cartesian for arbitrary Y , ie η is $\text{Ar}_{\text{act}}(\mathcal{O})$ -equifibered.

Proof For an arbitrary active morphism $\omega: X \rightsquigarrow Y$ consider the commutative cube

$$\begin{array}{ccccc} & & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} F(\omega_{\alpha}! X) & \xrightarrow{\lim F(\omega_{\alpha})} & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} F(E) \\ & \nearrow & \downarrow & & \downarrow \lim \eta_E \\ F(X) & \xrightarrow{\quad F(\omega) \quad} & F(Y) & & \\ \downarrow \eta_X & & \downarrow \lim \eta_{\omega_{\alpha}! X} & & \\ & \nearrow & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} G(\omega_{\alpha}! X) & \xrightarrow{\lim G(\omega_{\alpha})} & \lim_{\alpha: Y \twoheadrightarrow E \in \mathcal{O}_{Y/}^{\text{el}}} G(E) \\ G(X) & \xrightarrow{\quad G(\omega) \quad} & G(Y) & & \end{array}$$

The back square is cartesian as it is a limit over squares that we have assumed to be cartesian. (Note that $\omega_{\alpha}: \omega_{\alpha}! X \rightsquigarrow E$ is an active morphism with elementary target.) The right face is cartesian because η is a relative Segal object, and so is the left face from this and Lemma 3.3.8. Therefore the front face is cartesian by the pullback pasting lemma. $\square$

Proof of Theorem 5.1.1 It follows from Proposition 3.1.16 that the functor

$$f^*: \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty}) \rightarrow \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})$$

is an equivalence. From Lemma 5.1.4 we have $\mathcal{A}_{\mathcal{O}} \simeq f^* \mathcal{A}_{\mathcal{P}}$ and that $\mathcal{A}_{\mathcal{O}}$ is Segal. Hence the induced functor

$$f^{\oplus}: \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{P}}} \rightarrow \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}}$$

is also an equivalence. This means in the commutative square

$$\begin{array}{ccc} \text{Fbrs}(\mathcal{P}) & \xrightarrow{f^*} & \text{Fbrs}(\mathcal{O}) \\ \downarrow \text{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}} & & \downarrow \text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \\ \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{P}}} & \xrightarrow{f^{\oplus}} & \text{Seg}_{\mathcal{O}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{O}}} \end{array}$$

from Proposition 4.2.1, the bottom horizontal functor $f^{\oplus}$ is an equivalence, while the vertical functors are fully faithful. It follows that the top horizontal functor f^* is also fully faithful. To prove that it is also essentially surjective, it suffices to show that an object of $\text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{P}}}$ is in the image of $\text{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}$ if its image under the equivalence $f^{\oplus}$ is in the image of $\text{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$.

Suppose we are given some $(\eta: F \Rightarrow \mathcal{A}_{\mathcal{P}}) \in \text{Seg}_{\mathcal{P}}(\text{Cat}_{\infty})_{/\mathcal{A}_{\mathcal{P}}}$ such that $f^{\oplus} F \Rightarrow \mathcal{A}_{\mathcal{O}}$ is equifibered. Equivalently, $\eta_{\circ} f: (F \circ f) \Rightarrow (\mathcal{A}_{\mathcal{P}} \circ f)$ is equifibered. By Lemma 5.1.5 it suffices to check that the naturality squares are cartesian for active morphisms $\omega: X \rightsquigarrow E \in \mathcal{P}$ ending in an elementary. Since $f: \mathcal{O}^{\text{el}} \rightarrow \mathcal{P}^{\text{el}}$ is an equivalence, we may write $E \simeq f(E')$ for $E' \in \mathcal{O}$. Moreover, since $f: \mathcal{O}_{/E'}^{\text{act}} \rightarrow \mathcal{P}_{/f(E')}^{\text{act}}$ is an equivalence, we can find $\rho: Y \rightsquigarrow E' \in \mathcal{O}$ such that $f(\rho) \simeq \omega$ as objects of $\text{Ar}_{\text{act}}(\mathcal{P})$. Now it follows that the naturality square of η at ω is cartesian since we assumed that the naturality square of $\eta_{\circ} f$ at ρ is equifibered. This shows that η is $\text{Ar}_{\text{act}}(\mathcal{P})$ -equifibered, and hence that $f^*: \text{Fbrs}(\mathcal{P}) \rightarrow \text{Fbrs}(\mathcal{O})$ is essentially surjective. $\square$

As a variant of Theorem 5.1.1, we get a useful criterion for identifying the effect of the pushforward functor $f_!: \text{Fbrs}(\mathcal{O}) \rightarrow \text{Fbrs}(\mathcal{P})$ for a map of patterns $f: \mathcal{O} \rightarrow \mathcal{P}$:

Corollary 5.1.6 Suppose we have a commutative diagram of patterns

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{g} & \mathcal{R} \\ \downarrow p & & \downarrow q \\ \mathcal{O} & \xrightarrow{f} & \mathcal{P} \end{array}$$

such that

- (i) $\mathcal{O}$ is sound and $\mathcal{Q}$ is a fibrous $\mathcal{O}$ -pattern,
- (ii) $\mathcal{P}$ is soundly extendable and $\mathcal{R}$ is a fibrous $\mathcal{P}$ -pattern,
- (iii) f is a strong Segal morphism,
- (iv) g satisfies the assumptions of Theorem 5.1.1.

Then the induced map of fibrous $\mathcal{O}$ -patterns $\mathcal{Q} \rightarrow f^* \mathcal{R}$ is adjoint to an equivalence $f_! \mathcal{Q} \xrightarrow{\sim} \mathcal{R}$.

Proof For a fibrous $\mathcal{P}$ -pattern $\mathcal{T}$, we have natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fbrs}(\mathcal{O})}(\mathcal{Q}, f^*\mathcal{T}) &\simeq \mathrm{Map}_{\mathrm{Fbrs}(\mathcal{O})/\mathcal{Q}}(\mathcal{Q}, p^*f^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathrm{Fbrs}(\mathcal{Q})}(g^*\mathcal{R}, g^*q^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathrm{Fbrs}(\mathcal{R})}(\mathcal{R}, q^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathrm{Fbrs}(\mathcal{P})}(\mathcal{R}, \mathcal{T}), \end{aligned}$$

where we have used Theorem 5.1.1 and Corollary 4.1.17. $\square$

Corollary 5.1.7 Suppose $\mathcal{O}$ is a sound pattern, $q: \mathcal{P} \rightarrow \mathbb{F}_*$ is a symmetric ∞ -operad, and $f: \mathcal{O} \rightarrow \mathcal{P}$ is a strong Segal morphism that satisfies the assumptions of Theorem 5.1.1. Then f exhibits $\mathcal{P}$ as the **symmetrization** of $\mathcal{O}$, in the sense that the induced map $(qf)_! \mathcal{O} \rightarrow \mathcal{P}$ is an equivalence. $\square$

Example 5.1.8 Let Ass be the (symmetric) associative ∞ -operad as defined in [23, Definition 4.1.1.1.], and let $\mathrm{Cut}: \Delta^{\mathrm{op}} \rightarrow \mathrm{Ass}$ denote the functor defined in [23, Construction 4.1.2.9.]. Then pullback along Cut gives an equivalence

$$\mathrm{Fbrs}(\Delta^{\mathrm{op},b}) \xleftarrow{\sim} \mathrm{Fbrs}(\mathrm{Ass}) \xrightarrow{\sim} \mathrm{Fbrs}(\mathbb{F}_*)/\mathrm{Ass}$$

between nonsymmetric ∞ -operads and symmetric ∞ -operads over Ass , where the second equivalence is that of Corollary 4.1.17. In other words, nonsymmetric ∞ -operads are equivalent to symmetric ∞ -operads over the associative ∞ -operad. Moreover, Ass is the symmetrization of $\Delta^{\mathrm{op},b}$.

The equivalence of Example 5.1.8 is also proved by Lurie as [23, Theorem 4.1.3.14], which is a special case of [23, Theorem 2.3.3.26]. This more general statement can also be proved by our methods; to see this, we first need to recall some definitions:

Definition 5.1.9 Let $\pi: \mathcal{O} \rightarrow \mathbb{F}_*$ be an ∞ -operad. We say a functor $f: \mathcal{C} \rightarrow \mathcal{O}$ is an *approximation* if the following conditions hold:

- (1) For $C \in \mathcal{C}$ over $\langle n \rangle$ in $\mathbb{F}_*$, there exists for $i = 1, \dots, n$ a locally cocartesian morphism $\rho_i^C: C \rightarrow C_i$ in $\mathcal{C}$ over $\rho_i: \langle n \rangle \rightarrow \langle 1 \rangle$. Moreover, the image of ρ_i^C in $\mathcal{O}$ is inert.
- (2) $\mathcal{C}$ has all f -cartesian lifts of active morphisms in $\mathcal{O}$.

Following [21], we say that f is a *strong approximation* if we additionally have:

- (3) The functor $\mathcal{C}_{\langle 1 \rangle} \rightarrow \mathcal{O}_{\langle 1 \rangle}$ is an equivalence.

Observation 5.1.10 Suppose $\mathcal{O}$ is an ∞ -operad and $f: \mathcal{C} \rightarrow \mathcal{O}$ is an approximation. We say a morphism in $\mathcal{C}$ is *inert* if its image in $\mathcal{O}$ is inert, and *active* if it is f -cartesian and its image in $\mathcal{O}$ is active. Then the inert and active morphisms in $\mathcal{C}$ give a factorization system. We think of $\mathcal{C}$ as an algebraic pattern using this factorization system, with the elementary objects being those that map to $\langle 1 \rangle$ in $\mathbb{F}_*$; then f is a morphism of algebraic patterns.

Proposition 5.1.11 Suppose $\mathcal{O}$ is an ∞ -operad and $f: \mathcal{C} \rightarrow \mathcal{O}$ is a strong approximation.

- (i) $\mathcal{C}^{\text{el}} \rightarrow \mathcal{O}^{\text{el}}$ is an equivalence.
- (ii) $\mathcal{C}_{C/}^{\text{el}} \rightarrow \mathcal{O}_{f(C)/}^{\text{el}}$ is an equivalence for all $C \in \mathcal{C}$, ie f is an iso-Segal morphism.
- (iii) $\mathcal{C}_{/C}^{\text{act}} \rightarrow \mathcal{O}_{/f(C)}^{\text{act}}$ is an equivalence for all $C \in \mathcal{C}$.

Proof For (i), observe that from the equivalence $\mathcal{C}_{\langle 1 \rangle} \xrightarrow{\sim} \mathcal{O}_{\langle 1 \rangle}$ it follows that a morphism in $\mathcal{C}$ over $\langle 1 \rangle$ is inert if and only if it is an equivalence (since the equivalences are precisely the inert morphisms in $\mathcal{O}_{\langle 1 \rangle}$). Hence $\mathcal{C}^{\text{el}} = \mathcal{C}_{\langle 1 \rangle}^{\sim}$, so the functor $\mathcal{C}^{\text{el}} \rightarrow \mathcal{O}^{\text{el}}$ is just the underlying morphism of ∞ -groupoids of the functor between fibers over $\langle 1 \rangle$ that is an equivalence by assumption.

To show (ii), we first observe that $\mathcal{C}_{C/}^{\text{el}}$ is an ∞ -groupoid, since morphisms are given by inert maps over $\langle 1 \rangle$ and these are invertible. Moreover, if C lies over $\langle n \rangle$ then the fiber of $\mathcal{C}_{C/}^{\text{el}}$ over ρ_i is contractible, since there by assumption exists a locally cocartesian morphism over ρ_i — this is then initial in the ∞ -category $(\mathcal{C}_{C/})_{\rho_i}$ and so in particular has no automorphisms.

We thus have a commutative triangle

$$\begin{array}{ccc} \mathcal{C}_{C/}^{\text{el}} & \xrightarrow{\quad} & \mathcal{O}_{f(C)/}^{\text{el}} \\ & \searrow \sim & \swarrow \sim \\ & (\mathbb{F}_*^{\text{el}})_{\langle n \rangle} & \end{array}$$

where both maps to $(\mathbb{F}_*^{\text{el}})_{\langle n \rangle}$ are equivalences, hence so is the top horizontal map.

To prove (iii), observe that by assumption $\mathcal{C}^{\text{act}} \rightarrow \mathcal{O}^{\text{act}}$ is the underlying right fibration of the cartesian fibration $\mathcal{C} \times_{\mathcal{O}} \mathcal{O}^{\text{act}} \rightarrow \mathcal{O}^{\text{act}}$. This gives the required equivalence of slices by [24, Tag 00TE]. $\square$

Corollary 5.1.12 Suppose $f: \mathcal{C} \rightarrow \mathcal{O}$ is a strong approximation to an ∞ -operad $q: \mathcal{O} \rightarrow \mathbb{F}_*$.

- (1) If $\mathcal{X}$ is an ∞ -category with finite products, then restriction along f gives an equivalence

$$f^*: \text{Seg}_{\mathcal{O}}(\mathcal{X}) \xrightarrow{\sim} \text{Seg}_{\mathcal{C}}(\mathcal{X}).$$

- (2) Pullback along f gives an equivalence

$$f^*: \text{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \text{Fbrs}(\mathcal{C}).$$

- (3) The map f exhibits $\mathcal{O}$ as the symmetrization of $\mathcal{C}$, ie $(qf)_! \mathcal{C} \simeq \mathcal{O}$.

Proof Combine Proposition 5.1.11 with Proposition 3.1.16, Theorem 5.1.1, and Corollary 5.1.7. $\square$

Remark 5.1.13 Lurie’s proof of [23, Theorem 2.3.3.26] uses envelopes for approximations to ∞ -operads, just as our proof of Theorem 5.1.1, and we do not claim that our proof is different in any essential way.

We end this section with a couple of examples that do not follow from Corollary 5.1.12 or [23, Theorem 2.3.3.26]. These involve patterns defined using spans, so we start with a general observation about comparisons of these:

Observation 5.1.14 Consider two adequate triples $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and $(\mathfrak{Y}, \mathfrak{Y}^b, \mathfrak{Y}^f)$ and a functor $F : \mathfrak{X} \rightarrow \mathfrak{Y}$ that preserves the two subcategories and also preserves pullbacks of backwards maps along forwards maps. Suppose further that we have full subcategories $\mathfrak{X}_0 \subset \mathfrak{X}$ and $\mathfrak{Y}_0 \subset \mathfrak{Y}$ such that $F(\mathfrak{X}_0) \subset \mathfrak{Y}_0$. Then F induces a morphism of patterns:

$$F : \text{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0) \rightarrow \text{Span}_{b,f}(\mathfrak{Y}; \mathfrak{Y}_0).$$

We may apply Theorem 5.1.1 to this if the following conditions hold:

- (1) $\text{Span}_{b,f}(\mathfrak{Y}; \mathfrak{Y}_0)$ is soundly extendable. (See Proposition 3.3.23.)
- (2) For all $x \in \mathfrak{X}$, the map $\mathfrak{X}_0^b \times_{\mathfrak{X}^b} \mathfrak{X}_{/x}^b \rightarrow \mathfrak{Y}_0^b \times_{\mathfrak{Y}^b} \mathfrak{Y}_{/F(x)}^b$ is cofinal.
- (3) $F : \mathfrak{X}_0^b \rightarrow \mathfrak{Y}_0^b$ is an equivalence of ∞ -categories.
- (4) $F : \mathfrak{X}_{/x}^f \rightarrow \mathfrak{Y}_{/F(x)}^f$ induces an equivalence on maximal subgroupoids for all $x \in \mathfrak{X}$.

Point (2) ensures that F is a strong Segal morphism since $\text{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)^{\text{int}} \simeq (\mathfrak{X}^b)^{\text{op}}$ with the elementaries being $(\mathfrak{X}_0^b)^{\text{op}}$.

Corollary 5.1.15 Pullback along the inclusion $i : \mathbb{F}_* \simeq \text{Span}_{\text{inj}, \text{all}}(\mathbb{F}) \rightarrow \text{Span}(\mathbb{F})$ gives an equivalence

$$i^* : \text{Fbrs}(\text{Span}(\mathbb{F})) \xrightarrow{\sim} \text{Fbrs}(\mathbb{F}_*) \simeq \text{Opd}_{\infty}.$$

Proof We check the conditions of Theorem 5.1.1 in the form stated in Observation 5.1.14:

- (1) The pattern is soundly extendable by Example 3.3.25.
- (2) For $A \in \mathbb{F}$ the relevant functor is the restriction of $\mathbb{F}_{/A}^{\text{inj}} \rightarrow \mathbb{F}_{/A}$ to elementaries. But every map out of a one-point set is injective, so this is an equivalence.
- (3) Similarly, the functor on backwards morphisms $\mathbb{F}^{\text{inj}} \rightarrow \mathbb{F}$ restricts to an equivalence on elementaries.
- (4) Both categories have the same forward morphisms. $\square$

More generally, we have:

Corollary 5.1.16 Pullback along the inclusion $i_m : \text{Span}_{(m-1)\text{-tr}, \text{all}}(\mathcal{S}_m) \rightarrow \text{Span}(\mathcal{S}_m)$ induces an equivalence

$$i_m^* : \text{Fbrs}(\text{Span}(\mathcal{S}_m)) \rightarrow \text{Fbrs}(\text{Span}_{(m-1)\text{-tr}, \text{all}}(\mathcal{S}_m)).$$

Proof We can apply Theorem 5.1.1: $\text{Span}(\mathcal{S}_m)$ is soundly extendable by Example 3.3.27 and in this example we also note that i_m is an iso-Segal morphism. Condition (i) of Theorem 5.1.1 holds because in both cases the elementary ∞ -category is the terminal ∞ -category. Condition (ii) holds because both span ∞ -categories have the same forward morphisms. $\square$

5.2 G -equivariant ∞ -operads

In this section we apply the theory of fibrous patterns and envelopes in the setting of G -equivariant ∞ -operads developed by Nardin and Shah in [26]. While their paper works in the generality of T -parametrized ∞ -operads, we will restrict to the special case of the orbit category $T = \text{Orb}_G$ for simplicity. Our main result is that the G - ∞ -operads of [26] are equivalent to fibrous $\text{Span}(\mathbb{F}_G)$ -patterns; we will also show that the sliced envelope for G - ∞ -operads is fully faithful and characterize the image, giving a third description of these objects.

First, we recall some constructions in equivariant higher algebra, which were pioneered in [3] and further developed in [25; 26]. Fix a finite group G throughout.

Definition 5.2.1 Let $\mathbb{F}_G$ be the category of finite G -sets, $\mathbb{F}_{G,*}$ the category of finite pointed G -sets, and $\text{Orb}_G \subset \mathbb{F}_G$ the full subcategory of G -orbits.

Definition 5.2.2 A G - ∞ -category is a functor $\text{Orb}_G^{\text{op}} \rightarrow \text{Cat}_\infty$ and a G -symmetric monoidal ∞ -category is a $\text{Span}(\mathbb{F}_G)$ -Segal object in Cat_∞ . We write

$$\text{Cat}_{G,\infty} := \text{Fun}(\text{Orb}_G^{\text{op}}, \text{Cat}_\infty) \quad \text{and} \quad \text{Cat}_{G,\infty}^{\otimes} := \text{Seg}_{\text{Span}(\mathbb{F}_G)}(\text{Cat}_\infty)$$

and define the forgetful functor $\text{Cat}_{G,\infty}^{\otimes} \rightarrow \text{Cat}_{G,\infty}$ by restricting to the elementaries $\text{Orb}_G^{\text{op}} \rightarrow \text{Span}(\mathbb{F}_G)$.

Notation 5.2.3 For a G - ∞ -category $\mathcal{C}: \text{Orb}_G^{\text{op}} \rightarrow \text{Cat}_\infty$ we denote its value at G/H by $\mathcal{C}^H$ and refer to it as the H -fixed point ∞ -category of $\mathcal{C}$. There are restriction maps $\mathcal{C}^H \rightarrow \mathcal{C}^K$ for $K \subset H \subset G$. Given a G -symmetric monoidal ∞ -category $\mathcal{D}: \text{Span}(\mathbb{F}_G) \rightarrow \text{Cat}_\infty$ we further have tensor products $\otimes: \mathcal{D}^H \times \mathcal{D}^H \rightarrow \mathcal{D}^H$ and so-called norm maps $\text{Nm}_K^H: \mathcal{D}^K \rightarrow \mathcal{D}^H$ for all $K \subset H \subset G$ coming from the span $(G/K \leftarrow G/K \rightarrow G/H)$.

Example 5.2.4 Since $\text{Span}(\mathbb{F}_G)$ is an extendable pattern (Example 3.3.26) $\mathcal{A}_{\text{Span}(\mathbb{F}_G)}$ is a Segal object in Cat_∞ . We denote this G -symmetric monoidal ∞ -category by

$$\underline{\mathbb{F}}_G := \mathcal{A}_{\text{Span}(\mathbb{F}_G)}(-) = \text{Span}(\mathbb{F}_G)_{/-}^{\text{act}}: \text{Span}(\mathbb{F}_G) \rightarrow \text{Cat}_\infty.$$

The H -fixed point category is the category of finite H -sets:

$$(\underline{\mathbb{F}}_G)^H = \text{Span}(\mathbb{F}_G)_{/(G/H)}^{\text{act}} \simeq (\mathbb{F}_G)_{/(G/H)} \simeq \mathbb{F}_H.$$

The restriction maps are given by restriction, the tensor product by disjoint union, and the norm maps are $(- \times H)_{/K}: \mathbb{F}_K \rightarrow \mathbb{F}_H$. In summary, $\underline{\mathbb{F}}_G$ is $\mathbb{F}_G$ with its natural structure as a G -symmetric monoidal ∞ -category.

Below we will see that fibrous $\text{Span}(\mathbb{F}_G)$ -patterns model G - ∞ -operads. We now explain how $\mathcal{N}_\infty$ -operads fit into this framework:

Example 5.2.5 Let $\mathbb{F}_G^f \subset \mathbb{F}_G$ be a wide subcategory closed under base change and disjoint union. Then the inclusion functor $\text{Span}_{\text{all},f}(\mathbb{F}_G) \rightarrow \text{Span}(\mathbb{F}_G)$ defines a fibrous $\text{Span}(\mathbb{F}_G)$ -pattern. To see that it has cocartesian lifts for inerts, note that any functor of the form $\text{Span}_{b,f}(\mathcal{C}) \rightarrow \text{Span}_{b,\text{all}}(\mathcal{C})$ has cocartesian lifts for backwards maps. For the second condition we need to show that

$$(\mathbb{F}_G^f)_{/A} \rightarrow \lim_{U \in (\text{Orb}_G)_{/A}} (\mathbb{F}_G^f)_{/U}$$

is an equivalence. The limit may be rewritten as a product over the set of orbits of A and then the equivalence follows because $\mathbb{F}_G^f$ is closed under base change and disjoint union.

Categories $\mathbb{F}_G^f$ that in addition to the above also contain all fold maps $\nabla: G/H \amalg G/H \rightarrow G/H$, are in bijection with the indexing systems of [6], see [26, Remark 2.4.12]. Under the equivalence $\text{Fbrs}(\text{Span}(\mathbb{F}_G)) \simeq \text{Opd}_{G,\infty}$ proved below the fibrous $\text{Span}(\mathbb{F}_G)$ -patterns described above are the “commutative G - ∞ -operads” from [26, Definition 2.4.10], which correspond to the $\mathcal{N}_\infty$ -operads of [6] by [26, Remark 2.4.12].

We now quickly recall the necessary notation from [26] to state their definition of G - ∞ -operads, but we refer the reader there for details.

Definition 5.2.6 Define $\underline{\mathbb{F}}_G^\vee \subset \text{Ar}(\mathbb{F}_G)$ as the full subcategory of those morphisms $(f: U \rightarrow V)$ where V is an orbit: $\underline{\mathbb{F}}_G^\vee := \text{Ar}(\mathbb{F}_G) \times_{\mathbb{F}_G} \text{Orb}_G$. We say that a morphism $f \rightarrow g$ given by

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ \downarrow f & & \downarrow g \\ V & \xrightarrow{k} & Y \end{array}$$

- lies in $(\underline{\mathbb{F}}_G^\vee)^{\text{si}}$ if it is a summand inclusion, ie $U \rightarrow X \times_Y V$ is injective,
- lies in $(\underline{\mathbb{F}}_G^\vee)^{\text{deg}}$ if it is target degenerate, ie $k: V \rightarrow Y$ is an equivalence.

Definition 5.2.7 Define $\underline{\mathbb{F}}_{G,*}$ as the algebraic pattern

$$\underline{\mathbb{F}}_{G,*} := \text{Span}_{\text{si,tdeg}}(\underline{\mathbb{F}}_G^\vee; \text{Orb}_G),$$

where the elementary objects are those in the essential image of the identity inclusion $\text{Orb}_G \rightarrow \text{Ar}(\text{Orb}_G) \subset \underline{\mathbb{F}}_G^\vee$.

Remark 5.2.8 The functor $\text{ev}_1: \underline{\mathbb{F}}_G^\vee \rightarrow \text{Orb}_G$ induces a cocartesian fibration

$$\underline{\mathbb{F}}_{G,*} = \text{Span}_{\text{si,tdeg}}(\underline{\mathbb{F}}_G^\vee) \xrightarrow{\text{ev}_1} \text{Span}_{\text{all,iso}}(\text{Orb}_G) \simeq \text{Orb}_G^{\text{op}}.$$

Straightening this yields a G - ∞ -category whose H -fixed point category is $(\underline{\mathbb{F}}_{G,*})^H \simeq \underline{\mathbb{F}}_{H,*}$, similarly to Example 5.2.4.

Observation 5.2.9 For $(U \rightarrow V) \in \underline{\mathbb{F}}_{G,*}$ the category of elementaries under $(U \rightarrow V)$ is equivalent to the opposite of the category of orbits over U (as in Remark 3.2.7):

$$(\underline{\mathbb{F}}_{G,*})_{(U \rightarrow V)}^{\text{el}} \simeq (\text{Orb}_G \times_{(\underline{\mathbb{F}}_G^\vee)} (\underline{\mathbb{F}}_G^{\vee,\text{si}})_{(U \rightarrow V)})^{\text{op}} \simeq (\text{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\text{op}}.$$

Here we used that any morphism $(Q \rightrightarrows Q) \rightarrow (U \rightarrow V)$ (where Q is an orbit) is automatically in $(\mathbb{F}_G^\vee)^{\text{si}}$ since $Q \rightarrow Q \times_V U$ is injective. Now consider the full subcategory on those $(Q \rightarrow U)$ that are injective. This subcategory is equivalent to the discrete set of orbits U/G and moreover the inclusion of the subcategory is a left adjoint:

$$U/G \hookrightarrow (\text{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\text{op}} \simeq (\mathbb{F}_{G,*})_{(U \rightarrow V)/}^{\text{cl}},$$

with right adjoint given by sending $(f : Q \rightarrow U)$ to $(f(Q) \hookrightarrow U)$. In particular, the inclusion of U/G is a coinital functor. This means that for any kind of (weak) Segal condition over $\mathbb{F}_{G,*}$ the limit involved can be rewritten as a product indexed by the finite set U/G .

Corollary 5.2.10 *The pattern $\mathbb{F}_{G,*}$ is sound.*⁶

Proof We check the conditions of Proposition 3.3.23. First we show that the backwards maps satisfy cancellation. Consider two morphisms in $\mathbb{F}_G^\vee$:

$$\begin{array}{ccccc} A & \xrightarrow{a} & U & \xrightarrow{h} & X \\ \downarrow e & & \downarrow f & & \downarrow g \\ B & \xrightarrow{b} & V & \xrightarrow{k} & Y \end{array}$$

such that $A \rightarrow B \times_Y X$ is injective. We can write this map as a composite $A \rightarrow B \times_V U \rightarrow B \times_Y X$, the first map of which then has to be injective. In other words $(a, b) : e \rightarrow f$ is in $\mathbb{F}_G^{\vee, \text{si}}$ as claimed.

We also need to show that the inclusion $\mathfrak{X}_{0/y}^b \hookrightarrow \mathfrak{X}_{0/y}$ is cofinal. In the case at hand this inclusion is $\text{Orb}_G \times_{\mathbb{F}_G^\vee} (\mathbb{F}_G^{\vee, \text{si}})_{/(U \rightarrow V)} \rightarrow \text{Orb}_G \times_{\mathbb{F}_G^\vee} (\mathbb{F}_G^\vee)_{/(U \rightarrow V)}$, which is an equivalence by the argument from Observation 5.2.9. $\square$

Definition 5.2.11 [26] A G - ∞ -operad is a weak Segal fibration over $\mathbb{F}_{G,*}$ in the sense of [7, Definition 9.6]; see also Proposition 4.1.7. Let $\text{Opd}_{G,\infty}$ denote the full subcategory of $\text{Cat}_{\infty/\mathbb{F}_{G,*}}^{\text{int-cocart}}$ on the G - ∞ -operads.

Observation 5.2.12 This agrees with the definition of [26]. First we note that given $p : \mathcal{P} \rightarrow \mathbb{F}_{G,*}$ with cocartesian lifts for inerts, the composite $\text{ev}_1 \circ p : \mathcal{P} \rightarrow \text{Orb}_G^{\text{op}}$ exhibits $\mathcal{P}$ as a cocartesian fibration over Orb_G^{op} , ie an Orb_G - ∞ -category, and p as an Orb_G -functor. This holds because the inert morphisms in $\mathbb{F}_{G,*}$ contain all the cocartesian lifts of $\text{ev}_1 : \mathbb{F}_{G,*} \rightarrow \text{Orb}_G^{\text{op}}$. We hence have an identification:

$$\text{Cat}_{\infty/\mathbb{F}_{G,*}}^{\text{int-cocart}} = (\text{Cat}_{G,\infty})_{/\mathbb{F}_{G,*}}^{\text{int-cocart}}.$$

It remains to see that their conditions (2) and (3) exactly amount to the weak Segal conditions (2) and (3) in [7, Definition 9.6]. Indeed, this follows by inspection using Observation 5.2.9 and [7, Remark 9.7].

⁶In fact this pattern is soundly extendable. This follows because the functor $\mathbb{F}_{G,*} \rightarrow \text{Span}(\mathbb{F}_G)$ discussed in Proposition 5.2.14 is iso-Segal and induces an equivalence on forward maps. However, the extendability of $\mathbb{F}_{G,*}$ will not be needed here.

Corollary 5.2.13 We have $\mathrm{Opd}_{G,\infty} = \mathrm{Fbrs}(\mathbb{F}_{G,*})$.

Proof The pattern $\mathbb{F}_{G,*}$ is sound by Corollary 5.2.10 and hence weak Segal fibrations and fibrous patterns are the same by Proposition 4.1.7. $\square$

Proposition 5.2.14 Restriction along the morphism of patterns $\mathbb{F}_{G,*} \xrightarrow{s} \mathrm{Span}(\mathbb{F}_G)$ induced by the functor $\mathbb{F}_G^\vee \rightarrow \mathbb{F}_G$ given by evaluation at 0 yields an equivalence

$$s^* : \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G)) \xrightarrow{\simeq} \mathrm{Fbrs}(\mathbb{F}_{G,*}) = \mathrm{Opd}_{G,\infty}.$$

Proof We need to show that the morphism of patterns

$$s : \mathbb{F}_{G,*} = \mathrm{Span}_{\mathrm{si}, \mathrm{tdeg}}(\mathbb{F}_G^\vee; \mathrm{Orb}_G) \rightarrow \mathrm{Span}(\mathbb{F}_G; \mathrm{Orb}_G)$$

satisfies the conditions of Theorem 5.1.1. Since this comes from a morphism of adequate triples, we can use the formulation in Observation 5.1.14. We check each of the conditions there in turn:

- (1) It was checked in Example 3.3.25 that $\mathrm{Span}(\mathbb{F}_G)$ is soundly extendable.
- (2) We need to show that

$$(\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G^{\vee, \mathrm{si}})_{/(U \rightarrow V)})^{\mathrm{op}} \rightarrow (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\mathrm{op}}$$

is cofinal. But we have already noted in Observation 5.2.9 that it is an equivalence.

- (3) This holds since the functor induces the identity on Orb_G .
- (4) For all $U \in \mathbb{F}_G$ the functor

$$(\mathbb{F}_G^{\vee, \mathrm{tdeg}})_{/(U \rightarrow V)} \rightarrow (\mathbb{F}_G)_{/U}$$

is an equivalence by inspection of the definition of $(\mathbb{F}_G^\vee)^{\mathrm{tdeg}}$. $\square$

As a consequence we obtain a fully faithful envelope into the ∞ -category of G -symmetric monoidal ∞ -categories over $\mathbb{F}_G$ and a characterization of the image.

Corollary 5.2.15 There is an adjunction

$$\mathrm{Env}_G : \mathrm{Opd}_{G,\infty} \rightleftarrows \mathrm{Cat}_{G,\infty}^\otimes : \mathrm{forget},$$

where the left adjoint may be lifted to a fully faithful functor

$$\mathrm{Env}_G : \mathrm{Opd}_{G,\infty} \hookrightarrow (\mathrm{Cat}_{G,\infty}^\otimes)_{/\mathbb{F}_G}.$$

This functor has both adjoints and its essential image consists of those G -symmetric monoidal functors $p : \mathcal{C} \rightarrow \mathbb{F}_G$ that are $\mathrm{Ar}_{\mathrm{act}}(\mathrm{Span}(\mathbb{F}_G))$ -equifibered.

Proof Using that $\mathrm{Opd}_{G,\infty} \simeq \mathrm{Fbrs}(\mathrm{Span}(\mathbb{F}_G))$ by Proposition 5.2.14, this is an instance of Theorem 4.2.6. The envelope of the terminal G - ∞ -operad is $\mathrm{Env}_{\mathrm{Span}(\mathbb{F}_G)}(*) = \mathcal{A}_{\mathrm{Span}(\mathbb{F}_G)} = \mathbb{F}_G$ by Example 5.2.4. $\square$

We elaborate further on the characterization of the image:

Lemma 5.2.16 *A G -symmetric monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is $\text{Ar}_{\text{act}}(\text{Span}(\mathbb{F}_G))$ -equifibered if and only if*

$$\begin{array}{ccc} \mathcal{C}^H \times \mathcal{C}^H & \xrightarrow{\otimes} & \mathcal{C}^H \\ \downarrow & & \downarrow \\ \mathcal{D}^H \times \mathcal{D}^H & \xrightarrow{\otimes} & \mathcal{D}^H \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C}^K & \xrightarrow{\text{Nm}_K^H} & \mathcal{C}^H \\ \downarrow & & \downarrow \\ \mathcal{D}^K & \xrightarrow{\text{Nm}_K^H} & \mathcal{D}^H \end{array}$$

are pullback squares of ∞ -categories for all subgroups $K \subset H \subset G$.

Proof F induces a natural transformation of functors $\mathbb{F}_G \rightarrow \text{Cat}$, defined by restricting to forwards maps in $\text{Span}(\mathbb{F}_G)$. Let $\mathcal{K} \subset \mathbb{F}_G$ denote the maximal subcategory such that the restriction of F to $\mathcal{K}$ is a cartesian natural transformation. Then F is $\text{Ar}_{\text{act}}(\text{Span}(\mathbb{F}_G))$ -equifibered if and only if $\mathcal{K} = \mathbb{F}_G$. Note that $\mathcal{K}$ is closed under composition and right-cancellation, since pullback squares are, and contains all equivalences. Moreover, $\mathcal{K}$ is closed under disjoint union since both functors $\mathcal{C}, \mathcal{D}: \mathbb{F}_G \rightarrow \text{Cat}$ send disjoint unions to products. Using this one can see that to show $\mathcal{K} = \mathbb{F}_G$, it suffices to check that $\mathcal{K}$ contains the morphisms

$$\nabla: G/H \amalg G/H \rightarrow G/H \quad \text{and} \quad G/K \rightarrow G/H$$

for all subgroups $K \subset H \subset G$. This is exactly the condition stated in the lemma. $\square$

Remark 5.2.17 One might hope that G - ∞ -operads are also equivalent to fibrous $\mathbb{F}_{G,*}$ -patterns, in analogy with what we showed in Corollary 5.1.15 for $G = \{e\}$, but this is false for nontrivial groups. Note that the orbit functor $(-)_G: \mathbb{F}_{G,*} \rightarrow \mathbb{F}_*$ exhibits $\mathbb{F}_{G,*}$ as a fibrous $\mathbb{F}_*$ -pattern, ie an ∞ -operad in the sense of Lurie. Therefore there is an equivalence $\text{Fbrs}(\mathbb{F}_{G,*}) \simeq (\text{Opd}_{\infty})_{/\mathbb{F}_{G,*}}$. We refer to this as the ∞ -category of *naive G - ∞ -operads*. There is an inclusion of patterns $\mathbb{F}_{G,*} \rightarrow \text{Span}(\mathbb{F}_G)$ similar to the one used in Corollary 5.1.15, and this is a strong Segal morphism by an argument as in Observation 5.2.9. Therefore there is a restriction functor:

$$\text{Opd}_{G,\infty} \simeq \text{Fbrs}(\text{Span}(\mathbb{F}_G)) \rightarrow \text{Fbrs}(\mathbb{F}_{G,*}) \simeq (\text{Opd}_{\infty})_{/\mathbb{F}_{G,*}},$$

which forgets from (genuine) G - ∞ -operads to naive G - ∞ -operads. However, we cannot apply the comparison Theorem 5.1.1 since $(\text{Orb}_G^{\text{op}})^{\simeq} \simeq \mathbb{F}_{G,*}^{\text{el}} \rightarrow \text{Span}(\mathbb{F}_G)^{\text{el}} \simeq \text{Orb}_G^{\text{op}}$ is not an equivalence.

5.3 Upgrading to $(\infty, 2)$ -categories

In this subsection we will upgrade our main results from ∞ -categories to $(\infty, 2)$ -categories: we will see that the comparison equivalence of Theorem 5.1.1 is an equivalence of $(\infty, 2)$ -categories and the fully faithful envelope functor of Proposition 4.2.1 is a fully faithful functor of $(\infty, 2)$ -categories. More precisely, we will show that these functors are compatible with natural Cat_{∞} -module structures on the ∞ -categories

involved. It then follows from results of Hinich [21] and Heine [20] that these ∞ -categories can be upgraded to $(\infty, 2)$ -categories and the functors to functors of $(\infty, 2)$ -categories. We will not comment further on this, however, as our primary interest is in showing that our equivalences are compatible with the natural ∞ -categories of maps, which is an immediate consequence of compatibility with the Cat_∞ -module structures. We begin by defining such module structures on the ∞ -categories and functors we studied in Section 2:

Construction 5.3.1 Let $\mathcal{B}$ be an ∞ -category equipped with a wide subcategory $\mathcal{B}_0$. The forgetful functor $\text{Cat}_{\infty/\mathcal{B}} \rightarrow \text{Cat}_\infty$ has a right adjoint, taking $\mathcal{C} \in \text{Cat}_\infty$ to the projection $\mathcal{C} \times \mathcal{B} \rightarrow \mathcal{B}$; this factors through the subcategory $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and thus gives symmetric monoidal functors

$$\text{Cat}_\infty \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$$

with respect to the cartesian products. It follows that both $\text{Cat}_{\infty/\mathcal{B}}$ and $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ are Cat_∞ -modules, with the tensoring in both cases simply given by cartesian product, ie

$$(\mathcal{C}, \mathcal{E} \rightarrow \mathcal{B}) \mapsto \mathcal{E} \times \mathcal{C} \rightarrow \mathcal{B},$$

and that the forgetful functor $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$ is a Cat_∞ -module functor. Moreover, both Cat_∞ -module structures are adjoint to an enrichment in Cat_∞ , given, respectively, by $\text{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-, -)$ and $\text{Fun}_{/\mathcal{B}}(-, -)$. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞ -category equipped with a factorization system, then the ∞ -categories $\text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ are Cat_∞ -modules, with the tensoring given by the cartesian product, and the forgetful functor $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ is a Cat_∞ -module functor; it is easy to see that this Cat_∞ -module structure on $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ corresponds under the equivalence with $\text{Fun}(\mathcal{B}, \text{Cat}_\infty)$ to that given by taking products with constant functors.

Proposition 5.3.2 (i) For any ∞ -category $\mathcal{B}$, the tensoring of $\text{Cat}_{\infty/\mathcal{B}}$ over Cat_∞ in Construction 5.3.1 is adjoint to a cotensoring, with the cotensor of $\mathcal{C} \in \text{Cat}_\infty$ and $\mathcal{E} \rightarrow \mathcal{B}$ given by the pullback

$$\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}} := \text{Fun}(\mathcal{C}, \mathcal{E}) \times_{\text{Fun}(\mathcal{C}, \mathcal{B})} \mathcal{B}$$

along the constant diagram functor $\mathcal{B} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{B})$.

(ii) If $\mathcal{B}_0$ is a wide subcategory of $\mathcal{B}$, then $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ is also cotensored over Cat_∞ , with the cotensor of $\mathcal{C} \in \text{Cat}_\infty$ and $\mathcal{E} \rightarrow \mathcal{B}$ again given by $\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$. In particular, the forgetful functor $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$ preserves the cotensoring.

Proof Part (i) follows from the natural equivalences

$$\text{Map}_{\text{Cat}_{\infty/\mathcal{B}}}(\mathcal{C} \times \mathcal{F}, \mathcal{E}) \simeq \left\{ \begin{array}{ccc} \mathcal{C} \times \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{C} \times \mathcal{B} & \xrightarrow{\text{proj}} & \mathcal{B} \end{array} \right\} \simeq \left\{ \begin{array}{ccc} \mathcal{F} & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{\text{const}} & \text{Fun}(\mathcal{C}, \mathcal{B}) \end{array} \right\} \simeq \text{Map}_{\text{Cat}_{\infty/\mathcal{B}}}(\mathcal{F}, \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}).$$

To prove (ii), we observe that if $\mathcal{E} \rightarrow \mathcal{B}$ is in $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, then so is $\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ by [22, Proposition 3.1.2.3], and a morphism $[1] \rightarrow \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ is cocartesian if and only if the corresponding map $[1] \times \mathcal{C} \rightarrow \mathcal{E}$ has cocartesian components at every $c \in \mathcal{C}$. Thus a morphism $\mathcal{F} \rightarrow \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ over $\mathcal{B}$ preserves cocartesian morphisms over $\mathcal{B}_0$ if and only if the corresponding map $\mathcal{F} \times \mathcal{C} \rightarrow \mathcal{E}$ preserves cocartesian morphisms over $\mathcal{B}_0$, so that the previous equivalence of mapping spaces restricts on subspaces to an equivalence

$$\text{Map}_{\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{C} \times \mathcal{F}, \mathcal{E}) \simeq \text{Map}_{\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{F}, \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}).$$

□

Observation 5.3.3 If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞ -category equipped with a factorization system, then the ∞ -categories $\text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ are similarly cotensored over Cat_{∞} , with the same cotensors as in Proposition 5.3.2, and the forgetful functor $\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ preserves the cotensoring.

Proposition 5.3.4 (i) Let $\mathcal{B}$ be an ∞ -category with a wide subcategory $\mathcal{B}_0$. Then the left adjoint

$$(-) \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) : \text{Cat}_{\infty/\mathcal{B}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$$

of the forgetful functor from Corollary 2.1.5 is a Cat_{∞} -module functor, with the adjunction being an adjunction of Cat_{∞} -modules.

(ii) If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞ -category equipped with a factorization system, then the left adjoint

$$(-) \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) : \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$$

of the forgetful functor from Corollary 2.2.5 is a Cat_{∞} -module functor, with the adjunction being an adjunction of Cat_{∞} -modules.

Proof The forgetful functor $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{B}}$ is a Cat_{∞} -module functor by Construction 5.3.1. By [17, Theorem 3.4.7], the left adjoint then has a canonical oplax Cat_{∞} -module structure, given for $\mathcal{C} \in \text{Cat}_{\infty}$ and $\mathcal{E} \rightarrow \mathcal{B}$ in $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ by the natural map

$$(\mathcal{C} \times \mathcal{B}) \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}) \rightarrow \mathcal{C} \times (\mathcal{B} \times_{\mathcal{B}} \text{Ar}_0(\mathcal{B}));$$

this is clearly an equivalence, so the adjunction of Corollary 2.1.5 lifts to an adjunction of Cat_{∞} -modules. This proves (i), and the proof of (ii) is the same. □

Remark 5.3.5 The Cat_{∞} -module structures on $\text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and $\text{Cat}_{\infty/\mathcal{B}}$ are adjoint to enrichments in Cat_{∞} , given, respectively, by $\text{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-, -)$ and $\text{Fun}_{/\mathcal{B}}(-, -)$; the equivalence of Proposition 2.1.4 is then precisely that induced by the Cat_{∞} -module adjunction from Proposition 5.3.4. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞ -category equipped with a factorization system, the equivalence of Proposition 2.2.4 is also induced by the Cat_{∞} -module adjunction above.

Lemma 5.3.6 (i) For any functor of ∞ -categories $f : \mathcal{A} \rightarrow \mathcal{B}$ the functor $f^* : \text{Cat}_{\infty/\mathcal{B}} \rightarrow \text{Cat}_{\infty/\mathcal{A}}$ given by pullback along f is a Cat_{∞} -module functor and also preserves the cotensoring with Cat_{∞} .

(ii) Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞ -categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Then the functor $f^* : \text{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \rightarrow \text{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}}$ given by pullback along f is a Cat_{∞} -module functor and also preserves the cotensoring with Cat_{∞} .

Proof We prove (i); the proof of (ii) is the same. The functor f^* fits in a commutative triangle

$$\begin{array}{ccc} & \text{Cat}_\infty & \\ \swarrow & & \searrow \\ \text{Cat}_{\infty/B} & \xrightarrow{f^*} & \text{Cat}_{\infty/A}, \end{array}$$

where all three functors preserve finite products, and so are symmetric monoidal with respect to the cartesian products. Hence $f^*: \text{Cat}_{\infty/B} \rightarrow \text{Cat}_{\infty/A}$ is a Cat_∞ -module functor. To see that f^* also preserves the cotensoring, observe that for $\mathcal{E} \rightarrow \mathcal{B}$ in $\text{Cat}_{\infty/B}^{\mathcal{B}_0\text{-cocart}}$ or $\text{Cat}_{\infty/B}$ and $\mathcal{C} \in \text{Cat}_\infty$ we have a natural commutative cube

$$\begin{array}{ccccc} (f^*\mathcal{E})_{/A}^{\mathcal{C}} & \longrightarrow & \text{Fun}(\mathcal{C}, f^*\mathcal{E}) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathcal{A} & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{A}) & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{B}), \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{B} & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{B}), \end{array}$$

where the front, back and right faces are cartesian. The left vertical square is therefore also cartesian, giving an equivalence

$$(f^*\mathcal{E})_{/A}^{\mathcal{C}} \xrightarrow{\sim} f^*(\mathcal{E}_{/B}^{\mathcal{C}}).$$

□

Observation 5.3.7 For $f: \mathcal{A} \rightarrow \mathcal{B}$ a functor that preserves wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, we have a commutative diagram

$$\begin{array}{ccc} \text{Cat}_\infty & & \\ \swarrow & \searrow & \\ \text{Cat}_{\infty/B}^{\mathcal{B}_0\text{-cocart}} & \xrightarrow{f^*} & \text{Cat}_{\infty/A}^{\mathcal{A}_0\text{-cocart}} \\ \downarrow & & \downarrow \\ \text{Cat}_{\infty/B} & \xrightarrow{f^*} & \text{Cat}_{\infty/A} \end{array}$$

of symmetric monoidal functors (with the cartesian monoidal structures). It follows that the commutative square on the bottom right (as in Observation 2.1.7) is a square of Cat_∞ -modules. Similarly, if f is compatible with factorization systems $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$, then the commutative square

$$\begin{array}{ccc} \text{Cat}_{\infty/B}^{\text{cocart}} & \xrightarrow{f^*} & \text{Cat}_{\infty/A}^{\text{cocart}} \\ \downarrow & & \downarrow \\ \text{Cat}_{\infty/B}^{L\text{-cocart}} & \xrightarrow{f^*} & \text{Cat}_{\infty/A}^{L\text{-cocart}} \end{array}$$

is a square of Cat_∞ -modules. It follows that for both squares the Beck–Chevalley map is a natural transformation of Cat_∞ -modules.

Proposition 5.3.8 *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a factorization system. Then there is a natural Cat_∞ -module structure on the ∞ -category $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$, with the tensoring given by cartesian products, and the adjunction*

$$E: \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} : Q$$

is compatible with the Cat_∞ -module structures. Moreover, $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$ is also cotensored over Cat_∞ , with the cotensor of $\mathcal{C} \in \text{Cat}_\infty$ and $\mathcal{E} \rightarrow \text{Ar}_R(\mathcal{B})$ in $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$ being $\mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}}$.

Proof The forgetful functor $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} \rightarrow \text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ has a right adjoint, which takes a cocartesian fibration $\mathcal{E} \rightarrow \mathcal{B}$ to the projection $\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B}) \rightarrow \text{Ar}_R(\mathcal{B})$. We thus have a commutative diagram

$$\begin{array}{ccccc} & & \text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} & & \\ & \nearrow^{(-) \times \mathcal{B}} & \downarrow^{(-) \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B})} & \searrow^{\text{forget}} & \\ \text{Cat}_\infty & & & & \text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \\ & \searrow_{(-) \times \text{Ar}_R(\mathcal{B})} & \downarrow & \nearrow_Q & \\ & & (\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})} & & \end{array}$$

of right adjoints, which are then symmetric monoidal functors with respect to cartesian products. This in particular shows that $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$ is a Cat_∞ -module, with the tensoring given by taking cartesian products, and the functor Q is compatible with the Cat_∞ -module structures. As in Construction 5.3.1, it follows that the left adjoint E is an oplax Cat_∞ -module functor, and that the oplax structure maps are equivalences; thus we have a Cat_∞ -module adjunction.

To identify the cotensor, we first observe that $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$ can be described as a subcategory of $\text{Cat}_{\infty/\text{Ar}_R(\mathcal{B})}$; the Cat_∞ -module structures on both are clearly compatible, and the latter has a cotensoring given by $(\mathcal{C}, \mathcal{E}) \mapsto \mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}}$ by Proposition 5.3.2. It thus suffices to show that $\mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}}$ is an object of $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$, ie that the composite to $\mathcal{B}$ is a cocartesian fibration, that the morphism to $\text{Ar}_R(\mathcal{B})$ preserves cocartesian morphisms over $\mathcal{B}$, and that a morphism $\mathcal{F} \rightarrow \mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}}$ preserves cocartesian morphisms over $\mathcal{B}$ if and only if the adjoint map $\mathcal{F} \times \mathcal{C} \rightarrow \mathcal{E}$ does so. To see this, consider the commutative cube

$$(11) \quad \begin{array}{ccccc} \mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}} & \xrightarrow{\quad} & \mathcal{E}^{\mathcal{C}} & & \\ & \searrow & \downarrow & \searrow & \\ & & \text{Ar}_R(\mathcal{B}) & \xrightarrow{\quad} & \text{Ar}_R(\mathcal{B})^{\mathcal{C}} \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{B} & \xrightarrow{\quad} & \mathcal{B}^{\mathcal{C}} & \xrightarrow{\quad} & \mathcal{B}^{\mathcal{C}} \\ & \Downarrow & \downarrow & \Downarrow & \\ & & \mathcal{B} & \xrightarrow{\quad} & \mathcal{B}^{\mathcal{C}} \end{array}$$

Here the top and bottom squares are cartesian, the vertical maps are cocartesian fibrations, and both maps to $\text{Ar}_R(\mathcal{B})^{\mathcal{C}}$ preserve cocartesian morphisms. It follows that $\mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}} \rightarrow \mathcal{B}$ is a cocartesian fibration, and a morphism here is cocartesian if and only if its images in $\text{Ar}_R(\mathcal{B})$ and $\mathcal{E}^{\mathcal{C}}$ are both cocartesian. Combining this with the description of cocartesian morphisms in $\mathcal{E}^{\mathcal{C}}$ from [22, Proposition 3.1.2.1] gives the required description of cocartesian morphisms in $\mathcal{E}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{C}}$. $\square$

Observation 5.3.9 Let us write $\text{Fun}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{B}\text{-cocart}}(-, -)$ for the enrichment adjoint to the Cat_∞ -module structure on $(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}$; this satisfies

$$\text{Map}_{\text{Cat}_\infty}(\mathcal{C}, \text{Fun}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{B}\text{-cocart}}(-, -)) \simeq \text{Map}_{(\text{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\text{Ar}_R(\mathcal{B})}}(\mathcal{C} \times -, -);$$

identifying the right-hand side as a fiber product we see that for $\alpha: \mathcal{E} \rightarrow \text{Ar}_R(\mathcal{B}), \beta: \mathcal{F} \rightarrow \text{Ar}_R(\mathcal{B})$ we have a natural cartesian square

$$\begin{array}{ccc} \text{Fun}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{B}\text{-cocart}}((\mathcal{E}, \alpha), (\mathcal{F}, \beta)) & \longrightarrow & \text{Fun}_{/\mathcal{B}}^{\text{cocart}}(\mathcal{E}, \mathcal{F}) \\ \downarrow & & \downarrow \\ \{\alpha\} & \longrightarrow & \text{Fun}_{/\mathcal{B}}^{\text{cocart}}(\mathcal{E}, \text{Ar}_R(\mathcal{B})). \end{array}$$

Since the functor E is fully faithful and compatible with the Cat_∞ -module structures we conclude that it gives a natural equivalence

$$\text{Fun}_{/\mathcal{B}}^{L\text{-cocart}}(-, -) \xrightarrow{\sim} \text{Fun}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{B}\text{-cocart}}(E(-), E(-)).$$

Observation 5.3.10 Suppose $f: \mathcal{A} \rightarrow \mathcal{B}$ is a functor compatible with specified factorization systems. Passing to vertical left adjoints in the commutative square of Observation 2.3.9 yields a Beck–Chevalley transformation

$$E_{\mathcal{A}} f^* \rightarrow f^{\otimes} E_{\mathcal{B}}.$$

Unwinding the definitions, this is given at $\mathcal{E} \rightarrow \mathcal{B}$ in $\text{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \text{Ar}_R(\mathcal{A}) \rightarrow (\mathcal{E} \times_{\mathcal{B}} \text{Ar}_R(\mathcal{B})) \times_{\text{Ar}_R(\mathcal{B})} \text{Ar}_R(\mathcal{A}),$$

which is an equivalence. The functors and transformations here are also compatible with the Cat_∞ -module structures, by the same argument as in Observation 5.3.7, so for $\mathcal{E}, \mathcal{F} \rightarrow \text{Ar}_R(\mathcal{B})$ we have a natural commutative square in which the vertical maps are equivalences:

$$(12) \quad \begin{array}{ccc} \text{Fun}_{/\mathcal{B}}^{L\text{-cocart}}(\mathcal{E}, \mathcal{F}) & \longrightarrow & \text{Fun}_{/\mathcal{A}}^{L\text{-cocart}}(f^* \mathcal{E}, f^* \mathcal{F}) \\ \downarrow \sim & & \downarrow \sim \\ \text{Fun}_{/\text{Ar}_R(\mathcal{B})}^{\mathcal{B}\text{-cocart}}(E_{\mathcal{B}} \mathcal{E}, E_{\mathcal{B}} \mathcal{F}) & \longrightarrow & \text{Fun}_{/\text{Ar}_R(\mathcal{A})}^{\mathcal{A}\text{-cocart}}(E_{\mathcal{A}} f^* \mathcal{E}, E_{\mathcal{A}} f^* \mathcal{F}). \end{array}$$

After these preliminaries we are finally ready to consider fibrous patterns and their envelopes. First, we want to show that the ∞ -categories $\text{Fbrs}(\mathcal{O})$ and $\text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_\infty)$ have Cat_∞ -module structures inherited from those we have already considered. This is slightly complicated by the fact that $\text{Fbrs}(\mathcal{O})$ may not be closed under tensors in $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$, and similarly for the relative Segal objects. (For example, for $\mathcal{O} \in \text{Fbrs}(\mathbb{F}_*)$ and $\mathcal{C} \in \text{Cat}_\infty$, the ∞ -category $\mathcal{C} \times \mathcal{O}$ is not an object of $\text{Fbrs}(\mathbb{F}_*)$ since its fiber over $\langle 0 \rangle$ is $\mathcal{C}$, not $*$; on the other hand, $\text{Fbrs}(\mathbb{F}_*)$ is closed under tensoring with Cat_∞ .) Luckily, cotensors are better behaved:

Proposition 5.3.11 *Let $\mathcal{O}$ be an algebraic pattern.*

- (i) *For $\mathcal{P} \in \text{Fbrs}(\mathcal{O})$ and $\mathcal{C} \in \text{Cat}$, the cotensor $\mathcal{P}_{/\mathcal{O}}^{\mathcal{C}}$ in $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is again fibrous.*
- (ii) *For $X \in \text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_{\infty})$ corresponding to $\mathcal{X} \in (\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})}$ and $\mathcal{C} \in \text{Cat}$, the cotensor $\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}}$ in $(\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})}$ again straightens to a relative Segal object.*

Proof To prove (i), first observe that we can identify $\mathcal{P}_{/\mathcal{O}}^{\mathcal{C}} \times_{\mathcal{O}} \mathcal{O}_{/\mathcal{O}}^{\text{act}}$ as the fiber product

$$\text{Fun}(\mathcal{C}, \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/\mathcal{O}}^{\text{act}}) \times_{\text{Fun}(\mathcal{C}, \mathcal{O}_{/\mathcal{O}}^{\text{act}})} \mathcal{O}_{/\mathcal{O}}^{\text{act}},$$

so that we have a commutative cube

$$\begin{array}{ccccc} \mathcal{P}_{/\mathcal{O}}^{\mathcal{C}} \times_{\mathcal{O}} \mathcal{O}_{/\mathcal{O}}^{\text{act}} & \xrightarrow{\quad} & \text{Fun}(\mathcal{C}, \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/\mathcal{O}}^{\text{act}}) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} \mathcal{P}_{/\mathcal{O}}^{\mathcal{C}} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}} & \xrightarrow{\quad} & \text{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}_{/E}^{\text{act}}) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathcal{O}_{/\mathcal{O}}^{\text{act}} & \xrightarrow{\quad} & \text{Fun}(\mathcal{C}, \mathcal{O}_{/\mathcal{O}}^{\text{act}}) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} \mathcal{O}_{/E}^{\text{act}} & \xrightarrow{\quad} & \text{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} \mathcal{O}_{/E}^{\text{act}}) & & \end{array}$$

where the front and back faces are cartesian. Here the right vertical face is also cartesian since $\mathcal{P}$ is $\mathcal{O}$ -fibrous. It then follows that the left vertical face is also cartesian, ie $\mathcal{P}_{/\mathcal{O}}^{\mathcal{C}}$ is also $\mathcal{O}$ -fibrous.

For (ii), we extract the following commutative diagram from the cube (11) that describes $\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}}$:

$$\begin{array}{ccccc} (\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}})_{\mathcal{O}} & \xrightarrow{\quad} & (\mathcal{X}^{\mathcal{C}})_{\mathcal{O}} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} (\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}})_E & \xrightarrow{\quad} & \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} (\mathcal{X}^{\mathcal{C}})_E & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \text{Ar}_{\text{act}}(\mathcal{O})_{\mathcal{O}} & \xrightarrow{\quad} & (\text{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_{\mathcal{O}} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} \text{Ar}_{\text{act}}(\mathcal{O})_E & \xrightarrow{\quad} & \lim_{E \in \mathcal{O}_{/\mathcal{O}}^{\text{el}}} (\text{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_E & & \end{array}$$

(Here we have also used $\mathcal{O}$ for the constant functor $\mathcal{C} \rightarrow \mathcal{O}$ with this value.) The front and back vertical faces in this cube are cartesian by the definition of $\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}}$, while the right vertical face is cartesian since $\mathcal{X}$ by assumption straightens to a relative Segal object (and we can identify $\mathcal{X}_{\mathcal{O}}^{\mathcal{C}}$ as $\text{Fun}(\mathcal{C}, \mathcal{X}_{\mathcal{O}})$ etc). Hence the left vertical face is also cartesian, and this is precisely the relative Segal condition for $\mathcal{X}_{/\text{Ar}_{\text{act}}(\mathcal{O})}^{\mathcal{C}}$. $\square$

Corollary 5.3.12 *Let $\mathcal{O}$ be an algebraic pattern.*

- (i) *The localization $L_{\text{fbrs}}: \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \rightarrow \text{Fbrs}(\mathcal{O})$ is a localization of Cat_{∞} -modules.*
- (ii) *The localization $L_{\text{rseg}}: (\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})} \rightarrow \text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_{\infty})$ is a localization of Cat_{∞} -modules.*

Proof We prove the first claim; the proof of the second is the same — in particular, both follow from [23, Proposition 2.2.1.9]. In order to apply this to L_{fbrs} , we must verify the required hypothesis, which amounts to checking that for $\mathcal{C} \in \text{Cat}_\infty$ and $\mathcal{E} \in \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$, the canonical map $\mathcal{C} \times \mathcal{E} \rightarrow \mathcal{C} \times L_{\text{fbrs}}(\mathcal{E})$ is taken to an equivalence by L_{fbrs} . Equivalently, we must show that for $\mathcal{P} \in \text{Fbrs}(\mathcal{O})$, the induced map

$$\text{Map}_{\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{C} \times L_{\text{fbrs}}(\mathcal{E}), \mathcal{P}) \rightarrow \text{Map}_{\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{C} \times \mathcal{E}, \mathcal{P})$$

is an equivalence. Using the cotensoring, this is the same as the map

$$\text{Map}_{\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(L_{\text{fbrs}}(\mathcal{E}), \mathcal{P}_{/\mathcal{O}}^{\mathcal{C}}) \rightarrow \text{Map}_{\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{E}, \mathcal{P}_{/\mathcal{O}}^{\mathcal{C}})$$

given by composition with the localization map $\mathcal{E} \rightarrow L_{\text{fbrs}}(\mathcal{E})$. This map is indeed an equivalence, since $\mathcal{P}_{/\mathcal{O}}^{\mathcal{C}}$ is fibrous by Proposition 5.3.11. $\square$

Corollary 5.3.13 *Let $\mathcal{O}$ be a sound pattern. Then we have a commutative square*

$$\begin{array}{ccc} \text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} & \xrightarrow{L_{\text{fbrs}}} & \text{Fbrs}(\mathcal{O}) \\ \downarrow \text{E} & & \downarrow \text{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}} \\ (\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})} & \xrightarrow{L_{\text{rseg}}} & \text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_\infty) \end{array}$$

of Cat_∞ -module functors. Moreover, the adjunction

$$\text{Env}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}} : \text{Fbrs}(\mathcal{O}) \rightleftarrows \text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_\infty) : \text{Un}_{\mathcal{O}}^{\text{int}}$$

of Proposition 4.2.5 is an adjunction of Cat_∞ -modules, with the right adjoint being a **lax** Cat_∞ -module functor.

Proof Let us use the universal property of $\text{Fbrs}(\mathcal{O})$ as a Cat_∞ -module localization to verify that the composite

$$\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \xrightarrow{\text{E}} (\text{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\text{Ar}_{\text{act}}(\mathcal{O})} \xrightarrow{L_{\text{rseg}}} \text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_\infty)$$

factors through L_{fbrs} , as a functor of Cat_∞ -modules. Thus we need to verify that if a morphism $\mathcal{E} \rightarrow \mathcal{F}$ in $\text{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is taken to an equivalence by L_{fbrs} , then $\text{E}\mathcal{E} \rightarrow \text{E}\mathcal{F}$ is taken to an equivalence by L_{rseg} . The latter condition is equivalent to the induced morphism

$$\text{Map}(\text{E}\mathcal{F}, \mathcal{X}) \rightarrow \text{Map}(\text{E}\mathcal{E}, \mathcal{X})$$

being an equivalence provided $\mathcal{X}$ is the unstraightening of an object in $\text{Seg}_{\mathcal{O}}^{/\mathcal{A}\mathcal{O}}(\text{Cat}_\infty)$. By adjunction this holds if and only if the map

$$\text{Map}(\mathcal{F}, \text{Q}\mathcal{X}) \rightarrow \text{Map}(\mathcal{E}, \text{Q}\mathcal{X})$$

is an equivalence for all such $\mathcal{X}$, but since $\mathcal{O}$ is sound the object $\text{Q}\mathcal{X}$ is fibrous, and hence this is indeed an equivalence as by assumption $\mathcal{E} \rightarrow \mathcal{F}$ is taken to an equivalence by L_{fbrs} . It follows that the right adjoint inherits a lax Cat_∞ -module structure. $\square$

Remark 5.3.14 For any pattern $\mathcal{O}$ the Segal envelope

$$\mathrm{Env}_{\mathcal{O}}^{/A_{\mathcal{O}}} : \mathrm{Fbrs}(\mathcal{O}) \rightarrow \mathrm{Seg}_{\mathcal{O}}^{/A_{\mathcal{O}}}(\mathrm{Cat}_{\infty})$$

is a lax Cat_{∞} -module functor, since it can be defined by restricting $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ to these full subcategories, the inclusions of which are lax Cat_{∞} -module functors. This suffices to upgrade the envelope to a functor of $(\infty, 2)$ -categories, and we can see that it is fully faithful since it is obtained by restricting the functor $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}} : \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}} \rightarrow \mathrm{Fun}(\mathcal{O}, \mathrm{Cat}_{\infty})_{/A_{\mathcal{O}}}$, which is a fully faithful functor of $(\infty, 2)$ -categories by Observation 5.3.9.

Proposition 5.3.15 Let $\mathcal{O}$ and $\mathcal{P}$ be algebraic patterns and $f : \mathcal{O} \rightarrow \mathcal{P}$ a strong Segal morphism.

- (i) The functor $f^* : \mathrm{Fbrs}(\mathcal{P}) \rightarrow \mathrm{Fbrs}(\mathcal{O})$ is a lax Cat_{∞} -module functor and its left adjoint $f_!$ is a Cat_{∞} -module functor.
- (ii) The functor $f^{\otimes} : \mathrm{Seg}_{\mathcal{P}}^{/A_{\mathcal{P}}}(\mathrm{Cat}_{\infty}) \rightarrow \mathrm{Seg}_{\mathcal{O}}^{/A_{\mathcal{O}}}(\mathrm{Cat}_{\infty})$ is a lax Cat_{∞} -module functor and its left adjoint $f_!$ is a Cat_{∞} -module functor.

Proof To prove (i), we observe that f^* is obtained by restricting $f^* : \mathrm{Cat}_{\infty/\mathcal{O}}^{\mathrm{int-cocart}} \rightarrow \mathrm{Cat}_{\infty/\mathcal{P}}^{\mathrm{int-cocart}}$, which is a Cat_{∞} -module functor by Observation 5.3.10, to full subcategories; it is therefore a lax Cat_{∞} -module functor. The left adjoint $f_!$ is then automatically an oplax Cat_{∞} -module functor, and the oplax structure map is an equivalence if and only if the right adjoint f^* preserves Cat_{∞} -cotensors, which we know from Lemma 5.3.6 and Proposition 5.3.11. The proof of (ii) is the same. $\square$

Remark 5.3.16 It follows that for $\mathcal{Q} \in \mathrm{Fbrs}(\mathcal{O})$ and $\mathcal{R} \in \mathrm{Fbrs}(\mathcal{P})$ we have a natural equivalence

$$\mathrm{Fun}_{/\mathcal{P}}^{\mathrm{int-cocart}}(f_! \mathcal{Q}, \mathcal{R}) \simeq \mathrm{Fun}_{/\mathcal{O}}^{\mathrm{int-cocart}}(\mathcal{Q}, f^* \mathcal{R}).$$

Corollary 5.3.17 Let $f : \mathcal{O} \rightarrow \mathcal{P}$ be a strong Segal morphism between soundly extendable patterns that satisfies the hypotheses of Theorem 5.1.1. Then pullback along f gives an equivalence

$$f^* : \mathrm{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathrm{Fbrs}(\mathcal{O})$$

of Cat_{∞} -modules. In particular, for any $\mathcal{Q}, \mathcal{Q}'$ in $\mathrm{Fbrs}(\mathcal{P})$, the induced functor

$$\mathrm{Fun}_{/\mathcal{P}}^{\mathrm{int-cocart}}(\mathcal{Q}, \mathcal{Q}') \rightarrow \mathrm{Fun}_{/\mathcal{O}}^{\mathrm{int-cocart}}(f^* \mathcal{Q}, f^* \mathcal{Q}')$$

is an equivalence. $\square$

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On the topology of character varieties of once-punctured torus bundles

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This paper presents, for the special case of once-punctured torus bundles, a natural method to study the character varieties of hyperbolic 3-manifolds that are bundles over the circle. The main strategy is to restrict characters to the fibre of the bundle, and to analyse the resulting branched covering map. This allows us to extend results of Steven Boyer, Erhard Luft and Xingru Zhang. Both $\mathrm{SL}(2, \mathbb{C})$ -character varieties and $\mathrm{PSL}(2, \mathbb{C})$ -character varieties are considered. As an explicit application of these methods, we build on work of Baker and Petersen to show that there is an infinite family of hyperbolic once-punctured bundles with canonical curves of $\mathrm{PSL}(2, \mathbb{C})$ -characters of unbounded genus. A version of Thurston's dimension bound for character varieties is stated and proven for $\mathrm{PSL}(2, \mathbb{C})$ -character varieties.

57K31, 57K32, 57M05; 14D20

1 Introduction

The first part of this paper extends results of Boyer, Luft and Zhang [4] concerning the $\mathrm{SL}(2, \mathbb{C})$ -character variety $X(M_\varphi)$ of M_φ , a hyperbolic once-punctured torus bundle over S^1 with monodromy φ , and we determine its relationship with the $\mathrm{PSL}(2, \mathbb{C})$ -character variety $\bar{X}(M_\varphi)$. Denote a once-punctured torus fibre by S and let λ be the associated longitude, ie the boundary of a compact core of S . The monodromy induces a polynomial automorphism $\bar{\varphi}: X(S) \rightarrow X(S)$ and we denote its *fixed point set* by $X_\varphi(S)$. This is the image of the *restriction map* $r: X(M_\varphi) \rightarrow X(S)$ (Lemma 11). We are interested in the topology of the Zariski components¹ of $X(M_\varphi)$ that contain the characters of irreducible representations. Each such component is one-dimensional (Proposition 14), and we let $X^{\mathrm{irr}}(M_\varphi)$ denote their union.

Boyer, Luft and Zhang [4] show that the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$ are all simple points of $X(M_\varphi)$, and they provide an exact count of them. Namely their number is

$$(1-1) \quad \frac{1}{2} |2 + \mathrm{tr}(\varphi_*)| - 2^{b_1(\varphi)-2},$$

where $b_1(\varphi) = b_1(M_\varphi; \mathbb{Z}_2)$ with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and φ_* represents the automorphism induced by φ on first homology of S . We show that these are the only irreducible characters of M_φ that restrict to reducible characters of S (Scholion 12).

Let $\varepsilon: \pi_1(M_\varphi) \rightarrow \mathbb{Z}_2$ be the nontrivial homomorphism that is trivial on $\pi_1(S)$.

⁰Our *varieties* are affine algebraic sets, and to avoid using the word *irreducible* in two different ways, we refer to irreducible components of these sets as Zariski components.

Theorem 1 Suppose M_φ is a hyperbolic once-punctured torus bundle. Then each Zariski component of $X^{\text{irr}}(M_\varphi)$ is one-dimensional and $r: X^{\text{irr}}(M_\varphi) \rightarrow X_\varphi(S)$ is a two-fold branched covering map. The ramification points of r are simple points, namely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$. The fibres of r correspond to the orbits of the action of $\langle \varepsilon \rangle \leq H^1(M_\varphi, \mathbb{Z}_2)$. In particular,

$$\chi(X^{\text{irr}}(M_\varphi)) = 2\chi(X_\varphi(S)) - \frac{1}{2}|2 + \text{tr}(\varphi_*)| + 2^{b_1(\varphi)-2},$$

where χ denotes the Euler characteristic in the above equation.

Implicit in the above statement is that the restriction map satisfies $r(X(M_\varphi)) = r(X^{\text{irr}}(M_\varphi))$. This is shown as follows. Let C be a Zariski component containing only reducible representations. Then C is birationally equivalent to $\mathbb{C}$ and the restriction of r to C is constant. We apply a criterion due to Heusener and Porti [20] to show that $C \cap X^{\text{irr}}(M_\varphi) \neq \emptyset$, and that each point of intersection is contained on a unique curve in $X^{\text{irr}}(M_\varphi)$. Moreover, the intersection point is a smooth point of both curves and the intersection is transverse (Proposition 17).

In our applications, we either use genus bounds that follow from the above theorem (Corollary 20) or well-known results linking genus to the Newton polygon (see Section 5.2). The proof of Theorem 1 is organised as follows. We first show that no ramification point in $X^{\text{irr}}(M_\varphi)$ is the character of a reducible representation (Section 3.4), and then characterise the fibres as orbits of ε and complete the proof with Proposition 19.

The natural map $X^{\text{irr}}(M_\varphi) \rightarrow \bar{X}^{\text{irr}}(M_\varphi)$ can be viewed as the quotient map

$$X^{\text{irr}}(M_\varphi) \rightarrow X^{\text{irr}}(M_\varphi)/H^1(M_\varphi, \mathbb{Z}_2) \subseteq \bar{X}^{\text{irr}}(M_\varphi).$$

There is a natural splitting $H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H$, where H is the subgroup generated by all homomorphisms that are only nontrivial on $\pi_1(S)$. We may view $H \leq H^1(S, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Theorem 1 and the analogous result for projective representations (Proposition 26) imply the following:

Corollary 2 Suppose M_φ is a hyperbolic once-punctured torus bundle. We have the following commutative diagram of maps whose degrees are indicated in the diagram:

$$\begin{array}{ccccc} X^{\text{irr}}(M_\varphi) & \xrightarrow[r]{2:1} & X_\varphi(S) & \subset & X(S) & \cong & \mathbb{C}^3 \\ q_1 \downarrow 2^{b_1(\varphi)}:1 & & q_2 \downarrow 2^{b_1(\varphi)-1}:1 & & q_2 \downarrow 4:1 & & \\ \bar{X}^{\text{irr}}(M_\varphi) & \xrightarrow{\bar{r}} & \bar{X}_\varphi(S) & \subset & \bar{X}(S) & & \end{array}$$

Here, each of the maps r, q_1, q_2 is a branched covering map. The map q_1 is the quotient map associated with the action of $H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H$, and the map q_2 is the quotient map associated with the action of $H^1(S, \mathbb{Z}_2)$ on $X(S)$. The map r is the quotient map associated with the action of $\langle \varepsilon \rangle$. The orbits of the action of $H^1(S, \mathbb{Z}_2)$ on $X_\varphi(S)$ are the same as the orbits of H except when $b_1(\varphi) = 2$ and one of the coordinate axes (with respect to the natural identification $X(S) = \mathbb{C}^3$) is contained in $X_\varphi(S)$.

The fibres of the map $\bar{r}$ may have size one, two or four, arising from irreducible representations with nontrivial centraliser. The restriction $\bar{r}: \bar{X} \rightarrow \bar{X}_\varphi(S)$ to a Zariski component $\bar{X} \subseteq \bar{X}^{\text{irr}}(M_\varphi)$ is of degree one, except for a special situation when $b_1(\varphi) = 2$ and the degree is two (see Proposition 26).

Corollary 3 *If $b_1(\varphi) = 1$, then we have a birational isomorphism $\bar{X}^{\text{irr}}(M_\varphi) \cong X_\varphi(S)$.*

Proof Since $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2$, we have $H^2(\pi_1(M); \mathbb{Z}_2) \cong 0$ and hence every representation into $\text{PSL}(2, \mathbb{C})$ lifts to a representation into $\text{SL}(2, \mathbb{C})$. See Section 2.1 for details. Hence we obtain the claimed birational isomorphism from the commutative diagram in Corollary 2 since the fibres of each of the surjective maps q_1 and r are the orbits of the action of $\langle \varepsilon \rangle$. $\square$

The hypothesis of Corollary 3 is equivalent to φ_* mapping to an element of order three under the natural epimorphism $\text{SL}(2, \mathbb{Z}) \rightarrow \text{SL}(2, \mathbb{Z}_2) \cong \text{Sym}(3)$ (see Section 2.2).

Section 3 also contains a number of interesting facts about $X(M_\varphi)$ and $X_\varphi(S)$ that follow from our set-up. For instance, M_φ always admits an irreducible representation whose restriction to the fibre has image the quaternionic group. If $H_1(X, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then this character always lies on one (or three) special line(s) in $X_\varphi(S)$. Section 4 provides analogous results for the $\text{PSL}(2, \mathbb{C})$ -character variety $\bar{X}(M_\varphi)$ and the associated fixed point set $\bar{X}_\varphi(S) \subset \bar{X}(S)$. We use a dimension bound due to Thurston, for which we provide a proof (Proposition 8) and which is not implied by the dimension bounds in the setting of more general Lie groups due to Falbel and Guilloux [11] and Porti [24].

As pointed out by a referee, the combination of the restriction map to the fibre and the factorisation of the monodromy in terms of Dehn twists has previously been used by Porti [23, Section 4.5] to describe a method to compute the Reidemeister torsion of once-punctured torus bundles. In Section 5, we apply our refined analysis of the restriction map to compute the character varieties of infinite families of once-punctured torus bundles and a number of small examples. We follow the standard convention letting $A, B \in \text{SL}(2, \mathbb{Z})$ correspond to right-handed Dehn twists (see Section 2.2).

First family Baker and Petersen [2] determined the genera of the character varieties of the family M_n of once-punctured torus bundles with monodromy $\varphi_n = AB^{n+2}$. We note that $b_1(\varphi_n) = 1$ if n is odd and $b_1(\varphi_n) = 2$ if n is even. In the case where $n \geq 3$ is odd, we show that $X_{\varphi_n}(S)$ is a curve of genus zero, and hence Corollary 3 implies that the genus of $\bar{X}^{\text{irr}}(M_n)$ is zero. This agrees with the computation in [2]. See Section 5.3.

Second family Let N_n be the once-punctured torus bundle with monodromy $\psi_n = AB^{n+2}A$. We note that $b_1(\psi_n) = 2$ if n is odd and $b_1(\psi_n) = 3$ if n is even. In the case where $n = 2k + 1 \geq 1$ is odd, we show that $X_{\psi_n}(S)$ is a curve of genus k and the genus of $\bar{X}^{\text{irr}}(N_n)$ is zero. See Section 5.4.

Third family Let L_n be the once-punctured torus bundle with monodromy $\omega_n = A^2 B^{n+2} A$. We note that $b_1(\omega_n) = 1$ if n is odd and $b_1(\omega_n) = 2$ if n is even. In the case where $n = 2k + 1 \geq 3$ is odd, we show that $X_{\omega_n}(S) \cong \bar{X}^{\text{irr}}(L_n)$ has $\frac{1}{2}(n + 3)$ components, one of which is a curve of genus k and the others have genus 0. The curves of genus k are all canonical. This answers a question posed by Alan Reid of whether the canonical curves in the $\text{PSL}(2, \mathbb{C})$ -character variety of once-punctured torus bundles always have genus zero. See Section 5.5.

Small examples We summarise computational results for monodromies that are short words in A and B , or in A and B^{-1} , in Section 5.6. These were executed with *Singular* [10].

We remark that in all examples produced in this paper, each fixed-point set $X_\varphi(S)$ has the property that there is at most one Zariski component of positive genus and each Zariski component is birationally equivalent to an elliptic or a hyperelliptic curve.

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2 Preliminaries

This section collects some basic facts about character varieties of 3-manifolds in Section 2.1 and about hyperbolic once-punctured torus bundles in Section 2.2. We also recall facts about mutation in Section 2.4 and the character variety of the free group of rank two in Section 2.5. None of the material in this section is new, and it mainly serves to set up notation. In Section 2.6 we also include a statement and proof of a dimension bound based on [25, Theorem 5.6] in Thurston's notes. For varieties of representations into $\text{SL}(2, \mathbb{C})$ this was given by Culler and Shalen [9, Proposition 3.2.1], but we could not find the statement of Proposition 8 for the case of representations into $\text{PSL}(2, \mathbb{C})$ in the literature.

2.1 Basic facts about character varieties

We start by summarising the discussion found in [22, pages 512–513]. Let M be a compact, connected and orientable 3-manifold. Let $R(M)$ be the variety of representations of $\pi_1(M)$ into $\text{SL}(2, \mathbb{C})$, and $X(M)$ be the associated character variety. Similarly, let $\bar{R}(M)$ and $\bar{X}(M)$ be the respective varieties of representations into $\text{PSL}(2, \mathbb{C})$ and their characters.

The quotient map $q: \text{SL}(2, \mathbb{C}) \rightarrow \text{PSL}(2, \mathbb{C})$ induces a regular map $q_\star: R(M) \rightarrow \bar{R}(M)$, and its image is a union of topological components of $\bar{R}(M)$. Indeed, $R(M) \rightarrow q_\star(R(M))$ is a regular covering map

with group $H^1(\pi_1(M); \mathbb{Z}_2)$. Following [22], we let

$$Q(M) = R(M)/H^1(\pi_1(M); \mathbb{Z}_2) \cong q_*(R(M)) \subseteq \bar{R}(M).$$

There also is a regular map $q_*: X(M) \rightarrow \bar{X}(M)$, and its image is a union of topological components of $\bar{X}(M)$. The fibres over points in the image are again the orbits of the natural action of $H^1(\pi_1(M); \mathbb{Z}_2)$ on $X(M)$. Let $QX(M) = X(M)/H^1(\pi_1(M); \mathbb{Z}_2) \cong q_*(X(M)) \subseteq \bar{X}(M)$. However, on the level of character varieties, the action may not be free. The points of $X(M)$ fixed by $h: \pi_1(M) \rightarrow \mathbb{Z}_2$ are characterised by $\chi(\gamma) = 0$ for all $\gamma \in \pi_1(M)$ with the property that $h(\gamma) = -1$. This defines an algebraic subset $S(h)$ of $X(M)$. Let $U(M)$ be the Zariski-open set $X(M) \setminus \bigcup_h S(h)$. Then on $U(M)$ the action of $H^1(\pi_1(M); \mathbb{Z}_2)$ is free and the quotient map $U(M) \rightarrow \bar{X}(M)$ onto its image is a regular covering map of its image.

To make these general notions explicit, we appeal to the following material from [17]. For each $\gamma \in \pi_1(M)$, there is a regular function $\tau_\gamma: R(M) \rightarrow \mathbb{C}$ defined by $\tau_\gamma(\rho) = \text{tr } \rho(\gamma)$ and a regular function $I_\gamma: X(M) \rightarrow \mathbb{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$.

Suppose $\gamma_1, \dots, \gamma_n$ generate $\pi_1(M)$, and consider the $m = \frac{1}{6}n(n^2 + 5)$ functions τ_γ as γ ranges over the set

$$G = \{\gamma_i, \gamma_i\gamma_j, \gamma_i\gamma_j\gamma_k \mid 1 \leq i \leq n, 1 \leq i < j \leq n, 1 \leq i < j < k \leq n\}$$

Then we obtain a regular map $\tau: R(M) \rightarrow \mathbb{C}^m$ defined by

$$\rho \mapsto (\tau_\gamma(\rho))_{\gamma \in G}$$

and it is shown in [17] that there is a natural identification $\tau(R(M)) = X(M)$. We therefore use the notation $\tau: R(M) \rightarrow X(M)$ for the natural map.

Projective characters have a similar interpretation, using the fact that the *square* of the trace of an element in $\text{PSL}(2, \mathbb{C})$ is well defined. Given a representation $\bar{\rho}: \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C})$, its character is the map

$$\bar{\chi}_{\bar{\rho}}: \pi_1(M) \rightarrow \mathbb{C} \quad \text{defined by} \quad \gamma \mapsto (\text{tr } \bar{\rho}(\gamma))^2$$

As above, let $\bar{\tau}: \bar{R}(M) \rightarrow \bar{X}(M)$ be the natural map from representations to characters.

Throughout this paper, we make use of the following trace identities. Let $A, B, C \in \text{SL}(2, \mathbb{C})$. Then

$$(2-1) \quad \text{tr } A^{-1} = \text{tr } A,$$

$$(2-2) \quad \text{tr}(B^{-1}AB) = \text{tr } A,$$

$$(2-3) \quad \text{tr } A \text{ tr } B = \text{tr}(AB) + \text{tr}(AB^{-1}),$$

$$(2-4) \quad \text{tr}(ABC) = \text{tr } A \text{ tr}(BC) + \text{tr } B \text{ tr}(AC) + \text{tr}(C) \text{ tr}(AB) - \text{tr } A \text{ tr } B \text{ tr } C - \text{tr}(ACB).$$

Recall that a representation into $\text{SL}(2, \mathbb{C})$ is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant under its image are $\{0\}$ and $\mathbb{C}^2$. Otherwise, a representation is *reducible*. If a representation is reducible, then it is conjugate into the upper triangular matrices, and hence there is an abelian representation with the same character. We make frequent use of the following results from [9]:

Lemma 4 Suppose $\rho \in R(M)$.

- (1) Then ρ is reducible if and only if $\chi_\rho(\gamma) = 2$ for each element γ of the commutator subgroup of $\pi_1(M)$.
- (2) Suppose ρ is irreducible and $\sigma \in R(M)$ satisfies $\chi_\rho = \chi_\sigma$. Then ρ and σ are conjugate and, in particular, σ is irreducible.

It follows from the above lemma that reducible representations form a closed subset of $R(M)$, and irreducible representations form an open subset of $R(M)$ with respect to the Euclidean topology. Moreover, the subset $X^{\text{red}}(M) \subseteq X(M)$ of all characters of reducible representations is an affine algebraic set, and it is the same as the subset of all characters of abelian representations.

Let $X^{\text{irr}}(M)$ be the union of those Zariski components of $X(M)$ that contain at least one (and hence a Zariski-open) subset of irreducible representations. The following is shown in [23, Lemma 3.9(iii)].

Lemma 5 Suppose $\chi \in X^{\text{irr}}(M) \cap X^{\text{red}}(M)$. Then each representation with character χ is reducible and there is a nonabelian reducible representation with this character.

The quotient $QX^{\text{irr}}(M) = X^{\text{irr}}(M)/H^1(\pi_1(M), \mathbb{Z}_2)$ is naturally identified with a union of topological components of the union $\bar{X}^{\text{irr}}(M)$ of Zariski-components of $\bar{X}(M)$ containing irreducible representations. The above lemma implies:

Corollary 6 The map $X^{\text{irr}}(M) \rightarrow \bar{X}(M)$ is a branched covering map onto its image and its ramification points are characters of irreducible or nonabelian reducible representations in the set $\bigcup_h S(h)$.

A point of a complex affine algebraic set X is a *simple point* if it is contained in a unique Zariski component X_0 of X and is a smooth point of X_0 .

Suppose the boundary of M is a union of pairwise disjoint tori and the interior of M admits a complete hyperbolic structure of finite volume. Then the holonomy for the complete hyperbolic structure defines a discrete and faithful $\text{PSL}(2, \mathbb{C})$ -character, which is unique up to complex conjugation (the ambiguity arises from the two different orientations possible on M). Each Zariski component of $\bar{X}(M)$ containing such a character is called a *canonical component*. The discrete and faithful characters lift to $\text{SL}(2, \mathbb{C})$, and again each Zariski component of $X(M)$ containing at least one of these lifts is called a *canonical component*. It is shown in [23, Corollary 3.28] that the discrete and faithful characters are simple points of the canonical components. (The proof in [23] is given in the setting of $\text{SL}(2, \mathbb{C})$ -characters, but descends to $\text{PSL}(2, \mathbb{C})$ -characters.)

We end this section with a general discussion of the obstruction for lifting characters from $\bar{X}(M)$ to $X(M)$. See [19, Section 4], [7], [16, Section 4.5], or [17, Section 2] for details. Suppose $\bar{\rho} \in \bar{R}(M)$. Then $\bar{\rho}$ determines a Stiefel–Whitney class $w_2(\bar{\rho}) \in H^2(\pi_1(M); \mathbb{Z}_2)$. This is zero if and only if $\bar{\rho}$ lifts

to a representation $\rho \in R(M)$. Moreover, the class is constant on topological components of $R(M)$. In the case of interest in this paper, M is aspherical and has boundary a single torus. Hence it is a $K(\pi_1(M), 1)$ and so $H^2(\pi_1(M); \mathbb{Z}_2) \cong H^2(M; \mathbb{Z}_2)$. Moreover, Poincaré–Lefschetz duality implies $H^2(M; \mathbb{Z}_2) \cong H_1(M, \partial M; \mathbb{Z}_2)$. In particular, if $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2$, then $H^2(\pi_1(M); \mathbb{Z}_2) \cong 0$ and hence every representation lifts.

2.2 Once-punctured torus bundles

Throughout the remainder of this paper, M_φ denotes a compact core of the hyperbolic once-punctured torus bundle over S^1 with monodromy φ , and S denotes a fixed fibre of M_φ with the property that the monodromy is the identity on ∂S . For our fundamental groups, we assume that the base point lies in $\partial S \subset \partial M_\varphi$. (Note that this notation is a slight variation on the introduction, where we referred to the noncompact manifolds.)

We also denote by φ the automorphism $\pi_1(S) \rightarrow \pi_1(S)$ induced by the monodromy (this will cause no confusion in this paper). Let $\varphi_*: H_1(S) \rightarrow H_1(S)$ be the induced automorphism on homology. Since M_φ is hyperbolic, the associated monodromy is pseudo-Anosov and we have $|\text{tr } \varphi_*| > 2$.

The fundamental group of M_φ admits the presentation

$$(2-5) \quad \pi_1(M_\varphi) = \langle t, a, b \mid t^{-1}at = \varphi(a), t^{-1}bt = \varphi(b) \rangle,$$

where t is a meridian of M_φ (that is, it corresponds to the trace of the base point of $\pi_1(M_\varphi)$ under the monodromy) and a, b are free generators of $\pi_1(S)$, chosen such that their commutator is the longitude of M_φ (that is, it corresponds to ∂S).

We choose the basis of $H_1(S)$ corresponding to the generators a, b of $\pi_1(S)$, and often implicitly identify φ_* with its corresponding matrix representation, $[\varphi_*] \in \text{SL}(2, \mathbb{Z})$. We may write φ as a product of the automorphisms α and β induced by the right-handed Dehn-twists about the curves corresponding to a and b , since these generate the group of isotopy classes of orientation preserving homeomorphisms of the punctured torus. The automorphisms are defined as follows (where we also include β^{-1} for convenience):

$$(2-6) \quad \alpha := \begin{cases} a \mapsto a, \\ b \mapsto ba, \end{cases} \quad \beta := \begin{cases} a \mapsto ab^{-1}, \\ b \mapsto b, \end{cases} \quad \beta^{-1} = \begin{cases} a \mapsto ab, \\ b \mapsto b. \end{cases}$$

This also gives a factorisation of φ_* in terms of the induced maps α_* and β_* on homology. We have

$$A := [\alpha_*] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := [\beta_*] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The matrices A and B generate $\text{SL}(2, \mathbb{Z})$ and satisfy the relations

$$ABA = BAB, \quad \text{and} \quad (ABA)^4 = I.$$

Using the second relation, we note that any element of $SL(2, \mathbb{Z})$ can be written as a word in positive powers of A and B by substituting

$$B^{-1} = A(ABA)^3A = A^2BA^2BA^2BA^2 \quad \text{and} \quad A^{-1} = B(BAB)^3B = B^2AB^2AB^2AB^2.$$

Hence any element of $SL(2, \mathbb{Z})$ (including the identity and powers of A or B) is conjugate to a word of the form

$$(2-7) \quad A^{a_1}B^{b_1}A^{a_2}B^{b_2} \dots A^{a_n}B^{b_n},$$

where $n > 0$ and the a_i and b_i are positive integers. Also note that $(ABA)^2 = -I$. The manifold M_φ is determined up to homeomorphism by $[\varphi_*] \in SL(2, \mathbb{Z})$. It is a result of Murasugi, see [8, Proposition 1.3.3], that M_φ and M_ψ are homeomorphic if and only if $[\varphi_*]$ is conjugate to either $[\psi_*]$ or $[\psi_*^{-1}]$.

The choice of a monodromy as a positive word in A and B is not unique, and there also is no simple criterion to ensure that the absolute value of the trace is greater than two. It is well-known, see [18, Proposition 2.1], that if $|\text{tr } \varphi_*| > 2$, then the conjugacy class of φ_* has a representative of the form

$$(2-8) \quad \pm A^{a_1}B^{-b_1}A^{a_2}B^{-b_2} \dots A^{a_n}B^{-b_n},$$

where $n > 0$, the a_i and b_i are positive integers, and the sign equals the sign of the trace of φ_* . This representative is unique up to cyclic permutation of the factors $A^{a_i}B^{-b_i}$.

If $\text{tr } \varphi_* \neq 2$, then $H_1(M_\varphi, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tor}(M_\varphi, \mathbb{Z})$. The *parity* of the trace contains information on the rank of homology with coefficients in $\mathbb{Z}_2$. We now describe this in more detail.

2.3 Order and rank

The natural homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_2$ gives an epimorphism from $H_1(S) \rightarrow H_1(S; \mathbb{Z}_2)$, and takes φ_* to a map $\varphi_2: H_1(S; \mathbb{Z}_2) \rightarrow H_1(S; \mathbb{Z}_2)$. On the level of matrices, $[\varphi_*] \mapsto [\varphi_2]$ under the natural epimorphism $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}_2)$. Since $SL(2, \mathbb{Z}_2) \cong \text{Sym}(3)$, the order $o(\varphi_2)$ of φ_2 is either one, two or three. If φ_2 is not the identity, then it has order two if $\text{tr}(\varphi_2) = 0$ and order three if $\text{tr}(\varphi_2) = 1$. In particular, $o(\alpha_2) = o(\beta_2) = 2$ and the word in (2-7) maps to an element of order two if and only if the sum of all exponents is odd.

A direct calculation shows that $b_1(M_\varphi; \mathbb{Z}_2) = \text{rank } H_1(M_\varphi, \mathbb{Z}_2) = 4 - o(\varphi_2) \in \{1, 2, 3\}$. To simplify notation, we write $b_1(\varphi) = b_1(M_\varphi; \mathbb{Z}_2)$. The maximal rank $b_1(M_\varphi; \mathbb{Z}_2) = 3$ is attained if φ_2 is the identity, and hence for at least one of φ , φ^2 or φ^3 . Hence each once-punctured torus bundle is either 2-fold or 3-fold covered by a once-punctured torus bundle with homology with coefficients in $\mathbb{Z}_2$ of maximal rank three.

We identify $H_1(M_\varphi, \mathbb{Z}_2) = \text{Hom}(\pi_1(M_\varphi), \mathbb{Z}_2) = H^1(M_\varphi, \mathbb{Z}_2)$, and denote $\varepsilon: \pi_1(M_\varphi) \rightarrow \mathbb{Z}_2$ the homomorphism defined by

$$(2-9) \quad \varepsilon(t) = -1, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 1.$$

Let $H \leq \text{Hom}(\pi_1(M_\varphi), \mathbb{Z}_2)$ be the subgroup consisting of all homomorphisms

$$h: \pi_1(M_\varphi) \rightarrow \mathbb{Z}_2$$

satisfying $h(t) = 1$. Note that H has rank zero if $\text{tr}(\varphi_*)$ is odd. In the following, we will make use of the splitting

$$(2-10) \quad H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H.$$

2.4 Mutation

The once-punctured torus S admits an orientation preserving involution $\iota: S \rightarrow S$ defined by

$$\iota = \begin{cases} a \mapsto ba^{-1}b^{-1}, \\ b \mapsto bab^{-1}a^{-1}b^{-1}. \end{cases}$$

We have $\iota_* = -I$. If one cuts M_φ along S and changes the monodromy by ι , one obtains $M_{\iota\varphi}$. This process is called *mutation*.

It follows from [26, Proposition 1] that there is a birational equivalence between Zariski components of $\bar{X}(M_\varphi)$ and $\bar{X}(M_{\iota\varphi})$ that contain the character of a representation $\bar{\rho}$ whose restriction to S is irreducible and has trivial centraliser. In particular, if one is interested in the topology of generic components of $\bar{X}(M_\varphi)$, then one may restrict attention to bundles where the trace of φ_* is positive. Moreover, components where all irreducible representations have nontrivial centraliser can be understood directly (see Section 4.2).

In the case of $\text{SL}(2, \mathbb{C})$ -character varieties, using the HNN-splitting for the fundamental group as in the proof of [26, Proposition 1], one sees that the restriction maps to the fibre result in the same fixed-point set; that is: $X_\varphi(S) = X_{\iota\varphi}(S)$. However, the topology of the $\text{SL}(2, \mathbb{C})$ -character varieties may be different. For instance, mutation of the figure eight knot complement (with $\varphi_* = \text{AB}^{-1}$) along the fibre results in the associated sister manifold, and the smooth projective models of their canonical components are a torus and a sphere respectively [26, Section 3.4].

2.5 The character variety of the free group of rank two

The fundamental group of the once-punctured torus fibre S is the free group $\langle a, b \rangle$, and the character variety $X(S)$ admits the affine coordinates $(x, y, z) \in \mathbb{C}^3$, where

$$x = \tau_a(\rho), \quad y = \tau_b(\rho), \quad z = \tau_{ab}(\rho)$$

as ρ ranges over $R(S)$. An elementary calculation, which goes back to classical work of Fricke and Klein [14; 15] and Vogt [28], shows that $X(S) = \mathbb{C}^3$.

Let $\rho \in R(S)$ be an irreducible representation. Up to conjugation, ρ is of the form

$$(2-11) \quad \rho(a) = \begin{pmatrix} s & 0 \\ 1 & s^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix},$$

where $u \neq 0$.

If $\rho \in R(S)$ is a reducible representation, then ρ is conjugate to one of the following:

$$(2-12) \quad \rho(a) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix},$$

$$(2-13) \quad \rho(a) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}.$$

Every point in $X^{\text{red}}(S)$ is the character of a reducible character as in (2-12) with $u = 0$. A representation $\rho \in R(S)$ is reducible if and only if its character satisfies the equation

$$x^2 + y^2 + z^2 - xyz - 4 = 0.$$

This is a consequence of the first part of Lemma 4. See [9] for details.

A parametrisation of the $\text{PSL}(2, \mathbb{C})$ -character variety $\bar{X}(S)$ is described in [19, Section 4.2] as follows. Since the fundamental group of S is free in two generators, every character in $\bar{X}(S)$ lifts to $X(S)$. By studying the invariant functions on $X(S)$, one obtains an identification

$$(2-14) \quad \bar{X}(S) \cong \{(X, Y, Z, W) \in \mathbb{C}^4 \mid W^2 = XYZ\},$$

where $X = \tau_a^2$, $Y = \tau_b^2$, $Z = \tau_{ab}^2$ and $W = \tau_a \tau_b \tau_{ab} = \frac{1}{2}(\tau_a^2 \tau_b^2 + \tau_{ab}^2 - \tau_{ab^{-1}}^2)$. The four-fold branched covering map $X(S) \rightarrow \bar{X}(S)$ thus has the following description in these coordinates:

$$(2-15) \quad \bar{X}(S) \ni (x, y, z) \mapsto (x^2, y^2, z^2, xyz) \in \bar{X}(S).$$

2.6 Thurston's dimension bound

Based on Thurston [25, Theorem 5.6], Culler and Shalen [9, Proposition 3.2.1] showed:

Proposition 7 (Thurston) *Let N be a compact orientable 3-manifold. Suppose $\rho: \pi_1(N) \rightarrow \text{SL}(2, \mathbb{C})$ is an irreducible representation such that for each torus component T of ∂N ,*

$$\rho(\text{im}(\pi_1(T) \rightarrow \pi_1(N))) \not\subseteq \{\pm 1\}.$$

Let R_0 be an irreducible component of $R(N)$ containing ρ . Then $X_0 = \tau(R_0)$ has dimension $\geq s - 3\chi(N)$, where s is the number of torus components of ∂N .

An analogous statement for representations into $\text{PSL}(2, \mathbb{C})$ is the following:

Proposition 8 (Thurston) *Let N be a compact orientable 3-manifold. Suppose $\bar{\rho}_0: \pi_1(N) \rightarrow \text{PSL}(2, \mathbb{C})$ is an irreducible representation such that for each torus component T of ∂N , $\bar{\rho}_0(\text{im}(\pi_1(T) \rightarrow \pi_1(N)))$ is nontrivial and not isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\bar{R}_0$ be an irreducible component of $\bar{R}(N)$ containing $\bar{\rho}_0$. Then $\bar{X}_0 = \bar{\tau}(\bar{R}_0)$ has dimension $\geq s - 3\chi(N)$, where s is the number of torus components of ∂N .*

We first interpret the statement of the proposition before giving its proof.

The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not occur as an exclusion in the statement of [25, Theorem 5.6] because Thurston was concerned with *holonomies of hyperbolic structures* and not arbitrary irreducible representations. Its appearance is rather natural in light of the following observations.

First, abelian subgroups of $\mathrm{PSL}(2, \mathbb{C})$ either have a global fixed point in $P^1(\mathbb{C})$, or they are isomorphic to the Klein four group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (and are irreducible but elementary). Up to conjugacy, the latter has the nontrivial elements²

$$(2-16) \quad \kappa_1 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \kappa_2 = \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \kappa_3 = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In particular, for representations of $\langle \mu, \lambda \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, the two cases are distinguished by $\mathrm{tr} \rho[\mu, \lambda] = 2$ and $\mathrm{tr} \rho[\mu, \lambda] = -2$, with the first equation determining a positive dimensional component in the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of $\mathbb{Z} \oplus \mathbb{Z}$ and the second an isolated point. It also follows that an equivalent requirement on $\bar{\rho}_0$ in the statement of Proposition 8 is that

the image of each peripheral torus subgroup is nontrivial and reducible.

Second, each lift of the Klein four group to $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to the quaternionic group and hence if a peripheral subgroup corresponding to a torus boundary component has this image under a projective representation, then it does not lift. This explains why this case does not appear in Proposition 7.

Proof of Proposition 8 We follow the wording of the proof of [9, Proposition 3.2.1] closely, and combine this with observations from Thurston's proof and results stated in [19].

By [19, Corollary 3.3.5], the conclusion is equivalent to the assertion that $\dim \bar{R}_0 \geq s - 3\chi(N) + 3$. We shall prove this by induction on s . First suppose that $s = 0$. We may assume that $\partial N \neq \emptyset$ since otherwise $\chi(N) = 0$ and there is nothing to prove. Now N has the homotopy type of a finite 2-dimensional CW-complex K with one 0-cell, and, say, m 1-cells and n 2-cells. Thus $\pi_1(N)$ has a presentation $\langle g_1, \dots, g_m \mid r_1 = \dots = r_n = 1 \rangle$. This presentation gives a natural identification of $\bar{R}(N)$ with an algebraic subset of $(\mathrm{PSL}(2, \mathbb{C}))^m$.

Suppose $\bar{\rho}_0(g_i) = \pm G_i$. Then for each r_j there exists $\epsilon_j \in \{\pm 1\}$ such that

$$r_j(G_1, \dots, G_m) = \epsilon_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that if $\epsilon_j = 1$ for each j , then $\bar{\rho}_0$ lifts to a representation to $\mathrm{SL}(2, \mathbb{C})$. Now consider the coordinate ring $\mathbb{C}[a_1, b_1, c_1, d_1, \dots, a_m, b_m, c_m, d_m]$. Let $\bar{R}_0$ be the variety with ideal defined by the m equations $a_i c_i - b_i d_i = 1$ and the $3n$ equations arising from the $(1, 1)$, $(1, 2)$ and $(2, 1)$ entries of the m matrix equations

$$r_j \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} \right) = \epsilon_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

²Following standard convention, we write elements of $\mathrm{PSL}(2, \mathbb{C})$ not as sets $\{\pm C\}$ but rather in the form $\pm C$.

The equations given by the $(2, 2)$ entries of the above matrix equations are a consequence of the other equations and the multiplicative property of determinants. Our set-up gives a natural algebraic embedding $\bar{R}_0 \rightarrow (\mathrm{PSL}(2, \mathbb{C}))^m$ with image in $\bar{R}(N)$. Hence we may identify $\bar{R}_0$ with a subvariety of $\bar{R}(N)$ containing $\bar{\rho}_0$. Now

$$\dim \bar{R}_0 \geq 4m - m - 3n = 3m - 3n = -3\chi(K) + 3 = -3\chi(N) + 3$$

gives the desired conclusion.

Now suppose $s > 0$. Let T be a torus component of ∂N . Since $\bar{\rho}_0(\mathrm{im}(\pi_1(T) \rightarrow \pi_1(N))) \not\subseteq \{1\}$, there is $\alpha \in \pi_1(N)$, represented by a simple closed curve on T , such that $\bar{\rho}_0(\alpha) \neq 1 \in \mathrm{PSL}(2, \mathbb{C})$. Since $\bar{\rho}_0$ is irreducible, there is $\gamma \in \pi_1(N)$ such that $\bar{\rho}_0$ restricted to the subgroup generated by α and γ is irreducible.

Following Thurston, we show that γ can be chosen such that $\mathrm{tr}^2 \bar{\rho}_0(\gamma) \neq 4$. Hence suppose that $\mathrm{tr}^2 \bar{\rho}_0(\gamma) = 4$. Since $\bar{\rho}_0(\alpha)$ and $\bar{\rho}_0(\gamma)$ have no common fixed point on $P^1(\mathbb{C})$, it follows that $\bar{\rho}_0(\gamma)$ is parabolic and has a unique fixed point on $P^1(\mathbb{C})$. We may conjugate $\bar{\rho}_0$ such that

$$\bar{\rho}_0(\gamma) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{\rho}_0(\alpha) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $c \neq 0$. Then

$$\bar{\rho}_0(\alpha\gamma^{2n}) = \pm \begin{pmatrix} a & b+2na \\ c & d+2nc \end{pmatrix}.$$

Hence $\mathrm{tr}^2 \bar{\rho}_0(\alpha\gamma^{2n}) = (a + d + 2nc)^2$ and since $c \neq 0$ we may choose n such that $\mathrm{tr}^2 \bar{\rho}_0(\alpha\gamma^{2n})$ does not equal 4. Now the parabolic $\bar{\rho}_0(\gamma^{2n})$ has the same fixed point on $P^1(\mathbb{C})$ as $\bar{\rho}_0(\gamma)$. Hence $\bar{\rho}_0(\alpha)$ and $\bar{\rho}_0(\gamma^{2n})$ have no common fixed point on $P^1(\mathbb{C})$, and so the representation restricted to $\langle \alpha, \gamma^{2n} \rangle = \langle \alpha, \alpha\gamma^{2n} \rangle$ is irreducible. This completes the argument that we may choose γ such that $\mathrm{tr}^2 \bar{\rho}_0(\gamma) \neq 4$.

Choose a base point for the fundamental group of N based on T and thus represent γ by a simple closed curve based at this point. Drilling out a regular open neighbourhood of γ gives a submanifold M of N such that N is obtained by adding a 2-handle to a genus two boundary component S of M . We may choose a standard basis $\alpha', \beta', \gamma', \delta'$ of $\pi_1(S)$ such that α' and γ' are mapped to α and γ under the natural surjection $i_*: \pi_1(M) \rightarrow \pi_1(N)$; the 2-handle is attached to S along a simple closed curve that represents the conjugacy class of δ ; and the relation for $\pi_1(S)$ is given by $[\alpha', \beta'] = [\gamma', \delta']$.

The representation $\bar{\rho}'_0 = \bar{\rho}_0 \circ i_*: \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is irreducible since i_* is a surjection. By the induction hypothesis, there is an irreducible component $\bar{R}'_0$ of $\bar{R}(M)$ that contains $\bar{\rho}'_0$ and has dimension

$$\dim \bar{R}'_0 \geq (s-1) - 3\chi(M) + 3 = s - 3\chi(N) + 5$$

since $\chi(M) = \chi(N) - 1$. Let W be the intersection of $\bar{R}'_0$ with the subvariety defined by the two equations

$$\mathrm{tr}^2 \bar{\rho}(\delta') = 4, \quad \mathrm{tr} \bar{\rho}[\gamma', \delta'] = 2.$$

By [19, Section 2.4], these are well defined invariant functions on $\bar{R}'_0$. Then the dimension of each component of W is at least $s - 3\chi(N) + 3$.

We first show that $\bar{\rho}'_0$ is contained in W . The kernel of i_* is the normal closure of δ' , and hence $\bar{\rho}'_0(\delta') = 1$. This implies $\text{tr}^2 \bar{\rho}'_0(\delta') = 4$ and $\text{tr} \bar{\rho}'_0[\gamma', \delta'] = 2$. Hence there is an irreducible component $\bar{R}_0$ of W that contains $\bar{\rho}'_0$. To obtain the desired conclusion that we can identify $\bar{R}_0$ with a subvariety of $\bar{R}(N)$ it remains to show that all representations $\bar{\rho} \in \bar{R}_0$ near $\bar{\rho}_0$ satisfy $\bar{\rho}(\delta') = 1$.

Hence suppose that for $\bar{\rho} \in \bar{R}_0$ near $\bar{\rho}_0$, we have $\bar{\rho}(\delta') \neq 1$. Then $\bar{\rho}(\delta')$ is a nontrivial parabolic. Since $\text{tr} \bar{\rho}[\gamma', \delta'] = 2$, it follows that $\bar{\rho}(\delta')$ and $\bar{\rho}(\gamma')$ share a fixed point on $P^1(\mathbb{C})$. Since $\bar{\rho}$ is near $\bar{\rho}_0$, $\bar{\rho}(\gamma')$ is not a parabolic. Hence $\bar{\rho}[\gamma', \delta']$ is a nontrivial parabolic that has the same fixed point as $\bar{\rho}(\delta')$ and shares a fixed point with $\bar{\rho}(\gamma')$.

Since $[\gamma', \delta'] = [\alpha', \beta']$ in $\pi_1(M)$, we have $\text{tr} \bar{\rho}[\alpha', \beta'] = \pm 2$. Since $\bar{\rho}$ is near $\bar{\rho}_0$, it follows from our hypothesis on the peripheral subgroups that $\text{tr} \bar{\rho}[\alpha', \beta'] = 2$. Hence $\bar{\rho}(\alpha')$ and $\bar{\rho}(\beta')$ have a common fixed point on $P^1(\mathbb{C})$. Hence the unique fixed point of the nontrivial parabolic $\bar{\rho}[\gamma', \delta'] = \bar{\rho}[\alpha', \beta']$ is in common with both $\bar{\rho}(\gamma')$ and $\bar{\rho}(\alpha')$ and thus $\text{tr} \bar{\rho}[\alpha', \gamma'] = 2$. But this is not possible since $\bar{\rho}$ is near $\bar{\rho}'_0$, and $\bar{\rho}'_0$ restricted to the subgroup generated by α' and γ' is irreducible. This is the final contradiction to our hypothesis that $\bar{\rho}(\delta') \neq 1$. $\square$

Example 9 The $\text{PSL}(2, \mathbb{C})$ -character variety of the Whitehead link complement contains 0-dimensional components that satisfy the condition that all peripheral subgroups have image isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. See [27, Appendix C].

Example 10 Baker and Petersen [2, Theorem 7.6] give zero-dimensional components in character varieties of once-punctured torus bundles that appear to contradict Proposition 8. A less sophisticated approach than given in [2] shows that these characters indeed lie on one-dimensional components. The presentation for the fundamental group of M_n with monodromy $\varphi_n = \text{AB}^{n+2}$ is given in [2, Section 2.1] as

$$\Gamma_n = \langle \alpha, \beta \mid \beta^{-n} = \alpha^{-1} \beta \alpha^2 \beta \alpha^{-1} \rangle,$$

where a meridian is the element $\mu = \beta \alpha$ and a corresponding longitude is $\lambda = (\alpha \beta \alpha^{-1}) \beta (\alpha \beta \alpha^{-1})^{-1} \beta^{-1}$; see [2, Lemma 2.3]. Suppose $|n| \geq 2$ is even. Then

$$\bar{\rho}(\alpha) = \pm \begin{pmatrix} -yx^n & -1 \\ \frac{1+x^{2+n}}{(1-x^2)(1-x^n)} & y \end{pmatrix} \quad \text{and} \quad \bar{\rho}(\beta) = \pm \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

gives a representation of Γ_n into $\text{PSL}(2, \mathbb{C})$ that does not lift to $\text{SL}(2, \mathbb{C})$ if

$$x^2 + x^n = y^2 x^n (1 - x^2)(1 - x^n).$$

Indeed, the above equation is equivalent to the determinant condition for $\bar{\rho}(\alpha)$, and it is also a sufficient condition for $\bar{\rho}(\alpha) \bar{\rho}(\beta)^{-n} \bar{\rho}(\alpha) + \bar{\rho}(\beta) \bar{\rho}(\alpha)^2 \bar{\rho}(\beta) = 0$ (noting that both summands are independent of the choice of sign).

The alleged isolated points in [2, Theorem 7.6] correspond to the points (x_0, y_0) satisfying $x_0^n = -1$ and $y_0 = \pm \frac{1}{\sqrt{2}}$ on this curve. At each of these points, the image of the commutator of meridian and longitude has trace equal to two (hence the representation restricted to the peripheral subgroup is reducible) and the image of the meridian has trace $\pm \frac{1}{\sqrt{2}}(x_0 + x_0^{-1})$ (hence the representation restricted to the peripheral subgroup is nontrivial since the absolute value of the trace is bounded from above by $\sqrt{2}$). The triple of squares of traces is indeed as stated in [2, Theorem 7.6]

$$(\mathrm{tr}^2 \bar{\rho}(\alpha), \mathrm{tr}^2 \bar{\rho}(\beta), \mathrm{tr}^2 \bar{\rho}(\alpha\beta)) = (2, (x_0 + x_0^{-1})^2, \frac{1}{2}(x_0 + x_0^{-1})^2).$$

We have intentionally used the notation from [2] in this example. With respect to the notation of (2-5), one has $t = \mu^{-1}$, $a = \beta^{-1}\mu\beta\mu^{-1}$ and $b = \beta$ and

$$\pi_1(M_n) = \langle t, a, b \mid t^{-1}at = a(a^{-1}b^{-1})^{n+2}, t^{-1}bt = ba \rangle.$$

3 Character varieties of once-punctured torus bundles

We continue with the notation introduced in Section 2 and analyse the $\mathrm{SL}(2, \mathbb{C})$ -character varieties of once-punctured torus bundles using the restriction to the fibre.

3.1 The restriction map and the fixed point set

The fundamental group of a once-punctured torus bundle M_φ with monodromy φ is

$$\pi_1(M_\varphi) = \langle t, a, b \mid t^{-1}at = \varphi(a), t^{-1}bt = \varphi(b) \rangle.$$

Throughout, we assume

$$[\varphi_*] = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Since we restrict our attention to hyperbolic once-punctured torus bundles, we have $|\mathrm{tr} \varphi_*| = |k_1 + k_4| > 2$.

As in Section 2.1, $X(M_\varphi)$ admits the affine coordinates (x, y, z, u, v, w, q) in $\mathbb{C}^7$, where

$$(3-1) \quad x = \tau_a(\rho), \quad y = \tau_b(\rho), \quad z = \tau_{ab}(\rho), \quad u = \tau_t(\rho), \quad v = \tau_{at}(\rho), \quad w = \tau_{bt}(\rho), \quad q = \tau_{abt}(\rho)$$

for $\rho \in R(M_\varphi)$. Consider the *restriction map*

$$r : X(M_\varphi) \rightarrow X(S), \quad (x, y, z, u, v, w, q) \mapsto (x, y, z).$$

Each representation $\rho \in R(M_\varphi)$ satisfies

$$\rho(t)^{-1} \rho(\gamma) \rho(t) = \rho(\varphi(\gamma)) \quad \text{for all } \gamma \in \pi_1(S).$$

Taking the traces on both sides,

$$\mathrm{tr}(\rho(\gamma)) = \mathrm{tr}(\rho(\varphi(\gamma))) \quad \text{for all } \gamma \in \pi_1(S).$$

Since χ_ρ in $X(S)$ is uniquely determined by $\text{tr } \rho(a)$, $\text{tr } \rho(b)$ and $\text{tr } \rho(ab)$, the image of r satisfies

$$\begin{aligned} \text{im}(r) \subseteq X_\varphi(S) &:= \{(\text{tr } \rho(a), \text{tr } \rho(b), \text{tr } \rho(ab)) \mid \text{tr } \rho(a) = \text{tr } \rho(\varphi_*(a)), \text{tr } \rho(b) = \text{tr } \rho(\varphi_*(b)), \\ &\quad \text{tr } \rho(ab) = \text{tr } \rho(\varphi_*(ab)), \rho \in R(S)\} \\ &\subseteq X(S). \end{aligned}$$

Since $\text{tr } \rho(\varphi_*(a))$, $\text{tr } \rho(\varphi_*(b))$ and $\text{tr } \rho(\varphi_*(ab))$ can be written as polynomials with rational coefficients in $x = \text{tr } \rho(a)$, $y = \text{tr } \rho(b)$ and $z = \text{tr } \rho(ab)$, and since φ_* is an automorphism of the free group in two letters, it follows that there is a polynomial automorphism $\bar{\varphi}: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ induced by φ with the property that $X_\varphi(S)$ is the set of fixed points of this polynomial automorphism:

$$(3-2) \quad X_\varphi(S) = \{(x, y, z) \in \mathbb{C}^3 \mid (x, y, z) = \bar{\varphi}(x, y, z)\}.$$

Lemma 11 We have $\text{im}(r) = X_\varphi(S)$. Moreover, there are at most finitely many reducible characters in $X_\varphi(S)$.

Proof By definition, $\text{im}(r) \subseteq X_\varphi(S)$ and for the first statement, it remains to show that the map is surjective.

If $(x, y, z) \in X_\varphi(S)$ is the character of an irreducible representation, then $(x, y, z) \in \text{im}(r)$ follows from the presentation of the group and the second statement in Lemma 4. We analyse this situation in more detail in Lemma 18.

Hence suppose $(x, y, z) \in X_\varphi(S)$ is the character of a reducible representation. Since this is also the character of an abelian representation, we may assume that it is the character of the representation ρ defined by

$$\rho(a) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}.$$

Since $\rho(\varphi_*(a))$ can be viewed as a word in these matrices, it is also diagonal. Similarly for $\rho(\varphi_*(b))$. Hence

$$\rho(\varphi_*(a)) = \begin{pmatrix} s_1 & 0 \\ 0 & s_1^{-1} \end{pmatrix}, \quad \rho(\varphi_*(b)) = \begin{pmatrix} p_1 & 0 \\ 0 & p_1^{-1} \end{pmatrix}.$$

Then the three trace conditions defining $X_\varphi(S)$ imply that we have the mutually exclusive cases,

- (1) $(s_1, p_1) = (s, p)$ or
- (2) $(s_1, p_1) = (s^{-1}, p^{-1})$ with either $s \neq s^{-1}$ or $p \neq p^{-1}$ (equivalently, $s \neq \pm 1$ or $p \neq \pm 1$).

In order to show that $(x, y, z) \in \text{im}(r)$, we need to determine $T \in \text{SL}(2, \mathbb{C})$ such that $T^{-1}\rho T = \rho\varphi_*$.

In the first case, this is satisfied by $T = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$ for arbitrary $m \in \mathbb{C}^*$. We note that we have $\rho(\varphi_*(\gamma)) = \rho(\gamma)$ for all $\gamma \in \pi_1(S)$, and the resulting representation of $\pi_1(M_\varphi)$ is abelian and hence reducible.

In the second case, this is satisfied by $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We note that we have $\rho(\varphi_*(\gamma)) = \rho(\gamma)^{-1}$ for all $\gamma \in \pi_1(S)$, and the resulting representation of $\pi_1(M_\varphi)$ is irreducible and corresponds to a *binary dihedral representation* in $R(M_\varphi(\lambda)) \subset R(M_\varphi)$ (see [4, Section 4]).

Hence in either case, this shows that $(x, y, z) \in \text{im}(r)$.

We now prove the second statement. The monodromy $\varphi_* = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in SL(2, \mathbb{Z})$ satisfies $|k_1 + k_4| > 2$. The pair of eigenvalues of the abelian representation ρ as above satisfies $(s^{k_1} p^{k_3}, s^{k_2} p^{k_4}) = (s^{\pm 1}, p^{\pm 1})$ since abelian representations factor through first homology and the latter is determined by φ_* . This implies that each of s and p raised to the power of $(k_1 \mp 1)(k_4 \mp 1) - k_2 k_3$ equals one. There are finitely many solutions unless $(k_1 \mp 1)(k_4 \mp 1) = k_2 k_3$. Using the determinant condition, this is equivalent with $k_1 + k_4 = \mp 2$, which contradicts the trace condition for φ_* . This proves the second statement. $\square$

Scholion 12 *The irreducible characters in $X(M_\varphi)$ that map to reducible characters in $X_\varphi(S)$ are precisely the characters of the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$.*

3.2 The origin and the coordinate axes

For the automorphisms α and β^{-1} given in (2-6), the induced polynomial automorphisms are

$$\bar{\alpha}(x, y, z) = (x, z, xz - y) \quad \text{and} \quad \bar{\beta}^{-1}(x, y, z) = (z, y, yz - x),$$

respectively. Given the standard representative φ_* as in (2-8), it follows from Section 2.2 that the associated polynomial automorphism $\bar{\varphi}: \mathbb{C}^3 \rightarrow \mathbb{C}^3$ induced by φ factors in terms of $\bar{\iota}$, $\bar{\alpha}$ and $\bar{\beta}^{-1}$.

This implies that the origin $(0, 0, 0) \in X(S)$ is always contained in $X_\varphi(S)$ for any M_φ . Restricted to S , any representation with this character has image in $SL(2, \mathbb{C})$ isomorphic to the quaternionic group, and image isomorphic to the Klein four group under the quotient map to $PSL(2, \mathbb{C})$ (see (4-5) below).

The examples in Section 5.4 show that the component containing $(0, 0, 0)$ in $X_\varphi(S)$ may have arbitrarily large genus if $b_1(\varphi) \neq 3$. We now show that in the maximal rank case, at least one of the coordinate axes is always contained in $X_\varphi(S)$.

For all $x, y, z \in \mathbb{C}$, we have $\bar{\iota}(x, y, z) = (x, y, z)$ and

$$\begin{aligned} \bar{\alpha}(x, 0, 0) &= (x, 0, 0), & \bar{\beta}^{-1}(x, 0, 0) &= (0, 0, -x), \\ \bar{\alpha}(0, y, 0) &= (0, 0, -y), & \bar{\beta}^{-1}(0, y, 0) &= (0, y, 0), \\ \bar{\alpha}(0, 0, z) &= (0, z, 0) & \bar{\beta}^{-1}(0, 0, z) &= (z, 0, 0). \end{aligned}$$

Denote the x , y , and z coordinate axes in $\mathbb{C}^3$ by L_1 , L_2 and L_3 , respectively. Then $\bar{\alpha}$ and $\bar{\beta}^{-1}$ permute these three lines, with a sign change for exactly one of them. Indeed, we may assume that the permutation of the lines corresponds to the permutation of the indices of the lines induced by $[\alpha_2]$ and $[\beta_2]$ via the isomorphism $SL(2, \mathbb{Z}_2) \cong \text{Sym}(3)$.

Lemma 13 *If $H_1(M_\varphi, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then $X_\varphi(S)$ contains either exactly one or all three of L_1 , L_2 and L_3 . Moreover, $X_{\varphi^2}(S)$ contains all three lines.*

Proof Since $\bar{\iota}(x, y, z) = (x, y, z)$, it suffices to restrict to the case where $\text{tr } \varphi_* > 0$. We may assume that φ_* is given as in (2-8). Recall from Section 2.3 that $H_1(M_\varphi, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$ is equivalent to $o(\varphi_2) = 1$. Then

$$\text{Id} = [\varphi_2] = [\alpha_2]^{a_1} [\beta_2]^{-b_1} [\alpha_2]^{a_2} [\beta_2]^{-b_2} \cdots [\alpha_2]^{a_n} [\beta_2]^{-b_n}.$$

It follows from the description of the action of α_2, β_2 on L_1, L_2 and L_3 , that the induced polynomial automorphism $\bar{\varphi}$ stabilises each line. Hence, $\bar{\varphi}(L_i) = L_i$. Note that when $\bar{\alpha}$ or $\bar{\beta}^{-1}$ permutes the lines, the sign of exactly one line changes. Denote the number of sign changes of L_i by n_i . We have

$$\bar{\varphi}(x, 0, 0) = ((-1)^{n_1}x, 0, 0), \quad \bar{\varphi}(0, y, 0) = (0, (-1)^{n_2}y, 0), \quad \bar{\varphi}(0, 0, z) = (0, 0, (-1)^{n_3}z).$$

As $o(\varphi_2) = 1$, the number $n_1 + n_2 + n_3 = \sum_{i=1}^n (a_i + b_i)$ is also even. Hence at least one n_i is even and at least one coordinate is fixed by $\bar{\varphi}$. We also see that either exactly one n_i is even or all three n_i are even. This proves the first statement. The second statement follows from the observation that

$$\bar{\varphi}^2(x, 0, 0) = ((-1)^{2n_1}x, 0, 0) = (x, 0, 0),$$

$$\bar{\varphi}^2(0, y, 0) = (0, (-1)^{2n_2}y, 0) = (0, y, 0),$$

$$\bar{\varphi}^2(0, 0, z) = (0, 0, (-1)^{2n_3}z) = (0, 0, z). \quad \square$$

If $b_1(M_\varphi) = 1$, then $[\varphi_2]$ has order 3. So $\bar{\varphi}$ does not stabilise any axis and $L_i \cap X_\varphi(S) = \{(0, 0, 0)\}$ for each i .

We also note that for any φ , $X_{\varphi^6}(S)$ contains all three lines.

As a last observation, we add that $(x, 0, 0) \in L_1$ is a reducible character if and only if $x^2 = 4$. In this case, the extension to $\pi_1(M_\varphi)$ is either reducible or binary dihedral according to Scholion 12.

3.3 Dimensions and cyclic covers

Proposition 14 *Let M_φ be a hyperbolic once-punctured torus bundle. Then every Zariski component of $X(M_\varphi)$ is one-dimensional.*

Proof We first note that since $H_1(M_\varphi, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tor}(M_\varphi, \mathbb{Z})$, every component containing only reducible characters is one-dimensional and of genus zero.

Suppose $X \subset X(M_\varphi)$ is a component containing the character of the irreducible representation $\rho \in R(M_\varphi)$. Since M_φ does not contain a closed essential surface (see [8; 12]), the dimension of X is at most one.

We now appeal to a result due to Thurston (see Proposition 7) which implies that the dimension of X is at least one if $\rho(\pi_1(\partial M))$ is not contained in $\{\pm I\}$. Hence suppose that $\rho(\pi_1(\partial M))$ is contained in $\{\pm I\}$; such a representation is said to have *trivial peripheral holonomy*. Since the longitude is $a^{-1}b^{-1}ab$, we note that Lemma 4(1) implies that ρ is reducible if $\rho(t) = \pm I$ and $\rho(a^{-1}b^{-1}ab) = I$. Hence $\rho(a^{-1}b^{-1}ab) = -I$. An elementary calculation setting $\rho(a)$ and $\rho(b)$ as in (2-11) and solving $\rho(ab) = -\rho(ba)$ shows that the resulting representation satisfies $\text{tr } \rho(a) = \text{tr } \rho(b) = \text{tr } \rho(ab) = 0$ and hence is unique up to conjugation.

In particular, ρ descends to an abelian irreducible representation $\bar{\rho}: \pi_1(\partial M) \rightarrow \text{PSL}(2, \mathbb{C})$ with $\bar{\rho}(t) = \{\pm I\}$ and image isomorphic to the Klein four group, generated by the image of $\pi_1(S)$. It follows that $H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3$. In particular, Lemma 13 implies that the character of ρ is contained in at least one one-dimensional component containing irreducible representations. $\square$

The natural n -fold cyclic covering $M_{\varphi^n} \rightarrow M_{\varphi}$ gives an embedding of $\pi_1(M_{\varphi^n})$ as a subgroup of index n in $\pi_1(M_{\varphi})$. Hence the restriction map induces a regular map $X(M_{\varphi}) \rightarrow X(M_{\varphi^n})$. Example 16 below shows that this may not be surjective. We can say more about this map:

Proposition 15 *The natural n -fold cyclic covering $M_{\varphi^n} \rightarrow M_{\varphi}$ induces a regular map $X(M_{\varphi}) \rightarrow X(M_{\varphi^n})$ of degree at most two. If n is odd, then the degree is one. Moreover, the map takes canonical components to canonical components.*

Proof The first sentence follows directly from [4, Corollary 3.5], and the second from [4, Proposition 3.3]. The last sentence follows since the restriction of a discrete and faithful representation to a subgroup gives a discrete and faithful representation of that subgroup. $\square$

We conclude with an example that ties the discussion in this section together and exhibits different behaviours.

Example 16 Let $\varphi = \alpha^2 \beta^{-2}$. Then

$$[\varphi_*] = A^2 B^{-2} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

and hence

$$H_1(M_{\varphi}; \mathbb{Z}_2) \cong \mathbb{Z}_2^3 \cong H_1(M_{\varphi^2}; \mathbb{Z}_2).$$

A computation with the set-up in the proof of Proposition 14 shows that $X(M_{\varphi})$ contains no characters of representations with trivial peripheral holonomy. Indeed, $\bar{X}(M_{\varphi})$ contains such a component, but it does not lift. This example fits in the family given in [19].

A calculation shows that $X_{\varphi}(S) = V \cup L_3$, where V is an irreducible curve, and $X(M_{\varphi}) = V' \cup L'_3$ also has two components. The preimage V' of V is the canonical curve and the preimage L'_3 of L_3 has the property that the trace of the meridian is identically zero on it (hence the meridian maps to an element of order four). All binary dihedral characters are contained in L'_3 and since they are simple points they are hence not contained in V' . Since these characters are the only branch points (as will be shown in general), it follows that $L'_3 \rightarrow L_3$ is a two-fold branched cover, and $V' \rightarrow V$ is a two-fold unbranched cover.

Now $X_{\varphi^2}(S) \supset X_{\varphi}(S) \cup L_1 \cup L_2 = V \cup L_1 \cup L_2 \cup L_3$. Hence this is an example where all three lines are contained in the fixed-point set. The preimage in $X(M_{\varphi^2})$ of V is the canonical curve, but the preimage of L_3 has two components L_3^+ and L_3^- that are characterised by whether the trace of the meridian is identically $+2$ or -2 . The component L_3^- is the image of L'_3 under the map $X(M_{\varphi}) \rightarrow X(M_{\varphi^2})$ since the square of an element of order four has order two. Hence L_3^+ is not contained in the image of the map. We note that each map $L_3^- \rightarrow L_3$ and $L_3^+ \rightarrow L_3$ is one-to-one.

3.4 Reducible characters

Since $H_1(M_{\varphi}, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tor}(M_{\varphi}, \mathbb{Z})$, it follows that $X^{\text{red}}(M_{\varphi})$ consists of finitely many affine lines.

From the set-up in the proof of Lemma 11, we see that the reducible characters in $X_\varphi(S)$ that are restrictions of reducible characters in $X(M_\varphi)$ are the same as characters of abelian representations ρ satisfying $\rho(\varphi_*(\gamma)) = \rho(\gamma)$ for all $\gamma \in \pi_1(S)$. Since the monodromy of M_φ is

$$\varphi_* = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \text{with } |k_1 + k_4| > 2,$$

all possible pairs (s, p) of eigenvalues of $\rho(a)$ and $\rho(b)$ corresponding to a common invariant subspace are determined by

$$(3-3) \quad s = s^{k_1} p^{k_3}, \quad p = s^{k_2} p^{k_4}.$$

The (not necessarily distinct) components of $X^{\text{red}}(M_\varphi)$ corresponding to the finitely many solutions to these equations are given in our affine coordinates (3-1) by

$$X_{s,p} = \left\{ (s+s^{-1}, p+p^{-1}, sp+s^{-1}p^{-1}, m+m^{-1}, sm+s^{-1}m^{-1}, pm+p^{-1}m^{-1}, spm+s^{-1}p^{-1}m^{-1}) \mid m \in \mathbb{C}^* \right\}.$$

Here, we continue to use the convention

$$x = \tau_a(\rho) = s + s^{-1}, \quad y = \tau_b(\rho) = p + p^{-1} \quad \text{and} \quad u = \tau_t(\rho) = m + m^{-1}$$

from Section 3.1.

In particular, each component of $X^{\text{red}}(M_\varphi)$ has image a single point in $X_\varphi(S)$.

The proof of the following proposition is an application of the main result of [20] and generalises the example given in that paper by using fundamental results due to Fox [13]. We thank Michael Heusener and Joan Porti for encouraging correspondence, and refer the reader to [20] for the required definitions of twisted Alexander polynomials. The reader is especially encouraged to read [20, Examples 3.2, 4.4, 8.2] before going through the proof below.

Proposition 17 *For every component X_{s_0,p_0} of $X^{\text{red}}(M_\varphi)$, there exists $m_0 \in \mathbb{C}^*$ such that X_{s_0,p_0} intersects $X^{\text{irr}}(M_\varphi)$ at χ_{m_0} , where χ_{m_0} is the character of the abelian representation ρ_{m_0} with*

$$\rho_{m_0}(t) = \text{Diag}(m_0, m_0^{-1}), \quad \rho_{m_0}(a) = \text{Diag}(s_0, s_0^{-1}), \quad \rho_{m_0}(b) = \text{Diag}(p_0, p_0^{-1}).$$

Moreover, χ_{m_0} is contained on exactly one curve C in $X^{\text{irr}}(M_\varphi)$, a smooth point of both X_{s_0,p_0} and C and the intersection at χ_{m_0} is transverse. Moreover, if $s_0, t_0 \in \{\pm 1\}$, then m_0 is the square root of a zero of the characteristic polynomial of φ_* and otherwise $m_0 = \pm 1$.

Proof Let $m = m_0^2$, $s = s_0^2$ and $p = p_0^2$. With this set up, the remainder of the proof applies verbatim to the $\text{PSL}(2, \mathbb{C})$ -character variety. We have a group homomorphism $\alpha: \pi_1(M_\varphi) \rightarrow \mathbb{C}^*$ with

$$\alpha(a) = s, \quad \alpha(b) = p, \quad \alpha(t) = m.$$

The pair (s, p) satisfies (3-3), that is, the restriction $\alpha|_{\text{Tor}(M_\varphi, \mathbb{Z})}$ is induced by a group homomorphism $h: \langle a, b \rangle \rightarrow U(1)$ such that $h \circ \varphi = h$ and $h(a) = s$ and $h(b) = p$.

We apply the criterion of deformations of reducible characters in [20, Theorem 1.3]. Fix the canonical splitting $H_1(M_\varphi, \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tor}(M_\varphi, \mathbb{Z}) \rightarrow \text{Tor}(M_\varphi, \mathbb{Z})$ where $p(a) = a$, $p(b) = b$ and $p(t) = 0$ and an element $\psi \in H^1(M_\varphi, \mathbb{Z})$ where $\psi(a) = \psi(b) = 0$ and $\psi(t) = 1$. To prove the statement, we only need to show that m is a simple zero of the twisted Alexander polynomial $\Delta_{M_\varphi}^{\psi_\alpha}$. A direct computation shows that the Jacobian matrix is

$$J_{M_\varphi} = \begin{pmatrix} t^{-1}(a^{-1} - 1) & t^{-1}a^{-1} \left(t \frac{\partial}{\partial a} \varphi(a) - 1 \right) & t^{-1}a^{-1} t \frac{\partial}{\partial b} \varphi(a) \\ t^{-1}(b^{-1} - 1) & t^{-1}b^{-1} t \frac{\partial}{\partial a} \varphi(b) & t^{-1}b^{-1} \left(t \frac{\partial}{\partial b} \varphi(b) - 1 \right) \end{pmatrix}.$$

Each $\varphi \in \text{Aut}(\langle a, b \rangle)$ linearly extends to a ring homomorphism of $\mathbb{Z}\langle a, b \rangle$, also denoted by φ . Under the abelianisation homomorphism, it also induces a homomorphism $\varphi^{\text{ab}} : \mathbb{Z}[a, b] \rightarrow \mathbb{Z}[a, b]$. The Jacobian of φ is

$$J_\varphi = \begin{pmatrix} \frac{\partial}{\partial a} \varphi(a) & \frac{\partial}{\partial b} \varphi(a) \\ \frac{\partial}{\partial a} \varphi(b) & \frac{\partial}{\partial b} \varphi(b) \end{pmatrix}.$$

Let J_φ^{ab} denotes its image under the abelianisation homomorphism. By the chain rule [13], $J_{\varphi_1 \varphi_2} = \varphi_1(J_{\varphi_2})J_{\varphi_1}$ for any $\varphi_1, \varphi_2 \in \text{Aut}(F_2)$. Applying the abelianisation homomorphism and taking determinants on both sides, we have

$$\det J_{\varphi_1 \varphi_2}^{\text{ab}} = \det \varphi_1^{\text{ab}}(J_{\varphi_2}^{\text{ab}}) \det J_{\varphi_1}^{\text{ab}} = \varphi_1^{\text{ab}}(\det J_{\varphi_2}^{\text{ab}}) \det J_{\varphi_1}^{\text{ab}}.$$

We may assume that φ_* is given as in (2-8). Then $\det J_\alpha^{\text{ab}} = \det J_{\beta^{-1}}^{\text{ab}} = 1$ implies that $\det J_\varphi^{\text{ab}} = 1$. Let J_φ^h denotes the image of J_φ under the map h . It is clear that $h(J_\varphi^{\text{ab}}) = J_\varphi^h$. Hence $\det J_\varphi^h = 1$.

By fundamental formula of free calculus [13], we have

$$\varphi(a) - 1 = \left(\frac{\partial}{\partial a} \varphi(a) \right) (a - 1) + \left(\frac{\partial}{\partial b} \varphi(a) \right) (b - 1).$$

Applying h on both sides, we have

$$\left(h \left(\frac{\partial}{\partial a} \varphi(a) \right) - 1 \right) (s - 1) + h \left(\frac{\partial}{\partial b} \varphi(a) \right) (p - 1) = 0$$

since $h \circ \varphi = h$. By symmetry,

$$\left(h \left(\frac{\partial}{\partial b} \varphi(b) \right) - 1 \right) (p - 1) + h \left(\frac{\partial}{\partial a} \varphi(b) \right) (s - 1) = 0.$$

Define $\psi_\alpha : \pi_1(M_\varphi) \rightarrow \mathbb{C}[x^{\pm 1}]$, where $\psi_\alpha(\gamma) = \alpha(\gamma)x^{\psi(\gamma)}$. The Jacobian $J_{M_\varphi}^{\psi_\alpha}$ is given by $(J_{M_\varphi}^{\psi_\alpha})_{ij} = \psi_\alpha((J_{M_\varphi})_{ij})$. Since Alexander invariants are obtained from determinants, we may multiply each row by an unit and work with the following matrix:

$$\begin{pmatrix} 1 - s & h \left(\frac{\partial \varphi(a)}{\partial a} \right) x - 1 & h \left(\frac{\partial \varphi(a)}{\partial b} \right) x \\ 1 - p & h \left(\frac{\partial \varphi(b)}{\partial a} \right) x & h \left(\frac{\partial \varphi(b)}{\partial b} \right) x - 1 \end{pmatrix}.$$

We now have the following cases:

(1) If $h(a) = h(b) = 1$, a direct computation shows that $J_\varphi^h = \varphi_*$. So $\Delta_{M_\varphi}^{\psi_\alpha}$ is the characteristic polynomial of φ_* , which has 2 distinct simple zeroes since the trace is distinct from ± 2 . Hence m equals one of these roots.

(2) If $h(a) = 1$ and $h(b) \neq 1$, from the above equations, we have

$$h\left(\frac{\partial}{\partial b}\varphi(a)\right) = 0 \quad \text{and} \quad h\left(\frac{\partial}{\partial b}\varphi(b)\right) = 1.$$

Combined with $\det J_\varphi^h = 1$, we have $h\left(\frac{\partial}{\partial a}\varphi(a)\right) = 1$ so $\Delta_{M_\varphi}^{\psi_\alpha} = x - 1$. By symmetry, $\Delta_{M_\varphi}^{\psi_\alpha} = x - 1$ if $h(a) \neq 1$ and $h(b) = 1$. Hence in each of these cases, $m = 1$.

(3) If $h(a) \neq 1$ and $h(b) \neq 1$, we have

$$\left(h\left(\frac{\partial}{\partial a}\varphi(a)\right) - 1\right)\left(h\left(\frac{\partial}{\partial b}\varphi(b)\right) - 1\right) = h\left(\frac{\partial}{\partial b}\varphi(a)\right)h\left(\frac{\partial}{\partial a}\varphi(b)\right)$$

by multiplying the two equations. Rearranging the above equation and substitute $\det J_\varphi^h = 1$ into it, we have

$$h\left(\frac{\partial}{\partial a}\varphi(a)\right) + h\left(\frac{\partial}{\partial b}\varphi(b)\right) = 2.$$

Then

$$h\left(\frac{\partial}{\partial b}\varphi(b)\right) - 1 = 1 - h\left(\frac{\partial}{\partial a}\varphi(a)\right)$$

and we can rewrite the second equation as

$$\left(h\left(\frac{\partial}{\partial a}\varphi(a)\right) - 1\right)(p - 1) = h\left(\frac{\partial}{\partial a}\varphi(b)\right)(s - 1).$$

This gives us that the determinant of the first two columns of $J_{M_\varphi}^{\psi_\alpha}$ is $(1 - p)(x - 1)$. We can perform similar computation for the other two determinants which both have a factor $x - 1$. This implies $\Delta_{M_\varphi}^{\psi_\alpha} = x - 1$ and $m = 1$. $\square$

A direct consequence of the above proposition is that $r(X(M_\varphi)) = r(X^{\text{irr}}(M_\varphi))$. We now show that there are no ramification points of $r: X^{\text{irr}}(M_\varphi) \rightarrow X_\varphi(S)$ at reducible characters contained on $X^{\text{irr}}(M_\varphi)$.

Consider the action of $H^1(M_\varphi, \mathbb{Z}_2)$ on $X^{\text{red}}(M_\varphi)$. We first characterise the points in $X^{\text{red}}(M_\varphi)$ with nontrivial stabilisers under this action. Suppose $h \in H^1(M_\varphi, \mathbb{Z}_2)$ with $h(t) = \epsilon_t$, $h(a) = \epsilon_a$, $h(b) = \epsilon_b$. Then a direct calculation shows that $S(h) \cap X^{\text{red}}(M_\varphi)$ consists of all points in $X^{\text{red}}(M_\varphi)$ satisfying

$$m^2 = \epsilon_t, \quad s^2 = \epsilon_a, \quad p^2 = \epsilon_b.$$

In particular, the trace functions take values in $\{-2, 0, 2\}$.

We now focus on $S(\epsilon)$. Recall that $\epsilon(t) = -1$, $\epsilon(a) = \epsilon(b) = 1$. In this case,

$$m^2 = -1, \quad s^2 = 1, \quad p^2 = 1$$

and hence $s, p \in \{\pm 1\}$. If such a character is in the intersection of $X^{\text{red}}(M_\varphi) \cap X^{\text{irr}}(M_\varphi)$, then Lemma 5

implies that there is such a character of a nonabelian reducible representation ρ . Up to conjugacy and the action of ε on $R(M_\varphi)$, we may assume that

$$\rho(t) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} s & u \\ 0 & s \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & v \\ 0 & p \end{pmatrix},$$

where $s, p \in \{\pm 1\}$ and $(u, v) \neq (0, 0)$. Then

$$\begin{pmatrix} s & -u \\ 0 & s \end{pmatrix} = \rho(t^{-1}at) = \rho(\varphi(a)) = \begin{pmatrix} s^{k_1} \epsilon_b^{k_3} & k_3 s^{k_1} p^{k_3-1} v + k_1 s^{k_1-1} p^{k_3} u \\ 0 & s^{k_1} p^{k_3} \end{pmatrix}$$

and

$$\begin{pmatrix} p & -v \\ 0 & p \end{pmatrix} = \rho(t^{-1}bt) = \rho(\varphi(b)) = \begin{pmatrix} s^{k_2} p^{k_4} & k_4 s^{k_2} p^{k_4-1} v + k_2 s^{k_2-1} p^{k_4} u \\ 0 & s^{k_2} p^{k_4} \end{pmatrix}.$$

This implies in particular the equations $u = -sp(k_3v + k_1u)$ and $v = -sp(k_4v + k_2u)$. Now $sp = \pm 1$, and hence these two equations give

$$\text{transpose}[\varphi_*] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} k_1 & k_3 \\ k_2 & k_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But then φ_* has eigenvalue ± 1 and hence $\text{tr}(\varphi_*) = \pm 2$, which is a contradiction. This shows that there are no ramification points of $r: X^{\text{irr}}(M_\varphi) \rightarrow X_\varphi(S)$ at reducible characters contained on $X^{\text{irr}}(M_\varphi)$.

3.5 Irreducible characters

The following lemma proves that every irreducible character in $X_\varphi(S)$ is the restriction of two distinct irreducible characters in $X(M_\varphi)$, and hence is not a branch point of $r: X(M_\varphi) \rightarrow X_\varphi(S)$. This also implies, via Proposition 17, that the degree of r is two on a Zariski dense subset. Moreover, Example 16 gives examples where these distinct characters lie in different Zariski components of $X(M_\varphi)$ and where they lie in the same Zariski component.

Lemma 18 *For any irreducible character χ_ρ of S such that $\chi_\rho \in X_\varphi(S)$, there exists $T \in \text{SL}(2, \mathbb{C})$ such that $T^{-1}\rho(\gamma)T = \rho(\varphi_*(\gamma))$ for all $\gamma \in \pi_1 S$. Moreover, T is unique up to sign and at least one of $\text{tr } T$, $\text{tr}(\rho(a)T)$, $\text{tr}(\rho(b)T)$ and $\text{tr}(\rho(ab)T)$ is nonzero.*

Proof Let $\rho \in R(S)$ be an irreducible representation with $\chi_\rho \in X_\varphi(S)$. We define $\rho_1 \in R(S)$ by $\rho_1(\gamma) = \rho(\varphi_*(\gamma))$ for $\gamma \in \pi_1(S)$. According to the definition of $X_\varphi(S)$, $\chi_\rho = \chi_{\rho_1}$. Since χ_ρ is the character of an irreducible representation, it follows that ρ and ρ_1 are conjugate. Hence there exists $T \in \text{SL}(2, \mathbb{C})$ such that $T^{-1}\rho T = \rho_1$. Since χ is irreducible, the centraliser of ρ is $\pm E$, where E is the identity matrix. Hence T is unique up to sign. This proves the first part of the lemma, and it remains to show that at least one of the stated traces is nonzero.

Up to conjugation, we may assume that

$$\rho(a) = \begin{pmatrix} s & 0 \\ 1 & s^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix},$$

where $u \neq 0$. Since $\chi_\rho \in X_\varphi(S)$, we have

$$\rho(\varphi_*(a)) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s + s^{-1} - s_1 \end{pmatrix}, \quad \rho(\varphi_*(b)) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p + p^{-1} - p_1 \end{pmatrix}$$

for some $s_1, s_2, s_3, p_1, p_2, p_3 \in \mathbb{C}$. Suppose $T = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ for some $m_1, m_2, m_3, m_4 \in \mathbb{C}$, and that $\text{tr } T = \text{tr}(\rho(a)T) = \text{tr}(\rho(b)T) = \text{tr}(\rho(ab)T) = 0$ holds. Then, we have

$$\begin{aligned} 0 &= \text{tr } T = m_1 + m_4, \\ 0 &= \text{tr}(\rho(a)T) = sm_1 + m_2 + s^{-1}m_4, \\ 0 &= \text{tr}(\rho(b)T) = pm_1 + um_3 + p^{-1}m_4, \\ 0 &= \text{tr}(\rho(ab)T) = spm_1 + pm_2 + usm_3 + (u + s^{-1}p^{-1})m_4. \end{aligned}$$

Note that the equations do not change if one replaces T with $-T$. From the first three equations, we have $m_4 = -m_1$, $m_2 = (s^{-1} - s)m_1$ and $m_3 = \frac{1}{u}(p^{-1} - p)m_1$, which gives

$$T = m_1 \begin{pmatrix} 1 & s^{-1} - s \\ \frac{p^{-1} - p}{u} & -1 \end{pmatrix}.$$

Substituting m_2, m_3, m_4 into the fourth equation, we have

$$(u + sp + s^{-1}p^{-1} - s^{-1}p - sp^{-1})m_1 = 0.$$

However, this implies the contradiction

$$\det T = -\frac{(u + sp + s^{-1}p^{-1} - s^{-1}p - sp^{-1})m_1}{u} = 0.$$

Hence at least one of $\text{tr } T$, $\text{tr}(\rho(a)T)$, $\text{tr}(\rho(b)T)$ and $\text{tr}(\rho(ab)T)$ is nonzero. $\square$

Proposition 19 *The fibres of $r: X^{\text{irr}}(M_\varphi) \rightarrow X_\varphi(S)$ are the orbits of ε and the branch points for r are contained in the set of reducible characters in $X_\varphi(S)$. Moreover, the ramification points are simple points of $X^{\text{irr}}(M_\varphi)$ and are precisely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$.*

Proof The action of ε with respect to the affine coordinates (x, y, z, u, v, w, q) in $\mathbb{C}^7$ is given by the the involution $(x, y, z, u, v, w, q) \rightarrow (x, y, z, -u, -v, -w, -q)$. It was shown in Lemma 18 that the preimage under r of each irreducible character in $X_\varphi(S)$ is the orbit under this involution and has precisely two elements. It follows by continuity that each fixed point of ε is a ramification point for r . Hence the branch points for r are contained in the set of reducible characters in $X_\varphi(S)$.

It was shown in Section 3.4 that there are no ramification points of r in $X^{\text{red}}(M_\varphi) \cap X^{\text{irr}}(M_\varphi)$. Hence all ramification points correspond to irreducible characters in $X^{\text{irr}}(M_\varphi)$ that restrict to reducible characters in $X_\varphi(S)$ and are fixed by ε . It was noted in Scholion 12 that the irreducible characters $X^{\text{irr}}(M_\varphi)$ that restrict to reducible characters in $X_\varphi(S)$ are precisely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$, and it is an elementary calculation to verify that they satisfy $u = v = w = q = 0$.

It is shown in [4, Proposition 5.3] that these binary dihedral characters are simple points of $X(M_\varphi)$. $\square$

Proof of Theorem 1 First note that Proposition 17 implies that $r(X(M_\varphi)) = r(X^{\text{irr}}(M_\varphi))$. The main statement of Theorem 1 is now the content of Propositions 14 and 19. The additional information about the number of branch points follows from [4], but the formula is different. We now justify our formulation. The number of binary dihedral characters in $X(M_\varphi)$ is given by [4, Propositions 5.3 and 4.5] as

$$\frac{1}{2}(|2 + \text{tr}(\varphi_*)| - \zeta_{\varphi_*}),$$

where $\zeta_{\varphi_*} \in \{1, 2, 4\}$ is the order of $\text{Hom}(\text{coker}(\varphi_* + 1_{H_1(S^1 \times S^1)}), \mathbb{Z}_2)$. Here, the once-punctured torus is identified with the complement of a point in $S^1 \times S^1$. A direct calculation shows that

$$\zeta_{\varphi_*} = 4 \iff o(\varphi_2) = 1 \iff b_1(\varphi) = 3,$$

$$\zeta_{\varphi_*} = 2 \iff o(\varphi_2) = 2 \iff b_1(\varphi) = 2,$$

$$\zeta_{\varphi_*} = 1 \iff o(\varphi_2) = 3 \iff b_1(\varphi) = 1.$$

Hence the identity $\zeta_{\varphi_*} = 2^{3-o(\varphi_2)} = 2^{b_1(\varphi)-1}$ gives

$$\frac{1}{2}(|2 + \text{tr}(\varphi_*)| - \zeta_{\varphi_*}) = \frac{1}{2}|2 + \text{tr}(\varphi_*)| - 2^{b_1(\varphi)-2}$$

as claimed in (1-1). □

The relationship between Euler characteristic and genus of curves gives the following.

Corollary 20 Suppose $X_\varphi(S)$ and $X^{\text{irr}}(M_\varphi)$ are irreducible, and that $X_\varphi(S)$ is a nonsingular affine curve. Let g_0 and g_1 denote the genera of the smooth projective models of $X^{\text{irr}}(M_\varphi)$ and $X_\varphi(S)$, respectively. Then

$$g_0 = 2g_1 - 1 + \frac{e_1 + e_\infty}{2},$$

where $e_1 = \frac{1}{2}|2 + \text{tr}(\varphi_*)| - 2^{b_1(\varphi)-2}$ is the number of branch points in $X_\varphi(S)$ and e_∞ is number of branch points at ideal points of $X_\varphi(S)$. In particular, if i_∞ is the total number of ideal points of $X_\varphi(S)$, then g_0 is bounded by

$$2g_1 - 1 + \frac{e_1}{2} \leq g_0 \leq 2g_1 - 1 + \frac{e_1}{2} + \frac{i_\infty}{2}.$$

4 Projective characters

Following the blueprint of Section 3, we now analyse the $\text{PSL}(2, \mathbb{C})$ -character varieties of once-punctured torus bundles. We make frequent use of Heusener and Porti [19] and refer the reader to original sources therein.

4.1 The restriction map

Let $\bar{X}_\varphi(S) = \{\chi_{\bar{\rho}} \in \bar{X}(S) \mid \chi_{\bar{\rho}} = \chi_{\bar{\rho}\varphi}\}$. We first show that the natural restriction map $\bar{r}: \bar{X}(M_\varphi) \rightarrow \bar{X}(S)$ again satisfies $\text{im}(\bar{r}) = \bar{X}_\varphi(S)$. Recall that we have the four-fold branched covering map $X(S) \rightarrow \bar{X}(S)$ from Section 2.5,

$$(4-1) \quad X(S) \ni (x, y, z) \mapsto (x^2, y^2, z^2, xyz) \in \bar{X}(S).$$

Let $\bar{\rho}: \pi_1(M_\varphi) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a representation. Let $T, A, B \in \mathrm{SL}(2, \mathbb{C})$ satisfying $\bar{\rho}(t) = \pm T$, $\bar{\rho}(a) = \pm A$, $\bar{\rho}(b) = \pm B$. To account for all possible choices of sign, we write $\varphi(a) = \varphi_a(a, b)$ for a fixed word given for $\varphi(a)$ in terms of the generators. Similarly for b . Then

$$(4-2) \quad T^{-1}AT = \varepsilon_a(A, B)\varphi_a(A, B),$$

$$(4-3) \quad T^{-1}BT = \varepsilon_b(A, B)\varphi_b(A, B),$$

where $\varepsilon_a(A, B), \varepsilon_b(A, B) \in \{\pm 1\}$. Then

$$T^{-1}ABT = \varepsilon_a(A, B)\varphi_a(A, B) \cdot \varepsilon_b(A, B)\varphi_b(A, B).$$

In particular, the product of the signs in the last coordinate of (4-1) equals one and hence $\mathrm{im}(\bar{r}) \subseteq \bar{X}_\varphi(S)$.

Conversely, let $\bar{\rho}: \pi_1(S) \rightarrow \pi_1(S)$ such that $\chi_{\bar{\rho}} \in \bar{X}_\varphi(S)$. Suppose $\bar{\rho}(a) = \pm A$ and $\bar{\rho}(b) = \pm B$, where $A, B \in \mathrm{SL}(2, \mathbb{C})$. Since $\chi_{\bar{\rho}} = \chi_{\bar{\rho}\varphi}$, the first three coordinates of the corresponding points in $\bar{X}(S)$ (with respect to the parametrisation in (4-1)) imply that there are $\varepsilon_a, \varepsilon_b, \varepsilon_{ab} \in \{\pm 1\}$ such that

$$\mathrm{tr} A = \varepsilon_a \mathrm{tr}(\varphi_a(A, B)), \quad \mathrm{tr} B = \varepsilon_b \mathrm{tr}(\varphi_b(A, B)), \quad \mathrm{tr} AB = \varepsilon_{ab} \mathrm{tr}(\varphi_a(A, B)\varphi_b(A, B)).$$

Now the fourth coordinate implies that $\varepsilon_a \varepsilon_b \varepsilon_{ab} = 1$ and hence $\varepsilon_{ab} = \varepsilon_a \varepsilon_b$. Hence

$$\mathrm{tr} A = \mathrm{tr}(\varepsilon_a \varphi_a(A, B)), \quad \mathrm{tr} B = \mathrm{tr}(\varepsilon_b \varphi_b(A, B)), \quad \mathrm{tr} AB = \mathrm{tr}(\varepsilon_a \varphi_a(A, B) \cdot \varepsilon_b \varphi_b(A, B)).$$

As in the proof of Lemma 11 there is $T \in \mathrm{SL}(2, \mathbb{C})$ with

$$T^{-1}AT = \varepsilon_a \varphi_a(A, B) \quad \text{and} \quad T^{-1}BT = \varepsilon_b \varphi_b(A, B).$$

This shows that $\chi_{\bar{\rho}} \in \mathrm{im}(\bar{r})$. Hence $\mathrm{im}(\bar{r}) = \bar{X}_\varphi(S)$ as claimed.

It follows as in the proof of Lemma 18 that if $\bar{\rho}: \pi_1(M_\varphi) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is irreducible, then

$$(4-4) \quad ((\mathrm{tr} \bar{\rho}(t))^2, (\mathrm{tr} \bar{\rho}(ta))^2, (\mathrm{tr} \bar{\rho}(tb))^2, (\mathrm{tr} \bar{\rho}(tab))^2) \neq (0, 0, 0, 0).$$

As in Section 3.4, it is easy to show that there are finitely many characters in $\bar{X}_\varphi(S)$ that arise from reducible representations of $\pi_1(M_\varphi)$. The proof of Proposition 17 shows that $\bar{r}: \bar{X}^{\mathrm{irr}}(M_\varphi) \rightarrow \bar{X}_\varphi(S)$ is surjective.

The proof of Corollary 3 and the statement of Proposition 14 now give the following:

Proposition 21 *Suppose M_φ is a hyperbolic once-punctured torus bundle with $b_1(\varphi) = 1$. Then we have the following commutative diagram of surjective maps whose degrees are indicated in the diagram:*

$$\begin{array}{ccccc} X^{\mathrm{irr}}(M_\varphi) & \xrightarrow[2:1]{r} & X_\varphi(S) & \subset & X(S) \cong \mathbb{C}^3 \\ q_1 \downarrow 2:1 & & q_2 \downarrow 1:1 & & q_2 \downarrow 4:1 \\ \bar{X}^{\mathrm{irr}}(M_\varphi) & \xrightarrow[1:1]{\bar{r}} & \bar{X}_\varphi(S) & \subset & \bar{X}(S) \end{array}$$

In particular, every component of $\bar{X}^{\mathrm{irr}}(M_\varphi)$ is one-dimensional.

For the bundles with $b_1(\varphi) > 1$, we have the following to consider. Let $\bar{\rho}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a representation with $\chi_{\bar{\rho}} \in \bar{X}_{\varphi}(S)$. We call the characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ *extensions* of $\chi_{\bar{\rho}}$, and similar for representations. Any two extensions of $\bar{\rho}$ to $\pi_1(M_{\varphi})$ differ by an element in the centraliser $C(\mathrm{im}(\bar{\rho}))$. It is well-known (see [19]) that if $\bar{\rho}$ is irreducible, then the centraliser $C(\mathrm{im}(\bar{\rho}))$ is either trivial, or cyclic of order two, or isomorphic to the Klein four group. This is unlike the situation for $\mathrm{SL}(2, \mathbb{C})$, where the centraliser of an irreducible representation is always central. In terms of our coordinate system for $\bar{X}(S)$ these possibilities are given in Section 4.2. We start with the following preliminary observation; the group $H \leq H^1(M_{\varphi}, \mathbb{Z}_2)$ in the statement was defined in Section 2.3 as those homomorphisms $h: \pi_1(M_{\varphi}) \rightarrow \mathbb{Z}_2$ satisfying $h(t) = 1$.

Lemma 22 *Suppose $\chi_{\rho}, \chi_{\sigma} \in X_{\varphi}(S)$ are in the same $H^1(S, \mathbb{Z}_2)$ -orbit, and that χ_{ρ} is irreducible. If χ_{ρ} and χ_{σ} are in the same H -orbit, then $q_1(r^{-1}(\chi_{\rho})) = q_1(r^{-1}(\chi_{\sigma}))$. If they are in distinct H -orbits, then $q_1(r^{-1}(\chi_{\rho})) \neq q_1(r^{-1}(\chi_{\sigma}))$. Moreover, $q_1(r^{-1}(\chi_{\rho}))$ and $q_1(r^{-1}(\chi_{\sigma}))$ are the characters of representations into $\mathrm{PSL}(2, \mathbb{C})$ that agree on $\pi_1(S)$ and whose image of t differs by a nontrivial element in the centraliser of the image of $\pi_1(S)$.*

Proof Let $\rho: \pi_1(S) \rightarrow \mathrm{SL}(2, \mathbb{C})$ be a representation with character χ_{ρ} . Then there is $h \in H^1(S, \mathbb{Z}_2)$ such that the character of $\sigma(\gamma) = h(\gamma)\rho(\gamma)$ is χ_{σ} .

Since $\chi_{\rho} \in X_{\varphi}(S)$, there is $T \in \mathrm{SL}(2, \mathbb{C})$ such that $T^{-1}\rho(\gamma)T = \rho(\varphi(\gamma))$ for all $\gamma \in \pi_1(S)$. In particular, ρ extends to a representation of $\pi_1(M_{\varphi})$ into $\mathrm{SL}(2, \mathbb{C})$ by letting $\rho(t) = T$ or $\rho(t) = -T$ and these are the only extensions according to Lemma 18. Note that these extensions are in the same $\langle \varepsilon \rangle$ -orbit.

If $h \in H$, then $h(\gamma) = h(\varphi(\gamma))$ for all $\gamma \in \pi_1(S)$. Hence

$$T^{-1}\sigma(\gamma)T = T^{-1}h(\gamma)\rho(\gamma)T = h(\gamma)T^{-1}\rho(\gamma)T = h(\gamma)\rho(\varphi(\gamma)) = h(\varphi(\gamma))\rho(\varphi(\gamma)) = \sigma(\varphi(\gamma))$$

for all $\gamma \in \pi_1(S)$. So as above, σ extends to a representation of $\pi_1(M_{\varphi})$ by letting $\sigma(t) = T$ or $\sigma(t) = -T$. Hence the extensions of ρ are in the same $H^1(M_{\varphi}, \mathbb{Z}_2)$ -orbit as the extensions of σ . This implies $q_1(r^{-1}(\chi_{\rho})) = q_1(r^{-1}(\chi_{\sigma}))$.

Suppose there is no $h \in H^1(S, \mathbb{Z}_2)$ such that the character of $\sigma(\gamma) = h(\gamma)\rho(\gamma)$ is χ_{σ} . Then $q_1(r^{-1}(\chi_{\rho})) \neq q_1(r^{-1}(\chi_{\sigma}))$. Since $q_2(\chi_{\rho}) = q_2(\chi_{\sigma})$, we have $\bar{r}(q_1(r^{-1}(\chi_{\rho}))) = q_2(\chi_{\rho}) = q_2(\chi_{\sigma}) = \bar{r}(q_1(r^{-1}(\chi_{\sigma})))$. So $q_1(r^{-1}(\chi_{\rho}))$ and $q_1(r^{-1}(\chi_{\sigma}))$ are the characters of nonconjugate representations that agree on $\pi_1(S)$. Hence the conclusion. $\square$

4.2 The origin and three coordinate axes (revisited)

Let $\bar{\rho}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be an irreducible representation. We have $C(\mathrm{im}(\bar{\rho})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if and only if $\chi_{\bar{\rho}} = (0, 0, 0, 0) \in \bar{X}(S)$. For every φ , we have $(0, 0, 0, 0) \in X_{\varphi}(S)$ and hence $(0, 0, 0, 0) \in \bar{X}_{\varphi}(S)$. Up to conjugation, we may assume

$$(4.5) \quad \bar{\rho}(a) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \kappa_1, \quad \bar{\rho}(b) = \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \kappa_2, \quad \bar{\rho}(ab) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \kappa_3.$$

These Möbius transformations represent rotations by π with respective axes $[-i, i]$, $[-1, 1]$ and $[0, \infty]$, and we have $C(\text{im}(\bar{\rho})) = \text{im}(\bar{\rho})$.

Lemma 23 *If $\bar{\rho}: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ is irreducible and $|C(\text{im}(\bar{\rho}))| = 4$, then*

$$|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 2^{b_1(\varphi)-1}.$$

Moreover, exactly one of the characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts to $X(M_\varphi)$. For each representation of $\pi_1(M_\varphi)$ with character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$, the image of the longitude is always trivial; the image of the meridian has order two or four if $b_1(\varphi) = 2$ and it has order two or is trivial if $b_1(\varphi) = 3$.

Proof Since $(0, 0, 0) \in X_\varphi(S)$ is the unique preimage of $\chi_{\bar{\rho}}$ in $X_\varphi(S)$, it follows that any two lifts to $\text{SL}(2, \mathbb{C})$ of the Klein four group are conjugate in $\text{SL}(2, \mathbb{C})$. Hence the uniqueness up to sign in Lemma 18 implies that exactly one of the characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts to $X(M_\varphi)$.

Note that the image of the longitude is the commutator of κ_1 and κ_2 and hence trivial.

If $b_1(\varphi) = 3$, then we have $\varphi(\bar{\rho}(a)) = \bar{\rho}(a)$ and $\varphi(\bar{\rho}(b)) = \bar{\rho}(b)$ and hence $\bar{\rho}(t) \in C(\text{im}(\bar{\rho}))$. This implies that $\bar{\rho}$ extends to $\pi_1(M_\varphi)$ with $\bar{\rho}(t)$ any element in the centraliser. The values taken in (4-4) are $(4, 0, 0, 0)$, $(0, 4, 0, 0)$, $(0, 0, 4, 0)$, $(0, 0, 0, 4)$. Hence, no two of these four extensions are conjugate, but all have image equal to $\text{im}(\bar{\rho})$. Example 16 shows that the character that lifts depends on φ . The four extensions are characterised by $\bar{\rho}(t) = 1$, $\bar{\rho}(t) = \bar{\rho}(a)$, $\bar{\rho}(t) = \bar{\rho}(b)$, and $\bar{\rho}(t) = \bar{\rho}(ab)$, respectively. This proves the claim about the order. We call these representations the *four extensions of the Klein four group* and denote them respectively by $\bar{\rho}_0, \bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3$ and their characters by $\bar{\chi}_0, \bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3$. Note that there is a natural action of the Klein four group on these characters defined by $\kappa_i \cdot \bar{\chi}_0 = \bar{\chi}_i$.

If $b_1(\varphi) = 2$, then φ interchanges two of $\bar{\rho}(a), \bar{\rho}(b), \bar{\rho}(ab)$ and fixes the third. Without loss of generality, assume $\varphi(\bar{\rho}(a)) = \bar{\rho}(b)$. Then a direct calculation shows that $\bar{\rho}$ extends to a representation of $\pi_1(M_\varphi)$ by letting

$$(4-6) \quad \bar{\rho}(t) = \pm \begin{pmatrix} 0 & \sqrt{i} \\ i\sqrt{i} & 0 \end{pmatrix}, \quad \bar{\rho}(a) = \kappa_1, \quad \bar{\rho}(b) = \kappa_2.$$

The remaining extensions are obtained from this by twisting by the elements κ_i . The above extension is conjugate to the twisted representation $t \mapsto \kappa_3 \bar{\rho}(t)$ via κ_1 , and their values taken in (4-4) are $(0, 2, 2, 0)$. Similarly, the two representations obtained by twisting $\bar{\rho}$ by κ_1 or κ_2 are conjugate via κ_1 , and their values taken in (4-4) are $(2, 0, 0, 2)$; for instance

$$t \mapsto \kappa_1 \bar{\rho}(t) = \pm \begin{pmatrix} i\sqrt{i} & 0 \\ 0 & -\sqrt{i} \end{pmatrix}.$$

Hence we obtain two nonconjugate extensions and the order of the meridian is as claimed.

If $b_1(\varphi) = 1$, then any extension lifts to $\text{SL}(2, \mathbb{C})$ and hence any two extensions are conjugate. $\square$

Let $\bar{\rho}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be an irreducible representation. We have $C(\mathrm{im}(\bar{\rho})) \cong \mathbb{Z}_2$ if and only if $0 \neq \chi_{\bar{\rho}} \in \bar{L}_i$, where

$$\bar{L}_1 = \{(x^2, 0, 0, 0) \mid x \in \mathbb{C}\} \subset \bar{X}(S),$$

$$\bar{L}_2 = \{(0, y^2, 0, 0) \mid y \in \mathbb{C}\} \subset \bar{X}(S),$$

$$\bar{L}_3 = \{(0, 0, z^2, 0) \mid z \in \mathbb{C}\} \subset \bar{X}(S)$$

are the images of the coordinate axes in $X(S)$. The points $(4, 0, 0, 0)$, $(0, 4, 0, 0)$ and $(0, 0, 4, 0)$ are the only reducible characters on these lines. In $\bar{R}(S)$ we may choose for each $\bar{L}_i$ a curve of representations that have constant centraliser, and such that the curves meet in the representation given in (4-5).

Write $(4p^2, 0, 0, 0) = (x, 0, 0, 0) \in \bar{L}_1$ and let $q \in \mathbb{C}$ such that $p^2 + q^2 = 1$. We may then choose

$$(4-7) \quad \bar{\rho}_1(a) = \begin{pmatrix} q & -p \\ p & q \end{pmatrix} \kappa_1, \quad \bar{\rho}_1(b) = \kappa_2, \quad \text{and hence} \quad C(\mathrm{im}(\bar{\rho}_1)) = \langle \kappa_1 \rangle.$$

Write $(0, -4p^2, 0, 0) = (0, y, 0, 0) \in \bar{L}_2$ and let $q \in \mathbb{C}$ such that $p^2 - q^2 = 1$. We may then choose

$$(4-8) \quad \bar{\rho}_2(a) = \kappa_1, \quad \bar{\rho}_2(b) = \kappa_2 \begin{pmatrix} q & p \\ p & q \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar{\rho}_2)) = \langle \kappa_2 \rangle.$$

Let $(0, 0, (p + p^{-1})^2, 0) = (0, 0, z, 0) \in \bar{L}_3$. We may choose

$$(4-9) \quad \bar{\rho}_3(a) = \kappa_1, \quad \bar{\rho}_3(b) = \kappa_2 \begin{pmatrix} ip & 0 \\ 0 & -ip^{-1} \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar{\rho}_3)) = \langle \kappa_3 \rangle.$$

Lemma 24 Suppose $\bar{\rho}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is irreducible. If $\chi_{\bar{\rho}} \in \mathrm{im}(\bar{r})$ and $|C(\mathrm{im}(\bar{\rho}))| = 2$, then $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 2$ and $b_1(\varphi) > 1$. Moreover, either no character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts to $X(M_\varphi)$; or exactly one lifts and $b_1(\varphi) = 3$; or two of them lift and $b_1(\varphi) = 2$.

Proof We have $0 \neq \chi_{\bar{\rho}} \in \bar{L}_i$. Without loss of generality, we may assume $\bar{L}_i = \bar{L}_3$, and $\chi_{\bar{\rho}} = (0, 0, z_0^2, 0)$. Hence

$$(\mathrm{tr} \bar{\rho}(a))^2 = (\mathrm{tr} \bar{\rho}(b))^2 = 0 \neq (\mathrm{tr} \bar{\rho}(ab))^2$$

In L_3 , this character has two pre-images, $\chi_+ = (0, 0, z_0)$ and $\chi_- = (0, 0, -z_0)$. Recall the action of $\bar{\varphi}$ on the coordinate axes. If $b_1(\varphi) = 1$, then the three axes are in the same orbit under $\bar{\varphi}$. Hence $z_0 \neq 0$, implies $b_1(\varphi) > 1$.

We next show that $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 2$. Up to conjugation, we have

$$\bar{\rho}(a) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\rho}(b) = \pm \begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar{\rho})) = \left\langle \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$$

where $z_0 = -(p + p^{-1}) \notin \{-2, 0, 2\}$ since $\bar{\rho}$ is reducible if and only if $z_0 = \pm 2$, and the centraliser has order four if and only if $z_0 = 0$. These two cases correspond to the characters $(x^2, y^2, z^2, xyz) = (0, 0, 4, 0)$ and $(0, 0, 0, 0)$, respectively, with respect to (2-14). Assume that $\bar{\rho}$ is the restriction of a representation of M_φ with $\bar{\rho}(t) = \pm T$ for some

$$T = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

We know that $|\bar{r}^{-1}(\chi_{\bar{\rho}})| \leq |C(\text{im}(\bar{\rho}))| = 2$. Suppose that $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 1$. Then the two possible extensions of the representation of S to M_φ defined by $\bar{\rho}(t) = \pm T$ or $\bar{\rho}(t) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} T$ are conjugate. In particular, their values taken in (4-4) are identical. This gives a system of 4 equations. A direct computation shows that $z_0 \notin \{-2, 0, 2\}$ implies that $m_1 = m_2 = m_3 = m_4 = 0$. This is a contradiction. Hence $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 2$.

If $b_1(\varphi) = 2$, then $\bar{\varphi}$ stabilises one axis and permutes the other two axes. Since $z_0 \neq 0$, it preserves L_3 . We therefore have either $\bar{\varphi}(0, 0, z) = (0, 0, z)$ for each $z \in \mathbb{C}$ or $\bar{\varphi}(0, 0, z) = (0, 0, -z)$ for each $z \in \mathbb{C}$. In the second case, $\chi_{\bar{\rho}}$ does not have a preimage in $X_\varphi(S)$ and hence no character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts to $X(M_\varphi)$. In the first case, the points $(0, 0, z_0)$ and $(0, 0, -z_0)$ are both in $X_\varphi(S)$. Since $\bar{\varphi}$ stabilises L_3 and permutes L_1 and L_2 , the group H is generated by $a \mapsto -1$, $b \mapsto -1$ and hence the points $(0, 0, \pm z_0)$ are fixed by H and thus in distinct H -orbits. Hence their extensions map to the two distinct characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$.

If $b_1(\varphi) = 3$, we also have either $\bar{\varphi}(0, 0, z) = (0, 0, z)$ for each $z \in \mathbb{C}$ or $\bar{\varphi}(0, 0, z) = (0, 0, -z)$ for each $z \in \mathbb{C}$. As above, in the second case, neither character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts. In the first case, the points $(0, 0, z_0)$ and $(0, 0, -z_0)$ are in $X_\varphi(S)$ and in the same H -orbit since $H = H^1(\pi_1(S), \mathbb{Z}_2)$. According to Lemma 22, the extensions of these characters are in the same $H^1(\pi_1(M), \mathbb{Z}_2)$ -orbit and hence have the same image in $\bar{r}^{-1}(\chi_{\bar{\rho}})$. So exactly one of the characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts and the other does not lift. $\square$

The proof of Lemma 24 supplied the missing details for the following:

Corollary 25 *Suppose M_φ is a hyperbolic once-punctured torus bundle. We have the following commutative diagram of maps whose degrees are indicated in the diagram:*

$$\begin{array}{ccccc} X^{\text{irr}}(M_\varphi) & \xrightarrow[2:1]{r} & X_\varphi(S) & \subset & X(S) & \cong & \mathbb{C}^3 \\ q_1 \downarrow & & q_2 \downarrow & & q_2 \downarrow & & \\ 2^{b_1(\varphi)}:1 & & 2^{b_1(\varphi)-1}:1 & & 4:1 & & \\ \bar{X}^{\text{irr}}(M_\varphi) & \xrightarrow{\bar{r}} & \bar{X}_\varphi(S) & \subset & \bar{X}(S) & & \end{array}$$

Here, each of the maps r , q_1 , q_2 is a branched covering map onto its image. The map q_1 is the quotient map associated with the action of $H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H$, and the map q_2 is the quotient map associated with the action of $H^1(S, \mathbb{Z}_2)$ on $X(S)$. The map r is the quotient map associated with the action of $\langle \varepsilon \rangle$. The restriction of q_2 to $X_\varphi(S)$ is the quotient map associated with the action of H unless $b_1(\varphi) = 2$ and one of the coordinate axes is contained in $X_\varphi(S)$.

Proof If $b_1(\varphi) = 1$ this is Proposition 21 and there is nothing to prove. If $b_1(\varphi) = 3$, we have $H = H^1(S, \mathbb{Z}_2)$ and there also is nothing to prove. Hence suppose $b_1(\varphi) = 2$ and that $\bar{L}_i$ is the unique coordinate axis in $\bar{X}_\varphi(S)$. It follows from Lemma 22 and the fact that all characters of representations with nontrivial centraliser are contained on the coordinate axes that if L_i is not contained in $X_\varphi(S)$, then q_2 is the quotient map of the H -action. If $L_i \subset X_\varphi(S)$, then for the same reasons, q_2 restricted to $\bar{X}_\varphi(S) \setminus \bar{L}_i$ is the quotient map of the H -action. However, restricted to L_i , it is the quotient map associated with the sign change on L_i , this has degree two. $\square$

Proposition 26 *Let M_φ be a hyperbolic once-punctured torus bundle. Let $\bar{X}$ be a Zariski component of $\bar{X}(M_\varphi)$ containing the character of an irreducible representation. If $\bar{r}: \bar{X} \rightarrow \bar{X}_\varphi(S)$ does not have degree one, then the degree is two and $b_1(\varphi) = 2$ and $\bar{X}$ is the preimage of one of the lines $\bar{L}_i \subset \bar{X}_\varphi(S)$.*

Proof The above classification of irreducible characters with nontrivial centraliser shows that

$$\bar{r}: \bar{X} \rightarrow \bar{X}_\varphi(S)$$

has degree one unless $\bar{L}_i \subset \bar{X}_\varphi(S)$ and $\bar{X} = \bar{r}^{-1}(\bar{L}_i)$ for some $i \in \{1, 2, 3\}$. In this case, the degree is two.

The conclusion for $b_1(\varphi) = 1$ now follows from the description of the action on the lines in the proof of Lemma 13, which gives $\bar{L}_i \cap \bar{X}_\varphi(S) = \{(0, 0, 0, 0)\}$ for each i . Alternatively, this follows from the fact that each representation lifts and from the uniqueness up to sign in Lemma 18.

Suppose $b_1(\varphi) = 3$ and that $\bar{X} = \bar{r}^{-1}(\bar{L}_i)$. It follows from Lemma 24 that $\bar{X} \rightarrow \bar{L}_i$ is a 2-fold (possibly branched) cover. Since $\bar{L}_i \cong \mathbb{C}$ and hence is simply connected it follows that the map $\bar{X} \rightarrow \bar{L}_i$ must have a branch point. Lemma 24 shows that there are only two potential branch points: the Klein four group or the reducible character on $\bar{L}_i$. It follows from the explicit description of the four extensions in the proof of Lemma 23 and the description of the covering map in the proof of Lemma 24 that the extensions of the Klein four group contained in $\bar{r}^{-1}(\bar{L}_i)$ are not ramification points. Hence it must be the reducible character on $\bar{L}_i$. Without loss of generality suppose $i = 3$ and use the parametrisation from (4-9):

$$(4-10) \quad \bar{\rho}(a) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\rho}(b) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and let} \quad C = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since $\bar{\rho}(a) = \bar{\rho}(b)$ and both have order two, the hypothesis that $b_1(\varphi) = 3$ implies that $\bar{\rho}(a)$ and $\bar{\rho}(t)$ commute. Hence there are two possibilities for $\bar{\rho}(t) = T_k$:

$$(4-11) \quad T_1 = \pm \begin{pmatrix} m_1 & m_2 \\ -m_2 & m_1 \end{pmatrix} \quad \text{or} \quad T_2 = \pm \begin{pmatrix} m_4 & m_3 \\ m_3 & -m_4 \end{pmatrix},$$

where $m_1^2 + m_2^2 = 1$ or $m_3^2 + m_4^2 = -1$.

The representation defined by $\bar{\rho}_1(t) = T_1$, $\bar{\rho}_1(a) = \bar{\rho}_1(b) = \kappa_1$ satisfies $\text{tr}[\bar{\rho}_1(t), \bar{\rho}_1(a)] = 2$ and hence is reducible. Indeed, this gives a 1-dimensional family of reducible representations with the quadruple in (4-4) equal to

$$(4m_1^2, 4(1 - m_1^2), 4(1 - m_1^2), 4m_1^2)$$

The representation defined by $\bar{\rho}_2(t) = T_2$, $\bar{\rho}_2(a) = \bar{\rho}_2(b) = \kappa_1$ satisfies $\text{tr}[\bar{\rho}_2(t), \bar{\rho}_2(a)] = -2$ and hence is irreducible. For any choice of $m_3^2 + m_4^2 = -1$, the quadruple in (4-4) equals

$$(0, 0, 0, 0).$$

Indeed, this has image isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and lifts to a binary dihedral representation into $\text{SL}(2, \mathbb{C})$.

The representations $\bar{\rho}_1$ and $\bar{\rho}_2$ are related by twisting by C (and possibly another element of the centraliser of $\bar{\rho}(a)$ that fixes the fixed points of $\bar{\rho}(a)$). In particular, the reducible character on $\bar{L}_i$ has exactly two preimages in $\bar{X} = \bar{r}^{-1}(\bar{L}_i)$. This shows that there are no ramification points. $\square$

Examples for the case in Proposition 26 where the degree is two are given in Section 5.4.1. In these examples, one extension of the Klein four character is on this line, and the other extension is on the canonical component.

In the case of maximal rank, we can give a complete picture of the Zariski components that are pre-images of the lines:

Proposition 27 *If $H_1(M_\varphi, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then $\bar{X}_\varphi(S)$ contains all three of $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$. Moreover, each line $\bar{L}_i$ has two pre-images $\bar{L}'_i$ and $\bar{L}''_i$ in $\bar{X}(M_\varphi)$ that are pairwise disjoint. There is a permutation $\sigma \in \text{Sym}(4)$ such that the labels can be chosen such that $\chi_{\sigma(i)} \in \bar{L}''_i$ for $i \in \{1, 2, 3\}$, and $\chi_{\sigma(0)} = \bar{L}'_1 \cap \bar{L}'_2 \cap \bar{L}'_3$ is the only pairwise intersection point of the six preimages and is a character that lifts to $\text{SL}(2, \mathbb{C})$. Up to passing to the degree two cover M_{φ^2} , we may assume that σ is the identity.*

Proof The fact that $\bar{X}_\varphi(S)$ contains all three of $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$ follows from the observation in the proof of Lemma 13, noting that the sign is now irrelevant, and the fact that irreducible representations of the fibre always extend. We know from Lemma 13 that $X_\varphi(S)$ either contains all three lines L_1 , L_2 , and L_3 or it contains exactly one of them.

First assume that $X_\varphi(S)$ contains all three lines. Then $X(M_\varphi)$ contains a character that extends the quaternionic group and is the common intersection of the pre-images L'_1 , L'_2 , and L'_3 in $X(M_\varphi)$ of the three lines. Under the map to the $\text{PSL}(2, \mathbb{C})$ -character variety, these map to Zariski components $\bar{L}'_1$, $\bar{L}'_2$, $\bar{L}'_3$. Hence their common intersection is one of the extensions χ_j of the Klein four group.

Consider the line $\bar{L}_3$ and the parametrisation given in (4-9) that is consistent with (4-5). Each point in $\bar{L}_3$ has precisely two preimages in $\bar{X}(M_\varphi)$, and they are related by the action of κ_3 . It follows that the preimage $\bar{r}^{-1}(\bar{L}_3)$ either has one or two Zariski components. Since the lifting obstruction is constant on topological components of the character variety and $\bar{r}^{-1}(\bar{L}_3)$ contains two distinct extensions of the Klein four group, namely χ_j and $\kappa_3 \cdot \chi_j$, it follows that $\bar{r}^{-1}(\bar{L}_3)$ is a disjoint union of two components $\bar{L}'_3$ and $\bar{L}''_3$ and we have $\chi_j \in \bar{L}'_3$ and $\kappa_3 \cdot \chi_j \in \bar{L}''_3$.

The same argument applies to the lines $\bar{L}_1$ and $\bar{L}_2$ where the respective action with respect to the appropriate parametrisations consistent with (4-5) is given by κ_1 and κ_2 . It now follows from the action of the Klein four group on the characters $\chi_0, \chi_1, \chi_2, \chi_3$ that no two of $\bar{L}''_i$ and $\bar{L}''_k$ have one of these characters in common. This implies that they are pairwise disjoint.

If $\chi_j \neq \chi_0$, then the meridian is mapped to an element of order two under the character corresponding to the triple intersection. Hence under the map $\bar{X}(M_\varphi) \rightarrow \bar{X}(M_{\varphi^2})$, the character $\chi_j \in \bar{X}(M_\varphi)$ is mapped to $\chi_0 \in \bar{X}(M_{\varphi^2})$, and each preimage of one of the lines $\bar{L}_i$ in $\bar{X}(M_\varphi)$ is mapped to the corresponding connected component of the preimage of $\bar{L}_i$ in $\bar{X}(M_{\varphi^2})$ that passes through $\chi_0 \in \bar{X}(M_{\varphi^2})$. This completes the proof of the lemma in the case where $X_\varphi(S)$ contains all three lines since $\kappa_i \cdot \chi_0 = \chi_i$.

Now suppose that $X_\varphi(S)$ only contains one of the three lines L_1, L_2 , and L_3 . Without loss of generality, we may assume this is L_1 . Then in $\bar{X}(M_\varphi)$ there is a component $\bar{L}_1''$ that lifts to L_1 . The same argument as above shows that $\bar{r}^{-1}(\bar{L}_1)$ has two disjoint components. Suppose $\chi_j \in \bar{L}_1'$. Then $\kappa_1 \cdot \chi_j \in \bar{L}_1''$. Since the lifting obstruction is constant on topological components, $\bar{r}^{-1}(\bar{L}_2)$ and $\bar{r}^{-1}(\bar{L}_3)$ do not pass through $\kappa_1 \cdot \chi_j$. Then $\bar{r}^{-1}(\bar{L}_2)$ also does not pass through $\kappa_2 \kappa_1 \cdot \chi_j = \kappa_3 \cdot \chi_j$, and $\bar{r}^{-1}(\bar{L}_3)$ does not pass through $\kappa_3 \kappa_1 \cdot \chi_j = \kappa_2 \cdot \chi_j$. Hence $\bar{r}^{-1}(\bar{L}_2)$ contains χ_j and $\kappa_2 \cdot \chi_j$; and $\bar{r}^{-1}(\bar{L}_3)$ contains χ_j and $\kappa_3 \cdot \chi_j$. This proves that each extensions of the Klein four group is contained in at least one of the pre-images of the lines $\bar{L}_i$. The proof is now completed by the observation in the proof of Proposition 26 that each $\bar{L}_i$ has preimage in $\bar{X}(M_\varphi)$ consisting of two disjoint Zariski components, each containing one of the two extensions of the Klein four group contained in $\bar{r}^{-1}(\bar{L}_i)$. The last statement now follows as in the previous paragraph. $\square$

Remark 28 In the examples we computed explicitly, it is always the case that $\chi_0 = \bar{L}_1' \cap \bar{L}_2' \cap \bar{L}_3'$.

Example 29 We continue the example of $\varphi = \alpha^2 \beta^{-2}$ discussed in Example 16. As $H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, $\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \in \bar{X}_\varphi(S)$. A direct calculation shows that each line in $\bar{X}_\varphi(S)$ is the preimage of two Zariski components in $\bar{X}(M_\varphi)$ under the map $\bar{r}$. Consider the preimages $\bar{L}_3', \bar{L}_3''$ of $\bar{L}_3$. Suppose that the component $\bar{L}_3' \in \bar{X}(M_\varphi)$ described in Example 16 is the preimage of the component $\bar{L}_3''$ in $\bar{X}(M_\varphi)$. The values in (4-4) on $\bar{L}_3''$ are $(0, 0, 0, 4 - z^2)$. Twisting by the generator of $C(\text{im}(\bar{\rho}))$, we obtain another component $\bar{L}_3' \subset \bar{X}(M_\varphi)$ where the corresponding values are $(4, 0, 0, z^2)$. We may choose labels for the preimages of the other lines such that the values taken in (4-4) for $\bar{L}_1''$ are $(x^2, (x^2 - 2)^2, 0, 0)$, for $\bar{L}_1'$ are $(4 - x^2, x^2(4 - x^2), 0, 0)$, for $\bar{L}_2''$ are $(y^2, 0, (y^2 - 2)^2, 0)$, and for $\bar{L}_2'$ are $(4 - y^2, 0, y^2(4 - y^2), 0)$. Note that $\bar{L}_1', \bar{L}_2'$ and $\bar{L}_3'$ intersect at the character with trivial peripheral holonomy, and each of the other 3 characters extending the Klein four group is contained in precisely one of $\bar{L}_1'', \bar{L}_2''$ and $\bar{L}_3''$.

4.3 Dimensions and lifting

As an application of the discussion in the previous section, we have the following:

Proposition 30 Let M_φ be a hyperbolic once-punctured torus bundle. Then every Zariski component of $\bar{X}(M_\varphi)$ is at most one-dimensional. Moreover, if $\{\chi_{\bar{\rho}}\} \subset \bar{X}(M_\varphi)$ is a zero-dimensional component, then this character does not lift to $\text{SL}(2, \mathbb{C})$, $b_1(\varphi) \neq 1$, $\bar{\rho}(\text{im}(\pi_1(T) \rightarrow \pi_1(M_\varphi))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\bar{\rho}$ is irreducible with trivial centraliser. Moreover, $\chi_{\bar{\rho}}$ maps to a character on a one-dimensional component in $\bar{X}(M_{\varphi^2})$.

Proof The same arguments as in the proof of Proposition 14 show that every component containing only reducible characters is one-dimensional and of genus zero, and that the dimension of each Zariski component of $\bar{X}(M_\varphi)$ is at most one since M_φ does not contain a closed essential surface. Suppose X is a Zariski component containing the character of an irreducible representation $\bar{\rho}$. Proposition 8 implies that the dimension of X is at least one unless $\bar{\rho}(\text{im}(\pi_1(T) \rightarrow \pi_1(M_\varphi)))$ is trivial or isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If $\bar{\rho}(\text{im}(\pi_1(T) \rightarrow \pi_1(M_\varphi)))$ is trivial, then $\text{tr } \bar{\rho}(a^{-1}b^{-1}ab) = \pm 2$ since $a^{-1}b^{-1}ab$ is the longitude. If the trace equals $+2$, then $\bar{\rho}(a)$ and $\bar{\rho}(b)$ have a common fixed point on $P^1(\mathbb{C})$. But since $\rho(t) = \pm I$ this implies that the representation is reducible. Hence $\text{tr } \bar{\rho}(a^{-1}b^{-1}ab) = -2$. In this case, $\bar{\rho}(\langle a, b \rangle) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows that $H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, and that $\chi_{\bar{\rho}} = \chi_0$ is the extension of the Klein four group with trivial peripheral holonomy. It now follows from Proposition 27 that this character is contained on a one-dimensional component.

Hence suppose that $\bar{\rho}(\text{im}(\pi_1(T) \rightarrow \pi_1(M_\varphi))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In this case, $\text{tr } \bar{\rho}(a^{-1}b^{-1}ab) = 0$ and therefore the representation restricted to $\pi_1(S)$ is irreducible and has trivial centraliser. Since the Klein four group lifts to the quaternionic group, this representation does not lift to $\text{SL}(2, \mathbb{C})$. In particular, $b_1(\varphi) \neq 1$. Now under the map to $\bar{X}(M_{\varphi^2})$, $\chi_{\bar{\rho}}$ maps to a character with the property that the peripheral subgroup has image isomorphic with $\mathbb{Z}_2$. Hence Proposition 8 implies that the image lies on a one-dimensional component of $\bar{X}(M_{\varphi^2})$. $\square$

Remark 31 We do not have examples of once-punctured torus bundles with zero-dimensional components in $\bar{X}(M_\varphi)$. The purported examples given in [2, Theorem 7.6] contradict Proposition 8 and a simple description of one-dimensional components containing them is given in Example 10.

Heusener and Porti [19, Section 4.2] give examples of hyperbolic once-punctured torus bundles that have arbitrarily many one-dimensional Zariski components in $\bar{X}(M_\varphi)$ that do not lift to $X(M_\varphi)$. All of these components are of genus zero. We remark that all these examples satisfy $b_1(\varphi) = 3$. We give similar examples with $b_1(\varphi) = 2$ in Section 5.4.1, and it was already remarked that if $b_1(\varphi) = 1$, then every representation lifts.

Components containing only reducible characters have an analogous characterisation as given in Section 3.4. Moreover, the proof of Proposition 17 gives the analogous result for the $\text{PSL}(2, \mathbb{C})$ -character variety. We state this here for completeness:

Proposition 32 *For every Zariski component X of $\bar{X}^{\text{red}}(M_\varphi)$, the restriction $\bar{r}: X \rightarrow \bar{X}_\varphi(S)$ is constant and $X \cap \bar{X}^{\text{irr}}(M_\varphi) \neq \emptyset$. Each character $\bar{\chi}$ in the intersection is contained on exactly one curve C in $\bar{X}^{\text{irr}}(M_\varphi)$, a smooth point of X and C and the intersection at $\bar{\chi}$ is transverse.*

4.4 An algebraic subset

For computations, it is useful to identify $\bar{X}_\varphi(S)$ with the image of a suitable algebraic subset of $X(S)$ under the four-fold branched covering map $X(S) \rightarrow \bar{X}(S)$. Let

$$(4-12) \quad X_\varphi^1(S) = \{(x, y, z) \in \mathbb{C}^3 \mid (-x, -y, -z) = \bar{\varphi}(x, y, z)\},$$

$$(4-13) \quad X_\varphi^2(S) = \{(x, y, z) \in \mathbb{C}^3 \mid (-x, y, -z) = \bar{\varphi}(x, y, z)\},$$

$$(4-14) \quad X_\varphi^3(S) = \{(x, y, z) \in \mathbb{C}^3 \mid (-x, -y, z) = \bar{\varphi}(x, y, z)\}.$$

Note that the intersection of any two of the sets $X_\varphi(S)$, $X_\varphi^1(S)$, $X_\varphi^2(S)$, $X_\varphi^3(S)$ is contained in a coordinate axis. Recall that the natural map $q_2: X(S) \rightarrow \bar{X}(S)$ is the quotient map of the $H^1(S, \mathbb{Z}_2)$ action. The definitions imply that

$$q_2^{-1}(\bar{X}_\varphi(S)) = X_\varphi(S) \cup X_\varphi^1(S) \cup X_\varphi^2(S) \cup X_\varphi^3(S).$$

For each $(x, y, z) \in q_2^{-1}(\bar{X}_\varphi(S))$, there are $A, B \in \mathrm{SL}(2, \mathbb{C})$ with $\mathrm{tr} A = x$, $\mathrm{tr} B = y$ and $\mathrm{tr} AB = z$ and satisfying (4-2) and (4-3), where $\varepsilon_a, \varepsilon_b \in \{\pm 1\}$ are determined by

$$(\varepsilon_a x, \varepsilon_b y, \varepsilon_a \varepsilon_b z) = \bar{\varphi}(x, y, z).$$

The above sets therefore give us a simple way to compute the representations of $\pi_1(M_\varphi)$ into $\mathrm{PSL}(2, \mathbb{C})$. We now determine redundancies that arise from the action of the elements of $H^1(S, \mathbb{Z}_2)$ that are not in the group $H \leq H^1(M_\varphi, \mathbb{Z}_2)$ that was defined in Section 2.3 as those homomorphisms $h: \pi_1(M_\varphi) \rightarrow \mathbb{Z}_2$ satisfying $h(t) = 1$.

View $h \in H^1(S, \mathbb{Z}_2)$ as a homomorphism $h: \pi_1(S) \rightarrow \{\pm 1\}$. Then $h \in H$ if and only if $h(a) = h(\varphi(a))$ and $h(b) = h(\varphi(b))$. Let $\bar{\rho}: \pi_1(M_\varphi) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ and use the set up from (4-2) and (4-3). Then $\bar{\rho}$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if there is $h \in H^1(S, \mathbb{Z}_2)$ with

$$h(a) = h(\varphi(a))\varepsilon_A, \quad h(b) = h(\varphi(b))\varepsilon_B.$$

Note that the signs ε_A and ε_B are uniquely determined by $\bar{\rho}$ if $H = H^1(S, \mathbb{Z}_2)$ and otherwise they depend on the choice of the matrices A and B . The action of $h \in H^1(S, \mathbb{Z}_2)$ is given by

$$h \cdot (x, y, z) \mapsto (h(a)x, h(b)y, h(ab)z).$$

This induces a permutation of the sets $X_\varphi(S)$, $X_\varphi^1(S)$, $X_\varphi^2(S)$, and $X_\varphi^3(S)$. The permutation of the sets is determined by the action on the defining equations,

$$\bar{\varphi}(x, y, z) = (h(a)h(\varphi(a))\varepsilon_A x, h(b)h(\varphi(b))\varepsilon_B y, h(ab)h(\varphi(ab))\varepsilon_A \varepsilon_B z).$$

Each set is stabilised by $H \leq H^1(S, \mathbb{Z}_2)$ and the complementary elements permute the sets $X_\varphi(S)$, $X_\varphi^1(S)$, $X_\varphi^2(S)$, $X_\varphi^3(S)$.

If $b_1(\varphi) = 3$, then $H = H^1(S, \mathbb{Z}_2)$ and hence each of the sets is fixed under this action. In particular, $\bar{\rho}$ does not lift if at least one of ε_A and ε_B equals -1 . Define

$$X_\varphi^\perp(S) = X_\varphi^1(S) \cup X_\varphi^2(S) \cup X_\varphi^3(S).$$

If $b_1(\varphi) = 2$, then the action of $H^1(S, \mathbb{Z}_2)$ has two orbits, each containing two sets. Let

$$X_\varphi^\perp(S) = X_\varphi^i(S),$$

where $X_\varphi^i(S)$ is one of the sets not in the orbit of $X_\varphi(S)$.

If $b_1(\varphi) = 1$, then there is just one orbit and we define $X_\varphi^\perp(S) = \emptyset$. This observation can be viewed as an elementary proof of the fact that every representation into $\mathrm{PSL}(2, \mathbb{C})$ lifts to $\mathrm{SL}(2, \mathbb{C})$ in this case.

These definitions imply that the map

$$X_\varphi(S) \cup X_\varphi^\perp(S) \rightarrow \bar{X}_\varphi(S)$$

is surjective, and corresponds to the quotient map of the action of $H^1(S, \mathbb{Z}_2)$ twisted via φ .

5 Examples

This section provides details about the three infinite families of once-punctured torus bundles mentioned in the introduction, as well as the beginnings of a census. We begin by stating some general results that will be used repeatedly. The main ingredients in the proofs are properties of a sequence of Fibonacci polynomials, and determining irreducibility and genus of a plane algebraic curve from the Newton polygon of a defining polynomial.

5.1 The Fibonacci polynomials

This section collects some facts about a family of recursive polynomials, which are used throughout the computation in [2]. The Fibonacci polynomials will be used to compute $X_\varphi(S)$ for our examples.

Definition 33 For every integer n , the n -th Fibonacci polynomial $f_n(u)$ is defined by the recursive relation

$$f_n(u) = uf_{n-1}(u) - f_{n-2}(u),$$

where $f_0(u) = 0$ and $f_1(u) = 1$.

A number of useful properties of f_n according to [2] are listed below.

Lemma 34 (1) If $u = s + s^{-1}$, then

$$f_n(u) = \begin{cases} \frac{s^n - s^{-n}}{s - s^{-1}} & \text{if } u \neq \pm 2, \\ n & \text{if } u = 2, \\ (-1)^{n+1}n & \text{if } u = -2. \end{cases}$$

(2) If $n \neq 0$, the degree of $f_n(u)$ is $|n| - 1$.

(3) $f_n(u)$ is divisible by u if and only if n is even. If $n \neq 0$, $f_n(u)$ is not divisible by u^2 .

(4) For any integer n , $f_n(u)$, $f_{n+1}(u) - f_n(u)$, $f_{n+1}(u) + f_n(u)$ and $f_{n+2}(u) - f_n(u)$ are separable except for $f_0(u) = 0$.

(5) $f_{n+2}(u) - f_{n+1}(u) = 0$ and $f_{n+1}(u) - f_n(u) = 0$ do not have a common root.

Proof Part (1) follows from the defining relation. Parts (2), (3) and (4) follow from [2, Lemmas 4.3, 4.4 and 4.11]. A direct calculation using (1) results in (5). $\square$

5.2 Genus and the Newton polygon

A classical link between the Newton polygon and the genus of an irreducible algebraic curve is known as Baker's formula:

Theorem 35 (Baker [1]) *Suppose that $F(x, y) = 0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. The genus of X is at most the number of lattice points in the interior of the Newton polygon of F .*

Khovanskii [21] showed that one generically has equality instead of an upper bound. We use a version of this result as implied by Beelen and Pellikaan [3]. A polynomial $F(x, y) \in \mathbb{C}[x, y]$ is said to be *nondegenerate* with respect to its Newton polygon if for every edge γ of its Newton Polygon with the corresponding polynomial F_γ , the ideal generated by F_γ , $x \frac{\partial F_\gamma}{\partial x}$ and $y \frac{\partial F_\gamma}{\partial y}$ has no zero in $(\mathbb{C}/\{0\})^2$.

Corollary 36 (Beelen–Pellikaan) *Suppose that $F(x, y) = 0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. If F is nondegenerate with respect to its Newton polygon and it is smooth at every point $(x, y) \in X$ where $xy \neq 0$, then the genus of its nonsingular model is equal to the number of lattice points in the interior of its Newton polygon.*

Proof We show that this is implied by [3, Theorem 4.2]. The only statement that requires proof is that the smoothness hypothesis implies that the singular points of the homogeneous curve with equation $F^*(x, y, z) = 0$ are among $(0 : 0 : 1)$, $(0 : 1 : 0)$ and $(1 : 0 : 0)$, where F^* denotes the homogenisation of F . If S is a singular point other than the three points, it is not in the line of infinity and corresponds to a singular point $s = (x, y) \in X$ such that $xy \neq 0$. This contradicts the smoothness condition. $\square$

We will also appeal to the following result [5, Lemma 5.1]. The authors thank Michael Joswig for pointing them to this reference.

Lemma 37 (Castrick–Voight [5]) *Suppose that $F(x, y) = 0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. Assume that F is nondegenerate with respect to its Newton polygon, and that there are at least two lattice points in the interior of the Newton polygon of F . Then X is hyperelliptic if and only if the interior lattice points of the Newton polygon are collinear.*

5.3 The family M_n

The genera of the Zariski components of the $\mathrm{SL}(2, \mathbb{C})$ - and $\mathrm{PSL}(2, \mathbb{C})$ -character varieties of the infinite family of once-punctured torus bundles M_n with monodromies $\varphi_n = AB^{n+2}$ were determined by Baker and Petersen [2, Theorem 5.1] via a birational isomorphism between $X^{\mathrm{irr}}(M_{\varphi_n})$ and a family of hyperelliptic curves. In this section, we compute the corresponding varieties $X_{\varphi_n}(S)$. Note that $\mathrm{tr}(\varphi_n) = -n$ and M_n is hyperbolic if $n \geq 3$ or $n \leq -3$. For computational simplicity, we only consider the case when $n \geq 3$ and n is odd, and so $b_1(\varphi_n) = 1$.

The corresponding family of framings φ_n admit the form

$$\varphi_n = \begin{cases} a \rightarrow a(a^{-1}b^{-1})^{n+2}, \\ b \rightarrow ba. \end{cases}$$

By a direct computation as in Lemma 11, the binary dihedral characters for $X(M_n)$ are as follows.

Fact 38 For every odd integer $n \geq 3$, the ramification points of r are

$$(\beta^{-2k} + \beta^{2k}, \beta^k + \beta^{-k}, \beta^k + \beta^{-k}, 0, 0, 0, 0)$$

where $\beta = e^{\frac{2\pi i}{n-2}}$, $\beta^{2k} \neq 1$ and $k = 0, 1, \dots, n-3$. There are

$$\frac{1}{2}(n-3) = \frac{1}{2}|n-2| - \frac{1}{2} = \frac{1}{2}|\text{tr } \varphi_n + 2| - 2^{b_1(\varphi_n)-2}$$

such binary dihedral characters in total, which agrees with (1-1).

5.3.1 Topology of $X_{\varphi_n}(S)$ In this subsection, we compute the fixed point set $X_{\varphi_n}(S)$. Substituting the automorphism φ_n into the definition of $X_{\varphi_n}(S)$ in Section 3.1, we obtain the three equations

$$\text{tr } \rho(a) = \text{tr } \rho(a(a^{-1}b^{-1})^{n+2}), \quad \text{tr } \rho(b) = \text{tr } \rho(ba), \quad \text{tr } \rho(ab) = \text{tr } \rho(a(a^{-1}b^{-1})^{n+1}).$$

The second equation gives $y = z$. Define polynomials $P_n(x, y, z) = \text{tr } \rho(a(a^{-1}b^{-1})^n)$ for $n \in \mathbb{Z}$. Using the trace identities in Section 2, P_n satisfies the recursive relation $P_n(x, y, z) = zP_{n-1}(x, y, z) - P_{n-2}(x, y, z)$ with $P_0(x, y, z) = x$ and $P_1(x, y, z) = y$. Then $X_{\varphi_n}(S)$ is defined by $P_{n+2} - x$, $P_{n+1} - z$ and $y - z$. Recall the Fibonacci polynomial f_n in Section 5.1. We observe that

$$(5-1) \quad P_n(x, y, z) = yf_n(z) - xf_{n-1}(z) \quad \text{for } n \in \mathbb{Z}.$$

To simplify the defining equations of $X_{\varphi_n}(S)$, we use the idea of Buchberger's algorithm [6] which is used to compute the Gröbner basis of an ideal of a polynomial ring. Denote $X_{n+2} = P_{n+2} - x = P_{n+2} - P_0$, $X_{n+1} = P_{n+1} - P_1$. Buchberger's algorithm involves the so-called S-polynomials (the S stands for subtraction — one eliminates the leading terms of two polynomials). We apply the algorithm here and refer to [6] for definitions and details. We compute the S-polynomial of the first two polynomials recursively to summarise X_{n+2} and X_{n+1} into one single polynomial.

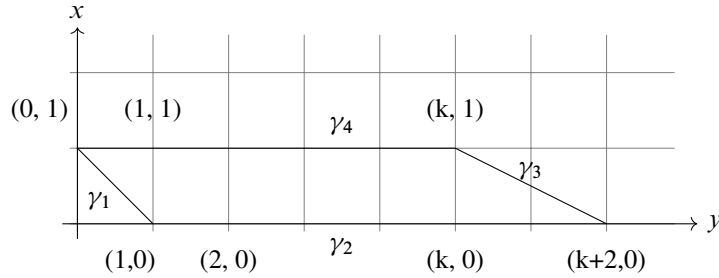
Lemma 39 Define $X_{i+2} = zX_{i+1} - X_i$ for $i \geq 0$. When $n = 2k + 1$ is a positive odd integer, $\langle X_{n+2}, X_{n+1} \rangle = \langle P_{k+2} - P_{k+1} \rangle$. When $n = 2k$ is a positive even integer, $\langle X_{n+2}, X_{n+1} \rangle = \langle P_{k+2} - P_k \rangle$.

Proof Using induction, we can prove that $X_i = P_i - P_{n+2-i}$ for $0 \leq i \leq n+2$. Since X_{i+2} is a linear combination of X_{i+1} and X_i , $\langle X_{i+2}, X_{i+1} \rangle \subseteq \langle X_{i+1}, X_i \rangle$. By definition, $\langle X_{i+1}, X_i \rangle \subseteq \langle X_{i+2}, X_{i+1} \rangle$. Thus, $\langle X_{i+2}, X_{i+1} \rangle = \langle X_{i+1}, X_i \rangle$.

Hence when $n = 2k + 1$,

$$\langle X_{n+2}, X_{n+1} \rangle = \dots = \langle X_{k+2}, X_{k+1} \rangle = \langle P_{k+2} - P_{k+1} \rangle.$$

The even case follows analogously. □

Figure 1: Newton polygon of g_k .

When n is odd, $X_{\varphi_n}(S) = V(\langle P_{k+2} - P_{k+1}, y - z \rangle)$, where we use the notation $V(I)$ for the variety defined by the ideal I . Under the restriction $(x, y, z) \mapsto (x, y)$, $X_{\varphi_n}(S)$ is homeomorphic to its image, denoted by U_n . Note that $U_n = V(\langle p_{k+2} - p_{k+1} \rangle)$ where $p_n(x, y) = P_n(x, y, y)$. Combining this with (5-1), U_n is defined by

$$g_k(x, y) := p_{k+2} - p_{k+1} = y(f_{k+2}(y) - f_{k+1}(y)) - x(f_{k+1}(y) - f_k(y)).$$

By Lemma 34, the Newton polygon of g_k is shown in Figure 1.

Lemma 40 $g_k(x, y)$ is irreducible in $\mathbb{C}[x, y]$ for any positive integer k .

Proof If $g_k(x, y)$ is reducible, then we can factorise it into $g_k(x, y) = s_k(x, y)t_k(x, y)$, where t_k, s_k are nonconstant polynomials in $\mathbb{C}[x, y]$. Since $g_k(x, y)$ is linear in x , without loss of generality, we assume that $t_k(x, y) = t_k(y)$ and $s_k(x, y) = x s_k^{(1)}(y) + s_k^{(2)}(y)$, where $s_k^{(i)}(y)$ and $t_k(y)$ are polynomials in $\mathbb{C}[y]$. Then

$$y(f_{k+2}(y) - f_{k+1}(y)) = s_k^{(2)}(y)t_k(y), \quad -(f_{k+1}(y) - f_k(y)) = s_k^{(1)}(y)t_k(y).$$

Since $t_k(y)$ is nonconstant, by the fundamental theorem of algebra, $t_k(y) = 0$ has a solution $\alpha \in \mathbb{C}$. Lemma 34 tells us that $f_{k+1}(y) - f_k(y)$ has a nonzero constant term, which implies that $\alpha \neq 0$. Then α is a common zero of $f_{k+2}(y) - f_{k+1}(y)$ and $f_{k+1}(y) - f_k(y)$. This contradicts Lemma 34(5). Hence $g_k(x, y)$ is irreducible. $\square$

Lemma 41 For $n = 2k + 1 \geq 3$, there is no singular point in $U_n = V(g_k)$ and g_k is nondegenerate.

Proof Suppose (x, y) is a singular point. Then

$$y(f_{k+2}(y) - f_{k+1}(y)) - x(f_{k+1}(y) - f_k(y)) = 0, \quad \frac{\partial g_k}{\partial x}(x, y) = f_{k+1}(y) - f_k(y) = 0.$$

This clearly contradicts Lemma 34(5). The nondegeneracy condition for edges γ_1 and γ_3 in Figure 1 is obvious. For the edge γ_2 , $F_{\gamma_2}(x, y) = y(f_{k+2}(y) - f_{k+1}(y))$. Suppose that $(x, y) \in (\mathbb{C}/\{0\})^2$ is a common zero of

$$F_{\gamma_2}, \quad x \frac{\partial F_{\gamma_2}}{\partial x} \quad \text{and} \quad y \frac{\partial F_{\gamma_2}}{\partial y}.$$

This results in

$$f_{k+2}(y) - f_{k+1}(y) = 0, \quad f'_{k+2}(y) - f'_{k+1}(y) = 0.$$

There is no such $y \neq 0$ satisfying the above two equations by Lemma 34(1). In detail, it is clear that $y = \pm 2$ are not zeros. When $y \neq \pm 2$, the first equation implies that $p^{2k+3} + 1 = 0$ and $p \neq -1$ given $y = p + p^{-1}$. Using the change of variable formula

$$\frac{df_n}{dp} = \frac{df_n}{dy} \frac{dy}{dp},$$

the second equation implies that

$$\frac{p^{-k}(k(p+1)(p^{2k+3}-1)-1-2p+2p^{2k+3}+p^{2k+4})}{(p^2-1)(1+p)^2} = 0.$$

Since $y \neq 0$,

$$k(p+1)(p^{2k+3}-1)-1-2p+2p^{2k+3}+p^{2k+4} = 0.$$

Combining the above equation with $p^{2k+3} = -1$ implies $p = -1$, which is a contradiction. The nondegeneracy condition for edge γ_4 can be checked using a similar argument. $\square$

Proposition 42 *If $n = 2k + 1 \geq 3$, then the variety $\bar{X}^{\text{irr}}(M_\varphi) \cong X_{\varphi_n}(S)$ is irreducible and a curve of genus 0.*

Proof By Lemmas 40 and 41, $g_k(x, y)$ satisfies assumptions in Corollary 36, which implies that the genus of U_n is 0. As $U_n \cong X_{\varphi_n}(S)$, this proves the statement by Corollary 3. $\square$

According to the Newton polygon Figure 1 and the theory of Puiseux expansions, there are k ideal points such that $x \rightarrow \infty$, $y \rightarrow c$ (a nonzero constant) and one ideal point such that $x \rightarrow \infty$, $y \rightarrow \infty$. Hence there are $k + 1 = \frac{1}{2}(n + 1)$ ideal points. Since the number $\frac{1}{2}(n - 3)$ of branch points is positive if $n > 3$, the variety $X^{\text{irr}}(M_n)$ is also irreducible, and the same can be confirmed by direct calculation if $n = 3$. Hence Corollary 20 implies:

Theorem 43 *Suppose that g_n denote the genus of $X^{\text{irr}}(M_n)$. When n is a positive odd integer and $n \geq 3$, g_n is bounded by*

$$\frac{1}{4}(n - 7) \leq g_n \leq \frac{1}{2}(n - 3).$$

Baker and Petersen [2, Theorem 5.1] show that $g_n = \frac{1}{2}(n - 3)$ for n odd. So there are $n - 1$ branch points in total. Hence in addition to the $\frac{1}{2}(n - 3)$ branch points at the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$, there are $\frac{1}{2}(n + 1)$ branch points at ideal points. In particular, every ideal point is a branch point (and in the case $n = 3$ these are the only branch points).

5.4 The family N_n

Let N_n be the once-punctured torus bundle with monodromy $\psi_n = AB^{n+2}A$. Note that $\text{tr}(\psi_n) = -2n - 2$ and N_n is hyperbolic when $n \geq 1$ or $n \leq -3$. The corresponding family of framings ψ_n admit the form

$$\psi_n = \begin{cases} a \rightarrow a(a^{-1}b^{-1})^{n+2}, \\ b \rightarrow ba^2(a^{-1}b^{-1})^{n+2}. \end{cases}$$

We restrict to $n = 2k + 1 \geq 1$ odd and hence $b_1(\psi_n) = 2$.

The corresponding fixed-point set $X_{\psi_n}(S)$ is defined by the equations

$$\begin{aligned} \text{tr } \rho(a) &= \text{tr } \rho(a(a^{-1}b^{-1})^{n+2}), \\ \text{tr } \rho(b) &= \text{tr } \rho(ba^2(a^{-1}b^{-1})^{n+2}), \\ \text{tr } \rho(ab) &= \text{tr } \rho(a(a^{-1}b^{-1})^{n+2}ba^2(a^{-1}b^{-1})^{n+2}). \end{aligned}$$

Using the trace identities in Section 2, the above equations can be simplified to $x = P_{n+2}(x, y, z)$, $y = P_{n+1}(x, y, z)$ and $z = xy - z$, where P_n is defined by (5-1). Then

$$X_{\psi_n}(S) = V(\langle P_{n+2} - x, P_{n+1} - y, xy - 2z \rangle).$$

By Lemma 39, $X_{\psi_n}(S) = V(\langle P_{k+2} - P_{k+1}, xy - 2z \rangle)$ when $n = 2k + 1$.

If $x = 0$, then $z = 0$ by second equation, and then $0 = P_{k+2} - P_{k+1} = y(f_{k+2}(z) - f_{k+1}(z))$ which implies $y = 0$ by Lemma 34(3). In particular, there is a unique point in $X_{\psi_n}(S)$ where $x = 0$.

When $x \neq 0$, we substitute $y = 2z/x$ into $P_{k+2}(x, y, z) - P_{k+1}(x, y, z) = 0$ and multiply by x on both sides, giving

$$h_k(x, z) := 2(f_{k+2}(z) - f_{k+1}(z))z - (f_{k+1}(z) - f_k(z))x^2 = 0.$$

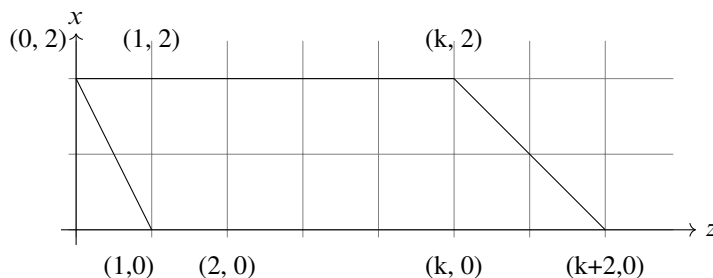
Let V_n be the variety in $\mathbb{C}^2$ defined by $h_k \in \mathbb{C}[x, z]$. Then $X_{\psi_n}(S)$ and V_n are birational via rational maps

$$\begin{aligned} (5-2) \quad r_1: X_{\psi_n}(S) &\dashrightarrow V_n, & (x, y, z) &\mapsto (x, z), \\ r_2: V_n &\dashrightarrow X_{\psi_n}(S), & (x, z) &\mapsto (x, 2z/x, z). \end{aligned}$$

Lemma 44 $h_k(x, z)$ is an irreducible polynomial in $\mathbb{C}[x, z]$ for every positive integer k .

Proof If $h_k(x, z)$ is reducible, then we can factorise it into $h_k(x, z) = s_k(x, z)t_k(x, z)$, where t_k, s_k are nonconstant polynomials in $\mathbb{C}[x, z]$. Then the Newton polygon of $h_k(x, z)$ is the Minkowski sum of the Newton polygons of $s_k(x, z)$ and $t_k(x, z)$. The Newton polygon of $h_k(x, z)$ is shown in Figure 2. The *boundary vector sequence* of a Newton polygon is obtained by choosing an orientation of its boundary and cyclically listing the vectors between consecutive integer lattice points on the boundary. Travelling along the boundary of the Newton polygon of $h_k(x, z)$, we obtain the boundary vector sequence

$$v(h_k) = ((1, -2), (1, 0), \dots, (1, 0), (-1, 1), (-1, 1), (-1, 0), \dots, (-1, 0)),$$


 Figure 2: Newton polygon of $h_k(x, z)$.

where the number of repetitions of $(1, 0)$ and $(-1, 0)$ in $v(h_k)$ are $k + 1$ and k , respectively. The sequence $v(h_k)$ can be partitioned into two disjoint nonempty subsequences $v(s_k)$ and $v(t_k)$, each of which sums to zero. Given $(1, -2)$, $(-1, 1)$ and $(-1, 1)$ are the only three vectors with nonzero second components, they must be in the same sequence. Without loss of generality, we assume that they are in $v(s_k)$. Noting that h_k has no linear term in x , we may write $t_k(x, z) = t_k(z)$ and $s_k(x, z) = x^2 s_k^{(1)}(z) + s_k^{(2)}(z)$, where $s_k^{(i)}(z)$ and $t_k(z)$ are polynomials in $\mathbb{C}[z]$. Hence

$$2(f_{k+2}(z) - f_{k+1}(z))z = s_k^{(2)}(z)t_k(z), \quad -(f_{k+1}(z) - f_k(z)) = s_k^{(1)}(z)t_k(z).$$

The remainder of the proof is analogous to the proof of Lemma 40, where one obtains a contradiction to Lemma 34(5). $\square$

Lemma 45 When $n = 2k + 1 \geq 1$, h_k is nondegenerate and there is no singular point in $V_n = V(h_k)$.

Proof The proof is almost identical to Lemma 41. $\square$

Combining Lemmas 44 and 45 with Corollary 36 and Lemma 37 implies:

Lemma 46 The curve $V_n \cong X_{\psi_n}(S)$ is hyperelliptic of genus k for each $n = 2k + 1 \geq 3$.

5.4.1 $\mathrm{PSL}(2, \mathbb{C})$ -character variety Since n is odd, $b_1(\psi_n) = 2$. If a representation $\bar{\rho} \in \bar{R}(N_n)$ lifts to $\mathrm{SL}(2, \mathbb{C})$, then it generically has 4 lifts. Let $\bar{X}_0(N_n)$ be the subvariety of $\bar{X}^{\mathrm{irr}}(N_n)$ consisting of the characters of all $\mathrm{PSL}(2, \mathbb{C})$ -representations that lift to representations into $\mathrm{SL}(2, \mathbb{C})$. By Corollary 2, $\bar{X}_0(N_n) \cong q_2(X_{\psi_n}(S))$. We have the following proposition.

Proposition 47 For every positive odd integer $n = 2k + 1$, the subvariety $\bar{X}_0(N_n)$ is birational to the affine line.

Proof Since $\bar{X}_0(N_n)$ is birational to $q_2(X_{\psi_n}(S))$, it suffices to show that the latter has genus 0.

Take a homomorphism $h \in \mathrm{Hom}(\pi_1(N_n), \mathbb{Z}_2)$. A direct calculation shows that either $(h(a), h(b)) = (1, 1)$ or $(h(a), h(b)) = (-1, -1)$ since n is odd. Hence $q_2(X_{\psi_n}(S))$ is the quotient of $X_{\psi_n}(S)$ by the involution $(x, y, z) \rightarrow (-x, -y, z)$.

Recall that $X_{\psi_n}(S)$ is birational to its restriction V_n via the maps given in (5-2). Let $\bar{V}_n$ be the variety corresponding to the identification of V_n under the involution $(x, z) \rightarrow (-x, z)$. Letting $(X, Z) = (x^2, z)$, $\bar{V}_n$ is defined by

$$\bar{h}_k(X, Z) := 2(f_{k+2}(Z) - f_{k+1}(Z))Z - (f_{k+1}(Z) - f_k(Z))X.$$

The genus of $\bar{V}_n$ is seen to be 0 by a simple modification of the argument of Proposition 42. $\square$

For each irreducible $\mathrm{PSL}(2, \mathbb{C})$ -representation $\bar{\rho}$ which does not lift, we may choose $T, A, B \in \mathrm{SL}(2, \mathbb{C})$ with the properties $\bar{\rho}(t) = \{\pm T\}$, $\bar{\rho}(a) = \{\pm A\}$, $\bar{\rho}(b) = \{\pm B\}$ and

$$T^{-1}AT = -\psi_n(A), \quad T^{-1}BT = \psi_n(B).$$

Using the convention from (4-12)–(4-14), the traces of these representatives under the restriction map r are contained in the set

$$X_{\psi_n}^2(S) = \{(x, y, z) \in \mathbb{C}^3 \mid (-x, y, -z) = \bar{\psi}_n(x, y, z)\} = V(P_{n+2} + x, P_{n+1} - y, xy).$$

This implies that the components of $X_{\psi_n}^2(S)$ are

$$(5-3) \quad L_3 = \{(0, 0, z) \mid z \in \mathbb{C}\},$$

$$(5-4) \quad C_\xi = \{(0, y, \xi) \mid y \in \mathbb{C}\}, \text{ where } \xi \text{ is a root of } f_{k+2}(z) - f_{k+1}(z) = 0,$$

$$(5-5) \quad C_\zeta = \{(x, 0, \zeta) \mid x \in \mathbb{C}\}, \text{ where } \zeta \text{ is a root of } f_{k+1}(z) - f_k(z) = 0.$$

For each of the components $C_z \subset X_{\psi_n}^2(S)$, the map $\bar{X}(N_n) \supset \bar{r}^{-1}q_2(C_z) \rightarrow \bar{X}_{\psi_n}(S)$ is a birational equivalence onto its image since the irreducible characters in (5-4) and (5-5) arise from $\mathrm{PSL}(2, \mathbb{C})$ -representations of $\pi_1(S)$ with trivial centraliser.

Points in $X_{\psi_n}(S)$ satisfy $2z = xy$, and none of the roots ξ or ζ equal zero. Hence the only point in $X_{\psi_n}(S) \cap X_{\psi_n}^2(S)$ is $(0, 0, 0)$, and the existence of this intersection point is due to the fact that we are not working with $\bar{X}(N_n)$, but with $\bar{X}_{\psi_n}(S)$. The two representations of N_n predicted by Lemma 23 are given by letting

$$\bar{\rho}(a) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\rho}(b) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$T_1 = \pm \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad T_2 = \bar{\rho}(a)T_1 = \pm \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then $\bar{\rho}$ extends to a representation into $\mathrm{PSL}(2, \mathbb{C})$ by letting either $\bar{\rho} = T_1$ or $\bar{\rho} = T_2$, and the former lifts to $\mathrm{SL}(2, \mathbb{C})$ whilst the latter doesn't. Denote the characters in $\bar{X}(N_n)$ of the 2 extensions as $\bar{\chi}_1$ and $\bar{\chi}_2$. According to the discussion in Section 2.1, $\bar{\chi}_1$ and $\bar{\chi}_2$ are on different topological components of $\bar{X}(N_n)$. Note that $\bar{\chi}_1$ is in $\bar{X}_0(N_n)$ while $\bar{\chi}_2$ is in $\bar{r}^{-1}q_2(L_3)$.

For the component $L_3 \subset X_{\psi_n}^2(S)$, we follow the same parametrisation of $\bar{L}_3$ in (4-9). For any point $(0, 0, (p + \frac{1}{p})^2, 0) \in \bar{L}_3$, a direct computation shows that it extends to 2 characters in $\bar{X}(N_n)$ where the quadruples in (4-4) are

$$\left(2 - p^{n+2} - \frac{1}{p^{n+2}}, 0, 0, 2 - p^{n+4} - \frac{1}{p^{n+4}}\right) \quad \text{and} \quad \left(2 + p^{n+2} + \frac{1}{p^{n+2}}, 0, 0, 2 + p^{n+4} + \frac{1}{p^{n+4}}\right).$$

Thus $\bar{r}^{-1}q_2(L_3) \rightarrow \bar{L}_3$ is a 2-fold branched cover with the only ramification point $\bar{\chi}_2$.

5.5 The family L_n

Let L_n be the once-punctured torus bundle with monodromy $\omega_n = A^2 B^{n+2} A$. Note that $\text{tr}(\omega_n) = -3n - 4$ so L_n is hyperbolic when n is positive. When n is odd, $b_1(\omega_n) = 1$ and hence $\bar{X}^{\text{irr}}(L_n) \cong X_{\omega_n}(S)$ according to Corollary 3. We now show:

Theorem 48 *Suppose $n = 2k + 1 \geq 3$ is odd. The $\text{PSL}(2, \mathbb{C})$ -character variety $\bar{X}^{\text{irr}}(L_n)$ consists of $k + 2$ components. The canonical curve $\bar{X}_0(L_n) \subset \bar{X}^{\text{irr}}(L_n)$ is birationally equivalent to a hyperelliptic curve of genus k while all the other Zariski components are birational to the affine line.*

We remark that if $n = 1$, then $\bar{X}^{\text{irr}}(L_n)$ has 3 components, each of genus zero.

The framing ω_n of L_n is

$$\omega_n = \begin{cases} a \rightarrow a(a^{-2}b^{-1})^{n+2}, \\ b \rightarrow ba^3(a^{-2}b^{-1})^{n+2}. \end{cases}$$

Then $X_{\omega_n}(S)$ is defined by

$$\begin{aligned} \text{tr } \rho(a) &= \text{tr } \rho(a(a^{-2}b^{-1})^{n+2}), \\ \text{tr } \rho(b) &= \text{tr } \rho(ba^3(a^{-2}b^{-1})^{n+2}), \\ \text{tr } \rho(ab) &= \text{tr } \rho(a(a^{-2}b^{-1})^{n+2}ba^3(a^{-2}b^{-1})^{n+2}). \end{aligned}$$

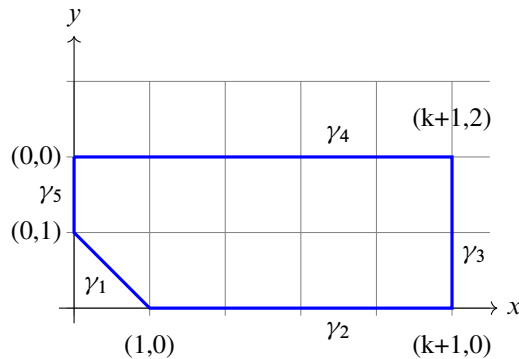
Using the trace identities in Section 2, the second equation is $\text{tr } \rho(b) = \text{tr } \rho(a(a^{-2}b^{-1})^{n+1})$ and the third equation is given by $z = xy - zx + y$. Let $Q_n(x, y, z) := \text{tr } \rho(a(a^{-2}b^{-1})^n)$. We have $Q_0 = x$, $Q_1 = z$ and $Q_n = (xz - y)Q_{n-1} - Q_{n-2}$. Note that the third equation is equivalent to $(y - z)(x + 1) = 0$ and hence naturally splits the variety $X_{\omega_n}(S)$ into the union of two algebraic sets:

$$X_{\omega_n}(S) = V(\langle Q_{n+2} - x, Q_{n+1} - y, x + 1 \rangle) \cup V(\langle Q_{n+2} - x, Q_{n+1} - y, y - z \rangle)$$

We denote the two algebraic sets in this union by R_n and W_n , respectively.

Lemma 49 *Each canonical component of $X^{\text{irr}}(L_n)$ is contained in the preimage of W_n .*

Proof We only need to show that each discrete and faithful character $\chi_{\rho_0} \in X^{\text{irr}}(L_n)$ is not in the preimage of R_n . Clearly ρ_0 is not binary dihedral, so we can write $\rho_0(a)$ and $\rho_0(b)$ as in (2-11). If $x = -1$, a simple calculation shows that $\rho_0(a)$ is of order 3, which contradicts the faithfulness of ρ_0 since a has infinite order in $\pi_1(M_{\omega_n})$. $\square$

Figure 3: The Newton polygon of r_k .

We will analyse W_n and R_n separately. Before doing so, we define a family of auxiliary polynomials $Q'_n \in \mathbb{C}[x, y, z]$, which follow the same recursive relation as Q_n but with $Q'_0 = x$ and $Q'_1 = y$. When $n = 2k + 1$ is odd, a similar argument as in Lemma 39 shows that

$$\langle Q_{n+2} - Q'_0, Q_{n+1} - Q'_1 \rangle = \langle Q_{k+2} - Q'_{k+1}, Q_{k+1} - Q'_{k+2} \rangle.$$

In addition, we observe that

$$Q_n = zf_n(xz - y) - xf_{n-1}(xz - y) \quad \text{and} \quad Q'_n = yf_n(xz - y) - xf_{n-1}(xz - y).$$

5.5.1 The canonical component Under the map $(x, y, z) \mapsto (x - 1, y)$, W_n is birationally equivalent with its image, denoted by $W'_n \subset \mathbb{C}^2$. As Q_n and Q'_n coincide when $y = z$, W'_n is defined by the single element $q_{k+2}(x, y) - q_{k+1}(x, y) \in \mathbb{C}[x, y]$, where

$$q_n(x, y) = yf_n(xy) - (x + 1)f_{n-1}(xy).$$

Let $\mathcal{W}_n$ be the variety in $\mathbb{C}^2$ defined by the polynomial

$$r_k(x, y) := y^2(f_{k+2}(x) - f_{k+1}(x)) - (x + y)(f_{k+1}(x) - f_k(x)).$$

There exists a birational isomorphism between W'_n and $\mathcal{W}_n$ via

$$\begin{aligned} r_1 : W'_n &\rightarrow \mathcal{W}_n, & (x, y) &\mapsto (xy, y), \\ r_2 : \mathcal{W}_n &\rightarrow W'_n, & (x, y) &\mapsto (x/y, y). \end{aligned}$$

Lemma 50 $r_k(x, y)$ defined as above is an irreducible polynomial in $\mathbb{C}[x, y]$ for every positive odd integer $n = 2k + 1 \geq 3$.

Proof If $r_k(x, y)$ is reducible, then we can factorise it into $r_k(x, y) = s_k(x, y)t_k(x, y)$, where t_n, s_n are nonconstant polynomials in $\mathbb{C}[x, y]$. The Newton polygon of r_k is shown in Figure 3. The boundary vector sequence of $r_k(x, y)$ is

$$v(r_k) = ((1, 0), \dots, (1, 0), (0, 1), (0, 1), (-1, 0), (-1, 0), \dots, (-1, 0), (0, -1), (1, -1)),$$

where the number of repetitions of $(1, 0)$ and $(-1, 0)$ in the sequence are k and $k + 1$, respectively. Given

$(0, 1)$, $(0, -1)$ and $(1, -1)$ are the only vectors with nonzero vertical component, there are two cases to split $v(r_k)$ into two disjoint nonempty subsequences $v(s_k)$ and $v(t_k)$, each of which sums to zero.

(1) All the four vectors with nonzero vertical component are in the same subsequence. Without loss of generality, let $s_k(x, y) = s_k(x)$ and $t_k(x, y) = t_k^{(2)}(x) y^2 + t_k^{(1)}(x) y + t_k^{(0)}(x)$, where the superscript i indicates the degree in y that the polynomial $t_k^{(i)} \in \mathbb{C}[x]$ is a coefficient of. Then

$$\begin{aligned} f_{k+2}(x) - f_{k+1}(x) &= t_k^{(2)}(x) s_k(x), \\ -(f_{k+1}(x) - f_k(x)) &= t_k^{(1)}(x) s_k(x), \\ -x(f_{k+1}(x) - f_k(x)) &= t_k^{(0)}(x) s_k(x). \end{aligned}$$

This is impossible due to Lemma 34(5).

(2) Hence $(1, -1)$, $(0, 1)$ belong to one sequence, and $(0, 1)$, $(0, -1)$ belong to the other sequence. Let $r_k(x, y) = (a(x)y + b(x))(c(x)y + d(x))$ where a, b, c, d are polynomials in $\mathbb{C}[x]$. Then

$$\begin{aligned} f_{k+2}(x) - f_{k+1}(x) &= a(x)c(x), \\ -(f_{k+1}(x) - f_k(x)) &= a(x)d(x) + b(x)c(x), \\ -x(f_{k+1}(x) - f_k(x)) &= b(x)d(x). \end{aligned}$$

Without loss of generality, take a root α of $f_{k+1}(x) - f_k(x)$ which is also a root of $b(x)$. The third equation and Lemma 34(4) imply $d(\alpha) \neq 0$. The first equation and Lemma 34(5) imply $a(\alpha) \neq 0$. But this gives a contradiction to the second equation.

Hence $r_k(x, y)$ is irreducible. $\square$

Remark 51 When $n = 1$, $W_1 = V(q_2 - q_1) = V(y + 1) \cup V(xy - x - 1)$. Both components have genus 0.

Lemma 52 When $n = 2k + 1 \geq 3$, r_k is nondegenerate and there is no singular point in $V_n = V(r_k)$. Hence V_k is a hyperelliptic curve of genus k .

Proof The proof is straight forward using the same arguments as in the proof of Lemma 41. $\square$

5.5.2 Other components When $n = 2k + 1$, $R_n = V(Q_{k+2} - Q'_{k+1}, Q_{k+1} - Q'_{k+2}, x + 1)$ is birationally equivalent with its image, denoted by R'_n , under the restriction $(x, y, z) \mapsto (y, z)$. The variety R'_n is defined by $q'_{k+2} - q_{k+1}$ and $q_{k+2} - q'_{k+1}$, where

$$q_n(y, z) = zf_n(-z - y) + f_{n-1}(-z - y), \quad q'_n(y, z) = yf_n(-z - y) + f_{n-1}(-z - y).$$

By a direct computation using the definition of f_n ,

$$R_n = \{(-1, -1)\} \cup V(f_{k+2}(-y - z) + f_{k+1}(-y - z)).$$

φ_*	$\text{tr}(\varphi_*)$	$o(\varphi_2)$	$\#X_\varphi(S)$	genera
$A^2 B^3$	-4	2	1	0
$A^3 B^2$	-4	2	1	0
AB^5	-3	3	1	0
$A^2 B^4$	-6	1	2	0, 0
$A^3 B^3$	-7	3	3	0, 0, 0
$A^4 B^2$	-6	1	2	0, 0

Table 1: Short words in A and B giving hyperbolic once-punctured torus bundles

The point $(-1, -1, -1)$ is already contained in W_n while the second algebraic set gives us $k+1$ components

$$\{(x, y, z) \mid y + z + \alpha = 0, x + 1 = 0\}$$

as α ranges over the $k+1$ distinct roots of $f_{k+2}(u) + f_{k+1}(u) = 0$. All these components are birationally equivalent with affine lines.

5.6 Experimental results

As stated in Section 2.2, the choice of a monodromy as a positive word in A and B is not unique. Also, there is no simple criterion to ensure that the absolute value of the trace is greater than two. We list in Table 1 the shortest words with this property. The computations were executed with Singular [10] using the following code template, and $\#X_\varphi(S)$ denotes the number of Zariski components of $X_\varphi(S)$. The map m below represents $\bar{\varphi}$ and is given as a composition of maps `alpha` and `betainv`:

```

ring r=0,(x,y,z),dp;
map alpha=r,x,z,xz-y;
map betainv=r,z,y,yz-x;
map m = [composition of alpha and betainv];
ideal I = m[1]-x,m[2]-y,m[3]-z;
def S = absPrimdecGTZ(I);
setring S;
absolute_primes;

```

The representative

$$\pm A^{a_1} B^{-b_1} A^{a_2} B^{-b_2} \dots A^{a_n} B^{-b_n},$$

where $n > 0$, the a_i and b_i are positive integers, and the sign equals the sign of the trace of φ_* , allows us to build up a census of examples more efficiently. As explained in Section 2.4, if one is only interested in the topology of the fixed-point set, then it suffices to restrict to the case of positive trace. We summarise our computational results in Table 2.

φ_*	$\mathrm{tr}(\varphi_*)$	$o(\varphi_2)$	$\#X_\varphi(S)$	genera
AB^{-1}	3	3	1	0
AB^{-2}	4	2	1	0
A^2B^{-1}	4	2	1	0
AB^{-3}	5	3	1	0
A^2B^{-2}	6	1	2	0, 0
A^3B^{-1}	5	3	1	0
$AB^{-1}AB^{-1}$	7	3	3	0, 0, 0
AB^{-4}	6	2	2	0, 0
A^2B^{-3}	8	2	1	1
A^3B^{-2}	8	2	1	1
A^4B^{-1}	6	2	2	0, 0
$AB^{-1}AB^{-2}$	10	2	3	0, 0, 0
$AB^{-1}A^2B^{-1}$	10	2	3	0, 0, 0
AB^{-5}	7	3	1	0
A^2B^{-4}	10	1	2	1, 0
A^3B^{-3}	11	3	2	1, 0
A^4B^{-2}	10	1	2	1, 0
A^5B^{-1}	7	3	1	0
$AB^{-1}AB^{-3}$	13	3	3	1, 0, 0
$AB^{-1}A^2B^{-2}$	15	3	1	2
$AB^{-1}A^3B^{-1}$	13	3	3	1, 0, 0
$AB^{-2}AB^{-2}$	14	1	5	0, 0, 0, 0, 0
$AB^{-2}A^2B^{-1}$	15	3	1	2
$AB^{-1}AB^{-1}AB^{-1}$	18	1	7	0, 0, 0, 0, 0, 0, 0

Table 2: Short words in A and B^{-1} giving hyperbolic once-punctured torus bundles.

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Segalification and the Boardman–Vogt tensor product

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We develop an analogue of Dugger and Spivak’s necklace formula, providing an explicit description of the Segal space generated by an arbitrary simplicial space. We apply this to obtain a formula for the Segalification of n -fold simplicial spaces, a new proof of the invariance of right fibrations, and a new construction of the Boardman–Vogt tensor product of ∞ -operads, for which we also derive an explicit formula.

18N60, 18N65, 18N70

Introduction

The nerve functor $N_\bullet: \text{Cat}_1 \rightarrow \text{sSet}$ has a left adjoint, which assigns to a simplicial set X its homotopy category $\text{ho}(X)$. The homotopy category $\text{ho}(X)$ has as objects the 0-simplices of X , and its morphisms are generated by the 1-simplices of X modulo the relations imposed by the 2-simplices. In the setting of ∞ -categories, the nerve $N_\bullet: \text{Cat}_\infty \rightarrow \text{PSh}(\Delta)$ is given by $N_n \mathcal{C} := \text{Fun}([n], \mathcal{C})^\simeq$ and participates in an adjunction

$$\mathbb{C}: \text{PSh}(\Delta) \rightleftarrows \text{Cat}_\infty : N_\bullet,$$

where the left adjoint $\mathbb{C}$ is left Kan extended from the inclusion $\Delta \subset \text{Cat}_\infty$. The purpose of this note is to give a formula for $\mathbb{C}(X)$.

The functor $N_\bullet$ is fully faithful and its essential image consists of the complete Segal spaces in the sense of Rezk [22]. Recall that a *Segal space* is a simplicial space $X: \Delta^{\text{op}} \rightarrow \mathcal{S}$ for which the natural map $X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an equivalence for all n . A Segal space is *complete* if the map $s_0: X_0 \rightarrow X_1$ induces an equivalence onto a certain union of components $X_1^{\text{eq}} \subset X_1$. Letting $\text{PSh}_{\text{CSS}}(\Delta) \subset \text{PSh}_{\text{seg}}(\Delta) \subset \text{PSh}(\Delta)$ denote the full subcategories of (complete) Segal spaces, we can factor the adjunction $\mathbb{C} \dashv N_\bullet$ as

$$\text{Cat}_\infty \xrightarrow[N_\bullet]{\simeq} \text{PSh}_{\text{CSS}}(\Delta) \xleftarrow[\mathbb{C}]{\mathbb{L}_\mathbb{C}} \text{PSh}_{\text{seg}}(\Delta) \xleftarrow[\mathbb{C}]{\mathbb{L}_\mathbb{S}} \text{PSh}(\Delta).$$

In his foundational work on complete Segal spaces [22], Rezk provides a formula for the Rezk-completion functor $\mathbb{L}_\mathbb{C}$. The purpose of this note is to provide a formula for the “Segalification” functor $\mathbb{L}_\mathbb{S}$. Combining the two, one obtains an explicit description of the ∞ -category generated by an arbitrary simplicial space.

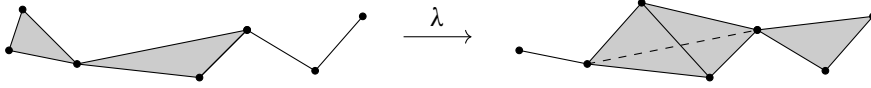


Figure 1: A morphism of necklaces $\lambda: \Delta^2 \vee \Delta^2 \vee \Delta^1 \vee \Delta^1 \rightarrow \Delta^1 \vee \Delta^3 \vee \Delta^2$ defined by collapsing the first edge and including the remaining necklace as indicated.

Necklaces and Segalification Our formula for $\mathbb{L}_S$ is heavily influenced by the work of Dugger and Spivak [9] on the rigidification of quasicategories. It will involve a colimit indexed by a certain category of “necklaces” [9, Section 3], which we now recall. A *necklace* is a simplicial set $N = \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$ obtained by joining standard simplices at their start- and endpoints, as indicated in Figure 1. Following Dugger and Spivak, we define Nec to be the (nonfull) subcategory of sSet whose objects are necklaces and whose morphisms are maps of simplicial sets $f: \Delta^{n_1} \vee \dots \vee \Delta^{n_k} \rightarrow \Delta^{m_1} \vee \dots \vee \Delta^{m_l}$ that preserve the minimal and maximal elements.

We can now state the formula for $\mathbb{L}_S$ in terms of necklaces. For the sake of simplicity, we state the formula here only for the case of 1-simplices $\mathbb{L}_S(X)_1$. This suffices to determine $\mathbb{L}_S(X)_n$ for all n by the Segal condition.

Theorem A *For every simplicial space $X \in \text{PSh}(\Delta)$, there is a canonical equivalence*

$$\mathbb{L}_S(X)_1 \simeq \text{colim}_{N \in \text{Nec}^{\text{op}}} \text{Map}_{\text{PSh}(\Delta)}(N, X) \simeq \text{colim}_{\Delta^{n_1} \vee \dots \vee \Delta^{n_r} \in \text{Nec}^{\text{op}}} X_{n_1} \times_{X_0} \dots \times_{X_0} X_{n_r}.$$

This indeed generalizes the formula for the homotopy category of a simplicial set $\text{ho}(X)$ mentioned above, as we shall see in Example 1.22. Theorem A can also be deduced from the results of Dugger–Spivak by passing through the various model categories for ∞ -categories, but we will instead give a “synthetic” proof, as we believe it to be insightful. A formula for monoidification analogous to Theorem A was obtained in [26].

Application: right fibrations Our first application is to the notion of right fibrations of simplicial spaces in the sense of Rezk [21, Remark 3.1]. Using the Segalification formula we show that right fibrations of simplicial spaces are invariant under $\mathbb{L}_{\text{CSS}}$ -equivalences: if $f: X \rightarrow Y$ is a map of simplicial spaces such that $\mathbb{L}_{\text{CS}}(f)$ is an equivalence, then base change along f induces an equivalence $f^*: \text{PSh}(\Delta)_{/Y}^{\text{r-fib}} \simeq \text{PSh}(\Delta)_{/X}^{\text{r-fib}}$. This implies that right fibrations over an arbitrary simplicial space X model presheaves on the associated ∞ -category $\mathbb{C}(X)$:

Corollary B (Rasekh) *For any simplicial space X the functor $\mathbb{C}(-)$ induces an equivalence*

$$\text{PSh}(\Delta)_{/X}^{\text{r-fib}} \xrightarrow[\simeq]{\mathbb{C}(-)} \text{Cat}_{\infty/\mathbb{C}(X)}^{\text{r-fib}} \simeq \text{PSh}(\mathbb{C}(X)).$$

A model-categorical version of this result was previously proven by Rasekh [21, Theorems 4.18 and 5.1]. Our proof has the advantage of being “synthetic” and also significantly shorter. An alternative formulation

of this corollary is to say that (the nerve of) the universal right fibration $\mathcal{S}_*^{\text{op}} \rightarrow \mathcal{S}^{\text{op}}$ classifies right fibrations of arbitrary simplicial spaces.¹

We shall now use the above result to give a formula for $\mathbb{C}(-): \text{PSh}(\Delta) \rightarrow \text{Cat}_\infty$. Let us write $\Delta_{\max} \subseteq \Delta$ for the wide subcategory spanned by morphisms which preserve the maximal element, and given a simplicial space $X: \Delta^{\text{op}} \rightarrow \mathcal{S}$ let us write $p_X: \Delta_{/X} \rightarrow \Delta$ for the associated right fibration.

Corollary C *For every simplicial space $X \in \text{PSh}(\Delta)$, the last vertex functor $\Delta_{/X} \rightarrow \mathbb{C}(X)$ induces an equivalence of ∞ -categories*

$$\Delta_{/X}[W_X^{-1}] \simeq \mathbb{C}(X),$$

where $W_X := p_X^{-1}(\Delta_{\max}) \subseteq \Delta_{/X}$.

In the case that X is levelwise discrete, ie a simplicial set, this recovers a result of Stevenson [25, Theorem 3] who attributes it to Joyal [17, Section 13.6]. A synthetic proof for the case that X is the nerve of an ∞ -category was given by Haugseng [14, Proposition 2.12], and extended to arbitrary X by Hebestreit–Steinebrunner [16, Corollary 3.8], motivated by the present paper.

Application: (∞, n) -categories The ∞ -category of (∞, n) -categories admits many equivalent descriptions including Rezk’s complete Segal Θ_n -spaces [23] and Barwick’s complete n -fold Segal spaces [3]. These were shown to be equivalent by Barwick–Schommer-Pries [4] and later, using different techniques, also by Bergner–Rezk [6] and Haugseng [12]. We shall now present an application of our main result to n -fold Segal spaces.

We say that an n -fold simplicial space $X: (\Delta^{\text{op}})^{\times n} \rightarrow \mathcal{S}$ is *reduced* if each of the $(n - k - 1)$ -fold simplicial spaces $X_{m_1, \dots, m_k, 0, \bullet, \dots, \bullet}$ is constant. We write $\text{PSh}^r(\Delta^{\times n}) \subseteq \text{PSh}(\Delta^{\times n})$ for the full subcategory of reduced n -fold simplicial spaces and let $\text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}} \subseteq \text{PSh}^r(\Delta^{\times n})$ denote the full subcategory spanned by the *n -fold Segal spaces*; that is, those reduced n -fold simplicial spaces that satisfy the Segal condition in each coordinate. This inclusion $\text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}} \hookrightarrow \text{PSh}^r(\Delta^{\times n})$ admits a left adjoint and we give a formula for it:

Theorem D *The left adjoint $\mathbb{L}: \text{PSh}^r(\Delta^{\times n}) \rightarrow \text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}}$ may be computed as $\mathbb{L} = \mathbb{L}_n \circ \dots \circ \mathbb{L}_1$ where $\mathbb{L}_j: \text{PSh}(\Delta^{\times n}) \rightarrow \text{PSh}(\Delta^{\times n})$ denotes the endofunctor that Segalifies the j^{th} coordinate:*

$$(\mathbb{L}_j X)_{m_1, \dots, m_{j-1}, 1, m_{j+1}, \dots, m_n} \simeq \text{colim}_{\Delta^{n-1} \vee \dots \vee \Delta^{n-r} \in \text{Nec}^{\text{op}}} X_{m_1, \dots, n_1, \dots, m_n} \times_{X_{m_1, \dots, 0, \dots, m_n}} \dots \times_{X_{m_1, \dots, 0, \dots, m_n}} X_{m_1, \dots, n_r, \dots, m_n}.$$

The order of the $\mathbb{L}_j$ is crucial: we need to first Segalify 1-morphisms, then 2-morphisms, etc. If one were to apply $\mathbb{L}_2$ and then $\mathbb{L}_1$ the result would not necessarily satisfy the Segal condition in the second coordinate.

¹The difficult part of this statement is the existence of a universal right fibration of simplicial spaces. Once existence is shown, it follows formally that the universal right fibration for simplicial space must agree with the one for complete Segal spaces.

Application: the Boardman–Vogt tensor product One of the many great achievements of Lurie’s book project on Higher algebra [20] is the construction of a homotopy coherent symmetric monoidal structure $\otimes_{\text{Lurie}}$ on the ∞ -category of ∞ -operads Op_∞ , generalizing the Boardman–Vogt tensor product [7, Section II.3]. The defining property of $\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}$ is that algebras over it are “ $\mathcal{O}$ -algebras in $\mathcal{P}$ -algebras”: for any symmetric monoidal ∞ -category $\mathcal{C}$ there is an equivalence

$$\text{Alg}_{\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}}(\mathcal{C}) \simeq \text{Alg}_{\mathcal{O}}(\text{Alg}_{\mathcal{P}}(\mathcal{C})).$$

The construction of $\otimes_{\text{Lurie}}$ is quite intricate as it involves a delicate mix of quasicategorical and model categorical techniques. We shall now describe how the necklace formula can be used to justify a simpler, alternative approach to the tensor product of ∞ -operads.

Theorem E *The tensor product of symmetric monoidal ∞ -categories uniquely restricts to a tensor product $\otimes_{\text{BV}}$ on Op_∞ such that the envelope $\text{Env}: (\text{Op}_\infty, \otimes_{\text{BV}}) \rightarrow (\text{Cat}_\infty^\otimes, \otimes)$ is a symmetric monoidal functor. For any two ∞ -operads $\mathcal{O}$ and $\mathcal{P}$ there is a canonical equivalence $\mathcal{O} \otimes_{\text{BV}} \mathcal{P} \simeq \mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}$.*

Remark Theorem E does *not* claim that $(\text{Op}_\infty, \otimes_{\text{BV}})$ and $(\text{Op}_\infty, \otimes_{\text{Lurie}})$ are equivalent as symmetric monoidal ∞ -categories. It does, however, reduce the question to whether the envelope can be constructed as a symmetric monoidal functor $(\text{Op}_\infty, \otimes_{\text{Lurie}}) \dashrightarrow (\text{Cat}_\infty^\otimes, \otimes)$. This is not entirely clear since the higher coherence data (associator, braiding, etc) of Lurie’s tensor product $\otimes_{\text{Lurie}}$ is tricky to access.² The authors are unaware of any applications in which the specific coherence of Lurie’s construction plays a role.

A novel consequence of Theorem E is that, at least in principle, the Boardman–Vogt tensor product is only as difficult to compute as necklace colimits. The resulting formula will be easiest to express in the language of symmetric sequences.

Outlook: symmetric sequences A *symmetric sequence* is a presheaf on the category of finite sets and bijections. The disjoint union $\sqcup$ and the product $\times$ of finite sets extend via Day convolution to symmetric monoidal structures on symmetric sequences $\text{SymSeq} := \text{PSh}(\text{Fin}^\cong)$ which we respectively denote by $\otimes$ and $\boxtimes$. Since $(\text{SymSeq}, \otimes)$ is the free presentably symmetric monoidal ∞ -category on a single generator $\underline{1} \in \text{SymSeq}$, evaluation induces an equivalence

$$\text{ev}_{\underline{1}}: \text{Fun}_{\text{CAlg}(\text{Pr}^L)}((\text{SymSeq}, \otimes), (\text{SymSeq}, \otimes)) \xrightarrow{\simeq} \text{SymSeq},$$

which endows SymSeq with yet another (nonsymmetric) monoidal structure $\circ$ coming from the composition of endofunctors on the left side. It is generally expected that 1-coloured (noncomplete) ∞ -operads are equivalent to associative algebras for $\circ$ in SymSeq , and for a different definition of $\circ$ this was shown in [13]. Given two such algebras $\mathcal{O}, \mathcal{P} \in \text{Alg}_{\mathbb{E}_1}(\text{SymSeq}, \circ)$, the necklace formula in this setting gives

$$\mathcal{O} \otimes_{\text{BV}} \mathcal{P} \simeq \text{colim}_{\Delta^{n_1} \vee \dots \vee \Delta^{n_r} \in \text{Nec}^{\text{op}}} (\mathcal{O}^{\circ n_1} \boxtimes \mathcal{P}^{\circ n_1}) \circ \dots \circ (\mathcal{O}^{\circ n_r} \boxtimes \mathcal{P}^{\circ n_r}).$$

²Lurie does give a model categorical construction of a (nonsymmetric) monoidal structure which does have a recognizable universal property as a certain localization of ∞ -categories over Fin_* . The symmetric monoidal structure however is constructed by hand, and apart from the binary operation the relation between the two is not commented on.

A formal proof does not fit in the scope of this paper, as it requires a good interface between ∞ -operads and symmetric sequences.

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1 Segalification

1.1 Necklace contexts

Let us fix an ∞ -category $\mathcal{C}$, which will be $\mathbf{\Delta}^{\text{op}}$ in later sections. In this section, we study the general problem of approximating a reflective localization functor $\mathbb{L} : \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}(\mathcal{C})$ in the sense of [19, Proposition 5.2.7.4] using a suitable auxiliary subcategory of $\text{PSh}(\mathcal{C})$.

Definition 1.1 A presheaf $X \in \text{PSh}(\mathcal{C})$ is called $\mathbb{L}$ -local if the unit map $X \rightarrow \mathbb{L}(X)$ is an equivalence, ie if X lies in the essential image of $\mathbb{L}$. A morphism of presheaves $f : Y \rightarrow Z \in \text{PSh}(\mathcal{C})$ is called an $\mathbb{L}$ -local equivalence if $\mathbb{L}(f)$ is an equivalence.

Definition 1.2 A necklace context is a triple $(\mathcal{C}, \mathbb{L}, \mathcal{N})$ where $\mathcal{C}$ and $\mathbb{L}$ are as above and $\mathcal{N} \subseteq \text{PSh}(\mathcal{C})$ is a full subcategory such that:

- (1) $\text{Yo}_{\mathcal{C}} := \text{Map}_{\mathcal{C}}(-, c)$ is $\mathbb{L}$ -local for all $c \in \mathcal{C}$.
- (2) $\text{Yo}_{\mathcal{C}} \in \mathcal{N}$ for all $c \in \mathcal{C}$ and $\mathbb{L}(N)$ is representable for all $N \in \mathcal{N}$.

Example 1.3 A compatible necklace category for a pair $(\mathcal{C}, \mathbb{L})$ as in Definition 1.2 exists if and only if the first condition holds. In this case, the minimal possible necklace category is given by the representable presheaves $\mathcal{N}_{\min} := \text{Yo}(\mathcal{C}) \subseteq \text{PSh}(\mathcal{C})$ and the maximal choice is given by $\mathcal{N}_{\max} := \mathbb{L}^{-1}(\mathcal{C}) \subseteq \text{PSh}(\mathcal{C})$, namely all $X \in \text{PSh}(\mathcal{C})$ such that $\mathbb{L}(X) \in \text{Yo}(\mathcal{C})$. The full subcategory $\mathcal{N}_{\text{sub}} \subseteq \text{PSh}(\mathcal{C})$ spanned by all subobjects $A \subseteq \text{Yo}_{\mathcal{C}}$ such that $\mathbb{L}(A) \simeq \text{Yo}_{\mathcal{C}}$ is another possible choice.

Given a necklace context $(\mathcal{C}, \mathbb{L}, \mathcal{N})$, the Yoneda embedding $\text{Yo} : \mathcal{C} \hookrightarrow \text{PSh}(\mathcal{C})$ lands in $\mathcal{N} \subseteq \text{PSh}(\mathcal{C})$ and thus gives rise to an adjunction

$$\ell := \mathbb{L}|_{\mathcal{N}} : \mathcal{N} \xrightleftharpoons[\perp]{\top} \mathcal{C} : \text{Yo} =: i.$$

Passing to presheaves we obtain a quadruple adjunction

$$\begin{array}{ccc} & \ell_! & \\ \swarrow & \perp & \searrow \\ \text{PSh}(\mathcal{C}) & i_! = \ell^* & \text{PSh}(\mathcal{N}) \\ \nwarrow & \perp & \nearrow \\ & i^* = \ell_* & \\ \swarrow & \perp & \searrow \\ & i_* & \end{array}$$

Lemma 1.4 *The natural transformation $\beta: i^* \rightarrow \ell_!$ defined by*

$$(\beta: i^* \xrightarrow{i^* \circ u} i^* \circ (\ell^* \circ \ell_!) \simeq (\ell \circ i)^* \circ \ell_! \simeq \ell_!) \in \text{Fun}(\text{PSh}(\mathcal{N}), \text{PSh}(\mathcal{C}))$$

is an $\mathbb{L}$ -local equivalence.

Proof The source and target of $\mathbb{L}(\beta): \mathbb{L}i^* \rightarrow \mathbb{L}\ell_!$ are both left adjoints, when thought of as functors $\text{PSh}(\mathcal{N}) \rightarrow \text{PSh}_{\mathbb{L}\text{-loc}}(\mathcal{C})$, so it suffices to check that $i^*(u): i^* \rightarrow i^*(\ell^* \ell_!)$ evaluates to an $\mathbb{L}$ -local equivalence at representable presheaves. To see this, note that the adjunction $\ell_! \dashv \ell^*$ agrees with $\ell \dashv i$ on representables and thus $u|_{\mathcal{N}}$ is the unit $\text{Id}_{\mathcal{N}} \rightarrow i\ell = \mathbb{L}$. Finally, we apply $i^*: \mathcal{N} \subset \text{PSh}(\mathcal{N}) \rightarrow \text{PSh}(\mathcal{C})$, which is simply the inclusion $\mathcal{N} \subset \text{PSh}(\mathcal{C})$. Therefore the restriction of $i^*(u)$ to representables is the canonical map $N \rightarrow \mathbb{L}(N)$ for all $N \in \mathcal{N}$. $\square$

Definition 1.5 Given a necklace category $(\mathcal{C}, \mathbb{L}, \mathcal{N})$ we define

$$Q_{\mathcal{N}}: \text{PSh}(\mathcal{C}) \xrightarrow{i_*} \text{PSh}(\mathcal{N}) \xrightarrow{\ell_!} \text{PSh}(\mathcal{C}).$$

This functor receives a canonical natural transformation from the identity

$$\lambda: \text{Id}_{\text{PSh}(\mathcal{C})} \xleftarrow{\simeq} i^* \circ i_* \xrightarrow{\beta \circ i_*} \ell_! \circ i_* = Q_{\mathcal{N}}.$$

Remark 1.6 The functor $i_*: \text{PSh}(\mathcal{C}) \rightarrow \text{PSh}(\mathcal{N})$ may be computed as $(i_* X)(N) \simeq \text{Map}_{\text{PSh}(\mathcal{C})}(N, X)$. By the pointwise formula for left Kan extensions we thus have

$$Q_{\mathcal{N}}(X)(c) = \text{colim}_{(N, \ell(N) \leftarrow c) \in (\mathcal{N} \times_{\mathcal{C}} \mathcal{C}_c)^{\text{op}}} \text{Map}_{\text{PSh}(\mathcal{C})}(N, X).$$

By Lemma 1.4, $\lambda: \text{id} \rightarrow Q_{\mathcal{N}}(X)$ is $\mathbb{L}$ -local and thus the unit transformation $\text{id} \rightarrow \mathbb{L}$ factors through λ . The resulting natural transformation $Q_{\mathcal{N}} \rightarrow \mathbb{L}$ is then $\mathbb{L}$ -local by cancellation. We thus conclude:

Corollary 1.7 *There exists a canonical $\mathbb{L}$ -local natural transformation $Q_{\mathcal{N}} \rightarrow \mathbb{L}$ such that for any $X \in \text{PSh}(\mathcal{C})$ the map $Q_{\mathcal{N}}(X) \rightarrow \mathbb{L}(X)$ is an equivalence if and only if $Q_{\mathcal{N}}(X)$ is $\mathbb{L}$ -local.*

1.2 Segalification

We now specialize to the setting of Segal spaces, where the localization $\mathbb{L}_S$ is defined as the left adjoint to the full inclusion

$$\text{Seg}_{\Delta^{\text{op}}}(\mathcal{S}) \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \mathcal{S})$$

of those simplicial spaces $X_{\bullet}$ for which the map $X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an equivalence. We will choose a necklace category and show that $Q_{\mathcal{N}} \simeq \mathbb{L}_S$.

Remark 1.8 The formula we will arrive at for $\mathbb{L}_S$ is closely related to the work of Dugger–Spivak [9], who construct a functor from the category of simplicial sets to the category of simplicial categories,

$$\mathfrak{C}^{\text{nec}}: \text{sSet} \rightarrow \text{sCat},$$

which they show to be weakly equivalent to the left adjoint $\mathfrak{C}$ of the coherent nerve. This gives a formula for the mapping spaces in $\mathfrak{C}(Z_\bullet)$ (as a colimit over the necklace category) when $Z_\bullet$ is a simplicial set in the Joyal model structure. The case of a simplicial space follows by using the left Quillen equivalence $t_1: \text{ssSet} \rightarrow \text{sSet}$ constructed by Joyal–Tierney [18].

Segal spaces Let us recall the category of necklaces, introduced by Dugger and Spivak [9].

Definition 1.9 The *concatenation* $A \vee B$ of two bipoointed simplicial sets $(A, a_{\min}, a_{\max})$ and $(B, b_{\min}, b_{\max})$ is defined as the pushout

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{b_{\min}} & B \\ a_{\max} \downarrow & \ulcorner & \downarrow \\ A & \longrightarrow & A \vee B \end{array}$$

which we point as $(A \vee B, a_{\min}, b_{\max})$. This defines a (nonsymmetric) monoidal structure on the category of bipoointed simplicial sets.

Definition 1.10 A *necklace* is a bipoointed simplicial set obtained by concatenating simplices $(\Delta^n, 0, n)$, ie it is of the form $N = \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$. We let Nec denote the category whose objects are necklaces and whose morphisms are maps of bipoointed simplicial sets.

While the category Nec will play the central role in the Segalification formula, we will need a slightly bigger category to set up the necklace context.

Definition 1.11 Let $\mathcal{N} \subset \text{PSh}(\Delta)$ denote the essential image of the faithful functor $\text{Nec} \rightarrow \text{PSh}(\Delta)$.

Lemma 1.12 Segalification $\mathbb{L}_S: \text{PSh}(\Delta) \rightarrow \text{Seg}_{\Delta^{\text{op}}}(\mathcal{S})$ restricts to a functor $\mathbb{L}_S|_{\mathcal{N}}: \mathcal{N} \rightarrow \Delta$. In particular, the triple $(\Delta, \mathbb{L}_S, \mathcal{N})$ is a necklace context.

Proof We will show by induction on n that the inclusion $\Delta^{\{0, \dots, n_1\}} \vee \dots \vee \Delta^{\{n_k, \dots, n\}} \hookrightarrow \Delta^n$ is a Segal equivalence, thereby proving the claim. Consider the nested inclusion

$$\Delta^{\{0, 1\}} \vee \dots \vee \Delta^{\{n-1, n\}} \hookrightarrow \Delta^{\{0, \dots, n_1\}} \vee \dots \vee \Delta^{\{n_k, \dots, n\}} \hookrightarrow \Delta^n.$$

The composite is a Segal equivalence by definition, and since $\mathbb{L}$ preserves colimits and $n_{j+1} - n_j < n$, the first map is a Segal equivalence by the induction hypothesis. The second map is therefore a Segal equivalence by cancellation. $\square$

Remark 1.13 The map $N \rightarrow \mathbb{L}_S(N) = \Delta^n$ is a monomorphism for each necklace N . In particular, for any two necklaces N, M , the map

$$\text{Map}_{\mathcal{N}}(N, M) \rightarrow \text{Map}_{\text{PSh}(\Delta)}(\mathbb{L}_S(N), \mathbb{L}_S(M)) \simeq \text{Map}_{\Delta}([n], [m])$$

is a monomorphism, ie $\mathbb{L}: \mathcal{N} \rightarrow \Delta$ is faithful.

The Segal condition Since $(\Delta, \mathbb{L}_S, \mathcal{N})$ is a necklace context we have by Lemma 1.12 a functor

$$Q: \text{PSh}(\Delta) \xrightarrow{i_*} \text{PSh}(\mathcal{N}) \xrightarrow{\ell_!} \text{PSh}(\Delta).$$

By Corollary 1.7 this comes with an $\mathbb{L}_S$ -local natural transformation $Q \rightarrow \mathbb{L}_S$. We may compute the functor $Q(-)$ using Remark 1.6 as

$$Q(X)_n = \text{colim}_{(N, I(N) \leftarrow [n]) \in (\mathcal{N} \times_{\Delta} \Delta_{[n]})^{\text{op}}} \text{Map}_{\text{PSh}(\Delta)}(N, X).$$

Below we show that $Q(X)$ is always a Segal space, and thus by Corollary 1.7 that $Q \simeq \mathbb{L}_S$.

Definition 1.14 For any necklace N we let $\iota_N: \Delta^1 \rightarrow \mathbb{L}_S(N)$ denote the unique map that preserves the extrema. Given $[n] \in \Delta$ we define the functor

$$J: \prod_{i=1}^n \text{Nec} \hookrightarrow \mathcal{N} \times_{\Delta} \Delta_{[n]}/,$$

by joining necklaces at their endpoints

$$(M_1, \dots, M_n) \mapsto (M_1 \vee \dots \vee M_n, [n]) \xrightarrow{\mathbb{L}_S(\iota_1 \vee \dots \vee \iota_n)} \mathbb{L}_S(\mathbb{L}_S(M_1) \vee \dots \vee \mathbb{L}_S(M_n)) \simeq \mathbb{L}_S(M_1 \vee \dots \vee M_n).$$

Lemma 1.15 *The functor J is fully faithful and admits a right adjoint. In particular, it is initial.*

Proof Fully-faithfulness follows by unwinding definitions. We claim that a right adjoint to J is given by

$$J^R: (N, \alpha: [n] \rightarrow \mathbb{L}_S(N)) \longmapsto (N_{\alpha(0), \alpha(1)}, \dots, N_{\alpha(n-1), \alpha(n)}),$$

where $N_{\alpha(j), \alpha(j+1)} := N \cap \Delta^{\{\alpha(j), \dots, \alpha(j+1)\}}$. To see this, note that a tuple of necklace morphisms

$$(M_1, \dots, M_n) \rightarrow (N_{\alpha(0), \alpha(1)}, \dots, N_{\alpha(n-1), \alpha(n)}) = J^R(N, \alpha)$$

is equivalent to a morphism $M_1 \vee \dots \vee M_n \rightarrow N_{\alpha(0), \alpha(1)} \vee \dots \vee N_{\alpha(n-1), \alpha(n)} \subset N$ such that M_i lands in $N_{\alpha(i-1), \alpha(i)}$. These can be identified with morphisms $J(M_1, \dots, M_n) \rightarrow (N, \alpha)$ in $\mathcal{N} \times_{\Delta} \Delta_{[n]}/$ and thus J^R is indeed right adjoint to J . $\square$

Observation 1.16 The finality in Lemma 1.15 implies that $Q(X)_n$ may be computed as

$$\begin{aligned} Q(X)_n &\simeq \text{colim}_{(M_1, \dots, M_n) \in (\text{Nec}^{\text{op}})^n} \text{Map}_{\text{PSh}(\Delta)}(M_1 \vee \dots \vee M_n, X) \\ &\simeq \text{colim}_{(M_1, \dots, M_n) \in (\text{Nec}^{\text{op}})^n} \text{Map}_{\text{PSh}(\Delta)}(M_1, X) \times_{X_0} \dots \times_{X_0} \text{Map}_{\text{PSh}(\Delta)}(M_n, X). \end{aligned}$$

In particular for $n = 0$ we just get $Q(X)_0 = X_0$. While this is a simplification of the general formula from Remark 1.6, it has the downside that the functoriality in $[n]$ is not clear in general. However, we can still see the functoriality in inert maps $\varphi: [m] \rightarrow [n]$, as it is simply given by restricting to the M_i that correspond to the image of φ . This functoriality will suffice to check the Segal condition.

Proposition 1.17 *For any simplicial space $X_{\bullet}$ the simplicial space $Q(X)_{\bullet}$ is a Segal space.*

Proof Consider the following diagram:

$$\begin{array}{ccc}
 \operatorname{colim}_{(M_1, \dots, M_n) \in (\operatorname{Nec}^{\operatorname{op}})^n} \operatorname{Map}(M_1, X) \times_{X_0} \cdots \times_{X_0} \operatorname{Map}(M_n, X) & \longrightarrow & Q(X)_n \\
 \downarrow & & \downarrow \\
 \operatorname{colim}_{M_1 \in \operatorname{Nec}^{\operatorname{op}}} \operatorname{Map}(M_1, X) \times_{X_0} \cdots \times_{X_0} \operatorname{colim}_{M_n \in \operatorname{Nec}^{\operatorname{op}}} \operatorname{Map}(M_n, X) & \longrightarrow & Q(X)_1 \times_{Q(X)_0} \cdots \times_{Q(X)_0} Q(X)_1
 \end{array}$$

The horizontal maps are equivalences by Lemma 1.15 and Observation 1.16. The left vertical map is an equivalence since the cartesian product in $\mathcal{S}_{/X_0}$ preserves colimits in each variable. $\square$

1.3 Variations on the Segalification formula

A formula for mapping spaces Given a Segal space $X \in \operatorname{Seg}_{\Delta^{\operatorname{op}}}(\mathcal{S})$ the mapping spaces in the associated ∞ -category $\mathbb{C}(X)$ may be computed as

$$\operatorname{Map}_{\mathbb{C}(X)}(x, y) \simeq \{x\} \times_{X_0} X_1 \times_{X_0} \{y\}$$

for any $x, y \in X_0$. Below we show how to use the results of the previous section to derive a formula for these mapping spaces when X is an arbitrary simplicial space. In the case where X is a simplicial set this recovers the formula of Dugger–Spivak [9], which inspired our Segalification formula.

Lemma 1.18 For any simplicial space $X \in \operatorname{PSh}(\Delta)$ and $x, y \in X$ there is a canonical equivalence

$$|\operatorname{Nec}_{/(X, x, y)}| \simeq \operatorname{Map}_{\mathbb{C}(X)}(x, y),$$

where $\operatorname{Nec}_{/(X, x, y)} \subseteq (\operatorname{PSh}(\Delta)_{\Delta^0 \sqcup \Delta^0 /})_{/(X, x, y)}$ denotes the full subcategory spanned by necklaces.

Proof Interpreting Nec as a full subcategory of $\operatorname{PSh}(\Delta)_{\Delta^0 \sqcup \Delta^0 /}$ by recording the minimal and maximal vertex, we can fit $\operatorname{Nec}_{/(X, x, y)}$ into a cartesian square:

$$\begin{array}{ccc}
 \operatorname{Nec}_{/(X, x, y)} & \longrightarrow & \operatorname{Nec} \times_{\operatorname{PSh}(\Delta)} \operatorname{PSh}(\Delta)_{/X} \\
 \downarrow & & \downarrow \\
 \{(x, y)\} & \longrightarrow & X_0 \times X_0
 \end{array}$$

The top right corner is a right fibration over Nec corresponding to the presheaf

$$\operatorname{Map}_{\operatorname{PSh}(\Delta)}(-, X): \operatorname{Nec}^{\operatorname{op}} \rightarrow \mathcal{S}.$$

The weak homotopy type of the top right corner is thus the colimit of this functor, which is precisely the definition of $Q(X)_1$. While the functor $|-|: \operatorname{Cat}_{\infty} \rightarrow \mathcal{S}$ does not generally preserve pullbacks, it does preserve those cartesian squares where the bottom arrow is a map of spaces (because $\Delta^{\operatorname{op}}$ -colimits in $\mathcal{S}$ are stable under base change). For the square at hand we obtain

$$|\operatorname{Nec}_{/(X, x, y)}| \simeq \{(x, y)\} \times_{X_0 \times X_0} Q(X)_1 \simeq \operatorname{Map}_{\mathbb{C}(X)}(x, y),$$

where the second equivalence holds since the Rezk-completion of $Q(X) \simeq \mathbb{L}_{\mathcal{S}}(X)$ is the nerve of $\mathbb{C}(X)$. $\square$

Remark 1.19 Given three points $x, y, z \in X$ the monoidal structure $\vee$ on Nec yields a functor

$$\vee : \text{Nec}/(X, x, y) \times \text{Nec}/(X, y, z) \rightarrow \text{Nec}/(X, x, z).$$

On weak homotopy types this yields the composition $\text{Map}_{\mathbb{C}(X)}(x, y) \times \text{Map}_{\mathbb{C}(X)}(y, z) \rightarrow \text{Map}_{\mathbb{C}(X)}(x, z)$ in $\mathbb{C}(X)$, as in [9, Equation (1.2)]. This can be seen by an argument similar to Lemma 1.18 using the necklace formula for $Q_2 X$.

1-categories Similar to how Δ^{op} -colimits in 1-categories can be computed as reflexive coequalizers, Nec^{op} colimits in a 1-category can be reduced to certain “thin” necklaces.

Definition 1.20 We say that a necklace $N = \Delta^{n_1} \vee \cdots \vee \Delta^{n_r} \in \text{Nec}$ is *thin* if $\sum_i n_i \leq r + 1$ and $n_i \geq 1$, in other words if it consists of 1-simplices and at most one 2-simplex. If N consists only of 1-simplices, we say that it is *very thin*. Let $\text{Nec}_{\text{thin}} \subset \text{Nec}$ denote the full subcategory of thin necklaces.

Lemma 1.21 The full inclusion $\text{Nec}_{\text{thin}}^{\text{op}} \hookrightarrow \text{Nec}^{\text{op}}$ is 1-final, that is, for any functor $\text{Nec}^{\text{op}} \rightarrow \mathcal{C}$ to a 1-category $\mathcal{C}$ the colimit may equivalently be computed over $\text{Nec}_{\text{thin}}^{\text{op}}$.

Proof We need to show that for any necklace $N = \Delta^{n_1} \vee \cdots \vee \Delta^{n_r} \in \text{Nec}$ the slice category $\text{Nec}_{\text{thin}/N}$ is *connected*. We enumerate the vertices of N in their canonical order as $0, \dots, n = \sum_i n_i$. A very thin necklace over N (a map $\Delta^1 \vee \cdots \vee \Delta^1 \rightarrow N$) may equivalently be encoded as a nondecreasing path $0 = a_0 \leq \cdots \leq a_k = n$ in $[n]$. These paths are subject to the condition that we never have strict inequalities $a_l < \sum_{i=1}^s n_i < a_{l+1}$ for any s and l . Suppose that $p = (0 = a_0 \leq \cdots \leq a_k = n)$ is such a path and s is such that $p' = (0 = a_0 \leq \cdots \leq \hat{a}_s \leq \cdots \leq a_k = n)$ is still an admissible path. Then there is a thin necklace M with a 2-simplex $(a_{s-1} \leq a_s \leq a_{s+1})$ that contains both of these paths. In particular, the paths are connected through a zigzag $p \rightarrow M \leftarrow p'$ as objects of $\text{Nec}_{\text{thin}/N}$. Proceeding by removing a vertex whenever possible, we see that every very thin necklace over N is connected in $\text{Nec}_{\text{thin}/N}$ to a very thin necklace that corresponds to a minimal path in N . But there is only one path in N that is minimal with respect to removing vertices, namely $(0 \leq n_1 \leq \cdots \leq \sum_{i=1}^{r-1} n_i \leq n)$. Therefore all the very thin objects in $\text{Nec}_{\text{thin}/N}$ are connected, and thus the category is connected as every (thin) necklace contains a very thin necklace. $\square$

Example 1.22 Suppose that $X_\bullet$ is a simplicial space and we want to compute the homotopy category $h_1(\mathbb{C}(X))$. For simplicity, let us assume that X_n is discrete for all n .³ Then the set of morphisms in $h_1(\mathbb{C}(X))$ is exactly $\pi_0(\mathbb{L}_S(X)_1)$ and we may compute it as the colimit

$$\text{Mor}(h_1(\mathbb{C}(X))) \cong \text{colim}_{N \in \text{Nec}^{\text{op}}} \text{Map}(N, X) \cong \text{colim}_{\Delta^{n_1} \vee \cdots \vee \Delta^{n_r} \in \text{Nec}^{\text{op}}} X(\Delta^{n_1}) \times_{X(\Delta^0)} \cdots \times_{X(\Delta^0)} X(\Delta^{n_r})$$

in the 1-category of sets. By Lemma 1.21 it suffices to take the colimit over $\text{Nec}_{\text{thin}}^{\text{op}}$. The very thin necklaces are 0-final, so the colimit may be expressed as a coproduct over the very thin necklaces modulo

³This is not a very restrictive assumption. Starting with a general simplicial space $Y_\bullet$, we may base change it along a π_0 -surjective map $Z_0 \rightarrow Y_0$ to get a simplicial space $Z_n = (Z_0)^{n+1} \times_{Y_0^{n+1}} Y_n$ such that the resulting functor $\mathbb{C}(Z_\bullet) \rightarrow \mathbb{C}(Y_\bullet)$ will be an equivalence. We may choose Z_0 to be discrete and define $X_n := \pi_0(Z_n)$. Then $\mathbb{C}(Z_\bullet) \rightarrow \mathbb{C}(X_\bullet)$ induces an equivalence on homotopy categories.

an equivalence relation. This leads to the formula

$$\mathrm{Mor}(h_1(\mathbb{C}(X))) \cong \left(\coprod_{n \geq 0} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \right) / \sim,$$

where the equivalence relation is generated by $(f_1, \dots, f_n) \sim (f_1, \dots, f_{i-1}, g, f_{i+2}, \dots, f_n)$ whenever there is a 2-simplex in X witnessing $f_{i+1} \circ f_i = g$, and $(f_1, \dots, f_n) \sim (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$ whenever f_i is a degenerate 1-simplex. This recovers the classical formula for the homotopy category of a simplicial set: namely, it is the free category on the edges of X modulo the relations generated by the 2-simplices and the degenerate 1-simplices.

Segalification in other categories We establish criteria on a presentable ∞ -category which guarantee that Segalification is given by the necklace formula. This is summarized by the following result, which we prove in the remainder of this section.

Proposition 1.23 *Let $\mathcal{V}$ be a presentable ∞ -category in which sifted colimits are stable under base change. Then the left adjoint to the inclusion $\mathrm{Seg}_{\Delta^{\mathrm{op}}}(\mathcal{V}) \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$ is given by the necklace formula:*

$$\mathbb{L}_S(X)_1 \simeq \mathrm{colim}_{N \in \mathrm{Nec}^{\mathrm{op}}} (i_* X)(N) \simeq \mathrm{colim}_{\Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \in \mathrm{Nec}^{\mathrm{op}}} X_{n_1} \times_{X_0} \cdots \times_{X_0} X_{n_k},$$

where i_* denotes the right Kan extension $i_*: \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathrm{Fun}(\mathcal{N}^{\mathrm{op}}, \mathcal{V})$.

Recall that if $\mathfrak{X}$ is an ∞ -topos then *all* colimits in $\mathfrak{X}$ are stable under base change (they are “universal”) [19, Proposition 6.1.3.19]. In particular, Proposition 1.23 applies to ∞ -topoi. A wider variety of examples is provided by passing to algebras over ∞ -operads.

Example 1.24 Let $\mathcal{V}$ be a presentably symmetric monoidal ∞ -category⁴ and $\mathcal{O}$ be an ∞ -operad. Then the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \rightarrow \mathrm{Fun}(\mathrm{col}(\mathcal{O}), \mathcal{V})$, which only remembers the object assigned to each colour $c \in \mathrm{col}(\mathcal{O})$, preserves and creates both limits and sifted colimits [20, Corollary 3.2.2.4 and Proposition 3.2.3.1]. Consequently, if sifted colimits in $\mathcal{V}$ are stable under base change, then the same holds for $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$.

Lemma 1.25 *Proposition 1.23 holds if we assume that $\mathrm{Nec}^{\mathrm{op}}$ -colimits in $\mathcal{V}$ are stable under base change.*

Proof The left adjoint $\mathbb{L}_S$ exists by the adjoint functor theorem. Since $\mathcal{V}$ is presentable we may find a small ∞ -category $\mathcal{E}$ and a fully faithful right adjoint $I: \mathcal{V} \hookrightarrow \mathrm{PSh}(\mathcal{E})$. We denote the resulting adjunction on presheaf categories by

$$I^{\Delta}: \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V}) \rightleftarrows \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{PSh}(\mathcal{E})) : L^{\Delta}.$$

We now define an endofunctor $Q^{\mathcal{V}} := L^{\Delta} \circ Q \circ I^{\Delta}: \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V}) \rightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{V})$, where Q is the endofunctor on $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{PSh}(\mathcal{E})) \simeq \mathrm{Fun}(\mathcal{E}, \mathrm{PSh}(\Delta))$, pointwise given by the usual formula (see Observation 1.16). This $Q^{\mathcal{V}}$ receives a natural transformation

$$\lambda^{\mathcal{V}}: L^{\Delta} \circ I^{\Delta} \xrightarrow{L^{\Delta} \circ \lambda \circ I^{\Delta}} L^{\Delta} \circ Q \circ I^{\Delta} = Q^{\mathcal{V}}$$

⁴In fact, it suffices to ask that the monoidal structure is compatible with sifted colimits.

coming from $\lambda: \text{id} \rightarrow Q$. (Note that the source of λ is $L^\Delta \circ I^\Delta \simeq \text{id}_{\text{Fun}(\Delta^{\text{op}}, \mathcal{V})}$.) This transformation is a Segal equivalence. Indeed, if $X, Y: \Delta^{\text{op}} \rightarrow \mathcal{V}$ and Y is Segal, then in the commutative square

$$\begin{array}{ccc} \text{Map}_{\text{Fun}(\Delta^{\text{op}}, \mathcal{V})}(Q^\vee X, Y) & \xrightarrow{(-) \circ \lambda_X^\vee} & \text{Map}_{\text{Fun}(\Delta^{\text{op}}, \mathcal{V})}((L^\Delta \circ I^\Delta)(X), Y) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Map}_{\text{Fun}(\Delta^{\text{op}}, \text{PSh}(\mathcal{E}))}((Q \circ I^\Delta)(X), I^\Delta(Y)) & \xrightarrow{(-) \circ \lambda_{I^\Delta(X)}} & \text{Map}_{\text{Fun}(\Delta^{\text{op}}, \text{PSh}(\mathcal{E}))}(I^\Delta(X), I^\Delta(Y)) \end{array}$$

the bottom map is an equivalence since $I^\Delta(Y)$ is Segal and thus so is the top map.

It remains to show that $Q^\vee(X)$ is Segal for all $X: \Delta^{\text{op}} \rightarrow \mathcal{V}$. This follows from the same proof as Proposition 1.17 by using that Nec^{op} -colimits are stable under base change. $\square$

In principle, it might be difficult to tell whether Nec^{op} -shaped colimits are stable under base change in a given ∞ -category, but fortunately Nec^{op} is a sifted category, colimits over which are well understood. We will deduce this from the following fact, to which it is intimately linked:

Lemma 1.26 *The Segalification functor $\mathbb{L}: \text{PSh}(\Delta) \rightarrow \text{PSh}(\Delta)$ preserves products.*

Proof The two functors

$$\text{PSh}(\Delta) \times \text{PSh}(\Delta) \rightarrow \text{Cat}_\infty$$

given by $(X, Y) \mapsto \mathbb{L}(X \times Y)$ and $\mathbb{L}(X) \times \mathbb{L}(Y)$ both preserve colimits in both variables. Therefore it suffices to check that the natural transformation between them is an equivalence on (Δ^n, Δ^m) . But in this case it is easy to check because Δ^n , Δ^m and $\Delta^n \times \Delta^m$ are all Segal spaces. $\square$

Lemma 1.27 *The category Nec^{op} is sifted.*

Proof We need to show that the diagonal functor $\Delta: \text{Nec}^{\text{op}} \rightarrow \text{Nec}^{\text{op}} \times \text{Nec}^{\text{op}}$ is final. Equivalently, we need that for all $A, B \in \text{Nec}$ the slice $\text{Nec}_{\text{Nec}^2} \text{Nec}_{/(A, B)}^2$ is weakly contractible. This category is equivalent to the full subcategory $\text{Nec}_{/A \times B} \subseteq (\text{PSh}(\Delta)_{\Delta^0 \sqcup \Delta^0})_{/A \times B}$ spanned by necklaces, where the product $A \times B$ is taken in the ∞ -category $\text{PSh}(\Delta)_{\Delta^0 \sqcup \Delta^0}$ of bipointed simplicial spaces. By Lemma 1.18 the weak homotopy type of this category computes the mapping space

$$|\text{Nec}_{/(A \times B, (a_{\min}, b_{\min}), (a_{\max}, b_{\max}))}| \simeq \text{Map}_{\mathbb{C}(A \times B)}((a_{\min}, b_{\min}), (a_{\max}, b_{\max})).$$

Since $\mathbb{C}(-)$ commutes with products by Lemma 1.26, we compute $\mathbb{C}(A \times B) \simeq \mathbb{C}(A) \times \mathbb{C}(B) = [n] \times [m]$. In particular, we see that the mapping space $\text{Map}_{[n] \times [m]}((0, 0), (n, m))$ is contractible. $\square$

2 Applications

2.1 Segalification and right fibrations

Throughout this section we fix a presentable ∞ -category $\mathcal{V}$ and a factorization system $(\mathcal{V}^L, \mathcal{V}^R)$.

Definition 2.1 We say that $X: \Delta^{\text{op}} \rightarrow \mathcal{V}$ is *right- $\mathcal{V}^R$ -fibred* if $d_0: X_n \rightarrow X_{n-1}$ is in $\mathcal{V}^R$ for all $n \geq 1$.

Observation 2.2 A Segal object $X: \Delta^{\text{op}} \rightarrow \mathcal{V}$ is right- $\mathcal{V}^R$ -fibered if and only if $d_0: X_1 \rightarrow X_0$ is in $\mathcal{V}^R$. Indeed, morphisms in $\mathcal{V}^R$ are closed under pullbacks in the arrow category [19, Proposition 5.2.8.6(8), page 369] and when X is Segal we can write $d_0: X_n \rightarrow X_{n-1}$ as a pullback in the arrow category of the following cospan:

$$\begin{array}{ccccc} X_{n-1} & \xrightarrow{d_1 \circ \dots \circ d_n} & X_0 & \xleftarrow{d_0} & X_1 \\ \downarrow = & & \downarrow = & & \downarrow d_0 \\ X_{n-1} & \xrightarrow{d_1 \circ \dots \circ d_n} & X_0 & \xleftarrow{=} & X_0 \end{array}$$

Under suitable assumptions the necklace formula can be used to show that Segalification preserves right- $\mathcal{V}^R$ -fibered objects.

Proposition 2.3 Suppose $\mathcal{V}$ and $(\mathcal{V}^L, \mathcal{V}^R)$ are such that:

- (1) Sifted colimits in $\mathcal{V}$ are stable under base change.
- (2) The full subcategory $\mathcal{V}_{/X}^{(R)} \subseteq \mathcal{V}_{/X}$ on those $Y \rightarrow X$ that are in $\mathcal{V}^R$ is closed under sifted colimits for all $X \in \mathcal{V}$.

Then Segalification $\mathbb{L}_S: \text{Fun}(\Delta^{\text{op}}, \mathcal{V}) \rightarrow \text{Seg}_{\Delta^{\text{op}}}(\mathcal{V})$ preserves right- $\mathcal{V}^R$ -fibered objects.

Proof By Observation 2.2 we only need to show that if $X: \Delta^{\text{op}} \rightarrow \mathcal{V}$ is right- $\mathcal{V}^R$ -fibered, then $d_0: \mathbb{L}(X)_1 \rightarrow \mathbb{L}(X)_0$ is in $\mathcal{V}^R$. We claim that for every necklace $N = \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$ the map $(i_* X)(N) \simeq X_{n_1} \times_{X_0} \dots \times_{X_0} X_{n_k} \rightarrow X_0$ induced by the inclusion of the terminal vertex $\Delta^0 \rightarrow N$ is in $\mathcal{V}^R$. Indeed, when N is a simplex this holds by assumption and the general case follows by taking pullbacks since morphisms in $\mathcal{V}^R$ are closed under base change and composition.

Using the necklace formula (see Proposition 1.23), we can write $d_0: \mathbb{L}(X)_1 \rightarrow \mathbb{L}(X)_0$ as

$$\mathbb{L}(X)_1 \simeq \text{colim}_{N \in \text{Nec}^{\text{op}}} (i_* X)(N) \rightarrow X_0 = \mathbb{L}(X)_0,$$

a colimit in $\mathcal{V}_{/X_0}$ of a diagram indexed by Nec^{op} of morphisms $(i_* X)(N) \rightarrow X_0$ that lie in $\mathcal{V}_{/X_0}^{(R)}$. Since Nec^{op} is sifted (Lemma 1.27) and $\mathcal{V}_{/X_0}^{(R)} \subseteq \mathcal{V}_{/X_0}$ is closed under sifted colimits, the colimit lies in $\mathcal{V}_{/X_0}^{(R)}$. $\square$

Right fibrations of simplicial spaces We shall now apply Proposition 2.3 to right fibrations of simplicial spaces in the sense of Rezk, whose definition we briefly recall.

Definition 2.4 A map of simplicial spaces $p: X \rightarrow Y$ is called a *right fibration* if the square

$$\begin{array}{ccc} X_n & \xrightarrow{d_0} & X_{n-1} \\ p \downarrow & & \downarrow p \\ Y_n & \xrightarrow{d_0} & Y_{n-1} \end{array}$$

is cartesian for all $n \geq 1$.

The Segalification formula implies the following:

Corollary 2.5 If $p: X \rightarrow Y$ is a right fibration, then so is the Segalification $\mathbb{L}(p): \mathbb{L}(X) \rightarrow \mathbb{L}(Y)$.

Proof The target map $t: \text{Ar}(\mathcal{S}) \rightarrow \mathcal{S}$ is a cartesian fibration and thus, by the opposite of [19, Example 5.2.8.15, page 370], we have a factorization system on $\text{Ar}(\mathcal{S})$ whose right part $\text{Ar}(\mathcal{S})^{\text{cart}} \subseteq \text{Ar}(\mathcal{S})$ consists of the cartesian edges, equivalently pullback squares. (The left part consists of morphisms which induce equivalence on the target.) Note that $(p: X \rightarrow Y) \in \text{Ar}(\text{PSh}(\Delta)) = \text{Fun}(\Delta^{\text{op}}, \text{Ar}(\mathcal{S}))$ is right- $\text{Ar}(\mathcal{S})^{\text{cart}}$ -fibered if and only if it is a right fibration; hence it suffices to verify the conditions of Proposition 2.3 for $\mathcal{V} = \text{Ar}(\mathcal{S})$ equipped with the aforementioned factorization system. The first condition holds since colimits in $\text{Ar}(\mathcal{S})$ are stable under base change. The second condition holds since colimits in $\mathcal{S}$ are stable under base change. $\square$

Remark 2.6 Examination of the proof of Corollary 2.5 shows that the same result holds if $\mathcal{S}$ is replaced with any presentable ∞ -category $\mathcal{V}$ in which sifted colimits are stable under base change.

Recall that the nerve $N_{\bullet}: \text{Cat}_{\infty} \rightarrow \text{PSh}(\Delta)$ is fully faithful and its essential image is precisely the complete Segal spaces. We write $\mathbb{C}(-): \text{PSh}(\Delta) \rightarrow \text{Cat}_{\infty}$ for the left adjoint of $N_{\bullet}$. We let $\mathbb{L}_{\text{CS}}: \text{PSh}(\Delta) \rightarrow \text{PSh}(\Delta)$ denote the localization onto the complete Segal spaces. With this notation we have for any $X \in \text{PSh}(\Delta)$ a canonical equivalence $N_{\bullet}\mathbb{C}(X) \simeq \mathbb{L}_{\text{CS}}X$. A model-categorical proof of the following proposition was given by Rasekh [21, Theorems 4.18 and 5.1].

Proposition 2.7 *For any simplicial space X , there is an adjoint equivalence of ∞ -categories:*

$$\mathbb{L}_{\text{S}}: \text{PSh}(\Delta)_{/X}^{\text{r-fib}} \xrightarrow[\simeq]{\simeq} (\text{Seg}_{\Delta^{\text{op}}})_{/\mathbb{L}_{\text{S}}(X)}^{\text{r-fib}}: X \times_{\mathbb{L}_{\text{S}}(X)} (-).$$

Proof Combining Corollary 2.5 and Lemma 2.8, we learn that if $E \rightarrow X$ is a right fibration and X is Segal, then E is also Segal. We claim that $X \times_{\mathbb{L}_{\text{S}}(X)} (-): (\text{Seg}_{\Delta^{\text{op}}})_{/\mathbb{L}_{\text{S}}(X)}^{\text{r-fib}} = \text{PSh}(\Delta)_{/\mathbb{L}_{\text{S}}(X)}^{\text{r-fib}} \rightarrow \text{PSh}(\Delta)_{/X}^{\text{r-fib}}$ is right adjoint to the functor $\mathbb{L}_{\text{S}}: \text{PSh}(\Delta)_{/X}^{\text{r-fib}} \rightarrow (\text{Seg}_{\Delta^{\text{op}}})_{/\mathbb{L}_{\text{S}}(X)}^{\text{r-fib}}$ afforded by Corollary 2.5. Indeed, for any $(E \rightarrow X) \in \text{PSh}(\Delta)_{/X}^{\text{r-fib}}$ and $(E' \rightarrow \mathbb{L}_{\text{S}}(X)) \in (\text{Seg}_{\Delta^{\text{op}}})_{/\mathbb{L}_{\text{S}}(X)}^{\text{r-fib}}$, we have

$$\text{Map}_{/X}(E, X \times_{\mathbb{L}_{\text{S}}(X)} E') \simeq \text{Map}_{/\mathbb{L}_{\text{S}}(X)}(E, E') \simeq \text{Map}_{/\mathbb{L}_{\text{S}}(X)}(\mathbb{L}_{\text{S}}(E), E').$$

It remains to check that the unit and counit, which are given respectively by $E \rightarrow X \times_{\mathbb{L}_{\text{S}}(X)} \mathbb{L}_{\text{S}}(E)$ and $\mathbb{L}_{\text{S}}(X \times_{\mathbb{L}_{\text{S}}(X)} E') \rightarrow E'$, are equivalences. Since $\mathbb{L}_{\text{S}}$ does not affect the 0-simplices, both maps evaluate to equivalences at $[0]$. The claim now follows from Lemma 2.8. $\square$

Lemma 2.8 *Suppose $X \rightarrow Y$ and $X' \rightarrow Y$ are right fibrations and $f: X \rightarrow X'$ is a map over Y . Then f is an equivalence if and only if $f_0: X_0 \rightarrow X'_0$ is an equivalence.*

Proof Since $f: X' \rightarrow X$ is a map of right fibrations, the map $f_n: X_n \rightarrow X'_n$ can be recovered as the base change of the map $f_0: X_0 \rightarrow X'_0$ over Y_0 along $(d_0)^n: Y_n \rightarrow Y_0$. $\square$

Corollary 2.9 *For any simplicial space X , there is an adjoint equivalence of ∞ -categories:*

$$\mathbb{L}_{\text{CS}}: \text{PSh}(\Delta)_{/X}^{\text{r-fib}} \xrightarrow[\simeq]{\simeq} (\text{CSeg}_{\Delta^{\text{op}}})_{/\mathbb{L}_{\text{CS}}(X)}^{\text{r-fib}}: X \times_{\mathbb{L}_{\text{CS}}(X)} (-).$$

Proof From Corollary 2.5 and [11, Proposition A.21] we learn that right fibrations are preserved by $\mathbb{L}_S: \text{PSh}(\Delta) \rightarrow \text{Seg}_{\Delta^{\text{op}}}$ and $\mathbb{L}_C: \text{Seg}_{\Delta^{\text{op}}} \rightarrow \text{CSeg}_{\Delta^{\text{op}}}$ respectively, so their composite yields a functor

$$\mathbb{L}_{CS}: \text{PSh}(\Delta)_{/X}^{\text{r-fib}} \xrightarrow{\mathbb{L}_S} \text{PSh}(\Delta)_{/\mathbb{L}_S X}^{\text{r-fib}} \xrightarrow{\mathbb{L}_C} \text{PSh}(\Delta)_{/\mathbb{L}_{CS} X}^{\text{r-fib}}.$$

The first is an equivalence by Proposition 2.7 and the second by [11, Proposition A.22]. $\square$

Under the equivalence $N_\bullet: \text{Cat}_\infty \simeq \text{CSeg}_{\Delta^{\text{op}}}(\mathcal{S})$ the functor $\mathbb{C}(-)$ is identified with $\mathbb{L}_{CS} \simeq \mathbb{L}_C \mathbb{L}_S$ from which we learn the following:

Corollary 2.10 *The functor $\mathbb{C}: \text{PSh}(\Delta) \rightarrow \text{Cat}_\infty$ induces for any simplicial space X an equivalence*

$$\text{PSh}(\Delta)_{/X}^{\text{r-fib}} \xrightarrow[\simeq]{\mathbb{C}(-)} \text{Cat}_{\infty/\mathbb{C}(X)}^{\text{r-fib}} \simeq \text{PSh}(\mathbb{C}(X)).$$

A formula for $\mathbb{C}(-)$ Let X be a simplicial space and write $\Delta_{/X}$ for its ∞ -category of simplices, ie the codomain of the associated right fibration $p_X: \Delta_{/X} \rightarrow \Delta$. Corollary 2.10 can be used to give a formula for $\mathbb{C}(X)$ as a certain localization of $\Delta_{/X}$. To do so we will need the “last vertex map” $N_\bullet(\Delta_{/X}) \rightarrow X$ (see eg [11, Section 4]) and the “last vertex functor” $e: \Delta_{/X} \rightarrow \mathbb{C}(X)$ obtained by applying $\mathbb{C}(-)$ to it. Write $\Delta_{\max} \subseteq \Delta$ for the wide subcategory spanned by morphisms which preserve the maximal element.

Proposition 2.11 *The “last vertex functor” $e: \Delta_{/X} \rightarrow \mathbb{C}(X)$ induces an equivalence of ∞ -categories*

$$\Delta_{/X}[W_X^{-1}] \simeq \mathbb{C}(X),$$

where $W_X := p_X^{-1}(\Delta_{\max}) \subseteq \Delta_{/X}$.

Proof First we observe that every object in $\Delta_{/X}$ is connected through a zigzag of last vertex maps to a 0-simplex and thus the maps $\Delta_{/\Delta^0}[W_{\Delta^0}^{-1}] \rightarrow \Delta_{/X}[W_X^{-1}]$ induced by $\Delta^0 \rightarrow X$ are jointly essentially surjective. By Lemma 2.12 it now suffices to construct an equivalence $\text{PSh}(\Delta_{/X}[W_X^{-1}]) \simeq \text{PSh}(\mathbb{C}(X))$ naturally in X . Right fibrations over X are precisely $\Delta_{\max}$ -equifibered simplicial spaces over X in the sense of [2, Lemma 4.1.8], and thus we have $\text{PSh}(\Delta_{/X}[W_X^{-1}]) \simeq \text{PSh}(\Delta)_{/X}^{\text{r-fib}}$. Combining with Corollary 2.10 yields the desired equivalence $\text{PSh}(\Delta_{/X}[W_X^{-1}]) \simeq \text{PSh}(\mathbb{C}(X))$. $\square$

Lemma 2.12 *Let $F: \text{PSh}(\Delta) \rightarrow \text{Cat}_\infty$ be a functor such that:*

- (1) *There is an equivalence $\text{PSh}(F(-)) \simeq \text{PSh}(\mathbb{C}(-))$ of functors $\text{PSh}(\Delta) \rightarrow \text{Pr}^{\text{L}}$.*
- (2) *For all X the functors $F(\Delta^0) \rightarrow F(X)$ induced by $x: \Delta^0 \rightarrow X$ are jointly surjective.*

Then F is naturally equivalent to $\mathbb{C}(-)$. Moreover, any natural transformation $F \Rightarrow \mathbb{C}(-)$ is an equivalence.

Proof The idempotent completion $\mathcal{C}^{\text{idem}}$ of any ∞ -category $\mathcal{C}$ can be constructed as the full subcategory of atomic⁵ objects in $\text{PSh}(\mathcal{C})$ [19, Proposition 5.1.6.8]. Restricting the equivalence from (1) to atomic objects yields a natural equivalence $\alpha_X: \mathbb{C}(X)^{\text{idem}} \simeq F(X)^{\text{idem}}$ of functors $\text{PSh}(\Delta) \rightarrow \text{Cat}_\infty$. In particular we have $* \simeq \mathbb{C}(\Delta^0)^{\text{idem}} \simeq F(\Delta^0)^{\text{idem}}$.

⁵An object $c \in \mathcal{C}$ is called *atomic* (or completely compact in [19]) if the copresheaf $\text{Map}_{\mathcal{C}}(c, -)$ preserves colimits.

The joint essential image of the functors $* = \mathbb{C}(\Delta^0)^{\text{idem}} \rightarrow \mathbb{C}(X)^{\text{idem}}$ induced from the maps $\Delta^0 \rightarrow X$ is precisely $\mathbb{C}(X) \subset \mathbb{C}(X)^{\text{idem}}$, and by (2) the joint essential image of $* = F(\Delta^0)^{\text{idem}} \rightarrow F(X)^{\text{idem}}$ is $F(X) \subset F(X)^{\text{idem}}$. Since α is a natural equivalence, both it and its inverse must preserve these subcategories and thus $F(-) \simeq \mathbb{C}(-)$.

It remains to show every endomorphism of $\mathbb{C}(-)$ is an equivalence. Since $\mathbb{C}(-)$ is the left Kan extension of the inclusion $\Delta \hookrightarrow \text{Cat}_\infty$ along the Yoneda embedding $\Delta \hookrightarrow \text{PSh}(\Delta)$, it suffices to observe that Id_Δ admits no nontrivial endomorphisms. $\square$

2.2 Segalification for (∞, n) -categories

By iterating the Segalification formula one can also obtain formulas for the Segalification for (∞, n) -categories. We recall the definition of n -fold Segal spaces due to Barwick [3]. See [8, Definitions 2.2 and 2.4; 12] for a reference.

Definition 2.13 Let $X : \Delta^{\text{op}, n} \rightarrow \mathcal{S}$ be an n -fold simplicial space.

- X is called *reduced* if for every $k \geq 0$ and $m_1, \dots, m_k \in \mathbb{N}$, the $(n - k - 1)$ -fold simplicial space $X_{m_1, \dots, m_k, 0, \bullet, \dots, \bullet}$ is constant. We denote by $\text{PSh}^r(\Delta^{\times n})$ the full subcategory spanned by reduced objects.
- X is called an *n -uple Segal space* if it is Segal in each coordinate, that is, if for every $k \geq 0$ and $m_1, \dots, m_{k-1}, m_{k+1}, \dots, m_n \in \mathbb{N}$, the simplicial space $X_{m_1, \dots, m_{k-1}, \bullet, m_{k+1}, \dots, m_n}$ is a Segal space.
- X is called an *n -fold Segal space* if it is an n -tuple Segal space and reduced. We denote by $\text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}} \subset \text{PSh}^r(\Delta^{\times n})$ the full subcategory spanned by the n -fold Segal spaces.

As we briefly explained in the introduction, complete n -fold Segal spaces model (∞, n) -categories. We will not discuss the issue of completeness here, but rather our goal will be to give a formula for the Segalification of reduced n -fold simplicial spaces. For $1 \leq j \leq n$ we denote by $\mathbb{L}_j : \text{PSh}(\Delta^{\times n}) \rightarrow \text{PSh}(\Delta^{\times n})$ the Segalification functor in the j^{th} coordinate.

Lemma 2.14 Suppose that $F : K \rightarrow \text{PSh}(\Delta)$ is a diagram of simplicial spaces such that K is sifted, each $F(k)$ is a Segal space, and the diagram $F(-)_0 : K \rightarrow \mathcal{S}$ is constant. Then the colimit $\text{colim}_{k \in K} F(k)$ is a Segal space.

Proof A simplicial space X is Segal if and only if the canonical map $X_n \rightarrow X_0 \times_{(X_0 \times X_0)} (X_{n-1} \times X_1)$ is an equivalence for all $n \geq 2$. In the case of $X = \text{colim}_{k \in K} F(k)$ we therefore want show that the outside rectangle in the following diagram is cartesian:

$$\begin{array}{ccccc}
 \text{colim}_{k \in K} F(k)_n & \longrightarrow & \text{colim}_{k \in K} F(k)_{n-1} \times F(k)_1 & \xrightarrow{\simeq} & \text{colim}_{k \in K} F(k)_{n-1} \times \text{colim}_{k \in K} F(k)_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{colim}_{k \in K} F(k)_0 & \xrightarrow{\Delta} & \text{colim}_{k \in K} F(k)_0 \times F(k)_0 & \xrightarrow{\simeq} & \text{colim}_{k \in K} F(k)_0 \times \text{colim}_{k \in K} F(k)_0
 \end{array}$$

The right horizontal maps are equivalences because K is sifted and hence it suffices to consider the left square. This square is a colimit of the cartesian squares that we have because $F(k)$ is Segal for all k .

The bottom row of the squares is a constant functor in k by assumption. So it follows that the colimit of the square is still cartesian because colimits in $\mathcal{S}$ are stable under base change. As observed above, this implies that $\operatorname{colim}_{k \in K} F(k)$ is Segal as claimed. $\square$

Lemma 2.15 *Let $X_{\bullet, \dots, \bullet}$ be a reduced n -fold simplicial space, then:*

- (1) $(\mathbb{L}_j X)_{\bullet, \dots, \bullet}$ is reduced for all j .
- (2) If $X_{\bullet, \dots, \bullet}$ satisfies the Segal condition in the first $j - 1$ coordinates, then $(\mathbb{L}_j X)_{\bullet, \dots, \bullet}$ satisfies the Segal condition in the first j coordinates.

Proof Claim (1) We need to check that $(\mathbb{L}_j X)_{m_1, \dots, m_k, 0, \bullet, \dots, \bullet}$ is a constant simplicial space. For $k < j - 1$ this is true because constant simplicial spaces are Segal and hence Segalifying in the j^{th} coordinate does not change $X_{m_1, \dots, m_k, 0, \bullet, \dots, \bullet}$. For $k = j - 1$ this is true because Segalifying never changes the 0-simplices. For $k \geq j$ consider the simplicial object $Y : \Delta^{\text{op}} \rightarrow \text{PSh}(\Delta^{\times \{k+2, \dots, n\}})$ defined by sending l to $X_{m_1, \dots, m_{j-1}, l, m_{j+1}, \dots, m_k, 0, \bullet, \dots, \bullet}$. By assumption Y_l is a constant $(n - k - 1)$ -fold simplicial space for all l . Since the full subcategory of those $(n - k - 1)$ -fold simplicial spaces that are constant is closed under all limits and colimits, it follows that $(\mathbb{L}_S Y)_l$ is still constant for all l .

Claim (2) To simplify notation, we will assume that $n = 2 = j$; the general case is analogous. Suppose that $X_{\bullet, \bullet}$ satisfies the Segal condition in the first coordinate. It suffices to show that $\mathbb{L}_2 X_{\bullet, \bullet}$ still satisfies the Segal condition in the first coordinate. In other words, we need to show that $(\mathbb{L}_2 X)_{\bullet, l}$ is a Segal space for all l . By the Segal condition in the second coordinate, it suffices to do so for $l = 0, 1$. For $l = 0$ there is nothing to show since Segalification does not change the 0-simplices. For $l = 1$ we have the necklace formula

$$(\mathbb{L}_2 X)_{\bullet, 1} \simeq \operatorname{colim}_{\Delta^{n_1} \vee \dots \vee \Delta^{n_k} \in \text{Nec}^{\text{op}}} X_{\bullet, n_1} \times_{X_{\bullet, 0}} \dots \times_{X_{\bullet, 0}} X_{\bullet, n_k},$$

where the pullbacks and colimit are computed in simplicial spaces. To complete the proof it suffices to show that the diagram on the right-hand side, whose colimit we are taking, satisfies the hypotheses of Lemma 2.14. The indexing category Nec^{op} is sifted by Lemma 1.27 and each of the terms in the diagram is a Segal space since Segal spaces are closed under pullbacks. It remains to observe that the diagram of 0-simplices is constant, as its value on any necklace $N = \Delta^{n_1} \vee \dots \vee \Delta^{n_k}$ is

$$X_{0, n_1} \times_{X_{0, 0}} \dots \times_{X_{0, 0}} X_{0, n_k} \simeq X_{0, 0} \times_{X_{0, 0}} \dots \times_{X_{0, 0}} X_{0, 0} \simeq X_{0, 0}$$

by the reduced assumption. $\square$

Proposition 2.16 *The left adjoint to the full inclusion of n -fold Segal spaces into reduced n -fold simplicial spaces may be computed as*

$$\mathbb{L} = \mathbb{L}_n \circ \dots \circ \mathbb{L}_1 : \text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}} \rightleftarrows \text{PSh}^r(\Delta^{\times n}) : \text{inc}.$$

Proof For any n -fold simplicial space the map $Y_{\bullet, \dots, \bullet} \rightarrow (\mathbb{L}_j Y)_{\bullet, \dots, \bullet}$ is local with respect to those n -fold simplicial spaces that are Segal in the j^{th} coordinate. Therefore, for any n -fold simplicial space $X_{\bullet, \dots, \bullet}$, all of the maps

$$X_{\bullet, \dots, \bullet} \rightarrow (\mathbb{L}_1 X)_{\bullet, \dots, \bullet} \rightarrow (\mathbb{L}_2 \circ \mathbb{L}_1)(X)_{\bullet, \dots, \bullet} \rightarrow \dots \rightarrow (\mathbb{L}_n \circ \dots \circ \mathbb{L}_1)(X)_{\bullet, \dots, \bullet}$$

are local with respect to the full subcategory of n -tuple Segal spaces. If we assume that $X_{\bullet, \dots, \bullet}$ is also reduced, then it follows by inductively applying Lemma 2.15 that $(\mathbb{L}_j \circ \dots \circ \mathbb{L}_1)(X)_{\bullet, \dots, \bullet}$ is reduced and satisfies the Segal condition in the first j coordinates. We have therefore shown that the map

$$X_{\bullet, \dots, \bullet} \rightarrow (\mathbb{L}_n \circ \dots \circ \mathbb{L}_1)(X)_{\bullet, \dots, \bullet}$$

is local with respect to n -fold Segal spaces and that its target is an n -fold Segal space. Consequently, it exhibits the target as the localization onto the full subcategory $\text{Seg}_{\Delta^{\text{op}}}^{n\text{-fold}}$. $\square$

Warning 2.17 In the context of Proposition 2.16, the order in which the Segalification functors are applied is crucial. It is important to Segalify the 1-morphisms first, then the 2-morphisms, and so on. If we were to apply $\mathbb{L}_2$ first and then $\mathbb{L}_1$, it would no longer be clear that the result is $\mathbb{L}_2$ -local as $\mathbb{L}_1$ can break the Segal condition in the second simplicial direction.

2.3 The Boardman–Vogt tensor product

We show how to use the Segalification formula to give a new construction of the Boardman–Vogt tensor product of ∞ -operads. We begin with a brief recollection on the tensor product of commutative monoids.

Recollection on tensor product of commutative monoids For an ∞ -category with products $\mathcal{C}$ we let $\text{CMon}(\mathcal{C}) \subset \text{Fun}(\text{Fin}_*, \mathcal{C})$ denote the ∞ -category of commutative monoids in $\mathcal{C}$; see [10, Section 1]. In the case $\mathcal{C} = \mathcal{S}$ we simply write $\text{CMon} := \text{CMon}(\mathcal{S})$. We let $\text{Cat}_{\infty}^{\otimes} := \text{CMon}(\text{Cat}_{\infty})$ denote the ∞ -category of symmetric monoidal ∞ -categories. By applying $\text{CMon}(-)$ to the adjunction $\mathbb{C}(-) \dashv \mathbb{N}_{\bullet}$ and using that $\text{CMon}(\text{PSh}(\Delta)) \simeq \text{Fun}(\Delta^{\text{op}}, \text{CMon})$, we obtain an adjunction

$$\mathbb{C}(-) : \text{Fun}(\Delta^{\text{op}}, \text{CMon}) \rightleftarrows \text{Cat}_{\infty}^{\otimes} : \mathbb{N}_{\bullet}.$$

The right adjoint here is the *symmetric monoidal nerve*, which is given by $\mathbb{N}_n(\mathcal{D}) = \text{Fun}([n], \mathcal{D})^{\simeq}$ in the n^{th} level, with the pointwise symmetric monoidal structure.

Let $\mathcal{C}$ be a cartesian closed presentable ∞ -category. Gepner, Groth and Nikolaus [10] show that $\text{CMon}(\mathcal{C})$ admits a unique symmetric monoidal structure $\otimes$ such that the free functor $\mathbb{F} : \mathcal{C} \rightarrow \text{CMon}(\mathcal{C})$ is symmetric monoidal. If $\mathcal{C}$ and $\mathcal{D}$ are presentable cartesian closed and $L : \mathcal{C} \rightarrow \mathcal{D}$ is a symmetric monoidal (ie finite product preserving) left adjoint, they show the induced functor $L : \text{CMon}(\mathcal{C}) \rightarrow \text{CMon}(\mathcal{D})$ is canonically symmetric monoidal [10, Lemma 6-3(ii)]. Segalification, completion, and $\mathbb{C}(-)$ are examples of such L . Using that $\text{Seg}_{\Delta^{\text{op}}}(\text{CMon}) \simeq \text{CMon}(\text{Seg}_{\Delta^{\text{op}}}(\mathcal{S}))$ (and similarly for complete Segal spaces), we record this for future use.

Corollary 2.18 *All of the functors in the following commutative diagram are canonically symmetric monoidal for the respective tensor product:*

$$\begin{array}{ccccccc} \text{Fun}(\Delta^{\text{op}}, \text{CMon}) & \xrightarrow{\mathbb{L}_{\mathcal{S}}} & \text{Seg}_{\Delta^{\text{op}}}(\text{CMon}) & \xrightarrow{\mathbb{L}_{\mathcal{C}}} & \text{CSeg}_{\Delta^{\text{op}}}(\text{CMon}) & \xleftarrow[\simeq]{\mathbb{N}_{\bullet}(-)} & \text{Cat}_{\infty}^{\otimes} \\ & & & & \mathbb{C}(-) & & \end{array}$$

While the characterization of the tensor product in [10] is an excellent tool for studying the symmetric monoidal structure on $\text{Cat}_\infty^\otimes$ as a whole, we will also need a more “local” description that gives a universal property for the tensor product of two fixed symmetric monoidal ∞ -categories. Below, in Proposition 2.28, we give such a description by closely following [5, Section 4.3].

Some equifibered theory A morphism of commutative monoids $f: M \rightarrow N$ is said to be *equifibered* if the canonical square

$$\begin{array}{ccc} M \times M & \xrightarrow{+} & M \\ f \times f \downarrow & & \downarrow f \\ N \times N & \xrightarrow{+} & N \end{array}$$

is cartesian [2, Definition 2.1.4]. Equifibered maps span a replete subcategory of commutative monoids $\text{CMon}^{\text{eqf}} \subset \text{CMon}$. This notion was introduced in op cit for the purpose of developing the theory of ∞ -properads. A quintessential feature of equifibered maps is that a morphism of free monoids $f: \mathbb{F}(X) \rightarrow \mathbb{F}(Y)$ is equifibered if and only if it is free, ie $f \simeq \mathbb{F}(g)$ for some map of spaces $g: X \rightarrow Y$. Equifibered maps can be thought of as a well-behaved generalization of free maps; for example they form the right part of a factorization system on CMon . Further details can be found in [2, Section 2].

Observation 2.19 Equifibered maps between free monoids are closed under the tensor product. Indeed, the free functor $\mathbb{F}: \mathcal{S} \rightarrow \text{CMon}$ is symmetric monoidal and by [2, Remark 2.1.8] induces an equivalence onto the subcategory $\text{CMon}^{\text{free,eqf}} \subseteq \text{CMon}$ of free monoids and equifibered maps.

Definition 2.20 A simplicial commutative monoid $M: \Delta^{\text{op}} \rightarrow \text{CMon}$ is called $\otimes$ -disjunctive if it is right- CMon^{eqf} -fibered.

Remark 2.21 By [2, Lemma 3.2.15], the nerve $N_\bullet \mathcal{C}$ of a symmetric monoidal ∞ -category $\mathcal{C} \in \text{Cat}_\infty^\otimes$ is $\otimes$ -disjunctive if and only if $\mathcal{C}$ is $\otimes$ -disjunctive in the sense of [2, Definition 3.2.14]. That is, if and only if for all $x, y \in \mathcal{C}$ the functor

$$\otimes: \mathcal{C}_{/x} \times \mathcal{C}_{/y} \rightarrow \mathcal{C}_{/x \otimes y}$$

is an equivalence.

Observation 2.22 Let $M: \Delta^{\text{op}} \rightarrow \text{CMon}$ be $\otimes$ -disjunctive. Then M is levelwise free if and only if M_0 is free. Indeed, evaluation at the last vertex $d_0 \circ \cdots \circ d_0: M_n \rightarrow M_0$ is equifibered, so if M_0 is free, the same holds for M_n [2, Corollary 2.1.16].

As a consequence of the necklace formula we have the following:

Lemma 2.23 Let $M \in \text{Fun}(\Delta^{\text{op}}, \text{CMon})$ be $\otimes$ -disjunctive. Then $\mathbb{L}_S(M)$ is $\otimes$ -disjunctive.

Proof It suffices to check the conditions of Proposition 2.3. The first condition was verified in Example 1.24. The second condition follows from [2, Lemma 2.1.28]. $\square$

We now give a description of ∞ -operads using equifibered maps.

Definition 2.24 A *preoperad* is a Segal commutative monoid $M \in \text{Seg}_{\Delta^{\text{op}}}(\text{CMon})$ which is $\otimes$ -disjunctive and levelwise free. A preoperad is called *complete* if its underlying Segal space is as well.

Warning 2.25 Preoperads in the sense of Definition 2.24 should not be confused with ∞ -preoperads in the sense of Lurie [20, Section 2.1.4]. Instead, in the language of [2], a preoperad is precisely a monic prepreoperad.

Preoperads are to ∞ -operads what Segal spaces are to ∞ -categories. Indeed, the envelope functor induces an equivalence between Lurie’s ∞ -operads and complete preoperads.

Theorem 2.26 *Lurie’s monoidal envelope functor $\text{Env}(-): \text{Op}_{\infty} \rightarrow \text{Cat}_{\infty}^{\otimes}$ is faithful (it induces a monomorphism on mapping spaces). Moreover, the composite*

$$\text{Op}_{\infty} \xrightarrow{\text{Env}(-)} \text{Cat}_{\infty}^{\otimes} \xrightarrow{\simeq} \text{N}^{\bullet} \text{CSeg}(\text{CMon})$$

identifies Op_{∞} with the (nonfull) subcategory of $\text{CSeg}(\text{CMon})$ whose objects are complete preoperads and whose morphisms are equifibered natural transformations.

The first instance of this theorem can be found in the work of Haugseng–Kock [15], who showed that the sliced functor $\text{Env}: \text{Op}_{\infty} \rightarrow \text{Cat}_{\infty/\text{Fin}}^{\otimes}$ is fully faithful and characterized its image. Barkan–Haugsgeng–Steinebrunner [1] then gave an alternative characterization of the image, closely related to preoperads. The above formulation was given in [2, Theorem 3.2.13].

Tensor products of ∞ -operads We can now apply the necklace formula for Segalification to show that preoperads are closed under the tensor product.

Proposition 2.27 *The replete subcategory $\text{pOp}_{\infty} \subseteq \text{Seg}_{\Delta^{\text{op}}}(\text{CMon})$ is closed under the tensor product.*

Proof First we claim that if $M: \Delta^{\text{op}} \rightarrow \text{CMon}$ is levelwise free and $\otimes$ -disjunctive then the same holds for $\mathbb{L}_S M$. Indeed, Lemma 2.23 shows that $\mathbb{L}_S(M)$ is $\otimes$ -disjunctive, and since $\mathbb{L}_S(M)_0 \simeq M_0$ is free, the claim follows from Observation 2.22.

To complete the proof it suffices to show that if $M, N \in \text{Fun}(\Delta^{\text{op}}, \text{CMon})$ are $\otimes$ -disjunctive and levelwise free, then the same holds for their tensor product $M \otimes N$. This follows from the fact that equifibered maps between free monoids are closed under the tensor product (Observation 2.19). $\square$

We are now in a position to prove Theorem E.

Proof of Theorem E For the first part it suffices by Theorem 2.26 to show that complete preoperads are closed under the tensor product. By Corollary 2.18, the equivalence $\text{Cat}_{\infty}^{\otimes} \simeq \text{CSeg}_{\Delta^{\text{op}}}(\text{CMon})$ identifies the tensor product of symmetric monoidal ∞ -categories with the bifunctor on complete Segal monoids

$$(M_{\bullet}, N_{\bullet}) \mapsto \mathbb{L}_{\text{CS}}(M_{\bullet} \otimes N_{\bullet}) \simeq \mathbb{L}_{\text{C}} \mathbb{L}_S(M_{\bullet} \otimes N_{\bullet}).$$

Suppose now that $M_{\bullet}$ and $N_{\bullet}$ are preoperads. By Proposition 2.27, $\mathbb{L}_S(M_{\bullet} \otimes N_{\bullet})$ is a preoperad and hence by [2, Proposition 3.3.8] so is the completion $\mathbb{L}_{\text{C}} \mathbb{L}_S(M_{\bullet} \otimes N_{\bullet})$.

For the second part we must compare $\otimes_{\text{BV}}$ to Lurie’s tensor product. It follows from Lemma 2.30 below that there is an equivalence $\text{Env}(\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}) \simeq \text{Env}(\mathcal{O} \otimes_{\text{BV}} \mathcal{P})$ for all ∞ -operads $\mathcal{O}$ and $\mathcal{P}$. And since Env is an equivalence onto a replete subcategory by Theorem 2.26, it follows that we already have such an equivalence before applying Env . $\square$

Comparison to Lurie’s Boardman–Vogt tensor product To complete Theorem E we need to compare Lurie’s Boardman–Vogt tensor product to the tensor product of symmetric monoidal ∞ -categories. Lurie defines the tensor product of two ∞ -operads $\mathcal{O} \rightarrow \text{Fin}_*$ and $\mathcal{P} \rightarrow \text{Fin}_*$ to be the universal ∞ -operad representing bifunctors of ∞ -operads. A bifunctor of ∞ -operads [20, Definition 2.2.5.3] is a functor $F: \mathcal{O} \times \mathcal{P} \rightarrow \mathcal{Q}$ together with a square

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{P} & \xrightarrow{F} & \mathcal{Q} \\ \downarrow & & \downarrow \\ \text{Fin}_* \times \text{Fin}_* & \xrightarrow{\wedge} & \text{Fin}_* \end{array}$$

such that F preserves cocartesian lifts of inert morphisms. We begin by giving an analogous characterization of the tensor product of symmetric monoidal ∞ -categories.

Let $\mathcal{V} \in \text{Pr}^{\text{L}}$ be a cartesian closed presentable ∞ -category, such as Cat_{∞} . The smash product of finite pointed sets $A_+ \wedge B_+ = (A \times B)_+$ induces via Day convolution [5, Section 3] a symmetric monoidal structure on the ∞ -category of functors $\text{Fun}(\text{Fin}_*, \mathcal{V})$. Writing $\mu := \wedge: \text{Fin}_* \times \text{Fin}_* \rightarrow \text{Fin}_*$ for the smash product functor, we can describe the Day convolution as $F \boxtimes_{\text{Day}} G = \mu_!(F \times G)$. This tensor product localizes to a symmetric monoidal structure on the full subcategory of commutative monoids $\text{CMon}(\mathcal{V}) \subset \text{Fun}(\text{Fin}_*, \mathcal{V})$:

Proposition 2.28 *The left adjoint in the localization adjunction*

$$\mathbb{L}: \text{Fun}(\text{Fin}_*, \mathcal{V}) \rightleftarrows \text{CMon}(\mathcal{V})$$

admits a symmetric monoidal structure with respect to the Day convolution on the functor category and the tensor product of commutative monoids on the right.

Proof First we argue that the Day convolution symmetric monoidal structure localizes to the full subcategory $\text{CMon}(\mathcal{V})$, ie that there is a (unique) symmetric monoidal structure on $\text{CMon}(\mathcal{V})$ for which $\mathbb{L}$ is symmetric monoidal. This follows by essentially the same argument as [5, Proposition 4.24], except that we need to check the analogue of the second part of [5, Lemma 4.22]. Indeed, if $X: \text{Fin}_* \rightarrow \mathcal{V}$ is a commutative monoid and $A_+ \in \text{Fin}_*$, then $X(A_+ \wedge -)$ is still a commutative monoid as can be seen by

$$X(A_+ \wedge B_+) \xrightarrow{\quad \quad \quad \simeq \quad \quad \quad} \prod_{b \in B} X(A_+ \wedge \{b\}_+) \xrightarrow{\quad \quad \quad \simeq \quad \quad \quad} \prod_{(a,b) \in A \times B} X(\{a\}_+ \wedge \{b\}_+).$$

Now proceed as in [5, Theorem 4.26] to argue that the localized Day convolution symmetric monoidal structure on $\text{CMon}(\mathcal{V})$ satisfies the universal property of [10, Theorem 5-1]. $\square$

Given a symmetric monoidal ∞ -category $\mathcal{C} \in \text{Cat}_{\infty}^{\otimes}$ we let $\mathcal{C}^{\otimes} := \text{Un}(\mathcal{C}: \text{Fin}_* \rightarrow \text{Cat}_{\infty}) \rightarrow \text{Fin}_*$ denote its associated cocartesian fibration. With this notation $\text{Map}_{\text{Cat}_{\infty}^{\otimes}}(\mathcal{C}, \mathcal{D}) = \text{Map}_{/\text{Fin}_*}^{\text{cocart}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ where the latter denotes the subspace of $\text{Map}_{/\text{Fin}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by functors which preserve cocartesian edges. (In [20] this is taken as a definition.) Similarly, we have $\text{Map}_{\text{Op}_{\infty}}(\mathcal{O}, \mathcal{P}) = \text{Map}_{/\text{Fin}_*}^{\text{int-cocart}}(\mathcal{O}, \mathcal{P})$. In this setting, the tensor product admits the following characterization:

Corollary 2.29 *If $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are (unstraightened) symmetric monoidal ∞ -categories, then there is an equivalence*

$$\text{Map}_{/\text{Fin}_*}^{\text{cocart}}(\mathcal{C}^{\otimes} \otimes \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}) \simeq \text{Map}_{/\text{Fin}_* \times \text{Fin}_*}^{\text{cocart}}(\mathcal{C}^{\otimes} \times \mathcal{D}^{\otimes}, \mu^* \mathcal{E}^{\otimes})$$

natural in the symmetric monoidal ∞ -category $\mathcal{E}^{\otimes}$.

This is entirely analogous to the universal property of the Boardman–Vogt tensor product of ∞ -operads in [20, Definition 2.2.5.3 and Remark 2.2.5.4], which in this language may be stated as

$$\text{Map}_{/\text{Fin}_*}^{\text{int-cocart}}(\mathcal{O} \otimes \mathcal{P}, \mathcal{Q}) \simeq \text{Map}_{/\text{Fin}_* \times \text{Fin}_*}^{\text{int-cocart}}(\mathcal{O} \times \mathcal{P}, \mu^* \mathcal{Q}).$$

Lemma 2.30 *Lurie’s tensor product satisfies that for any two ∞ -operads $\mathcal{O}$ and $\mathcal{P}$ there is a canonical equivalence*

$$\text{Env}(\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}) \simeq \text{Env}(\mathcal{O}) \otimes \text{Env}(\mathcal{P}).$$

The proof of Lemma 2.30 will use a variant of Lurie’s symmetric monoidal envelope construction in which Fin_* is replaced by $\text{Fin}_* \times \text{Fin}_*$. More precisely, let $\mathcal{Q}$ be an ∞ -category with a factorization system $(\mathcal{Q}^{\text{int}}, \mathcal{Q}^{\text{act}})$ and let $p: \mathcal{O} \rightarrow \mathcal{Q}$ be a functor with cocartesian lifts for inerts. Then the envelope of p is defined as

$$\text{Env}_{\mathcal{Q}}(p: \mathcal{O} \rightarrow \mathcal{Q}) := \mathcal{O} \times_{\mathcal{Q}} \text{Ar}^{\text{act}}(\mathcal{Q}),$$

where $\text{Ar}^{\text{act}}(\mathcal{Q}) \subset \text{Ar}(\mathcal{Q})$ denotes the full subcategory of the arrow category spanned by active morphisms. This functor was studied extensively in [24, Section 3; 1, Section 2]. In particular, for any cocartesian fibration $\mathcal{E} \rightarrow \mathcal{Q}$, [1, Proposition 2.2.4] provides an equivalence

$$(\star) \quad \text{Fun}_{/\mathcal{Q}}^{\text{int-cocart}}(\mathcal{O}, \mathcal{E}) \simeq \text{Fun}_{/\mathcal{Q}}^{\text{cocart}}(\text{Env}_{\mathcal{Q}}(p: \mathcal{O} \rightarrow \mathcal{Q}), \mathcal{E}).$$

Proof As before, let $\mu: \text{Fin}_* \times \text{Fin}_* \rightarrow \text{Fin}_*$ denote the smash product. For any $\mathcal{C} \in \text{Cat}_{\infty}^{\otimes}$ we have

$$\begin{aligned} \text{Fun}^{\otimes}(\text{Env}(\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}), \mathcal{C}) &\simeq \text{Alg}_{\mathcal{O} \otimes_{\text{Lurie}} \mathcal{P}}(\mathcal{C}) \\ &\simeq \text{Fun}_{/(\text{Fin}_* \times \text{Fin}_*)}^{\text{int-cocart}}(\mathcal{O} \times \mathcal{P}, \mu^* \mathcal{C}) \\ &\simeq \text{Fun}_{/(\text{Fin}_* \times \text{Fin}_*)}^{\text{cocart}}(\text{Env}_{\text{Fin}_* \times \text{Fin}_*}(\mathcal{O} \times \mathcal{P}), \mu^* \mathcal{C}) \quad (\text{by } (\star)) \\ &\simeq \text{Fun}_{/(\text{Fin}_* \times \text{Fin}_*)}^{\text{cocart}}(\text{Env}_{\text{Fin}_*}(\mathcal{O}) \times \text{Env}_{\text{Fin}_*}(\mathcal{P}), \mu^* \mathcal{C}) \\ &\simeq \text{Fun}_{/(\text{Fin}_* \times \text{Fin}_*)}^{\text{cocart}}(\text{Env}(\mathcal{O}) \times \text{Env}(\mathcal{P}), \mu^* \mathcal{C}) \\ &\simeq \text{Fun}^{\otimes}(\text{Env}(\mathcal{O}) \otimes \text{Env}(\mathcal{P}), \mathcal{C}), \end{aligned}$$

where the fourth equivalence uses the identification $\text{Ar}^{\text{act}}(\text{Fin}_* \times \text{Fin}_*) \simeq \text{Ar}^{\text{act}}(\text{Fin}_*) \times \text{Ar}^{\text{act}}(\text{Fin}_*)$. $\square$

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Small cancellation and outer automorphisms of Kazhdan groups acting on hyperbolic spaces

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We show that every finite group is realized as the outer automorphism group of a hyperbolic group with Kazhdan property (T) and trivial finite radical. This result complements the well-known theorem of Paulin stating that the outer automorphism group of every hyperbolic group with property (T) is finite. We also show that, for every countable group Q , there exists an acylindrically hyperbolic group G with property (T) such that $\text{Out}(G) \cong Q$. The proofs employ strengthened versions of some previously known results in small cancellation theory.

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1 Introduction

It has long been known that the combination of hyperbolic geometry and property (T) implies rigidity of the algebraic structure of the group. The ultimate manifestation of this phenomenon is the following theorem:

Theorem 1.1 (Paulin [27]) *The outer automorphism group of a hyperbolic group with property (T) is finite.*

Informally speaking, Paulin shows that every group G admitting infinitely many “sufficiently distinct” isometric actions on a hyperbolic space also acts on an $\mathbb{R}$ -tree without globally fixed points. For a hyperbolic group G , these “sufficiently distinct” actions can be constructed by precomposing the natural action of G on its Cayley graph with elements of $\text{Out}(G)$. The relevance of property (T) in this context stems from the observation that every action of a Kazhdan group on an $\mathbb{R}$ -tree has a fixed point; see de la Harpe and Valette [14].

Recall that every hyperbolic group G contains a unique maximal finite normal subgroup, denoted by $K(G)$ and called the *finite radical* of G ; see, for example, Olshanskii [21]. It is natural to ask the following question, reminiscent of the inverse Galois problem:

Question 1.2 Which finite groups can be realized as $\text{Out}(G)$ for a hyperbolic group G with property (T) and trivial finite radical?

Without the condition $K(G) = \{1\}$, Question 1.2 admits an easy answer since every finite group can be realized as the group of outer automorphisms of another finite group; see de Cornulier [9]. One can also consider direct products of the form $G = K \times H$, where K is an appropriate finite group and H is a nontrivial, torsion-free, hyperbolic group with property (T) such that $\text{Out}(H) = \{1\}$. Obviously $\text{Out}(G) \cong \text{Out}(K)$ in this case. However, these examples are not satisfactory since the outer automorphisms of such a group G do not reflect the symmetries of its “truly hyperbolic” part.

We give a complete answer to Question 1.2 by proving the following:

Theorem 1.3 For every finite group Q , there exists a hyperbolic group G with property (T) and trivial finite radical such that $\text{Out}(G) \cong Q$.

Another natural question prompted by Paulin’s theorem and its generalizations to relatively hyperbolic groups (see Belegarde and Szczepański [5] and Druţu and Sapir [11]) is whether the existence of a “nice” action of a group G on a hyperbolic space imposes any restrictions on $\text{Out}(G)$. We address this question for the class of acylindrically hyperbolic groups, which was introduced by Osin [25] and received considerable attention in recent years; see Osin [26]. Informally, a group G is *acylindrically hyperbolic* if it admits a nonelementary action on a hyperbolic space S such that the induced action of G on $S \times S$ satisfies a certain properness assumption. For the precise definition, we refer the reader to Section 2.3.

Theorem 1.4 For every countable group Q , there exists a finitely generated acylindrically hyperbolic group G with property (T) such that $\text{Out}(G) \cong Q$.

Remark 1.5 Every acylindrically hyperbolic group also contains a unique maximal finite normal subgroup; see Dahmani, Guirardel, and Osin [10, Theorem 2.24]. The same terminology and notation as in the hyperbolic case is used here. Our approach allows us to additionally ensure that the group G from Theorem 1.4 has trivial finite radical (see Theorem 4.6). Note, however, that Theorem 1.4 (unlike Theorem 1.3) is new even without the condition $K(G) = \{1\}$.

Theorem 1.4 strengthens several previously known results, all of which were obtained using some sort of small cancellation theory. Minasyan [17] proved that every countable group is realized as $\text{Out}(G)$ for a property-(T) group G . However, groups constructed in [17] are far from being acylindrically hyperbolic. Ollivier and Wise [19] showed that every countable group Q embeds in $\text{Out}(G)$ for some acylindrically hyperbolic group G with property (T). (Although acylindrical hyperbolicity is not mentioned in [19], groups constructed in [19] satisfy a small cancellation condition, which implies acylindrical hyperbolicity

by the work of Gruber and Sisto [13].) In addition, one can ensure that Q is of finite index in $\text{Out}(G)$ if Q is finitely generated, using the approach suggested in [4]. However, the technique used by Belegradek and Osin [4] is not sufficient to prove the equality $\text{Out}(G) = Q$. A significant part of this paper is devoted to the development of an improved version of small cancellation theory that allows us to obtain a more precise result. Since our work in this direction is likely to have other applications, we discuss it below in more detail.

The classical small cancellation theory studies quotient groups of the form

$$G = F(X)/\langle\langle \mathcal{R} \rangle\rangle,$$

where $F(X)$ is the free group with basis X and $\mathcal{R}$ is a set of reduced words in the alphabet $X \cup X^{-1}$ with “small overlaps”. The central result about such groups is the Greendlinger lemma, stating that every reduced word in $X \cup X^{-1}$ representing the identity in G contains a “long” subword of some relation $R \in \mathcal{R}$.

A generalization of this theory to quotients of groups acting on hyperbolic spaces was suggested by Gromov [12] and elaborated by Olshanskii [21]. The main technical tools and ideas employed in [21] go back to the geometric method of studying groups via van Kampen diagrams developed by Olshanskii in the late 1970s and early 1980s, which enabled him to construct examples of groups with unexpected properties and to give a short proof of the Novikov–Adian theorem [1] on groups of finite exponent. For more details, we refer to Olshanskii [20].

Although the paper [21] only deals with hyperbolic groups, many results obtained there hold for relatively hyperbolic and, more generally, acylindrically hyperbolic groups with little modification; see Hull [15] and Osin [24]. In these settings, the Greendlinger lemma is no longer true and is replaced with a weaker conclusion, which is formalized using the key notion of a *contiguity subdiagram* suggested by Olshanskii [20; 21]. In Section 3.4, we propose a new small cancellation condition, which allows us to eliminate contiguity subdiagrams and obtain a stronger result similar to the original Greendlinger lemma (see Proposition 3.18). At the core of our approach are hyperbolically embedded collections of subgroups and a new notion of an *attracting geodesic*, which seems to be of independent interest. We also take the opportunity to generalize and strengthen some results of [15; 21; 24].

The paper is organized as follows. In the next section, we review the necessary background on hyperbolic groups and their generalizations. In Section 3 we develop the small cancellation toolbox necessary for proving Theorems 1.3, and 1.4. The proofs of the latter two theorems are given in Section 4.

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2 Preliminaries

2.1 Hyperbolic spaces

Recall that metric space S with a distance function d is said to be *geodesic* if every two points $a, b \in S$ can be connected by a path of length $d(a, b)$. A geodesic metric space S is δ -*hyperbolic* for some $\delta \geq 0$ if, for any geodesic triangle Δ in S , every side of Δ is contained in the union of the closed δ -neighborhoods of the other two sides [12].

For a path p in a metric space, we denote by p_- and p_+ its origin and terminal point, respectively. If p is rectifiable, we denote by $\ell(p)$ its length. We will need the following standard results about geodesic polygons in hyperbolic spaces:

Lemma 2.1 *Let (S, d) be a δ -hyperbolic space, and $Q = pqrs$ a quadrilateral in S with geodesic sides p, q, r , and s .*

- (a) *Every side of Q belongs to the union of the closed 2δ -neighborhoods of the other three sides.*
- (b) *For any point $x \in p$, we have $d(x, r) \leq 2\delta + \max\{\ell(q), \ell(s)\}$.*
- (c) *Suppose that $\ell(p) > \ell(q) + \ell(s) + 4\delta$. Then there exists a subpath t of p such that*

$$\ell(t) = \ell(p) - \ell(q) - \ell(s) - 4\delta$$

$$\text{and } \max\{d(t_-, r), d(t_+, r)\} \leq 2\delta.$$

Proof The first part can be easily derived from the definition of a hyperbolic space by drawing a diagonal in Q . Part (b) follows from (a) by the triangle inequality. To prove (c), it suffices to take the subpath t of p such that $d(p_-, t_-) = 2\delta + \ell(q)$ and $d(p_+, t_+) = 2\delta + \ell(s)$; see Figure 1. Since $\ell(p) > \ell(q) + \ell(s) + 4\delta$, the distances from t_- and t_+ to the union of the interiors of s and q is greater than 2δ . Combining this with (a), we obtain $\max\{d(t_-, r), d(t_+, r)\} \leq 2\delta$. $\square$

The next lemma can be thought of as a generalization of the previous one to more general polygons. It is a simplification of [21, Lemma 10].

Lemma 2.2 (Olshanskii) *Let (S, d) be a δ -hyperbolic space. Suppose that the set of all sides of a geodesic polygon $P = p_1 p_2 \dots p_n$ is partitioned into two subsets, A and B . Let α (respectively, β) denote the sum of lengths of sides from A (respectively, B). Assume, in addition, that $\alpha > \max\{cn, 10^3\beta\}$ for some $c \geq 3 \cdot 10^4\delta$. Then there exist two distinct sides $p_i, p_j \in A$ and a subpath t of p_i of length greater than $10^{-3}c$ such that*

$$\max\{d(t_-, p_j), d(t_+, p_j)\} \leq 13\delta.$$

Throughout this paper, we often think of graphs as metric spaces. Given a connected graph Γ , we identify every open edge of Γ with $(0, 1)$ and define the distance between two points $a, b \in \Gamma$ to be the length of the shortest path in Γ connecting a to b .

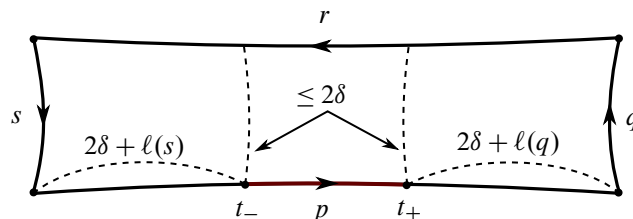


Figure 1: Proof of Lemma 2.1(c).

Remark 2.3 If Γ is a hyperbolic graph, we can (and will) always assume that its hyperbolicity constant is a nonnegative integer. Further, if the sides of the geodesic polygons considered in Lemmas 2.1 and 2.2 are (combinatorial) paths starting and ending at some vertices of Γ , elementary arguments show that we can additionally require the endpoints of the path t in each of these lemmas to be vertices of Γ as well.

A group G is *hyperbolic* if it is generated by a finite set X and its Cayley graph $\Gamma(G, X)$ is a hyperbolic metric space. This definition is independent of the choice of a particular finite generating set X . A hyperbolic group is called *elementary* if it contains a cyclic subgroup of finite index. For examples and basic properties of hyperbolic groups, we refer the reader to Chapters III.H and III. Γ of [7].

2.2 Relative hyperbolicity and hyperbolically embedded subgroups

Hyperbolically embedded collections of subgroups play a crucial role in our paper. To formulate the definition, it is convenient to work with generating alphabets instead of generating sets of groups. By a *generating alphabet* $\mathcal{A}$ of a group G we mean an abstract set given together with a map $\mathcal{A} \rightarrow G$ whose image generates G ; to simplify our notation, we do not distinguish between elements of $\mathcal{A}$ and their images in G whenever no confusion is possible.

By the *Cayley graph* of G with respect to a generating alphabet $\mathcal{A}$, written $\Gamma(G, \mathcal{A})$, we mean a graph with the vertex set G and the set of edges defined as follows. For every $a \in \mathcal{A}$ and every $g \in G$, there is an oriented edge e going from g to ga in $\Gamma(G, \mathcal{A})$ and labeled by a . Given a combinatorial path p in $\Gamma(G, \mathcal{A})$, we denote by $\text{Lab}(p)$ its label and by p^{-1} the combinatorial inverse of p . We use the notation $d_{\mathcal{A}}$ and $|\cdot|_{\mathcal{A}}$ to denote the standard metric on $\Gamma(G, \mathcal{A})$ and the length function on G with respect to the (image of) $\mathcal{A}$.

This terminology will be used in the following situation. Suppose that we have a group G , a collection of subgroups $\{H_i\}_{i \in I}$ of G , and a subset $X \subseteq G$ such that X and the union of all H_i together generate G . In this case we say that X is a *relative generating set* of G with respect to $\{H_i\}_{i \in I}$. We think of X and the subgroups H_i as abstract sets and consider the disjoint unions

$$(1) \quad \mathcal{H} = \bigsqcup_{i \in I} H_i \quad \text{and} \quad \mathcal{A} = X \sqcup \mathcal{H}.$$

Let $\mathcal{A}^*$ denote the free monoid on $\mathcal{A}$, ie the set of all words in the alphabet $\mathcal{A}$. For a word $W \in \mathcal{A}^*$, let $\|W\|$ denote its length. Given a word $a_1 \dots a_k \in \mathcal{A}^*$, we say that it *represents* an element $g \in G$ if $g = a_1 \dots a_k$ in G .

Convention 2.4 Henceforth, we assume that all generating sets and relative generating sets are symmetric, that is, closed under inversion.

This convention implies that the alphabet $\mathcal{A}$ defined in (1) is also symmetric, since so is X and each H_i is a subgroup. To take care of the possible ambiguity arising from the fact that distinct letters of $\mathcal{A}$ may represent the same element of G , we agree to think of a^{-1} as a letter from X (respectively, from H_i) if $a \in X$ (respectively, $a \in H_i$). Thus every element of G can be represented by a word from $\mathcal{A}^*$.

In the settings described above, we can think of the Cayley graphs $\Gamma(H_i, H_i)$ as subgraphs of $\Gamma(G, \mathcal{A})$. For every $i \in I$, we introduce a (generalized) metric $\hat{d}_{H_i}: H_i \times H_i \rightarrow [0, +\infty]$ as follows.

Definition 2.5 Given $g, h \in H_i$, let $\hat{d}_{H_i}(g, h)$ be the length of a shortest path in $\Gamma(G, \mathcal{A})$ that connects g to h and contains no edges of $\Gamma(H_i, H_i)$. If no such path exists, we set $\hat{d}_{H_i}(h, k) = \infty$.

Clearly $\hat{d}_{H_i}$ satisfies the triangle inequality (with addition extended to $[0, +\infty]$ in the natural way). We are now ready to define hyperbolically embedded collections of subgroups introduced in [10].

Definition 2.6 A collection of subgroups $\{H_i\}_{i \in I}$ of G is *hyperbolically embedded in G with respect to a subset $X \subseteq G$* , denoted by $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$, if the group G is generated by the alphabet $\mathcal{A}$ defined by (1) and the following conditions hold:

- (a) The Cayley graph $\Gamma(G, \mathcal{A})$ is hyperbolic.
- (b) For every $n \in \mathbb{N}$ and every $i \in I$, the set $\{h \in H_i \mid \hat{d}_{H_i}(1, h) \leq n\}$ is finite.

Further, we say that $\{H_i\}_{i \in I}$ is *hyperbolically embedded in G* and write $\{H_i\}_{i \in I} \hookrightarrow_h G$ if $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$.

To help the reader become familiar with these notions, we discuss three elementary examples.

Example 2.7 (a) For any group G we have $G \hookrightarrow_h G$. Indeed we can take $X = \emptyset$ in this case. Then $\Gamma(G, \mathcal{A}) = \Gamma(G, G)$ and $\hat{d}_G(g, h) = \infty$ for any $g \neq h$.

(b) If H is a finite subgroup of a group G , then $H \hookrightarrow_h G$. Indeed, it is straightforward to check that $H \hookrightarrow_h (G, X)$ for $X = G$.

(c) Let $G = H \times \mathbb{Z}$, let $X = \{x\}$, where x is a generator of $\mathbb{Z}$, and let $\mathcal{A} = \{x\} \sqcup H$. It is easy to see that the graph $\Gamma(G, \mathcal{A})$ is quasi-isometric to a line, and hence it is hyperbolic. However, every two elements $g, h \in H$ can be connected by a path of length at most 3 in $\Gamma(G, \mathcal{A})$ that avoids edges of $\Gamma(H, H)$; see Figure 2. Thus $\hat{d}_H(g, h) \leq 3$ and H is not hyperbolically embedded in G with respect to X whenever H is infinite.

Hyperbolically embedded collection of subgroups generalize peripheral structures of relatively hyperbolic groups. Indeed, we have the following proposition. Readers unfamiliar with relative hyperbolicity can regard it as a definition.

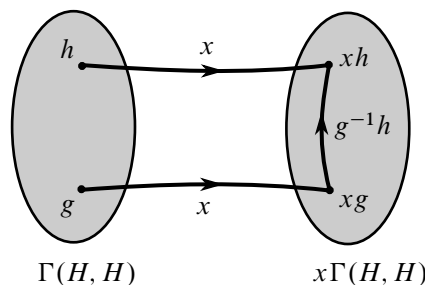


Figure 2: A path of length 3 in the Cayley graph $\Gamma(G, \mathcal{A})$, where $G = \langle x \rangle \times H$ and $\mathcal{A} = \{x\} \sqcup H$, connecting any elements $g, h \in H$ and avoiding edges of $\Gamma(H, H)$.

Proposition 2.8 [10, Proposition 4.28] *A group G is hyperbolic relative to a finite collection of subgroups $\{H_i\}_{i \in I}$ if and only if $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some (equivalently, any) finite subset $X \subseteq G$.*

If G is hyperbolic relative to $\{H_i\}_{i \in I}$, the subgroups H_i are called the *peripheral subgroups* of G . For more on relative hyperbolicity, we refer to [23].

We now turn to properties of hyperbolically embedded subgroups. An elaborated version of the argument from Example 2.7(c) can be used to derive the following:

Proposition 2.9 [10, Proposition 4.33] *Let G be a group, $\{H_i\}_{i \in I}$ a hyperbolically embedded collection of subgroups of G .*

- (a) *For any $i \in I$ and any $g \in G \setminus H_i$, we have $|H_i \cap g^{-1}H_i g| < \infty$.*
- (b) *For any distinct $i, j \in I$ and any $g \in G$, we have $|H_i \cap g^{-1}H_j g| < \infty$.*

We mention one useful corollary. Recall that a group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every nontrivial element of G is infinite.

Corollary 2.10 *Let G be a group containing two subgroups H_1, H_2 such that $\{H_1, H_2\} \hookrightarrow_h G$ and $H_1 \cap H_2 = \{1\}$. Suppose that $S \leq G$ and $S \cap H_i$ is infinite for $i = 1, 2$. Then, for every $g \in G \setminus \{1\}$, the set $g^S = \{s^{-1}gs \mid s \in S\}$ is infinite. In particular, S is ICC.*

Proof If g^S is finite, then g commutes with a finite-index subgroup of S . Since both $S \cap H_1$ and $S \cap H_2$ are infinite, Proposition 2.9 implies that $g \in H_1 \cap H_2 = \{1\}$. $\square$

The following result allows one to modify collections of hyperbolically embedded subgroups:

Proposition 2.11 [10, Theorem 4.35] *Let G be a group and $\{H_i\}_{i \in I}$ a finite collection of subgroups of G . Suppose that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$ and, for each $i \in I$, there is a collection of subgroups $\{K_{ij}\}_{j \in J_i}$ of H_i and a subset $Y_i \subseteq H_i$ such that $\{K_{ij}\}_{j \in J_i} \hookrightarrow_h (H_i, Y_i)$. Then we have $\bigcup_{i \in I} \{K_{ij}\}_{j \in J_i} \hookrightarrow_h (G, Z)$, where*

$$(2) \quad Z = X \cup \left(\bigcup_{i \in I} Y_i \right).$$

Note that the set Z defined by (2) is finite whenever X , I , and all Y_i are. Combining this with Proposition 2.8, we obtain the following:

Corollary 2.12 *Let G be a group hyperbolic relative to a finite collection of subgroups $\{H_i\}_{i \in I}$. Suppose that each H_i is hyperbolic relative to a finite collection of subgroups $\{K_{ij}\}_{j \in J_i}$. Then G is hyperbolic relative to $\bigcup_{i \in I} \{K_{ij}\}_{j \in J_i}$.*

Further, let G be a group generated by a finite set X . Note that the empty collection of subgroups is hyperbolically embedded in G with respect to X if and only if G is a hyperbolic group. Thus hyperbolic groups can always be excluded from finite hyperbolically embedded collections. More precisely, we have the following:

Corollary 2.13 *Let G be a group, and $\{H_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$ two finite collections of subgroups of G . Suppose that $\{H_i\}_{i \in I} \cup \{K_j\}_{j \in J} \hookrightarrow_h (G, X)$ for some $X \subseteq G$. Assume also that each subgroup H_i is hyperbolic and let Y_i denote a finite generating set of H_i . Then $\{K_j\}_{j \in J} \hookrightarrow_h (G, Z)$, where Z is defined by (2).*

Proof We apply the proposition to the empty collection of subgroups of each H_i and the subgroups $K_j \hookrightarrow_h (K_j, \emptyset)$ for $j \in J$. $\square$

Using Proposition 2.8, we obtain the following corollary for relatively hyperbolic groups; see Theorem 2.40 of [23] for a direct proof.

Corollary 2.14 *Let G be a group hyperbolic relative to a finite collection of subgroups $\{H_i\}_{i \in I} \cup \{K_j\}_{j \in J}$. If H_i is hyperbolic for every $i \in I$, then G is hyperbolic relative to $\{K_j\}_{j \in J}$. In particular, a group hyperbolic relative to a finite collection of hyperbolic subgroups is itself hyperbolic.*

2.3 Acylindrical hyperbolicity

An isometric action of a group G on a metric space S is said to be *acylindrical* if, for every $\varepsilon > 0$, there exist $R, N > 0$ such that, for every two points $x, y \in S$ with $d(x, y) \geq R$, there are at most N elements $g \in G$ satisfying the inequalities

$$d(x, gx) \leq \varepsilon \quad \text{and} \quad d(y, gy) \leq \varepsilon.$$

Informally, one can think of this condition as a kind of properness of the action on $S \times S$ minus a “thick diagonal”.

We begin with a classification of acylindrical group actions on hyperbolic spaces. Recall that an action of a group G on a hyperbolic space S is *nonelementary* if the limit set of G on the Gromov boundary ∂S has infinitely many points. The following classification of acylindrical actions is a simplification of [25, Theorem 1.1].

Theorem 2.15 *Let G be a group acting acylindrically on a hyperbolic space. Then G satisfies exactly one of the following three conditions:*

- (a) G has bounded orbits.
- (b) G has unbounded orbits and is virtually cyclic.
- (c) The action of G is nonelementary.

Thus an acylindrical action of a group G on a hyperbolic space is nonelementary if and only if G is not virtually cyclic and has infinite orbits. Readers unfamiliar with the notions of Gromov's boundary and the limit set can accept this as the definition of a nonelementary action.

The following theorem relates hyperbolically embedded subgroups to acylindrical actions. It can be easily derived from results of [25] as explained below.

Theorem 2.16 *Let G be a group and $\{H_i\}_{i \in I}$ a finite collection of subgroups of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$. Let also $\mathcal{H}$ and $\mathcal{A}$ be the alphabets defined by (1).*

- (a) *There exists $Y \subseteq G$ such that $X \subseteq Y$, $\{H_i\}_{i \in I} \hookrightarrow_h (G, Y)$, and the action of G on $\Gamma(G, \mathcal{H} \sqcup Y)$ is acylindrical.*
- (b) *If $|X| < \infty$, then the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical.*
- (c) *Suppose that the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical and there exists $i \in I$ such that H_i is infinite and $H_i \neq G$. Then the action of G on $\Gamma(G, \mathcal{A})$ is nonelementary.*

Proof We explain how to derive (a)–(c) from results in the literature. Part (a) is [25, Theorem 5.4]. It is worth noting that enlarging the relative generating set in (a) is necessary as the action of G on $\Gamma(G, \mathcal{A})$ may not be acylindrical if X is infinite (see the discussion before Theorem 5.4 in [25]).

By Proposition 2.8, G is hyperbolic relative to $\{H_i\}_{i \in I}$ if $|X| < \infty$. This lets us apply [25, Proposition 5.2], which gives us (b).

Finally, (c) can be proved as follows. Suppose that H_i is infinite and $H_i \neq G$ for some $i \in I$. Let $a \in G \setminus H_i$. By Proposition 2.9, $|a^{-1}H_i a \cap H_i| < \infty$. The existence of such an element a and the assumption $|H_i| = \infty$ allow us to apply [10, Theorem 6.11], which implies that the group G contains an element acting on $\Gamma(G, \mathcal{A})$ with unbounded orbits. Furthermore, the assumptions $|H_i| = \infty$ and $H_i \neq G$ imply that G contains a noncyclic free subgroup by [10, Theorem 2.24(c)]. In particular, G is not virtually cyclic. Therefore the action of G on $\Gamma(G, \mathcal{A})$ is nonelementary by Theorem 2.15. $\square$

Every group has an acylindrical action on a hyperbolic space, namely the trivial action on the point. For this reason, we want to avoid elementary actions in the definition below.

Definition 2.17 A group G is *acylindrically hyperbolic* if it admits a nonelementary acylindrical action on a hyperbolic space.

We mention some equivalent characterizations.

Theorem 2.18 [25, Theorem 1.2] *For any group G , the following conditions are equivalent:*

- (A₁) G is acylindrically hyperbolic.
- (A₂) *There exists a (possibly infinite) generating set X of G such that the Cayley graph $\Gamma(G, X)$ is hyperbolic and the action of G on $\Gamma(G, X)$ is nonelementary and acylindrical.*
- (A₃) G contains a proper infinite hyperbolically embedded subgroup.

The class of acylindrically hyperbolic groups includes all nonelementary hyperbolic and relatively hyperbolic groups, mapping class groups of closed surfaces of nonzero genus, $\text{Out}(F_n)$ for $n \geq 2$, nonvirtually cyclic groups acting properly on proper CAT(0) spaces and containing a rank-1 element, groups of deficiency at least 2, most 3-manifold groups, automorphism groups of some algebras (eg the Cremona group of birational transformations of the complex projective plane), and many other examples. For more details we refer to the survey [26].

We use the next result several times. It is a simplification of [25, Lemma 7.1].

Lemma 2.19 *Let G be a group acting acylindrically and nonelementarily on a hyperbolic space S . For any infinite normal subgroup $N \triangleleft G$, the induced action of N on S is nonelementary; in particular, N is acylindrically hyperbolic.*

Sizes of finite normal subgroups in every acylindrically hyperbolic group are uniformly bounded. Moreover, we have the following:

Definition 2.20 Every acylindrically hyperbolic group contains a unique maximal finite normal subgroup; see the first part of [10, Theorem 2.24]. We denote it by $K(G)$ and call it the *finite radical* of G .

Theorem 2.21 [10, Theorem 2.35] *An acylindrically hyperbolic group G is ICC if and only if $K(G) = \{1\}$.*

We conclude this section with the definition of a loxodromic element and examples of hyperbolically embedded subgroups, which we use many times.

Definition 2.22 An element g of a group G acting on a hyperbolic space S is called *loxodromic* if it acts as a translation along a bi-infinite quasigeodesic in S . If the action of G on S is acylindrical, this is equivalent to the requirement that $\langle g \rangle$ has unbounded orbits; see [6, Lemma 2.2].

If G is hyperbolic relative to a finite collection $\{H_i\}_{i \in I}$ and X is a finite relative generating set of G with respect to $\{H_i\}_{i \in I}$, then the action of G on $\Gamma(G, \mathcal{A})$ (where $\mathcal{A}$ is defined by (1)) is acylindrical by Proposition 2.8 and Theorem 2.16(b). In these settings, an element $g \in G$ is loxodromic if and only if g has infinite order and is not conjugate to an element of one of the peripheral subgroups; see [23, Theorem 4.23]. Thanks to this characterization, whether or not an element of a relatively hyperbolic group G is loxodromic is independent of the choice of a particular finite relative generating set of G . Thus we can simply talk about loxodromic elements of a relatively hyperbolic group if the peripheral collection is understood.

If G is hyperbolic, it is hyperbolic relative to the trivial subgroup and every infinite-order element is loxodromic with respect to this collection. For more details on loxodromic elements, we refer the reader to [25, Section 6].

The following is proved in [10, Lemma 6.5]:

Lemma 2.23 *Suppose that a group G acts acylindrically on a hyperbolic space S . Then every loxodromic element $g \in G$ is contained in a unique maximal virtually cyclic subgroup of G .*

Definition 2.24 In the setting of Lemma 2.23, we denote the unique maximal virtually cyclic subgroup of G containing g by $E(g)$.

Recall also that two elements a and b of a group G are said to be *commensurable* if some nonzero powers of a and b are conjugate in G . The following theorem was proved in [2, Corollary 3.12] (it is worth noting that the WPD condition assumed there follows immediately from acylindricity of the action). Similar but less precise statements were also proved in [10; 15].

Theorem 2.25 (Antolín, Minasyan, and Sisto) *Let G be a group and $\{H_i\}_{i \in I}$ a finite collection of subgroups of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$. Let also $\mathcal{A}$ be the alphabet defined by (1). Suppose that the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical. Then for any collection of pairwise noncommensurable loxodromic (with respect to the action on $\Gamma(G, \mathcal{A})$) elements $g_1, \dots, g_n \in G$, we have $\{E(g_1), \dots, E(g_n)\} \cup \{H_i\}_{i \in I} \hookrightarrow_h (G, X)$.*

Combining Theorem 2.25 with Proposition 2.8, Theorem 2.16(b), Lemma 2.23, and the characterization of loxodromic elements of relatively hyperbolic groups discussed above, we obtain the following particular case. For a direct proof see [22].

Corollary 2.26 *Let G be a group hyperbolic relative to a collection of subgroups $\{H_i\}_{i \in I}$. Suppose that $g_1, \dots, g_n \in G$ are pairwise noncommensurable elements of infinite order that are not conjugate to elements of the peripheral subgroups. Then every g_i is contained in a unique maximal virtually cyclic subgroup $E(g_i)$ of G and G is hyperbolic relative to $\{H_i\}_{i \in I} \cup \{E(g_1), \dots, E(g_n)\}$.*

3 Small cancellation in groups with hyperbolically embedded subgroups

3.1 Olshanskii's small cancellation conditions

We begin by reviewing the necessary background from [21]. Let G be a group generated by an alphabet $\mathcal{A}$. Consider any presentation

$$(3) \quad G = \langle \mathcal{A} \mid \mathcal{O} \rangle.$$

We do not assume that this presentation is finite; in particular, we can take $\mathcal{O}$ to be the set of all words in $\mathcal{A}$ representing the identity in G . We write $W_1 \equiv W_2$ to express letter-for-letter equality of two words $W_1, W_2 \in \mathcal{A}^*$.

In what follows, we use the standard terms *vertices*, *edges*, and *faces* for 0-cells, 1-cells, and 2-cells of planar CW-complexes. Recall that a *van Kampen diagram* Δ over (3) is a finite oriented connected simply connected planar CW-complex endowed with a labeling function $\text{Lab}: E(\Delta) \rightarrow \mathcal{A}$, where $E(\Delta)$ denotes the set of edges of Δ , such that $\text{Lab}(e^{-1}) \equiv (\text{Lab}(e))^{-1}$ for all $e \in E(\Delta)$.

Given a face Π of Δ , we denote by $\partial\Pi$ the boundary of Π ; similarly, $\partial\Delta$ denotes the boundary of Δ . The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to a cyclic permutation. An additional requirement is that for any face Π of Δ , the boundary label $\text{Lab}(\partial\Pi)$ is equal to (a cyclic permutation of) a word $P^{\pm 1}$, where $P \in \mathcal{O}$.

By the van Kampen lemma, a word w in the alphabet $\mathcal{A}$ represents the identity in the group G given by (3) if and only if there exists a van Kampen diagram Δ over (3) such that $\text{Lab}(\partial\Delta) \equiv W$.

Given a set of words $\mathcal{R}$ in the alphabet $\mathcal{A}$, we consider the quotient group of G given by

$$(4) \quad \bar{G} = G / \langle\langle \mathcal{R} \rangle\rangle = \langle \mathcal{A} \mid \mathcal{O} \cup \mathcal{R} \rangle.$$

To analyze properties of the quotient group $\bar{G}$ we will make use of a simplified version of the $C(\mu, \varepsilon, \lambda, c, \rho)$ and $C_1(\mu, \varepsilon, \lambda, c, \rho)$ small cancellation conditions introduced in [21]. The difference is that we require the relators to be geodesic rather than (λ, c) -quasigeodesic. Thus the conditions $C(\mu, \varepsilon, \rho)$ and $C_1(\mu, \varepsilon, \rho)$ correspond to $C(\mu, \varepsilon, 1, 0, \rho)$ and $C_1(\mu, \varepsilon, 1, 0, \rho)$, respectively, in the notation used in [21] and the subsequent papers [15; 16; 24].

More precisely, we say that a word W in the alphabet $\mathcal{A}$ is $(G, \mathcal{A})$ -geodesic if any path in the Cayley graph $\Gamma(G, \mathcal{A})$ labeled by W is geodesic. Recall that $\mathcal{R}$ is said to be *symmetric* if for any $R \in \mathcal{R}$, $\mathcal{R}$ contains all cyclic shifts of $R^{\pm 1}$. If a word $W \in \mathcal{A}^*$ decomposes as $W \equiv UV$ for some $U, V \in \mathcal{A}^*$, we say that U is an *initial subword* of W . The length of a word $W \in \mathcal{A}^*$ (ie the number of letters in W) is denoted by $\|W\|$.

Definition 3.1 A symmetric set of words $\mathcal{R}$ satisfies the $C(\varepsilon, \mu, \rho)$ -condition for some $\varepsilon \geq 0$ and $\mu, \rho > 0$ if the following conditions hold:

- (a) All words in $\mathcal{R}$ are $(G, \mathcal{A})$ -geodesic and have length at least ρ .
- (b) Suppose that words $R, R' \in \mathcal{R}$ have initial subwords U and U' , respectively, such that

$$(5) \quad \max\{\|U\|, \|U'\|\} \geq \mu \min\{\|R\|, \|R'\|\}$$

and $U' = YUZ$ in G for some words Y and Z of length

$$(6) \quad \max\{\|Y\|, \|Z\|\} \leq \varepsilon.$$

Then $YRY^{-1} = R'$ in G .

Further, $\mathcal{R}$ satisfies the $C_1(\varepsilon, \mu, \rho)$ -condition if, in addition to (a) and (b), we have the following:

- (c) Suppose that a word $R \in \mathcal{R}$ contains two disjoint subwords U and U' such that $U' = YUZ$ or $U' = YU^{-1}Z$ in G for some words Y and Z , and the inequality (6) holds. Then

$$\max\{\|U\|, \|U'\|\} < \mu\|R\|.$$

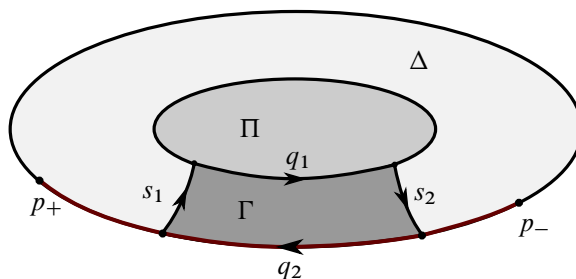


Figure 3: A contiguity subdiagram.

Let Δ be a van Kampen diagram over (4). A face in Δ is called an $\mathcal{R}$ -face if its boundary label is a word from $\mathcal{R}$. Suppose that p is a subpath of $\partial\Delta$ and there is a simple closed path

$$(7) \quad c = s_1 q_1 s_2 q_2$$

in Δ , where q_1 is a subpath of $\partial\Pi$, q_2 is a subpath of p , and

$$(8) \quad \max\{\ell(s_1), \ell(s_2)\} \leq \varepsilon$$

for some constant $\varepsilon > 0$. Let Γ denote the subdiagram of Δ bounded by c . If Γ contains no $\mathcal{R}$ -faces, we say that Γ is an ε -contiguity subdiagram of Π to the segment p of $\partial\Delta$. The ratio $\ell(q_1)/\ell(\partial\Pi)$ is called the *contiguity degree* of Π to $\partial\Delta$ and is denoted by $(\Pi, \Gamma, \partial\Delta)$. Since Γ contains no $\mathcal{R}$ -faces, it can be considered a diagram over (3).

The analogue of the classical Greendlinger lemma for small cancellation quotients of hyperbolic groups was proved by Olshanskii in [21, Lemma 6.6]. As observed in [24, Lemma 4.4], the same proof works in the more general context of groups with hyperbolic (but not necessarily locally finite) Cayley graphs. Below we reproduce a simplified version of this result, replacing the more general $C(\varepsilon, \mu, \lambda, c, \rho)$ condition considered in [21; 24] with the stronger condition $C(\varepsilon, \mu, \rho)$.

Lemma 3.2 (Olshanskii) *Suppose the Cayley graph $\Gamma(G, \mathcal{A})$ of a group G given by the presentation (3) is hyperbolic. Then for every $\mu \in (0, \frac{1}{16}]$, there exist $\varepsilon, \rho > 0$ such that the following holds.*

Let $\mathcal{R}$ be a symmetric set of words in the alphabet $\mathcal{A}$ satisfying $C(\varepsilon, \mu, \rho)$. Assume that $W \equiv W_1 W_2 W_3 W_4$ is a word in the alphabet $\mathcal{A}$ representing 1 in $\bar{G} = G/\langle\langle \mathcal{R} \rangle\rangle$ such that each of the words $W_1, \dots, W_4$ is $(G, \mathcal{A})$ -geodesic and $W \neq 1$ in G . Then there exist a van Kampen diagram Δ over (4) with boundary $\partial\Delta = p_1 p_2 p_3 p_4$, where $\text{Lab}(p_i) \equiv W_i$ for $i = 1, \dots, 4$, an $\mathcal{R}$ -face Π of Δ , and ε -contiguity subdiagrams Γ_i of Π to the segments p_i of $\partial\Delta$ (see Figure 4) such that

$$\sum_{i=1}^4 (\Pi, \Gamma_i, p_i) \geq 1 - 13\mu.$$

Remark 3.3 The lemma can be proved for the concatenation of n geodesic words for any $n \in \mathbb{N}$. However, the case $n = 4$ is sufficient for most applications.

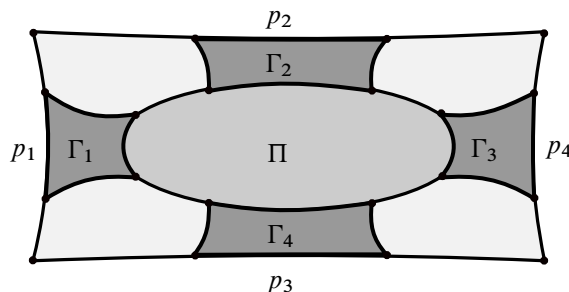


Figure 4

3.2 Centralizers in small cancellation quotients

In this section, we study centralizers of elements in small cancellation quotients of groups acting on hyperbolic spaces. Our main result is Lemma 3.4 below. It plays an important role in the proofs of Theorems 1.3 and 1.4. We begin with an auxiliary result.

Lemma 3.4 *Let G be a group given by presentation (3) such that the Cayley graph $\Gamma(G, \mathcal{A})$ is δ -hyperbolic for some $\delta \geq 0$. For any $\varepsilon \geq 4\delta$, any $\mu, \rho > 0$, and any set of words $\mathcal{R}$ satisfying the $C_1(\varepsilon, \mu, \rho)$ condition, the following holds.*

Let U and V be disjoint subwords of some $R \in \mathcal{R}$, and let p be a geodesic path in $\Gamma(G, \mathcal{A})$. Suppose that $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are quadrilaterals in $\Gamma(G, \mathcal{A})$ such that $\text{Lab}(b_1) \equiv U$, $\text{Lab}(b_2) \equiv V^{\pm 1}$, d_1 and d_2 are subpaths of p , and $\ell(a_i), \ell(c_i) \leq \varepsilon$ for $i = 1, 2$. Then the overlap between d_1 and d_2 has length less than $\mu\|R\| + 10\varepsilon$.

Proof Let o denote the overlap of d_1 and d_2 (see Figure 5). Assume that $\ell(o) \geq \mu\|R\| + 10\varepsilon$.

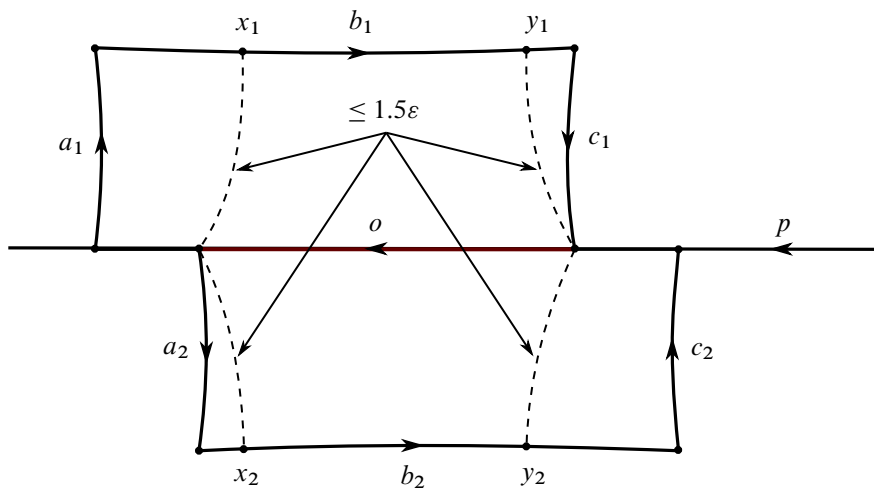


Figure 5: The proof of Lemma 3.4.

By Lemma 2.1(b), we can choose points x_1 and y_1 on b_1 such that

$$(9) \quad \max\{d(o_-, y_1), d(o_+, x_1)\} \leq 2\delta + \max\{\ell(a_1), \ell(c_1)\} \leq 2\delta + \varepsilon \leq 1.5\varepsilon.$$

Similarly, we choose x_2 and y_2 on b_2 satisfying $d(o_+, x_2) \leq 1.5\varepsilon$ and $d(o_-, y_2) \leq 1.5\varepsilon$. (If the quadrilaterals $a_1b_1c_1d_1$ and $a_2b_2c_2d_2$ are positioned as shown in Figure 5, we can take $y_1 = (c_1)_-$, $x_2 = (a_2)_+$, and improve the upper bounds for $d(o_-, y_1)$ and $d(o_+, x_2)$ to ε ; however, we choose to make our proof independent of the relative position of the quadrilaterals at the cost of a slight increase of the constants.)

By the triangle inequality, $d(x_1, x_2) \leq 3\varepsilon$ and $d(y_1, y_2) \leq 3\varepsilon$. Since p is geodesic, so is o . Using (9), we obtain

$$d(x_1, y_1) \geq \ell(o) - d(o_+, x_1) - d(o_-, y_1) \geq \mu\|R\| + 7\varepsilon \geq \mu\|R\| + d(x_1, x_2) + d(y_1, y_2) + 4\delta.$$

Applying Lemma 2.1(c) to the geodesic quadrilateral with the consecutive vertices x_1, y_1, y_2 and x_2 , we obtain a subpath t of $b_1^{\pm 1}$ such that $d(t_{\pm}, b_2) \leq 2\delta \leq \varepsilon$ and

$$\ell(t) \geq d(x_1, y_1) - d(x_1, x_2) - d(y_1, y_2) - 4\delta \geq \mu\|R\|.$$

Since $\text{Lab}(b_1) \equiv U$ and $\text{Lab}(b_2) \equiv V^{\pm 1}$ are disjoint subwords of R , this contradicts the $C_1(\varepsilon, \rho, \mu)$ small cancellation condition. $\square$

Lemma 3.5 *Let G be a group given by the presentation (3) such that the Cayley graph $\Gamma(G, \mathcal{A})$ is hyperbolic. For any $N \geq 0$, there exist $\varepsilon, \rho > 0$ such that the following holds.*

Let $\mathcal{R}$ be a set of words in the alphabet $\mathcal{A}$ satisfying the $C_1(\varepsilon, \frac{1}{100}, \rho)$ small cancellation condition and $\bar{G}$ the quotient group defined by (4). For any $g \in G$ of length $|g|_{\mathcal{A}} \leq N$, we have $C_{\bar{G}}(\gamma(g)) = \gamma(C_G(g))$, where $\gamma: G \rightarrow \bar{G}$ is the natural homomorphism.

Proof For hyperbolic groups, the lemma was proved in [21]. For relatively hyperbolic groups, this result can be derived from [16, Lemma 5.5]. Our proof is based on the same idea as the proof of [16, Lemma 5.5].

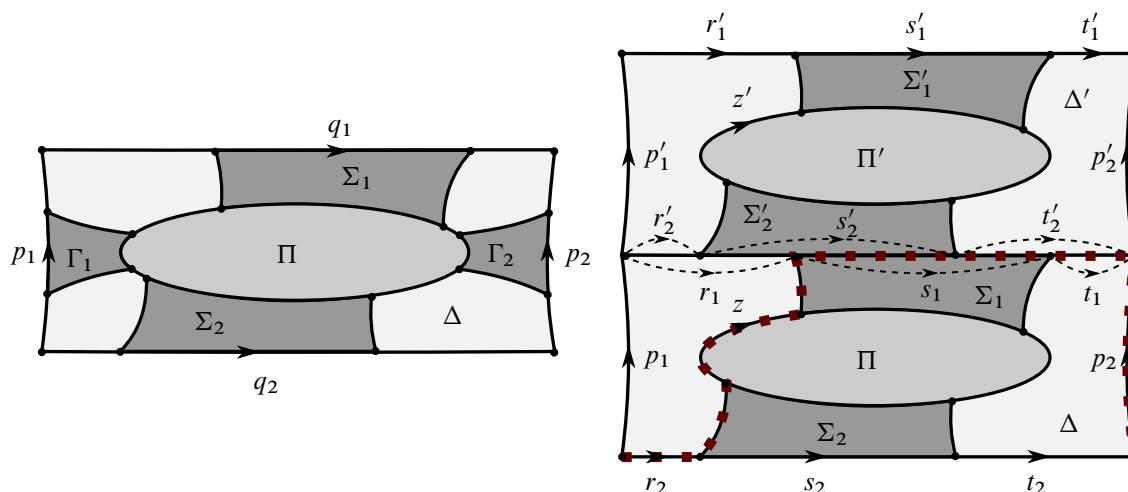
Fix some $N \in \mathbb{N}$ and let $\mu = \frac{1}{100}$. Note that the $C(\varepsilon, \mu, \rho)$ condition becomes stronger as ε and ρ increase. Therefore we can choose a sufficiently large value of ε and then a sufficiently large value of ρ such that the conclusions of Lemmas 3.2 and 3.4 simultaneously hold and, in addition,

$$(10) \quad \rho \geq 100(2\varepsilon + N).$$

In particular, $\varepsilon/\rho < \frac{1}{200}$.

Let $g \in G$ be an element of length $|g|_{\mathcal{A}} \leq N$ and let U be a $(G, \mathcal{A})$ -geodesic word representing g in G . The inclusion $\gamma(C_G(g)) \leq C_{\bar{G}}(\gamma(g))$ is trivial, so we only need to prove the inclusion $C_{\bar{G}}(\gamma(g)) \leq \gamma(C_G(g))$. Let $c \in C_{\bar{G}}(\gamma(g))$ and let V be a $(\bar{G}, \mathcal{A})$ -geodesic word representing c in $\bar{G}$ (in particular, V is $(G, \mathcal{A})$ -geodesic). We will show that

$$(11) \quad c \in \gamma(C_G(g)).$$

Figure 6: The diagrams Δ and Ω .

Clearly, the word $UVU^{-1}V^{-1}$ represents 1 in $\bar{G}$. To prove the lemma, it suffices to show that $UVU^{-1}V^{-1} = 1$ in G . Indeed, then V represents an element of $C_G(g)$ and we obtain (11).

Arguing by contradiction, assume that $UVU^{-1}V^{-1} \neq 1$ in G . By Lemma 3.2, there exist a van Kampen diagram Δ over (4) whose boundary decomposes as $\partial\Delta = p_1q_1p_2^{-1}q_2^{-1}$, where

$$\text{Lab}(p_1) \equiv \text{Lab}(p_2) \equiv U, \quad \text{Lab}(q_1) \equiv \text{Lab}(q_2) \equiv V,$$

an $\mathcal{R}$ -face Π of Δ , and ε -contiguity subdiagrams Γ_1 and Γ_2 (respectively, Σ_1 and Σ_2) of Π to the segments p_1 and p_2 (respectively, q_1 and q_2) of $\partial\Delta$ such that

$$(12) \quad \sum_{i=1}^2 (\Pi, \Gamma_i, p_i) + \sum_{i=1}^2 (\Pi, \Sigma_i, q_i) \geq 1 - 13\mu = 0.87$$

(see Figure 6, left).

Since $\text{Lab}(\partial\Pi)$ is $(G, \mathcal{A})$ -geodesic, the length of the common segment of Γ_1 and Π is bounded from above by the sum of lengths of the other three sides of Γ_1 , which is at most $2\varepsilon + \ell(p_1) \leq 2\varepsilon + N$. Hence $(\Pi, \Gamma_1, p_1) \leq (2\varepsilon + N)/\rho \leq \frac{1}{100}$ by (10). Similarly, $(\Pi, \Gamma_2, p_2) \leq \frac{1}{100}$. Combining these inequalities with (12), we obtain

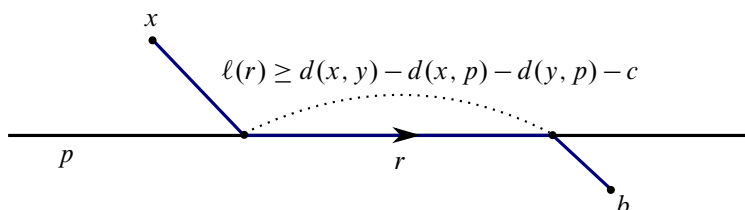
$$(13) \quad \sum_{i=1}^2 (\Pi, \Sigma_i, q_i) \geq 0.85.$$

For $i = 1, 2$, let $q_i = r_i s_i t_i$, where s_i is the common segment of Σ_i and q_i . We claim that

$$(14) \quad \ell(s_i) \geq 0.3|\partial\Pi|$$

for $i = 1, 2$. Indeed, suppose that $\ell(s_1) < 0.3|\partial\Pi|$. Since $\text{Lab}(\partial\Pi)$ is $(G, \mathcal{A})$ -geodesic and $|\partial\Pi| \geq \rho \geq 200\varepsilon$ by (10), we have $(\Pi, \Sigma_1, q_1) < (0.3|\partial\Pi| + 2\varepsilon)/|\partial\Pi| \leq 0.31$. Using (13), we obtain

$$(\Pi, \Sigma_2, q_2) \geq 0.85 - (\Pi, \Sigma_1, q_1) > 0.54.$$

Figure 7: A c -attracting geodesic.

Since V is $(\bar{G}, \mathcal{A})$ -geodesic, the path q_2 is geodesic in Δ ; therefore, so is s_2 . Comparing the length of s_2 to the path going along the “ ε -sides” of Σ_2 and around the face Π , we obtain

$$\ell(s_2) \leq 2\varepsilon + |\partial\Pi|(1 - (\Pi, \Sigma_2, q_2)) \leq 2\varepsilon + 0.46|\partial\Pi| \leq 0.47|\partial\Pi|.$$

Recall that the sides of Δ are $(G, \mathcal{A})$ -geodesic. Applying the triangle inequality to the sides of Σ_2 , we obtain

$$\ell(s_2) \geq |\partial\Pi|(\Pi, \Sigma_2, q_2) - 2\varepsilon > 0.54|\partial\Pi| - 2\varepsilon \geq 0.53|\partial\Pi|,$$

which contradicts the previous inequality. Thus $\ell(s_1) \geq 0.3|\partial\Pi|$, and similarly $\ell(s_2) \geq 0.3|\partial\Pi|$.

Let Δ' denote a copy of Δ ; we use primes to distinguish between parts of Δ' and the corresponding parts of Δ . Eg p'_1 will denote the path in Δ' corresponding to the path p_1 in Δ , Σ'_1 will denote the subdiagram of Δ' corresponding to Σ_1 , etc. Consider the van Kampen diagram Ω formed by gluing Δ and Δ' along q_1 and q'_2 (see Figure 6, right). For definiteness, we will assume that

$$(15) \quad \ell(r_2) \leq \ell(r_1)$$

(the other case is similar). By Lemma 3.4, the overlap between s_1 and s'_2 in Ω is at most $0.01|\partial\Pi| + 10\varepsilon \leq 0.06|\partial\Pi|$. Using (13)–(15), we obtain that the length of the path z given by the dotted path in Figure 6, right, satisfies

$$\begin{aligned} \ell(z) &\leq \ell(r_2) + \varepsilon + 0.15|\partial\Pi| + \varepsilon + \ell(s_1 t_1) + \ell(p_2) \leq 2\varepsilon + 0.15|\partial\Pi| + \ell(q_2) - (\ell(s_2) - 0.06|\partial\Pi|) + \ell(p_2) \\ &\leq 2\varepsilon + 0.15|\partial\Pi| + \ell(q_2) - 0.24|\partial\Pi| + N \leq \ell(q_2) + 2\varepsilon + N - 0.09\rho < \ell(q_2). \end{aligned}$$

This contradicts the assumption that V is $(\bar{G}, \mathcal{A})$ -geodesic, since $\text{Lab}(z)$ and $\text{Lab}(q_2) \equiv V$ represent the same element of $\bar{G}$. $\square$

3.3 Attracting geodesics

In this section we introduce the notion of an attracting geodesic, which is crucial for the proof of Proposition 3.18. By abuse of terminology, we identify a path $p: [0, 1] \rightarrow S$ in a metric space S with its image. Given a path p , we denote by p_- and p_+ its origin and terminal point, respectively. The inverse path is denoted by p^{-1} . By $d(s, p)$ we denote the distance from a point $s \in S$ to the path p defined in the usual way. We say that two paths p and q share an unoriented segment r if $p = arb$ and $q = cr^{\pm 1}d$.

Definition 3.6 Let Γ be a graph. We say that a (combinatorial) geodesic path p in Γ is c -attracting for some $c \in [0, \infty)$ if, for any vertices x and y of Γ such that $d(x, y) - d(x, p) - d(y, p) \geq c$, every geodesic in Γ going from x to y shares an unoriented segment of length at least $d(x, y) - d(x, p) - d(y, p) - c$ with p . (By definition, a segment of length 0 is a point.)

This definition is motivated by the obvious fact that in a tree, all geodesics are 0-attracting. On the other hand, in hyperbolic graphs and even in quasitrees, long c -attracting geodesics may not exist (note that every geodesic of length less than c in any graph is c -attracting though).

Our next goal is to construct arbitrarily long 2-attracting geodesics in Cayley graphs associated to hyperbolically embedded collections of subgroups. Throughout the rest of this section, we make the following assumptions:

(*) G is a group, $\{H_i\}_{i \in I}$ is a collection of subgroups of G , and X is a relative generating set of G with respect to $\{H_i\}_{i \in I}$.

We keep the notation $\mathcal{A}$ and $\hat{d}_{H_i}$ introduced in Section 2.2; see (1). In addition, we assume that

(**) $\Gamma(G, \mathcal{A})$ is hyperbolic.

Note that, in this section, we do not require $\{H_i\}_{i \in I}$ to be hyperbolically embedded in G with respect to X .

We make use of the following terminology introduced in [10]—in the particular case of relatively hyperbolic groups it goes back to [23].

Definition 3.7 Let p be a path in the Cayley graph $\Gamma(G, \mathcal{A})$. A subpath q of p is called an H_i -component (or simply a *component*) of p for some $i \in I$ if q has at least one edge, the label of q is a word in the alphabet $H_i \subseteq \mathcal{A}$, and q is not contained in any longer subpath of p with these properties. Two H_i -components q_1 and q_2 of p are *connected* if all vertices of q_1 and q_2 belong to the same coset of H_i in G . An H_i -component q of a path p is called *isolated* if no distinct H_i -component of p is connected to q .

Geometrically, the fact that H_i -components q_1 and q_2 of p are connected means that any two vertices u and v of q_1 and q_2 , respectively, can be connected by an edge of $\Gamma(G, \mathcal{A})$ labeled by an element of H_i .

Remark 3.8 If p is a geodesic in $\Gamma(G, \mathcal{A})$, then every H_i -component of p is isolated in p and consists of a single edge. This observation will be used many times in this section.

By a *geodesic n -gon* in a metric space we mean a loop consisting of n geodesic segments. The lemma below is a simplification of [10, Proposition 4.14].

Lemma 3.9 Under the assumptions (*) and (**), there exists a constant D satisfying the following condition. Let p be a geodesic n -gon in $\Gamma(G, \mathcal{A})$. Then for every $i \in I$ and every isolated H_i -component q of p , the element $h \in H_i$ represented by the label of q satisfies $\hat{d}_{H_i}(1, h) \leq Dn$.

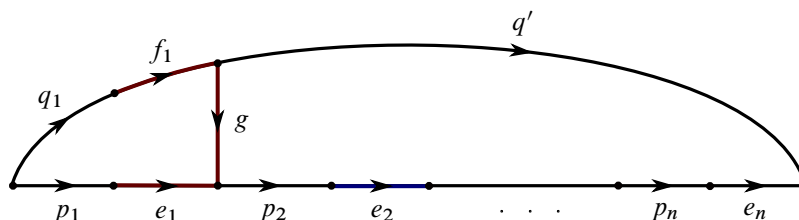


Figure 8: Proof of Lemma 3.10.

We will need the following result, which is an improved version of [10, Lemma 4.21]. Recall that the length of a word $u \in \mathcal{A}^*$ is denoted by $\|u\|$.

Lemma 3.10 Suppose that p is a path in $\Gamma(G, \mathcal{A})$ such that

$$\text{Lab}(p) \equiv U_1 a_1 U_2 a_2 \dots U_n a_n U_{n+1},$$

and the following conditions hold:

- (a) For every $j = 1, \dots, n$, a_j is a letter in $H_{i(j)}$ for some $i(j) \in I$, and $\widehat{d}_{H_{i(j)}}(1, a_j) > 5D$, where D is the constant provided by Lemma 3.9.
- (b) For every $j = 1, \dots, n+1$, U_j is a (possibly empty) word in the alphabet $\mathcal{A}$ such that for any element $g \in G$ satisfying $H_{i(j-1)}gH_{i(j)} = H_{i(j-1)}U_jH_{i(j)}$, we have $\|U_j\| \leq |g|_{\mathcal{A}}$. Here we assume $H_{i(0)} = H_{i(n+1)} = \{1\}$ for convenience.
- (c) If U_j is the empty word for some $j = 2, \dots, n$, then $H_{i(j-1)} \neq H_{i(j)}$.

Then p is geodesic.

Proof Arguing by contradiction, we fix the minimal $n \in \mathbb{N}$ such that there is a nongeodesic path p satisfying all the assumptions. We fix the notation

$$p = p_1 e_1 p_2 e_2 \dots p_n e_n p_{n+1}$$

for the segments of p , where

$$\text{Lab}(p_j) \equiv U_j \quad \text{and} \quad \text{Lab}(e_j) \equiv a_j$$

for all j (if U_j is the empty word, p_j is the trivial path). We will first prove the following:

Lemma 3.11 For every j , e_j is an isolated $H_{i(j)}$ -component of p .

Proof That each e_j is an $H_{i(j)}$ -component of p follows immediately from (b) and (c). Let us prove that it is isolated. For the sake of contradiction, assume that e_j is connected to another $H_{i(j)}$ -component f of p . For definiteness, let $p = r_1 e_j r_2 f r_3$ (see Figure 9). Also let g denote the edge of $\Gamma(G, \mathcal{A})$ labeled by an element of $H_{i(j)}$ and going from $(e_j)_+$ to f_- . Without loss of generality we can assume that the pair e_j, f is chosen in such a way that $\ell(r_2)$ is minimal possible. If r_2 contains e_{j+1} , then e_{j+1} cannot

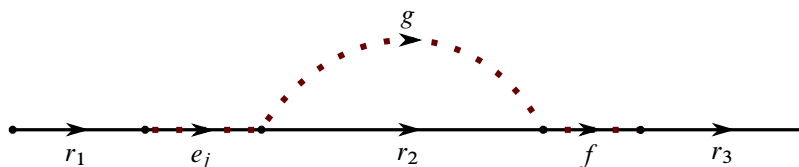


Figure 9: Proof of Lemma 3.11. Edges marked with a dotted line are labeled by elements of $H_{i(j)}$.

be isolated in the bigon $r_2 g^{-1}$ by (a) and Lemma 3.9, and we get a contradiction with the choice of e_j and f . Therefore r_2 is an initial subpath of p_{j+1} . Reading the label of the bigon $r_2 g^{-1}$ we obtain

$$(16) \quad \text{Lab}(r_2) \in H_{i(j)}.$$

If f is an edge of p_{j+1} , (16) contradicts (b); if $f = e_{j+1}$, then $U_2 = r_2$ is empty by (16) and (b), which contradicts (c). $\square$

We now return to the proof of Lemma 3.10. Let q be a geodesic in $\Gamma(G, \mathcal{A})$ connecting p_- to p_+ . By our assumption, the subpath $p_2 e_2 \dots p_n e_n p_{n+1}$ of p is geodesic. Thus we can think of $p q^{-1}$ as a geodesic quadrilateral with sides p_1 , e_1 , $p_2 e_2 \dots p_n e_n p_{n+1}$, and q^{-1} . By Lemma 3.9 and condition (a), the $H_{i(1)}$ -component e_1 cannot be isolated in $p q^{-1}$; by Lemma 3.11, it is connected to a component f_1 of q (see Figure 8).

Furthermore, let $q = q_1 f_1 q'$ and let g be an edge of $\Gamma(G, \mathcal{A})$ connecting $(f_1)_+$ to $(e_1)_+$. Arguing as above, we conclude that e_2 cannot be isolated in the geodesic pentagon with sides g , p_2 , e_2 , $q_3 e_3 \dots q_n e_n p_{n+1}$, and $(q')^{-1}$. By Lemma 3.11, e_2 is connected to a component f_2 of q' . Thus q decomposes as $q_1 f_1 q_2 f_2 q''$. Continuing this process, we get a decomposition

$$q = q_1 f_1 q_2 f_2 \dots q_n f_n q_{n+1},$$

where each f_j is an $H_{i(j)}$ -component connected to e_j . Note that every q_j has length at least $\ell(p_j) = \|U_j\|$ by (b) as $H_{i(j-1)} \text{Lab}(q_j) H_{i(j)} = H_{i(j-1)} U_j H_{i(j)}$. It follows that $\ell(q) \geq \ell(p)$, which contradicts the assumption that p is not geodesic. $\square$

Proposition 3.12 *Let p be a path in $\Gamma(G, \mathcal{A})$. Suppose that $\text{Lab}(p) \in \mathcal{H}^*$ and the following conditions hold for all $i, j \in I$:*

- (a) *If a letter $a \in H_i$ occurs in $\text{Lab}(p)$, then $\hat{d}_{H_i}(1, a) > 5D$, where D is the constant provided by Lemma 3.9.*
- (b) *If $a \in H_i$ and $b \in H_j$ are labels of two consecutive edges in p , then $H_i \cap H_j = \{1\}$ (in particular, $i \neq j$).*

Then p is a 2-attracting geodesic.

Proof It is easy to see that conditions (a) and (b) imply all the assumptions of Lemma 3.10 (paths p_j are trivial in this case). Thus p is geodesic.

To show that p is 2-attracting, let $x, y \in G$ be any vertices of $\Gamma(G, \mathcal{A})$ such that

$$(17) \quad d_{\mathcal{A}}(x, y) - d_{\mathcal{A}}(x, p) - d_{\mathcal{A}}(y, p) > 2.$$

Let q be a geodesic going from x to y . Let u and v be vertices on p such that

$$(18) \quad d_{\mathcal{A}}(x, u) = d_{\mathcal{A}}(x, p) \quad \text{and} \quad d_{\mathcal{A}}(y, v) = d_{\mathcal{A}}(y, p).$$

Without loss of generality, we can assume that p first passes through u and then through v ; we denote by p_0 the segment of p between u and v . Furthermore, we assume that u and v are chosen among all vertices satisfying (18) so that $\ell(p_0) = d_{\mathcal{A}}(u, v)$ is minimal. We denote by s and t geodesics in $\Gamma(G, \mathcal{A})$ connecting x to u and y to v , respectively.

Note that every edge of p_0 is an H_i -component for some i , which cannot be isolated in the geodesic quadrilateral $sp_0t^{-1}q^{-1}$ by (a). Since p is geodesic, every component of p_0 is isolated in p_0 . Further, suppose that a component e of p_0 is connected to a component g of s . Then $d_{\mathcal{A}}(g-, e+) \leq 1 \leq d_{\mathcal{A}}(g-, u)$ and, consequently, $d_{\mathcal{A}}(x, u) = d_{\mathcal{A}}(x, e+)$, which contradicts our assumption that the choice of u and v minimizes $\ell(p_0)$. Therefore no component of p_0 is connected to a component of s . Similarly, we show that no component of p_0 is connected to a component of t .

Let

$$p_0 = e_1 e_2 \dots e_n,$$

where every e_i is an edge. Note that

$$d_{\mathcal{A}}(x, y) \leq d_{\mathcal{A}}(x, u) + d_{\mathcal{A}}(u, v) + d_{\mathcal{A}}(v, y) = d_{\mathcal{A}}(x, p) + n + d_{\mathcal{A}}(y, p).$$

Combining with (17) we obtain

$$(19) \quad n \geq d_{\mathcal{A}}(x, y) - d_{\mathcal{A}}(x, p) - d_{\mathcal{A}}(y, p) \geq 2.$$

As explained in the previous paragraph, e_1 is a component of p_0 connected to a certain component f_1 of q . Assume for definiteness that e_1 and f_1 are H_{i_1} -components for some $i_1 \in I$ and let c_1 denote the edge of $\Gamma(G, \mathcal{A})$ labeled by a letter from H_{i_1} that connects $(f_1)_+$ to $(e_1)_+$.

Let $q = q' f_1 q_1$. The edge e_2 is a component of p , which cannot be isolated in the geodesic quadrilateral $c_1(e_2 \dots e_n)t^{-1}(q_1)^{-1}$ by (a). Obviously e_2 cannot be connected to c_1 by (b). As we showed above, e_2 cannot be connected to a component of t either. Therefore e_2 is connected to a component f_2 of q_1 . Note that there exist edges in $\Gamma(G, \mathcal{A})$ connecting $(f_1)_-$ to $(e_1)_+ = (e_2)_-$ and $(e_1)_+ = (e_2)_-$ to $(f_2)_+$ (see Figure 10). Therefore $d_{\mathcal{A}}((f_1)_-, (f_2)_+) = 2$. Since q is geodesic, f_2 must be the first edge of q_1 . Assume for definiteness that e_2 and f_2 are H_{i_2} -components for some $i_2 \in I$. Since e_1 is connected to f_1 and e_2 is connected to f_2 , the vertices $a = (e_1)_+ = (e_2)_-$ and $b = (f_1)_+ = (f_2)_-$ belong to the same left H_{i_1} -coset and the same left H_{i_2} -coset of G . Therefore $a^{-1}b \in H_{i_1} \cap H_{i_2}$. Using (b), we obtain $a = b$.

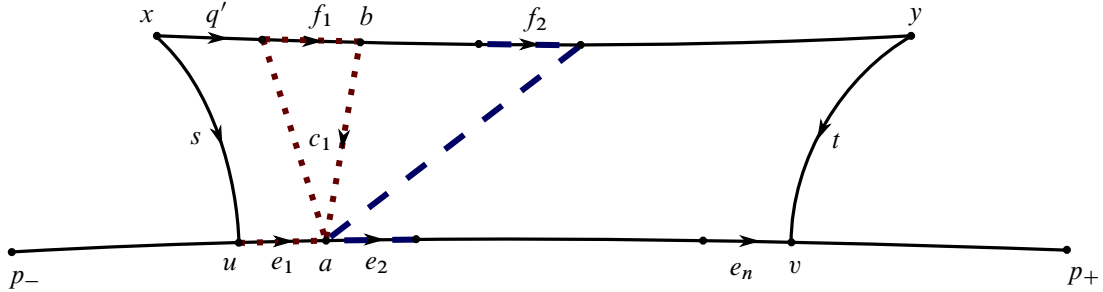


Figure 10: Proof of Proposition 3.12. Edges with the same style are labeled by letters from the same H_i .

If $n > 2$, we argue as above and show that e_3 is connected to the edge f_3 of q next to f_2 and f_2 shares its endpoints with e_2 . The latter condition means $e_2 = f_2$. Continuing this way, we obtain a decomposition

$$q = q' f_1 \dots f_n q'',$$

where for each $2 \leq j \leq n-1$, we have $f_j = e_j$. Thus p and q share a common subpath of length at least

$$n-2 \geq d_{\mathcal{A}}(x, y) - d_{\mathcal{A}}(x, p) - d_{\mathcal{A}}(y, p) - 2$$

(see (19)). □

3.4 The $W(\xi, \sigma)$ small cancellation condition

Constructing words satisfying the $C(\mu, \varepsilon, \rho)$ and $C_1(\mu, \varepsilon, \rho)$ small cancellation conditions discussed in Section 3.1 is not an easy task. Following [24], we now introduce an easily verifiable combinatorial condition that implies $C_1(\mu, \varepsilon, \rho)$.

Throughout this section, we fix a group G , a hyperbolically embedded collection of subgroups $\{H_i\}_{i \in I}$, and some $X \subseteq G$ such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$. Note that this last assumption is stronger than condition (*) from the previous section. Furthermore, we let $\mathcal{H}$ and $\mathcal{A}$ denote the alphabets defined by (1). We say that two letters a and b of a word $W \in \mathcal{A}^*$ are *cyclically consecutive* if they are consecutive or if a and b are the last and first letters of W , respectively.

Definition 3.13 A set of words $\mathcal{R} \subseteq \mathcal{H}^*$ satisfies the $W(\xi, \sigma)$ condition for some $\xi, \sigma \geq 0$ if the following hold:

(W₁) If $a \in H_i$ and $b \in H_j$ are cyclically consecutive letters of some word from $\mathcal{R}$, then $H_i \cap H_j = \{1\}$.

(W₂) If a letter $a \in H_i$ occurs in some word from $\mathcal{R}$, then $\hat{d}_{H_i}(1, a) \geq \xi$.

(W₃) For each letter $a \in \mathcal{H}$, there is at most one occurrence of $a^{\pm 1}$ in all words from $\mathcal{R}$. More precisely, let $R, S \in \mathcal{R}$. Suppose that $R \equiv R_1 a R_2$ and $S \equiv S_1 a^\varepsilon S_2$ for some $a \in \mathcal{H}$, $R_1, R_2, S_1, S_2 \in \mathcal{A}$, and $\varepsilon = \pm 1$. Then $\varepsilon = 1$, and $R_i \equiv S_i$ for $i = 1, 2$; in particular, $R \equiv S$.

(W₄) For every $R \in \mathcal{R}$, we have $\|R\| \geq \sigma$.

Remark 3.14 Although the condition $W(\xi, \sigma)$ makes sense for all groups with hyperbolically embedded subgroups, it is not really useful for groups with nontrivial finite radical. Indeed, in nondegenerate cases (eg if all H_i are infinite), one can use Corollary 2.10, (W_1) , and (W_4) to derive that the existence on a nonempty set $\mathcal{R}$ satisfying $W(\xi, \sigma)$ for some $\xi \geq 1$ and $\sigma \geq 2$ implies $K(G) = \{1\}$.

We consider a particular example. Suppose that $\{H_i\}_{i \in I} = \{A, B, \dots\}$, $A \cap B = \{1\}$, and there exist infinite-order elements $a \in A$ and $b \in B$. For any $m, n \in \mathbb{N}$, we think of a^m and b^n as single letters from the alphabet $\mathcal{H}$ that belong to A and B , respectively. For any $l \in \mathbb{N}$, let $\mathcal{P}(l)$ denote the set of all words

$$(20) \quad P_i = a^{il+1}b^{il+1}a^{il+2}b^{il+2} \dots a^{il+l}b^{il+l} \quad \text{for } i \in \mathbb{N}.$$

Lemma 3.15 For every $\xi, \sigma > 0$ there exists $N \in \mathbb{N}$ such that for any integer $l \geq N$, the set $\mathcal{P}(l)$ satisfies the $W(\xi, \sigma)$ condition.

Proof Conditions (W_1) and (W_3) obviously hold for each P_i . Since $\{H_i\}_{i \in I} \hookrightarrow_h G$ and the orders of a and b are infinite, we have

$$\lim_{n \rightarrow \infty} \hat{d}_A(1, a^n) = \lim_{n \rightarrow \infty} \hat{d}_B(1, b^n) = \infty$$

by part (b) of Definition 2.6. Therefore we can ensure that (W_2) holds for any given $\xi > 0$ by taking l large enough. Clearly, $\|P_i\| = 2l$. Thus we can also ensure (W_4) for any given σ by taking $l \geq \frac{1}{2}\sigma$. $\square$

The following theorem relates $W(\xi, \sigma)$ to the small cancellation conditions of Definition 3.1. Recall that a set of words in some alphabet is said to be *symmetric* if it is closed under taking inverses and cyclic shifts. The *symmetrization* of a set of words $\mathcal{R}$ is the smallest symmetric set of words containing $\mathcal{R}$; clearly, it coincides with the set of all cyclic shifts of $R \in \mathcal{R}$ and their inverses.

Lemma 3.16 For any positive constants ε, μ , and ρ , there exist positive ξ and σ such that, for any set of words $\mathcal{W} = \{W_j\}_{j \in J} \subseteq \mathcal{A}^*$ satisfying $W(\xi, \sigma)$, the following hold:

- (a) The symmetrization of $\mathcal{W}$ satisfies $C_1(\varepsilon, \mu, \rho)$.
- (b) For every $j \in J$, suppose the first (respectively, last) letter of W_j belongs to H_{j_1} (respectively, H_{j_2}) for some $j_1, j_2 \in I$. Let $\{x_j\}_{j \in J}$ be a set of letters from X such that the element represented by x_j in G does not belong to $H_{j_2}H_{j_1}$. Then the symmetrization of the set $\{x_j W_j\}_{j \in J}$ satisfies $C_1(\varepsilon, \mu, \rho)$.

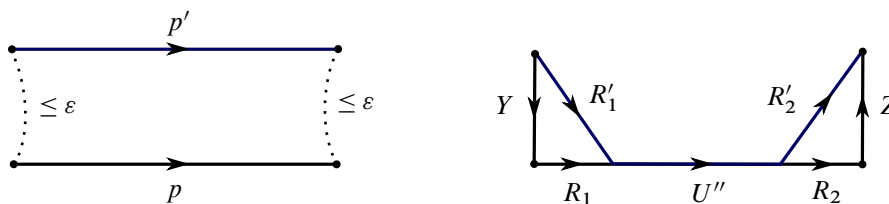
Remark 3.17 Recall that $C_1(\varepsilon, \mu, \rho)$ implies $C(\varepsilon, \mu, \rho)$. Thus the sets described in (a) and (b) also satisfy $C(\varepsilon, \mu, \rho)$.

Proof We first prove (a). Let ε, μ , and ρ be any positive numbers. Let ξ be any constant satisfying

$$(21) \quad \xi > 5D,$$

where D is the constant from Lemma 3.9. Further, we choose large enough σ so that

$$(22) \quad \sigma > \max\left\{\frac{2\varepsilon+2}{\mu}, \rho\right\}.$$

Figure 11: The proof of the implication $W(\xi, \sigma) \Rightarrow C_1(\varepsilon, \mu, \rho)$.

Let $\mathcal{W}$ be as above and let $\mathcal{R}$ denote the symmetrization of $\mathcal{W}$. Conditions (W_1) , (W_2) , and inequality (21) ensure that every path in $\Gamma(G, \mathcal{A})$ labeled by a word $R \in \mathcal{R}$ satisfies the assumptions of Proposition 3.12. Therefore every such R is $(G, \mathcal{A})$ -geodesic. Combining this with the inequality $\sigma > \rho$ and (W_4) , we conclude that condition (a) from Definition 3.1 holds.

Further, suppose that there are two relations $R, R' \in \mathcal{R}$ such that $R \equiv UV$, $R' \equiv U'V'$, and $U' = YUZ$ in G for some words $Y, Z \in \mathcal{A}^*$, and inequalities (5)–(6) hold. Without loss of generality, we can assume that $\|U'\| \geq \|U\|$; therefore (6) can be rewritten as

$$(23) \quad \|U'\| \geq \mu \min\{\|R\|, \|R'\|\}.$$

Translating these assumptions to geometric language, we can find paths p and p' in $\Gamma(G, \mathcal{A})$ such that

$$\text{Lab}(p) \equiv U, \quad \text{Lab}(p') \equiv U',$$

and

$$\max\{d_{\mathcal{A}}(p_-, p'_-), d_{\mathcal{A}}(p_+, p'_+)\} \leq \varepsilon$$

(see Figure 11, left). By (W_1) , (W_2) , and Proposition 3.12, p is a 2-attracting geodesic. Therefore p' and p share a common subpath p'' of length at least

$$(24) \quad \begin{aligned} \ell(p'') &\geq \ell(p') - d_{\mathcal{A}}(p'_-, p) - d_{\mathcal{A}}(p'_+, p) - 2 \geq \|U'\| - 2\varepsilon - 2 \geq \mu \min\{\|R\|, \|R'\|\} - 2\varepsilon - 2 \\ &\geq \mu\sigma - 2\varepsilon - 2 > 0. \end{aligned}$$

(We use (23) and (22) here.)

Let $\text{Lab}(p'') \equiv U''$, $\text{Lab}(p) \equiv R_1 U'' R_2$, and $\text{Lab}(p') \equiv R'_1 U'' R'_2$; see Figure 11, right. Condition (W_3) implies that the cyclic shifts of R and R' starting from U'' coincide; that is, $U'' R_2 V R_1 \equiv U'' R'_2 V' R'_1$. Reading the label of the leftmost triangle in Figure 11, right, we obtain $Y R_1 = R'_1$ in G . Therefore

$$Y R = Y R_1 U'' R_2 V = R'_1 U'' R_2 V R_1 R_1^{-1} = R'_1 U'' R'_2 V' R'_1 R_1^{-1} = R' Y$$

in G . Thus $Y R Y^{-1} = R'$ in G and $\mathcal{R}$ satisfies condition (b) in Definition 3.1.

Finally, suppose that some word $R \in \mathcal{R}$ contains two disjoint subwords U and U' such that $U' = YUZ$ or $U' = YU^{-1}Z$ in G for some words Y and Z , and (5)–(6) hold. Arguing as above, we can find two disjoint occurrences of the same word U'' in R , which contradicts (W_3) . Thus condition (c) in Definition 3.1 holds.

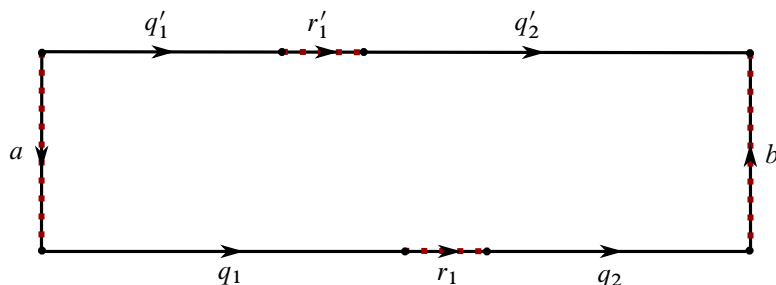


Figure 12: The geodesic octagon $aq_1rq_2b(q'_1r'_1q'_2)^{-1}$. Sides from the set B are given by the dotted paths.

To prove (b), we increase the parameter σ so that, in addition to (22), it satisfies

$$(25) \quad \sigma\mu > 1 + \max\{10^3(2\varepsilon + 2), 8c\},$$

where δ is the hyperbolicity constant of $\Gamma(G, \mathcal{A})$ and

$$c = \max\{10^3(26\delta + 2), 3 \cdot 10^4\delta\}.$$

Let $\mathcal{S}$ denote the symmetrization of the set $\{x_j W_j\}_{j \in J}$. We first observe that every $S \in \mathcal{S}$ satisfies the assumptions of Lemma 3.10 (note that $x_j \notin H_{j_2} H_{j_1}$ guarantees condition (b) of the lemma). Therefore every $S \in \mathcal{S}$ is $(G, \mathcal{A})$ -geodesic.

Further, suppose that there are two relations $S, S' \in \mathcal{S}$ such that

$$S \equiv UV, \quad S' \equiv U'V',$$

$U' = YUZ$ in G for some words $Y, Z \in \mathcal{A}^*$, and inequalities (5)–(6) hold. As above, we assume that $\|U'\| \geq \|U\|$, and rewrite (6) as in (23).

The difference with (a) is that the words U and U' may contain letters from the set $\{x_j\}_{i \in I}$ and thus the corresponding paths p and p' are not necessarily 2-attracting. However, we can overcome this difficulty as follows. Let $aqb(q')^{-1}$ be a geodesic quadrilateral in $\Gamma(G, \mathcal{A})$ such that

$$\text{Lab}(q) \equiv U, \quad \text{Lab}(q') \equiv U', \quad \text{Lab}(a) \equiv Y, \quad \text{Lab}(b) \equiv Z.$$

By (6),

$$\max\{\ell(a), \ell(b)\} \leq \varepsilon.$$

Note that q and q' are geodesic. We subdivide them as follows. If U contains some $x \in \{x_j^{\pm 1}\}_{i \in I}$, ie $U \equiv U_1 x U_2$, we let $q = q_1 r q_2$, where $\text{Lab}(q_1) \equiv U_1$, $\text{Lab}(r) \equiv x$, and $\text{Lab}(q_2) \equiv U_2$; otherwise, we let $q_1 = q$ and let r and q_2 be the trivial paths. Similarly, we define a subdivision $q' = q'_1 r' q'_2$ depending on whether U' contains a letter from $\{x_j^{\pm 1}\}_{i \in I}$ or not (see Figure 12).

Further, we divide the sides of the geodesic octagon $aq_1rq_2b(q'_1r'_1q'_2)^{-1}$ (some sides may be trivial) into the sets

$$A = \{q_1, q_2, q'_1, q'_2\} \quad \text{and} \quad B = \{a, b, r, r'\},$$

and let α (respectively, β) denote the sum of lengths of sides from the set A (respectively, B). By our construction, $\beta \leq 2\varepsilon + 2$. By (23), (W_4) , and (25), we have

$$\alpha \geq \ell(q') - 1 = \|U'\| - 1 \geq \mu \min\{\|R\|, \|R'\|\} - 1 \geq \mu\sigma - 1 > \max\{10^3\beta, 8c\}.$$

Thus we can apply Lemma 2.2 and find two segments p and p' of two distinct sides from A such that

$$\ell(p') > 10^{-3}c \geq 26\delta + 2$$

and

$$\max\{d(p_-, p'_-), d(p_+, p'_+)\} \leq 13\delta.$$

Since a letter from $\{x_j^{\pm 1}\}_{i \in I}$ can occur in $\text{Lab}(q)$ at most once, the word $\text{Lab}(p)$ does not contain any such letters. As in (a), we obtain that p is a 2-attracting geodesic by Proposition 3.12. Therefore p and p' share a common subpath p'' of length at least

$$\ell(p'') \geq \ell(p') - d_A(p'_-, p) - d_A(p'_+, p) - 2 > 26\delta + 2 - 13\delta - 13\delta - 2 = 0.$$

Since q and q' are geodesic, p and p' cannot simultaneously be subpaths of q or q' . Without loss of generality, we can assume that p is a subpath of q and p' is a subpath of q' . Now, the equality $YRY^{-1} \equiv R'$ in G is derived exactly as in (a). This gives us condition (b) in Definition 3.1.

The proof of condition (c) in Definition 3.1 is similar. Assuming that some word $S \in \mathcal{S}$ contains two disjoint subwords U and V such that $U' = YUZ$ or $U' = YU^{-1}Z$ in G and (5)–(6) hold, we argue as above and find two disjoint occurrences of a certain nonempty word in R . This contradicts (W_3) . $\square$

We conclude this section with the analogue of the Greendlinger lemma for the $W(\xi, \rho)$ small cancellation condition.

Proposition 3.18 *Let G be a group, $\{H_i\}_{i \in I}$ a collection of subgroups of G , and X a subset of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$. Let $\mathcal{H}$ and $\mathcal{A}$ denote the alphabets defined by (1). For every $\nu > 0$, there exist constants $\xi, \sigma > 0$ such that the following holds.*

Let $\mathcal{R}$ be a set of words in the alphabet $\mathcal{H}$ satisfying $W(\xi, \sigma)$. Suppose that a nontrivial $(G, \mathcal{A})$ -geodesic word $W \in \mathcal{A}^*$ represents the identity in $G/\langle\langle \mathcal{R} \rangle\rangle$. Then there exists $R \in \mathcal{R}$ such that a cyclic shift of $R^{\pm 1}$ and W have a common subword U of length $\|U\| \geq (1 - \nu)\|R\|$.

Proof We choose any $\mu \in (0, \frac{1}{16}]$ such that $\mu < \frac{1}{13}\nu$. Further, let ρ and ε be the corresponding constants provided by Lemma 3.2 for the group G . By Lemma 3.16, we can choose ξ and σ such that, for any set of words $\mathcal{R}$ in the alphabet $\mathcal{H}$ satisfying $W(\xi, \sigma)$, the symmetrization of $\mathcal{R}$ satisfies $C(\varepsilon, \mu, \rho)$. Note that increasing ξ and σ only makes the condition $W(\xi, \sigma)$ stronger. Thus we can assume that $\xi > 5D$, where D is provided by Lemma 3.9, $\sigma \geq \rho$, and

$$(26) \quad \mu < \frac{1}{13}\nu - \frac{4\varepsilon + 2}{13\sigma}.$$

We fix any presentation of G of the form (3).

By Lemma 3.2, for every $(G, \mathcal{A})$ -geodesic word $W \in \mathcal{A}^*$ representing the identity in $G/\langle\langle \mathcal{R} \rangle\rangle$, there exists a van Kampen diagram Δ over the presentation (4), an $\mathcal{R}$ -face Π in Δ , and an ε -contiguity diagram Γ of Π to $\partial\Delta$ such that $(\Pi, \Gamma, \partial\Delta) \geq 1 - 13\mu$. We keep the notation introduced in Figure 3 for the boundary sides of Γ . In this notation,

$$(27) \quad \ell(q_1) = (\Pi, \Gamma, \partial\Delta)\ell(\partial\Pi) > (1 - 13\mu)\ell(\partial\Pi) \geq \left(1 - \nu + \frac{4\varepsilon + 2}{\sigma}\right)\ell(\partial\Pi) \geq (1 - \nu)\ell(\partial\Pi) + 4\varepsilon + 2$$

(to derive the last two inequalities, we use (26) and (W_4)).

Recall that Γ is a diagram over (3). Therefore $\partial\Gamma$ can be mapped to $\Gamma(G, \mathcal{A})$ by a combinatorial map preserving labels and orientation of edges. We keep the notation $s_1q_1s_2q_2$ for the image of $\partial\Gamma$ in $\Gamma(G, \mathcal{A})$. Since q_2 is labeled by a subword of W , it is geodesic in $\Gamma(G, \mathcal{A})$. By Proposition 3.12, q_1 is a 2-attracting geodesic in $\Gamma(G, \mathcal{A})$. Therefore q_1 and q_2^{-1} share a subpath q of length at least

$$\ell(q) \geq \ell(q_2) - \ell(s_1) - \ell(s_2) - 2 \geq \ell(q_1) - 2\ell(s_1) - 2\ell(s_2) - 2 \geq \ell(q_1) - 4\varepsilon - 2.$$

Combining this inequality with (27), we obtain that W and $\text{Lab}(\partial\Pi)$ share a common subword $U \equiv \text{Lab}(q)$ of length at least $(1 - \nu)\|\text{Lab}(\partial\Pi)\|$. $\square$

3.5 Suitable subgroups and quotients

In this section, we summarize results from [18; 15; 24] and obtain some new results necessary to prove Theorems 1.3 and 1.4. We begin by recalling the notion of a suitable subgroup. It was originally introduced in [24] for relatively hyperbolic groups and then generalized to groups with hyperbolically embedded subgroups in [15]. Here we will use this notion in both settings.

We begin with a definition from [15, Section 5]:

Definition 3.19 Let G be a group and $\mathcal{A}$ a generating alphabet such that the Cayley graph $\Gamma(G, \mathcal{A})$ is hyperbolic. A subgroup $S \leq G$ is said to be $\mathcal{A}$ -suitable (suitable with respect to $\mathcal{A}$ in the terminology of [15]) if the action of S on $\Gamma(G, \mathcal{A})$ is nonelementary and S does not normalize any nontrivial finite subgroup of G .

Recall that two elements a and b of a group are commensurable if a^m is conjugate to b^n for some nonzero integers m and n .

Lemma 3.20 [15, Corollary 5.7] Let G be a group, and $\mathcal{A}$ a generating alphabet of G such that $\Gamma(G, \mathcal{A})$ is hyperbolic and the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical. Suppose that S is an $\mathcal{A}$ -suitable subgroup of G . Then, for every $n \in \mathbb{N}$, there exist pairwise noncommensurable elements $g_1, \dots, g_n \in S$ such that g_i acts loxodromically on $\Gamma(G, \mathcal{A})$ and $E(g_i) = \langle g_i \rangle$ for all $i = 1, \dots, n$.

For relatively hyperbolic groups, there is a similar notion of suitability defined as follows [3; 24]:

Definition 3.21 Let G be a group hyperbolic relative to a collection of subgroups $\{H_i\}_{i \in I}$. A subgroup $S \leq G$ is suitable with respect to the peripheral collection $\{H_i\}_{i \in I}$ if it is not virtually cyclic, contains a loxodromic element, and does not normalize any nontrivial finite subgroup of G .

Recall that, in the setting of Definition 3.21, an element $g \in G$ is loxodromic with respect to the action on $\Gamma(G, \mathcal{A})$ if and only if $|g| = \infty$ and g is not conjugate to an element of one of the peripheral subgroups. The definition of a suitable subgroup of a relatively hyperbolic group was first formulated in [24] in a slightly different way. Later it was shown to be equivalent to the above Definition 3.21 in [3] (see Lemma 3.3 and Proposition 3.4 there). Furthermore, Definitions 3.19 and 3.21 are equivalent in the following sense:

Lemma 3.22 *Let G be a group, hyperbolic relative to a finite collection of subgroups $\{H_i\}_{i \in I}$, X a finite relative generating set of G with respect to $\{H_i\}_{i \in I}$, and $\mathcal{A}$ the alphabet defined by (1). A subgroup $S \leq G$ is suitable with respect to the peripheral collection $\{H_i\}_{i \in I}$ if and only if S is $\mathcal{A}$ -suitable.*

Proof By Theorem 2.16, the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical. If S is $\mathcal{A}$ -suitable, it contains a loxodromic element by Lemma 3.20 and is not virtually cyclic because its action on $\Gamma(G, \mathcal{A})$ is nonelementary (see Theorem 2.15). Thus S is suitable with respect to the peripheral collection $\{H_i\}_{i \in I}$. Conversely, if S is suitable with respect to the peripheral collection $\{H_i\}_{i \in I}$, then its action on $\Gamma(G, \mathcal{A})$ must be nonelementary by Theorem 2.15; hence S is $\mathcal{A}$ -suitable. $\square$

The next result provides a sufficient condition for a subgroup to be suitable:

Lemma 3.23 *Let G be a group and $\mathcal{A}$ a generating alphabet of G such that $\Gamma(G, \mathcal{A})$ is hyperbolic and the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical and nonelementary. If $K(G) = \{1\}$, then every nontrivial normal subgroup of G is $\mathcal{A}$ -suitable.*

Proof Let S be a nontrivial normal subgroup of G . Since G contains no nontrivial finite normal subgroups, S must be infinite. By Lemma 2.19, the action of S on $\Gamma(G, \mathcal{A})$ is nonelementary. Acylindricity of the action of G on the Cayley graph $\Gamma(G, \mathcal{A})$ allows us to apply [15, Lemma 5.5], which claims the existence of a (unique) maximal finite subgroup $E \leq G$ normalized by S . Since S is normal in G , so is E . In particular, $E \leq K(G)$, which implies $E = \{1\}$. Thus S is $\mathcal{A}$ -suitable. $\square$

Lemma 3.24 *Let G be a group, $\{H_i\}_{i \in I}$ a collection of subgroups of G , X a subset of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$, $\mathcal{A}$ an alphabet defined by (1), and S a subgroup of G . Suppose that the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical and there exist $i_0, j_0 \in I$ such that $H_{i_0} \cap H_{j_0} = \{1\}$ and the intersections $S \cap H_{i_0}$ and $S \cap H_{j_0}$ are infinite. Then S is $\mathcal{A}$ -suitable.*

Proof Let D be the constant provided by Lemma 3.9 applied to $\{H_i\}_{i \in I} \hookrightarrow_h (G, Y)$ and let $\hat{d}_{H_i}$ be the corresponding generalized metrics on groups H_i . Since the intersections $S \cap H_{i_0}$ and $S \cap H_{j_0}$ are infinite, there exist $a \in S \cap H_{i_0}$ and $b \in S \cap H_{j_0}$ such that $\hat{d}_{H_{i_0}}(1, a) > 5D$ and $\hat{d}_{H_{j_0}}(1, b) > 5D$. By Lemma 3.10, every path in $\Gamma(G, \mathcal{A})$ labeled by $(ab)^n$ for some $n \in \mathbb{N}$ is geodesic. Therefore the action of S on $\Gamma(G, \mathcal{A})$ has unbounded orbits. Note also that S cannot be virtually cyclic; otherwise, any two infinite subgroups of S would have nontrivial intersection, which contradicts the assumption $H_{i_0} \cap H_{j_0} = \{1\}$. Therefore the action of S on $\Gamma(G, \mathcal{A})$ is nonelementary by Theorem 2.15. Finally, we note that S does not normalize any nontrivial finite subgroup of G by Corollary 2.10. $\square$

To make our paper as self-contained as possible, we summarize some results from [15; 18] used below. The first one is proved in [18, Proposition 3.3]. We simplify it and change the notation a bit to better fit the other results discussed below.

Lemma 3.25 *Let G be a group, and let $\{H_i\}_{i \in I}$ and $\{A, B\}$ be two collections of subgroups of G . Suppose that $A \cap B = \{1\}$ and $\{H_i\}_{i \in I} \cup \{A, B\} \hookrightarrow_h (G, X)$ for some $X \subseteq G$. Then there exists $n \in \mathbb{N}$ and finite subsets $\mathcal{F}_A \subseteq A$ and $\mathcal{F}_B \subseteq B$ such that the following holds.*

Suppose that $\mathcal{W} = \{W_m\}_{m \in \mathbb{N}}$ is an arbitrary set of words in the alphabet $X \sqcup A \sqcup B$ of the form

$$(28) \quad W_m \equiv x_m a_{m1} b_{m1} \dots a_{mn} b_{mn}$$

satisfying the following conditions for every $m \in \mathbb{N}$:

- (a) $x_m \in X$.
- (b) $a_{ml} \in A \setminus \mathcal{F}_A$ and $b_{ml} \in B \setminus \mathcal{F}_B$ for all $1 \leq l \leq n$.
- (c) *If a letter $c \in A \sqcup B$ occurs in W_m for some $m \in \mathbb{N}$, then it occurs only in W_m and only once; in addition, c^{-1} does not occur in any word from $\mathcal{W}$.*

Then the restriction of $\gamma: G \rightarrow G/\langle\langle \mathcal{W} \rangle\rangle$ to each H_i is injective and $\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h G/\langle\langle \mathcal{W} \rangle\rangle$.

The next result combines Lemmas 4.4 and 4.9 from [15] (note that the result about relative hyperbolicity in (b) follows from $\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h (\bar{G}, \gamma(X))$ and Proposition 2.8; it was first proved in [24, Lemma 5.1]). As everywhere in this paper, we reproduce the simplified versions of the abovementioned results for the $C(\varepsilon, \mu, \rho)$ small cancellation condition instead of the more general condition $C(\varepsilon, \mu, \lambda, c, \rho)$ considered in [15; 24]. Note that, unlike in the previous proposition, the set $\mathcal{R}$ is assumed to be finite here.

Lemma 3.26 *Let G be a group, $\{H_i\}_{i \in I}$ a collection of subgroups of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$, and $\mathcal{A}$ an alphabet defined by (1). There exist $\varepsilon \geq 0$ and $\mu, \rho > 0$ such that, for any finite symmetric set of words $\mathcal{R}$ in the alphabet $\mathcal{A}$ satisfying $C(\varepsilon, \mu, \rho)$, the following hold:*

- (a) *The restriction of the natural homomorphism $\gamma: G \rightarrow \bar{G} = G/\langle\langle \mathcal{R} \rangle\rangle$ to the set*

$$(29) \quad B_N = \{g \in G \mid |g|_{\mathcal{A}} \leq N\}$$

is injective. In particular, the restriction of γ to $\bigcup_{i \in I} H_i$ is injective.

- (b) $\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h (\bar{G}, \gamma(X))$. *In particular, if G is hyperbolic relative to $\{H_i\}_{i \in I}$ and $|X| < \infty$, then $\bar{G}$ is hyperbolic relative to $\{\gamma(H_i)\}_{i \in I}$.*
- (c) *Every finite-order element of $\bar{G}$ is the image of a finite-order element of G .*

It is worth noting that in the statement of [15, Lemma 4.4], Hull only claims that $\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h \bar{G}$ in the setting of Lemma 3.26. However, the proof (which is essentially the same as the one of [24, Lemma 5.1]) actually shows that $\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h (\bar{G}, \gamma(X))$.

We are now ready to state the main result of this section. It generalizes and strengthens analogous results of [15; 21; 24]. Compared to [15], the main improvement is that we allow the set $\mathcal{F}$ to be infinite. This generalization is essential for some applications, eg for the proof of Theorem 1.4. The result about centralizers (d) is also new in this setting. We provide a complete proof modulo results of [15; 18] discussed above to make sure that these new additions are consistent with previously known parts.

Theorem 3.27 *Let G be a group, $\{H_i\}_{i \in I}$ a collection of subgroups of G such that $\{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ for some $X \subseteq G$, and $\mathcal{A}$ an alphabet defined by (1). Suppose that the action of G on $\Gamma(G, \mathcal{A})$ is acylindrical. For any $\mathcal{A}$ -suitable subgroup $S \leq G$, any countable subset $\mathcal{F} \subseteq X$, and any $N \in \mathbb{N}$, there exists a subset $\{s_f \mid f \in \mathcal{F}\} \subseteq S$ such that the quotient group*

$$(30) \quad \bar{G} = G / \langle\langle \mathcal{R} \rangle\rangle \quad \text{where } \mathcal{R} = \{fs_f \mid f \in \mathcal{F}\}$$

satisfies the following conditions:

- (a) *The restriction of the natural homomorphism $\gamma: G \rightarrow \bar{G}$ to the set B_N defined by (29) is injective. In particular, γ is injective on $\bigcup_{i \in I} H_i$.*
- (b) *$\{\gamma(H_i)\}_{i \in I} \hookrightarrow_h \bar{G}$.*
- (c) *Every finite-order element of $\bar{G}$ is the image of a finite-order element of G .*
- (d) *For every $g \in B_N$, we have $C_{\bar{G}}(\gamma(g)) = \gamma(C_G(g))$.*
- (e) *If, in addition, the sets I , $\mathcal{F}$, and X are finite, then $\bar{G}$ is hyperbolic relative to $\{\gamma(H_i)\}_{i \in I}$.*

Remark 3.28 In the notation of Theorem 3.27, $\gamma(\mathcal{F}) \subseteq \gamma(S)$ since $s_f \in S$ for all $f \in \mathcal{F}$.

Proof By Lemma 3.20, there exist noncommensurable elements $a, b \in S$ acting loxodromically on $\Gamma(G, \mathcal{A})$ such that $E(a) = \langle a \rangle$ and $E(b) = \langle b \rangle$. Note that $E(a) \cap E(b) = \{1\}$ by Proposition 2.9. Furthermore, $\{\langle a \rangle, \langle b \rangle\} \cup \{H_i\}_{i \in I} \hookrightarrow_h (G, X)$ by Theorem 2.25. We use the notation $A = \langle a \rangle$, and $B = \langle b \rangle$, which is consistent with Lemma 3.26.

Let $\mathcal{F} = \{f_1, f_2, \dots\}$. Since $C_1(\varepsilon, \mu, \rho)$ implies $C(\varepsilon, \mu, \rho)$ and both conditions become stronger as ε and ρ increase and μ decreases, we can choose sufficiently large ε and ρ and sufficiently large μ such that the conclusions of Lemmas 3.5 and 3.26 simultaneously hold. Further, let ξ and σ be the constants provided by Lemma 3.16 for the chosen values of ε , μ , and ρ . By Lemma 3.15, there exists $l \in \mathbb{N}$ such that the set of words $\mathcal{P}(l) = \{P_i \mid i \in \mathbb{N}\}$ defined by (20) satisfies $W(\xi, \sigma)$. Combining this with Lemma 3.16(b), we conclude that the symmetrization of the set $\{f_1 P_1, f_2 P_2, \dots\}$ satisfies the $C_1(\varepsilon, \mu, \rho)$ small cancellation condition. In addition, increasing l if necessary, we can ensure that all letters in the words P_i come from the sets $A \setminus \mathcal{F}_A$ and $B \setminus \mathcal{F}_B$, where $\mathcal{F}_A$ and $\mathcal{F}_B$ are provided by Lemma 3.25 (we use the condition $\{A, B\} \hookrightarrow_h G$ and Definition 2.6(b) here).

Since $a, b \in S$, each P_i represents an element of S in G . Thus it suffices to show that claims (a)–(e) hold for the group

$$\bar{G} = G / \langle\langle f_1 P_1, f_2 P_2, \dots \rangle\rangle.$$

Thanks to the choice of the parameters explained in the previous paragraph, we can apply Lemmas 3.5, 3.25, and 3.26 to this quotient. Parts (a) and (c) of Lemma 3.26 give us parts (a) and (c) of the theorem, Lemma 3.25 gives (b), and Lemma 3.5 gives (d).

Since $\mathcal{F}$ is finite, so is $\mathcal{R}$. Therefore $\bar{G}$ is hyperbolic relative to $\{H_i\}_{i \in I} \cup \{A, B\}$ by part (b) of Lemma 3.26. Since A and B are cyclic, $\bar{G}$ is hyperbolic relative to $\{H_i\}_{i \in I}$ by Corollary 2.14. $\square$

4 Automorphisms of property-(T) groups acting on hyperbolic spaces

4.1 Auxiliary groups

The main goal of this section is to construct examples of groups with certain special properties, which will be used in the proofs of Theorems 1.3 and 1.4. We begin with a variant of the famous Rips construction [28]. The modification considered here is similar to those suggested in [4; 19].

Lemma 4.1 *Let Q be a countable group and H a noncyclic torsion-free hyperbolic group. There exists a short exact sequence $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ such that the following conditions hold:*

- (a) G is torsion-free.
- (b) N is an infinite quotient of H .
- (c) *There are elements $a, b \in N$ such that $\langle a \rangle \cap \langle b \rangle = \{1\}$ and $\{\langle a \rangle, \langle b \rangle\} \hookrightarrow_h G$. In particular, G is acylindrically hyperbolic.*
- (d) *If in addition Q is finitely presented, then G is hyperbolic relative to $\{\langle a \rangle, \langle b \rangle\}$; in particular, G is a hyperbolic group in this case.*

Proof Suppose that Q is given by a (possibly infinite) presentation $\langle Y \mid \mathcal{R} \rangle$. The free product $P = H * F(Y)$, where $F(Y)$ is the free group with basis Y , is hyperbolic relative to $\{H, F(Y)\}$. Since H is noncyclic, torsion-free, and hyperbolic, it contains two noncommensurable elements a and b such that $E(a) = \langle a \rangle$ and $E(b) = \langle b \rangle$; see, for example, [21, Lemma 3.8]. Applying Corollary 2.26, we obtain that H is hyperbolic relative to $\{\langle a \rangle, \langle b \rangle\}$. By Corollary 2.12, the group P is hyperbolic relative to $\{\langle a \rangle, \langle b \rangle, F(Y)\}$.

Since a and b are noncommensurable, $\langle a \rangle \cap \langle b \rangle = \{1\}$. Therefore H is a suitable subgroup of P with respect to $\{\langle a \rangle, \langle b \rangle, F(Y)\}$ by Lemmas 3.24 and 3.22.

Let X be a finite generating set of H and let

$$\mathcal{F} = \{y^{-1}xy, yxy^{-1} \mid x \in X, y \in Y\} \cup \mathcal{R}.$$

Note that $\{\langle a \rangle, \langle b \rangle\} \hookrightarrow_h (H, X)$ by Proposition 2.8. So $\{\langle a \rangle, \langle b \rangle, F(Y)\} \hookrightarrow_h (P, X)$ by Proposition 2.11. This allows us to apply Theorem 3.27 and conclude that, for every $f \in \mathcal{F}$, there exists $s_f \in H$ such that the quotient group

$$(31) \quad G = P / \langle\langle fs_f \mid f \in \mathcal{F} \rangle\rangle$$

is torsion-free, the restriction of the natural homomorphism $P \rightarrow G$ to each of the subgroups $\langle a \rangle$, $\langle b \rangle$, and $F(Y)$ is injective, and the (isomorphic) images of these subgroups are hyperbolically embedded in G . In particular, G is acylindrically hyperbolic by Theorem 2.18. Let N be the image of H in G . The choice of the set $\mathcal{F}$ ensures that $N \triangleleft G$. Since N contains the (isomorphic) image of $\langle a \rangle$, it is infinite.

Finally, if Q is finitely presented, we can assume that Y and $\mathcal{R}$ are finite. This implies that $|\mathcal{F}| < \infty$. Thus Theorem 3.27(e) applies and G is hyperbolic relative to the isomorphic images of the subgroups $\langle a \rangle$, $\langle b \rangle$, and $F(Y)$. Since each of these subgroups is hyperbolic in this case (recall that $|Y| < \infty$), G is a hyperbolic group by Corollary 2.14. $\square$

We are now ready to prove the main result of this section. Note that any pair of groups $N \triangleleft G$ satisfying conditions (d) and (e) (see below) has the following rigidity property: every automorphism φ of N preserves the conjugacy classes of H_1 and H_2 in G . It follows that φ maps x^2 and y^3 to their conjugates (or possibly conjugates of their inverses) in G . This property can be used to control the outer automorphism group of appropriate quotients of N .

Lemma 4.2 *Let Q be countable group and H a noncyclic torsion-free hyperbolic group. There exists a short exact sequence*

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

and subgroups

$$(32) \quad H_1 = \langle s, x \mid s^2, [s, x] \rangle \cong \mathbb{Z}_2 \times \mathbb{Z} \quad \text{and} \quad H_2 = \langle t, y \mid t^3, [t, y] \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}$$

of N satisfying the following conditions:

- (a) $H_1 \cap H_2 = \{1\}$ and $\{H_1, H_2\} \hookrightarrow_h G$. In particular, G is acylindrically hyperbolic.
- (b) N is a quotient of H .
- (c) N is generated by $\{x^2, y^3\}$.
- (d) $H_1 = C_G(s)$ and $H_2 = C_G(t)$.
- (e) Every finite-order element of N is conjugate in G to one of the elements $1, s, t$, or t^2 .
- (f) If Q is finitely presented, then G is hyperbolic relative to $\{H_1, H_2\}$; in particular, G is a hyperbolic group in this case.

Proof Let $1 \rightarrow N_0 \rightarrow G_0 \rightarrow Q \rightarrow 1$ be the short exact sequence provided by Lemma 4.1. That is, G_0 is torsion-free and acylindrically hyperbolic, N_0 is an infinite quotient of H , and G_0 is hyperbolic whenever Q is finitely presented.

The group

$$P = G_0 * H_1 * H_2$$

is hyperbolic relative to $\{G_0, H_1, H_2\}$. Using the standard facts about free products and the structure of H_1 and H_2 , it is easy to show that the following conditions hold:

(+) $H_1 = C_P(s)$ and $H_2 = C_P(t)$.

(++) Every finite-order element in P is conjugate to 1, s , t , or t^2 .

By Theorem 2.18, there exists a generating set $\mathcal{A}_0$ of G_0 such that the Cayley graph $\Gamma(G_0, \mathcal{A}_0)$ is hyperbolic, and the action of G_0 on $\Gamma(G_0, \mathcal{A}_0)$ is nonelementary and acylindrical. Since G_0 is torsion-free, it contains no nontrivial finite subgroups. Therefore N_0 is $\mathcal{A}_0$ -suitable in G_0 by Lemma 3.23.

Let $a, b \in N_0$ be the elements provided by Lemma 4.1. Thus we have $\{E(a), E(b)\} \hookrightarrow_h (G_0, \mathcal{A}_0)$. Since $\{G_0, H_1, H_2\} \hookrightarrow_h P$, we obtain that $\{E(a), E(b), H_1, H_2\} \hookrightarrow_h (P, Y)$ for some $Y \subseteq P$ by Proposition 2.11. Let $\mathcal{B} = Y \sqcup E(a) \sqcup E(b) \sqcup H_1 \sqcup H_2$. By Theorem 2.16(a), we can assume that the action of P on $\Gamma(P, \mathcal{B})$ is acylindrical. By Lemma 3.24, we conclude that N_0 is a $\mathcal{B}$ -suitable subgroup of P .

Let $s_1, \dots, s_k \in N_0$ be the elements provided by Theorem 3.27 applied to the group P , the hyperbolically embedded collection of subgroups $\{E(a), E(b), H_1, H_2\} \hookrightarrow_h (P, Y)$, the $\mathcal{B}$ -suitable subgroup N_0 , and a finite generating set $\mathcal{F} = \{f_1, \dots, f_k\}$ of $H_1 * H_2$. Let

$$(33) \quad G_1 = P / \langle\langle f_1 s_1, \dots, f_k s_k \rangle\rangle.$$

We keep the notation $E(a)$, $E(b)$, H_1 , and H_2 for the isomorphic images of these subgroups in G_1 and denote the image of N_0 in G_1 by N_1 . By Theorem 3.27, $\{E(a), E(b), H_1, H_2\} \hookrightarrow_h (G_1, X_1)$ for some $X_1 \subseteq G$. We let

$$\mathcal{A}_1 = X_1 \sqcup E(a) \sqcup E(b) \sqcup H_1 \sqcup H_2.$$

As above, we can assume that the action of G_1 on $\Gamma(G_1, \mathcal{A}_1)$ is acylindrical by Theorem 2.16(a). Since $s_1, \dots, s_k \in N_0$, we have

$$(34) \quad H_1, H_2 \leq N_1.$$

In particular, the restriction of the natural homomorphism $\gamma_0: P \rightarrow G_1$ to G_0 is surjective. Furthermore, $\text{Ker } \gamma_0 \leq \langle\langle N_0 \cup H_1 \cup H_2 \rangle\rangle$ (see (33)) and $\gamma_0(\langle\langle N_0 \cup H_1 \cup H_2 \rangle\rangle) = N_1$. Therefore,

$$G_1/N_1 \cong P/\langle\langle N_0 \cup H_1 \cup H_2 \rangle\rangle \cong G_0/N_0 \cong Q.$$

By Lemma 3.24, the subgroup $S = \langle x^2, y^3 \rangle \leq G_1$ is $\mathcal{A}_1$ -suitable in G_1 . Note that N_1 is finitely generated being a quotient of H . We can apply Theorem 3.27 to the group G_1 with the collection of hyperbolically embedded subgroups $\{E(a), E(b), H_1, H_2\}$, the suitable subgroup S , and a finite set $\{g_1, \dots, g_l\}$ of generators of N_1 . Let $t_1, \dots, t_l \in S$ be the elements provided by the theorem and let

$$G = G_1 / \langle\langle g_1 t_1, \dots, g_l t_l \rangle\rangle.$$

Let γ denote the natural homomorphism $G_1 \rightarrow G$ and let $N = \gamma(N_1)$. Since $g_1, \dots, g_l \in N_1$ and $t_1, \dots, t_l \in S \leq N_1$, we have $\text{Ker}(\gamma) \leq N_1$. Therefore $G/N \cong G_1/N_1 \cong Q$. Our construction can be

summarized by the following commutative diagram, where all rows are exact and all vertical maps are surjective:

$$\begin{array}{ccccccc}
 & & H & & & & \\
 & & \downarrow & & & & \\
 1 & \longrightarrow & N_0 & \longrightarrow & G_0 & \longrightarrow & Q \longrightarrow 1 \\
 & & \downarrow & & \downarrow \gamma_0 & & \parallel \\
 1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \longrightarrow & Q \longrightarrow 1 \\
 & & \downarrow & & \downarrow \gamma & & \parallel \\
 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q \longrightarrow 1
 \end{array}$$

We claim that the short exact sequence at the bottom satisfies the conclusion of the theorem. Indeed, the groups H_1 and H_2 embed in G since the restriction of γ to $H_1 \cup H_2$ is injective by Theorem 3.27(a). Moreover, we have $H_1, H_2 \leq N$ by (34). Further, (a), (d), and (e) follow from the corresponding parts of Theorem 3.27, and (+) and (++). Part (b) is obvious from the construction. Part (c) is ensured by passing from G_1 to G .

Finally, (f) can be proved as follows. If Q is finitely presented, G_0 is hyperbolic relative to $\{E(a), E(b)\}$ by Lemma 4.1 and P is hyperbolic relative to $\{E(a), E(b), H_1, H_2\}$ by Corollary 2.12. In particular, $\{E(a), E(b), H_1, H_2\} \hookrightarrow_h (P, Y)$ for some finite Y and the action of P on $\Gamma(P, \mathcal{B})$ is acylindrical by Proposition 2.8. Thus Theorem 3.27(e) applies and we obtain that G_1 is hyperbolic relative to (the isomorphic image of) $\{E(a), E(b), H_1, H_2\}$. Similarly, choosing finite X_1 allows us to conclude that G is hyperbolic relative to $\{E(a), E(b), H_1, H_2\}$. Since each of the subgroups in this collection is hyperbolic, so is G by Corollary 2.14. $\square$

4.2 Proofs of Theorems 1.3 and 1.4

We begin by generalizing some observations used in [4; 19].

Definition 4.3 To each short exact sequence of groups

$$(35) \quad 1 \rightarrow N \rightarrow G \xrightarrow{\varepsilon} Q \rightarrow 1,$$

we associate homomorphisms

$$\iota: G \rightarrow \text{Aut}(N) \quad \text{and} \quad \kappa: Q \rightarrow \text{Out}(N)$$

as follows. For every $g \in G$, $\iota(g)$ is the automorphism of N given by $n \mapsto gng^{-1}$. Further, given an element $q \in Q$, let g be any preimage of q under ε . We define

$$(36) \quad \kappa(q) = \iota(g) \text{Inn}(N).$$

Note that the map κ is well defined. Indeed the right side of (36) is independent of the choice of a particular preimage g .

Lemma 4.4 *In the notation of Definition 4.3, suppose that G is acylindrically hyperbolic and $K(G) = \{1\}$. If N is nontrivial, then the maps $\iota: G \rightarrow \text{Aut}(N)$ and $\kappa: Q \rightarrow \text{Out}(N)$ associated to the short exact sequence (35) are injective.*

Proof By Theorem 2.18, there exists a generating set X of G such that $\Gamma(G, X)$ is hyperbolic and the action of G on $\Gamma(G, X)$ is nonelementary and acylindrical. By Lemma 3.23, N is an X -suitable subgroup of G . By Lemma 3.20 and Theorem 2.25, there exist elements $a, b \in N$ of infinite order such that $\{\langle a \rangle, \langle b \rangle\} \hookrightarrow_h G$.

Suppose that $g \in \text{Ker}(\iota)$. Then g commutes with both a and b . By Proposition 2.9, $g \in \langle a \rangle \cap \langle b \rangle = \{1\}$. Thus ι is injective.

Further, suppose that $q \in \text{Ker}(\kappa)$. This means that $\iota(g)$ is an inner automorphism of N for some preimage g of q under ε . That is, there is $h \in N$ such that $\iota(g) = \iota(h)$. By injectivity of ι , we have $g = h$. This implies that $q = \varepsilon(g) = \varepsilon(h) = 1$. $\square$

We are now ready to prove our main result. Theorems 1.3 and 1.4 can be easily derived from this result, as explained below.

Theorem 4.5 *For every countable group Q and every noncyclic torsion-free hyperbolic group H , there exists an ICC quotient group B of H with the following properties:*

- (a) $\text{Out}(B) \cong Q$.
- (b) *The groups B and $\text{Aut}(B)$ are acylindrically hyperbolic. Moreover, if Q is finitely presented, then $\text{Aut}(B)$ is hyperbolic.*

Proof Let Q be a countable group. We fix an exact sequence $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ satisfying conditions Lemma 4.2(a)–(f). In particular, N is generated by the elements

$$(37) \quad a = x^2 \quad \text{and} \quad b = y^3.$$

By Lemma 4.2(a), $\{H_1, H_2\} \hookrightarrow_h (G, X)$ for some $X \subseteq G$. We keep the standard notation

$$\mathcal{A} = X \sqcup H_1 \sqcup H_2.$$

Let D be the constant provided by Lemma 3.9 applied to the hyperbolically embedded collection $\{H_1, H_2\} \hookrightarrow_h G$.

Since $C_1(\varepsilon, \mu, \rho)$ implies $C(\varepsilon, \mu, \rho)$ and both conditions become stronger as ε and ρ increase and μ decreases, we can choose ε and ρ such that the conclusions of Lemmas 3.5 and 3.26 simultaneously hold

for $N = 1$. Further, let ξ and σ be the constants provided by Lemma 3.16 for the chosen values of ε , μ , and ρ . Since $W(\xi, \sigma)$ becomes stronger as ξ and σ increase, we can additionally assume (after increasing ξ and σ if necessary) that

$$(38) \quad \xi > 5D,$$

and the conclusion of Proposition 3.18 holds for the group G , the collection of peripheral subgroups $\{H_1, H_2\}$, the subset $X \subseteq G$, and $\nu = \frac{1}{3}$.

Further, by Lemma 3.15, we can choose $l > 5$ such that the word

$$R = a^{l+1}b^{l+1}a^{l+2}b^{l+2} \dots a^{2l}b^{2l},$$

where a and b are defined by (37), satisfies $W(\xi, \sigma)$.

Let

$$\bar{G} = G/\langle\langle R \rangle\rangle$$

and let B be the image of N in $\bar{G}$. Clearly $B \triangleleft \bar{G}$ and $\bar{G}/B \cong G/N \cong Q$. By the choice of the parameters, we can apply Lemma 3.26 to the quotient group $\bar{G}$. Thus the restriction of the natural homomorphism $G \rightarrow \bar{G}$ to $H_1 \cup H_2$ is injective. Henceforth we identify groups H_1 and H_2 with their (isomorphic) images in $\bar{G}$ and keep the same notation s, t, x, y, a , and b for their elements. By Lemma 3.26(b), $\{H_1, H_2\} \hookrightarrow_h \bar{G}$. In particular, $\bar{G}$ is acylindrically hyperbolic. Since $H_1 \cap H_2 = \{1\}$ in G , the images of H_1 and H_2 intersect trivially in $\bar{G}$. Note that these images actually belong to B since $H_1, H_2 \leq N$. By Corollary 2.10, this implies that B is ICC. Further, by Lemma 3.5 (applied to $N = 1$) and Lemma 4.2(d),

$$(39) \quad H_1 = C_{\bar{G}}(s) \quad \text{and} \quad H_2 = C_{\bar{G}}(t) = C_{\bar{G}}(t^{-1}).$$

Finally, every element of finite order in $\bar{G}$ is conjugate to one of the elements $1, s, t$, or t^{-1} by Lemma 4.2(e) and Lemma 3.26(c).

Our next goal is to show that every automorphism of the group B is induced by conjugation in $\bar{G}$. We fix any $\varphi \in \text{Aut}(B)$. Using the description of finite-order elements of $\bar{G}$, we conclude that φ maps s and t to some conjugates of s and $t^{\pm 1}$ in $\bar{G}$, respectively. Composing φ with an inner automorphism of $\bar{G}$, we can assume that $\varphi(s) = s$ and $\varphi(t) = zt^{\pm 1}z^{-1}$ for some $z \in \bar{G}$. Combining this with (39), we obtain $\varphi(H_1) = H_1$ and $\varphi(H_2) = zH_2z^{-1}$. This easily implies that x is mapped to $x^{\pm 1}$ or $x^{\pm 1}s$ and y is mapped to $y^{\pm 1}$, $y^{\pm 1}t$, or $y^{\pm 1}t^2$; see (32). In all these cases, we have

$$\varphi(a) = a^\alpha \quad \text{and} \quad \varphi(b) = zb^\beta z^{-1},$$

where $\alpha, \beta = \pm 1$.

Note that if $H_1gH_2 = H_1zH_2$ for some $g \in H$, then $z = h_1gh_2$, where $h_1 \in H_1$ and $h_2 \in H_2$. Hence, the pair (a, zbz^{-1}) is conjugate to the pair (a, gbz^{-1}) by the element h_1 . Thus composing φ with another inner automorphism if necessary, we can assume that z is a shortest element in the double coset H_1zH_2 . It follows that z^{-1} is a shortest element in the double coset $H_2z^{-1}H_1$.

Let U be a shortest word in the alphabet $\mathcal{A}$ representing z . Consider the word

$$(40) \quad R' = a^{(l+1)\alpha} U b^{(l+1)\beta} U^{-1} a^{(l+2)\alpha} U b^{(l+2)\beta} U^{-1} \dots a^{2l\alpha} U b^{2l\beta} U^{-1}.$$

Here, as usual, we think of powers of a and b as single letters. Clearly R' represents $\varphi(R) = 1$ in $\bar{G}$. Thanks to (38) and the assumption about z explained in the previous paragraph, we can apply Lemma 3.10 to the word R' and conclude that R' is $(G, \mathcal{A})$ -geodesic. By Proposition 3.18, R' and a cyclic shift of $R^{\pm 1}$ have a common subword S of length at least $(1 - \nu)\|R\| = \frac{2}{3}\|R\|$. We will show that this can happen only if U is the empty word.

Note that the letters corresponding to powers of a and b can occur inside the subwords $U^{\pm 1}$ of the word R' ; the occurrences of these letters that are not inside the subwords $U^{\pm 1}$ will be called *regular*. Suppose first that S contains at most one regular letter. Note that this letter must be the first or the last one in S . Indeed, otherwise S would have common subwords with both U and U^{-1} and hence would contain letters corresponding to both positive and negative powers of a or b , which is impossible for a subword of $R^{\pm 1}$. Thus S shares a subword S' of length at least $\|S\| - 1 \geq \frac{2}{3}\|R\| - 1$ with a single occurrence of the subword $U^{\pm 1}$ in R' . Since $l > 5$, we have $\|R\| > 10$ and $\frac{2}{3}\|R\| - 1 > \frac{1}{2}\|R\|$. Thus $U = U_1 S' U_2$ and there is a cyclic shift of R of the form $S' S''$ such that $\|S''\| < \|S'\|$. The word $U_1 (S'')^{-1} U_2$ is shorter than U and represents the same element of the group $\bar{G}$ as U ; this contradicts the assumption that U is the shortest word representing the element z .

Further, assume that S contains at least two consecutive regular letters. Note that every letter from $H_1 \sqcup H_2$ occurs in the words R and R^{-1} at most once and no letter occurs in both. This easily implies that S is a subword of R (not R^{-1}) and these letters must be next to each other in R . Therefore $\alpha = \beta = 1$, U is the empty word, and $z = 1$ in this case.

Thus $\varphi(a) = a$ and $\varphi(b) = b$ up to conjugation in $\bar{G}$. Being a quotient of N , the group B is generated by a and b ; see Lemma 4.2(c). Therefore every automorphism of B is induced by a conjugation in $\bar{G}$. Combining this with Lemma 4.4, we obtain that $\text{Aut}(B) \cong \bar{G}$ and $\text{Out}(B) \cong \bar{G}/B$. In particular, $\text{Aut}(B)$ is acylindrically hyperbolic and $\text{Out}(B) \cong Q$.

Finally, if Q is finitely presented, then G is hyperbolic relative to $\{H_1, H_2\}$ by Lemma 4.2(f). This allows us to apply the second claim in Lemma 3.26(b) and conclude that $\bar{G}$ is hyperbolic relative to (the isomorphic images of) H_1 and H_2 . Since H_1 and H_2 are hyperbolic groups, $\bar{G}$ is hyperbolic by Corollary 2.14. $\square$

Proof of Theorem 1.3 Let H be a nontrivial, torsion-free, hyperbolic group with property (T). It is worth noting that such groups abound; for example, every property-(T) group is a quotient of a torsion-free hyperbolic property-(T) group by a result of Cornuier [8]. Also let Q be a finite group. By Theorem 4.5, there exists an ICC quotient group G of H such that $\text{Aut}(G)$ is hyperbolic and $\text{Out}(G) \cong Q$. Since G is ICC, we have $G \cong \text{Inn}(G)$. Since Q is finite, $\text{Inn}(G)$ has finite index in $\text{Aut}(G)$. It is well known that a

finite-index subgroup of a hyperbolic group is hyperbolic. Thus G is hyperbolic. It also has property (T), being a quotient of H . $\square$

Finally, we state and prove a more precise version of Theorem 1.4:

Theorem 4.6 *For every countable group Q , there exists a finitely generated acylindrically hyperbolic group G with property (T) and trivial finite radical such that $\text{Out}(G) \cong Q$.*

Proof As above, we apply Theorem 4.5 to a torsion-free hyperbolic group H with property (T) and a countable group Q . Let G be the group provided by the theorem. It has property (T), being a quotient of H . Since G is ICC, it has trivial finite radical and so $\text{Inn}(G) \cong G$. This equality implies that G is isomorphic to a normal subgroup of $\text{Aut}(G)$. The group $\text{Aut}(G)$ is acylindrically hyperbolic by Theorem 4.5(b). Therefore G is also acylindrically hyperbolic by Lemma 2.19. $\square$

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The structure of relatively hyperbolic groups in convex real projective geometry

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We prove a general structure theorem for relatively hyperbolic groups (with arbitrary peripheral subgroups) acting naive convex cocompactly on properly convex domains in real projective space. We also establish a characterization of such groups in terms of the existence of an invariant collection of closed unbounded convex subsets with good isolation properties. This is a real projective analogue of results of Hindawi–Hruska–Kleiner for CAT(0) spaces. We also obtain an equivariant homeomorphism between the Bowditch boundary of the group and a quotient of the ideal boundary.

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1 Introduction

Let $\mathbb{H}^d$ denote real hyperbolic d -space and recall that a discrete subgroup $\Gamma \leq \text{Isom}(\mathbb{H}^d)$ is called *convex cocompact* if there exists a nonempty Γ -invariant geodesically convex closed subset $\mathcal{C} \subset \mathbb{H}^d$ where the quotient $\Gamma \backslash \mathcal{C}$ is compact. The Beltrami–Klein model realizes $\mathbb{H}^d$ as a properly convex domain $\mathbb{B}^d \subset \mathbb{P}(\mathbb{R}^{d+1})$ (namely the Euclidean unit ball in a standard affine chart) in such a way that the isometry group $\text{Isom}(\mathbb{H}^d)$ coincides with the subgroup of $\text{PGL}_{d+1}(\mathbb{R})$ which preserves $\mathbb{B}^d$ (namely $\text{PO}(d, 1)$ up to conjugation). Further, in this model, a subset being geodesically convex is equivalent to being convex in some (hence any) affine chart that contains $\mathbb{B}^d$.

The Beltrami–Klein model perspective allows one to naturally generalize the classical notion of convex cocompact groups. In particular, one can consider a general properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and the group $\text{Aut}(\Omega) \leq \text{PGL}_d(\mathbb{R})$ of projective automorphisms which preserve Ω . Then a discrete subgroup $\Gamma \leq \text{Aut}(\Omega)$ is called *naive convex cocompact* if there exists a nonempty Γ -invariant closed convex subset $\mathcal{C} \subset \Omega$ where the quotient $\Gamma \backslash \mathcal{C}$ is compact. In this case, we say that $(\Omega, \mathcal{C}, \Gamma)$ is a *naive convex cocompact triple*.

When the group Γ is word hyperbolic (eg in the classical real hyperbolic setting), there is a close connection between this notion of naive convex cocompact groups and Anosov representations/higher Teichmüller theory; see [7; 27]. As one moves beyond the word hyperbolic case, the structure of these discrete groups becomes more mysterious.

In this paper, we study naive convex cocompact groups which are (intrinsically) relatively hyperbolic groups. This is a rich class with many examples, for instance where:

- (a) Γ is a projective reflection group generated by reflection along faces of a projective Coxeter polytope à la Vinberg and Γ is irreducible as a Coxeter group; see [8; 19].
- (b) Γ is isomorphic to $\pi_1(M)$ where M is a closed three-manifold such that each geometric component in its JSJ decomposition supports a hyperbolic structure; see [1; 17].

In both of these cases, the groups are relatively hyperbolic with respect to peripheral subgroups which are virtually abelian of rank at least two. However, there also exist examples where Γ is relatively hyperbolic with respect to non-virtually abelian subgroups; see the discussion in [23, Section 2.6.3]. The primary goal of this paper is to describe the structure of such examples. One of our main results is proving that the relative hyperbolicity of Γ is equivalent to the existence of a so-called *peripheral family*, a Γ -invariant collection of closed convex subsets with good isolation properties.

This investigation extends some recent work. Previously in [15], we considered the special case where the peripheral subgroups were virtually abelian of rank at least two. For such groups, we proved that relative hyperbolicity is equivalent to the existence of a collection of properly embedded simplices with good isolation properties. This geometric description is analogous to the case of CAT(0) spaces with isolated flats [13]. We also showed that the boundary of these simplices are, in a technical sense, the only places where the boundary is irregular; see [15, Theorem 1.19(6), Theorem 1.8(7) and (8)]. Shortly after, Weisman [23] considered convex cocompact groups (a more restrictive class than naive convex cocompact) who were relatively hyperbolic and whose peripheral subgroups were also convex cocompact, but not necessarily virtually abelian. For such groups he established a similar result about the irregular boundary points and also showed that the Bowditch boundary could be realized as a quotient of the boundary.

There are a number of other results in the literature concerning relatively hyperbolic groups acting on properly convex domains; see for instance [3; 4; 5; 6]. These results consider the case when $\Gamma \backslash \mathcal{C}$ is noncompact and characterize when Γ is relatively hyperbolic with respect to the fundamental groups of the ends (under some geometric assumptions on the ends and $\mathcal{C}$). There is some similarity between the statements in this paper and the statements in [3; 4; 5], but to the best of our knowledge, there is no actual overlap between the results.

We extend the results in [15; 23] to the case of general peripheral subgroups and the case of general naive convex cocompact groups. The proofs build upon ideas from both papers.

We will now introduce the notation required to precisely state our main results. Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, we will let d_Ω denote the Hilbert metric, which is a natural and classical $\text{Aut}(\Omega)$ -invariant, proper, and complete metric on Ω ; see Section 2.2 below. This allows us to speak of the diameter $\text{diam}_\Omega(A)$ and r -open neighborhood $\mathcal{N}_\Omega(A, r) \subset \Omega$ of a subset $A \subset \Omega$ relative to the Hilbert metric.

Definition 1.1 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\mathcal{C} \subset \Omega$ is a closed¹ convex subset, $\Gamma \leq \text{Aut}(\Omega)$, and $\mathcal{X}$ is a collection of closed unbounded convex subsets of $\mathcal{C}$. Then we say that:

- (1) $\mathcal{X}$ is Γ -invariant if $\Gamma \cdot \mathcal{X} = \mathcal{X}$.
- (2) $\mathcal{X}$ is *strongly isolated* if for every $r > 0$ there exists $D_1(r) > 0$ such that if $X_1, X_2 \in \mathcal{X}$ are distinct, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)) \leq D_1(r).$$

- (3) $\mathcal{X}$ *coarsely contains the properly embedded simplices of $\mathcal{C}$* if there exists $D_2 > 0$ such that if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset \mathcal{N}_\Omega(X, D_2)$.

When $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ satisfies all three of the above conditions, we say that $\mathcal{X}$ is a *peripheral family of $(\Omega, \mathcal{C}, \Gamma)$* .

Given a closed convex subset $\mathcal{C} \subset \Omega$ of a properly convex domain, the *ideal boundary* of $\mathcal{C}$ is $\partial_i \mathcal{C} := \bar{\mathcal{C}} \cap \partial \Omega$ (see Section 2.3). Also, given $x \in \bar{\Omega}$ we will let $F_\Omega(x)$ denote the open face of x in $\bar{\Omega}$ and given $A \subset \bar{\Omega}$ we will let $F_\Omega(A) = \bigcup_{x \in A} F_\Omega(x)$. Using these boundary faces and a peripheral family, one can define a natural quotient of the ideal boundary:

Definition 1.2 Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$. Let

$$[\partial_i \mathcal{C}]_\mathcal{X} := \partial_i \mathcal{C} / \sim$$

be the topological quotient where $x \sim y$ if and only if

- (1) $x, y \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$, or
- (2) $F_\Omega(y) = F_\Omega(x)$.

Remark 1.3 We remark that similar boundary quotients are considered by Choi [5] and Weisman [23].

We need one more piece of notation to state our main result: the *limit set* of a subgroup $G \leq \text{Aut}(\Omega)$ is

$$\mathcal{L}_\Omega(G) := \partial \Omega \cap \bigcup_{p \in \Omega} \overline{G \cdot p}.$$

This definition ensures that $\mathcal{L}_\Omega(G)$ is base point independent—unlike real hyperbolic geometry, the accumulation points of a single orbit in Ω may depend on the base point.

Our main result—Theorem 1.4 below—is essentially an equivalence statement for a naive convex cocompact triple $(\Omega, \mathcal{C}, \Gamma)$: the relative hyperbolicity of the group Γ is equivalent to the existence of a peripheral family $\mathcal{X}$. Theorem 1.4(1) shows that the group-theoretic property of relative hyperbolicity implies the geometric property of admitting a peripheral family. Theorem 1.4(2) shows the converse.

¹We make it our convention that, “ $Y \subset \Omega$ is a closed subset” always means that Y is closed in the subspace topology inherited from Ω .

Theorem 1.4 Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple. Then the following hold:

- (1) If Γ is relatively hyperbolic with respect to $\mathcal{P} = \{P_1, \dots, P_N\}$ and X_j is the closed convex hull of $\mathcal{L}_\Omega(P_j) \cap \partial_i \mathcal{C}$ in Ω , then

$$\mathcal{X} := \Gamma \cdot \{X_1, \dots, X_N\}$$

is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$.

- (2) If $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ and $\mathcal{P} := \{P_1, \dots, P_N\}$ is a set of representatives of the Γ -conjugacy classes in $\{\text{Stab}_\Gamma(X) : X \in \mathcal{X}\}$, then Γ is relatively hyperbolic with respect to $\mathcal{P}$.

Moreover, when either of the above conditions are satisfied, then:

- (a) $(\mathcal{C}, d_\Omega)$ is relatively hyperbolic with respect to $\mathcal{X} = \Gamma \cdot \{X_1, \dots, X_N\}$.
- (b) There is a Γ -equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_i \mathcal{C}]_\mathcal{X}$.
- (c) Each (Ω, X_j, P_j) is a naive convex cocompact triple.
- (d) There exists $L > 0$ such that if $x \in \partial_i \mathcal{C}$ and $\text{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L$, then $x \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$.
- (e) There exists $R > 0$ such that if $X \in \mathcal{X}$ and $x \in \partial_i X$, then

$$d_{F_\Omega(x)}^{\text{Haus}}(F_\Omega(x) \cap \partial_i X, F_\Omega(x) \cap \partial_i \mathcal{C}) \leq R.$$

Remark 1.5 Properties (d) and (e) are somewhat technical. Informally, property (d) states that any boundary face of Ω that $\partial_i \mathcal{C}$ intersects in a “large set” must intersect the ideal boundary of an element in $\mathcal{X}$. Property (e) informally states that if $X \in \mathcal{X}$, then $\partial_i X$ coarsely contains any boundary face of $\partial_i \mathcal{C}$ that it intersects.

Theorem 1.4(1) and (2) can be viewed as a real projective analogue of the results of Hruska–Kleiner and Hindawi–Hruska–Kleiner [13] in the setting of CAT(0)-geometry. This earlier work motivated the results in this paper, but the methods of proof are very different. In particular, we should also note that an old result of Kelly–Straus [18] says that a Hilbert geometry (Ω, d_Ω) is CAT(0) if and only if it is isometric to real hyperbolic $(d-1)$ -space (in which case Ω coincides, up to a change of coordinates, with the Beltrami–Klein model of real hyperbolic $(d-1)$ -space).

As mentioned above, in previous work [15] we proved a version of Theorem 1.4 in the special case when $\mathcal{X}$ consisted of properly embedded simplices of dimension at least two and the subgroups in $\mathcal{P}$ were virtually abelian of rank at least two. In this case, using the simple structure of simplices and abelian subgroups in naive convex cocompact groups (see [14]), one can weaken the strongly isolated assumption to only assuming that $\mathcal{X}$ is closed and discrete. In fact, a substantial portion of this earlier work was building a strongly isolated collection of properly embedded simplices from a closed and discrete collection. It seems unlikely to us that such a weakening is possible in the general case.

1.1 Convex cocompact groups

As mentioned above, the class of naive convex cocompact groups includes the more restrictive, but still interesting, class of convex cocompact groups. For this class of groups, Theorem 1.4 can be restated in a much simpler way.

Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a discrete subgroup $\Gamma \leq \text{Aut}(\Omega)$, let $\mathcal{C}_\Omega(\Gamma) \subset \Omega$ denote the closed convex hull of $\mathcal{L}_\Omega(\Gamma)$ in Ω .

Definition 1.6 A discrete subgroup $\Gamma \leq \text{Aut}(\Omega)$ is called *convex cocompact* if $\mathcal{C}_\Omega(\Gamma)$ is nonempty and the quotient $\Gamma \backslash \mathcal{C}_\Omega(\Gamma)$ is compact.

Every convex cocompact subgroup is clearly naive convex cocompact, but the converse is not true; see [15, Section 2.3] for some examples. One key difference between the two definitions is that if $\Gamma \leq \text{Aut}(\Omega)$ is convex cocompact, then any open boundary face of Ω which intersects $\partial_i \mathcal{C}_\Omega(\Gamma)$ is actually contained in $\partial_i \mathcal{C}_\Omega(\Gamma)$. On the other hand, if $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple, then it is possible for $\partial_i \mathcal{C}$ to intersect a boundary face of Ω in a small set. This seemingly small difference makes convex cocompact groups much easier to study.

For convex cocompact groups, Theorem 1.4(a)–(e) can be restated and expanded as follows.

Theorem 1.7 (see Section 10) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a convex cocompact subgroup. If Γ is relatively hyperbolic with respect to $\mathcal{P} = \{P_1, \dots, P_N\}$, then:*

(a) $(\mathcal{C}_\Omega(\Gamma), d_\Omega)$ is relatively hyperbolic with respect to

$$\mathcal{X} := \Gamma \cdot \{\mathcal{C}_\Omega(P_1), \dots, \mathcal{C}_\Omega(P_N)\}.$$

(b) Let $[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ denote the quotient of $\partial_i \mathcal{C}_\Omega(\Gamma)$ obtained by collapsing each limit set $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ to a point (where $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$). Then there is a Γ -equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$.

(c) Each P_j is a convex cocompact subgroup of $\text{Aut}(\Omega)$.

(d) If $x \in \partial_i \mathcal{C}_\Omega(\Gamma)$ is not a $\mathcal{C}^1$ -smooth point of Ω (ie Ω does not have a unique supporting hyperplane at x), then $x \in \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.

(e) If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.

We should note that the main new content of Theorem 1.7 is part (c). In particular:

- (1) Part (a) is an immediate consequence of part (c).
- (2) In [15] we previously proved parts (d) and (e) in the case where each P_j is virtually abelian with rank at least two, and once part (c) is known, the same argument works in the more general setting.
- (3) Weisman [23] established parts (b) and (e) with part (c) as an assumption.

In the context of convex cocompact groups, peripheral families can also be defined in terms of their boundary behavior.

Proposition 1.8 (see Section 10) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a convex cocompact subgroup. If $\mathcal{X}$ is a Γ -invariant collection of closed unbounded convex subsets of Ω , then the following are equivalent:*

- (1) $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}_\Omega(\Gamma), \Gamma)$.
- (2) $\mathcal{X}$ has the following properties:
 - (a) $\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric d_Ω .
 - (b) If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \emptyset$.
 - (c) If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.

Moreover, when the above conditions are satisfied, then

$$(1-1) \quad [\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}} = [\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{X}},$$

where $\mathcal{P}$ is a set of representatives of the Γ -conjugacy classes in $\{\text{Stab}_\Gamma(X) : X \in \mathcal{X}\}$.

Previously, Weisman [23, Theorem 1.16] characterized when a convex cocompact subgroup is hyperbolic relative to a collection of convex cocompact subgroups in terms of the behavior of the limit sets of the subgroups. We note that combining Proposition 1.8 and Theorem 1.4 provides a “subset” version of this characterization.

1.2 The case of Gromov hyperbolic groups

A word hyperbolic group is relatively hyperbolic with respect to the empty set. Thus if $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and Γ is a word hyperbolic group, then Theorem 1.4 holds with $\mathcal{X} = \emptyset$.

However, the choice of this peripheral family $\mathcal{X}$ is not canonical even for a convex cocompact group Γ (unlike our work in [15] where $\mathcal{X}$ was the set of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two). For instance, suppose that $\Gamma := \langle a, b \rangle$ is a convex cocompact subgroup of $\text{PSO}(2, 1)$ that is isomorphic to a free group on two generators. Then Γ is relatively hyperbolic with respect to $\mathcal{P} := \{\langle g \rangle\}$ where $g := aba^{-1}b^{-1}$. Then $\mathcal{X} := \Gamma \cdot (g^+, g^-)$ where (g^+, g^-) is the unique g -invariant projective line in the Beltrami–Klein model of $\mathbb{H}^2$. On the other hand, we can also choose $\mathcal{X} = \emptyset$ (when $\mathcal{P} = \emptyset$).

1.3 Outline of the paper and proofs

Sections 2 and 3 are expository. In Section 2 we recall the basic definitions and results about properly convex domains that we will require and in Section 3 we recall some properties of relatively hyperbolic spaces and groups.

Sections 4 through 9 are devoted to the proof of Theorem 1.4. The difficult part of Theorem 1.4 is part (2). Our general strategy is based on combining ideas from [15] and [23]. In particular, as in [23], we will use Yaman’s characterization of relatively hyperbolicity to show that Γ is relatively hyperbolic. We will verify the conditions in Yaman’s theorem by further developing the ideas used in [15] to study closed and discrete collections of properly embedded simplices.

We should also note that Choi [5] (which predates the work of Weisman) used Yaman’s characterization to verify that the fundamental groups of certain noncompact convex real projective manifolds were relatively hyperbolic with respect to the fundamental groups of their ends.

The arguments used to verify the conditions of Yaman’s theorem in this paper are somewhat similar to the analogous ones in [5; 23]. But given the different setups, there does not seem to be an easy way to reduce the proofs in this paper to any lemmas in these two works.

The content of these sections is as follows:

- (1) In Section 4 we prove a quantitative result which informally states that long line segments in an open boundary face (relative to the Hilbert metric of the face) imply the existence of nearby properly embedded simplices.
- (2) In Section 5 we establish a number of useful properties of peripheral families.
- (3) In Section 6 we prove that peripheral subgroups of a relatively hyperbolic naive convex cocompact subgroup are themselves naive convex cocompact.
- (4) In Section 7 we prove a technical result which will allow us to show that nonconical limit points are bounded parabolic points. A key tool here is a “closest subset” projection map.
- (5) In Section 8 we prove some basic properties of the quotient space $[\partial_i \mathcal{C}]_{\mathcal{X}}$, including a sufficient condition for a point to be a conical limit point.
- (6) In Section 9 we put everything together and prove Theorem 1.4.

Finally, in Section 10, we use Theorem 1.4 to prove Theorem 1.7.

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2 Preliminaries

2.1 Notation

If $V \subset \mathbb{R}^d$ is a nonzero linear subspace, we will let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^d)$ denote its projectivization. In most other cases, we will use $[o]$ to denote the projective equivalence class of an object o , for instance:

- (1) If $v \in \mathbb{R}^d \setminus \{0\}$, then $[v]$ denotes the image of v in $\mathbb{P}(\mathbb{R}^d)$.
- (2) If $\phi \in \text{GL}_d(\mathbb{R})$, then $[\phi]$ denotes the image of ϕ in $\text{PGL}_d(\mathbb{R})$.
- (3) If $T \in \text{End}(\mathbb{R}^d) \setminus \{0\}$, then $[T]$ denotes the image of T in $\mathbb{P}(\text{End}(\mathbb{R}^d))$.

We also standardize some metric notations. If (X, d) is a metric space, $p \in X$, $A \subset X$, and $r > 0$, then:

- (1) $\mathcal{N}_X(A, r) := \{x \in X : d(x, A) < r\}$.
- (2) $\mathcal{B}_X(p, r) := \{x \in X : d(x, p) < r\}$.
- (3) $\text{diam}_X(A) := \sup\{d(x, x') : x, x' \in A\}$.

2.2 Convexity and the Hilbert metric

A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is

- (1) *properly convex* if there exists an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a bounded convex subset;
- (2) a *properly convex domain* if C is properly convex and open in $\mathbb{P}(\mathbb{R}^d)$.

Given a properly convex set $C \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset \bar{C}$, we define its *convex hull* as

$$\text{ConvHull}_C(X) := \bigcap \{Y : Y \text{ is a closed convex subset such that } X \subset Y \subset \bar{C}\}.$$

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$ there is no canonical line segment with endpoints x and y , but we will use the following convention: if $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \bar{C}$, then (when the context is clear) we will let $[x, y]$ denote the closed line segment joining x to y which is contained in $\bar{C}$. In this case, we will also let $(x, y) = [x, y] \setminus \{x, y\}$, $[x, y) = [x, y] \setminus \{y\}$, and $(x, y] = [x, y] \setminus \{x\}$.

Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex domain. If $x, y \in \Omega$ are distinct, let $[a, b] := \mathbb{P}(\text{Span}\{x, y\}) \cap \bar{\Omega}$ where a and b are labeled such that $x \in [a, y]$ (ie the points are ordered a, x, y, b along $[a, b]$). Then the *Hilbert distance* between x and y is defined to be

$$d_\Omega(x, y) := \frac{1}{2} \log[a, x, y, b],$$

where

$$[a, x, y, b] := \frac{|x - b||y - a|}{|x - a||y - b|}$$

is the cross-ratio (here $|\cdot|$ is some (any) norm in some (any) affine chart which contains a, x, y, b). Then (Ω, d_Ω) is a complete geodesic metric space and $\text{Aut}(\Omega)$ acts properly and by isometries on Ω ; see for instance [2, Section 28]. Further, the projective line segment $[x, y]$ is a geodesic for the Hilbert distance.

2.3 Boundaries and faces

Suppose that $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set. The *relative interior* of C , denoted by $\text{rel-int}(C)$, is the interior of C in $\mathbb{P}(\text{Span } C)$. We will say that C is *open in its span* if $C = \text{rel-int}(C)$.

The boundary of C is $\partial C := \bar{C} \setminus \text{rel-int}(C)$, the ideal boundary of C is $\partial_i C := \partial C \setminus C$, and the nonideal boundary of C is $\partial_n C := \partial C \cap C$.

Definition 2.1 If $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set which is open in its span and $x \in \bar{C}$, let $F_C(x)$ denote the open face of x :

$$F_C(x) = \{x\} \cup \{y \in \bar{C} : \text{there exists an open line segment in } \bar{C} \text{ containing } x \text{ and } y\}.$$

We have the following observation about faces; see [16, Appendix] for a proof.

Observation 2.2 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.

- (1) $F_\Omega(x) = \Omega$ when $x \in \Omega$.
- (2) $F_\Omega(x)$ is open in its span.
- (3) $y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$.
- (4) If $x, y \in \bar{\Omega}$, $z \in (x, y)$, $p \in F_\Omega(x)$, and $q \in F_\Omega(y)$, then

$$(p, q) \subset F_\Omega(z).$$

In particular, $(p, q) \subset \Omega$ if and only if $(x, y) \subset \Omega$.

If $B \subset C \subset \mathbb{P}(\mathbb{R}^d)$ are properly convex sets, then we say that B is *properly embedded* in C if $B \hookrightarrow C$ is a proper map with respect to the subspace topology. Note that B is properly embedded in C if and only if $\partial_i B \subset \partial_i C$.

2.4 Limits of automorphisms

Every $T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$ induces a map

$$\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T) \rightarrow \mathbb{P}(\mathbb{R}^d)$$

defined by $x \mapsto T(x)$. We will frequently use the following observation.

Observation 2.3 If $\{T_n\}$ is a sequence in $\mathbb{P}(\text{End}(\mathbb{R}^d))$ converging to $T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$, then

$$T(x) = \lim_{n \rightarrow \infty} T_n(x)$$

for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$. The convergence is uniform on compact subsets of $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$.

We can view $\mathbb{P}(\text{End}(\mathbb{R}^d))$ as a compactification of $\text{PGL}_d(\mathbb{R})$ and then consider limits of automorphisms in this compactification.

Proposition 2.4 [14, Proposition 5.6] Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, and $\{g_n\}$ is a sequence in $\text{Aut}(\Omega)$ such that:

- (1) $g_n(p_0) \rightarrow x \in \partial\Omega$.
- (2) $g_n^{-1}(p_0) \rightarrow y \in \partial\Omega$.
- (3) $g_n \rightarrow T$ in $\mathbb{P}(\text{End}(\mathbb{R}^d))$.

Then image $T \subset \text{Span } F_\Omega(x)$, $\mathbb{P}(\ker T) \cap \Omega = \emptyset$, and $y \in \mathbb{P}(\ker T)$.

In the case of “nontangential” convergence we can say more.

Proposition 2.5 [14, Proposition 5.7] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, $x \in \partial\Omega$, $\{p_n\}$ is a sequence in $[p_0, x)$ converging to x , and $\{g_n\}$ is a sequence in $\text{Aut}(\Omega)$ such that*

$$\sup_{n \geq 1} d_\Omega(g_n(p_0), p_n) < +\infty.$$

If $g_n \rightarrow T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$, then $T(\Omega) = F_\Omega(x)$.

Proposition 5.7 in [14] is stated differently and a proof of the statement above can be found in [28, Proposition 2.13].

2.5 Projective simplices

A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is called a *k-dimensional simplex* in $\mathbb{P}(\mathbb{R}^d)$ if there exists $g \in \text{PGL}_d(\mathbb{R})$ such that

$$gS = \{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1, \dots, x_{k+1} > 0\}.$$

In this case, we call the $k + 1$ points

$$g^{-1}[1 : 0 : \cdots : 0], g^{-1}[0 : 1 : 0 : \cdots : 0], \dots, g^{-1}[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \partial S$$

the *vertices* of S .

The Hilbert metric on a simplex can be explicitly computed (see [11; 20, Proposition 1.7; 22]) and from this explicit form one obtains the following.

Proposition 2.6 *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex, then (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$.*

Remark 2.7 By the definition of the Hilbert metric, $d_S = d_\Omega|_{S \times S}$. So the quasi-isometry constants only depend on $\dim S$.

2.6 The Hausdorff distance and local Hausdorff topology

When (X, d) is a metric space, the *Hausdorff distance* between two subsets $A, B \subset X$ is defined by

$$d_X^{\text{Haus}}(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\right\}.$$

When (X, d) is a complete metric space, d_X^{Haus} is a complete metric on the set of nonempty compact subsets of X .

The local Hausdorff topology is a natural topology on the set of nonempty closed sets in X . For a nonempty closed set C_0 , a base point $x_0 \in X$, and $r_0, \epsilon_0 > 0$, define the set $U(C_0, x_0, r_0, \epsilon_0)$ to consist of all closed subsets $C \subset X$ where

$$d_X^{\text{Haus}}(C_0 \cap \mathcal{B}_X(x_0, r_0), C \cap \mathcal{B}_X(x_0, r_0)) < \epsilon_0.$$

The *local Hausdorff topology* on the set of nonempty closed subsets of X is the topology generated by the sets $U(\cdot, \cdot, \cdot, \cdot)$. This is the same topology as the one called the *local Hausdorff convergence topology* in [16, Section 2.5].

2.7 Distance estimates for the Hilbert metric

The asymptotic behavior of the Hilbert distance connects naturally with the structure of open faces in the boundary.

Proposition 2.8 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\{p_n\}, \{q_n\}$ are sequences in Ω where $p_n \rightarrow p \in \bar{\Omega}$ and $q_n \rightarrow q \in \bar{\Omega}$. If*

$$\liminf_{n \rightarrow \infty} d_{\Omega}(p_n, q_n) < +\infty,$$

then $F_{\Omega}(p) = F_{\Omega}(q)$ and

$$d_{F_{\Omega}(p)}(p, q) \leq \liminf_{n \rightarrow \infty} d_{\Omega}(p_n, q_n).$$

Proof This is a well-known property of the Hilbert metric, but we provide the proof for the readers' convenience.

Passing to a subsequence we can suppose that

$$\liminf_{n \rightarrow \infty} d_{\Omega}(p_n, q_n) = \lim_{n \rightarrow \infty} d_{\Omega}(p_n, q_n).$$

Since d_{Ω} is proper, if at least one of p, q is in Ω , then both must be in Ω . Further, if $p, q \in \Omega$, then $F_{\Omega}(p) = F_{\Omega}(q) = \Omega$ and the continuity of the Hilbert metric implies that

$$d_{F_{\Omega}(p)}(p, q) = d_{\Omega}(p, q) = \lim_{n \rightarrow \infty} d_{\Omega}(p_n, q_n).$$

Hence we can assume that $p, q \in \partial\Omega$. It suffices to consider $p \neq q$ since otherwise $d_{F_{\Omega}(p)}(p, q) = 0$.

For each $n \geq 1$, let $a_n, b_n \in \partial\Omega$ be the points where $p_n, q_n \in (a_n, b_n)$ and labeled so that the points are in the order a_n, p_n, q_n, b_n along (a_n, b_n) . Passing to a subsequence, we can assume that $a_n \rightarrow a$ and $b_n \rightarrow b$ in $\partial\Omega$. Then $p, q \in [a, b]$. Since $p \neq q$ and

$$\lim_{n \rightarrow \infty} \frac{1}{2} \log[a_n, p_n, q_n, b_n] = \lim_{n \rightarrow \infty} d_{\Omega}(p_n, q_n) < +\infty,$$

the definition of the cross-ratio implies that a, p, q, b are all distinct. Hence $p, q \in (a, b)$ and so $F_{\Omega}(p) = F_{\Omega}(q)$. Finally, since $(a, b) \subset F_{\Omega}(p)$, the monotonicity of the Hilbert metric implies that

$$d_{F_{\Omega}(p)}(p, q) \leq d_{(a,b)}(p, q) = \liminf_{n \rightarrow \infty} d_{\Omega}(p_n, q_n). \quad \square$$

We will frequently use the following fact about the Hausdorff distance between line segments.

Proposition 2.9 [14, Proposition 5.3] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $p_1, p_2, q_1, q_2 \in \bar{\Omega}$ are such that $(p_1, p_2) \subset \Omega$, $F_\Omega(p_1) = F_\Omega(q_1)$, and $F_\Omega(p_2) = F_\Omega(q_2)$, then*

$$d_\Omega^{\text{Haus}}([p_1, p_2] \cap \Omega, [q_1, q_2] \cap \Omega) \leq \max\{d_{F_\Omega(p_1)}(p_1, q_1), d_{F_\Omega(p_2)}(p_2, q_2)\}.$$

Using induction, Proposition 2.9 can be upgraded as follows.

Proposition 2.10 *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $q_1, \dots, q_m \in \bar{\Omega}$, and*

$$z \in \text{rel-int}(\text{ConvHull}_\Omega(q_1, \dots, q_m)).$$

If $p_1, \dots, p_m \in \bar{\Omega}$ and $F_\Omega(p_j) = F_\Omega(q_j)$ for all $1 \leq j \leq m$, then

$$d_F^{\text{Haus}}(\text{ConvHull}_\Omega(q_1, \dots, q_m) \cap F, \text{ConvHull}_\Omega(p_1, \dots, p_m) \cap F) \leq \max_{1 \leq j \leq m} d_{F_\Omega(q_j)}(q_j, p_j),$$

where $F := F_\Omega(z)$.

Proof For $1 \leq k \leq m$, let

$$S_k := \text{rel-int}(\text{ConvHull}_\Omega(q_1, \dots, q_k)) \quad \text{and} \quad S'_k := \text{rel-int}(\text{ConvHull}_\Omega(p_1, \dots, p_k)).$$

Since S_k, S'_k are open in their span, there exist faces F_k, F'_k of Ω such that $S_k \subset F_k$ and $S'_k \subset F'_k$.

For each $1 \leq k \leq m$, we claim that $F_k = F'_k$ and

$$d_{F_k}^{\text{Haus}}(S_k, S'_k) \leq \max_{1 \leq j \leq k} d_{F_\Omega(q_j)}(q_j, p_j).$$

We induct on k . The base case $k = 1$ is by definition.

Fix $k > 1$. Then fix $w \in S_k$. We will show that $F_k = F'_k$ and

$$(2-1) \quad d_{F_k}(w, S'_k) \leq \max_{1 \leq j \leq k} d_{F_\Omega(q_j)}(q_j, p_j).$$

If $w = q_k$, then we must have $q_k \in S_{k-1}$. Then $q_k \in F_{k-1}$, which implies that $p_k \in F_{k-1}$. So $S_k \cup S'_k \subset F_{k-1}$ and $F_k = F_{k-1} = F'_k$ by the induction hypothesis. Further,

$$d_{F_k}(w, S'_k) \leq d_{F_k}(q_k, p_k) = d_{F_\Omega(q_k)}(q_k, p_k).$$

If $w \neq q_k$, then there exists $q' \in S_{k-1}$ such that $w \in (q', q_k)$. Indeed, if $q_k \in \bar{S}_{k-1}$, then $S_k = S_{k-1}$ and then this is obvious as we can extend $(q_k, w]$ beyond $w \in S_k$. Otherwise, if $q_k \notin \bar{S}_{k-1}$, then every point in $\bar{S}_k$ is a nontrivial convex combination of a point $q'' \in \bar{S}_{k-1}$ and q_k . Among all such points in $\bar{S}_k$, $S_k = \text{rel-int}(\bar{S}_k)$ is precisely the set of points for which $q'' \in S_{k-1}$. In particular, as $w \in S_k$, there exists $q' \in S_{k-1}$ such that $w \in (q_k, q')$.

Since $q' \in S_{k-1}$, by induction there exists $p' \in S'_{k-1}$ with

$$d_{F_{k-1}}(q', p') \leq \max_{1 \leq j \leq k-1} d_{F_\Omega(q_j)}(q_j, p_j).$$

There are two cases to consider depending on whether p' equals p_k or not.

Case 1 Assume $p' \neq p_k$. Then Observation 2.2(4) implies that $(q', q_k) \cup (p', p_k) \subset F_k$. Since $(p', p_k) \subset S_k$, we then have $F_k = F'_k$. Further, by Proposition 2.9,

$$d_{F_k}(w, S'_k) \leq d_{F_k}^{\text{Haus}}((q', q_k), (p', p_k)) \leq \max_{1 \leq j \leq k} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

Case 2 Assume $p' = p_k$. Then $p_k = p' \in S'_{k-1} \subset F_{k-1}$, which implies that $q_k \in F_{k-1}$. Therefore $S_k \cup S'_k \subset F_{k-1}$ and $F_k = F_{k-1} = F'_k$ by the induction hypothesis. Further, by Proposition 2.9,

$$d_{F_k}(w, S'_k) \leq d_{F_k}([q', q_k], [p_k, p_k]) \leq \max(d_{F_k}(q', p_k), d_{F_k}(q_k, p_k)) \leq \max_{1 \leq j \leq k} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

Thus in all cases $F_k = F'_k$ and (2-1) holds. Since $w \in S_k$ was arbitrary, we see that

$$\sup_{w \in S_k} d_{F_k}(w, S'_k) \leq \max_{1 \leq j \leq k} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

Repeating the same argument with $w' \in S'_k$ shows that

$$\sup_{w' \in S'_k} d_{F_k}(w', S_k) \leq \max_{1 \leq j \leq k} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

Hence,

$$d_{F_k}^{\text{Haus}}(S_k, S'_k) \leq \max_{1 \leq j \leq k} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

This completes the proof of the claim.

Now, to prove the proposition, we apply the claim when $k = m$. Since $z \in S_m$, we have $F = F_{\Omega}(z) = F_m$.

By the claim, $S_m \cup S'_m \subset F_m$ and

$$d_{F_m}^{\text{Haus}}(S_m, S'_m) \leq \max_{1 \leq j \leq m} d_{F_{\Omega}(q_j)}(q_j, p_j).$$

Finally, S_m (resp. S'_m) is dense in $\text{ConvHull}_{\Omega}(q_1, \dots, q_m) \cap F$ (resp. $\text{ConvHull}_{\Omega}(p_1, \dots, p_m) \cap F$). So the result follows. $\square$

3 Relatively hyperbolic spaces and groups

In this expository section we recall some basic properties of relatively hyperbolic groups and spaces. We define relative hyperbolic spaces and groups in terms of Druţu and Sapir's tree-graded spaces; see [10, Definition 2.1].

Definition 3.1 (1) A complete geodesic metric space (X, d) is said to be *relatively hyperbolic with respect to a collection of subsets $\mathcal{S}$* if all its asymptotic cones, with respect to a fixed nonprincipal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of $\mathcal{S}$.

(2) A finitely generated group G is said to be *relatively hyperbolic with respect to a family of subgroups $\{H_1, \dots, H_N\}$* if the Cayley graph of G with respect to some (hence any) finite set of generators is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G, i = 1, \dots, N\}$.

Remark 3.2 These are one among several equivalent definitions of relatively hyperbolic spaces/groups; see [10] for more details.

Recall that if (X, d) is a metric space, $A \subset X$, and $r > 0$, then

$$\mathcal{N}_X(A, r) := \{x \in X : d(x, A) < r\}.$$

We will frequently use the following properties of relatively hyperbolic spaces. These results are taken from Druţu–Sapir [10].

Theorem 3.3 *Suppose that (X, d_X) is relatively hyperbolic with respect to $\mathcal{S}$.*

- (1) [10, Corollary 5.8] *For any $A \geq 1$ and $B \geq 0$ there exists $M = M(A, B)$ such that if $k \geq 2$ and $f : \mathbb{R}^k \rightarrow X$ is an (A, B) -quasi-isometric embedding, then there exists some $S \in \mathcal{S}$ such that*

$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S, M).$$

- (2) [10, Lemma 4.15] *For any $A \geq 1$ and $B \geq 0$ there exists $t = t(A, B)$ such that if $S \in \mathcal{S}$, $r \geq 1$, and $\gamma : [0, T] \rightarrow X$ is an (A, B) -quasigeodesic with $\gamma(0), \gamma(T) \in \mathcal{N}_X(S, r)$, then*

$$\gamma([0, T]) \subset \mathcal{N}_X(S, tr).$$

- (3) [10, Theorem 4.1] *For any $r > 0$ there exists $D = D(r) > 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then*

$$\text{diam}_X(\mathcal{N}_X(S_1, r) \cap \mathcal{N}_X(S_2, r)) \leq D.$$

3.1 Immediate consequences in convex projective geometry

As an immediate consequence of the general theory of relatively hyperbolic spaces, we have the following.

Proposition 3.4 *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ is a Γ -invariant collection of closed unbounded convex subsets of $\mathcal{C}$. If $(\mathcal{C}, d_\Omega)$ is relatively hyperbolic with respect to $\mathcal{X}$, then $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ (ie Γ -invariant, strongly isolated, and coarsely contains the properly embedded simplices of $\mathcal{C}$).*

Proof Theorem 3.3(3) implies that $\mathcal{X}$ is strongly isolated. Proposition 2.6 and Theorem 3.3(1) imply that $\mathcal{X}$ coarsely contains the properly embedded simplices of $\mathcal{C}$. $\square$

3.2 Yaman’s characterization

We recall Yaman’s characterization of relatively hyperbolic groups in terms of convergence group actions [21; 26].

Recall that when M is a nonempty compact Hausdorff metrizable space, a subgroup $G \leq \text{Homeo}(M)$ is called a *convergence group* if for every sequence $\{g_n\}$ of distinct elements in G there exists a subsequence $\{g_{n_j}\}$ and points $x, y \in M$ such that $g_{n_j}|_{M-\{y\}}$ converges locally uniformly to the constant map x . For such a subgroup:

- (1) The *limit set* $\mathcal{L}(G) \subset M$ is the set of points $x \in M$ where there exist $y \in M$ and a sequence $\{g_n\}$ in G where $g_n|_{M-\{y\}}$ converges locally uniformly to the constant map x .
- (2) A point $x \in \mathcal{L}(G)$ is a *conical limit point* if there exist distinct points $a, b \in M$ and a sequence of elements $\{g_n\}$ in G where $\lim_{n \rightarrow \infty} g_n(x) = a$ and $\lim_{n \rightarrow \infty} g_n(y) = b$ for all $y \in M - \{x\}$.
- (3) A point $x \in \mathcal{L}(G)$ is a *parabolic point* if $\text{Stab}_G(x)$ is infinite and nonloxodromic (ie x is the unique fixed point in $\mathcal{L}(G)$ for every $g \in \text{Stab}_G(x)$ of infinite order).
- (4) A parabolic point $x \in \mathcal{L}(G)$ is a *bounded parabolic point* if $\text{Stab}_G(x)$ acts cocompactly on $\mathcal{L}(G) - \{x\}$.

Finally, we say that a convergence group $G \leq \text{Homeo}(M)$ is *geometrically finite* if $\mathcal{L}(G)$ is a nonempty perfect set (ie $\# \mathcal{L}(G) \geq 3$) and every point in $\mathcal{L}(G)$ is either a conical limit point or a bounded parabolic point.

Theorem 3.5 (Yaman [26, Theorem 0.1]) *Suppose that $G \leq \text{Homeo}(M)$ is a geometrically finite convergence group and $B \subset \mathcal{L}(G)$ is the set of bounded parabolic fixed points. If B has finitely many G -orbits and $\text{Stab}_G(b) := \{h \in G : h(b) = b\}$ is finitely generated for every $b \in B$, then:*

- (1) G is relatively hyperbolic with respect to $\mathcal{P} = \{\text{Stab}_G(p_1), \dots, \text{Stab}_G(p_N)\}$ where $B = \bigsqcup_{i=1}^N G(p_i)$.
- (2) The Bowditch boundary $\partial(G, \mathcal{P})$ is equivariantly homeomorphic to $\mathcal{L}(G)$.

If Γ is relatively hyperbolic with respect to $\mathcal{P}$, then Γ acts as a geometrically finite convergence group on the Bowditch boundary.

4 Finding properly embedded simplices

For convex cocompact groups, if there is a line segment in the ideal boundary of the convex hull, then there is a properly embedded simplex in the convex hull [7, Theorem 1.15]. This fails for naive convex cocompact subgroups; see [15, Section 2.3]. Instead we will show that if there exists a line segment in the ideal boundary which is sufficiently long (relative to the Hilbert metric of the face containing it), then there exists a properly embedded simplex.

In the following proposition, $d_{\mathbb{P}}$ denotes some distance on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric.

Proposition 4.1 *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $q \in \mathcal{C}$. For any $r, \epsilon > 0$ there exists $L = L(q, r, \epsilon) \geq 0$ with the following property: if*

- (1) $a, b \in \partial_i \mathcal{C}$ are contained in a boundary face F of Ω with $d_F(a, b) \geq L$,
- (2) $m \in (a, b)$ is the midpoint of $[a, b]$ relative to d_F , and
- (3) $p \in \mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{C}$ is sufficiently close to m (in the metric $d_{\mathbb{P}}$),

then there exists a properly embedded simplex $S \subset \mathcal{C}$ of dimension at least two with

$$\mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{B}_{\Omega}(p, r) \subset \mathcal{N}_{\Omega}(S, \epsilon).$$

Proof Suppose for a contradiction that the proposition is false for some choice of $r, \epsilon > 0$. Then for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \partial_i \mathcal{C}$ and a sequence $\{p_{n,j}\}_{j \geq 1}$ where:

- (1) a_n, b_n are contained in a boundary face F_n and $d_{F_n}(a_n, b_n) > n$.
- (2) If $m_n \in (a_n, b_n)$ is the midpoint of $[a_n, b_n]$ relative to d_{F_n} , then $m_n = \lim_{j \rightarrow \infty} p_{n,j}$.
- (3) $p_{n,j} \in \mathbb{P}(\text{Span}\{a_n, b_n, q\}) \cap \mathcal{C}$ for all $j \geq 1$.
- (4) $\mathbb{P}(\text{Span}\{a_n, b_n, q\}) \cap \mathcal{B}_\Omega(p_{n,j}, r)$ is not contained in the ϵ -neighborhood of any properly embedded simplex in $\mathcal{C}$ of dimension at least two.

We first claim that for every n , there exists j_n such that

$$(4-1) \quad d_\Omega(p_{n,j_n}, (a_n, q] \cup [q, b_n)) \geq \frac{1}{2}n.$$

Fix n and suppose not. Then for every $j \in \mathbb{N}$ we can find $a_{n,j} \in (a_n, q]$ and $b_{n,j} \in (b_n, q]$ with

$$d_\Omega(p_{n,j}, \{a_{n,j}, b_{n,j}\}) \leq \frac{1}{2}n.$$

Since $\lim_{j \rightarrow \infty} p_{n,j} = m_n$ and d_Ω is proper, we must have $\lim_{j \rightarrow \infty} a_{n,j} = a_n$ and $\lim_{j \rightarrow \infty} b_{n,j} = b_n$. Further, by the definition of the Hilbert metric,

$$\frac{1}{2}n \geq \limsup_{j \rightarrow \infty} d_\Omega(p_{n,j}, \{a_{n,j}, b_{n,j}\}) \geq d_{F_n}(m_n, \{a_n, b_n\}) > \frac{1}{2}n.$$

So we have a contradiction and hence for each n such a j_n exists.

After possibly passing to a subsequence we can find a sequence $\{\gamma_n\}$ in Γ such that as $n \rightarrow \infty$,

$$\gamma_n(p_{n,j_n}) \rightarrow p_\infty \in \mathcal{C}, \quad \gamma_n(a_n) \rightarrow a_\infty \in \bar{\mathcal{C}}, \quad \gamma_n(b_n) \rightarrow b_\infty \in \bar{\mathcal{C}}, \quad \gamma_n(q) \rightarrow q_\infty \in \bar{\mathcal{C}}.$$

By construction, $[a_\infty, b_\infty] \subset \partial_i \mathcal{C}$, and by (4-1), we see that

$$[a_\infty, q_\infty] \cup [q_\infty, b_\infty] \subset \partial_i \mathcal{C}.$$

Hence $a_\infty, b_\infty, q_\infty$ are the vertices of a properly embedded two-dimensional simplex $S \subset \mathcal{C}$ which contains p_∞ . So for n large, we have

$$\mathbb{P}(\text{Span}\{\gamma_n(a_n), \gamma_n(b_n), \gamma_n(q)\}) \cap \mathcal{B}_\Omega(\gamma_n(p_{n,j_n}), r) \subset \mathcal{N}_\Omega(S, \epsilon),$$

which implies that

$$\mathbb{P}(\text{Span}\{a_n, b_n, q\}) \cap \mathcal{B}_\Omega(p_{n,j_n}, r) \subset \mathcal{N}_\Omega(\gamma_n^{-1}S, \epsilon).$$

Thus we have a contradiction. □

5 Properties of peripheral families

Theorem 5.1 Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ (ie $\mathcal{X}$ is Γ -invariant, strongly isolated, and coarsely contains all properly embedded simplices in $\mathcal{C}$). Then:

- (1) Γ has finitely many orbits in $\mathcal{X}$.
- (2) If $X \in \mathcal{X}$, then $\text{Stab}_\Gamma(X)$ acts cocompactly on X . In particular, $\text{Stab}_\Gamma(X)$ is finitely generated.

- (3) If $X_1, X_2 \in \mathcal{X}$ are distinct, then $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) = \emptyset$.
- (4) There exists $L > 0$ such that if $x \in \partial_i \mathcal{C}$ and $\text{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L$, then $x \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$.
- (5) There exists $R > 0$ such that if $X \in \mathcal{X}$ and $x \in \partial_i X$, then
- $$d_{F_\Omega(x)}^{\text{Haus}}(F_\Omega(x) \cap \partial_i \mathcal{C}, F_\Omega(x) \cap \partial_i X) \leq R.$$
- (6) If we have $X \in \mathcal{X}$, $x \in \partial_i X$, and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$.
- (7) If $[x, y] \subset \partial_i \mathcal{C}$, then either $F_\Omega(x) = F_\Omega(y)$, or there exists $X \in \mathcal{X}$ such that $x, y \in F_\Omega(\partial_i X)$.
- (8) If $X \in \mathcal{X}$, then $\text{Stab}_\Gamma(X)$ acts cocompactly on

$$\text{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$$

and this set is nonempty.

For the rest of the section, fix $\Omega, \mathcal{C}, \Gamma$, and $\mathcal{X}$ satisfying the hypotheses of the theorem.

Since $\mathcal{X}$ coarsely contains all properly embedded simplices in $\mathcal{C}$, there exists $D_2 > 0$ such that if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with

$$(5-1) \quad S \subset \mathcal{N}_\Omega(X, D_2).$$

Lemma 5.2 *If $K \subset \mathcal{C}$ is compact, then the set $\{X \in \mathcal{X} : X \cap K \neq \emptyset\}$ is finite.*

Proof Suppose not. Then there exist an infinite sequence of pairwise distinct elements $\{X_n\}$ in $\mathcal{X}$ where $X_n \cap K \neq \emptyset$ for all n . As each X_n is unbounded, we can fix $k_n \in X_n \cap K$ and $x_n \in \partial_i X_n$. Passing to a subsequence we can suppose that $k_n \rightarrow k \in K$ and $x_n \rightarrow x \in \partial_i \mathcal{C}$. Then

$$\liminf_{n, \ell \rightarrow \infty} \text{diam}_\Omega(\mathcal{N}_\Omega(X_n, 1) \cap \mathcal{N}_\Omega(X_\ell, 1)) \geq \liminf_{n, \ell \rightarrow \infty} \text{diam}_\Omega(\mathcal{N}_\Omega([k_n, x_n], 1) \cap \mathcal{N}_\Omega([k_\ell, x_\ell], 1)) = \infty,$$

and so by the strong isolation property $X_n = X_\ell$ for sufficiently large n, ℓ . So we have a contradiction. $\square$

Lemma 5.3 (part (1)) *Γ has finitely many orbits in $\mathcal{X}$.*

Proof Since Γ acts cocompactly on $\mathcal{C}$, there exists a compact set $K \subset \mathcal{C}$ such that $\Gamma \cdot K = \mathcal{C}$. Then for each $X \in \mathcal{X}$ there exists $g \in \Gamma$ such that $K \cap gX \neq \emptyset$. So by Lemma 5.2, there exist $X_1, \dots, X_m$ such that $\mathcal{X} = \bigsqcup_{i=1}^m \Gamma \cdot X_i$. $\square$

Lemma 5.4 (part (2)) *If $X \in \mathcal{X}$, then $\text{Stab}_\Gamma(X)$ acts cocompactly on X .*

Proof This argument is standard; see [12, Theorem 3.7; 13, Section 3.1; 25, Proposition 4.0.4].

Fix a compact set $K \subset \mathcal{C}$ such that $\Gamma \cdot K = \mathcal{C}$. Let

$$\mathcal{G} := \{g \in \Gamma : X \cap gK \neq \emptyset\}.$$

Then $X \subset \mathcal{G} \cdot K$. Also, by the previous lemma

$$\{g^{-1}X : g \in \mathcal{G}\} = \{h_1^{-1}X, \dots, h_k^{-1}X\}$$

for some $h_1, \dots, h_k \in \mathcal{G}$. Notice that $g^{-1}X = h_j^{-1}X$ if and only if $gh_j^{-1} \in \text{Stab}_\Gamma(X)$; hence

$$\mathcal{G} = \bigcup_{j=1}^k \text{Stab}_\Gamma(X)h_j.$$

Finally, if $\hat{K} := \bigcup_{j=1}^k h_j K$, then $\hat{K}$ is compact and $X \subset \text{Stab}_\Gamma(X) \cdot \hat{K}$. So $X = \text{Stab}_\Gamma(X) \cdot (X \cap \hat{K})$. Since X is closed, $X \cap \hat{K}$ is compact and thus $\text{Stab}_\Gamma(X)$ acts cocompactly on X .

Now, since X is convex, the metric space (X, d_Ω) is geodesic. So by the fundamental lemma of geometric group theory, $\text{Stab}_\Gamma(X)$ is finitely generated. $\square$

Lemma 5.5 (part (3)) *If $X_1, X_2 \in \mathcal{X}$ are distinct, then $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) = \emptyset$.*

Proof Suppose $X_1, X_2 \in \mathcal{X}$ and $x \in F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2)$. There exists $x_1 \in \partial_i X_1$ and $x_2 \in \partial_i X_2$ with

$$x_1, x_2 \in F_\Omega(x).$$

Fix $q_1 \in X_1$ and $q_2 \in X_2$. Then Proposition 2.9 implies that

$$d_\Omega^{\text{Haus}}([q_1, x_1], [q_2, x_2]) \leq \max\{d_\Omega(q_1, q_2), d_{F_\Omega(x)}(x_1, x_2)\}.$$

So for $r > \max\{d_\Omega(q_1, q_2), d_{F_\Omega(x)}(x_1, x_2)\}$, we have

$$\text{diam}_\Omega(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)) = \infty.$$

Since $\mathcal{X}$ is strongly isolated, then $X_1 = X_2$. $\square$

Recall that D_2 is the constant in (5-1).

Lemma 5.6 *There exists $L > 0$ such that if*

- (1) $a, b \in \partial_i \mathcal{C}$ are in the same boundary face F of Ω ,
- (2) $d_F(a, b) \geq L$, and
- (3) m is the midpoint of $[a, b]$ with respect to d_F ,

then there exists $X \in \mathcal{X}$ and $x \in \partial_i X \cap F$ such that

$$d_F(m, x) \leq D_2 + 1.$$

Proof Since $\mathcal{X}$ is strongly isolated, there exists $r > 0$ such that

$$(5-2) \quad \text{diam}_\Omega(\mathcal{N}_\Omega(X_1, D_2 + 1) \cap \mathcal{N}_\Omega(X_2, D_2 + 1)) < r$$

for all distinct pairs $X_1, X_2 \in \mathcal{X}$. Fix $q \in \mathcal{C}$ and let $L > 0$ satisfy Proposition 4.1 for $q \in \mathcal{C}$ and constants $\epsilon = 1$ and r as above. Now suppose

- (1) $a, b \in \partial_i \mathcal{C}$ are in the same boundary face F of Ω ,
- (2) $d_F(a, b) \geq L$, and
- (3) m is the midpoint of $[a, b]$ with respect to d_F .

By our choice of L , there exists $q' \in (m, q]$ such that if $p \in (m, q']$ then there exists some properly embedded simplex $S_p \subset \mathcal{C}$ of dimension at least two such that

$$\mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{B}_\Omega(p, r) \subset \mathcal{N}_\Omega(S_p, 1).$$

Then, by our choice of D_2 in (5-1), there exists $X_p \in \mathcal{X}$ such that

$$\mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{B}_\Omega(p, r) \subset \mathcal{N}_\Omega(X_p, D_2 + 1).$$

We claim that X_p does not depend on $p \in (m, q']$. To verify this it is enough to fix $p_1, p_2 \in (m, q']$ with $d_\Omega(p_1, p_2) < r$ and show that X_{p_1} and X_{p_2} coincide. For such $p_1, p_2 \in (m, q']$ we have

$$\mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{B}_\Omega(p_1, r) \cap \mathcal{B}_\Omega(p_2, r) \subset \mathcal{N}_\Omega(X_{p_1}, D_2 + 1) \cap \mathcal{N}_\Omega(X_{p_2}, D_2 + 1),$$

and

$$\text{diam}_\Omega(\mathbb{P}(\text{Span}\{a, b, q\}) \cap \mathcal{B}_\Omega(p_1, r) \cap \mathcal{B}_\Omega(p_2, r)) \geq r.$$

So $X_{p_1} = X_{p_2}$ by our choice of r ; see (5-2).

Now let $X := X_p$ for $p \in (m, q']$. Then

$$(m, q'] \subset \mathcal{N}_\Omega(X, D_2 + 1)$$

which implies, by Proposition 2.8, that there exists $x \in \partial_i X \cap F$ such that

$$d_F(m, x) \leq D_2 + 1. \quad \square$$

Lemma 5.7 (part (4)) *If $x \in \partial_i \mathcal{C}$ and*

$$\text{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L,$$

then $x \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$.

Proof This follows immediately from Lemma 5.6. $\square$

Lemma 5.8 (part (5)) *If $X \in \mathcal{X}$ and $x \in \partial_i X$, then*

$$d_{F_\Omega(x)}^{\text{Haus}}(F_\Omega(x) \cap \partial_i \mathcal{C}, F_\Omega(x) \cap \partial_i X) \leq \max\{L, 2D_2 + 2\}.$$

Proof Fix $X \in \mathcal{X}$ and $x \in \partial_i X$. Since $F_\Omega(x) \cap \partial_i X \subset F_\Omega(x) \cap \partial_i \mathcal{C}$, it suffices to show that

$$\sup_{y \in F_\Omega(x) \cap \partial_i \mathcal{C}} d_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) \leq \max\{L, 2D_2 + 2\}.$$

Fix $y \in F_\Omega(x) \cap \partial_i \mathcal{C}$ and suppose for a contradiction that

$$d_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) > \max\{L, 2D_2 + 2\}.$$

By changing x we may suppose that

$$d_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) = d_{F_\Omega(x)}(y, x).$$

Let m be the midpoint of $[x, y]$ with respect to $d_{F_\Omega(x)}$. Then by Lemma 5.6 there exist $X' \in \mathcal{X}$ and $x' \in \partial_i X'$ such that

$$d_{F_\Omega(x)}(m, x') \leq D_2 + 1.$$

However, then $x \in F_\Omega(\partial_i X) \cap F_\Omega(\partial_i X') \neq \emptyset$ and so by Lemma 5.5 we have $X = X'$. So

$$D_2 + 1 < \frac{1}{2} d_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) = d_{F_\Omega(x)}(m, F_\Omega(x) \cap \partial_i X) \leq d_{F_\Omega(x)}(m, x') \leq D_2 + 1,$$

and we have a contradiction. $\square$

Lemma 5.9 (part (7)) *If $[x, y] \subset \partial_i \mathcal{C}$, then either $F_\Omega(x) = F_\Omega(y)$, or there exists $X \in \mathcal{X}$ such that $x, y \in F_\Omega(\partial_i X)$.*

Proof We suppose that $F_\Omega(x) \neq F_\Omega(y)$ and show that $x, y \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$. Fix $z \in (x, y)$. Since $F_\Omega(x) \neq F_\Omega(y)$,

$$\text{diam}_{F_\Omega(z)}((x, y)) = \infty.$$

So by Lemma 5.7 there exists $X \in \mathcal{X}$ with $z \in F_\Omega(\partial_i X)$. Then Lemma 5.8 implies that

$$(x, y) \subset \mathcal{N}_{F_\Omega(z)}(F_\Omega(z) \cap \partial_i X, \max\{L, 2D_2 + 2\}).$$

Finally Proposition 2.8 implies that $x, y \in F_\Omega(\partial_i X)$. $\square$

Lemma 5.10 (part (6)) *If we have $X \in \mathcal{X}$, $x \in \partial_i X$, and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$.*

Proof This follows immediately from Lemmas 5.9 and 5.5. $\square$

Observation 5.11 *If $X \in \mathcal{X}$, then $\text{ConvHull}_\Omega(\partial_i X) \cap \Omega$ is nonempty.*

Proof By assumption X is unbounded and so $\partial_i X \neq \emptyset$. Fix $w_2 \in \partial_i X$ and $w_1 \in X$. Also fix a sequence $\{w'_n\} \subset [w_1, w_2)$ such that $w'_n \rightarrow w_2$. As $\text{Stab}_\Gamma(X)$ acts cocompactly on X , for each $n \geq 1$ there exists $\gamma_n \in \text{Stab}_\Gamma(X)$ such that $\{\gamma_n w'_n\}$ lies in a compact subset of X . Up to passing to a subsequence, we can assume that $\gamma_n w'_n \rightarrow z' \in X$ and $\gamma_n w_i \rightarrow z_i \in \bar{X}$ for $i = 1, 2$. Then $z_2 \in \partial_i X$ as $\partial_i X$ is compact and $z_1 \in \partial_i X$ as $d_\Omega(z', z_1) = \lim_{n \rightarrow \infty} d_\Omega(\gamma_n w'_n, \gamma_n w_1) = \infty$. Then $(z_1, z_2) \subset \Omega$, as $z' \in (z_1, z_2) \cap \Omega$. $\square$

Lemma 5.12 (part (8)) *If $X \in \mathcal{X}$, then $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed in $\mathbb{P}(\mathbb{R}^d)$ and $\text{Stab}_\Gamma(X)$ acts cocompactly on the nonempty set $\text{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$.*

Proof We first verify that $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed. Suppose $\{x_n\}$ is a sequence in $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ converging to some $x \in \partial_i \mathcal{C}$ (note $\partial_i \mathcal{C}$ is closed). Then for each n there exists $x'_n \in \partial_i X$ with $x_n \in F_\Omega(x'_n)$. In particular, $[x_n, x'_n] \subset \partial_i \mathcal{C}$. Passing to a subsequence we can suppose that $x'_n \rightarrow x'$. Then $x' \in \partial_i X$ and $[x, x'] \subset \partial_i \mathcal{C}$. Then Lemma 5.10 implies that $x \in F_\Omega(\partial_i X)$. Hence $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed.

Then the set

$$\mathcal{C}_X := \text{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$$

is closed in $\mathcal{C}$. As $\text{ConvHull}_\Omega(\partial_i X) \cap \Omega \subset \mathcal{C}_X$, Observation 5.11 implies that $\mathcal{C}_X$ is nonempty. As $\text{Stab}_\Gamma(X)$ acts cocompactly on X , it suffices to show that $\mathcal{C}_X$ is contained in a bounded neighborhood of X . Fix $p \in \mathcal{C}_X$. Then there exist $p_1, \dots, p_\ell \in F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ such that $p \in \text{ConvHull}_\Omega(p_1, \dots, p_\ell)$. Then by Lemma 5.8 there exist $p'_1, \dots, p'_\ell \in \partial_i X$ such that

$$d_{F_\Omega(p_j)}(p_j, p'_j) \leq \max\{L, 2D_2 + 2\} \quad \text{for } j = 1, \dots, \ell.$$

Then by Proposition 2.10,

$$\begin{aligned} d_{\Omega}(p, X) &\leq d_{\Omega}^{\text{Haus}}(\text{ConvHull}_{\Omega}(p_1, \dots, p_{\ell}) \cap \Omega, \text{ConvHull}_{\Omega}(p'_1, \dots, p'_{\ell}) \cap \Omega) \\ &\leq \max_{1 \leq j \leq \ell} d_{F_{\Omega}(p_j)}(p_j, p'_j) \leq \max\{L, 2D_2 + 2\}. \end{aligned}$$

Since $p \in \mathcal{C}_X$ was arbitrary, $\mathcal{C}_X \subset \overline{\mathcal{N}_{\Omega}(X, \max\{L, 2D_2 + 2\})}$. $\square$

6 Naive convex cocompactness of peripheral subgroups

Proposition 6.1 Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and Γ is relatively hyperbolic with respect to $\{P_1, \dots, P_N\}$. For $1 \leq j \leq N$, let

$$X_j := \text{ConvHull}_{\Omega}(\mathcal{L}_{\Omega}(P_j) \cap \partial_i \mathcal{C}) \cap \Omega.$$

Then:

- (1) Each (Ω, X_j, P_j) is a naive convex cocompact triple.
- (2) $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\mathcal{X} := \Gamma \cdot \{X_1, \dots, X_N\}$.

The rest of the section is devoted to the proof of this proposition. Suppose $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and Γ is relatively hyperbolic with respect to $\{P_1, \dots, P_N\}$.

Fix a point $p_0 \in \mathcal{C}$. The fundamental lemma of geometric group theory implies that $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\Gamma \cdot \{P_1(p_0), \dots, P_N(p_0)\}$.

For each $1 \leq j \leq N$, let $\mathcal{L}_j \subset \partial_i \mathcal{C}$ denote the accumulation points of the orbit $P_j(p_0)$ and let

$$\hat{X}_j = \text{ConvHull}_{\Omega}(\mathcal{L}_j) \cap \Omega.$$

Lemma 6.2 $\hat{X}_j$ is nonempty and P_j acts cocompactly on $\hat{X}_j$.

Proof For $n \geq 1$ let $Y_j^{(n)}$ be the points which are contained in the convex hull of at most n points in $P_j(p_0)$. By Carathéodory's convex hull theorem, $Y_j^{(n)} = Y_j^{(d)}$ for all $n \geq d$. Further, the closure of $Y_j^{(d)}$ in Ω contains $\hat{X}_j$.

By Theorem 3.3(2) there exists $t > 0$ such that if $r \geq 1$ and $p, q \in \mathcal{C} \cap \mathcal{N}_{\Omega}(P_j(p_0), r)$, then we have $[p, q] \subset \mathcal{N}_{\Omega}(P_j(p_0), tr)$.

We claim that

$$(6-1) \quad Y_j^{(n)} \subset \mathcal{N}_{\Omega}(P_j(p_0), (n-1)t)$$

for all $n \geq 2$. The base case follows by our choice of t . Suppose (6-1) holds for some n . If $p \in Y_j^{(n+1)}$, then there exist $p_1 \in P_j(p_0)$ and $p_2 \in Y_j^{(n)}$ such that $p \in [p_1, p_2]$. Then, by induction there exists $q \in P_j(p_0)$ with $d_{\Omega}(p_2, q) < (n-1)t$. Then

$$[p_1, q] \subset Y_j^{(2)} \subset \mathcal{N}_{\Omega}(P_j(p_0), t),$$

and by Proposition 2.9,

$$d_{\Omega}(p, [p_1, q]) \leq d_{\Omega}^{\text{Haus}}([p_1, p_2], [p_1, q]) \leq d_{\Omega}(p_2, q) < (n-1)t.$$

So $p \in \mathcal{N}_{\Omega}(P_j(p_0), nt)$. Since $p \in Y_j^{(n+1)}$ was arbitrary, this completes the induction step. Thus (6-1) holds for all $n \geq 2$.

Then

$$\hat{X}_j \subset \overline{Y_j^{(d)}} \subset \overline{\mathcal{N}_{\Omega}(P_j(p_0), (d-1)t)},$$

and so

$$\hat{X}_j = P_j \cdot (\overline{\mathcal{B}_{\Omega}(p_0, (d-1)t)} \cap \hat{X}_j).$$

Thus P_j acts cocompactly on $\hat{X}_j$.

We now show that $\hat{X}_j \neq \emptyset$. Since P_j is infinite and acts properly on Ω , there exists $\{g_k\}$ in P_j such that

$$\lim_{k \rightarrow \infty} d_{\Omega}(p_0, g_k(p_0)) = \infty.$$

Let m_k be the midpoint of $[p_0, g_k(p_0)]$ with respect to d_{Ω} . By (6-1),

$$m_k \in Y_j^{(2)} \subset \mathcal{N}_{\Omega}(P_j(p_0), t).$$

Thus there exists $\{h_k\}$ in P_j such that $h_k(m_k) \in \mathcal{B}_{\Omega}(p_0, t)$. Then, up to passing to a subsequence, we can assume that $h_k(m_k) \rightarrow m \in \Omega$ and $h_k[p_0, g_k(p_0)] \rightarrow [x, y]$. Then $x, y \in \mathcal{L}_j$ and so

$$m \in (x, y) \subset \hat{X}_j. \quad \square$$

The fundamental lemma of geometric group theory implies that $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\Gamma \cdot \{\hat{X}_1, \dots, \hat{X}_N\}$. In particular, by Proposition 3.4 and Theorem 5.1(5) there exists $R > 0$ such that if $1 \leq j \leq N$ and $x \in \partial_i \hat{X}_j$, then

$$(6-2) \quad d_{F_{\Omega}(x)}^{\text{Haus}}(F_{\Omega}(x) \cap \partial_i \mathcal{C}, F_{\Omega}(x) \cap \partial_i \hat{X}_j) \leq R.$$

Lemma 6.3 *If $1 \leq j \leq N$, then*

$$d_{F_{\Omega}(x)}^{\text{Haus}}(\hat{X}_j, X_j) \leq R.$$

Hence P_j acts cocompactly on X_j and $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\Gamma \cdot \{X_1, \dots, X_N\}$.

Proof First note that $\hat{X}_j \subset X_j$, as $\mathcal{L}_j \subset \mathcal{L}_{\Omega}(P_j)$. By the previous lemma, P_j acts cocompactly on $\hat{X}_j$. Thus it suffices to show that

$$X_j \subset \overline{\mathcal{N}_{\Omega}(\hat{X}_j, R)}.$$

Fix $q \in X_j$. Then there exist $q_1, \dots, q_{\ell} \in \mathcal{L}_{\Omega}(P_j) \cap \partial_i \mathcal{C}$ where

$$q \in \text{ConvHull}_{\Omega}(q_1, \dots, q_{\ell}).$$

For each $1 \leq i \leq \ell$, there exist $p_i \in \Omega$ and a sequence $\{\gamma_{i,n}\}_{n \geq 1}$ in P_j with $\gamma_{i,n}(p_i) \rightarrow q_i$. Passing to subsequences we can suppose that

$$\gamma_{i,n}(p_0) \rightarrow q'_i \in \mathcal{L}_j \subset \partial_i \hat{X}_j,$$

for all $1 \leq i \leq \ell$. By Proposition 2.8, $F_\Omega(q_i) = F_\Omega(q'_i)$. So by (6-2) there exist $q''_i \in \partial_i \hat{X}_j$ such that $d_{F_\Omega(q_i)}(q_i, q''_i) \leq R$. Then by Proposition 2.10,

$$d_\Omega(q, \hat{X}_j) \leq d_\Omega^{\text{Haus}}(\text{ConvHull}_\Omega(q_1, \dots, q_\ell) \cap \Omega, \text{ConvHull}_\Omega(q''_1, \dots, q''_\ell) \cap \Omega) \leq \max_{1 \leq i \leq \ell} d_{F_\Omega(q_i)}(q_i, q''_i) \leq R. \quad \square$$

This finishes the proof of Proposition 6.1.

7 Cocompact boundary actions

Given a naive convex cocompact triple (Ω, X, G) , it seems natural to ask if G acts cocompactly on $\partial\Omega - F_\Omega(\partial_i X)$ or more generally on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$ when $\mathcal{C} \subset \Omega$ is a closed G -invariant convex subset containing X . The next theorem gives a sufficient condition for this to occur.

Theorem 7.1 *Suppose that (Ω, X, G) is a naive convex cocompact triple and $\mathcal{C} \subset \Omega$ is a closed G -invariant convex subset containing X with the property that if $x \in \partial_i X$ and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$. Then G acts cocompactly on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$.*

Letting $\mathcal{C} = \Omega$ and changing notation, we obtain:

Corollary 7.2 *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple with the property that if $x \in \partial_i \mathcal{C}$ and $[x, y] \subset \partial\Omega$, then $[x, y] \subset F_\Omega(\partial_i \mathcal{C})$. Then Γ acts cocompactly on $\partial\Omega - F_\Omega(\partial_i \mathcal{C})$.*

Remark 7.3 The main application of Theorem 7.1 will be in Section 9 as we now explain. Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple, $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$, and $X \in \mathcal{X}$. Then $(\Omega, X, \text{Stab}_\Gamma(X))$ and $\mathcal{C}$ satisfy the conditions in Theorem 7.1; see Theorem 5.1(2) and (6). We will use this in the proof of Theorem 1.4(2). In particular, we use this to show that nonconical points are bounded parabolic points.

For the rest of the section, fix $\Omega, \mathcal{C}, X, G$ satisfying the hypotheses of Theorem 7.1. The key idea will be to study the following “closest point projection”.

Definition 7.4 Define $\pi_X: \bar{\mathcal{C}} \rightarrow \{\text{subsets of } X\}$ as follows:

- For $p \in \mathcal{C}$, let
$$\pi_X(p) := \{x \in X : d_\Omega(p, x) = d_\Omega(p, X)\}$$
 denote the points in X closest to p .
- For $y \in \partial_i \mathcal{C}$ let $\pi_X(y) \subset X$ denote the set of all points $x \in X$ where there exist sequences $\{p_n\}$ and $\{x_n\}$ such that $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \rightarrow y$, and $x_n \rightarrow x$.

We start by observing the following.

Observation 7.5 (1) *If $p \in \mathcal{C}$, then $\pi_X(p)$ is a compact convex subset of X .*

(2) *If $p \in \mathcal{C}$ and $x \in \pi_X(p)$, then $x \in \pi_X(q)$ for all $q \in [x, p]$.*

(3) *If $g \in G$, then $\pi_X \circ g = g \circ \pi_X$.*

Proof Notice that if $p \in \mathcal{C}$, then

$$\pi_X(p) = X \cap \{x \in \Omega : d_\Omega(p, x) \leq d_\Omega(p, X)\}.$$

Part (1) is then a consequence of the fact that closed metric balls in the Hilbert metric are convex and compact; see Proposition 2.9 above. Part (2) follows from the fact that $[x, p]$ can be parametrized to be a geodesic in the Hilbert metric and part (3) is by definition. $\square$

Lemma 7.6 *If $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$, then $\pi_X(y) \subset X$ is a nonempty bounded set.*

Proof It suffices to fix sequences $\{p_n\}$ and $\{x_n\}$ such that $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \rightarrow y$, and $x_n \rightarrow x \in \bar{X}$, then prove that $x \in X$.

Suppose for a contradiction that $x \in \partial_i X$. Since $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$ our hypothesis on X says that $(x, y) \subset \Omega$. Fix $v \in (x, y)$. Then pick a sequence $v_n \in (x_n, p_n)$ with $v_n \rightarrow v$. Then $x_n \in \pi_X(v_n)$ by Observation 7.5(2). However, we then have

$$d_\Omega(v, X) = \lim_{n \rightarrow \infty} d_\Omega(v_n, X) = \lim_{n \rightarrow \infty} d_\Omega(v_n, x_n) = \lim_{n \rightarrow \infty} d_\Omega(v, x_n) = \infty,$$

and we have a contradiction. $\square$

Lemma 7.7 *If $y \in \partial_i \mathcal{C} \cap F_\Omega(\partial_i X)$, then $\pi_X(y) = \emptyset$.*

Proof Suppose for a contradiction that there exist sequences $\{p_n\}$ and $\{x_n\}$ where $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \rightarrow y$, and $x_n \rightarrow x \in X$.

By hypothesis, there exists $w \in \partial_i X$ with $y \in F_\Omega(w)$. Fix $q \in (x, y)$ such that $d_\Omega(q, x) > d_{F_\Omega(w)}(y, w)$. Then fix $q_n \in (x_n, p_n)$ such that $q_n \rightarrow q$. By Observation 7.5(2),

$$d_\Omega(q_n, X) = d_\Omega(q_n, x_n),$$

and so

$$d_\Omega(q, X) = \lim_{n \rightarrow \infty} d_\Omega(q_n, X) = d_\Omega(q, x) > d_{F_\Omega(w)}(y, w).$$

However, this is impossible since Proposition 2.9 implies that

$$d_\Omega(q, X) \leq d_\Omega(q, (x, w)) \leq d_\Omega^{\text{Haus}}((x, y), (x, w)) \leq d_{F_\Omega(w)}(y, w).$$

So we have a contradiction. $\square$

Lemma 7.8 *If $K \subset X$ is compact, then the set*

$$\hat{K} := \{y \in \partial_i \mathcal{C} : \pi_X(y) \cap K \neq \emptyset\}$$

is compact.

Proof Consider a sequence $\{y_n\}$ in $\hat{K}$ where $y_n \rightarrow y \in \partial_i \mathcal{C}$. Fix $x_n \in \pi_X(y_n) \cap K$. Passing to a subsequence, we can suppose that $x_n \rightarrow x \in K$. By the definition of π_X , we can also find sequences $\{p_{n,m}\}$ and $\{x_{n,m}\}$ such that $p_{n,m} \in \mathcal{C}$, $x_{n,m} \in \pi_X(p_{n,m})$, and

$$y_n = \lim_{m \rightarrow \infty} p_{n,m} \quad \text{and} \quad x_n = \lim_{m \rightarrow \infty} x_{n,m}.$$

Then we can pick subsequences $\{p_{n_j, m_j}\}$ and $\{x_{n_j, m_j}\}$ such that

$$y = \lim_{j \rightarrow \infty} p_{n_j, m_j} \quad \text{and} \quad x = \lim_{j \rightarrow \infty} x_{n_j, m_j}.$$

So $x \in \pi_X(y) \cap K$ and hence $y \in \hat{K}$. □

Proof of Theorem 7.1 Fix a compact set $K \subset X$ such that $G \cdot K = X$. Then Lemmas 7.8 and 7.7 imply that

$$\hat{K} := \{y \in \partial_i \mathcal{C} : \pi_X(y) \cap K \neq \emptyset\}$$

is compact and contained in $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$.

We claim that

$$G \cdot \hat{K} = \partial_i \mathcal{C} - F_\Omega(\partial_i X).$$

Fix $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$. By Lemma 7.6 there exists $x \in \pi_X(y)$. There exists $g \in G$ such that $g(x) \in K$. Then $g(x) \in \pi_X(g(y))$ and so $g(y) \in \hat{K}$. So $y \in G \cdot \hat{K}$. Since y was arbitrary, this proves the claim and the theorem. □

8 Basic properties of boundary quotients

For the rest of the section fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a closed convex subset $\mathcal{C} \subset \Omega$, and a discrete subgroup $\Gamma \leq \text{Aut}(\Omega)$ which preserves $\mathcal{C}$. We do not assume that Γ acts cocompactly on $\mathcal{C}$.

Also fix a Γ -invariant equivalence relation $\sim$ on $\partial_i \mathcal{C}$ such that the set

$$R := \{(x, y) \in \partial_i \mathcal{C} \times \partial_i \mathcal{C} : x \sim y\}$$

is closed and $\#(\partial_i \mathcal{C} / \sim) \geq 3$. For each $x \in \partial_i \mathcal{C}$, let $[x]$ denote the equivalence class of x and let

$$(8-1) \quad \mathcal{C}_x := \text{ConvHull}_\Omega([x]) \cap \Omega.$$

Notice that it is possible for $\mathcal{C}_x$ to be empty.

We consider two conditions:

Condition (1) Whenever $x, y \in \partial_i \mathcal{C}$ and $[x, y] \subset \partial\Omega$, we have $x \sim y$.

Condition (2) There exist $r, D > 0$ such that if $x \not\sim y$, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(\mathcal{C}_x, r) \cap \mathcal{N}_\Omega(\mathcal{C}_y, r)) < D.$$

We will prove the following results about the quotient space $\partial_i \mathcal{C} / \sim$. These arguments are similar to analogous arguments of Choi [5] and Weisman [23].

Proposition 8.1 *The quotient $\partial_i \mathcal{C} / \sim$ is a compact Hausdorff metrizable space.*

Proposition 8.2 *If Condition (1) holds, then Γ acts as a convergence group on $\partial_i \mathcal{C} / \sim$.*

To state the final result, we need a definition.

Definition 8.3 A point $x \in \partial\Omega$ is a *uniformly conical limit point* of Γ acting on Ω if for any $p_0 \in \Omega$ the image of $[p_0, x)$ in $\Gamma \backslash \Omega$ is relatively compact.

Proposition 8.4 *If Conditions (1) and (2) hold, $x \in \partial_i \mathcal{C}$ is a uniformly conical limit point of Γ acting on Ω , and $[x] \subset F_\Omega(x)$, then $[x] \in \partial_i \mathcal{C}/\sim$ is a conical limit point of Γ (in the convergence group sense; see Section 3.2).*

8.1 Proof of Proposition 8.1

The proof is an exercise in point set topology. We include it for the convenience of the reader. We also note that similar arguments appear in [5, Proposition 10.3.8; 23, Proposition 8.10]. For the rest of this proof, let $\pi: \partial_i \mathcal{C} \rightarrow \partial_i \mathcal{C}/\sim$ denote the quotient map.

We begin the proof with an useful observation:

Observation 8.5 *Let $U \subset \partial_i \mathcal{C}$ be an open set, $U^* := \{x \in \partial_i \mathcal{C} : [x] \subset U\}$, and $V := \pi(U^*)$. Then U^* and V are open subsets of $\partial_i \mathcal{C}$ and $\partial_i \mathcal{C}/\sim$ respectively.*

Proof of Observation 8.5 If U^* is empty, then V is empty and the statement is true. So assume that U^* is nonempty. To show that U^* is open, we show that $\partial_i \mathcal{C} - U^*$ is closed. Consider any sequence $\{z_n\}$ in $\partial_i \mathcal{C} - U^*$ such that $z_n \rightarrow z \in \partial_i \mathcal{C}$. Since $z_n \notin U^*$, there exists $z'_n \notin U$ such that $(z_n, z'_n) \in R$. As $\partial_i \mathcal{C} - U$ is compact, we can pass to a subsequence and assume that $z'_n \rightarrow z' \in \partial_i \mathcal{C} - U$. Since R is closed and $(z_n, z'_n) \rightarrow (z, z')$, we have $(z, z') \in R$. Thus, $z \in \partial_i \mathcal{C} - U^*$ and hence $\partial_i \mathcal{C} - U^*$ is closed.

Since U^* is a union of equivalence classes, we have

$$\pi^{-1}(V) = \pi^{-1}(\pi(U^*)) = U^*,$$

and so, by the definition of the quotient topology, V is open. $\square$

$\partial_i \mathcal{C}/\sim$ is Hausdorff Since $\partial_i \mathcal{C}$ is a compact Hausdorff space and R is closed, it is well known that $\partial_i \mathcal{C}/\sim$ is Hausdorff; eg [9, Proposition 1.4.4]. To see this, fix $x, y \in \partial_i \mathcal{C}$ such that $\pi(x) \neq \pi(y)$. Since R is closed, $[w] = \{z \in \partial_i \mathcal{C} : (w, z) \in R\}$ is a closed, and hence compact, subset of $\partial_i \mathcal{C}$ for any $w \in \partial_i \mathcal{C}$. Then $[x]$ and $[y]$ are disjoint compact subsets of a Hausdorff space $\partial_i \mathcal{C}$. So we can separate $[x]$ and $[y]$ using disjoint open sets $U_x, U_y \subset \partial_i \mathcal{C}$.

Let $U_w^* := \{z \in \partial_i \mathcal{C} : [z] \subset U_w\}$ and $V_w := \pi(U_w^*)$ for $w = x, y$. By Observation 8.5, U_w^* are disjoint open sets with $[w] \subset U_w$ for $w = x, y$. Then, by Observation 8.5, V_x and V_y are disjoint nonempty open subsets of $\partial_i \mathcal{C}/\sim$ that separate $\pi(x)$ and $\pi(y)$. So $\partial_i \mathcal{C}/\sim$ is Hausdorff.

$\partial_i \mathcal{C}/\sim$ is compact and metrizable Since $\partial_i \mathcal{C}/\sim$ is Hausdorff and $\partial_i \mathcal{C}/\sim = \pi(\partial_i \mathcal{C})$ is the continuous image of a compact metric space, it is compact and metrizable; eg [24, Corollary 23.2]. Once again, we include the argument.

Since $\partial_i \mathcal{C}$ is compact and π is continuous, then $\partial_i \mathcal{C}/\sim$ is compact. Then, by the Uryshon metrization theorem, it suffices to show that $\partial_i \mathcal{C}/\sim$ has a countable basis to prove that it is metrizable. Since $\partial_i \mathcal{C}$ is

a compact metrizable space, we can choose a countable basis $\mathcal{U} := \{U_n : n \in \mathbb{N}\}$ for $\partial_i \mathcal{C}$. Without loss of generality, we can assume that $\mathcal{U}$ is closed under finite unions. For $n \geq 1$, let

$$U_n^* := \{x \in \partial_i \mathcal{C} : [x] \subset U_n\} \quad \text{and} \quad V_n := \pi(U_n^*).$$

By Observation 8.5, these are open sets. Further, since any equivalence class is compact and $\mathcal{U}$ is closed under finite unions, for any $x \in \partial_i \mathcal{C}$ there exists U_n with $[x] \subset U_n$. Then $x \in [x] \subset U_n^*$ and $[x] \in V_n$. Thus $\mathcal{U}^* := \{U_n^* : U_n^* \neq \emptyset, n \in \mathbb{N}\}$ is an open cover of $\partial_i \mathcal{C}$ and $\mathcal{V} := \{V_n : U_n^* \neq \emptyset, n \in \mathbb{N}\}$ is an open cover of $\partial_i \mathcal{C}/\sim$.

We will show that $\mathcal{V}$ is a basis for the topology on $\partial_i \mathcal{C}/\sim$. Let $w \in W \subset \partial_i \mathcal{C}/\sim$ where W is an open set. Then $\pi^{-1}(w) = [w]$ is an equivalence class and hence compact. Then, since $\mathcal{U}$ is closed under finite unions, there exists U_n with $[w] \subset U_n \subset \pi^{-1}(W)$. Then $w \in V_n^* \subset W$. Thus $\mathcal{V}$ is a basis for the topology.

8.2 Proof of Proposition 8.2

We also note that similar arguments appear in [5, Theorem 10.3.1; 23, Proposition 8.8].

Notice that Condition (1) implies that

$$\partial_i \mathcal{C} \cap \overline{F_\Omega(x')} \subset [x']$$

for all $x' \in \partial_i \mathcal{C}$.

Suppose that $\{\gamma_n\}$ is a sequence of distinct elements in Γ . Fix $p_0 \in \mathcal{C}$. Passing to a subsequence we can suppose that $\gamma_n(p_0) \rightarrow x \in \partial_i \mathcal{C}$, $\gamma_n^{-1}(p_0) \rightarrow y \in \partial_i \mathcal{C}$, and $\gamma_n \rightarrow S$ in $\mathbb{P}(\text{End}(\mathbb{R}^d))$. Then, by Observation 2.3,

$$\lim_{n \rightarrow \infty} \gamma_n(z) = S(z)$$

for all $z \in \mathbb{P}(\mathbb{R}^d) - \mathbb{P}(\ker S)$ and the convergence is locally uniform. By Proposition 2.4,

$$S(\bar{\Omega} - \mathbb{P}(\ker S)) \subset \bar{\Omega} \cap \text{Span } F_\Omega(x) = \overline{F_\Omega(x)}.$$

Then

$$S(\bar{\mathcal{C}} - \mathbb{P}(\ker S)) \subset \partial_i \mathcal{C} \cap \overline{F_\Omega(x)} \subset [x].$$

Proposition 2.4 also implies that $\mathbb{P}(\ker S) \cap \Omega = \emptyset$ and $y \in \mathbb{P}(\ker S)$. So if $z \in \partial_i \mathcal{C} \cap \mathbb{P}(\ker S)$, then $[y, z] \subset \partial \Omega$ and then Condition (1) implies that $z \in [y]$. Hence

$$\gamma_n|_{(\partial_i \mathcal{C}/\sim) - [y]}$$

converges locally uniformly to the constant map $[x]$.

8.3 Proof of Proposition 8.4

We also note that similar arguments appear in [5, Theorem 10.3.1; 23, Proposition 8.17].

Recall the notation $\mathcal{C}_x$ from (8-1). We start with a lemma.

Lemma 8.6 *If $y \in \partial_i \mathcal{C}$ and $z \in F_\Omega(\partial_i \mathcal{C}_y) \cap \partial_i \mathcal{C}$, then $z \sim y$.*

Proof By hypothesis, there exists some $z' \in \partial_i \mathcal{C}_y$ with $z \in F_\Omega(z')$.

Since $[y]$ is closed, the extreme points of $\text{ConvHull}_\Omega([y])$ are contained in $[y]$. Then there exist $y_1, \dots, y_m \in [y]$ such that

$$z' \in \text{rel-int}(\text{ConvHull}_\Omega(\{y_1, \dots, y_m\})).$$

Thus $y_1, \dots, y_m \in \overline{F_\Omega(z')}$. Then $[z, y_1] \subset \partial\Omega$ where $z, y_1 \in \partial_i \mathcal{C}$. Condition (1) implies that $z \sim y_1$. Thus $z \sim y$. $\square$

Now suppose $x \in \partial_i \mathcal{C}$ is a uniform conical limit point of Γ acting on Ω . Fix $p_0 \in \mathcal{C}$ and a sequence $\{p_n\}$ in $[p_0, x)$ with $p_n \rightarrow x$.

By Condition (2), there exist $r, D > 0$ such that if $y_1 \not\sim y_2$, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(\mathcal{C}_{y_1}, r) \cap \mathcal{N}_\Omega(\mathcal{C}_{y_2}, r)) < D.$$

Lemma 8.7 *For every $n \in \mathbb{N}$, there exists $q_n \in [p_n, x)$ such that*

$$\mathcal{B}_\Omega(q_n, 2D) \cap (p_0, x) \not\subset \mathcal{N}_\Omega(\mathcal{C}_y, r)$$

for all $y \in \partial_i \mathcal{C}$.

Proof Fix $n \in \mathbb{N}$ and suppose not. Then for every $q \in [p_n, x)$ there exists $y(q) \in \partial_i \mathcal{C}$ such that

$$\mathcal{B}_\Omega(q, 2D) \cap (p_0, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(q)}, r).$$

Note that this implies that $\mathcal{C}_{y(q)}$ is a nonempty set. We claim that $\mathcal{C}_{y(q)}$ does not depend on q . To show this it is enough to fix $q', q'' \in [p_n, x)$ with $d_\Omega(q', q'') \leq D$ and show that $\mathcal{C}_{y(q')} = \mathcal{C}_{y(q'')}$. In this case, $\mathcal{B}_\Omega(q', D) \subset \mathcal{B}_\Omega(q'', 2D)$; hence

$$\mathcal{B}_\Omega(q', D) \cap (p_0, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(q')}, r) \cap \mathcal{N}_\Omega(\mathcal{C}_{y(q'')}, r).$$

Then by our choice of D , we have $\mathcal{C}_{y(q')} = \mathcal{C}_{y(q'')}$. Thus $\mathcal{C}_{y(q)}$ does not depend on $q \in [p_n, x)$. Thus

$$(p_n, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(p_n)}, r)$$

Then Proposition 2.8 implies that $x \in F_\Omega(\partial_i \mathcal{C}_{y(p_n)})$. So by the previous lemma $x \sim y(p_n)$. Then $\mathcal{C}_x = \mathcal{C}_{y(p_n)}$ is nonempty, which contradicts the assumption that $[x] \subset F_\Omega(x)$. $\square$

We finish the proof of Proposition 8.4. Fix a sequence $\{q_n\}$ as in the above lemma. Since x is a uniformly conical limit point, there exists a sequence $\{\gamma_n\}$ in Γ such that $\{\gamma_n(q_n)\}$ is relatively compact in Ω . Passing to a subsequence we can suppose that $\gamma_n(p_0) \rightarrow b$, $\gamma_n^{-1}(p_0) \rightarrow c$, and $\gamma_n(x) \rightarrow a$. By the proof of Proposition 8.2,

$$\gamma_n|_{(\partial_i \mathcal{C}/\sim)-[c]}$$

converges locally uniformly to the constant map $[b]$. Also $\gamma_n[x] \rightarrow [a]$. So, to show that $[x]$ is a conical limit point in the convergence group sense, we need to prove that $[x] = [c]$ and $[a] \neq [b]$.

Suppose for a contradiction that $[a] = [b]$. Since $\{\gamma_n(q_n)\}$ is relatively compact in Ω and $\gamma_n(q_n) \in (\gamma_n(x), \gamma_n(p_0))$, we have $(a, b) \subset \Omega$. Hence $(a, b) \subset \mathcal{C}_a$ and so for n large,

$$\mathcal{B}_\Omega(\gamma_n(q_n), 2D) \cap (\gamma_n(p_0), \gamma_n(x)) \subset \mathcal{N}_\Omega(\mathcal{C}_a, r).$$

This implies that

$$\mathcal{B}_\Omega(q_n, 2D) \cap (p_0, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{\gamma_n^{-1}(a)}, r),$$

which is a contradiction. So $[a] \neq [b]$.

Since $\gamma_n([x]) \rightarrow [a]$ and $[a] \neq [b]$, we must have $[x] = [c]$.

9 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple.

Theorem 1.4(1) Suppose that Γ is relatively hyperbolic with respect to $\mathcal{P} := \{P_1, \dots, P_N\}$ and

$$X_j := \text{ConvHull}_\Omega(\mathcal{L}_\Omega(P_j) \cap \partial_i \mathcal{C}) \cap \Omega, \quad \text{where } j = 1, \dots, N.$$

Then Proposition 6.1 implies that $(\mathcal{C}, d_\Omega)$ is relatively hyperbolic with respect to

$$\mathcal{X} = \Gamma \cdot \{X_1, \dots, X_N\}.$$

Proposition 3.4 implies that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$.

Theorem 1.4(2) Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ and $\mathcal{P}$ is a set of representatives of the Γ -conjugacy classes in $\{\text{Stab}_\Gamma(X) : X \in \mathcal{X}\}$. Let

$$[\partial_i \mathcal{C}]_\mathcal{X} := \partial_i \mathcal{C} / \sim$$

be as in Definition 1.2. We claim that the action of Γ on $[\partial_i \mathcal{C}]_\mathcal{X}$ satisfies Theorem 3.5.

Lemma 9.1 The set $R := \{(x, y) \in \partial_i \mathcal{C} \times \partial_i \mathcal{C} : x \sim y\}$ is closed.

Proof Suppose that $\{(x_n, y_n)\}$ is a sequence in R converging to (x, y) in $\partial_i \mathcal{C} \times \partial_i \mathcal{C}$.

Case 1 Assume $(x, y) \subset \Omega$. Then $(x_n, y_n) \subset \Omega$ for n sufficiently large, and for such n there exists $X_n \in \mathcal{X}$ with $x_n, y_n \in F_\Omega(\partial_i X_n)$. By Theorem 5.1(5) there is some $R > 0$ such that for every n there exist $x'_n, y'_n \in \partial_i X_n$ where

$$d_{F_\Omega(x_n)}(x_n, x'_n) \leq R \quad \text{and} \quad d_{F_\Omega(y_n)}(y_n, y'_n) \leq R.$$

Then by Proposition 2.9,

$$d_\Omega^{\text{Haus}}((x_n, y_n), (x'_n, y'_n)) \leq R.$$

Since $(x_n, y_n) \rightarrow (x, y)$, this implies that

$$\lim_{n, k \rightarrow \infty} \text{diam}_\Omega(\mathcal{N}_\Omega(X_n, R+1) \cap \mathcal{N}_\Omega(X_k, R+1)) = \infty.$$

Since $\mathcal{X}$ is strongly isolated, $X := X_n = X_k$ for n, k sufficiently large. Then

$$(x, y) \subset \mathcal{N}_\Omega(X, R + 1),$$

and so $x, y \in F_\Omega(\partial_i X)$ by Proposition 2.8. Thus $x \sim y$.

Case 2 Assume $[x, y] \subset \partial\Omega$. Then $x \sim y$ by Theorem 5.1(7). $\square$

Let

$$B := \{[F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] : X \in \mathcal{X}\} \subset [\partial_i \mathcal{C}]_\mathcal{X}.$$

Lemma 9.2 (1) $[\partial_i \mathcal{C}]_\mathcal{X}$ is a compact Hausdorff metrizable space.

(2) Γ acts as a convergence group on $[\partial_i \mathcal{C}]_\mathcal{X}$.

(3) If $z \in [\partial_i \mathcal{C}]_\mathcal{X} - B$, then z is a conical limit point of Γ .

Proof We use the results of Section 8. Part (1) follows from the previous lemma and Proposition 8.1.

Notice that Theorem 5.1(7) implies that the quotient $[\partial_i \mathcal{C}]_\mathcal{X}$ satisfies Condition (1) from Section 8. We claim that the quotient $[\partial_i \mathcal{C}]_\mathcal{X}$ satisfies Condition (2) from Section 8. First notice that if $z \in [\partial_i \mathcal{C}]_\mathcal{X} - B$, then $\mathcal{C}_z = \emptyset$. So it suffices to consider elements in B . Further, if $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$, then by Theorem 5.1(2) and (8), $\text{Stab}_\Gamma(X)$ acts cocompactly on both $\mathcal{C}_b$ and X . So $\mathcal{C}_b$ is contained in a bounded neighborhood of X . By Theorem 5.1(1), we can choose this bound to be independent of b . Then since $\mathcal{X}$ is strongly isolated, $[\partial_i \mathcal{C}]_\mathcal{X}$ satisfies Condition (2).

Then Γ acts as a convergence group on $[\partial_i \mathcal{C}]_\mathcal{X}$ by Proposition 8.2. If $z \in [\partial_i \mathcal{C}]_\mathcal{X} - B$, then we have $z = [F_\Omega(x) \cap \partial_i \mathcal{C}]$ for some $x \in \partial_i \mathcal{C}$ and so Proposition 8.4 implies that z is a conical limit point of Γ . This proves parts (2) and (3). $\square$

Lemma 9.3 (1) Γ has finitely many orbits in B .

(2) If $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$, then $\text{Stab}_\Gamma(b) = \text{Stab}_\Gamma(X)$. In particular, $\text{Stab}_\Gamma(b)$ is finitely generated.

(3) If $b \in B$, then b is a bounded parabolic fixed point.

Proof (1) This follows immediately from Theorem 5.1(1).

(2) It is clear that $\text{Stab}_\Gamma(b) \supset \text{Stab}_\Gamma(X)$. For the other inclusion, if $\gamma \in \text{Stab}_\Gamma(b)$, then

$$F_\Omega(\partial_i(\gamma X)) = \gamma F_\Omega(\partial_i X) = F_\Omega(\partial_i X).$$

So $\gamma X = X$ by Theorem 5.1(3). Hence $\gamma \in \text{Stab}_\Gamma(X)$ and so $\text{Stab}_\Gamma(b) = \text{Stab}_\Gamma(X)$. The “in particular” part then follows from Theorem 5.1(2).

(3) The proof is an application of Theorem 7.1, as outlined in Remark 7.3. We now provide the details.

Fix $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$. By Theorem 5.1(2), $\text{Stab}_\Gamma(X)$ acts cocompactly on X . Furthermore, by Theorem 5.1(6), if $x \in \partial_i X$ and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \in F_\Omega(\partial_i X)$.

Then Theorem 7.1 implies that $\text{Stab}_\Gamma(X)$ acts cocompactly on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$. Thus $\text{Stab}_\Gamma(b)$ acts cocompactly on $[\partial_i \mathcal{C}]_\mathcal{X} - \{b\}$. $\square$

Then Theorem 3.5 implies that Γ is relatively hyperbolic with respect to $\mathcal{P}$ and there exists a Γ -equivariant homeomorphism $\partial(\Gamma, \mathcal{P}) \rightarrow [\partial_i \mathcal{C}]_{\mathcal{X}}$.

Theorem 1.4(a)–(e) Now suppose that at least one of the conditions is satisfied. Proposition 6.1 implies parts (a) and (c). Part (b) was established in the proof of Theorem 1.4(2); see the last paragraph. Parts (d) and (e) follow from Theorem 5.1.

10 The convex cocompact case

We prove Theorem 1.7 and Proposition 1.8. As mentioned in the introduction, the key result which allows our results to simplify in the convex cocompact case is the following observation; see [7, Corollaries 4.10 and 4.13].

Observation 10.1 Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \text{Aut}(\Omega)$ is a convex cocompact subgroup.

- (1) If $x \in \partial\Omega$ and $F_{\Omega}(x) \cap \partial_i \mathcal{C}_{\Omega}(\Gamma) \neq \emptyset$, then $F_{\Omega}(x) \subset \partial_i \mathcal{C}_{\Omega}(\Gamma)$.
- (2) $\mathcal{L}_{\Omega}(\Gamma) = \partial_i \mathcal{C}_{\Omega}(\Gamma)$. In particular, $\mathcal{L}_{\Omega}(\Gamma)$ is closed.

Proof (1) Fix $x \in \partial\Omega$ with $F_{\Omega}(x) \cap \partial_i \mathcal{C}_{\Omega}(\Gamma) \neq \emptyset$. Then Proposition 2.5 implies that $F_{\Omega}(x) \subset \mathcal{L}_{\Omega}(\Gamma)$ and we have $\mathcal{L}_{\Omega}(\Gamma) \subset \partial_i \mathcal{C}_{\Omega}(\Gamma)$ by the definition of $\mathcal{C}_{\Omega}(\Gamma)$.

(2) Since Γ acts cocompactly on $\mathcal{C}$, we have $\partial_i \mathcal{C}_{\Omega}(\Gamma) \subset \mathcal{L}_{\Omega}(\Gamma)$ and the reverse inclusion is by definition of $\mathcal{C}_{\Omega}(\Gamma)$. $\square$

10.1 Proof of Theorem 1.7

Suppose that $\Gamma \leq \text{Aut}(\Omega)$ is a convex cocompact group such that Γ is relatively hyperbolic with respect to $\mathcal{P} := \{P_1, \dots, P_N\}$. Then by definition $(\Omega, \mathcal{C}_{\Omega}(\Gamma), \Gamma)$ is a naive convex cocompact triple. Let $\mathcal{C} = \mathcal{C}_{\Omega}(\Gamma)$ and

$$\mathcal{X} := \Gamma \cdot \{\mathcal{C}_{\Omega}(P_1), \dots, \mathcal{C}_{\Omega}(P_N)\}.$$

Lemma 10.2 (parts (a) and (c)) Each P_j is a convex cocompact subgroup of $\text{Aut}(\Omega)$ and $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\mathcal{X}$.

Proof Since $\mathcal{L}_{\Omega}(P_j) \subset \mathcal{L}_{\Omega}(\Gamma) \subset \partial_i \mathcal{C}$, then

$$\mathcal{C}_{\Omega}(P_j) = \text{ConvHull}_{\Omega}(\mathcal{L}_{\Omega}(P_j)) \cap \Omega = \text{ConvHull}_{\Omega}(\mathcal{L}_{\Omega}(P_j) \cap \partial_i \mathcal{C}) \cap \Omega.$$

Proposition 6.1 implies that P_j acts cocompactly on $\mathcal{C}_{\Omega}(P_j)$. Then $(\mathcal{C}, d_{\Omega})$ is relatively hyperbolic with respect to $\mathcal{X}$. $\square$

If $X \in \mathcal{X}$, then $X = \gamma \mathcal{C}_{\Omega}(P_j) = \mathcal{C}_{\Omega}(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $1 \leq j \leq N$. Then, since P_j is a convex cocompact subgroup of $\text{Aut}(\Omega)$, Observation 10.1 implies that

$$(10-1) \quad F_{\Omega}(\partial_i X) = \partial_i X = \mathcal{L}_{\Omega}(\gamma P_j \gamma^{-1}).$$

Lemma 10.3 (part (e)) *If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

Proof Fix $x \in \text{rel-int}(\ell)$. Then $\dim F_\Omega(x) \geq 1$ and so, by Observation 10.1,

$$\text{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) = \text{diam}_{F_\Omega(x)} F_\Omega(x) = \infty.$$

Thus by Theorem 1.4(d), there exist $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$ such that

$$x \in F_\Omega(\partial_i X),$$

where $X = \mathcal{C}_\Omega(\gamma P_j \gamma^{-1}) = \gamma \mathcal{C}_\Omega(P_j)$. Then (10-1) implies that

$$\text{rel-int}(\ell) \subset F_\Omega(x) \subset F_\Omega(\partial_i X) = \mathcal{L}_\Omega(\gamma P_j \gamma^{-1}).$$

Since $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ is closed (see Observation 10.1), this completes the proof. $\square$

Recall that, by definition, $[\partial_i \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}$ is obtained from $\partial_i \mathcal{C}_\Omega(\Gamma)$ by collapsing each $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ to a point. Hence (10-1) and Lemma 10.3 imply that

$$(10-2) \quad [\partial_i \mathcal{C}]_\mathcal{X} = [\partial_i \mathcal{C}]_\mathcal{P}.$$

Lemma 10.4 (part (b)) *There is a Γ -equivariant homeomorphism*

$$\partial(\Gamma, \mathcal{P}) \rightarrow [\partial_i \mathcal{C}_\Omega(\Gamma)]_\mathcal{P}.$$

Proof This follows immediately from Theorem 1.4(b) and (10-2). $\square$

To prove Theorem 1.7(d) we will use the following lemma.

Lemma 10.5 [15, Lemma 15.5] *Assume that $x \in \partial_i \mathcal{C}$ is not a C^1 -smooth point of $\partial\Omega$ and $q \in \mathcal{C}$. For any $r > 0$ and $\epsilon > 0$ there exists $q_{r,\epsilon} \in (x, q]$ with the following property: if $p \in (x, q_{r,\epsilon}]$, then there exists a properly embedded simplex $S = S(p) \subset \mathcal{C}$ of dimension at least two such that*

$$(10-3) \quad \mathcal{B}_\Omega(p, r) \cap (x, q] \subset \mathcal{N}_\Omega(S, \epsilon).$$

Let us now prove part (d).

Lemma 10.6 (part (d)) *If $x \in \partial_i \mathcal{C}_\Omega(\Gamma)$ is not a $\mathcal{C}^1$ -smooth point of Ω , then $x \in \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

Proof Fix $q \in \mathcal{C}$. By Theorem 1.4, $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$. So there exist constants $D_1, D_2 > 0$ such that:

- (a) If $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset \mathcal{N}_\Omega(X, D_2)$.
- (b) If $X_1, X_2 \in \mathcal{X}$ are distinct, then

$$\text{diam}_\Omega(\mathcal{N}_\Omega(X_1, D_2 + 1) \cap \mathcal{N}_\Omega(X_2, D_2 + 1)) < D_1.$$

Let $q' \in (x, q]$ satisfy Lemma 10.5 for the constants $r = D_1 + 1$ and $\epsilon = 1$. Fix a sequence $\{p_n\}$ in $(x, q']$ such that $p_n \rightarrow x$ and $d_\Omega(p_n, p_{n+1}) \leq 1$. For each $n \geq 1$, fix $S_n \subset \mathcal{C}$ a properly embedded simplex of dimension at least two such that

$$\mathcal{B}_\Omega(p_n, D_1 + 1) \cap (x, q] \subset \mathcal{N}_\Omega(S_n, 1).$$

Then fix $X_n \in \mathcal{X}$ such that

$$S_n \subset \mathcal{N}_\Omega(X_n, D_2).$$

Then

$$\begin{aligned} \mathcal{B}_\Omega(p_n, D_1) \cap (x, q] &\subset (\mathcal{B}_\Omega(p_n, D_1 + 1) \cap \mathcal{B}_\Omega(p_{n+1}, D_1 + 1)) \cap (x, q] \\ &\subset \mathcal{N}_\Omega(X_n, D_2 + 1) \cap \mathcal{N}_\Omega(X_{n+1}, D_2 + 1). \end{aligned}$$

Since $\text{diam}_\Omega(\mathcal{B}_\Omega(p_n, D_1) \cap (x, q]) \geq D_1$, this implies that there exists $X \in \mathcal{X}$ such that $X = X_n$ for all n large enough. Then

$$(x, q'] \subset \mathcal{N}_\Omega(X, D_2 + 1).$$

Now $X = \mathcal{C}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$. Finally, Proposition 2.8 and (10-1) imply that $x \in \partial_i \mathcal{C}_\Omega(\gamma P_j \gamma^{-1}) = \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$. $\square$

10.2 Proof of Proposition 1.8

Suppose that $\Gamma \leq \text{Aut}(\Omega)$ is a convex cocompact subgroup and $\mathcal{X}$ is a Γ -invariant collection of closed unbounded convex subsets of Ω .

(1) $\implies$ (2) Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}_\Omega(\Gamma), \Gamma)$.

Lemma 10.7 $\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric d_Ω .

Proof This follows immediately from Lemma 5.2. $\square$

Lemma 10.8 If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \emptyset$.

Proof We prove the contrapositive. Suppose $x \in \partial_i X_1 \cap \partial_i X_2$. Fix $q_1 \in X_1$ and $q_2 \in X_2$. Then

$$d_\Omega^{\text{Haus}}((x, q_1], (x, q_2]) \leq d_\Omega(q_1, q_2)$$

by Proposition 2.9. So

$$(x, q_1] \subset \mathcal{N}_\Omega(X_1, d_\Omega(q_1, q_2) + 1) \cap \mathcal{N}_\Omega(X_2, d_\Omega(q_1, q_2) + 1),$$

and hence $X_1 = X_2$. $\square$

Lemma 10.9 If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.

Proof Fix $x \in \text{rel-int}(\ell)$. Then $\dim F_\Omega(x) \geq 1$ and so by Observation 10.1,

$$\text{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) = \text{diam}_{F_\Omega(x)} F_\Omega(x) = \infty.$$

Thus by Theorem 5.1(4) there exists $X \in \mathcal{X}$ with $x \in F_\Omega(\partial_i X)$. We claim that $F_\Omega(x) \subset \partial_i X$ which will imply the lemma.

To show that $F_\Omega(x) \subset \partial_i X$, it suffices to fix an extreme point $e \in \partial F_\Omega(x)$ of $F_\Omega(x)$ and show that $\partial_i X$ contains e . By Observation 10.1 and Theorem 5.1(5) there exists $R > 0$ such that

$$d_{F_\Omega(x)}^{\text{Haus}}(F_\Omega(x), \partial_i X \cap F_\Omega(x)) \leq R.$$

Since $F_{F_\Omega(x)}(e) = \{e\}$, then Proposition 2.8 implies that $e \in \partial_i X$. $\square$

(2) $\implies$ (1) Suppose that $\mathcal{X}$ has the following properties:

- (a) $\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric d_Ω .
- (b) If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \emptyset$.
- (c) If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.

Lemma 10.10 *If $X_1, X_2 \in \mathcal{X}$ and $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) \neq \emptyset$, then $X_1 = X_2$.*

Proof By hypothesis there exists $x_1 \in \partial_i X_1$ and $x_2 \in \partial_i X_2$ with $F_\Omega(x_1) = F_\Omega(x_2)$. If $x_1 = x_2$, then property (b) implies that $X_1 = X_2$. Otherwise, $[x_1, x_2] \subset F_\Omega(x_1) \subset \partial \Omega$ and so by property (c) there exists X_3 with $[x_1, x_2] \subset \partial_i X_3$. But then by property (b), $X_1 = X_3 = X_2$. $\square$

Lemma 10.11 *$\mathcal{X}$ is discrete in the local Hausdorff topology induced by the Hilbert metric d_Ω .*

Proof Fix a sequence $\{X_n\}$ in $\mathcal{X}$ converging to some closed subset X . Since $\mathcal{X}$ is closed, $X \in \mathcal{X}$. Fix $p \in X$ and $x \in \partial_i X$.

We claim that $X_n = X$ when n is sufficiently large. Suppose not. Then after passing to a subsequence we can suppose that $X_n \neq X$ for all n . For each n fix $p_n \in X_n$ such that $p_n \rightarrow p$. Passing to a tail of our sequence we can suppose that $d_\Omega(p_n, p) < 1$ for all n . The previous lemma implies that $x \notin F_\Omega(\partial_i X_n)$ and so by Proposition 2.8,

$$\lim_{q \in [p_n, x], q \rightarrow x} d_\Omega(q, X_n) = +\infty.$$

So for each n there exists $q_n \in [p_n, x)$ with $d_\Omega(q_n, X_n) = 1$. Notice that

$$\lim_{n \rightarrow \infty} d_\Omega(q_n, X) \leq \lim_{n \rightarrow \infty} d_\Omega^{\text{Haus}}((x, p_n], (x, p]) \leq \lim_{n \rightarrow \infty} d_\Omega(p_n, p) = 0$$

by Proposition 2.9. Then, since $X_n \rightarrow X$ in the local Hausdorff topology and $d_\Omega(q_n, X_n) = 1$, the sequence $\{q_n\}$ must leave every compact subset of Ω . So $q_n \rightarrow x$.

Since $q_n \in \mathcal{C}$, there exists $\{\gamma_n\}$ in Γ such that $\{\gamma_n(q_n)\}$ is relatively compact in Ω . Passing to a subsequence, we can suppose that $\gamma_n(q_n) \rightarrow q_\infty \in \mathcal{C}$, $\gamma_n(p) \rightarrow p_\infty \in \bar{\mathcal{C}}$, $\gamma_n(X) \rightarrow X_\infty$, and $\gamma_n(X_n) \rightarrow Y_\infty$. Since $q_n \rightarrow x$, we must have $p_\infty \in \partial_i \mathcal{C}$. Since $\mathcal{X}$ is Γ -invariant and closed in the local Hausdorff topology, $X_\infty, Y_\infty \in \mathcal{X}$.

By construction, $q_\infty \in X_\infty$ and $d_\Omega(q_\infty, Y_\infty) = 1$. So $X_\infty \neq Y_\infty$. Also

$$\lim_{n \rightarrow \infty} d_\Omega(\gamma_n(p), \gamma_n(p_n)) = \lim_{n \rightarrow \infty} d_\Omega(p_n, p) = 0,$$

so Proposition 2.8 implies that $\gamma_n(p_n) \rightarrow p_\infty$. Then, by Proposition 2.9,

$$[q_\infty, p_\infty) \subset \mathcal{N}_\Omega(Y_\infty, 2),$$

since $[q_n, p_n] \subset \mathcal{N}_\Omega(X_n, 2)$ for all n . So Proposition 2.8 implies that $p_\infty \in F_\Omega(\partial_i Y_\infty)$. However, by construction, $p_\infty \in \partial_i X_\infty$ and so Lemma 10.10 implies that $X_\infty = Y_\infty$. So we have a contradiction. $\square$

Lemma 10.12 $\mathcal{X}$ is strongly isolated.

Proof Fix $r > 0$ and suppose for a contradiction that for every $n \in \mathbb{N}$ there exist $X_n, Y_n \in \mathcal{X}$ distinct such that

$$\text{diam}_\Omega(\mathcal{N}_\Omega(X_n, r) \cap \mathcal{N}_\Omega(Y_n, r)) \geq n.$$

Proposition 2.9 implies that $C_n := \mathcal{N}_\Omega(X_n, r) \cap \mathcal{N}_\Omega(Y_n, r)$ is convex.

We claim that each C_n is bounded in $(\mathcal{C}, d_\Omega)$. If not, there exists $x \in \bar{C}_n \cap \partial\Omega$. Then Proposition 2.8 implies that $x \in F_\Omega(\partial_i X_n) \cap F_\Omega(\partial_i Y_n)$. So $X_n = Y_n$ by Lemma 10.10. Thus we have a contradiction and so each C_n is bounded in $(\mathcal{C}, d_\Omega)$.

For each n let $(p_n, q_n) \subset C_n$ denote an open line segment with maximal length (with respect to the Hilbert metric). Then $d_\Omega(p_n, q_n) \geq n$.

Now fix a sequence $\{\gamma_n\}$ in Γ such that $\{\gamma_n(p_n)\}$ is relatively compact in Ω . Passing to a subsequence, we can assume that $\gamma_n(p_n) \rightarrow p \in \Omega$, $\gamma_n(q_n) \rightarrow q \in \bar{\Omega}$, $\gamma_n(X_n) \rightarrow X$, and $\gamma_n(Y_n) \rightarrow Y$. Then $q \in \partial_i \mathcal{C}$, as $d_\Omega(p, q_n) \rightarrow \infty$ and $q_n \in \mathcal{C}$. Furthermore,

$$(q, p) \subset \mathcal{N}_\Omega(X, r+1) \cap \mathcal{N}_\Omega(Y, r+1),$$

and $X, Y \in \mathcal{X}$. Proposition 2.8 implies that

$$q \in F_\Omega(\partial_i X) \cap F_\Omega(\partial_i Y).$$

So Lemma 10.10 implies that $X = Y$. Then Lemma 10.11 implies that

$$\gamma_n(X_n) = X = Y = \gamma_n(Y_n)$$

for n large. So $X_n = Y_n$ for n large and we have a contradiction. $\square$

The following result implies that $\mathcal{X}$ coarsely contains the properly embedded simplices of $\mathcal{C}$.

Lemma 10.13 If $S \subset \mathcal{C}$ is a properly embedded simplex with dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset X$.

Proof Fix a properly embedded simplex $S \subset \mathcal{C}$ with dimension at least two. Let $v_1, \dots, v_k$ denote the vertices of S . By property (c), for each $2 \leq j \leq k$ there exists some $X_j \in \mathcal{X}$ with $[v_1, v_j] \subset \partial_i X_j$. Then by property (b), $X_2 = X_3 = \dots = X_k$. So by convexity $S \subset X_2$. $\square$

Proof of (1-1) in Proposition 1.8 Suppose that (1) and (2) hold, and let $\mathcal{P}$ be a set of representatives of the Γ -conjugacy classes in $\{\text{Stab}_\Gamma(X) : X \in \mathcal{X}\}$. To show that the quotients $[\partial_i \mathcal{C}]_\mathcal{X}$ and $[\partial_i \mathcal{C}]_\mathcal{P}$ coincide it suffices to prove the following.

Lemma 10.14 *If $X \in \mathcal{X}$, then $\partial_i X = F_\Omega(\partial_i X) = \mathcal{L}_\Omega(\text{Stab}_\Gamma(X))$.*

Proof Let $P = \text{Stab}_\Gamma(X)$.

We first observe that $\partial_i X = F_\Omega(\partial_i X)$. By definition $\partial_i X \subset F_\Omega(\partial_i X)$. For the other inclusion, fix $x \in F_\Omega(\partial_i X)$. Then fix $x' \in \partial_i X$ with $x \in F_\Omega(x')$. Observation 10.1 implies that $x \in \partial_i \mathcal{C}$. So, if $x' \neq x$, then there exists $Y \in \mathcal{X}$ with $[x', x] \subset \partial_i Y$. But then $\partial_i X \cap \partial_i Y \neq \emptyset$ which implies that $X = Y$. So $x \in \partial_i X$. Thus $\partial_i X = F_\Omega(\partial_i X)$.

Next we show that $\mathcal{L}_\Omega(P)$ is a subset of $\partial_i X$. Fix $x \in \mathcal{L}_\Omega(P)$, then there exist $p \in \Omega$ and a sequence $\{g_n\}$ in P such that $g_n(p) \rightarrow x$. Fix $q \in X$. By passing to a subsequence we may suppose that $g_n(q) \rightarrow x' \in \partial_i X$. Then Proposition 2.8 implies that $x \in F_\Omega(x') \subset \partial_i X$.

Finally, Theorem 5.1(2) implies that P acts cocompactly on X and so $\partial_i X \subset \mathcal{L}_\Omega(P)$ by Proposition 2.5. $\square$

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Characterizations of stability via Morse limit sets

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Subgroup stability is a strong notion of quasiconvexity that generalizes convex cocompactness in a variety of settings. In this paper, we characterize stability of a subgroup by properties of its limit set on the Morse boundary. Given $H < G$, both finitely generated, H is stable exactly when all the limit points of H are conical, or equivalently when all the limit points of H are horospherical, as long as the limit set of H is a compact subset of the Morse boundary for G . We also demonstrate an application of these results in the settings of the mapping class group for a finite-type surface, $\text{Mod}(S)$.

20F65, 20F67, 20F69

1 Introduction

An important example of Kleinian groups are called convex cocompact groups. These are exactly the discrete subgroups $H < \text{Iso}^+(\mathbb{H}^3)$ whose orbit in $\mathbb{H}^3$ is convex cocompact. Additionally the quotient of $\mathbb{H}^3$ by each of these groups is a compact Kleinian manifold, and every infinite-order element of a convex cocompact group is loxodromic. We highlight some of the other interesting properties of convex cocompact groups in the following theorem.

Theorem 1.1 [23; 27] *A Kleinian group $H < \text{Iso}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C})$ is called convex cocompact if one of the following equivalent conditions holds:*

- (1) *H acts cocompactly on the convex hull of its limit set ΛH .*
- (2) *Any H -orbit in $\mathbb{H}^3$ is quasiconvex.*
- (3) *Every limit point of H is conical.*
- (4) *H acts cocompactly on $\mathbb{H}^3 \cup \Omega$, where $\Omega = \partial\mathbb{H}^3 \setminus \Lambda H$.* □

However other more recent versions of this relationship have been shown. Swenson showed a generalization of this theorem for Gromov hyperbolic groups equipped with their visual boundaries [28], and there has been recent interest in generalizing these relationships beyond the setting of word-hyperbolic groups. For example, convex cocompact subgroups of mapping class groups acting on Teichmüller space, equipped with the Thurston compactification, have been characterized by Farb and Mosher [15] as exactly the subgroups which determine Gromov hyperbolic surface group extensions. There has also been recent work done in this direction for subgroups of $\text{Out}(F_n)$, relating convex cocompact subgroups to hyperbolic extensions of free groups [3; 12; 18].

There has also been interest in creating generalizations which are applicable for any finitely generated group. An important generalization comes from [13], where Durham and Taylor introduced *stability* (see Definition 2.32) to characterize convex cocompact subgroups of a mapping class group in a way which is intrinsic to the geometry of the mapping class group, and in fact, generalizes the notions of convex cocompactness to any finitely generated group. A subgroup of isometries is stable when the orbit map $H \rightarrow X$ is a quasi-isometric embedding into a hyperbolic subset of X . The concept of stability was later generalized to *strongly quasiconvex subgroup*, introduced in [29]. A subgroup is stable when it is undistorted and strongly quasiconvex.

In the Kleinian, hyperbolic, and mapping class group settings, convex cocompactness is characterized by properties of the limit set on an appropriate boundary, as shown by Kent and Leininger in [20], and independently by Hamenstädt in [17]. For an arbitrary finitely generated group, it is possible to construct a (quasi-isometric invariant) boundary called the Morse boundary, which was introduced by Cordes in [8] and expanded by Cordes and Hume in [10]. A generalization of convex cocompactness developed by Cordes and Durham [9], called *boundary convex cocompactness*, is an exact generalization of Theorem 1.1(1), in the case where H is a proper action on an arbitrary proper geodesic metric space with a nonempty and compact limit set in the Morse boundary; see Definition 2.33.

The purpose of this paper is to fully generalize Theorem 1.1(3) to the setting of finitely generated groups, thereby answering [9, Question 1.15]. In fact, we additionally generalize some other characterizations from the hyperbolic setting found in [28]. We summarize the results of this paper in the following theorem:

Theorem 1.2 *Let H be a finitely generated group acting by isometries on a proper geodesic metric space X . The following are equivalent:*

- (1) *Any H -orbit in X is a stable embedding of $H \rightarrow X$.*
- (2) *H acts boundary convex cocompactly on X .*
- (3) *Every point in ΛH is a conical limit point of H , $\Lambda H \neq \emptyset$, and ΛH is a compact subset of the Morse boundary of H .*
- (4) *Every point in ΛH is a horospherical limit point of H , $\Lambda H \neq \emptyset$, and ΛH is a compact subset of the Morse boundary of H .*

Remark 1.3 The result (1) $\Leftrightarrow$ (2) is found in the main theorem of [9]. We show (3) $\Rightarrow$ (4) in a combination of Proposition 3.4 and Theorem 3.8, using methods similar to [28]. We show (4) $\Rightarrow$ (2) in Theorem 4.3, by first showing that noncobounded actions on the weak convex hull of ΛH admit a sequence of points p_n which diverge quickly from the orbit (see Lemma 4.1), but then showing that the p_n converge to an element of ΛH , which ultimately contradicts the conical assumption. We give an alternate proof to (2) $\Rightarrow$ (3) in Proposition 4.5 which does not use the main theorem from [9].

A limit point in ΛH is *conical* if the limit point is accumulated by the orbit in a strong way: every geodesic ray representing the limit point gets boundedly close to the orbit. See Definitions 2.9 and 3.2. In general, a geodesic ray which is constructed from geodesic segments $[x, hx]$ need not stay close to the orbit of H , even when H is stably embedded in the hyperbolic setting. For an example, see [28, Lemma 3]. A limit point in ΛH is *horospherical* if it is accumulated by the orbit in a similar way: every horoball around a geodesic ray representing the limit point intersects the orbit; see Definition 3.2.

We take a moment to provide a broad overview of stability in the recent literature. In addition to results for the mapping class group from above in [13; 15; 17; 20], it is also known that infinite-index Morse subgroups of the mapping class group exactly coincide with stable subgroups [21], and stable subgroups of mapping class groups (and more generally, stable subgroups of Morse local-to-global groups) have interesting combination theorems [25]. Stability has also been studied in the context of Morse local-to-global groups [11], relatively hyperbolic groups [3], and hierarchically hyperbolic groups [1; 26]. It is also known that stable subgroups admit finite height [2] and that the growth series of a stable subgroup is rational [25]. There has also been recent work on recognizing spaces, ie spaces where the orbit map induces a quasi-isometric embedding, for stable subgroups [4; 30].

Comparing Theorem 1.2 to Theorem 1.1, we see a cocompact action involving a domain of discontinuity in Theorem 1.1 which does not appear in Theorem 1.2. This is because the standard methods used for showing this property rely on the fact that the (Gromov-)hyperbolic boundary for a word hyperbolic group is a compactification, and thus finding the requisite compact set needed for a cocompact action boils down to finding an appropriate closed subset. In contrast, the Morse boundary usually does not compactify the underlying group, in fact the Morse boundary compactifies a finitely generated group H if and only if H is word hyperbolic; see [8, Theorem 3.10; 9, Lemma 4.1]. This leads to an open question:

Open Question 1 Does there exist an appropriate classification of boundary convex cocompactness via an appropriate action on a domain of discontinuity analog?

For other properties in Theorem 1.2, we are able to address the need for some compactness in the Morse boundary by assuming that the limit set of the group, ΛH , is compact. See Definition 2.25 and Corollary 2.27. It is not possible to remove the compactness condition in either (3) or (4) of Theorem 1.2. For example, consider the group $G = \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle * \langle c \rangle * \langle d \rangle$ with subgroup $H = \langle a, b, c \rangle$. As discussed in [9, Remark 1.8], H is isometrically embedded and convex in G , and so every point of ΛH is conical with respect to H . In fact all rays representing a point in ΛH travel through H infinitely often. However H is not hyperbolic, so H is not stable. See [9, Section 1.2] for a complete discussion.

1.1 Applications

Convex cocompact subgroups of mapping class groups have been well studied, see [15; 17], but in particular conical limit point characterizations have been analyzed before. Let S be a finite-type surface, $\text{Mod}(S)$ its associated mapping class group, and let $\mathcal{T}(S)$ be its associated Teichmüller space.

In [20, Theorem 1.2], it is shown that a subgroup H of $\text{Mod}(S)$ is convex cocompact if and only if all the limit points of H in the Thurston compactification of $\mathcal{T}(S)$ are conical. A combination of Theorem 1.2 and [9, Theorem 1.18; 13, Theorem 1.1] gives the following direct comparison, which uses the intrinsic geometry of $\text{Mod}(S)$ instead of the geometry of $\mathcal{T}(S)$.

Theorem 1.4 *Let S be a finite-type surface, and let $H < \text{Mod}(S)$ be finitely generated. Then H is a convex-cocompact subgroup of $\text{Mod}(S)$ if and only if every point in ΛH is a conical limit point of $H \curvearrowright \text{Mod}(S)$, $\Lambda H \neq \emptyset$, and ΛH is compact in the Morse boundary of $\text{Mod}(S)$.*

This theorem, combined with the above result of [20], gives the following immediate corollary, which shows that conicality is a strong condition in the setting of mapping class groups:

Corollary 1.5 *Let S be a finite-type surface, and let $H < \text{Mod}(S)$ be finitely generated. The following are equivalent:*

- (1) *Every limit point of H in the Morse boundary of $\text{Mod}(S)$ is a conical limit point of $H \curvearrowright \text{Mod}(S)$ and ΛH is compact.*
- (2) *Every limit point of H in the Thurston compactification of $\mathcal{T}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

We also show that there exists a natural $\text{Mod}(S)$ -equivariant map from $\text{Mod}(S)$ to $\mathcal{T}(S)$ which sends conical limit points of $H < \text{Mod}(S)$ in the Morse boundary of $\text{Mod}(S)$ to conical limit point of H in the Thurston compactification of $\mathcal{T}(S)$. This directly proves the implication (1) $\Rightarrow$ (2) in Corollary 1.5 without requiring results of [20], and in fact, does not require H to be a convex cocompact subgroup. See Theorem 5.2 for details.

Recall that $\text{Out}(F_n)$ denotes the group of outer automorphisms on the free group F_n with n generators. Hamenstädt and Hensel defined convex cocompact subgroups of $\text{Out}(F_n)$ as subgroups which have quasiconvex orbits on the free factor graph [18, Definition 2]. In [13, Theorem 1.3], it is shown that if $H \leq \text{Out}(F_n)$ is convex cocompact then H is a stable subgroup of $\text{Out}(F_n)$. Combining this fact with Theorem 1.2, we get the following relationship.

Theorem 1.6 *Let $n \geq 3$. Suppose H is a convex cocompact subgroup of $\text{Out}(F_n)$ in the sense of [18, Definition 2]. Then every limit point of H in the Morse boundary of $\text{Out}(F_n)$ is a conical limit point of $H \curvearrowright \text{Out}(F_n)$ and ΛH is compact.*

However, in contrast of Theorem 1.4, it is unlikely that the converse holds. Due to an announcement by Hamenstädt in 2015, there is a classification of stable subgroups of $\text{Out}(F_n)$ which shows the converse of [13, Theorem 1.3] does not hold.

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2 Background

We first begin by setting some notation and basic definitions. We recall that a metric space X is *proper* if closed balls are compact. A path $\alpha: I \rightarrow X$ is a *geodesic* if $I \subseteq \mathbb{R}$ is a closed (potentially unbounded) interval and α preserves distances, ie, if for all $s, t \in I$, $d_{\mathbb{R}}(s, t) = d_X(\alpha(s), \alpha(t))$. If $I = [a, b]$, we call α a *geodesic segment*, if $I = [a, \infty)$, we call α a *geodesic ray*, and if $I = (-\infty, \infty)$ then we call α a *geodesic line*. Given two points $x, y \in X$, we use $[x, y]$ to denote a geodesic segment starting at x and ending at y . If there exists a geodesic segment between any pair of points in X , we say X is a *geodesic metric space*.

Given two geodesic segments $\alpha = [x, y]$ and $\beta = [y, z]$, we denote the (length preserving) concatenation between them as $[x, y] * [y, z]$. Formally, given $\alpha: [0, a] \rightarrow X$ and $\beta: [0, b] \rightarrow X$ with $\alpha(a) = \beta(0)$, we have $\alpha * \beta: [0, a + b] \rightarrow X$ given by

$$\alpha * \beta(t) = \begin{cases} \alpha(t), & t \in [0, a], \\ \beta(t - a), & t \in [a, a + b]. \end{cases}$$

We define the concatenation analogously in the case where α is a geodesic segment and β is a geodesic ray.

We use $B_K(p)$ to denote the closed ball of radius K centered at p , ie $B_K(p) = \{x \in X : d(p, x) \leq K\}$. Given $A \subseteq X$ and $K \geq 0$, we denote the K -neighborhood of A by $\mathcal{N}_K(A) = \{x \in X : d(x, A) \leq K\}$. Given two closed sets $A, B \subseteq X$, we denote the *Hausdorff distance* between A and B as

$$d_{\text{Haus}}(A, B) = \min\{K : A \subseteq \mathcal{N}_K(B) \text{ and } B \subseteq \mathcal{N}_K(A)\}.$$

Finally, given a closed set $A \subseteq X$, and a point $p \in X$, we denote the *closest point projection* of x to A as

$$\pi_A(p) = \{a \in A : d(a, p) = d(A, p)\}.$$

We now take a moment to give the definition of a quasigeodesic, since this term will appear frequently.

Definition 2.1 Let $I \subseteq \mathbb{R}$ be a closed interval X be a metric space, and let $\varphi: I \rightarrow X$. Let $K \geq 1$ and $C \geq 0$. We call φ a (K, C) -*quasigeodesic* if, for every $s, t \in I$, we have

$$\frac{1}{K}d(s, t) - C \leq d(\varphi(s), \varphi(t)) \leq Kd(s, t) + C.$$

We call φ a *quasigeodesic* if there exists a pair (K, C) so that φ is a (K, C) -quasigeodesic.

For a more thorough treatment of quasigeodesics and their properties, we refer the reader to [7].

2.1 Hyperbolic geometry

Here we provide a brief overview of the main result of [28], which is a direct analog of Theorem 1.1 in the setting of (Gromov-)hyperbolic geometry. Although our main results are not in the setting of hyperbolicity, many of the tools and constructions we use are inspired by the results in this setting. We begin with the definition of a δ -hyperbolic space.

Definition 2.2 Let X be a geodesic metric space. We call X a δ -hyperbolic metric space if every geodesic triangle is δ -slim, ie, if for every $x, y, z \in X$, we have $[x, z] \subseteq \mathcal{N}_\delta([x, y] \cup [y, z])$. We call X a *hyperbolic space* if X is δ -hyperbolic for some $\delta \geq 0$.

One of the most useful facts in a δ -hyperbolic space is that quasigeodesics fellow-travel geodesics. This is known as the Morse lemma. A detailed proof of this lemma can be found in [6, Theorem III.H.1.7].

Lemma 2.3 (Morse lemma) *Let X be a proper, geodesic δ -hyperbolic space. There is a (nondecreasing) function $N : [1, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that, for any geodesic α and any (K, C) -quasigeodesic $\varphi : [a, b] \rightarrow X$ such that $\varphi(a), \varphi(b) \in \alpha$, we have that $\varphi \subseteq \mathcal{N}_{N(K, C)}(\alpha)$.*

An important construction associated with δ -hyperbolicity is the visual boundary. For more information on the visual boundary of a hyperbolic space and its uses, we direct the reader to [6; 19].

Definition 2.4 Let X be a proper geodesic space, and let $\mathfrak{o} \in X$. Let $R_{\mathfrak{o}}(X)$ be the collection of all geodesic rays $\alpha : [a, \infty) \rightarrow X$ such that $\alpha(a) = \mathfrak{o}$. Then we can define an equivalence relation on $R_{\mathfrak{o}}(X)$ by setting $\alpha \sim \beta$ whenever the Hausdorff distance between α and β is bounded. The *visual boundary* of X based at $\mathfrak{o}$ is defined to be $\partial_\infty X_{\mathfrak{o}} = R_{\mathfrak{o}}(X) / \sim$. We use $\alpha(\infty)$ to refer to the equivalence class of α in $\partial_\infty X_{\mathfrak{o}}$. We equip $\partial_\infty X_{\mathfrak{o}}$ with the topology generated by the neighborhood basis for α ,

$$U(\alpha, r, n) = \{\beta \in \partial_\infty X_{\mathfrak{o}} : d(\alpha(t), \beta(t)) \leq r \text{ for all } t \leq n\}.$$

We present another, equivalent definition for two rays to be in the same equivalence class $\alpha(\infty)$. This definition does not require either α or β to be based at $\mathfrak{o}$.

Definition 2.5 Let (X, d) be a proper, geodesic metric space, and let $\alpha : [a, \infty) \rightarrow X$ and $\beta : [b, \infty) \rightarrow X$ be two geodesic rays. We say α and β *K -asymptotically fellow-travel*, denoted by $\alpha \sim_K \beta$, if there exists $T \in \mathbb{R}$ so that whenever $t \geq T$, we have $d(\alpha(t), \beta(t)) \leq K$.

Importantly, in the context of a δ -hyperbolic space, Definition 2.5 classifies the tail-end fellow traveling distance in terms of only δ , as expressed in the following lemma of Swenson, and is important for the definition of a horoball in a δ -hyperbolic space.

Lemma 2.6 [28, Lemma 4] *Suppose α and β are geodesic rays with $d_{\text{Haus}}(\alpha, \beta) < \infty$, that is, with $\alpha(\infty) = \beta(\infty)$. Then there exists an isometry $\rho : \mathbb{R} \rightarrow \mathbb{R}$ so that $\alpha \sim_{6\delta} \beta \circ \rho$.*

Definition 2.7 [28] Let X be a proper, geodesic, δ -hyperbolic metric space, and let $\alpha: [a, \infty) \rightarrow X$ and $\beta: [b, \infty) \rightarrow X$ be geodesic rays.

- We denote the *horoball about α* by $H(\alpha)$ and define it as $H(\alpha) = \bigcup \{\beta([b, \infty)) : \beta \sim_{6\delta} \alpha, b \geq a\}$.
- We denote the *funnel about α* by $F(\alpha)$ and define it as $F(\alpha) = \{x \in X : d(x, \alpha) \leq d(\pi_\alpha(x), \alpha(a))\}$.

Remark 2.8 By Lemma 2.6, $H(\alpha)$ is well defined.

Using these definitions, we now construct what it means for a point $x \in \partial_\infty X$ to be a horospherical limit point or a funneled limit point of some subset $A \subseteq X$. Heuristically, x is a horospherical limit point if every horoball around x intersects A . The corresponding statement is true for funneled limit points. We also take a moment to define a conical limit point.

Definition 2.9 Let X be a proper, geodesic δ -hyperbolic space.

- Given a point $x \in \partial_\infty X_0$ and a subset $A \subseteq X$, we say x is a *horospherical limit point of A* if, for every geodesic ray α with $\alpha(\infty) = x$, we have $H(\alpha) \cap A \neq \emptyset$.
- Given a point $x \in \partial_\infty X_0$ and a subset $A \subseteq X$, we say x is a *funneled limit point of A* if, for every geodesic ray α with $\alpha(\infty) = x$, we have $F(\alpha) \cap A \neq \emptyset$.
- Given a point $x \in \partial_\infty X_0$ and a subset $A \subseteq X$, we say x is a *conical limit point of A* if there exists $K > 0$ such that, for every geodesic ray α with $\alpha(\infty) = x$, we have $\mathcal{N}_K(\alpha) \cap A \neq \emptyset$.

We present here for completeness a relaxed version of a claim in [28, page 125] which shows that every horoball of a geodesic ray contains a funnel of an equivalent geodesic ray in a δ -hyperbolic space.

Lemma 2.10 Let (X, d) be a proper, geodesic, δ -hyperbolic metric space, and let $\alpha: [0, \infty) \rightarrow X$ be a geodesic ray. Define $\alpha': [0, \infty) \rightarrow X$ by $\alpha'(t) = \alpha(t + 6\delta)$. Then $F(\alpha') \subseteq H(\alpha)$.

Proof See Figure 1. Let $p \in F(\alpha')$. We construct a geodesic ray $\beta: [b, \infty) \rightarrow X$ such that $\beta \sim_{6\delta} \alpha$ and $\beta(b) = p$: Let β_n be a geodesic segment which begins at p and ends at $\alpha(n)$. Then, after potentially passing to a subsequence, the β_n converge to a geodesic ray β by the Arzelà–Ascoli theorem. Then as shown in [6, page 427–428], $\beta \sim_{6\delta} \alpha$.

Let $q \in \pi_{\alpha'}(p)$ such that $d(\alpha(0), q) = \min\{d(\alpha(0), x) : x \in \pi_{\alpha'}(p)\}$, ie, so that q is the point in $\pi_{\alpha'}(p)$ closest to $\alpha(0)$. Since $p \in F(\alpha')$ we have that $d(p, q) \leq d(q, \alpha'(0))$. Choose $T \geq 6\delta$ so that $q \in [\alpha'(0), \alpha'(T)]$ and so that for all $t \geq T$, we have $d(\alpha(t), \beta(t)) < 6\delta$. Then

$$\begin{aligned} T - b &= d(\beta(T), p) \leq d(\beta(T), \alpha(T)) + d(\alpha(T), q) + d(q, p) \\ &\leq 6\delta + d(\alpha'(T - 6\delta), q) + d(q, \alpha'(0)) = 6\delta + (T - 6\delta). \end{aligned}$$

This shows that $b \geq 0$, and so $p = \beta(b) \in H(\alpha)$. □

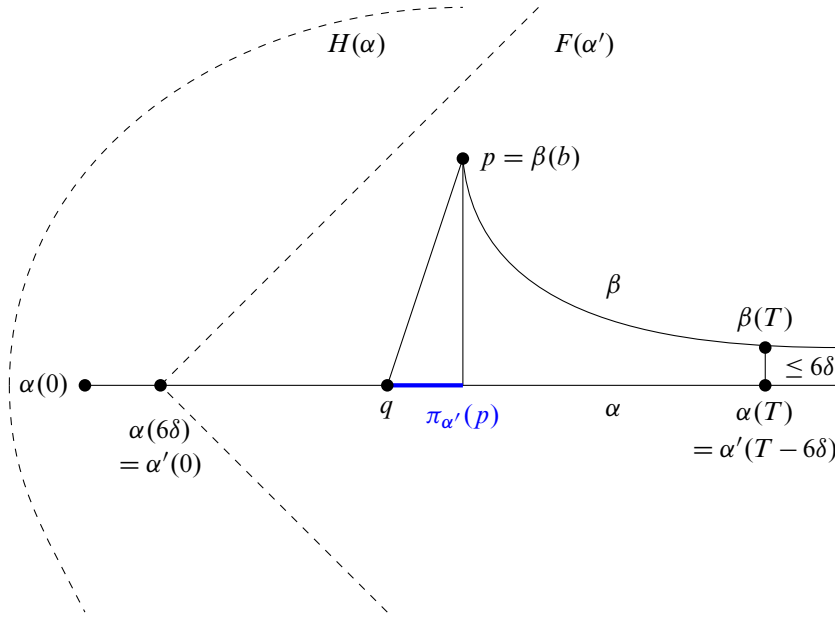


Figure 1: Diagram for Lemma 2.10.

We also include the complementary statement that every funnel of a geodesic ray contains a horoball of an equivalent geodesic ray.

Lemma 2.11 [28, Lemma 5] *Let (X, d) be a proper, geodesic, δ -hyperbolic metric space, and let $\alpha: [0, \infty) \rightarrow X$ be a geodesic ray. Define $\alpha': [0, \infty) \rightarrow X$ by $\alpha'(t) = \alpha(t + 12\delta)$. Then $H(\alpha') \subseteq F(\alpha)$. $\square$*

The combination of Lemmas 2.10 and 2.11 give the following relationship, which was originally stated as a corollary in [28].

Corollary 2.12 *In a proper, geodesic, δ -hyperbolic metric space, the funneled limit points are exactly the horospherical limit points. $\square$*

2.2 Morse boundary and Morse rays

The fact that X is δ -hyperbolic is an important part of the definition of a horoball in Definition 2.7, as we note in Remark 2.8. Since our main goal of Theorem 1.2 does not have X as a δ -hyperbolic space, we will need to develop some analog to Lemma 2.6 which does not use hyperbolicity. Our strategy for creating such an analog will be to use properties of Morse rays. We begin by recalling the definition.

Definition 2.13 [8, Definition 1.3] A (quasi)geodesic γ in a metric space is called N -Morse, where N is a function $[1, \infty) \times [0, \infty) \rightarrow [0, \infty)$, if for any (K, C) -quasigeodesic φ with endpoints on γ , we have $\varphi \subset \mathcal{N}_{N(K,C)}(\gamma)$. We call the function N a *Morse gauge*. We say γ is *Morse* if there exists a Morse gauge N so that γ is N -Morse.

Comparing this definition with Lemma 2.3 shows that Morse rays are the rays in X which have hyperbolic like properties. In the next definition, Cordes uses the visual boundary (see Definition 2.4 above) to construct a boundary on proper, geodesic X without requiring X to be hyperbolic.

Definition 2.14 [8; 10] Given a Morse gauge N and a basepoint $\mathfrak{o} \in X$, the N -Morse stratum, denoted by $X_{\mathfrak{o}}^N$, is defined as the set of all points x such that $[\mathfrak{o}, x]$ is an N -Morse geodesic. Each such stratum is δ -hyperbolic for δ depending only on N [10, Proposition 3.2], and thus has a well-defined visual boundary $\partial_{\infty} X_{\mathfrak{o}}^N$. If $\mathcal{M}$ is the set of all Morse gauges, then there is a natural partial order on $\mathcal{M}$: $N \leq N'$ if $N(K, C) \leq N'(K, C)$ for all K and C . The natural inclusion $\partial_{\infty} X_{\mathfrak{o}}^N \hookrightarrow \partial_{\infty} X_{\mathfrak{o}}^{N'}$ is continuous whenever $N \leq N'$ by [8, Corollary 3.2]. We define the *Morse boundary based at $\mathfrak{o}$* as

$$\partial X_{\mathfrak{o}} = \varinjlim_{\mathcal{M}} \partial_{\infty} X_{\mathfrak{o}}^N$$

with the induced direct limit topology. Given a Morse geodesic ray α , we denote the associated point in $\partial X_{\mathfrak{o}}$ as $\alpha(\infty)$.

Remark 2.15 Often when studying the Morse boundary, the basepoint is suppressed from the notation, as the Morse boundary is basepoint independent [8, Proposition 2.5]. However, we will often make use of the basepoint explicitly in the arguments to come, thus we keep it in the notation.

Remark 2.16 When X is a δ -hyperbolic space, $\partial_{\infty} X_{\mathfrak{o}} = \partial X_{\mathfrak{o}}$. This is because, by the Morse lemma (Lemma 2.3), there exists a maximum Morse gauge N so that $X = X_{\mathfrak{o}}^N$. See [8] for details.

The following fact states that subrays of Morse rays are also Morse. This will be especially useful in Section 3, as many of the arguments which describe the relationships between horoballs, funnels, and cones require restriction to a subray, as illustrated in the proof of Lemma 2.10.

Lemma 2.17 [22, Lemma 3.1] *Let X be a geodesic metric space. Let $\alpha: I \rightarrow X$ be an N -Morse (λ, ϵ) -quasigeodesic where I is an interval of $\mathbb{R}$. Then for any interval $I' \subseteq I$, the (λ, ϵ) -quasigeodesic $\alpha' = \alpha|_{I'}$ is N' -Morse where N' depends only on λ, ϵ , and N .* $\square$

We now present a combination of statements which will show that, given one Morse ray and another ray which fellow-travels with the first, eventually the fellow-traveling constant is determined only by the Morse gauge of the first ray. We begin by recalling two relevant facts from [8].

Proposition 2.18 [8, Proposition 2.4] *Let X be a geodesic metric space. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray. Let $\beta: [0, \infty) \rightarrow X$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for $t \in [A, A + D]$ for some $A \in [0, \infty)$ and $D \geq 6K$. Then for all $t \in [A + 2K, A + D - 2K]$,*

$$d(\alpha(t), \beta(t)) < 4N(1, 2N(5, 0)) + 2N(5, 0) + d(\alpha(0), \beta(0)).$$

$\square$

Corollary 2.19 [8, Corollary 2.6] *Let X be a geodesic metric space. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray. Let $\beta: [0, \infty) \rightarrow X$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for all $t \in [0, \infty)$ (ie $\beta(\infty) = \alpha(\infty)$). Then for all $t \in [2K, \infty)$,*

$$d(\alpha(t), \beta(t)) < \max\{4N(1, 2N(5, 0)) + 2N(5, 0), 8N(3, 0)\} + d(\alpha(0), \beta(0)). \quad \square$$

The proof of Proposition 2.18, as presented in [8], shows the following additional fact:

Corollary 2.20 *Let X be a geodesic metric space. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray. Let $\beta: [0, \infty) \rightarrow X$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for $t \in [A, A + D]$ for some $A \in [0, \infty)$ and $D \geq 6K$. Then there exists $x, y \in [0, A + 2K]$ such that $d(\alpha(y), \beta(x)) < N(5, 0)$.*

Proof The value x from the first paragraph of the proof in [8, Proposition 2.4] is an element of $[\max\{0, A - 2K\}, A + 2K]$, and as $[\max\{0, A - 2K\}, A + 2K] \subseteq [0, A + 2K]$, we get $x \in [0, A + 2K]$. The third to last paragraph defines y so that $\alpha(y) \in \pi_\alpha(\beta(x))$, and shows $y \leq A + 2K$ and that $d(\alpha(y), \beta(x)) < N(5, 0)$. $\square$

Corollaries 2.20 and 2.19 combine to give the following generalization of [6, Chapter 3, Lemma 3.3].

Proposition 2.21 *Let X be a geodesic metric space. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray. Let $\beta: [0, \infty) \rightarrow X$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for all $t \in [0, \infty)$ (ie $\beta(\infty) = \alpha(\infty)$). Then there exists $T_1, T_2 > 0$ such that for all $t \in [0, \infty)$,*

$$d(\alpha(T_1 + t), \beta(T_2 + t)) < \max\{4N(1, 2N(5, 0)) + 2N(5, 0), 8N(3, 0)\} + N(5, 0). \quad \square$$

Proof By Corollary 2.20, there exists $x, y \geq 0$ so that $d(\alpha(x), \beta(y)) < N(5, 0)$. Define $\alpha'(t) = \alpha(x + t)$ and $\beta'(t) = \beta(y + t)$, and note in particular that $\alpha'(0) = \alpha(x)$ and $\beta'(0) = \beta(y)$. Applying Corollary 2.19 to α' and β' produces the desired result. $\square$

For convenience, we will write

$$\delta_N = \max\{4N(1, 2N(5, 0)) + 2N(5, 0), 8N(3, 0)\} + N(5, 0).$$

Using this notation, Proposition 2.21 leads to the following generalization of [28, Lemma 4].

Corollary 2.22 *Let X be a geodesic metric space. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray. Let $\beta: [0, \infty) \rightarrow X$ be a geodesic ray such that $\beta(\infty) = \alpha(\infty)$. Then there exists $a \in \mathbb{R}$ and an isometry $\rho: [a, \infty) \rightarrow [0, \infty)$ so that $\alpha \sim_{\delta_N} \beta \circ \rho$.*

Proof Apply Proposition 2.21 to find $T_1, T_2 > 0$ so that for all $t \in [0, \infty)$, $d(\alpha(T_1 + t), \beta(T_2 + t)) < \delta_N$. Then let $\rho: [a, \infty) \rightarrow [0, \infty)$ be the unique isometry such that $\rho(T_1) = T_2$. $\square$

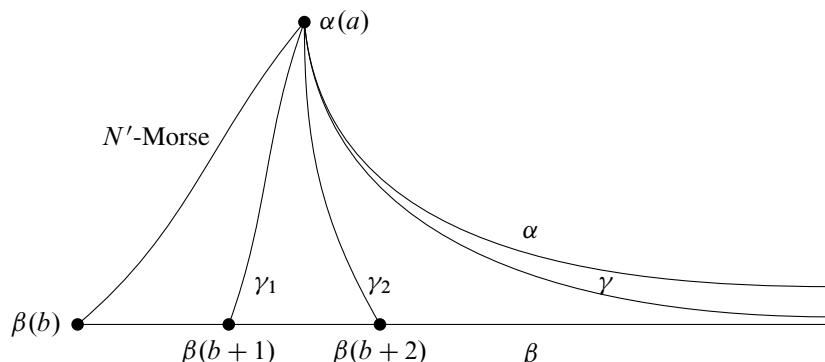


Figure 2: Diagram for Proposition 2.24.

Proposition 2.23 Suppose $\alpha: [a, \infty) \rightarrow X$ is an N -Morse geodesic ray and $\beta: [b, \infty) \rightarrow X$ is a geodesic ray such that $\beta \sim_{\delta_N} \alpha$ and $\alpha(a) = \beta(b)$. Then β is M -Morse where M depends only on N .

Proof It suffices to show that $d_{\text{Haus}}(\alpha, \beta) \leq K$ where $K \geq 0$ depends only on N . Choose $T > 0$ so that $d(\alpha(t), \beta(t)) \leq \delta_N$ for all $t \geq T$. Note that $[\beta(b), \beta(t)] * [\beta(t), \alpha(t)]$ is a $(1, 2\delta_N)$ quasigeodesic, so by [8, Lemma 2.1], $d_{\text{Haus}}([\alpha(a), \alpha(t)], [\beta(b), \beta(t)] * [\beta(t), \alpha(t)]) \leq L$ for some L depending only on N . But since $d(\alpha(t), \beta(t)) \leq \delta_N$, we have $d_{\text{Haus}}([\alpha(a), \alpha(t)], [\beta(b), \beta(t)]) \leq L + \delta_N$. $\square$

The above statement leads to the following generalization, which is very similar to [8, Lemma 2.8]. This statement will be useful for showing a generalization of Corollary 2.12, since our horoballs and funnels will be restricted to a single Morse stratum. See Theorem 3.8.

Proposition 2.24 Suppose $x \in \partial X_0^N$ for a Morse gauge N . Then any geodesic ray $\alpha: [a, \infty) \rightarrow X$ with $\alpha(\infty) = x$ is M -Morse, where M depends only on N and the Morse gauge of $[\alpha(a), o]$.

Proof See Figure 2. Let $\beta: [b, \infty) \rightarrow X$ be N -Morse with $\beta(b) = o$, $\beta(\infty) = \alpha(\infty)$, and let N' be the Morse gauge of $[\alpha(a), \beta(b)]$. For each $n \in \mathbb{N}$, let $\gamma_n = [\alpha(a), \beta(b+n)]$. Note that $\beta|_{[b, b+n]}$ is Morse for some Morse gauge depending only on N by Lemma 2.17, and so by [8, Lemma 2.3], γ_n is N'' -Morse for N'' depending only on $\max\{N, N'\}$. By potentially restricting to a subsequence for Arzelà–Ascoli and by [8, Lemma 2.10], there exists an N'' -Morse geodesic ray γ with $\gamma_n \rightarrow \gamma$ (uniformly on compact sets) and $\gamma(\infty) = \beta(\infty)$. Then Proposition 2.23 shows that α is Morse for an appropriate Morse gauge. $\square$

2.3 Limit sets and weak convex hulls

We now introduce limit sets and weak convex hulls, and give some useful properties that these sets have. We use these constructions to turn subsets of X into subsets of the Morse boundary, and vice versa.

Definition 2.25 [9, Definition 3.2] Let X be a proper, geodesic metric space and let $A \subseteq X$. The *limit set of A* , denoted as ΛA , is the set of points in ∂X_o such that, for some Morse gauge N , there exists a sequence of points $(a_k) \subset A \cap X_o^N$ such that $[o, a_k]$ converges (uniformly on compact sets) to a geodesic ray α with $\alpha(\infty) = x$. (Note α is N -Morse by [8, Lemma 2.10].) In the case where H acts properly by isometries on X , we use ΛH to denote the limit set of $H o$.

Remark 2.26 By [9, Lemma 3.3], ΛH can be defined as the limit set of *any* orbit of H , we merely choose the orbit $H o$ for convenience and simplicity in future arguments.

We also prove a small fact about limit sets, which is similar to [9, Lemma 4.1, Proposition 4.2].

Corollary 2.27 Let X be a proper, geodesic metric space and suppose $A \subseteq X$. If $\Lambda A \subseteq \partial X_o^N$ for some Morse gauge N , then ΛA is compact.

Proof By [8, Proposition 3.12], this is equivalent to the condition that ΛH is closed. $\square$

Remark 2.28 By Corollary 2.27 and by [9, Lemma 4.1], the requirement that ΛH is compact is equivalent to the requirement that ΛH is contained in the boundary of a single Morse stratum.

Definition 2.29 [9; 28] Let X be a proper, geodesic metric space, and let $A \subseteq X \cup \partial X_o$. Then the *weak convex hull* of A , denoted by $\text{WCH}(A)$, is the union of all geodesics (segments, rays, or lines) of X which have both endpoints in A .

We take a moment to highlight some nice interactions between the weak convex hull of a compact limit set with the Morse boundary.

Lemma 2.30 [9, Proposition 4.2] Let X be a proper geodesic metric space and let $A \subseteq X$ such that $\Lambda A \subseteq \partial X_o^N$ for some Morse gauge N . Then there exists a Morse gauge N' , depending only on N , such that $\text{WCH}(\Lambda A) \subset X_o^{N'}$. $\square$

Lemma 2.31 Let X be a proper geodesic metric space and let $A \subseteq X$ such that $\Lambda A \subseteq \partial X_o^N$ for some Morse gauge N . Then $\Lambda(\text{WCH}(\Lambda A)) \subseteq \Lambda A$.

Proof We may assume $|\Lambda A| > 1$. Let $x \in \Lambda(\text{WCH}(\Lambda A))$. By Definition 2.25, there is $x_n \in \text{WCH}(\Lambda A)$ such that $[o, x_n]$ converges to a geodesic ray γ with $\gamma(\infty) = x$. We show that there exists $K > 0$ so that for all n there exists a_n with $[o, x_n] \subseteq \mathcal{N}_K([o, a_n])$. Thus, (a subsequence of) the geodesics $[o, a_n]$ converge to a geodesic ray $\alpha: [0, \infty) \rightarrow X$ with $\alpha(0) = o$, and $\alpha(\infty) = \gamma(\infty) = x$ and so $x \in \Lambda A$. It remains to find K so that $[o, x_n] \subseteq \mathcal{N}_K([o, a_n])$.

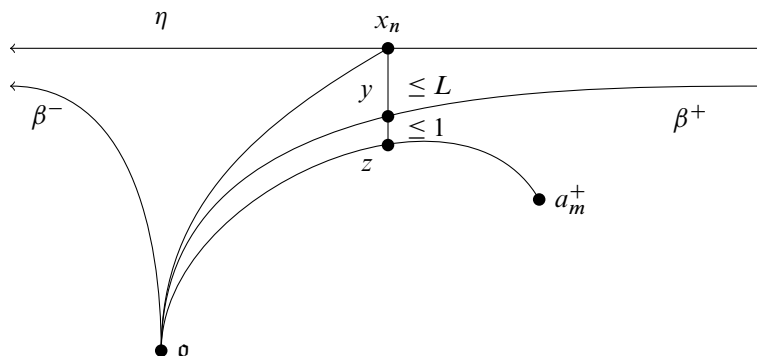


Figure 3: Diagram for Lemma 2.31.

Fix n . Since $x_n \in \text{WCH}(\Lambda A)$, $x \in \eta$ where $\eta: (-\infty, \infty) \rightarrow X$ is a geodesic with $\eta(\pm\infty) \in \Lambda A$. So, by Definition 2.25, there exists $a_k^+, a_k^- \in A \cap X_0^N$ so that $[o, a_k^+]$ and $[o, a_k^-]$ converge to geodesics β^+ and β^- , respectively, with $\beta^+(\infty) = \eta(\infty)$ and $\beta^-(-\infty) = \eta(-\infty)$. Since $\Lambda A \subseteq \partial X_0^N$, the triangle $\eta \cup \beta^+ \cup \beta^-$ is L -slim for L depending only on N by [9, Proposition 3.6], and as $x \in \eta$, there exists $y \in \beta^+ \cup \beta^-$ so that $d(x_n, y) \leq L$. Without loss of generality, assume $y \in \beta^+$. Since $[o, a_k^+]$ converges to β^+ uniformly on compact sets, choose m large enough so that $d(y, [o, a_m^+]) \leq 1$. Let $z \in [o, a_m^+]$ so that $d(y, z) \leq 1$. See Figure 3.

The concatenation $[o, x_n] * [x_n, z]$ is a $(1, L+1)$ -quasigeodesic with endpoints on $[o, a_m^+]$. Since $[o, a_m^+]$ is N -Morse, we have that $[o, x_n] \subseteq [o, x_n] * [x_n, z] \subseteq \mathcal{N}_{N(1, L+1)}([o, a_m^+])$. Since $K := N(1, L+1)$ did not depend on the choice of n , this completes the proof. $\square$

Finally, we finish this section by stating the definitions of stability and boundary convex cocompactness here for reference.

Definition 2.32 [9, Definition 1.3; 13] If $f: X \rightarrow Y$ is a quasi-isometric embedding between geodesic metric spaces, we say X is a *stable* subspace of Y if there exists a Morse gauge N such that every pair of points in X can be connected by an N -Morse quasigeodesic in Y ; we call f a *stable embedding*.

If $H < G$ are finitely generated groups, we say H is *stable in G* if the inclusion map $i: H \hookrightarrow G$ is a stable embedding.

Definition 2.33 [9, Definition 1.4] We say that H acts *boundary convex cocompactly* on X if the following conditions hold:

- (1) H acts properly on X .
- (2) ΛH is nonempty and compact.
- (3) The action of H on $\text{WCH}(\Lambda H)$ is cobounded.

3 Limit point characterizations in the Morse boundary

The goal of this section is show that, given a set $A \subseteq X$, if $x \in \partial X_o$ is a Morse conical limit point of A , then x is a Morse horospherical limit point of A . This was first shown in the hyperbolic case in [28], here we generalize this fact into the setting of proper geodesic metric spaces. We begin by introducing definitions which generalize horospheres and funnels for Morse rays.

Definition 3.1 (horoballs, funnels) Let X be a proper, geodesic metric space and let $o \in X$ be some designated point. Let $\alpha: [a, \infty) \rightarrow X$ be an N' -Morse geodesic ray, and let N be some, potentially different, Morse gauge. We define the N -Morse horoball around α based at o as

$$H_o^N(\alpha) = \{x \in X_o^N \mid \exists \beta: [b, \infty) \rightarrow X \text{ with } \beta \sim_{\delta_{N'}} \alpha \text{ and } b \geq a \text{ and } \beta(b) = x\}.$$

We define the N -Morse funnel around α based at o as

$$F_o^N(\alpha) = \{x \in X_o^N \mid d(x, \pi_\alpha(x)) \leq d(\alpha(a), \pi_\alpha(x))\}.$$

Comparing these definitions to Definition 2.7 shows that a Morse horoball is a horoball about a Morse geodesic intersected with an appropriate Morse stratum, and similarly, a Morse funnel is a funnel about a Morse geodesic intersected with an appropriate Morse stratum. The following three definitions classify points on the Morse boundary by asking if every horoball, funnel, or cone intersects a given subset of X .

Definition 3.2 Let X be a proper, geodesic metric space and let $o \in X$ be some designated point. Let $A \subset X$.

- We say that $x \in \partial X_o$ is a *Morse horospherical limit point* of A if for every Morse geodesic α with $\alpha(\infty) = x$, there exists a Morse gauge N such that $H_o^N(\alpha) \cap A \neq \emptyset$.
- We say that $x \in \partial X_o$ is a *Morse funneled limit point* of A if for every Morse geodesic α with $\alpha(\infty) = x$, there exists a Morse gauge N such that $F_o^N(\alpha) \cap A \neq \emptyset$.
- We say that $x \in \partial X_o$ is a *Morse conical limit point* of A if there exists $K > 0$ such that, for every Morse geodesic α with $\alpha(\infty) = x$, we have that $\mathcal{N}_K(\alpha) \cap A \neq \emptyset$.

Remark 3.3 In the case where X is a δ -hyperbolic space, these definitions agree with the definitions given in Definition 2.9, as every geodesic in a δ -hyperbolic space is N -Morse for N depending only on δ . In light of this, we will use “conical limit point” instead of “Morse conical limit point” for the rest of this paper, except in cases where the difference between these definitions causes confusion. We similarly reduce “Morse horospherical limit point” and “Morse funneled limit point” to “horospherical limit point” and “funneled limit point,” respectively.

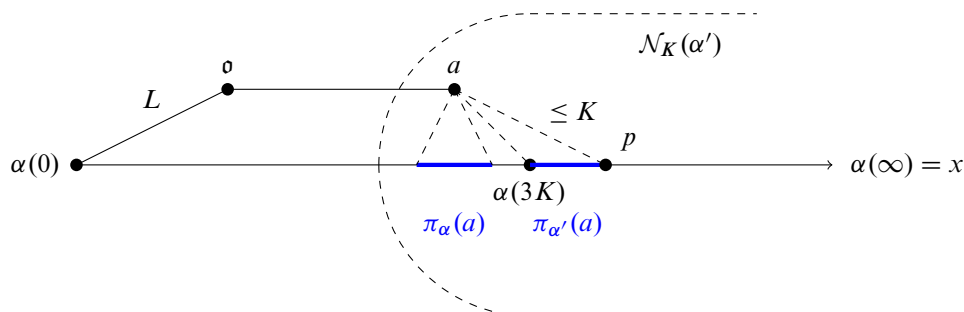


Figure 4: Diagram for Proposition 3.4.

We now begin proving the new implications found in Theorem 1.2. We will first show that every conical limit point of A is a funneled limit point of A , and then we will show that the funneled limit points of A exactly coincide with the horospherical limit points of A . These arguments generalize the arguments found in [28].

Proposition 3.4 *Let X be a proper, geodesic metric space and let $o \in X$. Let $A \subseteq X$. If $x \in \partial X_o$ is a conical limit point of A , then x is a funneled limit point of A .*

Proof See Figure 4. Let $x \in \partial X_o$ be a conical limit point of $A \subseteq X$. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic with $\alpha(\infty) = x$. By Lemma 2.17, there exists a Morse gauge M so that every geodesic subray of α is M -Morse. Thus by Definition 3.2, there exists $K \geq 0$ so that every subray of α gets at least K close to A .

Now define $\alpha' = \alpha|_{[3K, \infty)}$, and let $a \in A$ such that $a \in \mathcal{N}_K(\alpha')$. Then $d(a, \pi_\alpha(a)) \leq d(a, \pi_{\alpha'}(a)) \leq K$, and so $d(\pi_\alpha(a), \pi_{\alpha'}(a)) \leq 2K$. By the triangle inequality, $\pi_\alpha(a) \subseteq \alpha|_{[K, \infty)}$. Therefore,

$$d(\pi_\alpha(a), a) \leq K = d(\alpha(0), \alpha(K)) \leq d(\alpha(0), \pi_{\alpha'}(a)).$$

It remains to show that $a \in X_o^{N'}$ for a Morse gauge N' which is independent of the choice of $a \in A$.

Let $L = d(o, \alpha(0))$, and let $p \in \pi_{\alpha'}(a)$, and note that $d(p, a) \leq K$. Thus, $[o, \alpha(0)]$ and $[p, a]$ are both N'' -Morse depending only on $\max\{K, L\}$, and $[o, p]$ is N''' -Morse depending only on N by Lemma 2.17. Since $[o, a]$ is one side of a quadrilateral whose other three sides are $\max\{N'', N'''\}$ -Morse, $[o, a]$ is N' -Morse where N' does not depend on choice of $a \in A$ by [8, Lemma 2.3]. $\square$

Our next goal is to show that the funneled limit points of A coincide with the horospherical limit points of A . Towards this end, we show that, given a point x in a horoball of a subray, the projection of x to the subray is coarsely the same as the projection to the base ray.

Lemma 3.5 *Suppose α is an N -Morse geodesic ray and let α' be a subray. Suppose $x \in H_o^{N'}(\alpha')$. If $\alpha(\infty) \in \partial X_o^{N''}$, then $d_{\text{Haus}}(\pi_\alpha(x), \pi_{\alpha'}(x)) \leq K$, where $K \geq 0$ depends only on N, N' , and N'' .*

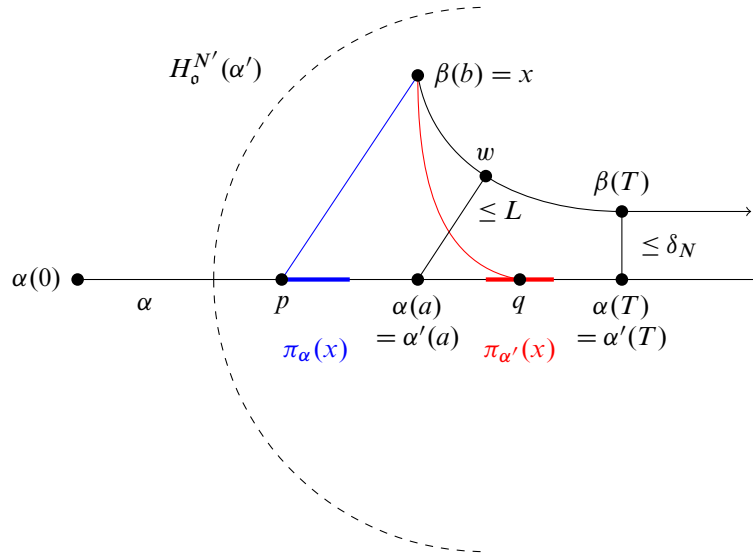


Figure 5: Diagram for Lemma 3.5.

Proof See Figure 5. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray and let $\alpha' = \alpha|_{[a, \infty)}$ for some $a \geq 0$. By Lemma 2.17, α' is M -Morse for M depending only on N . Let $x \in H_o^{N''}(\alpha')$, thus there exists $\beta: [b, \infty) \rightarrow X$ a geodesic ray with $b \geq a$, $\beta(b) = x$, and $\beta \sim_{\delta_M} \alpha'$. By Proposition 2.24, β is M' -Morse for M' depending only on N , N' , and N'' . If $\pi_\alpha(x) \subseteq \alpha'$, then $\pi_\alpha(x) = \pi_{\alpha'}(x)$. So, we assume that $\pi_\alpha(x) \not\subseteq \alpha'$. We shall show that in this case, $d(x, \pi_\alpha(x))$ and $d(x, \pi_{\alpha'}(x))$ are both bounded above by an appropriate constant, and this gives the desired result.

Let $p \in \pi_\alpha(x) \setminus \alpha'$, and let $q \in \pi_{\alpha'}(x)$. Without loss of generality, let T be large enough so that $q \in [\alpha'(a), \alpha'(T)]$ and $d(\alpha'(T), \beta(T)) \leq \delta_N$.

Put $\gamma = [\beta(b), \alpha'(a)] * [\alpha'(a), \alpha'(T)] * [\alpha'(T), \beta(T)]$, and note that γ is a $(3, 4\delta_N)$ quasigeodesic. Thus there exists $w \in [\beta(b), \beta(T)]$ and $L \geq 0$ such that $d(\alpha'(a), w) \leq L$, where L depends only on M' by [8, Lemma 2.1]. Notice now that

$$|d(\alpha'(a), \alpha'(T)) - d(w, \beta(T))| \leq \delta_N + L.$$

However, since $b \geq a$ and $w \in [\beta(b), \beta(T)]$, we know

$$|d(\alpha'(a), \alpha'(T)) - d(w, \beta(T))| = d(\alpha'(a), \alpha'(T)) - d(w, \beta(T)).$$

But then by the definition of the nearest point projection and the triangle inequality, we have

$$\begin{aligned} d(x, p) &\leq d(x, q) \leq d(x, \alpha'(a)) \leq d(x, w) + d(w, \alpha'(a)) = d(x, \beta(T)) - d(w, \beta(T)) + d(w, \alpha'(a)) \\ &= d(\alpha'(b), \alpha'(T)) - d(w, \beta(T)) + d(w, \alpha'(a)) \leq d(\alpha'(a), \alpha'(T)) - d(w, \beta(T)) + L \leq \delta_N + L + L. \end{aligned}$$

Therefore, $d(\pi_\alpha(x), x)$ and $d(\pi_{\alpha'}(x), x)$ are both bounded above by L , which is a constant depending only on N , N' , and N'' , as desired. $\square$

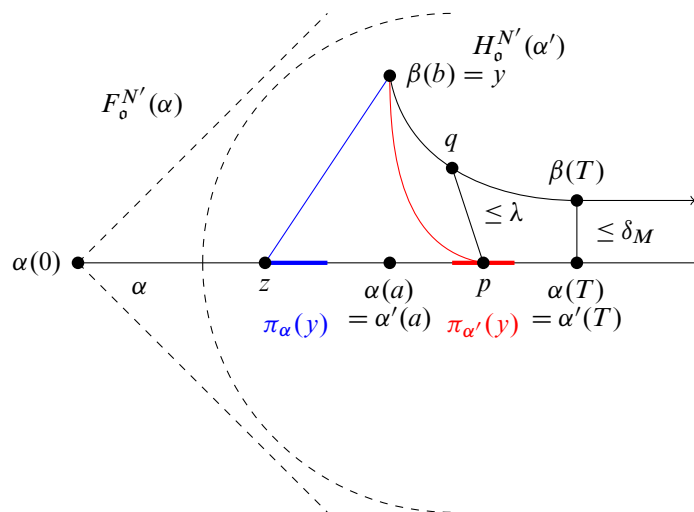


Figure 6: Diagram for Proposition 3.6.

Proposition 3.6 *Let $x \in \partial X_0^N$. Let $\alpha: [0, \infty) \rightarrow X$ be an N' -Morse geodesic with $\alpha(\infty) = x$. Then for every Morse gauge N'' , there exists $T \geq 0$ such that, for any subray α' of α with $d(\alpha(0), \alpha') \geq T$, we have $H_0^{N''}(\alpha') \subseteq F_0^{N''}(\alpha)$.*

Proof See Figure 6. Let $\alpha' = \alpha|_{[a, \infty)}$ be a subray of α . By Lemma 2.17, α' is M -Morse where M depends only on N' . Let $y \in H_0^{N''}(\alpha')$. Thus there exists $\beta: [b, \infty) \rightarrow X$ be a geodesic ray such that $b \geq a$, $\beta(b) = y$, and $\beta \sim_{\delta_M} \alpha'$. Note that β is M' -Morse where M' depends only on N , N' , and N'' by Proposition 2.24. Choose $z \in \pi_\alpha(y)$ such that $d(\alpha(0), z) = d(\alpha(0), \pi_\alpha(y))$, ie, so that z is closest to $\alpha(0)$. By Lemma 3.5, there exists $p \in \pi_{\alpha'}(x)$ so that $d(z, p) \leq L$ for some L depending only on N , N' , and N'' . Choose t large enough so that $d(\alpha'(t), \beta(t)) = d(\alpha(t), \beta(t)) \leq \delta_M$ and $p, \alpha(b) \in [\alpha(a), \alpha(t)]$. Note that $[y, p] * [p, \alpha(t)] * [\alpha(t), \beta(t)]$ is a $(3, 4\delta_M)$ -quasigeodesic, thus there exists $q \in [\beta(b), \beta(t)]$ and $\lambda \geq 0$ such that $d(p, q) \leq \lambda$ where λ depends only on M' by [8, Lemma 2.1]. It suffices to show that $d(y, z) \leq d(\alpha(0), z)$.

Using the triangle inequality and the definition of π_α , we find

$$\begin{aligned} d(y, z) &\leq d(y, p) \leq d(y, q) + d(q, p) \leq d(y, q) + \lambda \\ &= d(y, \beta(t)) - d(q, \beta(t)) + \lambda = d(\alpha(b), \alpha(t)) - d(q, \beta(t)) + \lambda \\ &\leq d(\alpha(a), \alpha(t)) - d(p, \alpha(t)) + \lambda + \delta_M + \lambda = d(\alpha(a), p) + 2\lambda + \delta_M \\ &\leq d(\alpha(a), z) + L + 2\lambda + \delta_M. \end{aligned}$$

So, if $a \geq L + 2\lambda + \delta_M$, we have

$$d(y, z) \leq d(\alpha(a), z) + L + 2\lambda + \delta_M \leq d(\alpha(a), z) + d(\alpha(0), \alpha(a)) = d(\alpha(0), z). \quad \square$$

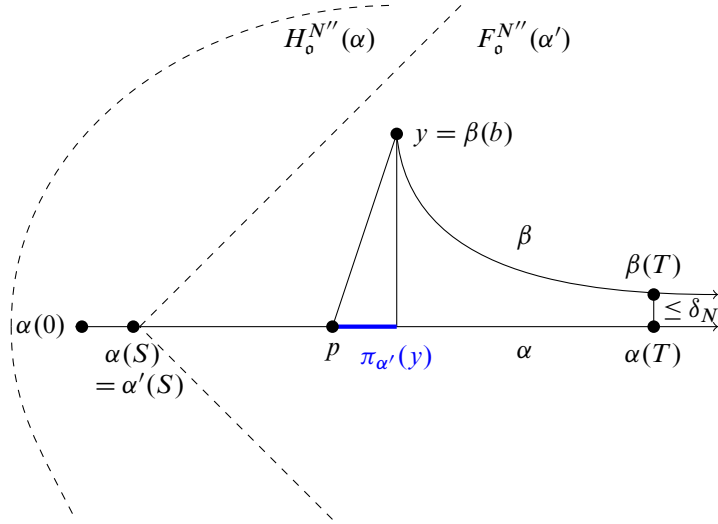


Figure 7: Diagram for Proposition 3.7.

Proposition 3.7 Let $x \in \partial X_0^N$ and let $\alpha: [0, \infty) \rightarrow X$ be an N' -Morse geodesic with $\alpha(\infty) = x$. Suppose $S = \delta_{N'}$. Define $\alpha' = \alpha|_{[S, \infty)}$. Then $F_0^{N''}(\alpha') \subseteq H_0^{N''}(\alpha)$ for any Morse gauge N'' .

Proof See Figure 7. Let $y \in F_0^{N''}(\alpha')$. By definition, $d(y, \pi_{\alpha'}(y)) \leq d(\alpha(S), \pi_{\alpha'}(y))$. Let $p \in \pi_{\alpha'}(y)$ such that $d(\alpha(S), p) = d(\alpha(S), \pi_{\alpha'}(y))$, ie, let p be the element of $\pi_{\alpha'}(y)$ which is closest to $\alpha(S)$. Then $d(y, p) \leq d(\alpha(S), p)$. Construct $\beta: [b, \infty) \rightarrow X$ such that $\beta(b) = y$ and $\beta \sim_{\delta_{N'}} \alpha$. We want to show that $b \geq 0$. Choose $T \geq 0$ so that $d(\beta(T), \alpha(T)) \leq \delta_{N'}$. Then

$$\begin{aligned} T - b &= d(y, \beta(T)) \leq d(y, p) + d(p, \alpha(T)) + d(\alpha(T), \beta(T)) \\ &\leq d(\alpha(S), p) + d(p, \alpha(T)) + \delta_{N'} = d(\alpha(S), \alpha(T)) + \delta_{N'} \\ &= d(\alpha(0), \alpha(T)) - d(\alpha(0), \alpha(S)) + \delta_{N'} = T - S + \delta_{N'} = T. \end{aligned}$$

In summary, $T - b \leq T$, but this immediately shows that $0 \leq b$, as desired. $\square$

Theorem 3.8 Let $x \in \partial X_0$. Then x is a Morse horospherical limit point of $A \subseteq X$ if and only if x is a Morse funneled limit point of A .

Proof Let $x \in \partial X_0$. Then there exists a Morse gauge N so that $x \in \partial X_0^N$. Let α be any Morse geodesic with $\alpha(\infty) = x$, and let $H_0^{N''}(\alpha)$ and $F_0^{N''}(\alpha)$ be a Morse horoball about α and a Morse funnel about α , respectively. By Propositions 3.6 and 3.7, there exists a subray α' so that $F_0^{N''}(\alpha') \subseteq H_0^{N''}(\alpha)$ and $H_0^{N''}(\alpha') \subseteq F_0^{N''}(\alpha)$.

Now suppose x is horospherical. Then there exists $a \in A$ so that $x \in H_0^{N''}(\alpha') \subseteq F_0^{N''}(\alpha)$, and as the funnel $F_0^{N''}(\alpha)$ was arbitrary, x is funneled. Similarly, suppose x is funneled. Then there exists $a \in A$ so that $x \in F_0^{N''}(\alpha') \subseteq H_0^{N''}(\alpha)$, and as the funnel $F_0^{N''}(\alpha)$ was arbitrary, x is horospherical. $\square$

4 Limit set conditions for stability

In this section, we show that the horospherical limit point condition, combined with the limit set being compact, is enough to show that the group action on the weak convex hull is cobounded. The main idea behind this argument is to show the contrapositive: when the group action is not cobounded, then geodesic rays in the space eventually end up very far from the orbit of the group. We begin by showing the following helpful fact, which states that if a group acts noncoboundedly on the weak convex hull of its limit set, there exists a sequence of points p_n in the weak convex hull that “maximally avoids” the orbit.

Lemma 4.1 *Suppose X is a proper geodesic metric space and suppose that H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$. If the action $H \curvearrowright \text{WCH}(\Lambda H)$ is not cobounded, then there exists an increasing sequence of positive integers, $(n_i)_i$, such that for each $i \in \mathbb{Z}_{\geq 1}$ there exists $p_i \in \text{WCH}(\Lambda H)$ satisfying*

- (1) $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$,
- (2) $d(p_i, \mathfrak{o}) \leq n_i + 1$.

Proof Set $n_0 = 1$. We define q_i and n_i for $i \geq 1$ via an inductive process. Since the action of $H \curvearrowright \text{WCH}(\Lambda H)$ is not cobounded, there exists a point $q_i \in \text{WCH}(\Lambda H)$ such that $n_{i-1} + 1 < d(H\mathfrak{o}, q_i)$. By the definition of $\text{WCH}(\Lambda)$, there exists a bi-infinite Morse geodesic γ with $\gamma(\pm\infty) \in \Lambda H$ such that $q_i \in \gamma$. Set n_i to be the unique positive integer such that $n_i < d(H\mathfrak{o}, q_i) \leq n_i + 1$. The sequence $(n_i)_i$ is increasing because $n_{i-1} + 1 \leq n_i$.

Since $d(H\mathfrak{o}, q_i) \leq n_i + 1$ there exists $h_i \in H$ so that $d(q_i, h_i\mathfrak{o}) \leq n_i + 1$. Recalling that the action of H on X is by isometries, we define $p_i = h_i^{-1}q_i$, and so $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$, and $d(\mathfrak{o}, p_i) \leq n_i + 1$. Finally, by [9, Lemma 3.3], $h_i^{-1}\gamma$ is a bi-infinite Morse geodesic with endpoints in ΛH , and so $p_i \in \text{WCH}(\Lambda H)$. $\square$

Under the additional assumption that ΛH is compact and that every point in ΛH is conical, we get a stronger conclusion to this lemma, namely, we can take $n_i = i$ for large i .

Lemma 4.2 (sliding spheres) *Suppose X is a proper geodesic metric space, and suppose that H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$, every point of ΛH is a conical limit point of $H\mathfrak{o}$, and that $\Lambda H \subseteq \partial X_{\mathfrak{o}}^N$ for some Morse gauge N . If the action $H \curvearrowright \text{WCH}(\Lambda H)$ is not cobounded, there exists a sequence of points $p_n \in \text{WCH}(\Lambda H)$ such that, for sufficiently large n , $B_n(p_n) \cap H\mathfrak{o} = \emptyset$ and $\mathfrak{o} \in B_{n+1}(p_n)$.*

Proof Let $K > 0$ be the conical limit point constant. Let $n \in \mathbb{N}$ with $n > K + 1$. By [22, Corollary 5.8], we may assume that ΛH has at least two distinct points. Since $H \curvearrowright \text{WCH}(\Lambda H)$ is not cobounded, there exists $p \in \text{WCH}(\Lambda H)$ with $d(p, H\mathfrak{o}) > n$. By definition, $p \in \gamma$ for some bi-infinite geodesic γ with $\gamma(\pm\infty) \in \Lambda H$. Since $\Lambda H \subseteq \partial X_{\mathfrak{o}}^N$, we have by [9, Proposition 4.2] that γ is Morse for some Morse gauge depending only on N . Since every point in ΛH is a conical limit point of $H\mathfrak{o}$, there exists $h' \in H$ such that $d(h'\mathfrak{o}, \gamma) < K$. Put $q \in \pi_{\gamma}(h'\mathfrak{o})$.

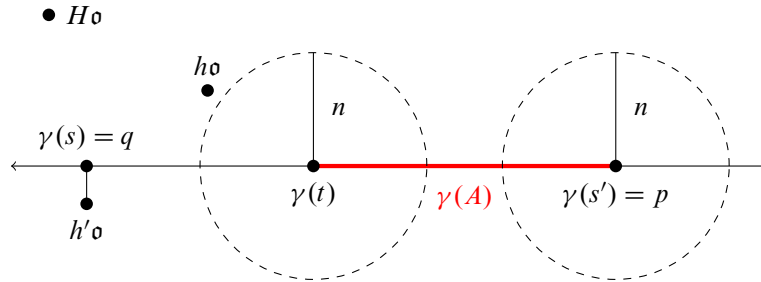


Figure 8: Diagram for Lemma 4.2. We can think of this proof as sliding the ball on the right towards the left until it is “up against” the orbit $H\mathfrak{o}$, such as the ball centered at $\gamma(t)$.

We may assume that $\gamma(s) = q$ and $\gamma(s') = p$ with $s < s'$. Let $A = \{r \in [s, s'] : n < d(\gamma(r), H\mathfrak{o})\}$. (Equivalently, one may define $A = \{r \in [s, s'] : B_n(\gamma(r)) \cap H\mathfrak{o} = \emptyset\}$.) Note that $s' \in A$. Put $t = \inf A$. By the definition of t , we have $n \leq d(\gamma(t), H\mathfrak{o})$. See Figure 8. We now claim that $d(\gamma(t), H\mathfrak{o}) < n + 1$.

Suppose for contradiction that $n + 1 \leq d(\gamma(t), H\mathfrak{o})$. By the triangle inequality, $n \leq d(\gamma(t - 1), H\mathfrak{o})$. So if $t - 1 \in [s, s']$, then $t - 1 \in A$, however $t = \inf A$. Thus $t - 1 \notin [s, s']$. Therefore, $t \in [s, s + 1]$, and so $n + 1 \leq d(\gamma(t), h\mathfrak{o}) \leq d(\gamma(t), h'\mathfrak{o}) \leq d(\gamma(t), q) + d(q, h'\mathfrak{o}) = d(\gamma(t), \gamma(s)) + d(q, h'\mathfrak{o}) \leq 1 + K \leq n$, a contradiction.

Therefore, there exists $h \in H$ such that $h\mathfrak{o} \in B_{n+1}(\gamma(t))$, but $B_n(\gamma(t)) \cap H\mathfrak{o} = \emptyset$. Put $p_n = h^{-1}(\gamma(t))$. By [9, Lemma 3.3], $h\gamma$ is a bi-infinite Morse geodesic with endpoints in ΛH , and since the action of H on X is by isometries, $B_n(p_n) \cap H\mathfrak{o} = \emptyset$, and $\mathfrak{o} \in B_{n+1}(p_n)$, as desired. $\square$

We now prove that (4) implies (2) in the language of Theorem 1.2. We show that, if the action is not cobounded on the weak convex hull, then using Lemma 4.1 we can find a sequence of points p_i which maximally avoid the orbit of H . However, this sequence of points defines a new ray γ with $\gamma(\infty) \in \Lambda H$. Then using the horospherical point assumption, we find an orbit point close to p_i , a contradiction.

Theorem 4.3 Suppose X is a proper geodesic metric space and suppose H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$, every point of ΛH is a horospherical limit point of $H\mathfrak{o}$, and that there exists a Morse gauge N such that $\Lambda H \subset \partial X_{\mathfrak{o}}^N$. Then the action of $H \curvearrowright \text{WCH}(\Lambda H)$ is cobounded.

Proof For contradiction, assume that $H \curvearrowright \text{WCH}(\Lambda H)$ is not a cobounded action. By Lemma 4.1, there exists a sequence of points $p_i \in \text{WCH}(\Lambda H)$ and an increasing sequence of positive integers $(n_i)_i$ such that $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$, and $\mathfrak{o} \in B_{n_i+1}(p_i)$. Let $\gamma_i : [0, d(0, p_i)] \rightarrow X$ be a geodesic connecting $\mathfrak{o}$ and p_i with $\gamma_i(0) = \mathfrak{o}$. Since $\Lambda H \subset \partial X_{\mathfrak{o}}^N$, we have that γ_i is N' -Morse for some N' depending only on N . By restricting to a subsequence, we may assume that γ_i converges, uniformly on compact subsets, to an N' -Morse geodesic ray γ with $\gamma(0) = \mathfrak{o}$.

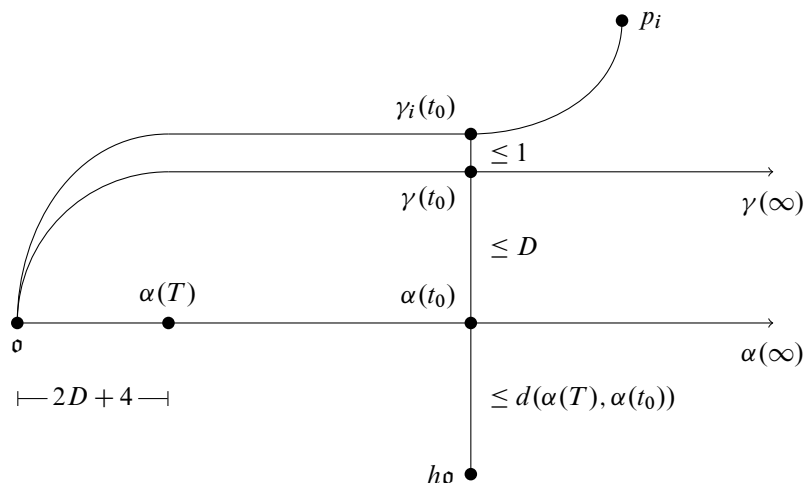


Figure 9: Diagram for Theorem 4.3.

By construction and by Lemma 2.31, $\gamma(\infty) \in \Lambda(\text{WCH}(\Lambda H)) \subseteq \Lambda H$. So, by [8, Corollary 2.6], there exists an N -Morse geodesic ray α with $\alpha(0) = o$ and $d(\alpha(t), \gamma(t)) < D$ for all $t \geq 0$, where $D \geq 0$ is a constant that depends only on N . Let $T = 2D + 4$, and put $\alpha' = \alpha_{[T, \infty)}$. Since $\alpha'(\infty) \in \Lambda H$, and so by Theorem 3.8, $\alpha'(\infty)$ is a funneled limit point of H . Thus there exists $h \in H$ so that $ho \in F_o^N(\alpha')$. Let $t_0 = \min\{s : \alpha'(s) \in \pi_{\alpha'}(ho)\}$. Since the sequence γ_i converges uniformly on compact sets to γ , we may choose i large enough so that $d(\gamma_i(t_0), \gamma(t_0)) \leq 1$. See Figure 9.

By the triangle inequality we have that $d(\gamma_i(t_0), \alpha'(t_0)) \leq D + 1$, and therefore

$$|d(o, \gamma_i(t_0)) - d(o, \alpha(t_0))| \leq D + 1.$$

Also, by construction we have that $d(ho, \alpha'(t_0)) \leq d(\alpha(T), \alpha(t_0))$. Therefore we have

$$\begin{aligned} d(p_i, ho) &\leq d(p_i, \gamma_i(t_0)) + d(\gamma_i(t_0), \alpha'(t_0)) + d(\alpha'(t_0), ho) \\ &\leq d(o, \gamma_i(t_0)) + (D + 1) + d(\alpha(T), \alpha(t_0)) \\ &= d(o, p_i) - d(o, \gamma_i(t_0)) + (D + 1) + d(o, \alpha(t_0)) - d(o, \alpha(T)) \\ &\leq (n_i + 1) + (D + 1) + (D + 1) - (2D + 4) \leq n_i - 1. \end{aligned}$$

However, this contradicts the assumption that $B_{n_i}(p_i) \cap Ho = \emptyset$. $\square$

We now present an alternate definition of a conical limit point which agrees with Definition 3.2 in the case where ΛA is compact, and requires us to only consider of the geodesic rays which emanate from the given basepoint. By Corollary 2.27 and by [9, Lemma 4.1], the requirement that ΛH is compact is equivalent to the requirement that ΛH is contained in the boundary of a single Morse stratum.

Proposition 4.4 *Let X be a proper, geodesic metric space. Let $Y \subseteq X$. Suppose $\Lambda Y \neq \emptyset$. Then the following are equivalent:*

- (1) $x \in \partial X_o$ is a conical limit point of Y .
- (2) There exists $K > 0$ such that, for every N -Morse geodesic ray $\alpha: [0, \infty) \rightarrow X$ with $\alpha(0) = o$ and $\alpha(\infty) = x$, and for every $T > 0$, there exists $y \in Y$ such that $y \in \mathcal{N}_K(\alpha')$, where $\alpha': [0, \infty) \rightarrow X$ is defined by $\alpha'(t) = \alpha(t + T)$.

Proof Showing that (1) implies (2) is a direct consequence of Lemma 2.17 and Definition 3.2.

Instead assume (2). Let $\beta: [b, \infty) \rightarrow X$ be an N' -Morse ray with $\beta(\infty) = x$. Let $\alpha: [0, \infty) \rightarrow X$ an N -Morse geodesic ray with $\alpha(0) = o$ and $\alpha(\infty) = x$. Without loss of generality, by Corollary 2.22 there exists $T > 0$ such that $d(\alpha(t), \beta(t)) < \delta_N$ for all $t > T$. Put $\alpha': [0, \infty) \rightarrow X$ via $\alpha'(t) = \alpha(t + T)$. By hypothesis, there exists $y \in Y$ such that $y \in \mathcal{N}_K(\alpha')$. Say $s \in [0, \infty)$ such that $d(\alpha'(s), y) < K$, so via the triangle inequality we have

$$d(\beta(s + T), y) \leq d(\beta(s + t), \alpha(s + t)) + d(\alpha'(s), y) \leq \delta_N + K.$$

Thus, $y \in \mathcal{N}_{K+\delta_N}(\beta)$, which shows (1). $\square$

We conclude this section by showing that (3) $\Rightarrow$ (2) for Theorem 1.2, which was first shown in [9, Corollary 1.14], however here we present a direct proof that does not rely on [9, Theorem 1.1].

Proposition 4.5 *Let X be a proper geodesic metric space and let H be a finitely generated group of isometries of X such that the orbit map $H \rightarrow X$ via $h \mapsto ho$ is a stable mapping. If there exists a Morse gauge N so that $\Lambda H \subseteq \partial X_o^N$, then every $x \in \Lambda H$ is a conical limit point of Ho .*

Proof Let $x \in \Lambda H$, and let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray with $\alpha(\infty) = x$, $\alpha(0) = o$. Let $\alpha' = \alpha|_{[a, \infty)}$ be a subray of α . Notice that α' is N' -Morse where N' depends only on N by Lemma 2.17. By Proposition 4.4, it suffices to show that there exists some $K \geq 0$, depending only on N' and H , so that $Ho \cap \mathcal{N}_K(\alpha') \neq \emptyset$.

Since H is a stable subgroup of isometries on X , we have that for any $h \in H$, there exists a (λ, λ) -quasigeodesic γ from o to ho such that, for any $p \in \gamma$, $B_{2\lambda}(p) \cap Ho \neq \emptyset$. (To find such a path γ , take a geodesic in a Cayley graph for H and embed it into X by extending the orbit map along appropriate geodesic segments.)

Now, since $x \in \Lambda H$, there exists a sequence $h_n \in H$ such that the sequence of geodesic segments, $\beta_n = [o, h_n o]$, converges (uniformly on compact subsets) to a geodesic ray $\beta: [b, \infty) \rightarrow X$ with $\beta(\infty) = x$ and $\beta(b) = o$. Since H is a stable group of isometries, β_n is N'' -Morse by Definition 2.32. Up to potentially reparameterizing β , there exists $T > a$ so that $d(\beta(T), \alpha(T)) < \delta_N$ by Corollary 2.22.

Since β_n converges to β uniformly on $\overline{B_{T+1}(o)}$, the ball of radius $T + 1$ centered at o , there exists $n \in \mathbb{N}$ and $p \in \beta_n$ so that $d(\beta(T), p) < 1$. Since γ_n is an (λ, λ) -quasigeodesic with endpoints on β_n , there exists $q \in \gamma_n$ so that $d(p, q) \leq N''(\lambda, \lambda)$. Finally, there exists $h \in H$ so that $d(ho, q) \leq \lambda$.

Therefore by the triangle inequality,

$$d(\alpha(T), h\alpha) \leq d(\alpha(T), \beta(T)) + d(\beta(T), p) + d(p, q) + d(q, h\alpha) \leq \delta_N + 1 + N''(\lambda, \lambda) + 2\lambda.$$

As $\alpha(T) \in \alpha'$, this completes the proof. $\square$

5 Applications to Teichmüller space

We conclude by illustrating applications of the above work in the setting of Teichmüller space for a finite-type surface S . We begin by setting some notation. Let $\text{Mod}(S)$ denote the mapping class group of S , ie the group of orientation-preserving homeomorphisms on S up to isotopy equivalence, which may permute punctures but fixes boundaries pointwise. Let $\mathcal{T}(S)$ denote the associated Teichmüller space, equipped with the Teichmüller metric. We will denote the set of projective measured foliations on S by $\text{PMF}(S)$. The Thurston compactification of Teichmüller space is $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \text{PMF}(S)$. For a thorough overview of the mapping class group, its associated Teichmüller space, and projective measured foliations, we refer the reader to [5; 14; 16; 24].

We take a moment to restate Corollary 1.5 using the above notation:

Corollary 5.1 (restatement of Corollary 1.5) *Let H be a finitely generated subgroup of $\text{Mod}(S)$. The following are equivalent:*

- (1) *Every element of $\Lambda H \subset \partial \text{Mod}(S)$ is a conical limit point of $H \curvearrowright \text{Mod}(S)$ and ΛH is compact (in the Morse boundary of $\text{Mod}(S)$).*
- (2) *Every element of $\Lambda H \subset \text{PMF}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

By work of Cordes, there exists a homeomorphism $g_\infty: \partial \text{Mod}(S) \rightarrow \partial \mathcal{T}(S)$ (where ∂ refers to the Morse boundary) [8, Theorem 4.12], and there exists a natural continuous injective map $h_\infty: \partial \mathcal{T}(S) \hookrightarrow \text{PMF}(S)$ [8, Proposition 4.14]. We denote the continuous inclusion formed by the composition of g_∞ and h_∞ as $f_\infty: \partial \text{Mod}(S) \hookrightarrow \text{PMF}(S)$. The purpose of this section is to prove the following theorem.

Theorem 5.2 *Let H be a subgroup of $\text{Mod}(S)$, and let $x_\infty \in \Lambda H \subseteq \partial \text{Mod}(S)$ be a conical limit point of $H \curvearrowright \text{Mod}(S)$. Then $f_\infty(x_\infty) \in \text{PMF}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

Remark 5.3 This theorem directly proves $(1) \Rightarrow (2)$ of Corollary 1.5.

Our proof of Theorem 5.2 uses several of the tools developed in [8], so we take a moment to recall the construction and definitions presented therein and from [24]. The *curve graph*, denoted by $\mathcal{C}(S)$, is a locally infinite simplicial graph whose vertices are isotopy classes of simple closed curves on S . We join two vertices with an edge if there exists representative from each class that are disjoint.

A set of (pairs of) curves $\mu = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_m, \beta_m)\}$ is called a complete clean *marking* of S if $\{\alpha_1, \dots, \alpha_m\}$ forms a pants decomposition of S , if each α_i is disjoint from β_j whenever $i \neq j$, and if each α_i intersects β_i once if the surface filled by α_i and β_i is a one-punctured torus. (Otherwise, α_i and β_i will intersect twice, and the filling surface is a four-punctured sphere.) We call $\{\alpha_1, \dots, \alpha_m\}$ the *base* of μ and we call β_i the *transverse curve to α_i in μ* . For the sake of completeness, we also define the marking graph, $\mathcal{M}(S)$, although the definition is not needed in this paper.

The marking graph $\mathcal{M}(S)$ is the simplicial graph whose vertices are markings as defined above, and two markings are joined by an edge of length one if they differ by an *elementary move*: either twisting β_i around α_i by a full, or when possible, by a half twist, or by swapping β_i and α_i . After performing an elementary move, one may need to replace the curves with isotopically equivalent curves to create a valid marking again. The marking graph $\mathcal{M}(S)$ is quasi-isometric to the mapping class group $\text{Mod}(S)$; see [5; 24].

For each $\sigma \in \mathcal{T}(S)$ there is a *short marking*, which is constructed inductively by picking the shortest curves in σ for the base and repeating for the transverse curves. Now define a map $\Upsilon: \mathcal{M}(S) \rightarrow \mathcal{T}(S)$ by taking a marking μ to the region in the ϵ -thick part of $\mathcal{T}(S)$, denoted by $\mathcal{T}_\epsilon(S)$, where μ is a short marking in that region. As stated in [8], it is a well-known fact that Υ is a coarsely well-defined map which is coarsely Lipschitz. We take a moment to prove that this map is coarsely equivariant.

Lemma 5.4 *Let $\Upsilon: \mathcal{M}(S) \rightarrow \mathcal{T}(S)$ be as above, and let $H < \text{Mod}(S)$ be finitely generated. Then there exists a constant $K \geq 0$ such that, for any marking $\mu \in \mathcal{M}$ and for any $h \in H$,*

$$d_{\mathcal{T}(S)}(h\Upsilon(\mu), \Upsilon(h\mu)) \leq K.$$

Proof Let $\mu = \{(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\} \in \mathcal{M}(S)$ and $h \in H$ be arbitrary. Let $\sigma \in \mathcal{T}(S)$ so that μ is a short marking on σ . (Equivalently, let $\sigma = \Upsilon(\mu)$.) Since the action of H on $\mathcal{T}(S)$ permutes the lengths of curves, the length of each pair (α_i, β_i) with respect to σ is the same as the length of the pair $(h\alpha_i, h\beta_i)$ with respect to $h\sigma$. Therefore as μ was a short marking for σ , this shows that $h\mu$ is a short marking for $h\sigma = h\Upsilon(\mu)$. However, by definition of Υ , $h\mu$ is also a short marking for $\Upsilon(h\mu)$. As Υ was a coarsely well-defined function, this shows that $d_{\mathcal{T}(S)}(h\Upsilon(\mu), \Upsilon(h\mu)) \leq K$ for some $K \geq 0$, as desired. $\square$

We now prove Theorem 5.2, using the above lemma and several tools from [8] to show that points in conical neighborhoods in $\mathcal{M}(S)$ end up in conical neighborhoods of $\mathcal{T}(S)$.

Proof Fix $\mu_0 \in \mathcal{M}(S)$. Let $x \in \partial\mathcal{M}(S)_{\mu_0}$ be a conical limit point of $H\mu_0$. Put $\sigma_0 = \Upsilon(\mu_0)$. We shall show that $f_\infty(x)$ is a conical limit point of $H\sigma_0$ by verifying the condition in Proposition 4.4. Let $T \geq 0$ be arbitrary, and let $\lambda: [0, \infty) \rightarrow \mathcal{T}(S)$ be an arbitrary Morse geodesic ray with $\lambda(0) = \sigma_0$ and $\lambda(\infty) = f_\infty(x)$.

Let $\alpha: \mathbb{N} \rightarrow \mathcal{M}(S)$ be an N -Morse geodesic with $\alpha(0) = \mu_0$ and $\alpha(\infty) = x$. By [8, Lemma 4.9], $\Upsilon(\alpha)$ is an N' -Morse (A, B) -quasigeodesic, for some A, B , and N' depending only on N . Put $\beta = \Upsilon(\alpha)$.

Notice that $\beta(0) = \sigma_0$ and, by the construction of f_∞ , we have $\beta(\infty) = f_\infty(x)$. (For details on the construction of f_∞ , we refer to [8], specifically Proposition 4.11, Theorem 4.12, and Proposition 4.14.)

Now let $\gamma_n = [\sigma_0, \beta(n)]$. Then each γ_n is N'' -Morse for N'' depending on N , and by Arzelà–Ascoli and [8, Lemma 2.10], a subsequence of the γ_n converges to a geodesic ray β which is N'' -Morse, and by [8, Lemma 4.9], β is bounded Hausdorff distance from γ , where the bound only depends on N . Say that $d_{\text{Haus}}(\beta, \gamma) \leq K_1$ for $K_1 \geq 0$. By [8, Corollary 2.6], $d_{\text{Haus}}(\gamma, \lambda) \leq K_2$ where $K_2 \geq 0$ depends only on N . Choose $S \geq 0$ so that, for all $s \geq S$, $d_{\mathcal{T}(S)}(\beta(s), \lambda|_{[T, \infty)}) \leq K_1 + K_2$.

By Proposition 4.4, there exists $L \geq 0$ where, for all $r \geq 0$, $d_{\mathcal{M}(S)}(h\mu_0, \alpha|_{[r, \infty)}) \leq L$ for some $h \in H$. Since $\beta = \Upsilon(\alpha)$ and Υ is coarse Lipschitz, there exists $K_3 \geq 0$ and $h \in H$ so that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), \beta|_{[S, \infty)}) \leq K_3$. Let $s_0 \in [S, \infty)$ so that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), \beta(s_0)) \leq K_3$. By Lemma 5.4, there exists $K_4 \geq 0$ such that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), h\Upsilon(\mu_0)) \leq K_4$.

By the triangle inequality, we have

$$\begin{aligned} d_{\mathcal{T}(S)}(h\sigma_0, \lambda|_{[T, \infty)}) &\leq d(h\Upsilon(\mu_0), \Upsilon(h\mu_0)) + d(\Upsilon(h\mu_0), \beta(s_0)) + d(\beta(s_0), \lambda|_{[T, \infty)}) \\ &\leq K_4 + K_3 + K_2 + K_1. \end{aligned}$$

By Proposition 4.4, $\lambda(\infty) = f_\infty(x)$ is a conical limit point of $H\sigma_0$. □

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Pre-Lie algebras with divided powers and the Deligne groupoid in positive characteristic

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We develop a deformation theory controlled by pre-Lie algebras with divided powers over a ring of positive characteristic. We show that every differential graded pre-Lie algebra with divided powers comes with operations, called weighted braces, which we use to generalize the classical deformation theory controlled by Lie algebras over a field of characteristic 0. Explicitly, we define the Maurer–Cartan set, as well as the gauge group, and prove that there is an action of the gauge group on the Maurer–Cartan set. This new deformation theory moreover admits a Goldman–Millson theorem which remains valid over the integers. As an application, we give the computation of the π_0 of a mapping space $\text{Map}(B^c(\mathcal{C}), \mathcal{P})$ with $\mathcal{C}$ and $\mathcal{P}$ suitable cooperad and operad, respectively.

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Introduction

An important result in deformation theory asserts that every deformation problem over a field of characteristic 0 can be encoded by a differential graded Lie algebra (see [15; 18]). More precisely, any deformation problem can be described by a solution of the *Maurer–Cartan equation*:

$$d(x) + \frac{1}{2}[x, x] = 0,$$

in some differential graded Lie algebra. The group obtained by the integration of the differential graded Lie algebra into a Lie group, called the *gauge group*, moreover acts on the Maurer–Cartan set. The orbits of this action give isomorphism classes of deformation problems.

In [6], Dotsenko, Shadrin and Vallette showed that if the differential graded Lie algebra comes from a differential graded pre-Lie algebra, then the Maurer–Cartan equation, the gauge group and its action on the Maurer–Cartan set can be described in terms of pre-Lie operations. A differential graded pre-Lie algebra is a vector space L with a bilinear operation $\star: L \otimes L \rightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = (-1)^{|y||z|}((x \star z) \star y - x \star (z \star y)),$$

and which satisfies the Leibniz rule with respect to the differential. Every differential graded pre-Lie algebra is in particular a differential graded Lie algebra with the graded commutator:

$$[x, y] = x \star y - (-1)^{|x||y|}y \star x.$$

Dotsenko, Shadrin and Vallette showed in particular that given a pre-Lie algebra L , the pre-Lie exponential map $\exp: L^0 \rightarrow (1 + L^0)$ induces an isomorphism between the gauge group and the group $(1 + L^0, \odot, 1)$ with $\odot$ the circular product defined by

$$a \odot (1 + b) = \sum_{n \geq 0} \frac{1}{n!} a \underbrace{\{b, \dots, b\}}_n,$$

where $-\{-, \dots, -\}$ denotes the *symmetric braces* determined by the pre-Lie structure $\star$, starting with $x\{y\} = x \star y$. Then, by writing the Maurer–Cartan equation as a zero-square equation, they prove that the action of the gauge group on the Maurer–Cartan set can be computed in terms of the circular product $\odot$ as

$$e^\lambda \cdot \alpha = (e^\lambda \star \alpha) \odot e^{-\lambda},$$

allowing us to have an easier way to compute the Deligne groupoid associated to any differential graded pre-Lie algebra over a field of characteristic 0.

The aim of this paper is to develop a deformation theory in positive characteristic which generalizes the deformation theory controlled by pre-Lie algebras over a field of characteristic 0 developed in [6]. Our idea is to use differential graded pre-Lie algebras with divided powers.

The notion of a pre-Lie algebra with divided powers (or $\Gamma(\text{PreLie}, -)$ -algebra) has been studied by Cesaro in [3]. He showed in particular that every pre-Lie algebra with divided powers comes equipped

with weighted brace operations $-\{-, \dots, -\}_{r_1, \dots, r_n}$, for each collection of integers $r_1, \dots, r_n \geq 0$, which satisfy similar identities as the quantities

$$x\{y_1, \dots, y_n\}_{r_1, \dots, r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}\}$$

in a pre-Lie algebra over a field of characteristic 0 (see [3, Propositions 5.9–5.10] for a precise list of these identities).

Every differential graded pre-Lie algebra with divided powers L comes equipped with analogous weighted brace operations $-\{-, \dots, -\}_{r_1, \dots, r_n}$ which satisfy a graded version of the identities satisfied by weighted braces in the nongraded framework. In this context, we have an analogue of the Maurer–Cartan equation:

$$d(x) + x\{x\}_1 = 0.$$

With suitable convergence hypothesis, we also get that the circular product $\odot$ can be written as

$$a \odot (1 + b) = \sum_{n \geq 0} a\{b\}_n$$

and gives rise to a group structure on $1 + L^0$. This group is called the *gauge group* of L . As in characteristic 0, we also show that this group acts on the Maurer–Cartan set of L .

Theorem A *Let $\mathbb{K}$ be a ring.*

- (i) *In any complete differential graded pre-Lie algebra with divided powers L , the circular product $\odot$, defined as above, endows the set $1 + L^0$ with a group structure.*
- (ii) *If we denote by d the differential of L , then this group acts on the Maurer–Cartan set via*

$$(1 + \mu) \cdot \alpha = (\alpha + \mu\{\alpha\}_1 - d(\mu)) \odot (1 + \mu)^{\odot -1}.$$

We prove that this new deformation theory satisfies an analogue of the Goldman–Millson theorem given in [12, Section 2.4]. Let $\text{Deligne}(L, A)$ be the Deligne groupoid of the complete dg pre-Lie algebra with divided powers $L \otimes \mathfrak{m}_A$, where L is a dg pre-Lie algebra with divided powers and $\mathfrak{m}_A$ the maximal ideal of a local artinian algebra A over the field of fraction $\mathbf{K}$ of $\mathbb{K}$. We precisely get the following result.

Theorem B *Let $\mathbb{K}$ be a noetherian integral domain and $\mathbf{K}$ its field of fractions. Let L and $\bar{L}$ be two positively graded $\Gamma(\text{PreLie}, -)$ -algebras. Let $\varphi: L \rightarrow \bar{L}$ be a morphism of $\Gamma(\text{PreLie}, -)$ -algebras such that $H^0(\varphi)$ and $H^1(\varphi)$ are isomorphisms and $H^2(\varphi)$ is a monomorphism. Then for every local artinian $\mathbf{K}$ -algebra A , the induced functor $\varphi_*: \text{Deligne}(L, A) \rightarrow \text{Deligne}(\bar{L}, A)$ is an equivalence of groupoids.*

Other approaches to generalize the usual deformation theory in the positive characteristic framework have been proposed recently in the literature. We have, for instance, a deformation theory in an associative context, via $\mathcal{A}_\infty$ -algebras, which is used to study deformations of group representations (see [16]). Another approach is given by (spectral) partition Lie algebras to get a full generalization of the Lurie–Pridham correspondence in the setting of a field with positive characteristic (see [1; 2]).

The main motivation for the approach developed in this paper is that operadic deformation problems are expressed in terms of pre-Lie structures. The goal is then to compute the π_0 of a mapping space $\text{Map}(B^c(\mathcal{C}), \mathcal{P})$, where we take any dg operad $\mathcal{P}$ on the target and the operad $B^c(\mathcal{C})$ given by the cobar of a dg coaugmented cooperad $\mathcal{C}$ on the source. Recall simply that $B^c(\mathcal{C})$ defines a cofibrant operad when $\mathcal{C}$ is cofibrant as a symmetric sequence (Σ_* -cofibrant). It is well known that, over a field of characteristic 0, the π_0 of this mapping space is the set of isomorphism classes of the Deligne groupoid of the Lie algebra $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$. Using the pre-Lie algebra structure of $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$, this can be seen as a consequence of the computations in [6]. To extend this result, we first show that $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ admits a structure of a dg pre-Lie algebra with divided powers. Then we get the following statement.

Theorem C *Let $\mathbb{K}$ be a field. Suppose that $\mathcal{C}$ is a Σ_* -cofibrant coaugmented dg cooperad and $\mathcal{P}$ an augmented dg operad. We then have an isomorphism:*

$$\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P})) \simeq \pi_0 \text{Deligne}(\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})),$$

where $\pi_0 \text{Deligne}(\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}}))$ denotes the set of isomorphism classes of the Deligne groupoid.

Theorem C gives a first step for the calculation of the homotopy groups of a mapping space $\text{Map}(B^c(\mathcal{C}), \mathcal{P})$ over any field.

In the first part of this paper, we recall some definitions and properties on pre-Lie algebras and pre-Lie algebras with divided powers: in Section 1.1 we briefly review the definition of the notion of a pre-Lie algebra and the construction of the corresponding operad; in Section 1.2, we review the definition of a pre-Lie algebra with divided powers and of the weighted brace operations.

In the second part, we develop the deformation theory for differential graded pre-Lie algebras with divided powers: in Section 2.1, we study pre-Lie algebras with divided powers in the dg framework; in Section 2.2, we define the circular product and prove assertion (i) of Theorem A; in Section 2.3, we define the Maurer–Cartan set and prove assertion (ii) of Theorem A; in Section 2.4, we finally prove our analogue of the Goldman–Millson theorem (Theorem B) for this new deformation theory.

We conclude this article with our application of this deformation theory for operadic deformation problems: in Section 3.1, we introduce some basic definitions on symmetric sequences and operads which will be useful to write our formulas; in Section 3.2, we study the structure of a differential graded pre-Lie algebra with divided powers of the convolution operad; in Section 3.3, we finally give a proof of Theorem C.

Conventions We denote the symmetric group on n letters by Σ_n . We recall that a permutation $\sigma \in \Sigma_{r_1+\dots+r_n}$ is a *shuffle permutation of type $(r_1, \dots, r_n)$* if σ preserves the order on the subsets $\{r_1 + \dots + r_i + 1 < \dots < r_1 + \dots + r_{i+1}\}$ of $\{1 < \dots < r_1 + \dots + r_n\}$. The shuffle permutation σ is *pointed* if we also have $\sigma(1) < \sigma(r_1 + 1) < \dots < \sigma(r_1 + \dots + r_{n-1} + 1)$.

We denote by $\text{Sh}(r_1, \dots, r_n)$ the subset of $\Sigma_{r_1+\dots+r_n}$ composed of shuffle permutations of type $(r_1, \dots, r_n)$ and by $\text{Sh}_*(r_1, \dots, r_n)$ the subset of $\text{Sh}(r_1, \dots, r_n)$ composed of pointed shuffle permutations.

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1 Recollections on pre-Lie algebras with divided powers

We first recall some definitions and basic properties on pre-Lie algebras and pre-Lie algebras with divided powers. Pre-Lie algebras were introduced in deformation theory by Gerstenhaber in [10], while pre-Lie algebras with divided powers were introduced by Cesaro in [3].

In Section 1.1, we give brief recollections on the notion of a pre-Lie algebra. We will more particularly see pre-Lie algebras as algebras over an operad introduced by Chapoton and Livernet in [5], the rooted tree operad, of which we also recall the definition in this subsection.

In Section 1.2, we give recollections on the notion of a pre-Lie algebra with divided powers. These objects can be seen as pre-Lie algebras with some extra operations. We will focus on some of these operations called weighted braces that will mimic the quantities which appear in the definition of the circular product.

1.1 Pre-Lie algebras and the rooted tree operad

We will use the following basic definitions.

Definition 1.1 A *pre-Lie algebra* over a ring $\mathbb{K}$ is a $\mathbb{K}$ -module L endowed with a bilinear morphism $\star: L \otimes L \rightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y).$$

The category of pre-Lie algebras is isomorphic to the category of symmetric braces algebras (see [17] or [13]). The symmetric braces $-\{-, \dots, -\}$ are defined by induction on the length of the brace by

$$\begin{aligned} a\{\} &= a, \\ a\{b_1\} &= a \star b_1, \\ \forall n \geq 1, \quad a\{b_1, \dots, b_n\} &= a\{b_1, \dots, b_{n-1}\}\{b_n\} - \sum_{i=1}^{n-1} a\{b_1, \dots, b_{i-1}, b_i\{b_n\}, b_{i+1}, \dots, b_{n-1}\}, \end{aligned}$$

for all $a, b_1, \dots, b_n \in L$.

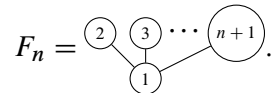
For our purpose, it will be more convenient to see pre-Lie algebras as algebras over an operad. This operad can be described in terms of rooted trees as follows.

Definition 1.2 (see [5, Section 1.5]) We call an n -rooted tree a nonplanar tree with n vertices equipped with a numbering from 1 to n , together with a distinguished vertex called the *root*. By convention, we choose to put the root at the bottom in any representation of a tree.

We let $\mathcal{RT}(n)$ to be the set of all trees with n vertices, and $\text{PreLie}(n) = \mathbb{K}[\mathcal{RT}(n)]$.

The collection PreLie is endowed with an operad structure. The action of Σ_n on $\text{PreLie}(n)$ for all $n \geq 1$ is given by the permutation of the indices attached to the vertices. The i^{th} partial composition $S \circ_i T \in \text{PreLie}(p + q - 1)$ of $S \in \mathcal{RT}(p)$ and $T \in \mathcal{RT}(q)$ is given by the sum of all the possible trees obtained by putting T in the vertex i of S , with the obvious choice of the numbering (see an example in [5, Section 1.5]). This operad is also called the *rooted tree operad*.

One can show that the algebras over the rooted tree operad are precisely the pre-Lie algebras (see [5, Section 1.9]). In particular, the symmetric braces are given by the trees F_n for $n \geq 0$ called *corollas with n leaves*:



1.2 Pre-Lie algebras with divided powers

In this part, we recall the notion of a pre-Lie algebra with divided powers. We obtain this definition as a particular case of a general construction, for algebras over an operad, which we briefly recall.

Every operad $\mathcal{P}$ on a suitable monoidal category $\mathcal{C}$ gives a functor $\mathcal{S}(\mathcal{P}, -): \mathcal{C} \rightarrow \mathcal{C}$, called the *Schur functor*, defined by

$$\mathcal{S}(\mathcal{P}, V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n},$$

where we consider, in the direct sum, the coinvariants of $\mathcal{P}(n) \otimes V^{\otimes n}$ under the diagonal action of Σ_n given by its action on $\mathcal{P}(n)$ and its action by permutation on the tensor product $V^{\otimes n}$. The image of a tensor product $p \otimes v_1 \otimes \cdots \otimes v_n \in \mathcal{P}(n) \otimes V^{\otimes n}$ in $\mathcal{P}(n) \otimes_{\Sigma_n} V^{\otimes n}$ will be denoted by $p(v_1, \dots, v_n)$. The Schur functor defines a monad and the category of algebras over this monad is the usual category of algebras over the operad $\mathcal{P}$. In particular, pre-Lie algebras in the sense of Definition 1.1 are identified with $\mathcal{S}(\text{PreLie}, -)$ -algebras.

In the above definition, one can chose to take invariants instead of coinvariants. We obtain a new functor $\Gamma(\mathcal{P}, -): \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\Gamma(\mathcal{P}, V) = \bigoplus_{n \geq 0} \mathcal{P}(n) \otimes^{\Sigma_n} V^{\otimes n}.$$

If $\mathcal{P}(0) = 0$, this functor also gives a monad (see [7, Section 1.1.18]). The algebras over this monad are called $\mathcal{P}$ -algebras with divided powers. The motivation for this terminology comes from the fact that $\Gamma(\text{Com}, -)$ -algebras are precisely the usual commutative and associative algebras over $\mathbb{K}$ with divided powers.

If C is a category whose objects are $\mathbb{K}$ -modules, then we have a morphism of monads $\text{Tr}: \mathcal{S}(\mathcal{P}, V) \rightarrow \Gamma(\mathcal{P}, V)$ called the *trace map* and defined by

$$\text{Tr}(p(v_1, \dots, v_n)) = \sum_{\sigma \in \Sigma_n} (\sigma \cdot p) \otimes v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}.$$

If $\mathbb{K}$ is a field of characteristic 0, the trace map is an isomorphism. It is no longer the case in general when $\text{char}(\mathbb{K}) \neq 0$.

In the case $C = \text{Mod}_{\mathbb{K}}$ of the category of $\mathbb{K}$ -modules and $\mathcal{P} = \text{PreLie}$, if V is a free $\mathbb{K}$ -module, we however have an isomorphism of modules given by the *orbit morphism* $\mathcal{O}: \mathcal{S}(\text{PreLie}, V) \rightarrow \Gamma(\text{PreLie}, V)$ defined as follows. Let $n \geq 1$ and $t \in \text{PreLie}(n) \otimes V^{\otimes n}$ be a basis element. We set

$$\mathcal{O}(t) = \sum_{\sigma \in \Sigma_n / \text{Stab}_{\Sigma_n}(t)} \sigma \cdot t,$$

where $\text{Stab}_{\Sigma_n}(t)$ is the stabilizer of t under the diagonal action of Σ_n on $\text{PreLie}(n) \otimes V^{\otimes n}$. The map $\mathcal{O}$ is then extended by linearity on $\text{PreLie}(n) \otimes V^{\otimes n}$.

Theorem 1.3 (A Cesaro [3]) *Every pre-Lie algebra with divided powers L comes equipped with operations $-\{-, \dots, -\}_{r_1, \dots, r_n}: L^{\times n+1} \rightarrow L$ called **weighted braces** which satisfy*

- (i) $x\{y_{\sigma(1)}, \dots, y_{\sigma(n)}\}_{r_{\sigma(1)}, \dots, r_{\sigma(n)}} = x\{y_1, \dots, y_n\}_{r_1, \dots, r_n},$
- (ii) $x\{y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_n\}_{r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_n} = x\{y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n\}_{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n},$
- (iii) $x\{y_1, \dots, \lambda y_i, \dots, y_n\}_{r_1, \dots, r_i, \dots, r_n} = \lambda^{r_i} x\{y_1, \dots, y_i, \dots, y_n\}_{r_1, \dots, r_i, \dots, r_n},$
- (iv) $x\{y_1, \dots, y_i, y_i, \dots, y_n\}_{r_1, \dots, r_i, r_{i+1}, \dots, r_n} = \binom{r_i + r_{i+1}}{r_i} x\{y_1, \dots, y_i, \dots, y_n\}_{r_1, \dots, r_{i-1}, r_i + r_{i+1}, r_{i+2}, \dots, r_n},$
- (v) $x\{y_1, \dots, y_i + \widetilde{y}_i, \dots, y_n\}_{r_1, \dots, r_i, \dots, r_n} = \sum_{s=0}^{r_i} x\{y_1, \dots, y_i, \widetilde{y}_i, \dots, y_n\}_{r_1, \dots, s, r_i - s, \dots, r_n},$
- (vi) $x\{y_1, \dots, y_n\}_{r_1, \dots, r_n} \{z_1, \dots, z_m\}_{s_1, \dots, s_m}$

$$= \sum_{s_i = \beta_i + \sum \alpha_i^{\bullet, \bullet}} \frac{1}{\prod_j (r_j)!} x\{y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,1}, \dots, \alpha_m^{1,1}}, \dots, y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,r_1}, \dots, \alpha_m^{1,r_1}},$$

$$\dots, y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,1}, \dots, \alpha_m^{n,1}}, \dots, y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,r_n}, \dots, \alpha_m^{n,r_n}}, z_1, \dots, z_m\}_{1, \dots, 1, \beta_1, \dots, \beta_m},$$

for all $n, m \geq 0$, $r_1, \dots, r_n, s_1, \dots, s_m \geq 0$, $1 \leq i \leq n$, $\sigma \in \Sigma_n$, $\lambda \in \mathbb{K}$ and $x, y_1, \dots, y_n, z_1, \dots, z_m \in L$.

The formula (vi) is written in a form that uses fractions for more convenience, but can be reduced to $\mathbb{Z}$ using the other formulas. The process works as follows. Let i such that $1 \leq i \leq n$. In the sum, we first fix $\beta_1, \dots, \beta_m$ and $\alpha_j^{p,q}$ for $1 \leq j \leq m$, $1 \leq q \leq r_j$ and $p \neq i$. We obtain a sum with $(\alpha_1^{i,1}, \dots, \alpha_m^{i,1}, \dots, \alpha_1^{i,r_i}, \dots, \alpha_m^{i,r_i})$ as variables. We identify this last tuple with a tuple of tuples of the form $((\alpha_1^{i,1}, \dots, \alpha_m^{i,1}); \dots; (\alpha_1^{i,r_i}, \dots, \alpha_m^{i,r_i}))$. Let u be one of these tuples and suppose

$$u = (\underbrace{\widetilde{u}_1, \dots, \widetilde{u}_1}_{t_1}, \dots, \underbrace{\widetilde{u}_q, \dots, \widetilde{u}_q}_{t_q})$$

up to permutation. If $\widetilde{u}_1, \dots, \widetilde{u}_q$ are given, we exactly have $r_i!/(t_1! \cdots t_q!)$ such terms occurring in the

sum. Then, by using the symmetry formula (i), the formula (iv) and by summing over all such tuples, we have, in the sum,

$$\frac{1}{\prod_{j \neq i} (r_j)!} \frac{r_i!}{t_1! \cdots t_q!} t_1! \cdots t_q! x \{y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,1}, \dots, \alpha_m^{1,1}}, \dots, y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,r_1}, \dots, \alpha_m^{1,r_1}}, \dots, \\ y_i \{z_1, \dots, z_m\}_{\tilde{u}_1}, \dots, y_i \{z_1, \dots, z_m\}_{\tilde{u}_q}, \dots, y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,1}, \dots, \alpha_m^{n,1}}, \dots, \\ y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,r_n}, \dots, \alpha_m^{n,r_n}}, z_1, \dots, z_m\}_{1, \dots, t_1, \dots, t_q, \dots, 1, \beta_1, \dots, \beta_m},$$

where we have set $y_i \{z_1, \dots, z_m\}_{\tilde{u}_k} = y_i \{z_1, \dots, z_m\}_{\alpha_1, \dots, \alpha_m}$ if $\tilde{u}_k = (\alpha_1, \dots, \alpha_m)$. Hence,

$$\frac{1}{\prod_{j \neq i} (r_j)!} x \{y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,1}, \dots, \alpha_m^{1,1}}, \dots, y_1 \{z_1, \dots, z_m\}_{\alpha_1^{1,r_1}, \dots, \alpha_m^{1,r_1}}, \dots, \\ y_i \{z_1, \dots, z_m\}_{\tilde{u}_1}, \dots, y_i \{z_1, \dots, z_m\}_{\tilde{u}_q}, \dots, y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,1}, \dots, \alpha_m^{n,1}}, \dots, \\ y_n \{z_1, \dots, z_m\}_{\alpha_1^{n,r_n}, \dots, \alpha_m^{n,r_n}}, z_1, \dots, z_m\}_{1, \dots, t_1, v, t_q, \dots, 1, \beta_1, \dots, \beta_m}.$$

By iterating this argument on the other terms, we obtain an expression over $\mathbb{Z}$.

The reader can find an example of such a reduction of the formula (vi) in [3, Example 5.11], as well as a proof of the previous theorem (see [3, Propositions 5.9–5.10]).

We give the explicit construction of the weighted braces.

Construction 1.4 We regard the weighted braces $x \{y_1, \dots, y_n\}_{r_1, \dots, r_n}$ as the action of the corolla $F_{\sum_i r_i}$ on the tensor

$$x \otimes \underbrace{y_1 \otimes \cdots \otimes y_1}_{r_1} \otimes \cdots \otimes \underbrace{y_n \otimes \cdots \otimes y_n}_{r_n}$$

where we regard the y_i 's as distinct variables (see below). If $y_i \neq y_j$ for all $i \neq j$, then we precisely set

$$x \{y_1, \dots, y_n\}_{r_1, \dots, r_n} = \gamma(\mathcal{O}_{F_{\sum_i r_i}}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})),$$

where γ is the $\Gamma(\text{PreLie}, -)$ -algebra structure on L .

In order to include the case where some of the y_i 's might be the same, let E_n be the free $\mathbb{K}$ -module generated by a basis $e, e_1, \dots, e_n$. Let $\psi_{x, y_1, \dots, y_n}: E_n \rightarrow L$ be the morphism which sends e to x and e_i to y_i for all $1 \leq i \leq n$. We obtain a morphism

$$\Gamma(\text{PreLie}, \psi_{x, y_1, \dots, y_n}): \Gamma(\text{PreLie}, E_n) \rightarrow \Gamma(\text{PreLie}, L).$$

We then take the orbit map at the source and apply this morphism next to have a good definition of the weighted braces.

Remark 1.5 The converse of the previous theorem is also true, provided that L is free as a $\mathbb{K}$ -module.

2 Deformation theory of $\Gamma(\text{PreLie}, -)$ -algebras

The main goal of this section is to extend the results proved in [6] in the context of a ring of positive characteristic. The main idea is that formulas which define the circular product and the gauge action can be written in terms of weighted brace operations.

In Section 2.1, we revisit the definition of pre-Lie algebras with divided powers in the dg framework. In particular, we give the analogue of the weighted brace operations. We then make explicit an example of differential graded pre-Lie algebras with divided powers given by differential graded brace algebras.

In Section 2.2, we define the circular product $\odot$ in terms of weighted brace operations that will generalize the one given in [6]. We then show that this induces a group called the gauge group associated to the $\Gamma(\text{PreLie}, -)$ -algebra.

In Section 2.3, we define the Maurer–Cartan equation in a $\Gamma(\text{PreLie}, -)$ -algebra, and then the Maurer–Cartan set. We also see that the gauge group acts on the Maurer–Cartan set by a similar formula given in [6].

In Section 2.4, we finally motivate this new deformation theory with an analogue of the Goldman–Millson theorem. This theorem, in particular, has the advantage to be true on integers.

2.1 Differential graded pre-Lie algebras with divided powers

As we are dealing with differential graded modules, our first goal is to define and study differential graded pre-Lie algebras with divided powers.

In the following sections, we assume that dg modules are equipped with a cohomological grading convention. We will denote by $\otimes$ the usual tensor product of graded modules over any ring $\mathbb{K}$. This induces a symmetric monoidal category that we will denote by $\text{dgMod}_{\mathbb{K}}$. If there is no possible confusion, then we will denote by $\pm$ any sign produced by the Koszul sign rule.

2.1.1 Weighted braces on $\Gamma(\text{PreLie}, -)$ -algebras Our goal here is to extend [3, Proposition 5.13] in the context of dg modules. We begin with a basic definition.

Definition 2.1 A differential graded pre-Lie algebra is an algebra over the monad

$$S(\text{PreLie}, -): \text{dgMod}_{\mathbb{K}} \rightarrow \text{dgMod}_{\mathbb{K}}.$$

Equivalently, we can see that a differential graded pre-Lie algebra is a graded module $L = \bigoplus_{k \in \mathbb{Z}} L^k$ endowed with a morphism of graded modules $\star: L \otimes L \rightarrow L$ such that

$$(x \star y) \star z - x \star (y \star z) = \pm((x \star z) \star y - x \star (z \star y))$$

and a differential $d: L^k \rightarrow L^{k+1}$, which satisfies

$$d(x \star y) = d(x) \star y \pm x \star d(y),$$

where $\pm$ is the sign yielded by the permutation of x and d .

We now define the notion of a pre-Lie algebra with divided powers in the dg framework.

Definition 2.2 A differential graded pre-Lie algebra with divided powers is an algebra over the monad $\Gamma(\text{PreLie}, -): \text{dgMod}_{\mathbb{K}} \rightarrow \text{dgMod}_{\mathbb{K}}$.

Let $L \in \text{dgMod}_{\mathbb{K}}$. Suppose that L^k is a free $\mathbb{K}$ -module for every $k \in \mathbb{Z}$. Let $\mathcal{L}$ be a basis of L composed of homogeneous elements. Then we have a basis on $\text{PreLie}(n) \otimes L^{\otimes n}$ for every $n \geq 1$ given by tensors $T \otimes e_1 \otimes \cdots \otimes e_n$ where $T \in \mathcal{RT}(n)$ and $e_1, \dots, e_n \in \mathcal{L}$. We denote by $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ this basis. If $\text{char}(\mathbb{K}) \neq 2$, then the action of Σ_n on $\text{PreLie}(n) \otimes L^{\otimes n}$ does not restrict to an action on $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ because of the Koszul sign rule. To handle things properly, we put this sign apart. We endow $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ with the diagonal action of Σ_n where Σ_n acts on $\mathcal{L}^{\otimes n}$ by the permutation of elements where we omit the Koszul sign. Given a tensor $\mathfrak{t} \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$, we denote by $X_{\mathfrak{t}}$ the orbit of $\mathfrak{t}$ under this action so that we have the following equality of graded $\mathbb{K}$ -modules:

$$\text{PreLie}(n) \otimes L^{\otimes n} = \bigoplus_{\bar{\mathfrak{t}} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}) / \Sigma_n} \mathbb{K}[X_{\mathfrak{t}}].$$

We recover the Koszul sign rule in the following way. Let $\mathfrak{t} \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$. For every $\sigma \in \Sigma_n$, we define $\varepsilon(\sigma, \mathfrak{t}) \in \{\pm 1\} \subset \mathbb{K}^{\times}$ as the Koszul sign which appears after the usual action of σ on $\mathfrak{t}$ in the graded module $\text{PreLie}(n) \otimes L^{\otimes n}$. For every $\sigma, \tau \in \Sigma_n$, we have

$$\varepsilon(\sigma\tau, \mathfrak{t}) = \varepsilon(\sigma, \tau \cdot \mathfrak{t})\varepsilon(\tau, \mathfrak{t}).$$

Equivalently, we can see ε as a functor from the groupoid with $X_{\mathfrak{t}}$ as set of objects and

$$\text{Hom}(\mathfrak{t}', \mathfrak{t}'') = \{\sigma \in \Sigma_n \mid \sigma \cdot \mathfrak{t}' = \mathfrak{t}''\}$$

for $\mathfrak{t}', \mathfrak{t}'' \in X_{\mathfrak{t}}$ to the groupoid, denoted by $\{\pm 1\}$, with only one object $*$ and $\text{Hom}(*, *) = \{\pm 1\} \subset \mathbb{K}^{\times}$. We can then define the Σ_n -representation $\mathbb{K}[X_{\mathfrak{t}}]^{\pm}$ as $\mathbb{K}[X_{\mathfrak{t}}]$ endowed with the action of Σ_n defined by

$$\sigma \cdot x^{\pm} = \varepsilon(\sigma, x)(\sigma \cdot x)^{\pm}$$

for every $\sigma \in \Sigma_n$ and $x \in X_{\mathfrak{t}}$. We thus have the following identity of Σ_n -representations:

$$\text{PreLie}(n) \otimes L^{\otimes n} = \bigoplus_{\bar{\mathfrak{t}} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}) / \Sigma_n} \mathbb{K}[X_{\mathfrak{t}}]^{\pm}.$$

Our purpose is to define an analogue of the orbit map, by using the above decomposition in set-theoretic orbits. We rely on the following lemma.

Lemma 2.3 Let G be a group and $H \subset G$ be a subgroup. We consider the action of G on G/H by the left translation. Let X be the groupoid with as set of objects G/H and with as morphisms $\text{Hom}(x, x') = \{g \in G \mid g \cdot x = x'\}$. Let $\varepsilon: X \rightarrow \{\pm 1\} \subset \mathbb{K}^{\times}$ be a functor. We denote by $\varepsilon(g, x)$ the image of the morphism $g: x \rightarrow g \cdot x$ under this functor. Consider the G -representation $\mathbb{K}[G/H]^{\pm} = \mathbb{K}[G/H]$ on which G acts by $g \cdot x^{\pm} = \varepsilon(g, x)(g \cdot x)^{\pm}$ for every $x \in G/H$. For every $g \in G$, we denote by $\bar{g}$ its class in G/H and by $[\bar{g}^{\pm}]$ the class of $\bar{g}^{\pm} \in \mathbb{K}[G/H]^{\pm}$ in $(\mathbb{K}[G/H]^{\pm})_G$.

- If there exists $h \in H$ such that $\varepsilon(h, \bar{1}) \neq 1$, then

$$(\mathbb{K}[G/H]^\pm)_G = (\mathbb{K}/2\mathbb{K})[[\bar{1}^\pm]]; \quad (\mathbb{K}[G/H]^\pm)^G = \text{Tor}_2(\mathbb{K}) \left[\sum_{\bar{g} \in G/H} \varepsilon(g, \bar{1}) \bar{g}^\pm \right],$$

where $\text{Tor}_2(\mathbb{K})$ denotes the set of 2-torsion elements of $\mathbb{K}$.

- Otherwise,

$$(\mathbb{K}[G/H]^\pm)_G = \mathbb{K}[[\bar{1}^\pm]]; \quad (\mathbb{K}[G/H]^\pm)^G = \mathbb{K} \left[\sum_{\bar{g} \in G/H} \varepsilon(g, \bar{1}) \bar{g}^\pm \right].$$

Proof We first compute $(\mathbb{K}[G/H]^\pm)_G$. For every $g \in G$, we have $\bar{g}^\pm = g \cdot (\varepsilon(g, \bar{1}) \bar{1}^\pm)$ so that the $\mathbb{K}$ -module $(\mathbb{K}[G/H]^\pm)_G$ is generated by $[\bar{1}^\pm]$. If there exists $h \in H$ such that $\varepsilon(h, \bar{1}) \neq 1$ (meaning that $\varepsilon(h, \bar{1}) = -1 \neq 1$ in $\mathbb{K}$), then, for every $\lambda \in \mathbb{K}$, we have that $\lambda[\bar{1}^\pm] = \lambda[h \cdot \bar{1}^\pm] = -\lambda[\bar{1}^\pm]$ which shows that $(\mathbb{K}[G/H]^\pm)_G = (\mathbb{K}/2\mathbb{K})[[\bar{1}^\pm]]$. If $\varepsilon(H, \bar{1}) \subset \{1\}$, then $(\mathbb{K}[G/H]^\pm)_G = \mathbb{K}[[\bar{1}^\pm]]$.

We now compute $(\mathbb{K}[G/H]^\pm)^G$. Let $x = \sum_{\bar{g} \in G/H} \lambda_{\bar{g}} \bar{g}^\pm \in (\mathbb{K}[G/H]^\pm)^G$. For every $g \in G$, the identity $g \cdot x = x$ gives $\lambda_{\bar{g}} = \varepsilon(g, \bar{1}) \lambda_{\bar{1}}$. If there exists $h \in H$ such that $\varepsilon(h, \bar{1}) \neq 1$, then

$$\varepsilon(g, \bar{1}) \lambda_{\bar{1}} = \lambda_{\bar{g}} = \lambda_{\overline{gh}} = \varepsilon(gh, \bar{1}) \lambda_{\bar{1}} = \varepsilon(g, \bar{1}) \varepsilon(h, \bar{1}) \lambda_{\bar{1}},$$

which gives $2\lambda_{\bar{1}} = 0$. Then $\lambda_{\bar{1}} \in \text{Tor}_2(\mathbb{K})$ which shows that $(\mathbb{K}[G/H]^\pm)^G = \text{Tor}_2(\mathbb{K}) \left[\sum_{\bar{g} \in G/H} \varepsilon(g, \bar{1}) \bar{g}^\pm \right]$. If $\varepsilon(H, \bar{1}) \subset \{1\}$ then $(\mathbb{K}[G/H]^\pm)^G = \mathbb{K} \left[\sum_{\bar{g} \in G/H} \varepsilon(g, \bar{1}) \bar{g}^\pm \right]$. $\square$

Lemma 2.4 In the situation of Lemma 2.3 and when $\varepsilon(H, \bar{1})$ is a subset of $\{1\}$, we define the **orbit map** $\mathcal{O}: (\mathbb{K}[G/H]^\pm)_G \rightarrow (\mathbb{K}[G/H]^\pm)^G$ by

$$\mathcal{O}([\bar{1}^\pm]) = \sum_{\bar{g} \in G/H} \varepsilon(g, \bar{1}) \bar{g}^\pm.$$

This map is an isomorphism.

Proof This is an immediate consequence of the second point of Lemma 2.3. $\square$

We can apply this lemma to our situation, by noting that for every $t \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$, the set X_t is in bijection with $\Sigma_n / \text{Stab}_{\Sigma_n}(t)$. In order to apply Lemma 2.4, we need to remove some elements of $\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$. These elements are given by tensors $t \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ such that there exists $\sigma \in \text{Stab}_{\Sigma_n}(t)$ with $\varepsilon(\sigma, t) \neq 1$. We denote by $(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^o$ the set of such elements and

$$(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r = (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}) \setminus (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^o.$$

In the case $\text{char}(\mathbb{K}) = 2$, we just have $(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r = \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$.

These sets are stable under the action of Σ_n . Indeed, if $t \in \mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}$ is such that there exists $\sigma \in \text{Stab}_{\Sigma_n}(t)$ with $\varepsilon(\sigma, t) \neq 1$, then for every $\tau \in \Sigma_n$, we have that $\tau \sigma \tau^{-1} \in \text{Stab}_{\Sigma_n}(\tau \cdot t)$ and

$$\varepsilon(\tau \sigma \tau^{-1}, \tau \cdot t) = \varepsilon(\tau, \sigma \cdot t) \varepsilon(\sigma, t) \varepsilon(\tau^{-1}, \tau \cdot t) = \varepsilon(\tau, t) \varepsilon(\sigma, t) \varepsilon(\tau^{-1}, \tau \cdot t) = \varepsilon(\sigma, t) \neq 1.$$

We deduce that the quotient $(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})/\Sigma_n$ is the disjoint union of $(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^o/\Sigma_n$ and $(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r/\Sigma_n$. We then define

$$\mathcal{S}^r(\text{PreLie}, L) = \bigoplus_{n \geq 1} (\mathbb{K}[(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r]^\pm)_{\Sigma_n} \subset \mathcal{S}(\text{PreLie}, L).$$

We deduce the following proposition.

Proposition 2.5 *If $\mathbb{K}$ is an integral domain and if L^k is a free $\mathbb{K}$ -module for every $k \in \mathbb{Z}$, then the map $\mathcal{O}: \mathcal{S}^r(\text{PreLie}, L) \rightarrow \Gamma(\text{PreLie}, L)$ is an isomorphism.*

Proof Let $\mathcal{L}$ be a basis of L composed of homogeneous elements. We adopt the same notation as before Lemma 2.3 and before the statement of Proposition 2.5. Then

$$\begin{aligned} \mathcal{S}^r(\text{PreLie}, L) &= \bigoplus_{n \geq 1} \bigoplus_{\mathfrak{t} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r / \Sigma_n} (\mathbb{K}[X_{\mathfrak{t}}]^\pm)_{\Sigma_n}; \\ \Gamma(\text{PreLie}, L) &= \bigoplus_{n \geq 1} \bigoplus_{\mathfrak{t} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n}) / \Sigma_n} (\mathbb{K}[X_{\mathfrak{t}}]^\pm)^{\Sigma_n}. \end{aligned}$$

If $\text{char}(\mathbb{K}) = 2$, then $2\mathbb{K} = 0$ and $\text{Tor}_2(\mathbb{K}) = \mathbb{K}$ so that the proposition is an immediate consequence of Lemma 2.4.

If $\text{char}(\mathbb{K}) \neq 2$, then by the first point of Lemma 2.3 and by noting that $\text{Tor}_2(\mathbb{K}) = 0$ because $\mathbb{K}$ is an integral domain, we have that $(\mathbb{K}[(\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^o]^\pm)^{\Sigma_n} = 0$. We are then reduced to analyze the orbit maps $\mathcal{O}: (\mathbb{K}[X_{\mathfrak{t}}]^\pm)_{\Sigma_n} \rightarrow (\mathbb{K}[X_{\mathfrak{t}}]^\pm)^{\Sigma_n}$ for $\mathfrak{t} \in (\mathcal{RT}(n) \otimes \mathcal{L}^{\otimes n})^r$, which are isomorphisms by the second point of Lemma 2.4.

In any case, we then obtain that $\mathcal{O}: \mathcal{S}^r(\text{PreLie}, L) \rightarrow \Gamma(\text{PreLie}, L)$ is an isomorphism. $\square$

Theorem 2.6 *Let $\mathbb{K}$ be a ring. A graded pre-Lie algebra with divided powers $L = \bigoplus_{k \in \mathbb{Z}} L^k$ over $\mathbb{K}$ comes equipped with operations, called **weighted braces**, which have the following form.*

- If $\text{char}(\mathbb{K}) = 2$, weighted braces are maps

$$-\{-, \dots, -\}_{r_1, \dots, r_n}: L^{\times n+1} \rightarrow L,$$

defined for any collection of integers $r_1, \dots, r_n \geq 0$, which satisfy all formulas of Theorem 1.3 and preserve the grading in the sense that

$$L^k \{L^{k_1}, \dots, L^{k_n}\}_{r_1, \dots, r_n} \subset L^{k+k_1 r_1 + \dots + k_n r_n}.$$

- If $\text{char}(\mathbb{K}) \neq 2$, by setting $L^{\text{ev}} = \bigoplus_{k \in \mathbb{Z}} L^{2k}$ and $L^{\text{odd}} = \bigoplus_{k \in \mathbb{Z}} L^{2k+1}$, weighted braces are maps

$$-\{\underbrace{\quad, \dots, \quad}_p, \underbrace{\quad, \dots, \quad}_q\}_{r_1, \dots, r_p, 1, \dots, 1}: L \times (L^{\text{ev}})^{\times p} \times (L^{\text{odd}})^{\times q} \rightarrow L,$$

defined for any collection of integers $p, q, r_1, \dots, r_p \geq 0$, which satisfy all formulas of Theorem 1.3 with a sign given by the Koszul sign rule (see Remark 2.7 below) and preserve the grading.

Conversely, if $\mathbb{K}$ is an integral domain, if L^k is a free $\mathbb{K}$ -module for every $k \in \mathbb{Z}$ and if L admits weighted brace operations, then L is a $\Gamma(\text{PreLie}, -)$ -algebra.

Remark 2.7 If $\text{char}(\mathbb{K}) \neq 2$, formulas (i) and (vi) of Theorem 1.3 differ by a sign given by the Koszul sign rule. The sign which appears in formula (i) is given by

$$y_{\sigma(1)}^{\otimes r_{\sigma(1)}} \otimes \cdots \otimes y_{\sigma(n)}^{\otimes r_{\sigma(n)}} \mapsto \pm y_1^{\otimes r_1} \otimes \cdots \otimes y_n^{\otimes r_n}.$$

In formula (vi), for each β_i 's and $\alpha_i^{\bullet, \bullet}$'s, the sign which appears in the relevant term is given by

$$z_1^{\otimes s_1} \otimes \cdots \otimes z_m^{\otimes s_m} \mapsto \pm z_1^{\otimes \alpha_1^{1,1}} \otimes \cdots \otimes z_m^{\otimes \alpha_m^{1,1}} \otimes \cdots \otimes z_1^{\otimes \alpha_1^{n,1}} \otimes \cdots \otimes z_m^{\otimes \alpha_m^{n,r_n}} \otimes z_1^{\otimes \beta_1} \otimes \cdots \otimes z_m^{\otimes \beta_m}.$$

These signs are induced by the permutation of the odd-degree elements between them. Since their associated weight are equal to 1, formula (vi) can still be written without rational coefficients by the same process as in the paragraph after Theorem 1.3.

In order to handle both of the cases, in the following, when taking elements with associated weights, we will tacitly suppose that if $\text{char}(\mathbb{K}) \neq 2$, then all odd-degree elements will have an associated weight equal to 1.

Proof We basically do the same thing as in [3, Proposition 5.10]. Let $x, y_1, \dots, y_n \in L$. Let $E_{x,y_1,\dots,y_n}$ be the graded $\mathbb{K}$ -module generated by $e, e_1, \dots, e_n$ with matching degrees. We have a morphism of graded modules from $\psi_{x,y_1,\dots,y_n}: E_{x,y_1,\dots,y_n} \rightarrow L$ which sends e to x and e_i to y_i for every $1 \leq i \leq n$. This gives rise by functoriality to a morphism $\Gamma(\text{PreLie}, \psi_{x,y_1,\dots,y_n}): \Gamma(\text{PreLie}, E_{x,y_1,\dots,y_n}) \rightarrow \Gamma(\text{PreLie}, L)$. We set

$$x\{y_1, \dots, y_n\}_{r_1, \dots, r_n} := l(\Gamma(\text{PreLie}, \psi_{x,y_1,\dots,y_n})(OF_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}))),$$

where l is the $\Gamma(\text{PreLie}, -)$ -algebra structure on L . One can check that all the desired formulas are satisfied.

Now suppose that $\mathbb{K}$ is an integral domain and that L admits weighted brace operations $-\{ \dots, -\}_{r_1, \dots, r_n}$. Every element in $\Gamma(\text{PreLie}, L)$ can be described as a sum of iterated monadic compositions of corollas in some basis of homogeneous elements of L . This can be proved by using Proposition 2.5 and by following the same proofs of [3, Theorem 5.1 and Lemma 5.2], which come from the computation of the monadic composition in $\Gamma(\text{PreLie}, -)$. We next pick elements $x, y_1, \dots, y_n$ of the chosen basis such that $y_i \neq y_j$ if $i \neq j$, and set

$$l(OF_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) := x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}.$$

Since every elements of $\Gamma(\text{PreLie}, L)$ can be described as a composite of corollas in $\Gamma(\text{PreLie}, -)$, we have defined $l: \Gamma(\text{PreLie}, L) \rightarrow L$. We see that this construction does not depend on the choice of the basis of the L^k 's. Indeed, we can apply the same arguments as in [3, Lemma 5.15], which only rely on computations and on the relations satisfied by weighted braces. The same proof of [3, Lemma 5.18] can also be applied to prove that the resulting morphism l endows L with a structure of a $\Gamma(\text{PreLie}, -)$ -algebra. $\square$

This theorem admits an analogue in the differential graded case.

Theorem 2.8 Every differential graded pre-Lie algebras with divided powers $L = \bigoplus_{k \in \mathbb{Z}} L^k$ admits weighted braces which satisfy the same formulas as in Theorem 2.6, and which satisfy

$$d(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) = d(x)\{y_1, \dots, y_n\}_{r_1, \dots, r_n} + \sum_{k=1}^n (-1)^{\varepsilon_k} x\{y_1, \dots, y_k, d(y_k), \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n},$$

where $\varepsilon_k = |x| + |y_1| + \dots + |y_{k-1}|$.

Conversely, if $\mathbb{K}$ is an integral domain, if L^k is a free $\mathbb{K}$ -module for every $k \in \mathbb{Z}$ and if L admits weighted brace operations which satisfy the previous identity, then L is a $\Gamma(\text{PreLie}, -)$ -algebra.

Proof Let $x, y_1, \dots, y_n \in L$. Let $E_{x, y_1, \dots, y_n}$ be the dg module generated by $e, e_1, \dots, e_n, f, f_1, \dots, f_n$ such that $|e| = |x|, |e_1| = |y_1|, \dots, |e_n| = |y_n|$ and $d(e) = f, d(e_1) = f_1, \dots, d(e_n) = f_n$. We then have a morphism $\psi_{x, y_1, \dots, y_n}: E_{x, y_1, \dots, y_n} \rightarrow L$ of dg modules defined by sending e to x , and e_i (resp f_i) to y_i (resp $d(y_i)$) for every $1 \leq i \leq n$. We set

$$x\{y_1, \dots, y_n\}_{r_1, \dots, r_n} := l(\Gamma(\text{PreLie}, \psi_{x, y_1, \dots, y_n})(\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}))),$$

where l is the $\Gamma(\text{PreLie}, -)$ -algebra structure on L . By forgetting the differentials and by applying Theorem 2.6, we have that the operations $-\{ -, \dots, - \}_{r_1, \dots, r_n}$ satisfy all the formulas of Theorem 1.3, with a sign. It only remains to prove the compatibility with the differential d :

$$\begin{aligned} d(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) &= dl(\Gamma(\text{PreLie}, \psi_{x, y_1, \dots, y_n})(\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}))) \\ &= l(\Gamma(\text{PreLie}, \psi_{x, y_1, \dots, y_n})(d\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}))), \end{aligned}$$

by commutation of d with the algebra structure l and $\psi_{x, y_1, \dots, y_n}$. Next, we claim that

$$\begin{aligned} d\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) &= \mathcal{O}F_{\sum_i r_i}(f, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) \\ &\quad + \sum_{k=1}^n \pm \mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_k, \dots, e_k}_{r_k-1}, f_k, \dots, \underbrace{e_n, \dots, e_n}_{r_n}). \end{aligned}$$

Indeed, recall that

$$\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) = \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_n)} \sigma \cdot (F_{\sum_i r_i} \otimes e \otimes e_1^{\otimes r_1} \otimes \dots \otimes e_n^{\otimes r_n}).$$

We then have

$$\begin{aligned} d\mathcal{O}F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) &= \mathcal{O}F_{\sum_i r_i}(f, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) \\ &\quad + \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_n)} \sum_{i=1}^{r_k} \pm \sigma \cdot (F_{\sum_i r_i} \otimes e \otimes e_1^{\otimes r_1} \otimes \dots \otimes e_k^{\otimes i-1} \otimes f_k \otimes e_k^{r_k-i} \otimes \dots \otimes e_n^{\otimes r_n}). \end{aligned}$$

Let $1 \leq k \leq n$. For every $1 \leq i \leq r_k$, we define $\tau_{k,i}$ as the permutation which permutes f_k with the block $e_k^{\otimes r_k-i}$. We then obtain

$$\begin{aligned} d \mathcal{O} F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) &= \mathcal{O} F_{\sum_i r_i}(f, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) \\ &+ \sum_{k=1}^n \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_n)} \sum_{i=1}^{r_k} \pm \sigma \tau_{k,i}^{-1} \cdot (F_{\sum_i r_i} \otimes e \otimes e_1^{\otimes r_1} \otimes \dots \otimes e_k^{\otimes r_k-1} \otimes f_k \otimes \dots \otimes e_n^{\otimes r_n}). \end{aligned}$$

For every $1 \leq i \leq r_k$, the permutation $\sigma \tau_{k,i}^{-1}$ is in $\text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)$. In the converse direction, if $\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)$, then there is a unique $1 \leq i \leq r_k$ such that $\tilde{\sigma} \tau_{k,i} \in \text{Sh}(1, r_1, \dots, r_n)$. We thus have proved that

$$\begin{aligned} d \mathcal{O} F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) &= \mathcal{O} F_{\sum_i r_i}(f, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) \\ &+ \sum_{k=1}^n \sum_{\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)} \pm \tilde{\sigma} \cdot (F_{\sum_i r_i} \otimes e \otimes e_1^{\otimes r_1} \otimes \dots \otimes e_k^{\otimes r_k-1} \otimes f_k \otimes \dots \otimes e_n^{\otimes r_n}), \end{aligned}$$

which gives

$$\begin{aligned} d \mathcal{O} F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) &= \mathcal{O} F_{\sum_i r_i}(f, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_n, \dots, e_n}_{r_n}) \\ &+ \sum_{k=1}^n \pm \mathcal{O} F_{\sum_i r_i}(e, \underbrace{e_1, \dots, e_1}_{r_1}, \dots, \underbrace{e_k, \dots, e_k}_{r_k-1}, f_k, \dots, \underbrace{e_n, \dots, e_n}_{r_n}). \end{aligned}$$

Applying $\Gamma(\text{PreLie}, \psi_{x, y_1, \dots, y_n})$ and l will then give the desired quantity, by definition of the weighted braces.

Suppose now that $\mathbb{K}$ is an integral domain, that L^k is free for every $k \in \mathbb{Z}$ and that L is endowed with weighted brace operations. By Theorem 2.6, and by forgetting the differential of L , we can define a morphism of graded modules $l: \Gamma(\text{PreLie}, L) \rightarrow L$ which is compatible with the monadic structure of $\Gamma(\text{PreLie}, -)$. We now prove that l commutes with the differential d . Since d commutes with the monadic structure, it is sufficient to prove the commutation with d when reducing to corollas. Let $x, y_1, \dots, y_n$ be some basis elements with $y_i \neq y_j$ if $i \neq j$ and $r_1, \dots, r_n \geq 1$. We then have

$$\begin{aligned} dl(\mathcal{O} F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) &= d(x \{y_1, \dots, y_n\}_{r_1, \dots, r_n}) \\ &= d(x) \{y_1, \dots, y_n\}_{r_1, \dots, r_n} \\ &+ \sum_{k=1}^n \pm x \{y_1, \dots, y_k, d(y_k), \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n}. \end{aligned}$$

We decompose $d(y_k)$ in the chosen basis, and write

$$d(y_k) = \sum_{\substack{i=1 \\ i \neq k}}^n \lambda_{k,i} y_i + f_k,$$

where $f_k = 0$ or $f_k \notin \text{Vect}(y_1, \dots, y_n)$ (y_k cannot appear in the decomposition of $d(y_k)$ for degree reasons). This gives

$$\begin{aligned} dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) &= d(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) \\ &= d(x)\{y_1, \dots, y_n\}_{r_1, \dots, r_n} \\ &\quad + \sum_{k=1}^n \sum_{\substack{i=1 \\ i \neq k}}^n \pm \lambda_{k,i} x\{y_1, \dots, y_k, y_i, \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n} \\ &\quad + \sum_{k=1}^n \pm x\{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n} \end{aligned}$$

and then, by using the symmetry relations,

$$\begin{aligned} dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) &= d(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) \\ &= d(x)\{y_1, \dots, y_n\}_{r_1, \dots, r_n} + \sum_{k=1}^n \sum_{\substack{i=1 \\ i \neq k}}^n \pm \lambda_{k,i} x\{y_1, \dots, y_i, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, r_i, 1, \dots, r_k-1, \dots, r_n} \\ &\quad + \sum_{k=1}^n \pm x\{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n}. \end{aligned}$$

The second sum can be simplified, depending on the parity of $|y_i|$ for every i . If $|y_i|$ is even, then by formula (iv) of Theorem 1.3, we have

$$\begin{aligned} x\{y_1, \dots, y_i, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, r_i, 1, \dots, r_k-1, \dots, r_n} \\ = (r_i + 1)x\{y_1, \dots, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, r_i+1, \dots, r_k-1, \dots, r_n}. \end{aligned}$$

If $|y_i|$ is odd, then since we have supposed that any odd-degree element has an associated weight equal to 1, we have that $r_i = 1$. By using the symmetry relation, and using that we have no 2-torsion elements since L is free and that $\mathbb{K}$ is an integral domain, we have that

$$x\{y_1, \dots, y_i, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, 1, 1, \dots, r_k-1, \dots, r_n} = 0.$$

We finally have that

$$\begin{aligned} dl(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) &= d(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) \\ &= d(x)\{y_1, \dots, y_n\}_{r_1, \dots, r_n} + \sum_{k=1}^n \sum_{\substack{i=1 \\ i \neq k}}^n \pm \delta_i \lambda_{k,i} (r_i + 1)x\{y_1, \dots, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, r_i+1, \dots, r_k-1, \dots, r_n} \\ &\quad + \sum_{k=1}^n \pm x\{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1, \dots, r_k-1, 1, \dots, r_n}, \end{aligned}$$

where $\delta_i = 0$ if $|y_i|$ is odd, and $\delta_i = 1$ otherwise.

We now compute

$$ld(\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})).$$

By the same computations as the beginning of the proof, we have that

$$\begin{aligned} d\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}) &= \mathcal{O}F_{\sum_i r_i}(d(x), \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}) \\ &+ \sum_{k=1}^n \sum_{\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)} \pm \tilde{\sigma} \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_k^{\otimes r_k-1} \otimes d(y_k) \otimes \dots \otimes y_n^{\otimes r_n}). \end{aligned}$$

We now fix $1 \leq k \leq n$. We aim to compute the sum

$$\sum_{\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)} \pm \tilde{\sigma} \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_k^{\otimes r_k-1} \otimes d(y_k) \otimes \dots \otimes y_n^{\otimes r_n}).$$

We use the decomposition of $d(y_k)$ in the chosen basis. The f_k part will precisely give

$$\mathcal{O}F_{\sum_i r_i}(x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_k, \dots, y_k}_{r_k-1}, f_k, \dots, \underbrace{y_n, \dots, y_n}_{r_n}).$$

We now look at the other terms. These terms give the sum

$$\sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_k-1, 1, \dots, r_n)} \pm \lambda_{k,i}(\tilde{\sigma} \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_k^{\otimes r_k-1} \otimes y_i \otimes \dots \otimes y_n^{\otimes r_n})).$$

By putting the only y_i with the others, we find that this sum is equal to

$$\sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_i, 1, \dots, r_k-1, \dots, r_n)} \pm \lambda_{k,i}(\tilde{\sigma} \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_i^{\otimes r_i} \otimes y_i \otimes \dots \otimes y_k^{\otimes r_k-1} \otimes \dots \otimes y_n^{\otimes r_n})).$$

We now use that for every $\tilde{\sigma} \in \text{Sh}(1, r_1, \dots, r_i, 1, \dots, r_k-1, \dots, r_n)$, there exists a unique permutation $\tau_{i,j}$, defined by inserting the last occurrence of y_i among

$$\underbrace{y_i \otimes \dots \otimes y_i}_{r_i}$$

in position j (so that $1 \leq j \leq r_i + 1$), such that $\tilde{\sigma} \tau_{i,j}^{-1} \in \text{Sh}(1, r_1, \dots, r_i + 1, \dots, r_k-1, \dots, r_n)$. In the other direction, for every $\sigma \in \text{Sh}(1, r_1, \dots, r_i + 1, \dots, r_k-1, \dots, r_n)$ and $1 \leq j \leq r_i + 1$, we have that $\sigma \tau_{i,j} \in \text{Sh}(1, r_1, \dots, r_i, 1, \dots, r_k-1, \dots, r_n)$. We thus obtain the sum

$$\begin{aligned} \sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_i+1, \dots, r_k-1, \dots, r_n)} \sum_{j=1}^{r_i+1} \pm \lambda_{k,i}(\sigma \tau_{i,j} \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_i^{\otimes r_i+1} \otimes \dots \otimes y_k^{\otimes r_k-1} \otimes \dots \otimes y_n^{\otimes r_n})). \end{aligned}$$

For a fixed $i \neq k$, we need to distinguish two cases: either $|y_i|$ is even, or $|y_i|$ is odd. In the first case, we have that this sum is

$$\sum_{\substack{i=1 \\ i \neq k}}^n \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_i + 1, \dots, r_k - 1, \dots, r_n)} \sum_{j=1}^{r_i + 1} \pm \lambda_{k,i} (\sigma \cdot (F_{\sum_i r_i} \otimes x \otimes y_1^{\otimes r_1} \otimes \dots \otimes y_i^{\otimes r_i + 1} \otimes \dots \otimes y_k^{\otimes r_k - 1} \otimes \dots \otimes y_n^{\otimes r_n})),$$

which is precisely

$$\sum_{\substack{i=1 \\ i \neq k}}^n \pm \lambda_{k,i} (r_i + 1) \mathcal{O} F_{\sum_i r_i} (x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_i, \dots, y_i}_{r_i + 1}, \dots, \underbrace{y_k, \dots, y_k}_{r_k - 1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}).$$

In the second case, since we have supposed that every odd-degree element has an associated weight equal to 1, we have that $r_i = 1$. We have that $\tau_{i,1}$ permutes the two y_i 's, which gives a sign, while $\tau_{i,2} = \text{id}$. The sum of the two obtained elements is then 0.

We thus have proved that

$$\begin{aligned} d \mathcal{O} F_{\sum_i r_i} (x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}) &= \mathcal{O} F_{\sum_i r_i} (d(x), \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}) \\ &+ \sum_{k=1}^n \sum_{\substack{i=1 \\ i \neq k}}^n \pm \delta_i \lambda_{k,i} (r_i + 1) \mathcal{O} F_{\sum_i r_i} (x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_i, \dots, y_i}_{r_i + 1}, \dots, \underbrace{y_k, \dots, y_k}_{r_k - 1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}) \\ &+ \sum_{k=1}^n \pm \mathcal{O} F_{\sum_i r_i} (x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_k, \dots, y_k}_{r_k - 1}, f_k, \dots, \underbrace{y_n, \dots, y_n}_{r_n}), \end{aligned}$$

where we have set $\delta_i = 0$ if $|y_i|$ is odd, and $\delta_i = 1$ otherwise. By definition of l , we have

$$\begin{aligned} l(d \mathcal{O} F_{\sum_i r_i} (x, \underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n})) \\ = d(x) \{y_1, \dots, y_n\}_{r_1, \dots, r_n} + \sum_{k=1}^n \sum_{\substack{i=1 \\ i \neq k}}^n \pm \delta_i \lambda_{k,i} (r_i + 1) x \{y_1, \dots, y_i, \dots, y_k, \dots, y_n\}_{r_1, \dots, r_i + 1, \dots, r_i - 1, \dots, r_n} \\ + \sum_{k=1}^n \pm x \{y_1, \dots, y_k, f_k, \dots, y_n\}_{r_1, \dots, r_k - 1, 1, \dots, r_n}, \end{aligned}$$

which proves that l commutes with d . □

We then deduce from Propositions 2.6 and 2.8 that every differential graded pre-Lie algebra with divided powers is in particular a differential graded pre-Lie algebra, with

$$x \star y = x \{y\}_1.$$

Remark 2.9 If $\mathbb{Q} \subset \mathbb{K}$ and if L is a differential graded pre-Lie algebra, then L is a differential graded pre-Lie algebra with divided powers whose weighted braces are explicitly given by

$$x\{y_1, \dots, y_n\}_{r_1, \dots, r_n} = \frac{1}{\prod_i r_i!} x\{\underbrace{y_1, \dots, y_1}_{r_1}, \dots, \underbrace{y_n, \dots, y_n}_{r_n}\}$$

in terms of symmetric braces.

Remark 2.10 Every morphism of $\Gamma(\text{PreLie}, -)$ -algebras preserves the weighted braces:

$$f(x\{y_1, \dots, y_n\}_{r_1, \dots, r_n}) = f(x)\{f(y_1), \dots, f(y_n)\}_{r_1, \dots, r_n}.$$

In order to perform infinite sums, we define the notion of a *complete* $\Gamma(\text{PreLie}, -)$ -algebra. We recall the following definition.

Definition 2.11 A *filtered* dg module is the data of a dg module L with inclusions of dg modules

$$\cdots \subset F_n L \subset F_{n-1} L \subset \cdots \subset F_1 L = L.$$

A filtered dg module is *complete* if the morphism $L \rightarrow \lim_{n \geq 1} L/F_n L$ is an isomorphism.

In general, for every filtered dg module L , the dg module $\widehat{L} = \lim_{n \geq 1} L/F_n L$ is a complete dg module with the filtration $F_n \widehat{L} = \text{Ker}(\widehat{L} \rightarrow L/F_n L)$, since $\widehat{L}/F_n \widehat{L} \simeq L/F_n L$.

We now define the notion of a filtered $\Gamma(\text{PreLie}, -)$ -algebra.

Definition 2.12 A *filtered* $\Gamma(\text{PreLie}, -)$ -algebra is a $\Gamma(\text{PreLie}, -)$ -algebra endowed with a filtration preserved by the weighted braces in the sense that

$$F_k L\{F_{k_1} L, \dots, F_{k_n} L\}_{r_1, \dots, r_n} \subset F_{k+k_1 r_1 + \cdots + k_n r_n} L.$$

A filtered $\Gamma(\text{PreLie}, -)$ -algebra is *complete* if L is complete as a filtered dg module.

If L is a filtered $\Gamma(\text{PreLie}, -)$ -algebra, then the weighted braces $-\{-, \dots, -\}_{r_1, \dots, r_n}$ induce weighted braces on the completion $\widehat{L} = \lim_{n \geq 1} L/F_n L$, which satisfy the formulas of Theorems 2.6 and 2.8, and preserve the filtration on $\widehat{L}$, so that $\widehat{L}$ forms a complete $\Gamma(\text{PreLie}, -)$ -algebra (provided that we work over a field).

2.1.2 Examples of $\Gamma(\text{PreLie}, -)$ -algebras We give examples of dg pre-Lie algebras with divided powers. The first examples are given by dg brace algebras, following the idea of the proof in the nongraded framework in [3].

Definition 2.13 A *differential graded brace algebra* is a differential graded module L endowed with brace operations

$$-\langle -, \dots, - \rangle: L^{\otimes n+1} \rightarrow L,$$

which are compatible with the differential d :

$$d(f\langle g_1, \dots, g_n \rangle) = d(f)\langle g_1, \dots, g_n \rangle + \sum_{k=1}^n \pm f\langle g_1, \dots, d(g_k), \dots, g_n \rangle,$$

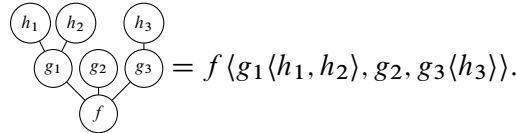
and such that $f\langle \rangle = f$ and

$$f\langle g_1, \dots, g_n \rangle \langle h_1, \dots, h_r \rangle = \sum \pm f\langle H_1, g_1\langle H_2 \rangle, \dots, H_{2n-1}, g_n\langle H_{2n} \rangle, H_{2n+1} \rangle,$$

where the sum is over all consecutive subsets $H_1 \sqcup H_2 \sqcup \dots \sqcup H_{2n+1} = \{h_1, \dots, h_r\}$, and the sign is yielded by the permutation of the g_i 's with the h_j 's.

The operad which governs brace algebras is denoted by $\mathcal{B}r$ ace, and is defined, in arity n , as the $\mathbb{K}$ -module spanned by the planar n -trees, ie trees with an order on the set of inputs for each vertex (see [3, Section 6.1] or [4, Section 2] for some details on the operad $\mathcal{B}r$ ace).

This operad allows us to represent all operations in brace algebras by the action of a planar tree, or by a planar tree labeled with the inputs. For instance, we have



Remark 2.14 Because the action of the symmetric groups on $\mathcal{B}r$ ace is free, we have that the trace map induces an isomorphism of monads $\text{Tr}: \mathcal{S}(\mathcal{B}r$ ace, $-$) $\rightarrow$ $\Gamma(\mathcal{B}r$ ace, $-$).

We have an inclusion

$$i: \mathcal{P}re\mathcal{L}ie \hookrightarrow \mathcal{B}r$$
ace

defined by the *symmetrization* of trees. Namely, i is obtained by summing over all possible ways to write a given tree t as a planar tree. For instance,

$$i\left(\begin{array}{c} (2) \quad (3) \quad (4) \\ \diagdown \quad \diagup \\ (1) \end{array}\right) = \begin{array}{c} (2) \quad (3) \quad (4) \\ \diagdown \quad \diagup \\ (1) \end{array} + \begin{array}{c} (3) \quad (2) \quad (4) \\ \diagdown \quad \diagup \\ (1) \end{array} + \begin{array}{c} (4) \quad (3) \quad (2) \\ \diagdown \quad \diagup \\ (1) \end{array} + \begin{array}{c} (2) \quad (4) \quad (3) \\ \diagdown \quad \diagup \\ (1) \end{array} + \begin{array}{c} (4) \quad (2) \quad (3) \\ \diagdown \quad \diagup \\ (1) \end{array} + \begin{array}{c} (3) \quad (4) \quad (2) \\ \diagdown \quad \diagup \\ (1) \end{array}.$$

The map i induces a morphism of monads that can be used to define a $\Gamma(\mathcal{P}re\mathcal{L}ie, -)$ -algebra structure on every dg brace algebra L , given by the composition

$$\Gamma(\mathcal{P}re\mathcal{L}ie, L) \xrightarrow{\Gamma(i, L)} \Gamma(\mathcal{B}r$$
ace, $L) \xleftarrow[\cong]{\text{Tr}} \mathcal{S}(\mathcal{B}r$ ace, $L) \xrightarrow{l} L,$

where we denote by $l: \mathcal{S}(\mathcal{P}re\mathcal{L}ie, L) \rightarrow L$ the $\mathcal{B}r$ ace-algebra structure. We aim to compute the weighted braces.

Theorem 2.15 Every dg brace algebra L is endowed with a $\Gamma(\text{PreLie}, -)$ -algebra structure. Moreover, weighted braces $-\{ \dots, -\}_{r_1, \dots, r_n}$ are explicitly given by

$$f\{g_1, \dots, g_n\}_{r_1, \dots, r_n} = \sum_{\sigma \in \text{Sh}(r_1, \dots, r_n)} \pm f(\bar{g}_{\sigma^{-1}(1)}, \dots, \bar{g}_{\sigma^{-1}(r)}),$$

where we have set $r = \sum_i r_i$ and $(\bar{g}_1, \dots, \bar{g}_r) = (\underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_n, \dots, g_n}_{r_n})$.

Proof Let L be a brace algebra. As seen before, we have a $\Gamma(\text{PreLie}, -)$ -algebra structure on L given by the composite

$$\Gamma(\text{PreLie}, L) \xrightarrow{\Gamma(i, L)} \Gamma(\text{Brace}, L) \xleftarrow[\simeq]{\text{Tr}} \mathcal{S}(\text{Brace}, L) \xrightarrow{l} L,$$

where $l: \mathcal{S}(\text{Brace}, L) \rightarrow L$ is the brace algebra structure. We now compute the weighted braces. Let $f, g_1, \dots, g_n \in L$ be homogeneous elements with $g_i \neq g_j$ whenever $i \neq j$ and $r_1, \dots, r_n \geq 0$ (recall that we have supposed that, in the situation $\text{char}(\mathbb{K}) \neq 2$, any odd-degree element has an associated weight equal to 0 or 1). We set $E = E_{f, g_1, \dots, g_n}$ and $\psi = \psi_{f, g_1, \dots, g_n}$ (see the proof of Theorem 2.8). We use the commutative diagram

$$\begin{array}{ccccc} \Gamma(\text{PreLie}, L) & \xrightarrow{\Gamma(i, L)} & \Gamma(\text{Brace}, L) & \xleftarrow[\simeq]{\text{Tr}} & \mathcal{S}(\text{Brace}, L) \xrightarrow{l} L \\ \Gamma(\text{PreLie}, \psi) \uparrow & & \Gamma(\text{Brace}, \psi) \uparrow & & \mathcal{S}(\text{Brace}, \psi) \uparrow \\ \Gamma(\text{PreLie}, E) & \xrightarrow{\Gamma(i, E)} & \Gamma(\text{Brace}, E) & \xleftarrow[\simeq]{\text{Tr}} & \mathcal{S}(\text{Brace}, E) \end{array}$$

We keep the notation $f, g_1, \dots, g_n$ for the corresponding elements in E . Then $f\{g_1, \dots, g_n\}_{r_1, \dots, r_n}$ is given by the image of

$$x = \mathcal{O}F_r(f, \underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_n, \dots, g_n}_{r_n}) \in \Gamma(\text{PreLie}, E)$$

under the composite

$$\Gamma(\text{PreLie}, E) \xrightarrow{\Gamma(\text{PreLie}, \psi)} \Gamma(\text{PreLie}, L) \xrightarrow{\Gamma(i, L)} \Gamma(\text{Brace}, L) \xleftarrow[\simeq]{\text{Tr}} \mathcal{S}(\text{Brace}, L) \xrightarrow{l} L.$$

Our goal is to compute the image of x under the bottom composite of the diagram, which is

$$\Gamma(\text{PreLie}, E) \xrightarrow{\Gamma(i, E)} \Gamma(\text{Brace}, E) \xleftarrow[\simeq]{\text{Tr}} \mathcal{S}(\text{Brace}, E) \xrightarrow{\mathcal{S}(\text{Brace}, \psi)} \mathcal{S}(\text{Brace}, L).$$

We set

$$(\bar{g}_1, \dots, \bar{g}_{r+1}) = (f, \underbrace{g_1, \dots, g_1}_{r_1}, \dots, \underbrace{g_n, \dots, g_n}_{r_n})$$

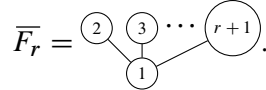
(we have added f here so that these $\bar{g}_i$'s are different from the $\bar{g}_i$'s of the theorem). We then precisely have

$$x = \sum_{\sigma \in \Sigma_{r+1} / \prod_i \Sigma_{r_i}} \pm (\sigma \cdot F_r) \otimes \bar{g}_{\sigma^{-1}(1)} \otimes \dots \otimes \bar{g}_{\sigma^{-1}(r+1)}$$

by definition of the orbit map (see Lemma 2.4). Now, because $\Sigma_{r+1}/\prod_i \Sigma_{r_i}$ is in bijection with $\text{Sh}(1, r_1, \dots, r_n)$, we can write x as

$$x = \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_n)} \pm(\sigma \cdot F_r) \otimes \bar{g}_{\sigma^{-1}(1)} \otimes \cdots \otimes \bar{g}_{\sigma^{-1}(r+1)}.$$

We now embed PreLie into Brace . The tree F_r can be seen in Brace as $\sum_{s \in \Sigma_{r+1}, s(1)=1} s \cdot \bar{F}_r$ where $\bar{F}_r$ is the planar tree



We then obtain, in $\Gamma(\text{Brace}, E)$,

$$x = \sum_{\sigma \in \text{Sh}(1, r_1, \dots, r_n)} \sum_{\substack{s \in \Sigma_{r+1} \\ s(1)=1}} \pm(\sigma s \cdot \bar{F}_r) \otimes \bar{g}_{\sigma^{-1}(1)} \otimes \cdots \otimes \bar{g}_{\sigma^{-1}(r+1)}.$$

We now need to compute $y = \text{Tr}^{-1}(x) \in \mathcal{S}(\text{Brace}, E)$. We claim that

$$y = \sum_{\substack{\omega \in \text{Sh}(1, r_1, \dots, r_n) \\ \omega(1)=1}} \pm \bar{F}_r(\bar{g}_{\omega^{-1}(1)}, \dots, \bar{g}_{\omega^{-1}(r+1)}).$$

We compute

$$\text{Tr}(y) = \sum_{\substack{\omega \in \text{Sh}(1, r_1, \dots, r_n) \\ \omega(1)=1}} \sum_{\tau \in \Sigma_{r+1}} \pm(\tau \cdot \bar{F}_r) \otimes \bar{g}_{\omega^{-1}\tau^{-1}(1)} \otimes \cdots \otimes \bar{g}_{\omega^{-1}\tau^{-1}(r+1)}.$$

The fact that $\text{Tr}(y) = x$ comes from the existence of a bijective correspondence

$$\varphi: \text{Sh}(1, r_1, \dots, r_n) \times \{s \in \Sigma_{r+1} \mid s(1) = 1\} \rightarrow \{\omega \in \text{Sh}(1, r_1, \dots, r_n) \mid \omega(1) = 1\} \times \Sigma_{r+1}.$$

Indeed, let $\sigma \in \text{Sh}(1, r_1, \dots, r_n)$ and $s \in \Sigma_{r+1}$ be such that $s(1) = 1$. We set $\tau = \sigma s$, and write the decomposition $\tau^{-1}\sigma = s^{-1} = \omega\mu$ where $\omega \in \text{Sh}(1, r_1, \dots, r_n)$ and $\mu \in \Sigma_1 \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_n}$. Since $s(1) = 1$, we have that $\omega(1) = 1$. We then set $\varphi(\sigma, s) := (\omega, \tau)$. Since the couple (ω, μ) uniquely depends on s , we have a well defined injective map φ between two sets with the same cardinal. The map φ is then a bijection.

Since we have $\bar{g}_{\mu\sigma^{-1}(i)} = \bar{g}_{\sigma^{-1}(i)}$ for every i , we obtain that $\text{Tr}(y) = x$. $\square$

Corollary 2.16 *Let $\mathcal{P}$ be a nonsymmetric dg operad with $\mathcal{P}(0) = 0$. We denote by $1 \in \mathcal{P}(1)$ the unit element of $\mathcal{P}$, by $p(q_1, \dots, q_n) = p \otimes q_1 \otimes \cdots \otimes q_n \in \mathcal{P} \circ \mathcal{P}$ and by $\gamma: \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ the operadic composition of $\mathcal{P}$. Then the dg module $\bigoplus_{r \geq 1} \mathcal{P}(r)$ admits a structure of a $\Gamma(\text{PreLie}, -)$ -algebra induced by the brace algebra structure*

$$p\langle q_1, \dots, q_n \rangle = \sum_{1 \leq i_1 < \cdots < i_n \leq r} \gamma(p(1, \dots, q_1, \dots, q_n, \dots, 1)).$$

We also set $p\langle q_1, \dots, q_n \rangle = 0$ if the operadic composition is not possible.

Proof We refer to [11] for the brace algebra structure of $\bigoplus_{r \geq 1} \mathcal{P}(r)$. It is endowed with a $\Gamma(\text{PreLie}, -)$ -algebra structure by Theorem 2.15. $\square$

In the symmetric context, we can recover an analogue of this corollary for $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$. However, the operations $-\langle -, \dots, - \rangle$ do not preserve $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$. We thus need to force the symmetry, and then to sum on every possible positions of $q_1, \dots, q_n$ in order to retrieve a $\Gamma(\text{PreLie}, -)$ -algebra structure.

Proposition 2.17 Suppose that $\mathbb{K}$ is a field and let $\mathcal{P}$ be a symmetric dg operad such that $\mathcal{P}(0) = 0$. Then $\mathcal{L}(\mathcal{P}) = \bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$ is endowed with a $\Gamma(\text{PreLie}, -)$ -algebra structure defined by

$$p\{q_1, \dots, q_n\}_{r_1, \dots, r_n} = \sum_{\sigma \in \text{Sh}(r_1, \dots, r_n)} \sum_{\substack{1 \leq i_1 < \dots < i_r \leq u \\ \omega \in \text{Sh}_*(1, \dots, \bar{s}_{\sigma^{-1}(1)}, \dots, \bar{s}_{\sigma^{-1}(r)}, \dots, 1) \\ i_1 \quad i_r}} \pm \omega \cdot \gamma(p(1, \dots, \bar{q}_{\sigma^{-1}(1)}, \dots, \bar{q}_{\sigma^{-1}(r)}, \dots, 1)),$$

for elements of homogeneous arity $p \in \mathcal{P}(m)^{\Sigma_m}$, $q_1 \in \mathcal{P}(s_1)^{\Sigma_{s_1}}, \dots, q_n \in \mathcal{P}(s_n)^{\Sigma_{s_n}}$ and where we have set

$$r = \sum_i r_i, \quad (\bar{q}_1, \dots, \bar{q}_r) = (\underbrace{q_1, \dots, q_1}_{r_1}, \dots, \underbrace{q_n, \dots, q_n}_{r_n}), \quad (\bar{s}_1, \dots, \bar{s}_r) = (\underbrace{s_1, \dots, s_1}_{r_1}, \dots, \underbrace{s_n, \dots, s_n}_{r_n}).$$

The sign is induced by the commutation of $\bar{q}_1, \dots, \bar{q}_r$ to $\bar{q}_{\sigma^{-1}(1)}, \dots, \bar{q}_{\sigma^{-1}(r)}$. We set $p\{q_1, \dots, q_n\}_{r_1, \dots, r_n} = 0$ if $u < r_1 + \dots + r_n$. The weighted brace operations are then extended to the sum $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$ by using formula (v) of Theorem 1.3.

Proof We first prove that these operations preserve $\mathcal{L}(\mathcal{P})$. Let $p \in \mathcal{P}(m)^{\Sigma_m}$, $q_1 \in \mathcal{P}(s_1)^{\Sigma_{s_1}}, \dots, q_n \in \mathcal{P}(s_n)^{\Sigma_{s_n}}$ and $r_1, \dots, r_n \geq 0$. Since we have

$$p\{q_1, \dots, q_n\}_{r_1, \dots, r_n} = p\{q_1, \dots, q_n, 1\}_{r_1, \dots, r_n, m-(r_1+\dots+r_n)},$$

we can suppose that $m = r_1 + \dots + r_n$. We then have

$$p\{q_1, \dots, q_n\}_{r_1, \dots, r_n} = \sum_{\sigma \in \text{Sh}(r_1, \dots, r_n)} \sum_{\omega \in \text{Sh}_*(\bar{s}_{\sigma^{-1}(1)}, \dots, \bar{s}_{\sigma^{-1}(r)})} \pm \omega \cdot \gamma(p(\bar{q}_{\sigma^{-1}(1)}, \dots, \bar{q}_{\sigma^{-1}(r)})).$$

Let $\mu \in \Sigma_{s_1 r_1 + \dots + s_n r_n}$. For a given $\sigma \in \text{Sh}(r_1, \dots, r_n)$, we write $\mu\omega = \tilde{\omega} \cdot \nu(\tau_1, \dots, \tau_r)$ where $\nu \in \Sigma_r$, $\tau_1 \in \Sigma_{\bar{s}_{\sigma^{-1}(1)}}, \dots, \tau_r \in \Sigma_{\bar{s}_{\sigma^{-1}(r)}}, \tilde{\omega} \in \text{Sh}_*(\bar{s}_{(\nu\sigma)^{-1}(1)}, \dots, \bar{s}_{(\nu\sigma)^{-1}(r)})$. We also have set $\nu(\tau_1, \dots, \tau_r)$ to be the composite of $\tau_1 \oplus \dots \oplus \tau_r$ with the corresponding blocks permutation given by $\nu \in \Sigma_r$. We obtain

$$\mu\omega \cdot \gamma(p(\bar{q}_{\sigma^{-1}(1)}, \dots, \bar{q}_{\sigma^{-1}(r)})) = \tilde{\omega} \cdot \gamma(p(\bar{q}_{(\nu\sigma)^{-1}(1)}, \dots, \bar{q}_{(\nu\sigma)^{-1}(r)})),$$

as $p, q_1, \dots, q_n$ are invariants. We now write $\nu\sigma = \tilde{\sigma} \cdot (\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_r)$ where $\tilde{\sigma} \in \text{Sh}(r_1, \dots, r_n)$, $\tilde{\tau}_1 \in \Sigma_{r_1}, \dots, \tilde{\tau}_r \in \Sigma_{r_n}$. We thus obtain

$$\mu\omega \cdot \gamma(p(\bar{q}_{\sigma^{-1}(1)}, \dots, \bar{q}_{\sigma^{-1}(r)})) = \tilde{\omega} \cdot \gamma(p(\bar{q}_{\tilde{\sigma}^{-1}(1)}, \dots, \bar{q}_{\tilde{\sigma}^{-1}(r)})).$$

We thus have proved that

$$\begin{aligned} \mu \cdot (p\{q_1, \dots, q_n\}_{r_1, \dots, r_n}) &= \sum_{\sigma \in \text{Sh}(r_1, \dots, r_n)} \sum_{\tilde{\omega} \in \text{Sh}_*(\bar{s}_{\tilde{\sigma}^{-1}(1)}, \dots, \bar{s}_{\tilde{\sigma}^{-1}(r)})} \pm \tilde{\omega} \cdot \gamma(p(\bar{q}_{\tilde{\sigma}^{-1}(1)}, \dots, \bar{q}_{\tilde{\sigma}^{-1}(r)})) \\ &= p\{q_1, \dots, q_n\}_{r_1, \dots, r_n}. \end{aligned}$$

The operations $-\{ -, \dots, - \}_{r_1, \dots, r_n}$ then preserve $\mathcal{L}(\mathcal{P})$.

We now prove formulas of Theorem 2.6. We can immediately check that formulas (i)–(v) are satisfied. The commutation with the differential is also satisfied since the operadic structure is compatible with the differential. It remains to prove formula (vi) of Theorem 2.6. We first note that the theorem holds if $\mathbb{K} = \mathbb{Q}$. Indeed, in that case, the trace map $\text{Tr}: \mathcal{S}(\text{PreLie}, -) \rightarrow \Gamma(\text{PreLie}, -)$ induces an isomorphism of monads. We thus only need to prove that the $\Gamma(\text{PreLie}, -)$ -algebra structure is induced by a pre-Lie algebra structure. This presumed pre-Lie algebra structure is given by

$$p\{q\}_1 = \sum_{i=1}^m \sum_{\omega \in \text{Sh}_*(1, \dots, \eta_i, \dots, 1)} \omega \cdot (p \circ_i q),$$

where $p \in \mathcal{P}(m)^{\Sigma_m}$ and $q \in \mathcal{P}(n)^{\Sigma_n}$. We then recover the pre-Lie algebra structure given in Section 5.3.16 of [14]. We now need to prove that the operations $-\{-, \dots, -\}_{1, \dots, 1}$ coincide with the symmetric braces $-\{-, \dots, -\}$ induced by the pre-Lie operation (see Definition 1.1). It is equivalent to prove

$$p\{q_1, \dots, q_{n+1}\}_{1, \dots, 1} = p\{q_1, \dots, q_n\}_{1, \dots, 1} \{q_{n+1}\}_1 - \sum_{k=1}^n \pm p\{q_1, \dots, q_k \{q_{n+1}\}_1, \dots, q_n\}_{1, \dots, 1}.$$

This follows from the associativity of the operadic composition. More precisely, $p\{q_1, \dots, q_n\}_{1, \dots, 1} \{q_{n+1}\}_1$ is composed of two types of operadic composition. Either q_{n+1} is in the same level as $q_1, \dots, q_n$, which will give exactly $p\{q_1, \dots, q_{n+1}\}_{1, \dots, 1}$ by definition, or q_{n+1} will be attached to one of $q_1, \dots, q_n$. These last terms are removed in order to retrieve $p\{q_1, \dots, q_{n+1}\}_{1, \dots, 1}$.

We now prove the general case. Consider elements $p, q_1, \dots, q_n, f_1, \dots, f_m$ in $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$ which are homogeneous in degrees and in arities, and $r_1, \dots, r_n, s_1, \dots, s_m \geq 0$. We need to compute

$$p\{q_1, \dots, q_n\}_{r_1, \dots, r_n} \{f_1, \dots, f_m\}_{s_1, \dots, s_m}$$

and to find the right-hand side of Theorem 2.6. We consider the symmetric sequence $M_{\mathbb{K}}$, defined over any ring $\mathbb{K}$, spanned by abstract variables $P, Q_1, \dots, Q_n, F_1, \dots, F_m$ of the same arities and degrees as $p, q_1, \dots, q_n, f_1, \dots, f_m$ and endowed with a trivial action of the symmetric groups. We have an obvious morphism of symmetric sequences $M_{\mathbb{K}} \rightarrow \mathcal{P}$ which sends P to p , the Q_i 's to the q_i 's and the F_j 's to the f_j 's. We thus have a unique morphism of operads $\mathcal{F}(M_{\mathbb{K}}) \rightarrow \mathcal{P}$ which extends the morphism $M_{\mathbb{K}} \rightarrow \mathcal{P}$, where $\mathcal{F}(M_{\mathbb{K}})$ is the free operad generated by the symmetric sequence $M_{\mathbb{K}}$. Because the presumed $\Gamma(\text{PreLie}, -)$ -algebra structure is written in terms of the operadic composition, if formula (vi) of Theorem 2.6 holds for $\mathcal{F}(M_{\mathbb{K}})$, then it holds also for $\mathcal{P}$.

We prove first that the formula is satisfied for $\mathcal{F}(M_{\mathbb{Z}})$. Since the morphism of rings $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces an injective morphism of operads $\mathcal{F}(M_{\mathbb{Z}}) \hookrightarrow \mathcal{F}(M_{\mathbb{Q}})$ and since the relation (vi) of Theorem 2.6 is true in $\mathcal{F}(M_{\mathbb{Q}})$, then it is also satisfied in $\mathcal{F}(M_{\mathbb{Z}})$. By using the morphism $\mathbb{Z} \rightarrow \mathbb{K}$ which gives rise to a morphism of operads $\mathcal{F}(M_{\mathbb{Z}}) \rightarrow \mathcal{F}(M_{\mathbb{K}})$, we find that it is also satisfied in $\mathcal{F}(M_{\mathbb{K}})$.

We thus have weighted braces operation on $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$. Since we work on a field $\mathbb{K}$, by Theorem 2.8, it implies that this structure endows $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$ with a structure of a $\Gamma(\text{PreLie}, -)$ -algebra. $\square$

Corollary 2.18 Suppose that $\mathbb{K}$ is a field and let $\mathcal{P}$ be a dg operad such that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{K}$. Then $\prod_{r \geq 2} \mathcal{P}(r)^{\Sigma_r}$ is a complete $\Gamma(\text{PreLie}, -)$ -algebra.

Proof Let $\mathcal{L}(\mathcal{P}) = \bigoplus_{r \geq 2} \mathcal{P}(r)^{\Sigma_r}$. Note that $\mathcal{L}(\mathcal{P})$ is a sub $\Gamma(\text{PreLie}, -)$ -algebra of $\bigoplus_{r \geq 1} \mathcal{P}(r)^{\Sigma_r}$. We have a filtration on it given by $F_k \mathcal{L}(\mathcal{P}) = \bigoplus_{r \geq k+1} \mathcal{P}(r)^{\Sigma_r}$ which is preserved by the weighted brace operations. The completion with respect to this filtration is exactly $\prod_{r \geq 2} \mathcal{P}(r)^{\Sigma_r}$. By the remark after Definition 2.12, we have weighted brace operations on $\prod_{r \geq 2} \mathcal{P}(r)^{\Sigma_r}$. Since we work over a field, this endows $\prod_{r \geq 2} \mathcal{P}(r)^{\Sigma_r}$ with a structure of a $\Gamma(\text{PreLie}, -)$ -algebra by Theorem 2.8. $\square$

2.2 The gauge group

We can now define an analogue of the circular product given in [6] using the weighted brace operations. Before doing so, we adopt the following notation. Let L be a complete $\Gamma(\text{PreLie}, -)$ -algebra and $L_+ = \mathbb{K}1 \oplus L$. We extend the weighted braces $-\{-, \dots, -\}_{r_1, \dots, r_n} : L^{\times n+1} \rightarrow L$ on $L_+ \times L^{\times n}$ by setting

$$1\{y_1, \dots, y_n\}_{r_1, \dots, r_n} = \begin{cases} y_i & \text{if } r_i = 1 \text{ and } \forall k \neq i, r_k = 0, \\ 0 & \text{if } r_1 + \dots + r_n > 1, \end{cases}$$

for every $y_1, \dots, y_n \in L$. We can check that all the formulas from Theorem 1.3 are still satisfied if we take $x \in L_+$.

Definition 2.19 Let $\alpha \in L_+$ and $\mu \in L^0$. We set

$$\alpha \odot (1 + \mu) = \sum_{n=0}^{+\infty} \alpha \{\mu\}_n.$$

This quantity is well defined since L is complete, and because $1\{y\}_n = 0$ as soon as $n \geq 2$.

By applying this definition in the case $\mathbb{Q} \subset \mathbb{K}$ and using the weighted braces given by Remark 2.9, we retrieve the usual circular product given in [6].

Remark 2.20 One can see that we have $1 \odot (1 + \mu) = 1 + \mu = (1 + \mu) \odot 1$ so that 1 is a unit element for $\odot$. We thus have

$$\forall \mu, \nu \in L^0, \quad (1 + \mu) \odot (1 + \nu) = 1 + \nu + \sum_{n=0}^{+\infty} \mu \{\nu\}_n,$$

which shows that $\odot$ preserves $1 + L^0$.

Lemma 2.21 The circular product $\odot$ is associative, in the sense that for all $\alpha \in L_+$ and $\mu, \nu \in L^0$,

$$(\alpha \odot (1 + \mu)) \odot (1 + \nu) = \alpha \odot ((1 + \mu) \odot (1 + \nu)).$$

Proof Let $\alpha \in L_+$ and $\mu, \nu \in L^0$. We first have

$$(\alpha \odot (1 + \mu)) \odot (1 + \nu) = \left(\sum_{n=0}^{+\infty} \alpha \{\mu\}_n \right) \odot (1 + \nu) = \sum_{n,p=0}^{+\infty} \alpha \{\mu\}_n \{\nu\}_p.$$

On the other hand, we have

$$\alpha \odot ((1 + \mu) \odot (1 + \nu)) = \alpha \odot \left(1 + \nu + \sum_{n=0}^{+\infty} \mu\{v\}_n\right) = \sum_{p=0}^{+\infty} \alpha\left\{\nu + \sum_{n=0}^{+\infty} \mu\{v\}_n\right\}_p.$$

We thus need to prove that

$$\sum_{n,p=0}^{+\infty} \alpha\{\mu\}_n\{v\}_p = \sum_{p=0}^{+\infty} \alpha\left\{\nu + \sum_{n=0}^{+\infty} \mu\{v\}_n\right\}_p.$$

To prove this identity, we use formula (vi) of Theorem 1.3:

$$\alpha\{\mu\}_n\{v\}_p = \sum_{p=\beta+\sum_{i=1}^n \alpha^i} \frac{1}{n!} \alpha\{\mu\{v\}_{\alpha^1}, \dots, \mu\{v\}_{\alpha^n}, v\}_{1,\dots,1,\beta},$$

which gives

$$\sum_{n,p=0}^{+\infty} \alpha\{\mu\}_n\{v\}_p = \sum_{n=0}^{+\infty} \sum_{p=0}^{+\infty} \sum_{\beta=0}^p \sum_{p-\beta=\sum_{i=1}^n \alpha^i} \frac{1}{n!} \alpha\{\mu\{v\}_{\alpha^1}, \dots, \mu\{v\}_{\alpha^n}, v\}_{1,\dots,1,\beta}.$$

In this sum, because of the symmetry, some terms occur several times. For a given p and β , we count the number of partitions of $p - \beta = \alpha^1 + \dots + \alpha^n$ of the particular form $r_1 \tilde{\alpha}^1 + \dots + r_q \tilde{\alpha}^q$. We get $n(\tilde{\alpha}^1, \dots, \tilde{\alpha}^n) = n!/(r_1! \dots r_q!)$ for this number. We then have

$$\frac{1}{r_1! \dots r_q!} \alpha\{\mu\{v\}_{\tilde{\alpha}^1}, \dots, \mu\{v\}_{\tilde{\alpha}^1}, \dots, \mu\{v\}_{\tilde{\alpha}^q}, \dots, \mu\{v\}_{\tilde{\alpha}^q}, v\}_{1,\dots,1,\beta} = \alpha\{\mu\{v\}_{\tilde{\alpha}^1}, \dots, \mu\{v\}_{\tilde{\alpha}^q}, v\}_{r_1,\dots,r_q,\beta}.$$

We conclude by formula (v) of Theorem 1.3. $\square$

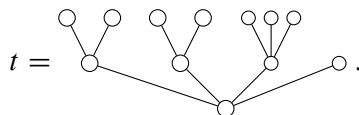
We now need to find an explicit inverse for a given element $1 - \mu$ with $\mu \in L^0$.

Definition 2.22 Let t be a nonlabeled tree with n vertices and $\mu \in L^0$. We set

$$\mathcal{O}t(\mu) = \gamma(\mathcal{O}t(\mu^{\otimes n})),$$

for some choice of labeling of t .

Because $\mathcal{O}$ is Σ -invariant, this quantity does not depend on the choice of a labeling for t . For example, let t be the nonlabeled tree



Then

$$\mathcal{O}t(\mu) = \mu\{\mu\{\mu\}_2, \mu\{\mu\}_3, \mu\}_{2,1,1}.$$

Lemma 2.23 For every $\mu \in L^0$, the element $1 - \mu$ has an inverse in $1 + L^0$ for the circular product $\odot$ given by

$$(1 - \mu)^{\odot -1} = 1 + \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu),$$

where rRT^* is the set of trees without any labeling and with at least one vertex.

Proof We first see that this defines a right-inverse for $1 - \mu$. Indeed, we first have that

$$(1 - \mu) \odot \left(1 + \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu)\right) = 1 + \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu) - \sum_{k=0}^{+\infty} \mu \left\{ \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu) \right\}_k.$$

Then, as every $t \in \text{rRT}^*$ can be uniquely described by its root and branches, we have that every term in the first sum at the right-hand side can be uniquely described by an element from the second sum, and vice versa. Formula (v) from Theorem 1.3 thus give the result.

We now need to prove that it is a left inverse, which is slightly more difficult. We compute

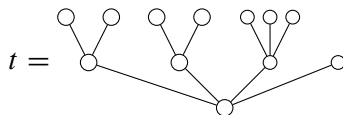
$$\left(1 + \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu)\right) \odot (1 - \mu) = 1 - \mu + \sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu) + \sum_{k \geq 1} \left(\sum_{t \in \text{rRT}^*} \mathcal{O}t(\mu) \right) \{-\mu\}_k.$$

We focus on one term $\mathcal{O}t(\mu)$ from the first sum, for some tree $t \in \text{rRT}^*$. Recall that a vertex of t is called a *leaf* if it is not the root of t and if it is connected to one and only one other vertex in t . We denote by m_t the number of leaves.

If $m_t = 0$, then t is the trivial tree: $\mathcal{O}t(\mu) = \mu$. This term does not appear in the second sum (because $k \geq 1$) and vanishes with $-\mu$.

If $m_t \neq 0$, the idea is to fix a number $1 \leq k \leq m_t$, and to see which trees we can obtain if we remove k leaves of t . These trees will occur in the second sum and give $(-1)^k \mathcal{O}t(\mu)$ by adding k copies of $-\mu$.

Let X_t be the set of leaves of t . Let $X_{t,k}$ be the set of nonordered subsets of X_t with k elements. When we remove k leaves, we need to take into account that we can obtain the same tree by removing a different nonordered set of k leaves. For example, if we take the previous tree



and if we look at the first branch, removing the vertex at the left gives the same tree as removing the vertex at the right.

Let $t_k^1, \dots, t_k^{p_k}$ be all the different trees that we can get from t by removing k leaves. We denote by $X_{t,k}^{t_k^i}$ the subset of $X_{t,k}$ formed by all the vertices that lead to t_k^i when removing them from t . We then have a disjoint union $X_{t,k} = \bigsqcup_{i=1}^{p_k} X_{t,k}^{t_k^i}$.

Each terms $\mathcal{O}t_k^i(\mu) \{-\mu\}_k$ will then give, among other terms, $(-1)^k \text{Card}(X_{t,k}^{t_k^i}) \mathcal{O}t(\mu)$. When we take the sum over i , we obtain $(-1)^k \text{Card}(X_{t,k}) \mathcal{O}t(\mu) = (-1)^k \binom{m_t}{k} \mathcal{O}t(\mu)$. By taking the sum over k , we therefore obtain $-\mathcal{O}t(\mu)$ which vanishes with $\mathcal{O}t(\mu)$ given by the first sum. $\square$

From Lemmas 2.21 and 2.23, we deduce assertion (i) of Theorem A:

Theorem 2.24 *The triple $G = (1 + L^0, \odot, 1)$ is a group called the gauge group of L .*

2.3 Maurer–Cartan elements and the Deligne groupoid

We now aim to prove assertion (ii) of Theorem A. We first make explicit the definition of the Maurer–Cartan set.

Definition 2.25 Let L be a dg $\Gamma(\text{PreLie}, -)$ -algebra. A given $\alpha \in L^1$ is a *Maurer–Cartan element* if it satisfies the *Maurer–Cartan equation*:

$$d(\alpha) + \alpha\{\alpha\}_1 = 0.$$

We let $\mathcal{MC}(L)$ to be the set of all Maurer–Cartan elements of L .

Remark 2.26 In the case $\mathbb{Q} \subset \mathbb{K}$, we retrieve the classical definition:

$$d(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0,$$

written with the dg Lie algebra structure on L .

As in the case of characteristic zero, we expect the gauge group to act on the Maurer–Cartan set. Before seeing that, we define a new operation.

Definition 2.27 Let $\alpha \in L_+$, $\beta \in L$ and $1 + \mu \in G$. We set

$$\alpha \odot (1 + \mu; \beta) = \sum_{n=0}^{+\infty} \alpha\{\mu, \beta\}_{n,1}.$$

Lemma 2.28 We have

$$\begin{aligned} (\alpha \odot (1 + \mu))\{\beta\}_1 &= \alpha \odot (1 + \mu; \beta + \mu\{\beta\}_1), \\ \alpha\{\beta\}_1 \odot (1 + \mu) &= \alpha \odot (1 + \mu; \beta \odot (1 + \mu)), \\ d(\alpha \odot (1 + \mu)) &= d(\alpha) \odot (1 + \mu) + (-1)^{|\alpha|} \alpha \odot (1 + \mu; d(\mu)). \end{aligned}$$

Proof By applying formula (vi) of Theorem 1.3, we find that

$$\begin{aligned} (\alpha \odot (1 + \mu))\{\beta\}_1 &= \sum_{n=0}^{+\infty} \alpha\{\mu\}_n \{\beta\}_1 = \sum_{n=0}^{+\infty} \alpha\{\mu, \beta\}_{n,1} + \sum_{n=1}^{+\infty} \alpha\{\mu, \mu\{\beta\}_1\}_{n-1,1} \\ &= \sum_{n=0}^{+\infty} \alpha\{\mu, \beta + \mu\{\beta\}_1\}_{n,1} = \alpha \odot (1 + \mu; \beta + \mu\{\beta\}_1), \end{aligned}$$

as well as

$$\begin{aligned} \alpha\{\beta\}_1 \odot (1 + \mu) &= \sum_{m=0}^{+\infty} \alpha\{\beta\}_1 \{\mu\}_m \\ &= \sum_{p,q=0}^{+\infty} \alpha\{\beta\{\mu\}_p, \mu\}_{1,q} \\ &= \alpha \odot (1 + \mu; \beta \odot (1 + \mu)). \end{aligned}$$

Finally, by using the compatibility of d with weighted braces, we obtain

$$\begin{aligned} d(\alpha \odot (1 + \mu)) &= \sum_{n=0}^{+\infty} d(\alpha)\{\mu\}_n + (-1)^{|\alpha|} \sum_{n=1}^{+\infty} \alpha\{\mu, d(\mu)\}_{n-1,1} \\ &= d(\alpha) \odot (1 + \mu) + (-1)^{|\alpha|} \alpha \odot (1 + \mu; d(\mu)). \end{aligned}$$

□

We can now prove assertion (ii) of Theorem A.

Theorem 2.29 *The gauge group G acts on the Maurer–Cartan set $\mathcal{MC}(L)$ by*

$$(1 + \mu) \cdot \alpha = (\alpha + \mu\{\alpha\}_1 - d(\mu)) \odot (1 + \mu)^{\odot -1}$$

for all $(1 + \mu) \in G$ and $\alpha \in \mathcal{MC}(L)$.

Proof We first need to prove that $\beta = (1 + \mu) \cdot \alpha$ is indeed a Maurer–Cartan element. For this, we first remark that applying d on each side of $d(\mu) = \alpha + \mu\{\alpha\}_1 - \beta \odot (1 + \mu)$, and by using that $d(\alpha) = -\alpha\{\alpha\}_1$ and the previous lemma, we have

$$d(\beta) \odot (1 + \mu) = -\alpha\{\alpha\}_1 - \mu\{\alpha\{\alpha\}_1\}_1 + d(\mu)\{\alpha\}_1 + \beta \odot (1 + \mu; d(\mu)).$$

Moreover, again by the previous lemma, we have

$$\begin{aligned} d(\mu)\{\alpha\}_1 &= \alpha\{\alpha\}_1 + \mu\{\alpha\}_1\{\alpha\}_1 - \beta \odot (1 + \mu)\{\alpha\}_1 \\ &= \alpha\{\alpha\}_1 + \mu\{\alpha\{\alpha\}_1\}_1 + \mu\{\alpha, \alpha\}_{1,1} - \beta \odot (1 + \mu; \alpha) - \beta \odot (1 + \mu; \mu\{\alpha\}_1). \end{aligned}$$

Then

$$d(\beta) \odot (1 + \mu) = \beta \odot (1 + \mu; d(\mu)) - \beta \odot (1 + \mu; \alpha) - \beta \odot (1 + \mu; \mu\{\alpha\}_1) + \mu\{\alpha, \alpha\}_{1,1}.$$

We note that $\mu\{\alpha, \alpha\}_{1,1} = 0$. Indeed, if we take the notation of the proof of Theorem 2.6, we have that $\mu\{\alpha, \alpha\}_{1,1} = \Gamma(\text{PreLie}, \psi_{\mu, \alpha, \alpha})(\mathcal{OF}_2(e, e_1, e_2))$ where $|e| = 0$ and $|e_1| = |e_2| = 1$ are formal elements. We explicitly have that

$$\begin{aligned} \mathcal{OF}_2(e, e_1, e_2) &= F_2(e, e_1, e_2) + ((12).F_2)(e_1, e, e_2) - ((13).F_2)(e_2, e_1, e) \\ &\quad - F_2(e, e_2, e_1) - ((12).F_2)(e_2, e, e_1) + ((13).F_2)(e_1, e_2, e). \end{aligned}$$

Applying $\psi_{\mu, \alpha, \alpha}$ will then give $\mu\{\alpha, \alpha\}_{1,1} = 0$. We then finally have

$$d(\beta) \odot (1 + \mu) = -\beta \odot (1 + \mu; \beta \odot (1 + \mu)).$$

By the previous lemma, this gives

$$d(\beta) \odot (1 + \mu) = -\beta\{\beta\}_1 \odot (1 + \mu)$$

and then $(d(\beta) + \beta\{\beta\}_1) \odot (1 + \mu) = 0$, that is to say $d(\beta) + \beta\{\beta\}_1 = 0$ by composing with $(1 + \mu)^{\odot -1}$ on the right. We thus have proved that $\beta \in \mathcal{MC}(L)$.

We now need to check that we have indeed an action of G on $\mathcal{MC}(L)$. We have that $1 + 0$ acts trivially on $\mathcal{MC}(L)$, so we just need to prove that $((1 + \nu) \odot (1 + \mu)) \cdot \alpha = (1 + \nu) \cdot ((1 + \mu) \cdot \alpha)$.

By hypothesis, we have

$$d(\mu) = \alpha + \mu\{\alpha\}_1 - \beta \odot (1 + \mu),$$

$$d(\nu) = \beta + \nu\{\beta\}_1 - \gamma \odot (1 + \nu).$$

Let $1 + \lambda = (1 + \nu) \odot (1 + \mu) = 1 + \mu + \nu \odot (1 + \mu)$. We compute

$$\begin{aligned} \alpha + \lambda\{\alpha\}_1 - \gamma \odot (1 + \lambda) &= \alpha + \mu\{\alpha\}_1 + \nu \odot (1 + \mu)\{\alpha\}_1 + d(\nu) \odot (1 + \mu) - \beta \odot (1 + \mu) - \nu\{\beta\}_1 \odot (1 + \mu) \\ &= d(\mu) + d(\nu) \odot (1 + \mu) + \nu \odot (1 + \mu; \alpha) + \nu \odot (1 + \mu; \mu\{\alpha\}_1) - \nu \odot (1 + \mu; \beta \odot (1 + \mu)) \end{aligned}$$

by the previous lemma. We then have

$$\alpha + \lambda\{\alpha\}_1 - \gamma \odot (1 + \nu) \odot (1 + \mu) = d(\mu) + d(\nu) \odot (1 + \mu) + \nu \odot (1 + \mu; d(\mu)) = d(\lambda). \quad \square$$

We end this section with the definition of the *Deligne groupoid*.

Proposition-Definition 2.30 *Let L be a complete $\Gamma(\text{PreLie}, -)$ -algebra. We let $\text{Deligne}(L)$ be the category with $\mathcal{MC}(L)$ as set of objects and $\text{Mor}_{\text{Deligne}(L)}(\alpha, \beta) = \{(1 + \mu) \in G \mid (1 + \mu) \cdot \alpha = \beta\}$. Then $\text{Deligne}(L)$ is a groupoid called the **Deligne groupoid** of L .*

Proof It is a corollary of the previous theorem. $\square$

2.4 An integral Goldman–Millson theorem

We conclude this part with an analogue of the Goldman–Millson theorem. This theorem allows us to give a link between two particular groupoids when changing a dg Lie algebra L to another one $\bar{L}$ which is quasi-isomorphic to L (see [12, Section 2.4]).

Let A be a local artinian K -algebra with maximal ideal $\mathfrak{m}_A$, where K is the field of fractions of some noetherian integral domain $\mathbb{K}$. Let L be a $\Gamma(\text{PreLie}, -)$ -algebra (without any convergence hypothesis). If $\otimes$ denotes the tensor product over $\mathbb{K}$, then $L \otimes A$ is also a $\Gamma(\text{PreLie}, -)$ -algebra with the definitions

$$(L \otimes A)^k = L^k \otimes A,$$

$$\gamma(\text{Ot}(x_1 \otimes a_1, \dots, x_n \otimes a_n)) = \gamma(\text{Ot}(x_1, \dots, x_n)) \otimes a_1 \cdots a_n,$$

$$d(x \otimes a) = d(x) \otimes a.$$

To retrieve our convergence hypothesis, we can consider the sub $\Gamma(\text{PreLie}, -)$ -algebra $L \otimes \mathfrak{m}_A$. This $\Gamma(\text{PreLie}, -)$ -algebra has a filtration given by

$$F_n(L \otimes \mathfrak{m}_A) = L \otimes \mathfrak{m}_A^n,$$

which is 0 for n big enough, because $\mathfrak{m}_A$ is nilpotent. In particular, our series will be reduced to finite sums.

Let $\text{Deligne}(L, A) = \text{Deligne}(L \otimes \mathfrak{m}_A)$ the associated Deligne groupoid. As in [12, Section 2.3], we remark that $\text{Deligne}(-, -)$ defines a bifunctor such that, for all morphisms of $\Gamma(\text{PreLie}, -)$ -algebras $\varphi: L \rightarrow \bar{L}$ and for all morphisms of algebras $\psi: A \rightarrow \bar{A}$, we have the diagram

$$\begin{array}{ccc} \text{Deligne}(L, A) & \xrightarrow{\varphi_*} & \text{Deligne}(\bar{L}, A) \\ \psi_* \downarrow & & \downarrow \psi_* \\ \text{Deligne}(L, \bar{A}) & \xrightarrow{\varphi_*} & \text{Deligne}(\bar{L}, \bar{A}) \end{array}$$

which is commutative.

We can now prove Theorem B.

Theorem 2.31 *Let $\mathbb{K}$ be a noetherian integral domain and K its field of fractions. Let L and $\bar{L}$ be two positively graded $\Gamma(\text{PreLie}, -)$ -algebras. Let $\varphi: L \rightarrow \bar{L}$ be a morphism of $\Gamma(\text{PreLie}, -)$ -algebras such that $H^0(\varphi)$ and $H^1(\varphi)$ are isomorphisms, and $H^2(\varphi)$ is a monomorphism. Then for every local artinian K -algebras A , the induced functor $\varphi_*: \text{Deligne}(L, A) \rightarrow \text{Deligne}(\bar{L}, A)$ is an equivalence of groupoids.*

Proof The proof is close to the one given in [12, Sections 2.5–2.11]. By the same arguments as in [12, Section 2.5], we are reduced to prove the following. Let A be a local artinian K -algebra with maximal ideal $\mathfrak{m}_A$ and $\mathfrak{J} \subset A$ be an ideal such that $\mathfrak{J} \cdot \mathfrak{m}_A = 0$. Then if the theorem holds for $A/\mathfrak{J}$, then it also holds for A .

Our first goal is to prove the same proposition given in [12, Section 2.6], which constructs three obstruction maps o_2, o_1 and o_0 . Let $\pi_*: L \otimes A \rightarrow L \otimes A/\mathfrak{J}$ be the map induced by the canonical projection $\pi: A \rightarrow A/\mathfrak{J}$.

We first define $o_2: \text{Obj}(\text{Deligne}(L, A/\mathfrak{J})) \rightarrow H^2(L \otimes \mathfrak{J})$ which is such that $o_2(\omega) = 0$ if and only if there exists $\tilde{\omega} \in \text{Obj}(\text{Deligne}(L, A))$ with $\pi_*(\tilde{\omega}) = \omega$.

Let $\omega \in \text{Obj}(\text{Deligne}(L, A/\mathfrak{J})) \subset \mathfrak{m}_A/\mathfrak{J}$. Let $\tilde{\omega} \in L^1 \otimes \mathfrak{m}_A$ be such that

$$\pi_*(\tilde{\omega}) = \omega \quad \text{and} \quad \mathcal{Q}(\tilde{\omega}) = d(\tilde{\omega}) + \tilde{\omega}\{\tilde{\omega}\}_1.$$

Then $\pi_*(\mathcal{Q}(\tilde{\omega})) = 0$ to that $\mathcal{Q}(\tilde{\omega}) \in L^2 \otimes \mathfrak{J}$. By using that $\mathfrak{J} \cdot \mathfrak{m}_A = 0$, we obtain

$$d(\mathcal{Q}(\tilde{\omega})) = d(\tilde{\omega})\{\tilde{\omega}\}_1 - \tilde{\omega}\{d(\tilde{\omega})\}_1 = -\tilde{\omega}\{\tilde{\omega}\}_1\{\tilde{\omega}\}_1 + \tilde{\omega}\{\tilde{\omega}\{\tilde{\omega}\}_1\}_1 = \tilde{\omega}\{\tilde{\omega}, \tilde{\omega}\}_{1,1} = 0.$$

This implies that $\mathcal{Q}(\tilde{\omega}) \in Z^2(L \otimes \mathfrak{J})$. Let $\tilde{\omega}' \in L^1 \otimes \mathfrak{m}_A$ be some other element such that $\pi_*(\tilde{\omega}') = \omega$. In particular, $\tilde{\omega} - \tilde{\omega}' \in L^1 \otimes \mathfrak{J}$. We then have, using again that $\mathfrak{J} \cdot \mathfrak{m}_A = 0$,

$$\mathcal{Q}(\tilde{\omega}') - \mathcal{Q}(\tilde{\omega}) = d(\tilde{\omega}' - \tilde{\omega}) + (\tilde{\omega}' - \tilde{\omega})\{\tilde{\omega}' - \tilde{\omega}\}_1 + \tilde{\omega}\{\tilde{\omega}' - \tilde{\omega}\}_1 + (\tilde{\omega}' - \tilde{\omega})\{\tilde{\omega}\}_1 = d(\tilde{\omega}' - \tilde{\omega}).$$

We then let $o_2(\omega)$ to be the class of $\mathcal{Q}(\tilde{\omega})$ in $H^2(L \otimes \mathfrak{J})$. Suppose that $o_2(\omega) = 0$. Then by definition, there exists some $\psi \in L^1 \otimes \mathfrak{J}$ such that $\mathcal{Q}(\tilde{\omega}) = d(\psi)$. We can check that $\tilde{\omega}' := \tilde{\omega} - \psi \in \text{Obj}(\text{Deligne}(L, A))$ and $\pi_*(\tilde{\omega}') = \omega$. In the converse direction, we obviously have that if $\omega = \pi_*(\tilde{\omega})$ with $\tilde{\omega} \in \text{Obj}(\text{Deligne}(L, A))$, then $o_2(\omega) = 0$.

We now prove the following analogue of the lemma given in [12, Section 2.8]. For all $\alpha \in L^1 \otimes \mathfrak{m}_A$, $\eta \in L^0 \otimes \mathfrak{m}_A$ and $u \in L^0 \otimes \mathfrak{I}$, we have

$$(1 + u + \eta) \cdot \alpha = (1 + \eta) \cdot \alpha - d(u).$$

Let $\beta = (1 + \eta) \cdot \alpha$. We have

$$\begin{aligned} (\beta - d(u)) \odot (1 + u + \eta) &= \sum_{n=0}^{+\infty} (\beta - d(u)) \{u + \eta\}_n \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^n (\beta - d(u)) \{u, \eta\}_{k, n-k}. \end{aligned}$$

Since $\mathfrak{I} \cdot \mathfrak{m}_A = 0$, and because $u \in L^1 \otimes \mathfrak{I}$ and $\eta \in L^0 \otimes \mathfrak{m}_A$, the terms with $n \neq 0$ and $k \neq 0$ are 0 by definition of the $\Gamma(\mathcal{P}\text{reLie}, -)$ -algebra structure in $L \otimes A$. We then have

$$\begin{aligned} (\beta - d(u)) \odot (1 + u + \eta) &= \beta - d(u) + \sum_{n=1}^{+\infty} \beta \{ \eta \}_n \\ &= \beta \odot (1 + \eta) - d(u) \\ &= \alpha + \eta \{ \alpha \}_1 - d(\eta) - d(u) \\ &= \alpha + (u + \eta) \{ \alpha \}_1 - d(u + \eta), \end{aligned}$$

which finally gives

$$(1 + \eta) \cdot \alpha - d(u) = (\alpha + (u + \eta) \{ \alpha \}_1 - d(u + \eta)) \odot (1 + u + \eta)^{\odot -1} = (1 + u + \eta) \cdot \alpha.$$

Let $\xi \in \text{Obj}(\text{Deligne}(L, A/\mathfrak{I}))$. We let $\pi_*^{-1}(\xi)$ be the category whose objects are $\omega \in \text{Obj}(\text{Deligne}(L, A))$ such that $\pi_*(\omega) = \xi$, and with morphisms the gauge group elements $\gamma \in \text{Deligne}(L, A)$ such that $\pi_*(\gamma) = 1$. We now construct a map $o_1: \text{Obj}(\pi_*^{-1}(\xi)) \times \text{Obj}(\pi_*^{-1}(\xi)) \rightarrow Z^1(L \otimes \mathfrak{I})$ such that $o_1(\alpha, \beta) = 0$ if and only if there exists a morphism γ in $\pi_*^{-1}(\xi)$ such that $\gamma(\alpha) = \beta$.

Let $\eta \in Z^1(L \otimes \mathfrak{I})$. For every $\alpha \in \text{Obj}(\pi_*^{-1}(\xi))$, we have that $\mathcal{Q}(\alpha + \eta) = \mathcal{Q}(\alpha) + \alpha \{ \eta \}_1 + \eta \{ \eta \}_1 = 0$, since $\mathfrak{I} \cdot \mathfrak{m}_A = 0$. Then $\alpha + \eta \in \text{Obj}(\text{Deligne}(L, A))$, and $\pi_*(\alpha + \eta) = \xi$. We thus have an action of the group $Z^1(L \otimes \mathfrak{I})$ on the set $\text{Obj}(\pi_*^{-1}(\xi))$. This action is simply transitive. Indeed, let $\alpha, \beta \in \text{Obj}(\text{Deligne}(L, A))$ be such that $\pi_*(\alpha) = \pi_*(\beta) = \xi$. Then $\alpha - \beta \in L^1 \otimes \mathfrak{I}$ and $d(\alpha - \beta) = \mathcal{Q}(\alpha) - \mathcal{Q}(\beta) = 0$. The element $\eta := \alpha - \beta$ is then an element of $Z^1(L \otimes \mathfrak{I})$ whose action on β is α . Since $\alpha - \beta = 0$ if and only if $\alpha = \beta$, the action is indeed simply transitive. We then can set $o_1(\alpha, \beta)$ to be the class of $\alpha - \beta$ in $H^1(L \otimes \mathfrak{I})$.

Now, if there exists some element $1 + u \in G$ in the gauge group of $L \otimes \mathfrak{m}_A$ such that $(1 + u) \cdot \alpha = \beta$ and $\pi_*(1 + u) = 1$, we then have, according to the analogue of the lemma given in [12, Section 2.8], that $\beta = (1 + u) \cdot \alpha = \alpha - d(u)$ so that $o_1(\alpha, \beta) = 0$. In the converse direction, if $o_1(\alpha, \beta) = 0$, then there exists $u \in L^0 \otimes \mathfrak{I}$ such that $\alpha - \beta = d(u)$. By using the lemma again, we find that $(1 + u) \cdot \alpha = \beta$.

Let $\tilde{\alpha}, \tilde{\beta} \in \text{Obj}(\text{Deligne}(L, A))$ and $\alpha = \pi_*(\tilde{\alpha}), \beta = \pi_*(\tilde{\beta})$ be such that there exists an element $1 + u$ of the gauge group of $L \otimes \mathfrak{m}_A/\mathfrak{I}$ with $(1 + u) \cdot \alpha = \beta$. Let $\pi_*^{-1}(1 + u)$ be the set of gauge group

elements $1 + \tilde{u}$ in $L \otimes \mathfrak{m}_A$ such that $(1 + \tilde{u}) \cdot \tilde{\alpha} = \tilde{\beta}$ and $\pi_*(1 + \tilde{u}) = 1 + u$. We finally construct a map $o_0: \pi_*^{-1}(1 + u) \times \pi_*^{-1}(1 + u) \rightarrow H^0(L \otimes \mathfrak{J})$ which satisfies the following. For every $1 + \tilde{u}, 1 + \tilde{u}' \in \pi_*^{-1}(1 + u)$, we have that $o_0(1 + \tilde{u}, 1 + \tilde{u}') = 0$ if and only if $\tilde{u} = \tilde{u}'$.

We define a simply transitive action of $H^0(L \otimes \mathfrak{J})$ on the set $\pi_*^{-1}(1 + u)$. Let $1 + \tilde{u} \in \pi_*^{-1}(1 + u)$ and let $w \in L \otimes \mathfrak{J}$. We have that

$$(1 + \tilde{u} + w) \cdot \tilde{\alpha} = (1 + \tilde{u}) \cdot \tilde{\alpha} - d(w) = \tilde{\beta} - d(w).$$

Therefore, if $w \in H^0(L \otimes \mathfrak{J}) = Z^0(L \otimes \mathfrak{J})$, then $(1 + \tilde{u} + w) \cdot \tilde{\alpha} = \tilde{\beta}$. This then defines an action of $H^0(L \otimes \mathfrak{J})$ on $\pi_*^{-1}(1 + u)$. Let $1 + \tilde{u}' \in \pi_*^{-1}(1 + u)$. We set $w = o_0(1 + \tilde{u}, 1 + \tilde{u}') := \tilde{u} - \tilde{u}' \in L \otimes \mathfrak{J}$. Then

$$d(w) = d(o_0(1 + \tilde{u}, 1 + \tilde{u}')) = (1 + \tilde{u}') \cdot \tilde{\alpha} - (1 + \tilde{u}' + w) \cdot \tilde{\alpha} = \tilde{\beta} - \tilde{\beta} = 0.$$

We then obtain that $w \in H^0(L \otimes \mathfrak{J})$ is the unique element which sends $1 + \tilde{u}$ to $1 + \tilde{u}'$. The action is then simply transitive, and we have constructed o_0 .

This then proves an analogue of the proposition given in [12, Section 2.6]. The other parts of the proof only use these three obstructions maps and do not directly use the structure of L . We can then follow exactly the same arguments in [12, Section 2.11] to obtain the result. $\square$

Definition 2.32 Two positively graded $\Gamma(\text{PreLie}, -)$ -algebras L and $\bar{L}$ are *quasi-isomorphic* if there exists a zig-zag of morphisms of $\Gamma(\text{PreLie}, -)$ -algebras

$$L = L_0 \rightarrow L_1 \leftarrow \cdots \rightarrow L_{m-1} \leftarrow L_m = \bar{L}$$

in which each morphism induces an isomorphism in cohomology.

Corollary 2.33 If L and $\bar{L}$ are quasi-isomorphic, then for all local artinian K -algebras A , the groupoids $\text{Deligne}(L, A)$ and $\text{Deligne}(\bar{L}, A)$ are equivalent. More precisely, we have a zig-zag of equivalence of groupoids

$$\text{Deligne}(L, A) \rightarrow \text{Deligne}(L_1, A) \leftarrow \cdots \rightarrow \text{Deligne}(L_{m-1}, A) \leftarrow \text{Deligne}(\bar{L}, A),$$

which is natural in A .

3 Application in homotopy theory for operads

The goal of this section is to establish Theorem C, which gives a computation of $\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P}))$ where $\mathcal{C}$ is a Σ_* -cofibrant coaugmented cooperad, $\mathcal{P}$ an augmented operad and B^c the cobar construction (see [9] or [14] for a definition of this construction). In the case of a field of characteristic 0, it can be expressed in terms of the Deligne groupoid with the structure of dg Lie algebra of $\text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})$. We extend this result using a structure of $\Gamma(\text{PreLie}, -)$ -algebra that underlies this dg Lie algebra structure.

In Section 3.1, we define infinitesimal k -compositions and k -decompositions that generalize the usual infinitesimal composition and decomposition operations given in [14, Section 6.1]. These operations will be used in the next section to write more easily weighted brace operations of the convolution operad.

In Section 3.2, we recall the definition of the convolution operad $\text{Hom}(\mathcal{C}, \mathcal{P})$, as given in [14, Section 6.4.1], and study the $\Gamma(\text{PreLie}, -)$ -algebra structure of $\text{Hom}_\Sigma(\mathcal{C}, \mathcal{P})$. In the same way that infinitesimal composition and decomposition can be used to express the pre-Lie algebra structure of the convolution operad (see [14, Proposition 6.4.5]), we will use infinitesimal k -compositions and k -decompositions to compute weighted brace operations of the convolution operad.

In Section 3.3, we just use a cylinder object of $B^c(\mathcal{C})$ given by Fresse in [8, Section 5.1] to get our result: the quotient of $\text{Hom}_\Sigma(\mathcal{C}, \mathcal{P})$ by the gauge action gives $\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P}))$.

3.1 Infinitesimal compositions and decompositions of an operad and a cooperad

We first introduce some definitions which will be useful for the computations.

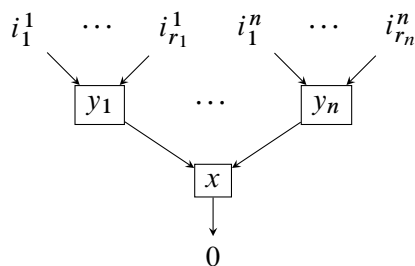
Let M and N be two symmetric sequences such that $N(0) = 0$. Recall that we have a monoidal structure on the category of symmetric sequences defined by

$$M \circ N(n) = \bigoplus_{k \geq 0} M(k) \otimes_{\Sigma_k} \left(\bigoplus_{i_1 + \dots + i_k = n} \text{Ind}_{\Sigma_{i_1} \times \dots \times \Sigma_{i_k}}^{\Sigma_n} (N(i_1) \otimes \dots \otimes N(i_k)) \right),$$

with as unit the symmetric sequence I defined by

$$I(n) = \begin{cases} \mathbb{K} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$$

Every element of $M \circ N(n)$ can be identified as a tree of the form



where $x \in M(n)$, $y_1 \in N(r_1), \dots, y_n \in N(r_n)$, and where $1 \leq i_p^q \leq \sum_i r_i$ are labels which represent a permutation of $\Sigma_{r_1 + \dots + r_n}$.

We can write $M \circ N(n)$ without quotients by the group permutations by taking a choice of set of representatives. This set is given by pointed shuffle permutations (see conventions):

$$M \circ N(n) = \bigoplus_{k \geq 0} M(k) \otimes \left(\bigoplus_{i_1 + \dots + i_k = n} N(i_1) \otimes \dots \otimes N(i_k) \otimes \mathbb{K}[\text{Sh}_*(i_1, \dots, i_k)] \right).$$

We now generalize the definition of the infinitesimal composition/decomposition defined in [14], in order to write some formulas in a more convenient way.

Definition 3.1 Let M and N be two symmetric sequences. For all $k \geq 0$, we define a new symmetric sequence denoted by $M \circ_{(k)} N$ called the k -infinitesimal composite of M and N defined, in each arity n , as the submodule of $M \circ (I \oplus N)(n)$ spanned by trees where exactly k elements at level 2 are in N , and the others in I .

Let M, M', N and N' be symmetric sequences. One can check that if we have morphisms of symmetric sequences $f: M \rightarrow M'$ and $g: N \rightarrow N'$, then we have a morphism $f \circ_{(k)} g: M \circ_{(k)} N \rightarrow M' \circ_{(k)} N'$ induced by $f \circ (\text{id}_I \oplus g)$.

Let $\mathcal{P}$ be an operad with composition $\gamma: \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}$ and unit $\eta: I \rightarrow \mathcal{P}$, and let $\mathcal{C}$ a cooperad with coproduct $\Delta: \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$ and counit $\varepsilon: \mathcal{C} \rightarrow I$. We will suppose that $\mathcal{P}$ is augmented, ie the unit $\eta: I \rightarrow \mathcal{P}$ admits a retraction $\pi: \mathcal{P} \rightarrow I$. We then have that there exists a symmetric sequence $\overline{\mathcal{P}}$ with $\mathcal{P} \simeq I \oplus \overline{\mathcal{P}}$ such that the first projection on $\mathcal{P}$ is given by π . Similarly, we suppose that $\mathcal{C}$ is coaugmented, ie the counit $\varepsilon: \mathcal{C} \rightarrow I$ admits a section $s: I \rightarrow \mathcal{C}$. We then have that there exists a symmetric sequence $\overline{\mathcal{C}}$ with $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ such that the first projection is given by ε .

In the following, we assume that $\mathcal{C}(0) = \mathcal{P}(0) = 0$ and $\mathcal{C}(1) = \mathcal{P}(1) = \mathbb{K}$.

We give an extension of the usual infinitesimal composition and decomposition operations given in [14, Section 6.1] for $k = 1$.

Definition 3.2 Let $k \geq 1$.

- We define the *infinitesimal k -composition* in $\mathcal{P}$ as

$$\gamma_{(k)}: \overline{\mathcal{P}} \circ_{(k)} \overline{\mathcal{P}}(n) \longrightarrow \mathcal{P} \circ \mathcal{P}(n) \xrightarrow{\gamma} \mathcal{P}(n),$$

where the first map is the inclusion of $\overline{\mathcal{P}} \circ_{(k)} \overline{\mathcal{P}}$ in $\mathcal{P} \circ \mathcal{P}$.

- We define the *infinitesimal k -decomposition* in $\mathcal{C}$ as

$$\Delta_{(k)}: \mathcal{C}(n) \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C}(n) \longrightarrow \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}(n),$$

where the last map is the projection of $\mathcal{C} \circ \mathcal{C}$ onto $\overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}$.

Because $\mathcal{C}$ is coaugmented, we have that the coproduct $\Delta: \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C}$ preserves the isomorphism $\mathcal{C} \simeq I \oplus \overline{\mathcal{C}}$ in the following sense. We have the isomorphism

$$\mathcal{C} \circ \mathcal{C} \simeq I \circ I \oplus \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}.$$

Then, we get that the restriction of Δ on I and on $\overline{\mathcal{C}}$ are such that

$$\Delta: I \rightarrow I \circ I \quad \text{and} \quad \Delta: \overline{\mathcal{C}} \rightarrow \overline{\mathcal{C}} \circ I \oplus I \circ \overline{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \overline{\mathcal{C}} \circ_{(k)} \overline{\mathcal{C}}.$$

We can then define the infinitesimal k -decompositions on $\bar{\mathcal{C}}$ by

$$\begin{aligned}\Delta_{(0)}: \bar{\mathcal{C}} &\xrightarrow{\Delta} \bar{\mathcal{C}} \circ I \oplus I \circ \bar{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \bar{\mathcal{C}} \circ_{(k)} \bar{\mathcal{C}} \longrightarrow \bar{\mathcal{C}} \circ I \oplus I \circ \bar{\mathcal{C}}, \\ \Delta_{(k)}: \bar{\mathcal{C}} &\xrightarrow{\Delta} \bar{\mathcal{C}} \circ I \oplus I \circ \bar{\mathcal{C}} \oplus \bigoplus_{k \geq 1} \bar{\mathcal{C}} \circ_{(k)} \bar{\mathcal{C}} \longrightarrow \bar{\mathcal{C}} \circ_{(k)} \bar{\mathcal{C}},\end{aligned}$$

for all $k \geq 1$.

3.2 $\Gamma(\text{PreLie}, -)$ -algebra structure of the convolution operad

Let M and N be two symmetric sequences of differential graded $\mathbb{K}$ -modules. We define a new symmetric sequence $\text{Hom}(M, N)$ in dg $\mathbb{K}$ -modules by $\text{Hom}(M, N)(n) = \text{Hom}(M(n), N(n))$, the differential graded module formed by the homogeneous morphisms $f: M(n) \rightarrow N(n)$. The differential on $\text{Hom}(M, N)$ is given by

$$d(f) = d_M \circ f - (-1)^{\deg(f)} f \circ d_N,$$

for all $f \in \text{Hom}(M, N)$. The action of Σ_n on $\text{Hom}(M(n), N(n))$ is defined by

$$\forall x \in M(n), \quad (\sigma \cdot f)(x) = \sigma \cdot f(\sigma^{-1} \cdot x),$$

for all $\sigma \in \Sigma_n$.

Proposition 3.3 (see [14]) *Let $\mathcal{C}$ be a cooperad and $\mathcal{P}$ be an operad. Then $\text{Hom}(\mathcal{C}, \mathcal{P})$ has the structure of a dg operad called the **convolution operad** of $\mathcal{C}$ and $\mathcal{P}$.*

We recall the operad structure on $\text{Hom}(\mathcal{C}, \mathcal{P})$. For elements $f \in \text{Hom}(\mathcal{C}, \mathcal{P})(k)$, $g_1 \in \text{Hom}(\mathcal{C}, \mathcal{P})(i_1), \dots, g_k \in \text{Hom}(\mathcal{C}, \mathcal{P})(i_k)$ the composition $\gamma(f \otimes g_1 \otimes \dots \otimes g_k)$ is given by the composite

$$\begin{aligned}\mathcal{C}(n) &\xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C}(n) \longrightarrow \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \dots \otimes \mathcal{C}(i_k) \otimes \mathbb{K}[\text{id}] \\ &\quad \downarrow f \otimes g_1 \otimes \dots \otimes g_k \otimes \text{id} \\ \mathcal{P}(k) \otimes \mathcal{P}(i_1) \otimes \dots \otimes \mathcal{P}(i_k) \otimes \mathbb{K}[\text{id}] &\hookrightarrow \mathcal{P} \circ \mathcal{P}(n) \xrightarrow{\gamma} \mathcal{P}(n),\end{aligned}$$

where $n = \sum_p i_p$.

We have $\text{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n)) = \text{Hom}(\mathcal{C}(n), \mathcal{P}(n))^{\Sigma_n}$, and we set

$$\begin{aligned}\text{Hom}_{\Sigma}(\mathcal{C}, \mathcal{P}) &= \prod_{n \geq 1} \text{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n)); \\ \text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}}) &= \prod_{n \geq 2} \text{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n)) \subset \text{Hom}_{\Sigma}(\mathcal{C}, \mathcal{P}).\end{aligned}$$

Then, according to Corollary 2.18, we have that $\text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ is endowed with a complete $\Gamma(\text{PreLie}, -)$ -algebra structure. We also have the isomorphism

$$\text{Hom}_{\Sigma}(\mathcal{C}, \mathcal{P}) \simeq \mathbb{K} \oplus \text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}}),$$

so that any morphism in $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ can be identified with a morphism in $\text{Hom}_\Sigma(\mathcal{C}, \mathcal{P})$ which is 0 on $\mathcal{C}(1) = \mathbb{K}$.

We can explicitly describe the weighted braces with one input of $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ in terms of infinitesimal decompositions and compositions.

Lemma 3.4 *Let $\bar{f}, \bar{g} \in \text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$. Then $\bar{f}\{\bar{g}\}_k$ is given by the composite*

$$\bar{\mathcal{C}} \xrightarrow{\Delta_{(k)}} \bar{\mathcal{C}} \circ_{(k)} \bar{\mathcal{C}} \xrightarrow{\bar{f} \circ_{(k)} \bar{g}} \bar{\mathcal{P}} \circ_{(k)} \bar{\mathcal{P}} \xrightarrow{\gamma_{(k)}} \bar{\mathcal{P}}.$$

In particular, we retrieve the well-known pre-Lie algebra structure on $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ given by the composite

$$\bar{\mathcal{C}} \xrightarrow{\Delta_{(1)}} \bar{\mathcal{C}} \circ_{(1)} \bar{\mathcal{C}} \xrightarrow{\bar{f} \circ_{(1)} \bar{g}} \bar{\mathcal{P}} \circ_{(1)} \bar{\mathcal{P}} \xrightarrow{\gamma_{(1)}} \bar{\mathcal{P}}$$

as shown in [14, Proposition 6.4.5] (here we consider left actions).

Proof By definition of the $\Gamma(\text{PreLie}, -)$ -algebra structure of $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ (see Corollary 2.18), it is sufficient to prove it on $\bigoplus_{r \geq 2} \text{Hom}_{\Sigma_r}(\mathcal{C}(r), \mathcal{P}(r))$. We write

$$\bar{g} = \bar{g}_1 + \cdots + \bar{g}_p,$$

where $\bar{g}_i \in \text{Hom}_{\Sigma_{n_i}}(\mathcal{C}(n_i), \mathcal{P}(n_i))$ with $n_i \neq n_j$ whenever $i \neq j$. Since the identity we need to prove is linear in $\bar{f}$, we can suppose that $\bar{f} \in \text{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n))$. We then have that

$$\bar{f}\{\bar{g}\}_k = \sum_{r_1 + \cdots + r_p = k} \bar{f}\{\bar{g}_1, \dots, \bar{g}_p\}_{r_1, \dots, r_p}.$$

If we denote by γ the operadic composition in the operad $\text{Hom}(\mathcal{C}, \mathcal{P})$, and if we set

$$\bar{h}_1, \dots, \bar{h}_k = \underbrace{\bar{g}_1, \dots, \bar{g}_1}_{r_1}, \dots, \underbrace{\bar{g}_p, \dots, \bar{g}_p}_{r_p} \quad \text{and} \quad s_1, \dots, s_k = \underbrace{n_1, \dots, n_1}_{r_1}, \dots, \underbrace{n_p, \dots, n_p}_{r_p},$$

then by definition of the $\Gamma(\text{PreLie}, -)$ -algebra structure of $\bigoplus_{r \geq 2} \text{Hom}_{\Sigma_r}(\mathcal{C}(r), \mathcal{P}(r))$ (Proposition 2.17), we have

$$\begin{aligned} & \bar{f}\{\bar{g}_1, \dots, \bar{g}_p\}_{r_1, \dots, r_p} \\ &= \sum_{\sigma \in \text{Sh}(r_1, \dots, r_p)} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq n \\ \omega \in \text{Sh}_*(1, \dots, s_{\sigma^{-1}(1)}, \dots, s_{\sigma^{-1}(k)}, \dots, 1) \\ i_1 \quad i_k}} \pm \omega \cdot \gamma(\bar{f}(1, \dots, \bar{g}_{\sigma^{-1}(1)}, \dots, \bar{g}_{\sigma^{-1}(k)}, \dots, 1)). \end{aligned}$$

For a given $\sigma \in \text{Sh}(r_1, \dots, r_p)$ and $\omega \in \text{Sh}_*(1, \dots, s_{\sigma^{-1}(1)}, \dots, s_{\sigma^{-1}(k)}, \dots, 1)$, we can see the corresponding term in the sum as the composite

$$\begin{aligned} \mathcal{C}(m) & \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C}(m) \longrightarrow \mathcal{C}(n) \otimes \mathcal{C}(s_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathcal{C}(s_{\sigma^{-1}(k)}) \otimes \mathbb{K}[\omega] \\ & \quad \downarrow \bar{f} \otimes \bar{h}_{\sigma^{-1}(1)} \otimes \cdots \otimes \bar{h}_{\sigma^{-1}(k)} \otimes \text{id} \\ \mathcal{P}(n) \otimes \mathcal{P}(s_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathcal{P}(s_{\sigma^{-1}(k)}) \otimes \mathbb{K}[\omega] & \hookrightarrow \mathcal{P} \circ \mathcal{P}(m) \xrightarrow{\gamma} \mathcal{P}(m), \end{aligned}$$

where $m = n + (n_1 - 1)r_1 + \cdots + (n_p - 1)r_p$, and where we omit some unit elements. Summing over the ω 's and using the decomposition of $\mathcal{C} \circ \mathcal{C}(m)$ and $\mathcal{P} \circ \mathcal{P}(m)$ just before Definition 3.1 will then give the composite

$$\mathcal{C}(m) \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C}(m) \xrightarrow{\bar{f} \otimes \bar{h}_{\sigma^{-1}(1)} \otimes \cdots \otimes \bar{h}_{\sigma^{-1}(k)}} \mathcal{P} \circ \mathcal{P}(m) \xrightarrow{\gamma} \mathcal{P}(m),$$

where we omit, again, the unit elements. Since $\bar{f}$ and the $\bar{h}_i$'s are 0 on $\mathcal{C}(1)$, this composite is also 0 on $\mathcal{C}(1)$. Summing on all $\sigma \in \text{Sh}(r_1, \dots, r_p)$ will give the desired composite. $\square$

Theorem 3.5 *The circular product of two elements $f = 1 + \bar{f}$, $g = 1 + \bar{g}$ of the gauge group of $\text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ is given by*

$$f \odot g : \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \xrightarrow{f \circ g} \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P}.$$

Proof Because $f|_I = g|_I = 1$, we have that $(f \odot g)|_I = 1$. We thus need to show the equality on $\bar{\mathcal{C}}$. Recall that we have infinitesimal decompositions on $\bar{\mathcal{C}}$ denoted by $\Delta_{(0)}$ and $\Delta_{(k)}$ for $k \geq 1$ such that $\Delta|_{\bar{\mathcal{C}}} = \Delta_{(0)} \oplus \bigoplus_{k \geq 1} \Delta_{(k)}$. The map $\Delta_{(0)}$ will give $\bar{f} + \bar{g}$, and each $\Delta_{(k)}$ will give $\bar{f}\{\bar{g}\}_k$ according to the previous lemma. We thus have that the composite in the statement of the theorem gives

$$1 + \bar{f} + \sum_{k \geq 0} \bar{f}\{\bar{g}\}_k,$$

which is exactly $f \odot g$. $\square$

3.3 Computation of $\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P}))$

We now extend the computation of $\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P}))$ on a field $\mathbb{K}$ with positive characteristic. In this last section, we chose to work with a homological convention to follow the conventions in the literature. This does not change anything on the results of the previous sections.

Recall that we can give an explicit cylinder object for $B^c(\mathcal{C})$, where B^c is the cobar construction of $\mathcal{C}$, when $\mathcal{C}$ is Σ_* -cofibrant (see, for instance, [8] or [14]). Explicitly, let $K = \mathbb{K}\sigma^0 \oplus \mathbb{K}\sigma^1 \oplus \mathbb{K}\sigma^{01}$ where $|\sigma^0| = |\sigma^1| = -1$, $|\sigma^{01}| = 0$ and $d(\sigma^{01}) = \sigma^1 - \sigma^0$. Then there exists a derivation of operads ∂ such that the free dg operad $(\mathcal{F}(K \otimes \bar{\mathcal{C}}), \partial)$ is a cylinder object for $B^c(\mathcal{C})$. We refer to [8, Section 5.1] for an explicit construction of ∂ and a proof of the previous statement.

Theorem 3.6 *Suppose that $\mathcal{C}$ is Σ_* -cofibrant. We then have a bijection:*

$$\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P})) \simeq \pi_0 \text{Deligne}(\text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})).$$

Proof Recall from Fresse (see [9, Theorem 3.2.14]) that $\pi_0(\text{Map}(B^c(\mathcal{C}), \mathcal{P})) \simeq (\text{Mor}(B^c(\mathcal{C}), \mathcal{P}), \sim_h)$ where $\sim_h$ is the homotopy relation in the category of symmetric operads. Recall also that the data of a Maurer–Cartan element α in $\text{Hom}_{\Sigma}(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ is equivalent to give a morphism of operads ϕ_{α} from $B^c(\mathcal{C})$

to $\mathcal{P}$ (see [8] or [14]). We just need to show that the action of the gauge group on the Maurer–Cartan set of $\text{Hom}_\Sigma(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ from one Maurer–Cartan α to an other one β is equivalent to give a homotopy from ϕ_α to ϕ_β .

Let $1 + \lambda$ be an element of the gauge group. We define a morphism $h : \text{Cyl}(B^c(\mathcal{C})) \rightarrow \mathcal{P}$ via $h : K \otimes \bar{\mathcal{C}} \rightarrow \mathcal{F}(K \otimes \bar{\mathcal{C}})$ by setting

$$h(\sigma^0 \otimes \gamma) = \alpha(\gamma), \quad h(\sigma^1 \otimes \gamma) = \beta(\gamma), \quad h(\sigma^{01} \otimes \gamma) = \lambda(\gamma),$$

where γ is some element of $\bar{\mathcal{C}}$. We claim that $(1 + \lambda) \cdot \alpha = \beta$ if and only if h is a homotopy from ϕ_α to ϕ_β . Accordingly, we must prove

$$d(\lambda) = \alpha + \lambda\{\alpha\} - \beta \odot (1 + \lambda) \Leftrightarrow d(h) = 0,$$

where d is the differential of $\text{Mor}(B^c(\mathcal{C}), \mathcal{P})$.

Because α and β are Maurer–Cartan elements, and by definition of ∂ , the second equality is always satisfied for $\sigma^\varepsilon \otimes \gamma$ with $\varepsilon = 0, 1$ and $\gamma \in \bar{\mathcal{C}}$. We just need to check this equality on terms $\sigma^{01} \otimes \gamma$ for any $\gamma \in \bar{\mathcal{C}}$:

$$\begin{aligned} d(h)(\sigma^{01} \otimes \gamma) &= d(h(\sigma^{01} \otimes \gamma)) - h(\partial(\sigma^{01} \otimes \gamma)) \\ &= d(\lambda(\gamma)) - \lambda(d(\gamma)) - \alpha(\gamma) + \beta(\gamma) - \lambda\{\alpha\}_1(\gamma) + (\beta \odot (1 + \lambda)(\gamma) - \beta(\gamma)) \\ &= d(\lambda)(\gamma) - \alpha(\gamma) - \lambda\{\alpha\}_1(\gamma) + \beta \odot (1 + \lambda)(\gamma). \end{aligned}$$

□

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A note on rational maps with three branching points on the Riemann sphere

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Studying the existence of rational maps with given branching datum is a classical problem in the field of complex analysis and algebraic geometry. This problem dates back to Hurwitz and remains open to this day. We utilize complex analysis to establish a property of rational maps with 3 branching points on the Riemann sphere.

Given two compact Riemann surfaces M and N , a pair $(d, \mathcal{D})$ of an integer $d \geq 2$ and a collection $\mathcal{D}$ of nontrivial partitions of d is called a candidate branching datum if it satisfies the Riemann–Hurwitz formula. A candidate branching datum is considered exceptional when there is no holomorphic map realization for it. As applications of our main theorem, we present some new types of exceptional branching datum which cover some previously known results. We also deduce the realizability of a certain type of candidate branching datum on the Riemann sphere.

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1 Introduction

Studying the existence of branched covers between two compact Riemann surfaces with given branching datum is an important problem in the field of complex analysis and algebraic geometry. This problem is commonly referred to as the Hurwitz existence problem. It asks whether, given two compact Riemann surfaces M and N , along with a collection $\mathcal{D}$ of partitions of a positive integer d , there exists a degree- d branched cover $f: M \rightarrow N$ with $\mathcal{D}$ as the branching datum.

Let M and N be compact Riemann surfaces, and let their respective Euler characteristics be denoted by $\chi(M)$ and $\chi(N)$. A smooth map $f: M \rightarrow N$ is a degree- d branched cover if for each $x \in N$ there is a partition $\mu(x) = [\alpha_1, \dots, \alpha_r]$ (here the square brackets are used to denote an unordered set with repetitions) of d such that, over a neighborhood of x in N , f is equivalent to the map $\tilde{f}: \{1, \dots, r\} \times \mathbb{D} \rightarrow \mathbb{D}$ where $\tilde{f}(j, z) = z^{\alpha_j}$ and x corresponds to 0 in the disc $\mathbb{D} = \{z \mid |z| < 1\} \subseteq \mathbb{C}$. The points $x \in N$ for which $\mu(x)$ is not the trivial partition $[1, 1, \dots, 1]$ of d constitute the branching set B_f of f , and it is a finite set. The collection $\mathcal{D} = \{\mu(x) \mid x \in B_f\}$ (with repetitions allowed) is called the branching datum of f . As is well known, the degree- d , Euler characteristics $\chi(M)$, $\chi(N)$ and the branching datum $\mathcal{D}$ of f should satisfy the Riemann–Hurwitz formula

$$(1) \quad v(\mathcal{D}) = d \cdot \chi(N) - \chi(M),$$

where $\nu(\mathcal{D})$ denotes the total branching of f , which is defined as follows. Let $B_f = \{x_1, \dots, x_n\} \subseteq N$ and $\mathcal{D} = \{[\alpha_1^1, \dots, \alpha_1^{r_1}], \dots, [\alpha_n^1, \dots, \alpha_n^{r_n}]\}$ represent the branching set and the branching datum of f , respectively. For each x_k , its preimage under f consists of a finite number of points $y_k^1, \dots, y_k^{r_k} \in M$, and near each y_k^j the map f is equivalent to $\tilde{f}(z) = z^{\alpha_k^j}$. The integer α_k^j is usually referred to as the local degree or multiplicity of f at the point y_k^j . The total branching of f is then defined as $\nu(\mathcal{D}) = \sum_{k=1}^n \sum_{j=1}^{r_k} (\alpha_k^j - 1)$. Since

$$(2) \quad \sum_{j=1}^{r_k} \alpha_k^j = d \quad \text{for all } k = 1, \dots, n,$$

the Riemann–Hurwitz formula (1) can be expressed as

$$(3) \quad \sum_{k=1}^n (d - r_k) = d \cdot \chi(N) - \chi(M).$$

Given two compact Riemann surfaces M and N , a pair $(d, \mathcal{D})$ of an integer $d \geq 2$ and a collection $\mathcal{D}$ of nontrivial partitions of d is called a candidate branching datum if it satisfies the Riemann–Hurwitz formula. In his classical work [7], Hurwitz reduced the Hurwitz problem to a problem involving partitions realized by suitable permutations in symmetric groups. In [5], Edmonds, Kulkarni, and Stong proved that all candidate branching datum is realizable when $\chi(N) \leq 0$. However, when $N = S^2$ the problem becomes much more complex. It is well known that there exist candidate branching datum $(d, \mathcal{D})$ that cannot be realized by a branched cover, hence called exceptional. For example, this is the case for $d = 4$ and $\mathcal{D} = \{[3, 1], [2, 2], [2, 2]\}$. Characterizing all of such exceptional candidate branching datum remains an open problem to this day. In [25], Zheng determined all exceptional candidate branching datum with 3 branching points and degree $d \leq 22$ by computer for the cases where $M = N = S^2$ and $M = T^2, N = S^2$.

Finding new types of exceptional candidate branching datum is of interest as it may provide insights towards establishing a universal criterion. However, the general pattern of realizable data remains unclear. Various approaches such as dessins d'enfant, Speiser graphs and the monodromy approach have been explored to attack the problem. For more details, see [1; 2; 4; 5; 6; 8; 9; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21; 22; 24; 25]. Particularly, we refer the reader to [4; 19; 21] for a review of available results and techniques. In [5], a conjecture proposing connections with number-theoretic facts has been put forward, which is supported by strong evidence in [15; 16].

Conjecture 1.1 (prime-degree conjecture) Suppose $(d, \mathcal{D})$ is a candidate branching datum. If d is a prime, then $(d, \mathcal{D})$ is realizable.

In [5], Edmonds, Kulkarni, and Stong reduced the prime-degree conjecture to the case of candidate branching datum with exactly 3 partitions. Up to now, this conjecture is still open. In this paper we investigate the properties of branched covers for the cases where $M = N = S^2$. Regarding $S^2 \cong \overline{\mathbb{C}}$ as the Riemann sphere, by Stoilow's theorem (see [23] or [10]), the Hurwitz problem reduces to the

existence of rational maps on $\bar{\mathbb{C}}$ with the given branching datum. Thus we can use complex analysis techniques to attack the problem. Motivation by the recent work of Song and Xu [22], we use their idea of expressing a rational map as a composition of a rational map and a power map. Our main result is as follows and can be viewed as a generalization of a result in [18, Theorems 1.4 and 1.5].

Theorem 1.2 Suppose $f: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ is a rational map of degree $d = rk$, $r \geq 2$, $k \geq 2$, with 3 branching points and associated branching datum

$$\{\mu_1 = [\alpha_1, \dots, \alpha_A], \mu_2 = [rx_1, \dots, rx_B], \mu_3 = [ry_1, \dots, ry_C]\}.$$

Then, up to two Möbius transformations on $\bar{\mathbb{C}}$, there is a rational map $g: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ of degree k such that $f = g^r$, with the property that its branching datum consists of $[x_1, \dots, x_B]$ and $[y_1, \dots, y_C]$, together with data arising by splitting the partition of $[\alpha_1, \dots, \alpha_A]$ into r partitions of k .

Remark In Theorem 1.2, the sentence “up to two Möbius transformations” means that $f = \varphi \circ g^k \circ \phi$, where $\varphi, \phi: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ are two Möbius transformations.

Remark In Theorem 1.2, if $x_1 = \dots = x_B = 1$ or $y_1 = \dots = y_C = 1$ then the corresponding partition of k should be expunged from the branching datum of g .

Then, under the assumption of Theorem 1.2, one would have immediate corollaries for realizable data.

Corollary 1.3 Each $\alpha_j \leq k$.

Corollary 1.4 If $A = 2$, then $r = 2$, $\mu_1 = [k, k]$.

Corollary 1.5 If $A = r$, then $\mu_1 = [k, \dots, k]$.

As applications of these results, we offer new proofs of some of the results presented in [18; 26] and give some new type of exceptional candidate branching datum (Propositions 1.9 and 1.10).

Proposition 1.6 [26] Suppose $d = 2k$ with $k \geq 2$. Then a candidate branching datum of the form

$$(d, \{[k_1, k_2], \underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2]}_k\})$$

is exceptional if $k_1 \neq k_2$. Suppose in addition $k \geq 3$. Then a candidate branching datum of the form

$$(d, \{\underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2]}_{j_1}, 2k - 2j_1, \underbrace{[2, \dots, 2]}_{j_2}, 2k - 2j_2\})$$

is exceptional if $j_1 \neq j_2$.

Proof Since $k_1 \neq k_2$ and $k_1 + k_2 = 2k$, $k_1, k_2 \neq k$. By Corollary 1.4, $(d, \mathcal{D})$ is exceptional.

Suppose $k \geq 3$, and note that $j_1 + j_2 = k$ and $j_1 \neq j_2$. Then, $2j_1 < k$ or $2j_2 < k$. If $2j_1 < k$, then $2k - 2j_1 > k$; If $2j_2 < k$, then $2k - 2j_2 > k$. Thus, by Corollary 1.3, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.7 [26] Suppose $d = 3k$ with $k \geq 3$. Then a candidate branching datum of the form

$$(d, \{\underbrace{[k-2, 2, \dots, 2]}_{k+1}, \underbrace{[3, \dots, 3]}_k, \underbrace{[3, \dots, 3]}_k\})$$

is exceptional if k is odd.

Proof Since k is odd and 2 is even, there is just one odd in $[k-2, 2, \dots, 2]$. Thus $[k-2, 2, \dots, 2]$ cannot be split into 3 partitions of k . Therefore, by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.8 [26] Suppose $d = rk$ with $r \geq 2, k \geq 2$. Then a candidate branching datum of the form

$$(d, \{\underbrace{[j_1, j_2, 1, \dots, 1]}_{(r-2)k}, \underbrace{[r, \dots, r]}_k, \underbrace{[r, \dots, r]}_k\})$$

is exceptional if $j_1 \neq j_2$.

Proof Since $j_1 + j_2 = 2k$ and $j_1 \neq j_2$, it follows that either $j_1 > k$ or $j_2 > k$. Thus, by Corollary 1.3, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.9 Suppose $d = 3k$ with $k \geq 3$ and $k \equiv 1 \pmod{2}$. Then a candidate branching datum of the form

$$(d, \{\underbrace{[j_1, j_2, 2, \dots, 2]}_k, \underbrace{[3, \dots, 3]}_k, \underbrace{[3, \dots, 3]}_k\})$$

is exceptional if j_1, j_2 have different parity.

Proof Since j_1, j_2 have different parity, there is just one odd in the partition

$$[j_1, j_2, \underbrace{2, \dots, 2}_k].$$

Since k is odd, $[k-2, 2, \dots, 2]$ cannot be split into 3 partitions of k . Thus by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. $\square$

Proposition 1.10 Suppose $d = 3k$ with $k = 2 + 3l, l \geq 1$. Then a candidate branching datum of the form

$$(d, \{\underbrace{[3, \dots, 3, 1, 1, 1]}_{k-1}, \underbrace{[3, \dots, 3]}_3, \underbrace{[3, \dots, 3]}_k, \underbrace{[3, \dots, 3]}_k\})$$

is exceptional.

Proof Since $k = 2 + 3l \geq 5$ and the number of 1's in

$$\underbrace{[3, \dots, 3, 1, 1, 1]}_{k-1}$$

equals 3,

$$\underbrace{[3, \dots, 3, 1, 1, 1]}_{k-1}$$

cannot be split into 3 partitions of k . Thus by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. $\square$

From the constructed exceptional datum mentioned above, we obtain an additional result regarding nonprime degrees on the case $M = N = S^2$, which has been proven in [5; 26] using different methods.

Corollary 1.11 [5; 26] *For every nonprime d , there exists at least one candidate branching datum $(d, \mathcal{D})$ that is exceptional.*

2 Proof of Theorem 1.2

The key to proving Theorem 1.2 is to use suitable Möbius transformations to simplify the representation of f . This simplification then facilitates the construction of the mapping g , derived from the transformed expression. The final step involves calculating the branching datum of g . To begin, we establish the following lemma.

Lemma 2.1 *Suppose $g: \mathbb{C} \rightarrow \mathbb{C}$ is a nonconstant holomorphic function. For a point $z_0 \in \mathbb{C}$, if $g(z_0) \neq 0$, then g and g^r , for every integer r , have the same local degree at z_0 .*

Proof Call n the local degree of g at z_0 . Then, knowing that $g(z_0) \neq 0$, the order- n Taylor expansion of g at z_0 has the form

$$g(z) = a + b(z - z_0)^n + o((z - z_0)^n),$$

with $a, b \neq 0$. It follows that

$$g^r(z) = a^r + rab(z - z_0)^n + o((z - z_0)^n),$$

and $rab \neq 0$, so g^r also has local degree n at z_0 . □

Now, the proof of Theorem 1.2 is as follows.

Since Möbius transformations act in a triply transitive fashion on $\overline{\mathbb{C}}$, one can assume that:

- (i) 0 in the target $\overline{\mathbb{C}}$ is the branching point corresponding to the partition $[rx_1, \dots, rx_B]$.
- (ii) ∞ in the target $\overline{\mathbb{C}}$ is the branching point corresponding to the partition $[ry_1, \dots, ry_C]$.
- (iii) 1 in the target $\overline{\mathbb{C}}$ is not the branching point, and $f(\infty) = 1$.

Thus, the expression for f can be written as

$$f(z) = \frac{(z - z_1)^{rx_1} (z - z_2)^{rx_2} \dots (z - z_B)^{rx_B}}{(z - w_1)^{ry_1} (z - w_2)^{ry_2} \dots (z - w_C)^{ry_C}},$$

where $z_1, \dots, z_B, w_1, \dots, w_C \in \mathbb{C}$ are distinct complex numbers.

Define

$$g(z) = \frac{(z - z_1)^{x_1} (z - z_2)^{x_2} \dots (z - z_B)^{x_B}}{(z - w_1)^{y_1} (z - w_2)^{y_2} \dots (z - w_C)^{y_C}}.$$

Then g is a rational map with degree k , and $f = g^r$ on $\overline{\mathbb{C}}$. It is evident that 0 is a branching point of g if and only if $x_j \geq 2$ for some index j , and ∞ is a branching point of g if and only if $y_l \geq 2$ for some index l .

For any $z_0 \in \mathbb{C} \setminus \{z_1, \dots, z_B, w_1, \dots, w_C\}$, by Lemma 2.1, g and $f = g^r$ have the same local degree at z_0 . Consequently, the branching datum of g is

$$\{[\alpha_1^1, \dots, \alpha_1^{l_1}], \dots, [\alpha_s^1, \dots, \alpha_s^{l_s}], [x_1, \dots, x_B], [y_1, \dots, y_C]\},$$

for some $1 \leq s \leq r$, where $\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}$ belong to $\alpha_1, \dots, \alpha_A$ which means that up to a permutation $(\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}, 1, \dots, 1) = (\alpha_1, \dots, \alpha_A)$.

In summary, we have completed the proof of the Theorem 1.2.

By Lemma 2.1, the proof of the following theorem is easy.

Theorem 2.2 Suppose $g: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a rational map with degree $d \geq 3$ and branching datum

$$\{[\alpha_1, \dots, \alpha_A], [\beta_1, \dots, \beta_B], [\gamma_1, \dots, \gamma_C]\}.$$

Then, for any integer $k \geq 2$, there is a Möbius transformation φ such that $f = (\varphi \circ g)^k$ is a rational map with degree kd and branching datum

$$\{[\alpha_1, \dots, \alpha_A, \underbrace{1, \dots, 1}_{(k-1)d}], [k\beta_1, \dots, k\beta_B], [k\gamma_1, \dots, k\gamma_C]\}.$$

3 Some existence results

We conclude the paper by presenting some existence results for a certain type of rational maps on $\overline{\mathbb{C}}$ with 3 branching points. First, we give a new proof of the following theorem which was proved in [5] and which is a special case in [4, Theorem 3.5].

Theorem 3.1 [5] There exists a rational map $f: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with branching datum

$$\{[\alpha_1, \alpha_2], [\underbrace{2, \dots, 2}_k], [\underbrace{2, \dots, 2}_k]\},$$

where $k \geq 2$ and $\alpha_1 + \alpha_2 = 2k$, if and only if $\alpha_1 = \alpha_2 = k$.

Proof Necessity follows from Corollary 1.4. The proof of sufficiency is as follows.

Define $g(z) = (z^k - 1)/(z^k + 1)$. By direct calculation, the branching points of g are -1 and 1 . Additionally, $g^{-1}(-1) = \{0\}$, $g^{-1}(1) = \{\infty\}$ and the branching datum of g is $\{[k], [k]\}$. Set $f = g^2$. By direct calculation, we find that the branching points of f are $1, 0, \infty$, and the branching datum of f is

$$\{[k, k], [\underbrace{2, \dots, 2}_k], [\underbrace{2, \dots, 2}_k]\}. \quad \square$$

Second, to give more existence results, we recall a result by Boccara [3].

Theorem 3.2 [3] *Let p and q be two positive integers. Consider a candidate branching datum*

$$\mathcal{D} = \{[\alpha_1, \dots, \alpha_p], [\beta_1, \dots, \beta_q], [p+q-1, 1, \dots, 1]\},$$

which contains 3 partitions of a positive integer d . Then there exists a rational map $f: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ with $\mathcal{D}$ being its branching datum if and only if

$$p+q-1 \leq \frac{d}{\text{GCD}(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q)}.$$

Theorem 3.3 *Suppose that $k \geq 3$. There exists a degree- $2k$ rational map $f: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ with branching datum*

$$\{[\alpha_1, \alpha_2, \alpha_3], \underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2, 4]}_{k-2}\}$$

if and only if one of the α_j 's is k .

Proof Sufficiency Without loss of generality, we can suppose that $\alpha_1 = k$. By Theorem 3.2, there exists a rational map $g: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ with branching datum

$$\{[k], [\alpha_2, \alpha_3], \underbrace{[1, \dots, 1, 2]}_{k-2}\}.$$

Using a suitable Möbius transformation, we may suppose that:

- (i) -1 is the branching point of g corresponding to the partition $[k]$.
- (ii) 1 is the branching point of g corresponding to the partition $[\alpha_2, \alpha_3]$.
- (iii) 0 is the branching point of g corresponding to the partition $\underbrace{[1, \dots, 1, 2]}_{k-2}$.

Then $f = g^2$ is the desired function.

Necessity Suppose $f: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ is a rational map with branching datum

$$\{[\alpha_1, \alpha_2, \alpha_3], \underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2, 4]}_{k-2}\}.$$

Then, by Theorem 1.2, $[\alpha_1, \alpha_2, \alpha_3]$ can be split into 2 partitions of k . Thus one of the α_j 's is k . $\square$

In general, we obtain the following theorem.

Theorem 3.4 *Suppose that $k \geq 3$ and $x \geq 1$. There exists a degree- $2k$ rational map $f: \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ with branching datum*

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+1}], \underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2, 2x]}_{k-x}\}$$

if and only if $[\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$ can be split into two partitions of k and $k/\text{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1}) \geq x$.

Proof Necessity Suppose $f: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a rational map with branching datum

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+1}], \underbrace{[2, \dots, 2]}_k, \underbrace{[2, \dots, 2, 2x]}_{k-x}\}.$$

Then, by Theorem 1.2, there exists a rational map g of degree k with branching datum

$$\{[\alpha_1^1, \dots, \alpha_1^{r_1}], [\alpha_2^1, \dots, \alpha_2^{r_2}], [1, \dots, 1, x]\}$$

such that $f = g^2$, where $[\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2}] = [\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$. By Theorem 3.2,

$$\frac{k}{\text{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1})} \geq x.$$

Sufficiency Suppose $[\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$ can be split into two partitions of k . Denote the two partitions by $[\alpha_1^1, \dots, \alpha_1^{r_1}]$ and $[\alpha_2^1, \dots, \alpha_2^{r_2}]$. Since

$$\text{GCD}(\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2}) = \text{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1}),$$

we have

$$\frac{k}{\text{GCD}(\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2})} \geq x.$$

Thus by a result of Boccara [3], there exists a rational map $g: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with branching datum

$$\{[\alpha_1^1, \dots, \alpha_1^{r_1}], [\alpha_2^1, \dots, \alpha_2^{r_2}], [1, \dots, 1, x]\}.$$

Using a suitable Möbius transformation, we may suppose that:

- (i) -1 is the branching point of g corresponding to the partition $[\alpha_1^1, \dots, \alpha_1^{r_1}]$.
- (ii) 1 is the branching point of g corresponding to the partition $[\alpha_2^1, \dots, \alpha_2^{r_2}]$.
- (iii) 0 is the branching point of g corresponding to the partition $[1, \dots, 1, x]$.

Then $f = g^2$ is the desired function. □

Finally, we provide more explicit examples to explain our results.

Example 3.5 There exists a rational map $f: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with branched data

$$\{[\alpha_1, \alpha_2, \alpha_3, 2, 2], [3, 3, 3], [3, 3, 3]\},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 5$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, if and only if $\alpha_1 = 3$.

Proof Necessity Since $\alpha_1 + \alpha_2 + \alpha_3 = 5$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, it follows that $\alpha_1 = 2$ or 3 . If $\alpha_1 = 2$, then $\alpha_2 = 2$ and $\alpha_3 = 1$. But $[\alpha_1, \alpha_2, \alpha_3, 2, 2] = [2, 2, 1, 2, 2]$ cannot be split into 3 partitions of 3. Thus $\alpha_1 = 3$.

Sufficiency If $\alpha_1 = 3$, then $\alpha_2 = \alpha_3 = 1$. By Theorem 3.2, there exists a rational map g with degree 3 and branching datum

$$\{[3], [2, 1], [2, 1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

- (i) 1 is the branching point of g corresponding to the partition [3].
- (ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 1].
- (iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 1].

Then $f = g^3$ is a rational map with branching datum

$$\{[3, 1, 1, 2, 2], [3, 3, 3], [3, 3, 3]\}.$$

□

Example 3.6 There exists a rational map $f: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with branching datum

$$\{[\alpha_1, \alpha_2, \alpha_3, 2, 2, 2, 2], [3, 3, 3, 3, 3], [3, 3, 3, 3, 3]\},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 7$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, if and only if $\alpha_1 = 5$ or $\alpha_1 = \alpha_2 = 3$.

Proof Necessity Since $\alpha_1 + \alpha_2 + \alpha_3 = 7$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3$, it follows that $\alpha_1 = 3, 4$ or 5 .

When $\alpha_1 = 3$, it follows that $(\alpha_2, \alpha_3) = (2, 2)$ or $(3, 1)$. When $\alpha_1 = 3, \alpha_2 = 2$ and $\alpha_3 = 2$, since $[3, 2, 2, 2, 2, 2, 2]$ cannot be split into 3 partitions of 5, we exclude this case. When $\alpha_1 = 3, \alpha_2 = 3$ and $\alpha_3 = 1$, it is evident that $[3, 3, 1, 2, 2, 2, 2]$ can be split into 3 partitions of 5.

When $\alpha_1 = 4$, since $\alpha_2 + \alpha_3 = 3$, there is just one odd in $[4, \alpha_2, \alpha_3, 2, 2, 2, 2]$. Thus $[4, \alpha_2, \alpha_3, 2, 2, 2, 2]$ cannot be split into 3 partitions of 5. Thus we exclude this case.

When $\alpha_1 = 5$, it follows that $\alpha_2 = \alpha_3 = 1$. It is evident that $[5, 1, 1, 2, 2, 2, 2]$ can be split into 3 partitions of 5.

In summary, $\alpha_1 = 5$ or $\alpha_1 = \alpha_2 = 3$.

Sufficiency When $\alpha_1 = 5$, it follows that $\alpha_2 = \alpha_3 = 1$. By Theorem 3.2, there exists a rational map g with branching datum

$$\{[5], [2, 2, 1], [2, 2, 1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

- (i) 1 is the branching point of g corresponding to the partition [5].
- (ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 2, 1].
- (iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 2, 1].

Then $f = g^3$ is a rational map with branching datum

$$\{[5, 1, 1, 2, 2, 2, 2], [3, 3, 3, 3, 3], [3, 3, 3, 3, 3]\}.$$

When $\alpha_1 = \alpha_2 = 3$, it follows that $\alpha_3 = 1$. By a result of Zheng [25] there exists a rational map g with branching datum

$$\{[3, 2], [3, 2], [2, 2, 1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

- (i) 1 is the branching point of g corresponding to the partition $[3, 2]$.
- (ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition $[3, 2]$.
- (iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition $[2, 2, 1]$.

Then $f = g^3$ is a rational map with branching datum

$$\{[3, 3, 1, 2, 2, 2], [3, 3, 3, 3, 3], [3, 3, 3, 3, 3]\}.$$

□

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Loop homology of moment-angle complexes in the flag case

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We develop a general homological approach to presentations of connected graded associative algebras, and apply it to the loop homology of moment-angle complexes Z_K that correspond to flag simplicial complexes K . For an arbitrary coefficient ring, we describe generators of the Pontryagin algebra $H_*(\Omega Z_K)$ and defining relations between them. We prove that such moment-angle complexes are coformal over $\mathbb{Q}$, give a necessary condition for rational formality, and compute their homotopy groups in terms of homotopy groups of spheres.

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1 Introduction

For a simply connected space X and a commutative ring k with unit, the *Pontryagin algebra* $H_*(\Omega X; k)$ is a connected graded associative k -algebra with respect to the Pontryagin product. We study the Pontryagin algebras of *moment-angle complexes* $X = Z_K := (D^2, S^1)^K$ that correspond to simplicial complexes K . Moment-angle complexes play an important role in toric topology [15], and they have interesting homotopical properties and surprising connections to several topics in algebra and combinatorics [5]. If K is a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$, there is an effective action of the m -dimensional torus $\mathbb{T}^m = (S^1)^{\times m}$ on Z_K . The homotopy quotient $E\mathbb{T}^m \times_{\mathbb{T}^m} Z_K$ (the Borel construction) is known as the *Davis–Januszkiewicz space* $DJ(K)$ and is homotopy equivalent to the polyhedral product $(\mathbb{C}P^\infty, *)^K$; see [15, Theorem 4.3.2].

Panov and Ray [35] reduced the study of corresponding Pontryagin algebras to an algebraic problem. Applying the based loops functor to the homotopy fibration

$$(1) \quad Z_K \rightarrow DJ(K) \rightarrow B\mathbb{T}^m,$$

they obtained a split fibration of H-spaces $\Omega Z_K \rightarrow \Omega DJ(K) \rightarrow \mathbb{T}^m$ and thus an extension of cocommutative Hopf algebras

$$k \rightarrow H_*(\Omega Z_K; k) \rightarrow H_*(\Omega DJ(K); k) \rightarrow \Lambda[u_1, \dots, u_m] \rightarrow k$$

over a field k . For any K , there is an isomorphism of Hopf algebras $H_*(\Omega DJ(K); k) \cong \text{Ext}_{k[K]}(k, k)$ [20; 35] (moreover, this is true for any principal ideal domain k such that $H_*(\Omega DJ(K); k)$ is a free

$\mathbf{k}$ -module). If $\mathcal{K}$ is a *flag* simplicial complex, this Hopf algebra is known completely: it is isomorphic to the partially commutative algebra

$$\mathbf{k}[\mathcal{K}]^! := T(u_1, \dots, u_m) / (u_i^2 = 0, i = 1, \dots, m; u_i u_j + u_j u_i = 0, \{i, j\} \in \mathcal{K}), \quad \deg u_i = 1.$$

Generators u_i are primitive and have degree $(-1, 2e_i)$ with respect to the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading introduced in [43]. In this case Grbić, Panov, Theriault and Wu [23] found a minimal generating set for the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$, and the author calculated the number of relations in any minimal presentation (by homogeneous generators and relations) of this algebra [43].

The last calculation relies on homological methods developed by Wall [44] and Lemaire [29] for connected graded associative algebras over a field. Namely, multiplicative generators of a connected $\mathbf{k}$ -algebra A correspond to additive generators of the graded $\mathbf{k}$ -module $\mathrm{Tor}_1^A(\mathbf{k}, \mathbf{k})$, and relations correspond to generators of $\mathrm{Tor}_2^A(\mathbf{k}, \mathbf{k})$. In order to study the integer Pontryagin algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})$, we generalise these results to the case of *arbitrary* commutative rings $\mathbf{k}$ with unit, and construct explicit presentations of connected $\mathbf{k}$ -algebras using cycles in the bar construction. These results are presented in Appendix A. We hope that they will be useful in other contexts.

Let us give a general description of our approach. Suppose that we are given a connected $\mathbf{k}$ -algebra A which is a free left module over its subalgebra S , $A \simeq S \otimes_{\mathbf{k}} V$. We wish to construct a presentation of S . Theorem A.6 does that, if we know a set of cycles in the bar construction $\bar{B}(S)$, such that their images generate the $\mathbf{k}$ -modules $H_i(\bar{B}(S)) \simeq \mathrm{Tor}_i^S(\mathbf{k}, \mathbf{k})$, $i = 1, 2$. The following algorithm computes such cycles:

- (i) Build a free resolution $(A \otimes M, d)$ of the left A -module $\mathbf{k}$.
- (ii) Interpret it as a free resolution $(S \otimes V \otimes M, \hat{d})$ of the left S -module $\mathbf{k}$. Compute the functor $\mathrm{Tor}^S(\mathbf{k}, \mathbf{k})$ as the homology of the complex $(V \otimes M, \bar{d})$. Find cycles in $(V \otimes M, \bar{d})$ such that their images generate $\mathrm{Tor}_i^S(\mathbf{k}, \mathbf{k})$, $i = 1, 2$.
- (iii) Construct a morphism $\varphi: (S \otimes V \otimes M, \hat{d}) \rightarrow (B(S), d_B)$ of free resolutions of the left S -module $\mathbf{k}$, using the contracting homotopy of the bar resolution (see Corollary 2.2). Obtain a morphism of chain complexes $\bar{\varphi}: (V \otimes M, \bar{d}) \rightarrow (\bar{B}(S), d_{\bar{B}})$ that induces an isomorphism on the homology.
- (iv) Applying $\bar{\varphi}$ to the cycles from (ii), obtain the required cycles in $\bar{B}(S)$.

This situation takes place if $\mathbf{k} \rightarrow S \rightarrow A \rightarrow V \rightarrow \mathbf{k}$ is an extension of connected Hopf algebras; see [2; 31, Proposition 4.9]. In that sense, our algorithm has similarities with the *Reidemeister–Schreier algorithm* that constructs a presentation of a subgroup, given a presentation of the whole group. See [30] for another approach to Hopf subalgebras in connected Hopf algebras. It is well known that extensions of Hopf algebras arise in the study of fibrations $F \rightarrow E \rightarrow B$ that have a section after looping (see Appendix B for the proof). For such “ Ω -split” fibrations, the proposed method allows us to study presentations of $H_*(\Omega F; \mathbf{k})$, if the algebras $H_*(\Omega E; \mathbf{k})$ and $H_*(\Omega B; \mathbf{k})$ are known.

Fibrations of this kind are studied by Theriault [41]; see also [9, Proposition 6.1]. (However, these works deal with cases when the algebra $H_*(\Omega F; \mathbf{k})$ is known better than $H_*(\Omega E; \mathbf{k})$.) We consider the case $F = \mathcal{Z}_{\mathcal{K}}$, $E = \mathrm{DJ}(\mathcal{K})$, $B = (\mathbb{C}P^\infty)^m$. The algorithm is also applicable to *partial quotients* of moment-angle complexes [15, Section 4.8] (we will consider their Pontryagin algebras in subsequent publications) and polyhedral products of the form $(PX, \Omega X)^{\mathcal{K}}$ (here we refer to the recent work [16] by Li Cai).

1.1 Main results

We give a presentation of the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ for a flag simplicial complex $\mathcal{K}$ and any ring $\mathbf{k}$. The presentation is explicit up to a rewriting process described in Algorithm 5.4. For $x \in H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$ and a subset $A = \{a_1 < \dots < a_k\} \subset [m]$, define

$$c(A, x) := [u_{a_1}, [u_{a_2}, \dots, [u_{a_k}, x], \dots]] \in H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}).$$

This element belongs to the subalgebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \subset H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$, if $x = u_i$ and $A \neq \emptyset$ (see Corollary 3.10). For every $J \subset [m]$, denote by $\Theta(J)$ the set of all vertices $i \in J$ such that

- vertices i and $\max(J)$ are in different path components of the complex $\mathcal{K}_J$;
- i is the smallest vertex in its path component.

Denote by $\tilde{b}_i(X; \mathbf{k})$ the minimal number of elements that generate the $\mathbf{k}$ -module $\tilde{H}_i(X; \mathbf{k})$. Clearly, $|\Theta(J)| = \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ for any principal ideal domain $\mathbf{k}$. Consider the $\tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ -element set

$$\{c(J \setminus \{i\}, u_i) : J \subset [m], i \in \Theta(J)\} \subset H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}).$$

We call its elements the *GPTW generators* (after Grbić, Panov, Theriault and Wu).

Theorem 1.1 *Let $\mathbf{k}$ be a commutative ring with unit and $\mathcal{K}$ be a flag simplicial complex without ghost vertices on vertex set $[m]$.*

(i) *For every $J \subset [m]$, choose a set of simplicial 1-cycles*

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} [\{i, j\}] \in C_1(\mathcal{K}_J; \mathbf{k})$$

that generate the $\mathbf{k}$ -module $H_1(\mathcal{K}_J; \mathbf{k})$. Then the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is generated by GPTW generators modulo the relations

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i, \max(B) > j}} \pm [\hat{c}(A, u_i), \hat{c}(B, u_j)] = 0$$

that correspond to the chosen 1-cycles. (Here $\hat{c}(A, u_i)$, $\hat{c}(B, u_j)$ are the elements $c(A, u_i)$, $c(B, u_j)$ of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ that are arbitrarily expressed through the GPTW generators, and $[x, y]$ is defined as $x \cdot y - (-1)^{|x| \cdot |y|} y \cdot x$.) In particular, $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ admits a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation by $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ generators and $\sum_{J \subset [m]} \tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations.

(ii) *If $\mathbf{k}$ is a principal ideal domain, then this presentation is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ contains at least $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J; \mathbf{k})$ generators and at least $\sum_{J \subset [m]} \tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations.*

This theorem follows from Theorems 5.1 and 5.6, proven in Section 5. For field coefficients, these results were partially obtained by Grbić, Panov, Theriault, Wu (the minimal set of generators [23, Theorem 4.3]) and the author (number of relations and their degrees [43, Corollary 4.5]). Sometimes the number of relations can be reduced, if we do not require them to be $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous (see Theorem 5.7).

We also present new results on the homotopy of moment-angle complexes that correspond to flag complexes. Using a result of Huang [26], we prove in Corollary 6.7 that in the flag case $\mathcal{Z}_{\mathcal{K}}$ is *coformal* over $\mathbb{Q}$ in the sense of rational homotopy theory. Results of Berglund [10] then give a necessary condition for such moment-angle complexes to be rationally formal (Theorem 6.13). Finally, we improve a recent result of Stanton [39] about the homotopy type of $\Omega \mathcal{Z}_{\mathcal{K}}$ by finding the explicit number of spheres in the product:

Theorem 1.2 *Let $\mathcal{K}$ be a $(d-1)$ -dimensional flag simplicial complex on $[m]$ with no ghost vertices. Then there is a homotopy equivalence*

$$(2) \quad \Omega \mathcal{Z}_{\mathcal{K}} \simeq \prod_{n \geq 3} (\Omega S^n)^{\times D_n},$$

where the numbers $D_n \geq 0$ are determined by

$$(3) \quad - \sum_{J \subset [m]} \bar{\chi}(\mathcal{K}_J) \cdot t^{|J|} = (1+t)^{m-d} h_{\mathcal{K}}(-t) = \prod_{n \geq 3} (1-t^{n-1})^{D_n},$$

where $\bar{\chi}(X) := \chi(X) - 1 = \sum_{i \geq 0} (-1)^i \dim \tilde{H}_i(X)$ is the reduced Euler characteristic and $h_{\mathcal{K}}(t) := \sum_{i=0}^d h_i(\mathcal{K}) \cdot t^i$ is the h -polynomial [15, Definition 2.2.5] of $\mathcal{K}$. In particular, for every $N \geq 1$ we have an isomorphism

$$(4) \quad \pi_N(\mathcal{Z}_{\mathcal{K}}) \simeq \bigoplus_{n=3}^N \pi_N(S^n)^{\oplus D_n}.$$

This theorem is proved in Section 6. Using (4), it is easy to describe the homotopy groups of corresponding Davis–Januszkiewicz spaces (using the fibration (1)) and partial quotients of moment-angle complexes, including quasitoric manifolds and smooth toric varieties (using similar fibrations; see [15, Proposition 7.3.13; 19, Section 4]).

1.2 Organisation of the paper

Section 2 consists of algebraic preliminaries. We highlight Corollary 2.2 that allows us to construct chain maps into the bar resolution. In Section 3 we recall notions from toric topology and discuss the properties of Pontryagin algebras $H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$ and $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$. Main calculations are carried in Section 4. In Section 5 we prove Theorem 1.1 and consider an example. Section 6 contains results about (co)formality and homotopy groups of moment-angle complexes in the flag case. In Appendix A we develop the homological tools for working with presentations of connected graded algebras over a commutative ring. In Appendix B we prove the following folklore fact: split fibrations of loop spaces correspond (by passing to homology) to extensions of Hopf algebras. Appendix C contains commutator identities that are used in Section 4.

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2 Preliminaries: algebra

2.1 Connected graded algebras

Fix a commutative associative ring k with unit. We consider associative k -algebras with unit that are graded by a commutative monoid G (usually $G = \mathbb{Z}$ or $\mathbb{Z}^k \times \mathbb{Z}_{\geq 0}^m$, $k = 0, 1, 2$.) Left A -modules are also G -graded. Elements of $\mathbb{Z}_{\geq 0}^m$ are denoted by $\alpha = (\alpha_1, \dots, \alpha_m) = \sum_{j=1}^m \alpha_j e_j$, $\alpha_j \geq 0$. Subsets $J \subset [m]$ are identified with elements $\sum_{j \in J} e_j \in \mathbb{Z}_{\geq 0}^m$. Define also

$$|\alpha| := \alpha_1 + \dots + \alpha_m, \quad \text{supp } \alpha := \{i \in [m] : \alpha_i > 0\}.$$

Every $\mathbb{Z}^k \times \mathbb{Z}_{\geq 0}^m$ -graded algebra A is considered as $\mathbb{Z}$ -graded with respect to the total grading

$$A_n := \bigoplus_{n=i_1+\dots+i_k+|\alpha|} A_{i_1, \dots, i_k, \alpha}.$$

The graded algebra A is *connected* if $A_{<0} = 0$ and $A_0 = k \cdot 1$. We have the canonical augmentation $\varepsilon: A \rightarrow A_0 = k$ and the augmentation ideal $I(A) := \text{Ker } \varepsilon$. Examples of connected k -algebras are

- the exterior algebra $\Lambda[m] := \Lambda[u_1, \dots, u_m]$, $\deg(u_i) = (-1, 2e_i) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$, with the basis $\{u_I := u_{i_1} \wedge \dots \wedge u_{i_k} : I = \{i_1 < \dots < i_k\}\}$;
- the polynomial algebra $k[m] := k[v_1, \dots, v_m]$, $\deg(v_i) = (0, 2e_i)$, with the basis

$$\left\{ v^\alpha := \prod_{i=1}^m v_i^{\alpha_i} : \alpha \in \mathbb{Z}_{\geq 0}^m \right\};$$

- the tensor algebra $T(x_1, \dots, x_N)$, where x_i are homogeneous elements of arbitrary positive degrees.

For a homogeneous element a , define $\bar{a} := (-1)^{1+\deg(a)} \cdot a$. Clearly, $\overline{a \cdot b} = -\bar{a} \cdot \bar{b}$ and $\bar{\bar{a}} = a$.

Let A be a G -graded algebra. Complexes of A -modules (M, d) are considered as $\mathbb{Z} \times G$ -graded modules with a differential of degree $(-1, 0)$. We use the Koszul sign rule with respect to the total grading: $d(a \cdot m) = (-1)^{\deg(a)} a \cdot d(m) = -\bar{a} \cdot d(m)$. Several formulas from [43] do not follow this rule and are corrected in this paper.

2.2 Bar resolution and bar construction

Let A be a connected k -algebra and $\varepsilon: A \rightarrow k$ be the augmentation. The resulting left A -module k has the *bar resolution*

$$\cdots \rightarrow B_2(A) \rightarrow B_1(A) \rightarrow B_0(A) \rightarrow k \rightarrow 0,$$

where $B_n(A) := A \otimes I(A)^{\otimes n}$. An element of the form $a \otimes a_1 \otimes \cdots \otimes a_n \in B_n(A)$ has bidegree $(n, \deg(a) + \sum_{i=1}^n \deg(a_i))$ and is traditionally written as $a[a_1 | \dots | a_n]$. The differential d_B has bidegree $(-1, 0)$ and is given by

$$-d_B(a[a_1 | \dots | a_n]) := \bar{a} \cdot a_1[a_2 | \dots | a_n] + \sum_{i=1}^{n-1} \bar{a}[\bar{a}_1 | \dots | \bar{a}_{i-1} \bar{a}_i \cdot a_{i+1} | a_{i+2} | \dots | a_n].$$

Consider also the contracting homotopy $s_n: B_n(A) \rightarrow B_{n+1}(A)$,

$$(5) \quad s_n(a[a_1 | \dots | a_n]) := \begin{cases} 0, & a \in A_0 \simeq k; \\ [a[a_1 | \dots | a_n], & \deg(a) > 0; \end{cases} \quad s_{-1}: k \rightarrow B_0(A), \quad 1 \mapsto 1[\cdot].$$

It is easy to show that $s \circ d_B + d_B \circ s = \text{id}$, $d_B^2 = 0$. Hence $(B(A), d_B)$ is a free resolution of the left A -module k , assuming that A is a free k module. In this case, we obtain

$$\text{Tor}_n^A(k, k) \cong H_n[\bar{B}(A), d_{\bar{B}}],$$

where $\bar{B}(A) := k \otimes_A B(A)$ is the *bar construction* of A . We have

$$(6) \quad \bar{B}_n(A) = I(A)^{\otimes n}, \quad \deg([a_1 | \dots | a_n]) = (n, \deg(a_1) + \dots + \deg(a_n)), \quad \deg d_{\bar{B}} = (-1, 0),$$

$$d_{\bar{B}}([a_1 | \dots | a_n]) = \sum_{i=1}^{n-1} [\bar{a}_1 | \dots | \bar{a}_{i-1} \bar{a}_i \cdot a_{i+1} | a_{i+2} | \dots | a_n] \in \bar{B}_{n-1}(A).$$

In particular, $d_{\bar{B}}([x|y]) = [\bar{x} \cdot y]$ and $d_{\bar{B}}([x|y|z]) = [\bar{x} \cdot y|z] + [\bar{x}|\bar{y} \cdot z]$.

2.3 Chain maps into resolutions with a contracting homotopy

Any map of modules can be extended to a map of their free resolutions. Moreover, this extension can be described in terms of the contracting homotopy for the latter resolution. This recursive construction seems to be known to specialists: its generalisations and applications are discussed in [12]. The author thanks Georgy Chernykh for the reference.

Lemma 2.1 *Let A be an associative k -algebra. Suppose that the commutative diagram of left A -modules and their homomorphisms*

$$\begin{array}{ccccc} C_n & \xrightarrow{\hat{d}_n} & C_{n-1} & \xrightarrow{\hat{d}_{n-1}} & C_{n-2} \\ & & \downarrow \varphi_{n-1} & & \downarrow \varphi_{n-2} \\ B_n & \xrightarrow{d_n} & B_{n-1} & \xrightarrow{d_{n-1}} & B_{n-2} \end{array}$$

satisfy the conditions

- (i) C_n is a free A -module with a basis $\{e_i\}$;
- (ii) $\hat{d}_{n-1} \circ \hat{d}_n = 0$;
- (iii) there are k -linear maps $s_{n-1}: B_{n-1} \rightarrow B_n$ and $s_{n-2}: B_{n-2} \rightarrow B_{n-1}$ such that

$$d_n \circ s_{n-1} + s_{n-2} \circ d_{n-1} = \text{id}_{B_{n-1}}.$$

Define an A -linear map $\varphi_n: C_n \rightarrow B_n$ on the basis by

$$\varphi_n(e_i) := s_{n-1}(\varphi_{n-1}(\hat{d}_n(e_i))) \in B_n.$$

Then $d_n \circ \varphi_n = \varphi_{n-1} \circ \hat{d}_n$.

Proof Since $d_n \circ \varphi_n$ and $\varphi_{n-1} \circ \hat{d}_n$ are maps of A -modules, it is sufficient to show that they agree on the basis of C_n . By definition,

$$d_n(\varphi_n(e_i)) = (d_n \circ s_{n-1} \circ \varphi_{n-1} \circ \hat{d}_n)(e_i).$$

Condition (iii) gives $d \circ s \circ \varphi \circ \hat{d} = \varphi \circ \hat{d} - s \circ d \circ \varphi \circ \hat{d}$. From the commutativity of the diagram and condition (ii) we obtain $s \circ d \circ \varphi \circ \hat{d} = s \circ \varphi \circ \hat{d} \circ \hat{d} = 0$. Hence

$$d_n(\varphi_n(e_i)) = \varphi_{n-1}(\hat{d}_n(e_i)) - 0. \quad \square$$

Corollary 2.2 Let A be a connected k -algebra, $(A \otimes V_\bullet, \hat{d}_\bullet)$ be a free resolution of the left A -module k . Let $\bar{\varphi}_0: V_0 \rightarrow k$ be a map of k -modules such that the diagram

$$\begin{array}{ccccc} A \otimes V_1 & \xrightarrow{\hat{d}_1} & A \otimes V_0 & \xrightarrow{\hat{d}_0} & k \\ & & \downarrow \text{id} \otimes \bar{\varphi}_0 & & \parallel \text{id} = \varphi_{-1} \\ B_1(A) & \xrightarrow{d_{B,1}} & A & \xrightarrow{\varepsilon} & k \end{array}$$

commutes. Choose bases $\{e_i^{(n)}\}$ of k -modules V_n , and define A -linear maps $\varphi_n: A \otimes V_n \rightarrow B_n(A)$ recursively as

$$\varphi_0 := \text{id}_A \otimes \bar{\varphi}_0, \quad \varphi_n(a \otimes e_i^{(n)}) := a \cdot s_{n-1}(\varphi_{n-1}(\hat{d}_n(e_i^{(n)}))),$$

where $s_{n-1}: B_{n-1}(A) \rightarrow B_n(A)$ is the contracting homotopy (5).

Then $\varphi_\bullet: (A \otimes V_\bullet, \hat{d}_\bullet) \rightarrow (B_\bullet(A), d_B)$ is a chain map.

Proof Induction on n . For $n = 0$ the identity

$$d_{B,n} \circ \varphi_n = \varphi_{n-1} \circ \hat{d}_n$$

holds, since the diagram commutes. The inductive step from $n - 1$ to n is supplied by Lemma 2.1. $\square$

2.4 Hopf algebra extensions and loop homology

If A is a Hopf algebra over k , we denote the comultiplication by $\Delta: A \rightarrow A \otimes A$ and the (co)unit maps by $\eta_A: k \rightarrow A$, $\varepsilon_A: A \rightarrow k$. A graded k -Hopf algebra A is *connected* if $A_{<0} = 0$, $A_0 = k \cdot 1$. The counit is then the standard augmentation $\varepsilon: A \rightarrow A_0 \simeq k$.

Definition 2.3 Let $\iota: A \rightarrow C$, $\pi: C \rightarrow B$ be morphisms of k -Hopf algebras. They form an *extension of Hopf algebras*, or a *short exact sequence of Hopf algebras*

$$k \rightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \rightarrow k,$$

if

- (i) ι is injective;
- (ii) π is surjective;
- (iii) $\pi \circ \iota = \varepsilon$;
- (iv) $\text{Ker } \pi = I(A) \cdot C$;
- (v) $\text{Im } \iota = \{x \in C : ((\text{id}_C \otimes \pi) \circ \Delta)(x) = x \otimes 1\}$.

See [2, Definition 1.2.0, Proposition 1.2.3] for an equivalent and more “symmetrical” definition. Extensions of connected Hopf algebras were studied implicitly in [31, Section 4].

Proposition 2.4 (see [31, Proposition 4.9]) Let $\iota: A \rightarrow C$, $\pi: C \rightarrow B$ be maps of connected k -Hopf algebras. Suppose that a map $\Phi: A \otimes B \rightarrow C$ is an isomorphism of left A -modules and right C -comodules, and suppose that

$$\iota = \Phi \circ (\text{id}_A \otimes \eta_B), \quad \pi \circ \Phi = \varepsilon_A \otimes \text{id}_B.$$

Then $k \rightarrow A \xrightarrow{\iota} C \xrightarrow{\pi} B \rightarrow k$ is an extension of Hopf algebras. Conversely, for every Hopf algebra extension there is a map Φ with the described properties. $\square$

Our main example of Hopf algebras are Pontryagin algebras (loop homology) of connected topological spaces. Let k be a commutative ring, Y be a topological space such that $H_*(Y; k)$ is a free k -module. Then $H_*(Y; k)$ is supplied with the cocommutative *cup coproduct* which is dual to the cup product on $H^*(Y; k)$: it is the composition

$$H_*(Y; k) \xrightarrow{\Delta_*} H_*(Y \times Y; k) \xrightarrow[\cong]{\text{AW}_*} H_*(C_*(Y; k) \otimes C_*(Y; k)) \xleftarrow[\cong]{\kappa} H_*(Y; k) \otimes H_*(Y; k),$$

where AW is the Alexander–Whitney map and κ is the Künneth isomorphism. If Y is also an H-space, the cup coproduct respects the Pontryagin product

$$m: H_*(Y; k) \otimes H_*(Y; k) \xrightarrow{\times} H_*(Y \times Y; k) \xrightarrow{\mu_*} H_*(Y; k)$$

and hence $H_*(Y; \mathbf{k})$ is a cocommutative Hopf algebra. In particular, $H_*(\Omega X; \mathbf{k})$ is a connected cocommutative $\mathbf{k}$ -Hopf algebra whenever X is a simply connected space such that $H_*(\Omega X; \mathbf{k})$ is free over $\mathbf{k}$ [31, 8.9]. Otherwise κ fails to be an isomorphism, hence the coproduct is not defined and $H_*(\Omega X; \mathbf{k})$ is merely a connected associative $\mathbf{k}$ -algebra with unit.

In Appendix B we describe a situation when a fibration $F \rightarrow E \rightarrow B$ of simply connected spaces gives rise to an extension $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$ of connected Hopf algebras.

3 Preliminaries: toric topology

3.1 Simplicial complexes and polyhedral products

A *simplicial complex* $\mathcal{K}$ on the vertex set W is a nonempty family of subsets $I \subset W$ that is closed under taking subsets. Elements $I \in \mathcal{K}$ are called *faces*. We suppose that $\mathcal{K}$ has no ghost vertices, ie $\{i\} \in \mathcal{K}$ for all $i \in W$. Usually $W \subset [m] := \{1, \dots, m\}$. Sometimes by properties of a complex $\mathcal{K}$ we mean properties of its *geometrical realisation*, of the topological space $|\mathcal{K}| := \bigcup_{I \in \mathcal{K}} \Delta_I \subset \Delta_W$.

For every $J \subset W$, a simplicial complex $\mathcal{K}_J := \{I \in \mathcal{K} : I \subset J\}$ on the vertex set J (a *full subcomplex* of $\mathcal{K}$) is defined.

Throughout the text, we write $I \setminus i := I \setminus \{i\}$ for $i \in I$ and $I \sqcup i := I \sqcup \{i\}$ for $i \in W \setminus I$. A subset $I \subset W$ is a *missing face* of $\mathcal{K}$ if $I \notin \mathcal{K}$, but $I \setminus i \in \mathcal{K}$ for all $i \in I$. A simplicial complex $\mathcal{K}$ is *flag* if all its missing faces consist of two elements.

For every complex $\mathcal{K}$ on a vertex set $[m]$, the $\mathbb{Z}_{\geq 0}^m$ -graded *Stanley–Reisner ring*

$$\mathbf{k}[\mathcal{K}] := \mathbf{k}[v_1, \dots, v_m] / \left(\prod_{i \in I} v_i = 0, I \notin \mathcal{K} \right), \quad \deg v_i := 2e_i \in \mathbb{Z}_{\geq 0}^m,$$

is defined. It has a homogeneous basis $\{v^\alpha := \prod_{i=1}^m v_i^{\alpha_i} \mid \text{supp } \alpha \in \mathcal{K}\}$ as a $\mathbf{k}$ -module. The dual $\mathbf{k}$ -module $\mathbf{k}\langle \mathcal{K} \rangle$ is called the *Stanley–Reisner coalgebra*. It has an additive basis $\{\chi_\alpha \mid \text{supp } \alpha \in \mathcal{K}\}$, $\deg \chi_\alpha = 2\alpha$, and commutative associative comultiplication $\Delta \chi_\alpha := \sum_{\alpha = \beta + \gamma} \chi_\beta \otimes \chi_\gamma$.

Now let $\mathcal{K}$ be a simplicial complex on $[m]$ and $(\underline{X}, \underline{A}) := ((X_1, A_1), \dots, (X_m, A_m))$ be a sequence of pairs of topological spaces. Their *polyhedral product* $(\underline{X}, \underline{A})^\mathcal{K}$ is the union

$$(\underline{X}, \underline{A})^\mathcal{K} := \bigcup_{I \in \mathcal{K}} (\underline{X}, \underline{A})^I \subset X^m, \quad (\underline{X}, \underline{A})^I = Y_1 \times \dots \times Y_m, \quad Y_j := \begin{cases} X_j, & j \in I; \\ A_j, & j \notin I. \end{cases}$$

The addition of a ghost vertex v to $\mathcal{K}$ replaces the space $(\underline{X}, \underline{A})^\mathcal{K}$ with $(\underline{X}, \underline{A})^\mathcal{K} \times A_v$. Hence in many cases it is sufficient to consider only complexes without ghost vertices.

Define $(X, A)^\mathcal{K} := (\underline{X}, \underline{A})^\mathcal{K}$ if $X_i = X$, $A_i = A$ for all $i \in [m]$. We consider two special cases of this construction: *moment-angle complexes* $\mathcal{Z}_\mathcal{K} := (D^2, S^1)^\mathcal{K}$ and *Davis–Januszkiewicz spaces*

$\mathrm{DJ}(\mathcal{K}) := (\mathbb{C}P^\infty, *)^{\mathcal{K}}$. It is well known that $H^*(\mathrm{DJ}(\mathcal{K}); \mathbf{k}) \cong \mathbf{k}[\mathcal{K}]$ and $H^*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[m]}(\mathbf{k}[\mathcal{K}], \mathbf{k})$ as graded rings. Moreover,

$$H^n(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) = \bigoplus_{n=-i+2|J|} H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}), \quad H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J; \mathbf{k}),$$

and the product has a geometric description in terms of maps $\mathcal{K}_{I \sqcup J} \hookrightarrow \mathcal{K}_I * \mathcal{K}_J$; see [15, Theorem 4.5.8].

3.2 Loop homology as Hopf algebras

Proposition 3.1 [15, Theorem 4.3.2, Section 8.4] *There is a homotopy fibration*

$$\mathcal{Z}_{\mathcal{K}} \rightarrow \mathrm{DJ}(\mathcal{K}) \xrightarrow{i} (\mathbb{C}P^\infty)^m$$

of simply connected spaces, where i is the standard inclusion. The map Ωi admits a homotopy section $\sigma: \mathbb{T}^m \rightarrow \Omega \mathrm{DJ}(\mathcal{K})$ that corresponds to the choice of generators in $\pi_2(\mathrm{DJ}(\mathcal{K})) \cong \mathbb{Z}^m$ and gives rise to a homotopy equivalence $\Omega \mathrm{DJ}(\mathcal{K}) \simeq \Omega \mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$. $\square$

The following description of $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$ was first given in [35, (8.4)] for $\mathbf{k} = \mathbb{Q}$, but the argument is easily generalised to the arbitrary coefficient ring. The main ingredients are integral *formality* of $\mathrm{DJ}(\mathcal{K})$ [34], Adams' cobar construction (see [18]) and a result of Fröberg [21].

Theorem 3.2 [43, Theorem 1.1] *For any simplicial complex $\mathcal{K}$ with no ghost vertices and any commutative ring $\mathbf{k}$, we have an isomorphism $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \cong \mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}(\mathbf{k}, \mathbf{k})$ of graded $\mathbf{k}$ -algebras (with respect to the Pontryagin product and to the Yoneda product). More precisely,*

$$H_n(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \cong \bigoplus_{-i+2|\alpha|=n} \mathrm{Ext}_{\mathbf{k}[\mathcal{K}]}^i(\mathbf{k}, \mathbf{k})_{2\alpha}.$$

This isomorphism defines the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading on $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$. The “diagonal” subalgebra

$$D = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^m} H_{-|\alpha|, 2\alpha}(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \subset H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$$

is isomorphic to the algebra

$$\mathbf{k}[\mathcal{K}]^! := T(u_1, \dots, u_m) / (u_i^2 = 0, i = 1, \dots, m; u_i u_j + u_j u_i = 0, \{i, j\} \in \mathcal{K}), \quad \deg u_i = (-1, 2e_i).$$

For a flag $\mathcal{K}$, the algebra $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$ coincides with D , and we have $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \cong \mathbf{k}[\mathcal{K}]^!$. $\square$

If $H_*(\Omega Y; \mathbf{k})$ is a free $\mathbf{k}$ -module, the cup coproduct is compatible with the Pontryagin product, hence this associative algebra is a cocommutative $\mathbf{k}$ -Hopf algebra. Similarly, if A is a commutative graded $\mathbf{k}$ -algebra such that $\mathrm{Ext}_A(\mathbf{k}, \mathbf{k})$ is a free $\mathbf{k}$ -module, then the shuffle product on the bar construction (see [28, Theorem X.12.2]) induces a commutative coproduct on $\mathrm{Ext}_A(\mathbf{k}, \mathbf{k})$ that is compatible with the Yoneda product. In our case, these coproduct coincide. This follows from a stronger formality result for Davis–Januszkiewicz spaces, the *hga formality* [20, Theorem 1.3].

Proposition 3.3 [20, Proposition 6.5] *Let $\mathcal{K}$ be a simplicial complex with no ghost vertices, and let k be a principal ideal domain such that $H_*(\Omega\mathrm{DJ}(\mathcal{K}); k)$ is a free k -module. Then*

$$H_*(\Omega\mathrm{DJ}(\mathcal{K}); k) \cong \mathrm{Ext}_{k[\mathcal{K}]}(k, k)$$

as Hopf algebras.

Outline of the proof Let A be a dga algebra. The *homotopy Gerstenhaber algebra* (hga) structure on A is a multiplication on its bar construction $\bar{B}(A)$ such that $\bar{B}(A)$ becomes a dga bialgebra [20, Section 4]. This structure arises naturally if A is commutative (then the multiplication is the shuffle product) or if $A = C^*(X; k)$ is the dga algebra of cochains of a 1-reduced simplicial set (then the multiplication was essentially constructed by Baues [8, Section 2]). Then $H^*(\Omega X; k) \cong H^*[\bar{B}(C^*(X; k))]$ as bialgebras. By a result of Franz [20, Theorem 1.3], hga algebras $C^*(\mathrm{DJ}(\mathcal{K}); k)$ and $k[\mathcal{K}]$ are quasi-isomorphic. The functor $\bar{B}$ preserves quasi-isomorphisms, so $H^*(\Omega\mathrm{DJ}(\mathcal{K}); k) \cong H^*(\bar{B}(k[\mathcal{K}]); k) \cong \mathrm{Tor}^{k[\mathcal{K}]}(k, k)$ as bialgebras. Since the Hopf algebra structure on a bialgebra is unique, it is an isomorphism of Hopf algebras. The statement for $H_*(\Omega\mathrm{DJ}(\mathcal{K}); k)$ follows by dualisation. $\square$

Remark 3.4 The algebra $H_*(\Omega\mathrm{DJ}(\mathcal{K}); k)$ is not always a free k -module. For example, let $\mathcal{K}$ be a minimal triangulation of $\mathbb{R}P^2$. Then $\mathcal{Z}_{\mathcal{K}}$ is a wedge of $\Sigma^7\mathbb{R}P^2$ and spheres [23, Example 3.3]. We have $\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$, hence $\Omega\Sigma^7\mathbb{R}P^2$ is a retract of $\Omega\mathrm{DJ}(\mathcal{K})$. It follows that $H_*(\Omega\mathrm{DJ}(\mathcal{K}); \mathbb{Z})$ has 2-torsion.

Recall that an element x is called *primitive* if $\Delta x = x \otimes 1 + 1 \otimes x$, and a Hopf algebra is *primitively generated* if it is multiplicatively generated by its primitive elements.

Conjecture 3.5 *The Hopf algebra $H_*(\Omega\mathrm{DJ}(\mathcal{K}); k)$ is primitively generated for every simplicial complex $\mathcal{K}$ and every ring k such that $H_*(\Omega\mathrm{DJ}(\mathcal{K}); k)$ is a free k -module.*

By deep results of André and Sjödén — see [4, Theorem 10.2.1(5)] — for every field k the Hopf algebra $\mathrm{Ext}_A(k, k)$ is the universal enveloping of a Lie algebra (of a 2-restricted Lie algebra, if $\mathrm{char} k = 2$). In particular, this Hopf algebra is primitively generated. (This also follows from results of Browder [11]; see [33, Theorem 10.4].) Hence Conjecture 3.5 holds if k is a field.

Remark 3.6 The Hopf algebra $H_*(\Omega X; k)$ is not always primitively generated, even if X is a suspension. For example, one can take $X = \Sigma\mathbb{C}P^2$, $k = \mathbb{Z}$ or $\mathbb{Z}/2$ (see [13, Section 4.2]). On the other hand, [25, Theorem B] implies that $H_*(\Omega\Sigma\mathbb{C}P^d; \mathbb{Z}/p)$ is primitively generated for $p > d$.

Now we describe the connection between the loop homology of Davis–Januszkiewicz spaces and of moment-angle complexes in the form of a Hopf algebra extension.

Proposition 3.7 *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and k be a commutative ring with unit, such that $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; k)$ is a free k -module. Then*

$$k \rightarrow H_*(\Omega\mathcal{Z}_{\mathcal{K}}; k) \xrightarrow{\iota} H_*(\Omega\mathrm{DJ}(\mathcal{K}); k) \xrightarrow{p} \Lambda[u_1, \dots, u_m] \rightarrow k$$

is an extension of connected $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded k -Hopf algebras. The projection p maps u_i to u_i . Its k -linear section $\sigma_* : \Lambda[u_1, \dots, u_m] \rightarrow H_*(\Omega \text{DJ}(\mathcal{K}); k)$ is given by

$$\sigma_*(u_I) = \hat{u}_I := u_{i_1} \cdots u_{i_k}, \quad I = \{i_1 < \cdots < i_k\}.$$

Therefore, $\Phi(a \otimes u_I) := \iota(a) \cdot \hat{u}_I$ defines an isomorphism of left $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ -modules and right $\Lambda[u_1, \dots, u_m]$ -comodules $\Phi : H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k) \otimes \Lambda[u_1, \dots, u_m] \rightarrow H_*(\Omega \text{DJ}(\mathcal{K}); k)$.

Proof By Theorem B.3, the fibration from Proposition 3.1 gives rise to the required Hopf algebra extension. The formula for p follows from functoriality, since the map $\text{DJ}(\mathcal{K}) \hookrightarrow \text{DJ}(\Delta_{[m]}) \cong (\mathbb{C}P^\infty)^m$ is induced by the inclusion $\mathcal{K} \hookrightarrow \Delta_{[m]}$. The formula for σ_* follows from the description of the homotopy section $\sigma : \mathbb{T}^m \simeq \Omega B\mathbb{T}^m = (\Omega \mathbb{C}P^\infty)^{\times m} \rightarrow \Omega \text{DJ}(\mathcal{K})$ as a concatenation of loops, $(\gamma_1, \dots, \gamma_m) \mapsto \gamma_1 \cdots \gamma_m$. The maps p and σ_* respect the $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading, hence the multigrading on $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is well defined. $\square$

Since ι is injective, we identify elements of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ with their images in $H_*(\Omega \text{DJ}(\mathcal{K}); k)$. Let us describe some of these elements. Recall that we define $[a, b] := ab + (-1)^{\deg(a)\deg(b)+1}ba$ and $c(I, x) := [u_{i_1}, [u_{i_2}, \dots, [u_{i_k}, x], \dots]] \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$ for $I = \{i_1 < \cdots < i_k\}$ and $x \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$. In particular, $c(\emptyset, x) := x$ and $c(\{i\}, u_j) = [u_i, u_j] = u_i u_j + u_j u_i$.

Corollary 3.8 *Let $x \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$ be a primitive element such that $p(x) = 0$. Then $x \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$.*

Proof This follows from Corollary B.4 applied to the Hopf algebra extension from Proposition 3.7. $\square$

Corollary 3.9 *Let $x \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$ be a primitive element and $I \subset [m]$, $I \neq \emptyset$. Then $c(I, x)$ is an element of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$.*

Proof Elements $u_1, \dots, u_m \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$ are primitive for dimension reasons. Primitive elements form a Lie algebra, hence $c(I, x) \in H_*(\Omega \text{DJ}(\mathcal{K}); k)$ is primitive. We have $p(c(I, x)) = c(I, p(x)) = 0$, since it is a commutator in the commutative algebra $\Lambda[m]$. Then $c(I, x) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ by Corollary 3.8. $\square$

Corollary 3.10 *Let $j \in [m]$ and $I \subset [m]$, $I \neq \emptyset$. Then $c(I, u_j)$ is an element of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$.* $\square$

3.3 The flag case

Let $\mathcal{K}$ be a flag complex with no ghost vertices. By Theorem 3.2, $H_*(\Omega \text{DJ}(\mathcal{K}); k) \cong k[\mathcal{K}]^!$ is a free k -module, hence the Hopf algebra structure on $H_*(\Omega \text{DJ}(\mathcal{K}); k)$ is well defined. Moreover, the connected k -algebra $k[\mathcal{K}]^!$ is generated by elements of degree 1. These conditions determine the Hopf algebra structure on $k[\mathcal{K}]^!$ uniquely: the elements $u_1, \dots, u_m$ are primitive. Therefore, in the flag case Conjecture 3.5 is true for any k .

The following important result was recently obtained by Stanton.

Theorem 3.11 [39, Corollary 1.5] *Let $\mathcal{K}$ be a flag simplicial complex or a skeleton of a flag complex. Then $\Omega \mathcal{Z}_{\mathcal{K}}$ is homotopy equivalent to a finite-type product of spaces of the form S^1 , S^3 , S^7 and ΩS^n for $n \geq 2$, $n \neq 2, 4, 8$.* $\square$

This gives a short proof of the fact that $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is free over $\mathbf{k}$.

Proposition 3.12 [23, Corollary 5.2] *If $\mathcal{K}$ is a flag simplicial complex, then $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is a free $\mathbf{k}$ -module of finite type.*

Proof By the Künneth formula (more precisely, by the collapse of the Künneth spectral sequence — see [37, Theorem 10.90]), $H_*(X \times Y; \mathbf{k}) \simeq H_*(X; \mathbf{k}) \otimes H_*(Y; \mathbf{k})$ if $H_*(X; \mathbf{k})$ and $H_*(Y; \mathbf{k})$ are free over $\mathbf{k}$. Hence $H_*(X \times Y; \mathbf{k})$ is also a free $\mathbf{k}$ -module.

Clearly, $H_*(S^n; \mathbf{k})$ and $H_*(\Omega S^n; \mathbf{k}) \simeq T(a_{n-1})$ are free $\mathbf{k}$ -modules. By Theorem 3.11 and the arguments above, the same holds for $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$. $\square$

Hence in the flag case we have a Hopf algebra extension

$$\mathbf{k} \rightarrow H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \rightarrow H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}) \rightarrow \Lambda[m] \rightarrow \mathbf{k}$$

from Proposition 3.7 for any $\mathbf{k}$.

4 Main calculations

In what follows, $\mathcal{K}$ is a flag simplicial complex on the vertex set $[m]$ with no ghost vertices, and $\mathbf{k}$ is a commutative ring with unit. We consider $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded $\mathbf{k}$ -algebras that are connected with respect to the total grading $A_n := \bigoplus_{n=-i+|\alpha|} A_{-i, \alpha}$.

4.1 Resolutions and formulas for differentials

By [43, Proposition 4.1], the left $H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k})$ -module $\mathbf{k}$ has a free resolution $(H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k} \langle \mathcal{K} \rangle, d)$, $\deg \chi_{\alpha} := (|\alpha|, -|\alpha|, 2\alpha)$, $\deg(d) = (-1, 0, 0)$, with the differential

$$d(1 \otimes \chi_{\alpha}) := \sum_{i \in \text{supp}(\alpha)} u_i \otimes \chi_{\alpha - e_i}.$$

The isomorphism of left $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -modules

$$\Phi: H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \rightarrow H_*(\Omega \text{DJ}(\mathcal{K}); \mathbf{k}), \quad a \otimes u_I \mapsto a \cdot \hat{u}_I,$$

from Proposition 3.7 allows us to consider this resolution as a free resolution $(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \hat{d})$ of the left $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ -module $\mathbf{k}$. We apply the functor $\mathbf{k} \otimes_{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})} (-)$ and obtain a chain complex

$(\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \bar{d})$ whose homology is isomorphic to $\mathrm{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})$. The differentials $\hat{d}$ and $\bar{d}$ are determined by the commutative diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{d} & H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}) \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \cdots \\
 & & \uparrow \Phi \otimes \mathrm{id} \simeq & & \uparrow \Phi \otimes \mathrm{id} \simeq & & \\
 \cdots & \longrightarrow & H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{\hat{d}} & H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \otimes \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \cdots \\
 & & \downarrow \varepsilon \otimes \mathrm{id} \otimes \mathrm{id} & & \downarrow \varepsilon \otimes \mathrm{id} \otimes \mathrm{id} & & \\
 \cdots & \longrightarrow & \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n)} & \xrightarrow{\bar{d}} & \Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle_{(n-1)} & \longrightarrow & \cdots
 \end{array}$$

Here $\mathbf{k}\langle \mathcal{K} \rangle_{(n)}$ is a $\mathbf{k}$ -submodule in $\mathbf{k}\langle \mathcal{K} \rangle$ with the basis $\{\chi_\alpha : |\alpha| = n\}$. With different signs, this construction was considered by the author in [43, Section 4]. Now we describe the differential $\hat{d}$ explicitly. For subsets $A, B \subset [m]$ define the Koszul sign $\theta(A, B) := |\{(a, b) \in A \times B : a > b\}|$.

Proposition 4.1 *The differential $\hat{d}$ is given by*

$$\begin{aligned}
 (7) \quad \hat{d}(1 \otimes u_I \otimes \chi_\alpha) &= \sum_{i \in \mathrm{supp}(\alpha)} (-1)^{|I|} \cdot 1 \otimes (u_I \wedge u_i) \otimes \chi_{\alpha - e_i} \\
 &\quad + \sum_{i \in \mathrm{supp}(\alpha)} \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |A|} c(A, u_i) \otimes u_B \otimes \chi_{\alpha - e_i}.
 \end{aligned}$$

The differential $\bar{d}$ is given by

$$(8) \quad \bar{d}(u_I \otimes \chi_\alpha) = (-1)^{|I|} \sum_{i \in \mathrm{supp}(\alpha)} (u_I \wedge u_i) \otimes \chi_{\alpha - e_i}.$$

Remark 4.2 We define $\max(\emptyset) := -\infty$, hence A cannot be empty.

Proof of the proposition Recall that $u_j^2 = 0 \in H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k})$. Therefore, by Proposition C.2,

$$\begin{aligned}
 \hat{u}_I \cdot u_i &= 1 \cdot \begin{cases} (-1)^{|I| > i} \hat{u}_{I \sqcup i}, & i \notin I \\ 0, & i \in I \end{cases} + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \cdot \hat{u}_B \\
 &= \Phi \left(1 \otimes (u_I \wedge u_i) + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \otimes u_B \right) \in H_*(\Omega \mathrm{DJ}(\mathcal{K}); \mathbf{k}).
 \end{aligned}$$

(Here $c(A, u_i) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ by Corollary 3.10.) Define $\Phi_0 = \Phi \otimes \mathrm{id}_{\mathbf{k}\langle \mathcal{K} \rangle}$. Then

$$\begin{aligned}
 \Phi_0(\hat{d}(1 \otimes u_I \otimes \chi_\alpha)) &= d(\Phi_0(1 \otimes u_I \otimes \chi_\alpha)) = d(\hat{u}_I \otimes \chi_\alpha) = (-1)^{|I|} \sum_{i \in \mathrm{supp}(\alpha)} \hat{u}_I u_i \otimes \chi_{\alpha - e_i} \\
 &= (-1)^{|I|} \sum_{i \in \mathrm{supp}(\alpha)} \Phi_0 \left(1 \otimes (u_I \wedge u_i) \otimes \chi_{\alpha - e_i} + \sum_{\substack{I = A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |B|} c(A, u_i) \otimes u_B \otimes \chi_{\alpha - e_i} \right).
 \end{aligned}$$

Applying Φ_0^{-1} , we obtain precisely (7). After the homomorphism $\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ it turns into (8), since $\varepsilon(1) = 1$ and $\varepsilon(c(A, u_i)) = 0$ for $A \neq \emptyset$. $\square$

4.2 Computation of Tor-modules

By [43, Theorem 1.2], for a flag $\mathcal{K}$ we have a $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded isomorphism of $\mathbf{k}$ -modules

$$(9) \quad \mathrm{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}_*(\mathcal{K}_J; \mathbf{k}), \quad \mathrm{Tor}_n^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}).$$

The homology of $\mathcal{Z}_{\mathcal{K}}$ admits a $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading, and for any $\mathcal{K}$ we have a similar additive isomorphism dual to [15, Theorem 4.5.8]:

$$H_*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}_*(\mathcal{K}_J; \mathbf{k}), \quad H_{n-|J|, 2J}(\mathcal{Z}_{\mathcal{K}}; \mathbf{k}) \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}).$$

Hence $\mathrm{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong H_*(\mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ for flag $\mathcal{K}$. Moreover, both modules are computed as the homology of $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, d)$.

Remark 4.3 In general, if X is simply connected and $H_*(\Omega X; \mathbf{k})$ is free over $\mathbf{k}$, there is *Milnor–Moore spectral sequence* $E_{p,q}^2 = \mathrm{Tor}_p^{H_*(\Omega X; \mathbf{k})}(\mathbf{k}, \mathbf{k})_q \Rightarrow H_{p+q}(X; \mathbf{k})$. We see that it collapses at E^2 for $X = \mathcal{Z}_{\mathcal{K}}$ if $\mathcal{K}$ is a flag complex. For $\mathbf{k} = \mathbb{Q}$, the collapse is explained by the *coformality* of $\mathcal{Z}_{\mathcal{K}}$; see Corollary 6.7 and the discussion after.

Now we construct a chain map g that induces the isomorphism (9). For any chain complex $(C_\bullet, d)$ of free $\mathbf{k}$ -modules, we have the *dual complex*

$$(C^\bullet, d_{\mathrm{dual}}), \quad C^n := \mathrm{Hom}_{\mathbf{k}}(C_n, \mathbf{k}), \quad d_{\mathrm{dual}}(f) : c \mapsto f(d(c)).$$

Dualisation preserves isomorphisms and chain homotopies. For a simplicial complex $\mathcal{K}$, the augmented complex of simplicial chains $\widetilde{C}_*(\mathcal{K}; \mathbf{k})$ has the basis $\{[I] : I \in \mathcal{K}\}$, $\deg[I] := |I| + 1$ and the differential

$$d([I]) := \sum_{i \in I} (-1)^{|I| < i|} [I \setminus \{i\}].$$

The dual complex is the augmented complex of simplicial cochains $(\widetilde{C}^*(\mathcal{K}; \mathbf{k}), d_{\mathrm{dual}})$, which has the basis $\{[I]^* : I \in \mathcal{K}\}$ and the differential

$$d_{\mathrm{dual}}([I]^*) = \sum_{\substack{i \notin I \\ I \sqcup i \in \mathcal{K}}} (-1)^{|I| < i|} [I \sqcup \{i\}]^*.$$

Proposition 4.4 For every $J \subset [m]$, consider the map

$$g_J : \widetilde{C}_{*-1}(\mathcal{K}_J; \mathbf{k}) \rightarrow (\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle)_{*, -|J|, 2J}, \quad [L] \mapsto \epsilon(L, J) \cdot u_{J \setminus L} \otimes \chi_L,$$

where $\epsilon(L, J) := (-1)^{\sum_{\ell \in L} |J < \ell|}$. Then g_J are chain maps, and the direct sum

$$g : \bigoplus_{J \subset [m]} \widetilde{C}_*(\mathcal{K}_J; \mathbf{k}) \rightarrow (\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \bar{d})$$

induces an isomorphism on homology. Therefore,

$$H_{n, -|J|, 2J}(\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \bar{d}) \cong \widetilde{H}_{n-1}(\mathcal{K}_J; \mathbf{k}), \quad J \subset [m], n \geq 0,$$

all the other graded components of $H_*(\Lambda[m] \otimes \mathbf{k}\langle \mathcal{K} \rangle, \bar{d})$ being zero.

Since $\mathrm{Tor}^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k}) \cong H(\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d})$, this proposition implies (9). The proof is the dualisation of arguments from [15, Section 3.2].

Proof of Proposition 4.4 Consider the dga algebra $(\Lambda[u_1, \dots, u_m] \otimes \mathbf{k}[\mathcal{K}], d)$ with the differential that is defined on generators by $d(u_i) = v_i$, $d(v_i) = 0$ and with the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading

$$\deg u_i := (0, -1, 2e_i), \quad \deg v_i := (1, -1, 2e_i), \quad \deg d := (1, 0, 0).$$

This complex has the basis $\{u_I v^\alpha : I \subset [m], \alpha \in \mathbb{Z}_{\geq 0}^m, \mathrm{supp}(\alpha) \in \mathcal{K}\}$ and the differentials

$$d(u_I v^\alpha) = \sum_{i \in I} (-1)^{|I_{<i}|} u_{I_{<i}} v_i u_{I_{>i}} v^\alpha = \sum_{i \in I} (-1)^{|I_{<i}|} u_{I \setminus i} v_i v^\alpha.$$

Then the dual complex $(\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*$ has the basis $\{(u_I v^\alpha)^* : I \subset [m], \mathrm{supp}(\alpha) \in \mathcal{K}\}$ and the differential

$$d_{\mathrm{dual}}((u_I v^\alpha)^*) = \sum_{\substack{i \in \mathrm{supp}(\alpha) \\ i \notin I}} (-1)^{|I_{<i}|} (u_{I \sqcup i} v^{\alpha - e_i})^*.$$

This formula is similar to (8). We obtain an isomorphism of chain complexes

$$\psi : (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \rightarrow ((\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*, d_{\mathrm{dual}}), \quad u_I \otimes \chi_\alpha \mapsto (u_I v^\alpha)^*.$$

Consider the dga algebra $R^*(\mathcal{K}) := (\Lambda[m] \otimes \mathbf{k}[\mathcal{K}]) / (u_i v_i = v_i^2 = 0, i = 1, \dots, m)$. It is well defined, since the ideal $(u_i v_i, v_i^2) \subset \Lambda[m] \otimes \mathbf{k}[\mathcal{K}]$ is d -invariant. The following facts are obtained in the proof of [15, Theorem 3.2.9].

Lemma 4.5 [15, Lemma 3.2.6] *The natural projection $\pi : \Lambda[m] \otimes \mathbf{k}[\mathcal{K}] \rightarrow R^*(\mathcal{K})$ is a chain homotopy equivalence.* $\square$

Lemma 4.6 *We have well-defined chain maps $f_J : \tilde{C}^*(\mathcal{K}_J; \mathbf{k}) \rightarrow R^*(\mathcal{K})$,*

$$f_J : \tilde{C}^{n-1}(\mathcal{K}_J; \mathbf{k}) \xrightarrow{\cong} R^{n, -n, 2J}(\mathcal{K}), \quad [L]^* \mapsto \epsilon(L, J) \cdot u_{J \setminus L} v^L, \quad \epsilon(L, J) := (-1)^{\sum_{\ell \in L} |J_{<\ell}|}.$$

The direct sum $f : \bigoplus_{J \subset [m]} \tilde{C}^*(\mathcal{K}_J; \mathbf{k}) \rightarrow R^*(\mathcal{K})$ is an isomorphism of chain complexes. $\square$

After dualisation, we obtain a chain homotopy equivalence π^* and an isomorphism f^* of chain complexes. It remains to show that the diagram

$$\begin{array}{ccc} \bigoplus_{J \subset [m]} \tilde{C}^*(\mathcal{K}_J; \mathbf{k}) & \xrightarrow{g} & (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \\ f^* \uparrow \simeq & & \uparrow \psi \simeq \\ (R^*(\mathcal{K}))^* & \xrightarrow[\sim]{\pi^*} & ((\Lambda[m] \otimes \mathbf{k}[\mathcal{K}])^*, d_{\mathrm{dual}}) \end{array}$$

is commutative. Indeed, $f^*((u_{J \setminus L} v^L)^*) = \epsilon(L, J) \cdot [L]$, hence

$$g(f^*((u_{J \setminus L} v^L)^*)) = \epsilon(L, J) \cdot \epsilon(L, J) u_{J \setminus L} \otimes \chi_L = \psi(\pi^*((u_{J \setminus L} v^L)^*)). \quad \square$$

Remark 4.7 In our notation, $\epsilon(L, J) = (-1)^n$, $n = \theta(J \setminus L, L) + |L|(|L| - 1)/2$ for $L \subset J$.

4.3 A chain map to the bar resolution

Theorem 4.8 *The identity map of the left $H_*(\Omega \mathcal{Z}_K; k)$ -module k can be extended to the map of free resolutions $\varphi_\bullet: (H_*(\Omega \mathcal{Z}_K; k) \otimes \Lambda[m] \otimes k \langle K \rangle, \hat{d}) \rightarrow (B_*(H_*(\Omega \mathcal{Z}_K; k)), d_B)$, given by*

$$\varphi_n(u_I \otimes \chi_\alpha) = (-1)^{|I|} \sum_{\substack{\alpha=e_{i_1}+\dots+e_{i_n} \\ I=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^{\sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})} [c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})].$$

Proof We apply Corollary 2.2 for $\bar{\varphi}_0(u_I) = \varepsilon(u_I)$. It is sufficient to show that

$$\varphi_{n+1}(u_I \otimes \chi_\alpha) = s(\varphi_n(\hat{d}(u_I \otimes \chi_\alpha)))$$

for $|\alpha| = n + 1$, $n \geq 0$. By (7) and by the $H_*(\Omega \mathcal{Z}_K; k)$ -linearity of φ_n , we have

$$\begin{aligned} \varphi_n(\hat{d}(u_I \otimes \chi_\alpha)) &= \sum_{i \in \text{supp}(\alpha)} (-1)^{|I|} \varphi_n((u_I \wedge u_i) \otimes \chi_{\alpha-e_i}) \\ &\quad + \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} (-1)^{\theta(A, B) + |A|} c(A, u_i) \varphi_n(u_B \otimes \chi_{\alpha-e_i}). \end{aligned}$$

The map s is trivial on summands of the first sum, since they belong to

$$\bar{B}(H_*(\Omega \mathcal{Z}_K; k)) \subset \text{Ker } s \subset B(H_*(\Omega \mathcal{Z}_K; k)).$$

Hence we have

$$\begin{aligned} s(\varphi_n(\hat{d}(u_I \otimes \chi_\alpha))) &= 0 + \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} \sum_{\substack{\alpha-e_i=e_{i_1}+\dots+e_{i_n} \\ B=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^\zeta s(c(A, u_i) [c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})]) \\ &= \sum_{i \in \text{supp}(\alpha)} \sum_{\substack{I=A \sqcup B \\ \max(A) > i}} \sum_{\substack{\alpha-e_i=e_{i_1}+\dots+e_{i_n} \\ B=A_1 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, \forall t \in [n]}} (-1)^\zeta [c(A, u_i) | c(A_1, u_{i_1}) | \dots | c(A_n, u_{i_n})], \end{aligned}$$

where $\zeta = |B| + \theta(A, B) + |A| + \sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})$. Defining $i = i_0$, $A = A_0$, we obtain

$$\begin{aligned} s(\varphi_n(\hat{d}(u_I \otimes \chi_\alpha))) &= \sum_{\substack{\alpha=e_{i_0}+\dots+e_{i_n} \\ I=A_0 \sqcup \dots \sqcup A_n \\ \max(A_t) > i_t, 0 \leq t \leq n}} (-1)^{\sum_{t=1}^n \theta(A_0, A_t) + |I| + \sum_{1 \leq t_1 < t_2 \leq n} \theta(A_{t_1}, A_{t_2})} [c(A_0, u_{i_0}) | \dots | c(A_n, u_{i_n})]. \end{aligned}$$

The right-hand side equals $\varphi_{n+1}(u_I \otimes \chi_\alpha)$ up to a shift of indices. $\square$

Theorem 4.9 *Let $J \subset [m]$. Let a class $\alpha \in \text{Tor}_n^{H_*(\Omega \mathcal{Z}_K)}(k, k)_{-|J|, 2J} \cong \widetilde{H}_{n-1}(\mathcal{K}_J; k)$ be represented by a cycle*

$$\kappa = \sum_{\substack{I \in \mathcal{K}_J \\ |I|=n}} \lambda_I \cdot [I] \in \widetilde{C}_{n-1}(\mathcal{K}_J; k).$$

Then the same class is represented by the cycle $\kappa' \in \bar{B}_n(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}))_{-|J|, 2J}$ in the bar construction,

$$\kappa' := \sum_{\substack{I \in \mathcal{K}_J \\ |I|=n}} \epsilon(I, J) \lambda_I \sum_{\substack{I=\{i_1, \dots, i_n\} \\ J \setminus I = J_1 \sqcup \dots \sqcup J_n \\ \max(J_t) > i_t, \forall t \in [n]}} (-1)^{\sum_{1 \leq t_1 < t_2 \leq n} \theta(J_{i_1}, J_{i_2})} [c(J_1, u_{i_1}) | \dots | c(J_n, u_{i_n})].$$

Proof The map $\tilde{H}_*(\mathcal{K}_J; \mathbf{k}) \rightarrow \text{Tor}_*^{H_*(\Omega \mathcal{Z}_{\mathcal{K}})}(\mathbf{k}, \mathbf{k})$ is induced by the composition

$$\bigoplus_{J \subset [m]} \tilde{C}_*(\mathcal{K}_J; \mathbf{k}) \xrightarrow[\sim]{g} (\Lambda[m] \otimes \mathbf{k} \langle \mathcal{K} \rangle, \bar{d}) \xrightarrow[\sim]{\bar{\varphi}} (\bar{B}(H_*(\Omega \mathcal{Z}_{\mathcal{K}})), d_{\bar{B}}),$$

of chain maps, where g is defined in Proposition 4.4 and $\bar{\varphi}$ is induced by the chain map φ from Theorem 4.8. We have $\kappa' = \bar{\varphi}(g(\kappa))$ by construction. $\square$

The formulas become simpler for $n = 1, 2$.

Corollary 4.10 Let $J \subset [m]$. Let a class

$$\alpha \in \text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \tilde{H}_1(\mathcal{K}_J)$$

be represented by a cycle

$$\kappa = \sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij} [\{i, j\}] \in \tilde{C}_1(\mathcal{K}_J; \mathbf{k}).$$

Then the same class is represented by the following cycle in the bar construction:

$$\kappa' = \sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J < i| + |J < j|} \lambda_{ij} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} [c(A, u_i) | c(B, u_j)] + (-1)^{\theta(B, A)} [c(B, u_j) | c(A, u_i)]. \quad \square$$

Corollary 4.11 Let $J \subset [m]$, and let the simplicial complex $\mathcal{K}_J$ have $t + 1$ path components. Let vertices $i_1, \dots, i_t, \max(J)$ be representatives of these components. Then a basis of the $\mathbf{k}$ -module $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J} \cong \tilde{H}_0(\mathcal{K}_J; \mathbf{k}) \simeq \mathbf{k}^t$ is represented by cycles

$$[c(J \setminus i_s, u_{i_s})] \in \bar{B}_1(H_*(\Omega \mathcal{Z}_{\mathcal{K}}))_{-|J|, 2J}, \quad s = 1, \dots, t.$$

Proof Define $j := \max(J)$. The cycles $\kappa_s = [\{j\}] - [\{i_s\}] \in \tilde{C}_0(\mathcal{K}_J; \mathbf{k})$, $1 \leq s \leq t - 1$, represent a basis in $\tilde{H}_0(\mathcal{K}_J; \mathbf{k})$. By Theorem 4.9, the basis in $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})_{-|J|, 2J}$ is represented by cycles

$$\kappa'_s = 0 \pm [c(J \setminus i_s, u_{i_s})], \quad s = 1, \dots, t.$$

(The summand $[\{j\}]$ in κ_s does not contribute to κ'_s , since the subset $J_1 := J \setminus \{j\}$ does not satisfy the condition $\max(J_1) > j$.) $\square$

5 Generators and relations in the flag case

5.1 Minimal sets of generators

Let M be a finitely generated k -module. Denote the smallest number of generators by $\text{gen}(M)$, and define $\tilde{b}_i(X; k) := \text{gen}(\tilde{H}_i(X; k))$. Since $\tilde{H}_0(X; k)$ is a free module, this number does not depend on k if k is a principal ideal domain, so we write $\tilde{b}_0(X; k) = \tilde{b}_0(X)$. (In fact, $\tilde{b}_0(X) + 1$ is the number of path components in X).

Theorem 5.1 *Let $\mathcal{K}$ be a flag simplicial complex on a vertex set $[m]$ and k be a commutative ring with unit. For every $J \subset [m]$, choose a $\tilde{b}_0(\mathcal{K}_J)$ -element subset $\Theta(J) \subset J \setminus \{\max(J)\}$ such that $\Theta(J) \sqcup \{\max(J)\}$ contains exactly one vertex from each path component of $\mathcal{K}_J$. Then $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is multiplicatively generated by the following set of $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J)$ elements:*

$$\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}, \quad c(J \setminus i, u_i) \in H_{-|J|, 2J}(\Omega \mathcal{Z}_{\mathcal{K}}; k).$$

If k is a principal ideal domain, this set is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ contains at least $\tilde{b}_0(\mathcal{K}_J)$ generators of degree $(-|J|, 2J)$; any $\mathbb{Z}$ -homogeneous presentation contains at least $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators of degree n .

Proof By Corollary 4.11, images of cycles $\{[c(J \setminus i, u_i)] : J \subset [m], i \in \Theta(J)\} \subset \bar{B}_1(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k))$ additively generate the k -module $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)$. Hence, by Theorem A.6(i), the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is multiplicatively generated by the elements in question. The lower bounds on the number of generators follow from (9) and from Theorem A.10(ii). $\square$

Definition 5.2 Let $\mathcal{K}$ be a simplicial complex on $[m]$, and let $J \subset [m]$. Choose $\Theta(J)$ as the set of the smallest vertices in corresponding path components. More precisely, define $\Theta(J)$ as the set of all vertices $i \in J$ such that

- (i) i and $\max(J)$ belong to different path components of the complex $\mathcal{K}_J$;
- (ii) i is the smallest vertex (has the smallest number) in its path component.

The corresponding set of generators $\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}$ will be called the *GPTW generators*.

Grbić, Panov, Theriault and Wu proved [23, Theorem 4.3] that GPTW generators minimally generate the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ if k is a field. The minimality was proved using topological methods. Our Theorem 5.1 gives a purely algebraic proof for any ring k .

5.2 Rewriting of nested commutators

Thus the GPTW generators are indeed multiplicative generators of the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ for any ring k and any flag complex $\mathcal{K}$.

Definition 5.3 Let $i \in J \subset [m]$. Express the element $c(J \setminus i, u_i) \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ as a noncommutative polynomial in GPTW generators (this expression may be nonunique). Any such expression will be denoted by $\hat{c}(J \setminus i, u_i)$.

These noncommutative polynomials can be computed recursively, following the proof of [23, Theorem 4.3]. We describe an explicit rewriting process.

Algorithm 5.4 Suppose that expressions $\hat{c}(A \setminus t, u_t)$, $|A| < |J|$, are already computed, and we compute $\hat{c}(J \setminus i, u_i)$. Three cases are possible:

(i) **$i = \max(J)$** Define $j = \max(J \setminus i)$. Then

$$c(J \setminus i, u_i) = c(J \setminus ij, [u_j, u_i]) = c(J \setminus j, u_j).$$

The task is reduced to the case $i \neq \max(J)$.

(ii) **i and $\max(J)$ belong to the same path component of $\mathcal{K}_J$** The length of the shortest path from i to $\max(J)$ along the edges of $\mathcal{K}_J$ will be called the *rank* of a vertex i . We proceed by induction on the rank. The case of rank zero is discussed above. If rank equals 1, we have $[u_{\max(J)}, u_i] = 0$, so

$$c(J \setminus i, u_i) = c(J \setminus \{i, \max(J)\}, [u_{\max(J)}, u_i]) = 0.$$

Suppose that rank is greater than one, and let $\{i, j\}$ be the first edge in (any) shortest path from i to $\max(J)$. Since $[u_i, u_j] = 0 \in H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$, the identity (16) expresses $c(J \setminus i, u_i)$ in terms of $c(J \setminus j, u_j)$ (this element has smaller rank) and commutators of smaller degree (expressions for which are already computed).

(iii) **i and $\max(J)$ are in different path components** Let i_0 be the smallest vertex of the component that contains i . The length of the shortest path from i to i_0 will be called the *rank* of a vertex i . If the rank is zero, then $i \in \Theta(J)$, so we can set $\hat{c}(J \setminus i, u_i) := c(J \setminus i, u_i)$. Otherwise we decrease the rank using (16), as in case (ii).

Remark 5.5 A similar argument works more generally: suppose that we have a set $\{x_{J,i} : i \in J \subset [m]\}$ such that, for any $\{i, j\} \in \mathcal{K}_J$, the linear combination $x_{J,i} \pm x_{J,j}$ is a noncommutative polynomial on elements of smaller degree. Then we can express each element $x_{A,t}$ through the “GPTW elements” $\{x_{J,i} : i \in \Theta(J), J \subset [m]\}$ by a similar rewriting process. In our case $x_{J,i} = c(J \setminus i, u_i)$, and the polynomial is given by the last summand in (16).

5.3 Minimal sets of relations

Theorem 5.6 Let $\mathcal{K}$ be a flag simplicial complex on a vertex set $[m]$, and let $\mathbf{k}$ be a commutative ring. For each $J \subset [m]$, choose a collection of simplicial 1-cycles

$$\sum_{\{i < j\} \in \mathcal{K}_J} \lambda_{ij}^{(\alpha)} [\{i, j\}] \in \tilde{\mathcal{C}}_1(\mathcal{K}_J; \mathbf{k})$$

that generate the $\mathbf{k}$ -module $H_1(\mathcal{K}_J; \mathbf{k})$. Then the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is presented by GPTW generators $\{c(J \setminus i, u_i) : i \in \Theta(J), J \subset [m]\}$ (see Definition 5.2) modulo the relations

$$(10) \quad \sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J_{<i}| + |J_{<j}|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus \{i, j\} = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B) + |A|} [\hat{c}(A, u_i), \hat{c}(B, u_j)] = 0.$$

In particular, $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ admits a presentation by $\sum_{J \subset [m]} \tilde{b}_0(\mathcal{K}_J)$ generators modulo $\sum_{J \subset [m]} \tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations: one should take the 1-cycles that correspond to minimal sets of generators.

If $\mathbf{k}$ is a principal ideal domain, this presentation is minimal: any $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation contains at least $\tilde{b}_1(\mathcal{K}_J; \mathbf{k})$ relations of degree $(-|J|, 2J)$ for every $J \subset [m]$.

Proof By Corollary 4.10, our 1-cycles correspond to the elements

$$\sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J_{<i}| + |J_{<j}|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus ij = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} [c(A, u_i) | c(B, u_j)] + (-1)^{\theta(B, A)} [c(B, u_j) | c(A, u_i)]$$

in bar construction, and their images additively generate $\text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})}(\mathbf{k}, \mathbf{k})$. We apply Theorem A.6(ii) to this situation. (In the notation of this theorem, we take GPTW generators as $a_1, \dots, a_N$. Their images freely generate $\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}})}(\mathbf{k}, \mathbf{k})$, so we can take $R = 0$. We take $\hat{c}(A, u_i)$ and $\hat{c}(B, u_j)$ as polynomials $P_{j, \alpha}$ and $Q_{j, \alpha}$.) It follows that $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is generated by GPTW generators and presented by the relations

$$\sum_{\{i < j\} \in \mathcal{K}_J} (-1)^{|J_{<i}| + |J_{<j}|} \lambda_{ij}^{(\alpha)} \sum_{\substack{J \setminus ij = A \sqcup B \\ \max(A) > i \\ \max(B) > j}} (-1)^{\theta(A, B)} \overline{\hat{c}(A, u_i)} \hat{c}(B, u_j) + (-1)^{\theta(B, A)} \overline{\hat{c}(B, u_j)} \hat{c}(A, u_i) = 0.$$

Define $x = \hat{c}(A, u_i)$, $y = \hat{c}(B, u_j)$. Since $\theta(A, B) + \theta(B, A) \equiv |A| \cdot |B|$, we have

$$\begin{aligned} (-1)^{\theta(A, B)} \overline{x} y + (-1)^{\theta(B, A)} \overline{y} x &= (-1)^{\theta(A, B)} ((-1)^{|A|} x y + (-1)^{|B| + |A| \cdot |B|} y x) \\ &= (-1)^{\theta(A, B) + |A|} (x y - (-1)^{(|A|+1)(|B|+1)} y x) = (-1)^{\theta(A, B) + |A|} [x, y]. \end{aligned}$$

Hence the obtained relations coincide with (10). Finally, the lower bound on the number of relations follows from (9) and Theorem A.10(ii). $\square$

Sometimes we can reduce the number of relations if the presentation is not required to be $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous. For example, suppose that for some $I, J \subset [m]$ we have $|I| = |J| = n$, $H_1(\mathcal{K}_I; \mathbb{Z}) = \mathbb{Z}/2$, $H_1(\mathcal{K}_J; \mathbb{Z}) = \mathbb{Z}/3$. Then the graded components of the module $\text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})}(\mathbb{Z}, \mathbb{Z})$ having multidegrees $(-n, 2I)$ and $(-n, 2J)$ are equal to $\mathbb{Z}/2$ and $\mathbb{Z}/3$. By Theorem A.10, every $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous presentation of $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbb{Z})$ should contain relations of these multidegrees. On the other hand, these $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded components contribute $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \simeq \mathbb{Z}/6$ to the $\mathbb{Z}$ -graded component of degree n . Hence we can take just one $\mathbb{Z}$ -homogeneous relation (for example, the sum of these $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -homogeneous relations). Let us give a general result.

Theorem 5.7 Let $\mathcal{K}$ be a flag simplicial complex and k be a principal ideal domain. Consider all homogeneous presentations of the $\mathbb{Z}$ -graded k -algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$.

- (i) There is a presentation that consists of, for each $n \geq 0$, exactly $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators and exactly $\text{gen}(\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k))$ relations of degree n . One can take GPTW generators as generators, and take linear combinations of identities from Theorem 5.6, corresponding to minimal generators of the k -module $\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k)$, as relations.
- (ii) For every $n \geq 0$, any presentation contains at least $\sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ generators and at least $\text{gen}(\bigoplus_{|J|=n} H_1(\mathcal{K}_J; k))$ relations of degree n .

Proof By Theorem A.10, $\text{gen Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n$ and $\text{gen Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n + \text{rel Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n$ are precise bounds on the number of generators and relations, respectively, of degree n . By (9), we have

$$\text{Tor}_1^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n = \bigoplus_{|J|=n} \tilde{H}_0(\mathcal{K}_J; k) \simeq k^{\oplus \sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)}, \quad \text{Tor}_2^{H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)}(k, k)_n = \bigoplus_{|J|=n} H_1(\mathcal{K}_J; k);$$

hence $\text{gen Tor}_1 = \sum_{|J|=n} \tilde{b}_0(\mathcal{K}_J)$ and $\text{rel Tor}_1 = 0$. One can take the GPTW generators since the images of corresponding cycles generate Tor_1 by Corollary 4.11. $\square$

5.4 Example: moment-angle complexes for m -cycles

Let $\mathcal{K}$ be the boundary of an m -gon. The corresponding moment-angle complex $\mathcal{Z}_{\mathcal{K}}$ is homeomorphic to a connected sum of sphere products, $\mathcal{Z}_{\mathcal{K}} \cong \#_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2)} \binom{m-2}{k-1}$, and hence $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; k)$ is a one-relator algebra. It was considered in [24; 42]. From the point of view of Theorem 5.6, the relation corresponds to the 1-cycle

$$\kappa = [\{1, m\}] - \sum_{i=1}^{m-1} [\{i, i+1\}] \in \tilde{C}_1(\mathcal{K}; k)$$

and has the form

$$\sum_{\substack{\{2, \dots, m-1\} = A \sqcup B \\ \max(A) > 1 \\ \max(B) > m}} (\dots) - \sum_{i=1}^{m-1} (-1)^{(i-1)+i} \sum_{\substack{[m] \setminus \{i, i+1\} = A \sqcup B \\ \max(A) > i \\ \max(B) > i+1}} (-1)^{\theta(A, B) + |A|} [\hat{c}(A, u_i), \hat{c}(B, u_{i+1})] = 0.$$

The first sum is empty, since $\max(B) \leq m-1$. Similarly, in the second sum the inner sum is empty for $i = m-1, m-2$. The simplified relation is

$$\sum_{i=1}^{m-3} \sum_{\substack{[m] \setminus \{i, i+1\} = A \sqcup B \\ \max(A), \max(B) \geq i+2}} (-1)^{\theta(A, B) + |A|} [\hat{c}(A, u_i), \hat{c}(B, u_{i+1})] = 0.$$

Some summands are immediately zero. For example, if $\max(B) = i+2$, then

$$c(B, u_{i+1}) = c(B \setminus \{i+2\}, [u_{i+2}, u_{i+1}]) = 0,$$

so we can take $\hat{c}(B, u_{i+1}) = 0$. Similarly, $c(A, u_1) = 0$ if $i = 1$ and $m \in A$. Other summands can be computed using Algorithm 5.4. We were not able to obtain a closed formula for this relation (as a polynomial of GPTW generators or other minimal generators). However, we at least have an effective algorithm that computes the relation for any given m .

Consider the case $m = 5$. Besides from the partitions $[5] \setminus \{i, i+1\} = A \sqcup B$ considered above, for $i = 1$ the allowed partitions are $\{3, 4, 5\} = \{3\} \sqcup \{4, 5\} = \{4\} \sqcup \{3, 5\} = \{3, 4\} \sqcup \{5\}$; for $i = 2$ the allowed partitions are $\{1, 4, 5\} = \{4\} \sqcup \{1, 5\} = \{1, 4\} \sqcup \{5\}$. The resulting relation has five summands:

$$\begin{aligned} & (-1)^{\theta(3,45)+1} [\hat{c}(3, u_1), \hat{c}(45, 2)] + (-1)^{\theta(4,35)+1} [\hat{c}(4, u_1), \hat{c}(35, 2)] + (-1)^{\theta(34,5)+2} [\hat{c}(34, u_1), \hat{c}(5, u_2)] \\ & + (-1)^{\theta(4,15)+1} [\hat{c}(4, u_2), \hat{c}(15, u_3)] + (-1)^{\theta(14,5)+2} [\hat{c}(14, u_2), \hat{c}(5, u_3)] = 0. \end{aligned}$$

All commutators, apart from $\hat{c}(14, u_2) = [u_1, [u_4, u_2]] = -[u_2, [u_4, u_1]] = -c(24, u_1)$, already are GPTW generators. We obtain the following identity between the generators:

$$\begin{aligned} & -[[u_3, u_1], [u_4, [u_5, u_2]]] + [[u_4, u_1], [u_3, [u_5, u_2]]] - [[u_5, u_2], [u_3, [u_4, u_1]]] \\ & + [[u_4, u_2], [u_1, [u_5, u_3]]] + [[u_5, u_3], [u_2, [u_4, u_1]]] = 0. \end{aligned}$$

This relation was first obtained by Veryovkin as a result of brute force [42, Theorem 3.2]. For $m = 6$, the analogous relation is initially the sum of $7 + 10 + 4 = 21$ commutators. After computing the elements $\hat{c}(J \setminus i, u_i)$ and changing the set of generators, it can be written as $\sum_{i=1}^{17} [a_i, b_i] = 0$ (see [42, Theorem 4.1]). This agrees with the homeomorphism $\mathcal{Z}_{\mathcal{K}} \cong (S^3 \times S^5)^{\#9} \# (S^4 \times S^4)^{\#8}$.

6 Homotopical properties in the flag case

6.1 Homotopy groups

As in [39], we denote by $\mathcal{P}$ the class of H-spaces which are homotopy equivalent to finite-type products of spheres and loops on simply connected spheres, and by $\mathcal{W}$ the class of topological spaces which are homotopy equivalent to finite-type wedges of simply connected spheres. The author thanks Lewis Stanton for providing a proof of the following lemma.

Lemma 6.1 *Let $A_1, \dots, A_m$ be connected topological spaces, $\mathcal{K}$ be a simplicial complex on $[m]$, and suppose that $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \in \mathcal{P}$. Then $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}}$ is homotopy equivalent to a finite-type product of loops on simply connected spheres.*

Proof By [40, Corollary 9.8], $\Omega(\underline{CA}, \underline{A})^{\mathcal{K}} \simeq \prod_{i=1}^m \Omega \Sigma Y_i$ for some spaces Y_i . Since the class $\mathcal{P}$ is closed under retracts [39, Theorem 3.10], $\Omega \Sigma Y_i \in \mathcal{P}$. By repeated use of the homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ and the James splitting $\Sigma \Omega \Sigma X \simeq \bigvee_{n \geq 1} \Sigma X^{\wedge n}$ [27], we have $\Sigma Z \in \mathcal{W}$ for $Z \in \mathcal{P}$. In particular, $\Sigma \Omega \Sigma Y_i \in \mathcal{W}$. On the other hand, ΣY_i is a retract of $\Sigma \Omega \Sigma Y_i$ by the James splitting. The class $\mathcal{W}$ is closed under retracts (see, for example, [1, Lemma 3.1]), so $\Sigma Y_i \in \mathcal{W}$. Now $\Omega \Sigma Y_i$ is homotopy equivalent to a product of loops on spheres by the Hilton–Milnor theorem. It follows that the same holds for $\prod_{i=1}^m \Omega \Sigma Y_i$. $\square$

Proof of Theorem 1.2 Since $\mathcal{K}$ is flag, we have $\Omega \mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$ by Theorem 3.11. Hence $\Omega \mathcal{Z}_{\mathcal{K}} = \Omega(CS^1, S^1)^{\mathcal{K}}$ is a product of loops on spheres by Lemma 6.1. It follows that for some $D_n \geq 0$ we have a homotopy equivalence

$$\Omega \mathcal{Z}_{\mathcal{K}} \simeq \prod_{n \geq 2} (\Omega S^n)^{\times D_n}.$$

The numbers D_n are finite, since $\dim_{\mathbf{k}} H_i(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}) < \infty$ for all i . (Here $\mathbf{k}$ is any field.) Also $D_2 = 0$, since $\mathcal{Z}_{\mathcal{K}}$ is 2-connected [15, Proposition 4.3.5]. In order to compute D_n , we calculate $\dim H_i(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ twice. Recall that the *Poincaré series* $F(V; t)$ of a graded $\mathbf{k}$ -vector space V are the formal power series

$$F(V; t) := \sum_{i \geq 0} \dim_{\mathbf{k}}(V_i) \cdot t^i \in \mathbb{Z}[[t]].$$

We have

$$F(V \oplus W) = F(V; t) + F(W; t) \quad \text{and} \quad F(V \otimes W; t) = F(V; t) \cdot F(W; t).$$

From $F(H_*(\Omega S^k; \mathbf{k}); t) = (1 - t^{k-1})^{-1}$ and the Künneth formula we have

$$F(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}); t) = \prod_{n \geq 3} (1 - t^{n-1})^{-D_n}.$$

On the other hand, it is known (see [15, Proposition 8.5.4] and [43, Theorem 4.8]) that

$$F(H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k}); t) = \frac{1}{(1+t)^{m-d} \cdot h_{\mathcal{K}}(-t)} = - \frac{1}{\sum_{J \subset [m]} \bar{\chi}(\mathcal{K}_J) t^{|J|}}$$

for a flag complex $\mathcal{K}$. We obtain the required identity (3). $\square$

Remark 6.2 In the proof above, the algebra $H_*(\Omega \mathcal{Z}_{\mathcal{K}}; \mathbf{k})$ is actually $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded. We expect that factors of the product (2) can be considered as “ $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -graded spheres”, and thus $\pi_*(\Omega \mathcal{Z}_{\mathcal{K}})$ admits a functorial $\mathbb{Z} \times \mathbb{Z}_{\geq 0}^m$ -grading as conjectured in [43, Remark 4.10].

Problem 6.3 Describe the Whitehead bracket in $\pi_*(\mathcal{Z}_{\mathcal{K}})$ in terms of the decomposition (4).

6.2 Rational coformality of moment-angle complexes

Let X be a simply connected space and ΩX be the space of Moore loops. Since ΩX is a strictly associative topological monoid, the chain complex $C_*(\Omega X; \mathbf{k})$ is a dga algebra with respect to the Pontryagin product for any $\mathbf{k}$. Also, the cochain complex $C^*(X; \mathbf{k})$ is a dga algebra with respect to the Kolmogorov–Alexander product (cup product).

Definition 6.4 A topological space X is *formal* over a ring $\mathbf{k}$ if the dga algebras $H^*(X; \mathbf{k})$ (with zero differential) and $C^*(X; \mathbf{k})$ are quasi-isomorphic (are connected by a zigzag of dga maps which induce isomorphisms on homology).

Definition 6.5 A simply connected space X is *coformal* over a ring $\mathbf{k}$ if the dga algebras $H_*(\Omega X; \mathbf{k})$ (with zero differential) and $C_*(\Omega X; \mathbf{k})$ are quasi-isomorphic.

The notions of formality and coformality (over a field of characteristic zero) arose in rational homotopy theory, and were initially formulated in terms of Sullivan and Quillen models. The rational homotopy type of a formal (coformal) space is fully determined by the algebra $H^*(X; \mathbb{Q})$ (by the algebra $H_*(\Omega X; \mathbb{Q})$). As proved by Saleh [38, Corollary 1.2, 1.4], our definitions are equivalent to the classical ones.

Notbohm and Ray proved [34, Theorem 4.8] that all Davis–Januszkiewicz spaces $\mathrm{DJ}(\mathcal{K})$ are formal over $\mathbb{Z}$. (In fact, the proof works over any ring k ; see also [20, Theorem 1.3] for a stronger formality result). Also, $\mathrm{DJ}(\mathcal{K})$ is coformal over $\mathbb{Q}$ if and only if $\mathcal{K}$ is flag [15, Theorem 8.5.6]. First examples of nonformal moment-angle complexes were constructed by Baskakov [6] using Massey products. See [14, Introduction] for a survey of further developments in this area.

The following result of Huang can be used to prove coformality over $\mathbb{Q}$.

Proposition 6.6 [26, Proposition 5.1] *Let $F \xrightarrow{i} E \rightarrow B$ be a fibration of nilpotent spaces of finite type, such that*

- *the map $i_*: \pi_*(F) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \pi_*(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective;*
- *E is coformal over $\mathbb{Q}$.*

Then F is coformal over $\mathbb{Q}$. □

Corollary 6.7 *Let $\mathcal{K}$ be a flag simplicial complex with no ghost vertices. Then $\mathcal{Z}_{\mathcal{K}}$ is coformal over $\mathbb{Q}$.*

Proof We apply Proposition 6.6 to the fibration $\mathcal{Z}_{\mathcal{K}} \rightarrow \mathrm{DJ}(\mathcal{K}) \rightarrow B\mathbb{T}^m$. By Proposition 3.1 and exact sequence of homotopy groups, $\pi_*(\mathcal{Z}_{\mathcal{K}}) \rightarrow \pi_*(\mathrm{DJ}(\mathcal{K}))$ is injective. The second condition holds by [15, Theorem 8.5.6]. □

It is natural to hope that Huang’s theorem admits the following generalisation.

Conjecture 6.8 *Let $F \rightarrow E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type, such that*

- *Ωp has a homotopy section;*
- *E is coformal over k .*

Then F is coformal over k .

Let X be a simply connected space such that $H_*(\Omega X; k)$ is a free k -module. The tensor filtration on the bar construction $\bar{B}(C_*(\Omega X; k))$ gives rise to the *Milnor–Moore spectral sequence*

$$E_{p,q}^2 = \mathrm{Tor}_p^{H_*(\Omega X; k)}(k, k)_q \Rightarrow \mathrm{Tor}_{p+q}^{C_*(\Omega X; k)}(k, k) \cong H_{p+q}(X; k).$$

(The last isomorphism is due to Eilenberg and Moore; see [18, Theorem IV]).

The differential Tor is preserved by quasi-isomorphisms. Hence the spectral sequence collapses at E^2 if X is coformal over k . On the other hand, it collapses for $X = \mathcal{Z}_{\mathcal{K}}$ in the flag case; see (9). This suggests the following conjecture.

Conjecture 6.9 *Let $\mathcal{K}$ be a flag simplicial complex. Then the spaces $\mathrm{DJ}(\mathcal{K})$ and $\mathcal{Z}_{\mathcal{K}}$ are coformal over any commutative ring with unit.*

6.3 A necessary condition for the rational formality in the flag case

The space X is *Koszul* if it is both formal and coformal over $\mathbb{Q}$. Hence $\mathrm{DJ}(\mathcal{K})$ is Koszul if and only if $\mathcal{K}$ is flag. Koszul spaces were introduced by Berglund [10].

Definition 6.10 Let k be a field, $A = \bigoplus_{n \in \mathbb{Z}} A^n$ be a graded k -algebra that admits an additional “weight” grading $A^n = \bigoplus_{j \geq 0} A^{n,(j)}$. The algebra A is *Koszul* with respect to the weight grading if $\mathrm{Ext}_A^i(k, k)^{n,(j)} = 0$ for all $i \neq j$.

For every Koszul algebra, there is a *quadratic dual* Koszul algebra $A^!$; see [22]. More explicitly, we set

$$A^! := \mathrm{Ext}_A(k, k), \quad (A^!)^{n,(i)} = \mathrm{Ext}_A^i(k, k)^{-i-n,(i)}.$$

Then it is known that $(A^!)^! \cong A$ as bigraded algebras.

Remark 6.11 In the classical theory of Koszul algebras [22; 36] the $\mathbb{Z}$ -grading $A = \bigoplus_{n \in \mathbb{Z}} A^n$ is absent, and only the weight grading $(A^!)^{(i)} = \mathrm{Ext}_A^i(k, k)^{(i)}$ is considered. Classical results are readily generalised to the graded case.

The following result is due to Berglund. We replace the Koszul Lie algebras with their universal enveloping algebras. Berglund considers a stronger version of the Koszul duality, the duality between Lie algebras and commutative algebras.

Theorem 6.12 [10, Theorems 2 and 3] *Let X be a simply connected space of finite type such that X is coformal over $\mathbb{Q}$. The following conditions are equivalent:*

- (a) X is formal over $\mathbb{Q}$.
- (b) The graded algebra $A = H_*(\Omega X; \mathbb{Q})$ admits a weight grading $A = \bigoplus_{i \geq 0} A^{(i)}$ such that A is Koszul with respect to it.

Moreover, if these conditions are met, then the $\mathbb{Z}$ -graded algebras $A^!$ and $H^{-*}(X; \mathbb{Q})$ are isomorphic: $H^n(X; \mathbb{Q}) \cong \bigoplus_{i \geq 0} (A^!)^{-n,(i)}$. $\square$

Theorem 6.13 *Let $\mathcal{K}$ be a flag simplicial complex on $[m]$ with no ghost vertices, such that $\mathcal{Z}_{\mathcal{K}}$ is rationally formal. Then $\Gamma = H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Q})$ is a Koszul algebra with respect to the grading*

$$\Gamma^{(i)} := \bigoplus_{J \subset [m]} H^{i-|J|, 2J}(\mathcal{Z}_{\mathcal{K}}; \mathbb{Q}) = \bigoplus_{J \subset [m]} \tilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q}).$$

In particular, Γ is generated by elements in $\tilde{H}^0(\mathcal{K}_J; \mathbb{Q})$ modulo the relations in $\tilde{H}^1(\mathcal{K}_J; \mathbb{Q})$.

Proof By Theorem 6.12, the algebra $A = H_*(\Omega \mathcal{Z}_K; \mathbb{Q})$ is Koszul with respect to a weight grading $A = \bigoplus_{i \geq 0} A^{(i)}$. From [43, Theorem 1.2] we have $\mathrm{Tor}_i^A(\mathbb{Q}, \mathbb{Q})_j = \bigoplus_{|J|=j} \widetilde{H}_{i-1}(\mathcal{K}_J; \mathbb{Q})$. Therefore, $\mathrm{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^j = \bigoplus_{|J|=j} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q})$. The algebra A is Koszul, hence

$$\mathrm{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^j = \mathrm{Ext}_A^i(\mathbb{Q}, \mathbb{Q})^{j, (i)} = (A^!)^{-i-j, (i)}.$$

Since $(A^!)^* \cong \Gamma^{-*}$ as graded algebras, we obtain a weight grading

$$\Gamma^{i+j, (i)} = \bigoplus_{|J|=j} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q}), \quad \Gamma^{(i)} = \bigoplus_{J \subset [m]} \widetilde{H}^{i-1}(\mathcal{K}_J; \mathbb{Q})$$

such that Γ is Koszul with respect to it. Finally, any Koszul algebra is generated by elements of weight 1 modulo relations of weight 2. $\square$

Conjecture 6.14 *If $\mathcal{K}$ is a flag complex and $H^*(\mathcal{Z}_K; \mathbb{Q})$ is Koszul with respect to the grading from Theorem 6.13, then $\mathcal{Z}_K$ is formal over $\mathbb{Q}$.*

Appendix A Presentations of connected graded algebras

In this section we prove Theorems A.1 and A.10 that generalise some results of Wall [44, Section 7]. We also prove Theorem A.6, which seems to be new. We use the notation from Section 2; some of which is recalled below.

A.1 Conventions

The ring k is assumed to be an arbitrary commutative associative ring with unit. All tensor products are over k .

We consider G -graded k -algebras, where G is a commutative monoid supplied with a homomorphism $G \rightarrow \mathbb{Z}$. It induces a $\mathbb{Z}$ -grading. Such an algebra A is *connected* if it is connected with respect to the $\mathbb{Z}$ -grading, ie $A_{<0} = 0$ and $A_0 = k \cdot 1$. Then the standard augmentation $\varepsilon: A \rightarrow A_0 \cong k$ makes k a left A -module and a right A -module.

Every complex of G -graded modules is considered as a $\mathbb{Z} \times G$ -graded module with a differential of degree $(-1, 0)$. Hence, A -linear differentials satisfy the following version of Leibniz's rule:

$$d(a \cdot x) = (-1)^{\deg(a)} a \cdot d(x) = -\bar{a} \cdot d(x),$$

where $\bar{a} := (-1)^{1+\deg(a)} a$.

A *presentation* of a connected k -algebra A is an isomorphism of the form $A \simeq T(x_1, \dots, x_N)/(r_1, \dots, r_M)$, sometimes written as

$$A \simeq T(x_1, \dots, x_N)/(r_1 = \dots = r_M = 0),$$

where $T(x_1, \dots, x_N)$ is a tensor algebra and $(r_1, \dots, r_M) \subset T(x_1, \dots, x_N)$ is the two-sided ideal generated by the set $\{r_1, \dots, r_M\}$. It is assumed that generators and relations are homogeneous and have positive degree, hence belong to $\text{Ker } \varepsilon$. Note that A is not required to be a free k -module, and M, N can be infinite of any cardinality.

A.2 Exact sequence of a presentation

Let $T(x_1, \dots, x_N)$ be a tensor algebra generated by homogeneous elements of positive degrees. Every element $w \in T(x_1, \dots, x_N)$ is uniquely represented as a sum

$$w = \varepsilon(w) + \sum_{i=1}^N w_i \cdot x_i, \quad w_i \in T(x_1, \dots, x_N).$$

In the next proposition we use this representation implicitly. For example, we assume that

$$r_j = \varepsilon(r_j) + \sum_{i=1}^N r_{ji} \cdot x_i.$$

Since $r_j \in \text{Ker } \varepsilon$, the first summand is zero.

Proposition A.1 *Let $A = T(x_1, \dots, x_N)/(r_1, \dots, r_M)$ be a presentation of a connected k -algebra,*

$$\pi: T(x_1, \dots, x_N) \twoheadrightarrow A$$

be the projection. Then the following sequence of graded free left A -modules is exact:

$$A \cdot \{R_1, \dots, R_M\} \xrightarrow{d_2} A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} k \rightarrow 0,$$

$$d_2(R_j) := - \sum_{i=1}^N \pi(\overline{r_{ji}}) \cdot X_i, \quad d_1(X_i) := x_i.$$

Proof We first prove that the sequence is a chain complex. Indeed, $\varepsilon(d_1(X_i)) = \varepsilon(x_i) = 0$ and

$$d_1(d_2(R_j)) = \sum_{i=1}^N \pi(r_{ji}) \cdot d_1(X_i) = \sum_{i=1}^N \pi(r_{ji})x_i = \pi\left(\sum_{i=1}^N r_{ji}x_i\right) = \pi(r_j) = 0 \in A$$

($r_j \in \text{Ker } \varepsilon$, hence $r_j = \sum_i r_{ji}x_i$). We check the exactness in the term A . Let $y \in \text{Ker } \varepsilon \subset A$. We have $y = \pi(w)$ for some element $w \in T(x_1, \dots, x_N)$ of positive degree, hence

$$y = \pi\left(\sum_{i=1}^N w_i x_i\right) = \sum_{i=1}^N \pi(w_i)x_i = d_1\left(-\sum_{i=1}^N \pi(\overline{w_i}) \cdot X_i\right) \in \text{Im } d_1.$$

We check the exactness in the term $A \cdot \{X_1, \dots, X_N\}$. Suppose that $\sum_{i=1}^N a_i \cdot X_i \in \text{Ker } d_1$, so $\sum_{i=1}^N \overline{a_i} x_i = 0$. We have $a_i = \pi(v_i)$ for some $v_i \in T(x_1, \dots, x_N)$. Then the element $w := \sum_{i=1}^N \overline{v_i} x_i \in T(x_1, \dots, x_N)$ belongs to $\text{Ker } \pi$. This kernel is a two-sided ideal generated by r_j . Hence $w = \sum_{j=1}^M \sum_{\alpha} u_{j,\alpha} r_j w_{j,\alpha}$ for some $u_{j,\alpha}, w_{j,\alpha} \in T(x_1, \dots, x_N)$. We can rewrite it as

$$w = \sum_{j=1}^M \sum_{\alpha} u_{j,\alpha} r_j \varepsilon(w_{j,\alpha}) + \sum_{j=1}^M \sum_{\alpha} \sum_{i=1}^N u_{j,\alpha} r_j w_{j,\alpha,i} x_i = \sum_{i=1}^N \sum_{j=1}^M \sum_{\alpha} (\varepsilon(w_{j,\alpha}) u_{j,\alpha} r_{ji} + u_{j,\alpha} r_j w_{j,\alpha,i}) x_i.$$

On the other hand, $w = \sum_{i=1}^N \bar{v}_i x_i$. Such a representation is unique, so we have

$$\bar{v}_i = \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) u_{j,\alpha} r_{ji} + u_{j,\alpha} r_j w_{j,\alpha,i}, \quad i = 1, \dots, N.$$

Applying π to both parts of this identity, we obtain $\bar{a}_i = \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(u_{j,\alpha}) \pi(r_{ji})$, since $\pi(v_i) = a_i$ and $\pi(r_j) = 0$. Finally,

$$\sum_{i=1}^N a_i \cdot X_i = - \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(\bar{u}_{j,\alpha}) \pi(\bar{r}_{ji}) \cdot X_i = d_2 \left(- \sum_{j=1}^M \sum_{\alpha} \varepsilon(w_{j,\alpha}) \pi(u_{j,\alpha}) \cdot R_j \right) \in \text{Im } d_2. \quad \square$$

Remark A.2 Proposition A.1 holds for presentations of *augmented* algebras such that $\varepsilon(x_i) = \varepsilon(r_j) = 0$. The corresponding exact sequence is called the “Koszul resolution” in [3, Section 2].

Corollary A.3 Let $A = T(x_1, \dots, x_N)/(r_1, \dots, r_M)$ be a presentation of a connected graded $\mathbf{k}$ -algebra, which is a free $\mathbf{k}$ -module. Then the $\mathbf{k}$ -module $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})$ is additively generated by images of cycles $[x_1], \dots, [x_N] \in \bar{B}_1(A)$.

Proof We extend the exact sequence from Proposition A.1 to a free resolution

$$\cdots \rightarrow A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \mathbf{k} \rightarrow 0, \quad d_1(X_i) = x_i,$$

of the left A -module $\mathbf{k}$. Consider the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A \cdot \{X_1, \dots, X_N\} & \xrightarrow{d_1} & A & \xrightarrow{\varepsilon} & \mathbf{k} \longrightarrow 0 \\ & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \parallel \\ \cdots & \longrightarrow & B_1(A) & \xrightarrow{d_{B,1}} & B_0(A) & \xrightarrow{\varepsilon} & \mathbf{k} \longrightarrow 0 \end{array}$$

It is commutative, since $d_1(a \otimes X_i) = -\bar{a}x_i$ and $d_{B,1}(a[x_i]) = -\bar{a}x_i[\cdot]$. Hence it can be extended to a map of resolutions (eg using Lemma 2.1). Apply the functor $\mathbf{k} \otimes_A (-)$. We obtain a map of chain complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbf{k} \cdot \{X_1, \dots, X_N\} & \xrightarrow{0} & \mathbf{k} & \longrightarrow & 0 \\ & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \\ \cdots & \longrightarrow & \bar{B}_1(A) & \xrightarrow{d_{\bar{B},1}} & \bar{B}_0(A) & \longrightarrow & 0 \end{array}$$

The homology of both complexes equals $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})$, and the induced map in homology is an isomorphism. The elements X_i in the first complex are cycles, and their images generate $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})$. $\square$

Corollary A.4 Let $A = T(x_1, \dots, x_N)$ be the tensor algebra over a ring $\mathbf{k}$, where $x_1, \dots, x_N$ are homogeneous elements of positive degrees. Then $\text{Tor}_1^A(\mathbf{k}, \mathbf{k})$ is a free $\mathbf{k}$ -module with the basis represented by cycles $[x_1], \dots, [x_N] \in \bar{B}_1(A)$. Moreover, $\text{Tor}_i^A(\mathbf{k}, \mathbf{k}) = 0$ for $i > 1$.

Proof By Proposition A.1, the sequence

$$0 \rightarrow A \cdot \{X_1, \dots, X_N\} \xrightarrow{d_1} A \xrightarrow{\varepsilon} \mathbf{k} \rightarrow 0, \quad d_1(X_i) = x_i,$$

is exact. As in the proof of Corollary A.3, we obtain a map of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{k} \cdot \{X_1, \dots, X_N\} & \xrightarrow{0} & \mathbf{k} & \longrightarrow & 0 \\ \downarrow & & \downarrow X_i \mapsto [x_i] & & \downarrow a \mapsto a[\cdot] & & \\ \cdots & \longrightarrow & \bar{\mathbf{B}}_1(A) & \xrightarrow{d_{\bar{\mathbf{B}},1}} & \bar{\mathbf{B}}_0(A) & \longrightarrow & 0 \end{array}$$

Homology of both complexes is equal to $\mathrm{Tor}^A(\mathbf{k}, \mathbf{k})$, and the induced map in homology is the identity. $\square$

A.3 A presentation that corresponds to cycles

Recall that $\mathrm{Tor}^A(\mathbf{k}, \mathbf{k}) \cong H(\bar{\mathbf{B}}(A))$ if A is a free $\mathbf{k}$ -module. The following lemma is proved by Lemaire [29, Corollaire 1.2.3] in the case of field coefficients.

Lemma A.5 *Let $f: A \rightarrow C$ be a morphism of connected $\mathbf{k}$ -algebras, where $\mathbf{k}$ is a commutative ring with unit.*

- (i) *Suppose that the map $f_{*,1}: H_1(\bar{\mathbf{B}}(A)) \rightarrow H_1(\bar{\mathbf{B}}(C))$ is surjective. Then $f: A \rightarrow C$ is surjective.*
- (ii) *Suppose that $f_{*,1}: H_1(\bar{\mathbf{B}}(A)) \rightarrow H_1(\bar{\mathbf{B}}(C))$ is bijective, and $f_{*,2}: H_2(\bar{\mathbf{B}}(A)) \rightarrow H_2(\bar{\mathbf{B}}(C))$ is surjective. Then $f: A \rightarrow C$ is an isomorphism.*

We prove by induction that the maps $f_n: A_n \rightarrow C_n$ are surjective (bijective). The base case is the bijection $A_0 \cong \mathbf{k} \cong C_0$. Recall that the bar construction $\bar{\mathbf{B}}(A)$ is the chain complex

$$\begin{aligned} \cdots \rightarrow \bar{\mathbf{B}}_3(A) \xrightarrow{d_3} \bar{\mathbf{B}}_2(A) \xrightarrow{d_2} \bar{\mathbf{B}}_1(A) \xrightarrow{0} \mathbf{k} \rightarrow 0, \\ \bar{\mathbf{B}}_k(A) = I(A)^{\otimes k}, \quad d_2(x \otimes y) = \bar{x}y, \quad d_3(x \otimes y \otimes z) = \bar{x}y \otimes z + \bar{x} \otimes \bar{y}z. \end{aligned}$$

We define $f_{\#}: \bar{\mathbf{B}}(A) \rightarrow \bar{\mathbf{B}}(C)$.

Proof of statement (i) Suppose that $f: A_i \rightarrow C_i$ is surjective for $i < n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \bar{\mathbf{B}}_2(A)_n & \xrightarrow{d_2} & \bar{\mathbf{B}}_1(A)_n \cong A_n & \longrightarrow & H_1(\bar{\mathbf{B}}(A))_n & \longrightarrow & 0 \\ \downarrow f_{\#,2} & & \downarrow f & & \downarrow f_{*,1} & & \parallel \\ \bar{\mathbf{B}}_2(C)_n & \xrightarrow{d_2} & \bar{\mathbf{B}}_1(C)_n \cong C_n & \longrightarrow & H_1(\bar{\mathbf{B}}(C))_n & \longrightarrow & 0 \end{array}$$

The map $f_{\#,2}$ is surjective, since it is a direct sum of maps $f \otimes f: A_i \otimes A_j \rightarrow C_i \otimes C_j$ for $i, j < n$, and f is surjective in these degrees by the inductive hypothesis. The surjectivity $f_{*,1}$ is given, and $0 \rightarrow 0$ is injective. Hence $f: A_n \rightarrow C_n$ is surjective by the “first half of five lemma” [37, Proposition 2.72(i)]. $\square$

Proof of statement (ii) Suppose that $f: A_i \rightarrow C_i$ is bijective for $i < n$. Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \bar{B}_3(A)_n & \xrightarrow{d_3} & \text{Ker } d_2 & \longrightarrow & H_2(\bar{B}(A))_n & \longrightarrow & 0 \\ \downarrow f_{\#,3} & & \downarrow \varphi & & \downarrow f_{*,2} & & \parallel \\ \bar{B}_3(C)_n & \xrightarrow{d_3} & \text{Ker } d_2 & \longrightarrow & H_2(\bar{B}(C))_n & \longrightarrow & 0 \end{array}$$

The map $f_{\#,3}$ is surjective, since it is a direct sum of maps $f \otimes f \otimes f: A_i \otimes A_j \otimes A_k \rightarrow C_i \otimes C_j \otimes C_k$ for $i, j, k < n$, and f is surjective in these degrees. The surjectivity of $f_{*,2}$ is given, and $0 \rightarrow 0$ is injective. Hence φ is surjective by the “first half of five lemma”. Now consider the following map of exact sequences:

$$\begin{array}{ccccccc} \text{Ker } d_{\bar{B},2} & \longrightarrow & \bar{B}_2(A)_n & \xrightarrow{d_2} & \bar{B}_1(A)_n \cong A_n & \longrightarrow & H_1(\bar{B}(A))_n \\ \downarrow \varphi & & \simeq \downarrow f_{\#,2} & & \downarrow f & & \downarrow f_{*,1} \\ \text{Ker } d_{\bar{B},2} & \longrightarrow & \bar{B}_2(C)_n & \xrightarrow{d_2} & \bar{B}_1(C)_n \cong C_n & \longrightarrow & H_1(\bar{B}(C))_n \end{array}$$

We proved that φ is surjective. The map $f_{\#,2}$ is bijective by the inductive hypothesis (it is a direct sum of maps $f \otimes f: A_i \otimes A_j \rightarrow C_i \otimes C_j$, $i, j < n$); in particular, it is injective. The injectivity of $f_{*,1}$ is given. Hence the map $f: A_n \rightarrow C_n$ is injective by the “second half of five lemma” [37, Proposition 2.72(ii)]. By (i), this map is also surjective. $\square$

The following theorem allows one to obtain a presentation of a connected k -algebra A , knowing the structure of k -modules $H_1(\bar{B}(A))$ and $H_2(\bar{B}(A))$. In the proof, we do not use the notation $[x|y|z]$ for elements of the bar construction, and write $x \otimes y \otimes z$ instead. Therefore, $[c]$ always denotes the class in $H(\bar{B}(\Gamma))$ represented by a cycle $c \in \bar{B}(\Gamma)$.

We also use the following notation. Let $a_1, \dots, a_N \in A$ be some homogeneous elements of positive degree and $K, L \in T(x_1, \dots, x_N)$ be homogeneous noncommutative polynomials that belong to the augmentation ideal. Then the elements $K(a_1, \dots, a_N), L(a_1, \dots, a_N) \in I(A)$ are defined, and hence we can consider the elements $K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N) \in I(A) \otimes I(A) = \bar{B}_2(A)$ and

$$d_{\bar{B},2}(K(a_1, \dots, a_N) \otimes L(a_1, \dots, a_N)) = \bar{K}(a_1, \dots, a_N) \cdot L(a_1, \dots, a_N) \in \bar{B}_1(A) = I(A).$$

Theorem A.6 Let A be a connected algebra over a commutative ring k with unit.

- (i) Suppose that, for homogeneous elements $a_1, \dots, a_N \in A_{>0}$, the k -module $H_1(\bar{B}(A))$ is additively generated by the classes $[a_1], \dots, [a_N] \in H_1(\bar{B}(A))$. Then A is multiplicatively generated by $a_1, \dots, a_N$.
- (ii) Suppose that the k -module $H_1(\bar{B}(A))$ is additively generated by N elements $[a_1], \dots, [a_N]$ modulo R relations

$$\sum_{i=1}^N \lambda_{ri} [a_i] = 0 \in H_1(\bar{B}(A)), \quad r = 1, \dots, R, \lambda_{ri} \in k.$$

Suppose that homogeneous polynomials $P_{j,\alpha}, Q_{j,\alpha}, K_{r,\beta}, L_{r,\beta} \in T(x_1, \dots, x_N)$ are such that

$$\sum_{i=1}^N \lambda_{ri} \cdot a_i = d_{\bar{B},2} \left(\sum_{\beta} K_{r,\beta}(a_1, \dots, a_N) \otimes L_{r,\beta}(a_1, \dots, a_N) \right) \in I(A), \quad r = 1, \dots, R,$$

and the cycles in bar construction

$$\sum_{\alpha} P_{j,\alpha}(a_1, \dots, a_N) \otimes Q_{j,\alpha}(a_1, \dots, a_N) \in I(A) \otimes I(A), \quad j = 1, \dots, M,$$

generate the k -module $H_2(\bar{B}(A))$. Then the algebra A has a presentation

$$A \cong T(x_1, \dots, x_N) / \left(\sum_{i=1}^N \lambda_{ri} x_i = \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta}, \quad r = 1, \dots, R; \sum_{\alpha} \bar{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0, \quad j = 1, \dots, M \right).$$

(Here N, M, R can be infinite of any cardinality.)

Proof of statement (i) Consider the morphism $f: T(x_1, \dots, x_N) \rightarrow A$, $x_i \mapsto a_i$, of connected algebras. The classes $[a_1], \dots, [a_N]$ generate $H_1(\bar{B}(A))$ and are images of classes $[x_1], \dots, [x_N]$ with respect to the map $f_{*,1}: H_1(\bar{B}(T(x_1, \dots, x_N))) \rightarrow H_1(\bar{B}(A))$. Hence $f_{*,1}$ is surjective. By Lemma A.5(i), f is surjective. $\square$

Proof of statement (ii) Consider the algebra

$$C := T(x_1, \dots, x_N) / \left(\sum_{i=1}^N \lambda_{ri} x_i = \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta}, \quad r = 1, \dots, R; \sum_{\alpha} \bar{P}_{j,\alpha} \cdot Q_{j,\alpha} = 0, \quad j = 1, \dots, M \right).$$

The following identities in A are given:

$$\sum_{i=1}^N \lambda_{ri} \cdot a_i = \sum_{\beta} \bar{K}_{r,\beta}(a_1, \dots, a_N) \cdot L_{r,\beta}(a_1, \dots, a_N), \quad 0 = \sum_{\alpha} \bar{P}_{j,\alpha}(a_1, \dots, a_N) \cdot Q_{j,\alpha}(a_1, \dots, a_N).$$

Hence the morphism $f: C \rightarrow A$, $x_i \mapsto a_i$, is well defined. The induced map $f_{*,1}: H_1(\bar{B}(C)) \rightarrow H_1(\bar{B}(A))$ is surjective, since the elements $[a_i] = f_{*,1}([x_i])$ generate $H_1(\bar{B}(A))$.

We prove that $f_{*,1}$ is injective. Let $\xi \in H_1(\bar{B}(C))$ and $f_{*,1}(\xi) = 0$. By Corollary A.4 and surjectivity of $T(x_1, \dots, x_N) \rightarrow C$, we have $\xi = \sum_{i=1}^N \mu_i \cdot [x_i]$ for some $\mu_i \in k$. Then $0 = f_*(\xi) = \sum_i \mu_i [a_i] \in H_1(\bar{B}(A))$. All linear relations between $[a_1], \dots, [a_N]$ follow from the relations $\sum_i \lambda_{ri} [a_i] = 0$, and therefore $\mu_i = \sum_{r=1}^R c_r \lambda_{ri}$ for some $c_r \in k$. It follows that ξ is represented by the cycle

$$\sum_{i=1}^N \sum_{r=1}^R c_r \lambda_{ri} \cdot x_i = \sum_{r=1}^R c_r \sum_{\beta} \bar{K}_{r,\beta} \cdot L_{r,\beta} = d_{\bar{B},2} \left(\sum_{r=1}^R c_r \sum_{\beta} K_{r,\beta} \otimes L_{r,\beta} \right) \in \bar{B}_1(C).$$

Hence $\xi = 0$. We proved that $f_{*,1}$ is bijective.

The elements $\sum_{\alpha} P_{i,\alpha} \otimes Q_{i,\alpha} \in I(C) \otimes I(C)$ are cycles in $\bar{B}_2(C)$, and their images generate $H_2(\bar{B}(A))$. Hence $f_{*,2}: H_2(\bar{B}(C)) \rightarrow H_2(\bar{B}(A))$ is surjective. Conditions of Lemma A.5(ii) are satisfied, so f is bijective. $\square$

A.4 Bounds on the number of homogeneous generators and relations

Let A be a connected k -algebra. Proposition A.1 gives a lower bound on the number of generators and relations in the homogeneous presentations of A , and Theorem A.6 gives an upper bound. These bounds coincide if k is a principal ideal domain, A is a free k -module, and graded components are finitely generated. We introduce some notation.

Definition A.7 Let M be a finitely generated module over a principal ideal domain k . By the structure theorem of such modules, we have

$$(11) \quad M \simeq k/(d_1) \oplus \cdots \oplus k/(d_r),$$

where $d_1, \dots, d_r \in k$ are noninvertible, and $d_i \mid d_{i+1}$ for all $i = 1, \dots, r-1$. The number r is determined uniquely, and the elements d_i — uniquely up to a multiplication by an invertible element. Hence,

$$\text{gen } M := r, \quad \text{rel } M := \max\{s : d_s \neq 0\}$$

are well defined. We get a short exact sequence $k^{\text{rel } M} \rightarrow k^{\text{gen } M} \rightarrow M \rightarrow 0$.

Lemma A.8 Let k be a principal ideal domain. Suppose that there exists a short exact sequence $k^A \xrightarrow{f} k^B \rightarrow M \rightarrow 0$ for some $A, B < \infty$. Then $A \geq \text{rel } M$ and $B \geq \text{gen } M$.

Proof We can assume that f is in the Smith normal form, that is, f is represented by a diagonal matrix with diagonal elements $d'_1, \dots, d'_s$ such that $d'_1 \mid d'_2 \mid \cdots \mid d'_s$. Remove all nonzero columns: this preserves cokernel and does not increase A . If d'_i is invertible, remove the i^{th} row and the i^{th} column: this preserves cokernel and diminish A and B by 1. We obtain a diagonal matrix $B' \times A'$ having no zero columns and no invertible elements on diagonal. Hence the cokernel is exactly of the form (11) for $B' = r = \text{gen } M$ and $A' = s = \text{rel } M$. $\square$

Lemma A.9 Let k be a principal ideal domain and $0 \rightarrow k^a \rightarrow k^b \xrightarrow{f} k^c \rightarrow 0$ be an exact sequence of k -modules for some $a, b, c < \infty$. Then $b = a + c$.

Proof We can assume that f is in a Smith normal form. In this basis, f is represented by a diagonal matrix $c \times b$. Since f is surjective, the matrix has no nonzero rows, and all diagonal elements are noninvertible. Hence $\text{Ker } f \simeq k^{b-c}$. We have $k^d \not\subseteq k^{d'}$ for $d \neq d'$, so $a = b - c$. $\square$

Recall that we consider G -graded algebras that are connected with respect to the $\mathbb{Z}$ -grading given by a map $G \rightarrow \mathbb{Z}$.

Theorem A.10 Let A be a connected associative algebra with unit over a principal ideal domain k . Suppose that k -modules $\text{Tor}_1^A(k, k)_n$ and $\text{Tor}_2^A(k, k)_n$ are finitely generated for all $n \in G$. Then:

- (i) If A is a free k -module, it admits a homogeneous presentation that contains (for every n) precisely $\text{gen } \text{Tor}_1^A(k, k)_n$ generators and $\text{gen } \text{Tor}_2^A(k, k)_n + \text{rel } \text{Tor}_1^A(k, k)_n$ relations of degree n .
 - (ii) If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n ,
- $$(12) \quad N_n \geq \text{gen } \text{Tor}_1^A(k, k)_n, \quad M_n \geq \text{gen } \text{Tor}_2^A(k, k)_n + \text{rel } \text{Tor}_1^A(k, k)_n.$$

Proof of statement (i) For every n , choose a set of $\text{gen}(\text{Tor}_1^A(k, k)_n)$ additive generators for the k -module $\text{Tor}_1^A(k, k)_n$, a set of $\text{rel}(\text{Tor}_1^A(k, k)_n)$ linear relations between them, and a set of $\text{gen}(\text{Tor}_2^A(k, k)_n)$ generators for $\text{Tor}_2^A(k, k)_n$. These elements are represented by cycles and boundaries in the bar construction. Applying Theorem A.6 to them, we obtain a presentation of required size. $\square$

Proof of statement (ii) Apply Proposition A.1 and continue the exact sequence to the free resolution of the left A -module k . It has the form

$$\dots \rightarrow A \otimes k^M \rightarrow A \otimes k^N \rightarrow A \xrightarrow{\varepsilon} k \rightarrow 0.$$

Applying the functor $k \otimes_A (-)$, we obtain a chain complex of graded k -modules

$$\dots \rightarrow k^M \xrightarrow{\partial} k^N \xrightarrow{0} k \rightarrow 0,$$

having $\text{Tor}^A(k, k)$ as homology. Therefore, for some $\partial_n: k^{M_n} \rightarrow k^{N_n}$ we have

$$\text{Coker } \partial_n \simeq \text{Tor}_1^A(k, k)_n, \quad \text{Ker } \partial_n \twoheadrightarrow \text{Tor}_2^A(k, k)_n.$$

In particular, $\text{Tor}_1^A(k, k)_n$ is generated by N_n elements, so $N_n \geq \text{gen } \text{Tor}_1^A(k, k)_n$.

If M_n is infinite, both inequalities (12) are true, since the right side is finite. If M_n is finite, then N_n is finite, since $\text{Coker } \partial_n$ is finitely generated. Thus $\text{Ker } \partial_n \subset k^{M_n}$, $\text{Im } \partial_n \subset k^{N_n}$ are submodules of finitely generated free modules, so these modules are free: $\text{Ker } \partial_n \simeq k^P$, $\text{Im } \partial_n \simeq k^Q$. We obtain exact sequences

$$k^P \rightarrow k^{N_n} \rightarrow \text{Tor}_1^A(k, k)_n \rightarrow 0, \quad k^Q \rightarrow \text{Tor}_2^A(k, k)_n \rightarrow 0, \quad 0 \rightarrow k^P \rightarrow k^{M_n} \rightarrow k^Q \rightarrow 0.$$

Then $N_n \geq \text{gen } \text{Tor}_1^A(k, k)_n$, $P \geq \text{rel } \text{Tor}_1^A(k, k)_n$, $Q \geq \text{gen } \text{Tor}_2^A(k, k)_n$ by Lemma A.8 and $P + Q = M_n$ by Lemma A.9. This proves (12). $\square$

As a corollary, we obtain a well known result by Wall [44, Section 7]:

Corollary A.11 *Let A be a connected associative algebra with unit over a field k . Then:*

- (i) *A admits a homogeneous presentation that contains (for every n) precisely $\dim_k \text{Tor}_1^A(k, k)_n$ generators and $\dim_k \text{Tor}_2^A(k, k)_n$ relations of degree n .*
- (ii) *If A admits a homogeneous presentation that contains N_n generators and M_n relations of degree n , then $N_n \geq \dim_k \text{Tor}_1^A(k, k)_n$ and $M_n \geq \dim_k \text{Tor}_2^A(k, k)_n$.* $\square$

We also obtain a criterion of freeness.

Corollary A.12 [32, Proposition 8.5.4] *Let A be a connected associative algebra with unit over a principal ideal domain k , which is a free k -module. The following conditions are equivalent.*

- (a) *A is a free algebra (a tensor algebra on homogeneous generators).*
- (b) *$\text{Tor}_1^A(k, k)$ is a free k -module, and $\text{Tor}_2^A(k, k) = 0$.*

Proof By Corollary A.4, (a) implies (b). Conversely, suppose that (b) holds. Then

$$\text{rel Tor}_1^A(\mathbf{k}, \mathbf{k}) = \text{gen Tor}_2^A(\mathbf{k}, \mathbf{k}) = 0.$$

By Theorem A.10(i), the algebra A admits a presentation with no relations. Hence A is free. $\square$

Appendix B Loop homology and extensions of Hopf algebras

Consider a homotopy fibration $F \rightarrow E \xrightarrow{p} B$ of simply connected spaces, such that $\Omega p: \Omega E \rightarrow \Omega B$ admits a homotopy section (ie there is a continuous map $\sigma: \Omega B \rightarrow \Omega E$ that preserves basepoints, and a homotopy $\Omega p \circ \sigma \sim \text{id}_{\Omega B}$). It is well known that then ΩE is homotopy equivalent to $\Omega F \times \Omega B$ (see [7, Proposition A.2; 17, Theorem 5.2]). If $\mathbf{k}$ -homology of these loop spaces is free, we obtain an extension of Hopf algebras $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$. In Theorem B.3 we give a full proof of this folklore result. We consider ordinary loop spaces instead of Moore loop spaces, so that $\Omega(X \times Y) \cong \Omega X \times \Omega Y$ is a strict isomorphism of H-spaces.

We have a natural isomorphism

$$\alpha: \pi_n(A \times B) \xrightarrow{\cong} \pi_n(A) \oplus \pi_n(B), \quad [f] \mapsto [\text{pr}_A \circ f] \oplus [\text{pr}_B \circ f],$$

for any A, B and $n \geq 1$. We denote basepoint inclusion by $\varepsilon: * \rightarrow Y$ and collapse map by $\eta: Y \rightarrow *$.

Lemma B.1 *Let X be a simply connected space and $\mu: \Omega X \times \Omega X \rightarrow \Omega X$ be the composition of loops. Then the following diagram is commutative:*

$$\begin{array}{ccc} \pi_n(\Omega X \times \Omega X) & \xrightarrow{\mu_*} & \pi_n(\Omega X) \\ \alpha \downarrow \cong & \nearrow (x,y) \mapsto x+y & \\ \pi_n(\Omega X) \oplus \pi_n(\Omega X) & & \end{array}$$

Proof Let elements $x, y \in \pi_n(\Omega X)$ be represented by maps $f, g: S^n \rightarrow \Omega X$. Consider the element $z = [f \times \eta] + [\eta \times g] \in \pi_n(\Omega X \times \Omega X)$.

The map $\mu \circ (f \times \eta)$ is the composition $S^n \xrightarrow{f} \Omega X \xrightarrow{\text{id} \times \eta} \Omega X \times \Omega X \xrightarrow{\mu} \Omega X$. The composition of two right maps is homotopic to the identity, hence $\mu \circ (f \times \eta) \sim f$. Passing to homotopy groups, we have $\mu_*([f \times \eta]) = x$. Similarly, $\mu_*([\eta \times g]) = y$, hence $\mu_*(z) = x + y$. On the other hand, $\alpha([f \times \eta]) = [\text{pr}_1 \circ (f \times \eta)] \oplus [\text{pr}_2 \circ (f \times \eta)] = [f] \oplus [\eta \varepsilon] = x \oplus 0$. Similarly, $\alpha([\eta \times g]) = 0 \oplus y$, hence $\alpha(z) = x \oplus y$. We obtained $\mu_*(\alpha^{-1}(x \oplus y)) = \mu_*(z) = x + y$, so the diagram commutes. $\square$

In the following lemma, we say that a diagram commutes if it homotopy commutes.

Lemma B.2 *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of simply connected spaces, and $\sigma: \Omega B \rightarrow \Omega E$ be a homotopy section for Ωp . Consider the composition*

$$f: \Omega F \times \Omega B \xrightarrow{\Omega i \times \sigma} \Omega E \times \Omega E \xrightarrow{\mu} \Omega E.$$

Then:

(i) f is a weak homotopy equivalence.

(ii) f respects the inclusion and the projection, that is, the following diagram commutes:

$$\begin{array}{ccccc} & & \Omega E & & \\ & \nearrow \Omega i & \uparrow f & \nwarrow \Omega p & \\ \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B \end{array}$$

(iii) f respects the left action of ΩF , that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times f} & \Omega E \times \Omega E \\ \downarrow \mu \times \text{id} & & \downarrow \mu \\ \Omega F \times \Omega B & \xrightarrow{f} & \Omega E \end{array}$$

(iv) f respects the right coaction of ΩB , that is, the following diagram commutes:

$$\begin{array}{ccc} \Omega F \times \Omega B & \xrightarrow{f} & \Omega E \\ \downarrow \text{id} \times \Delta & & \downarrow \Delta \\ \Omega F \times \Omega B \times \Omega B & \xrightarrow{f \times \text{id}} & \Omega E \times \Omega E \\ & & \downarrow \text{id} \times \Omega p \\ & & \Omega E \times \Omega B \end{array}$$

Proof We have an exact sequence

$$\cdots \rightarrow \pi_n(\Omega F) \xrightarrow{(\Omega i)_*} \pi_n(\Omega E) \xrightarrow{(\Omega p)_*} \pi_n(\Omega B) \rightarrow \cdots,$$

where the map $(\Omega p)_*$ has a section σ_* . For every $n \geq 1$, we obtain an isomorphism of groups

$$\varphi: \pi_n(\Omega F) \oplus \pi_n(\Omega B) \xrightarrow{\cong} \pi_n(\Omega E), \quad \varphi(x, y) = (\Omega i)_*(x) + \sigma_*(y).$$

(We use that $\pi_1(\Omega X)$ is abelian.) By Lemma B.1 and naturality of

$$\alpha: \pi_n(\Omega F \times \Omega B) \rightarrow \pi_n(\Omega F) \oplus \pi_n(\Omega B)$$

we have

$$\varphi \circ \alpha = (\mu \circ (\Omega i \times \sigma))_* = f_*: \pi_n(\Omega F \times \Omega B) \rightarrow \pi_n(\Omega E).$$

Hence f_* is an isomorphism for all n , so f is a weak homotopy equivalence. Now consider the diagram

$$\begin{array}{ccccccc} & & \Omega E & \xrightarrow{\Omega p} & \Omega B & & \\ & \nearrow \text{id} & \uparrow \mu & & \uparrow \mu & & \\ \Omega E \times * & \xrightarrow{\text{id} \times \eta} & \Omega E \times \Omega E & \xrightarrow{\Omega p \times \Omega p} & \Omega B \times \Omega B & & \\ \uparrow \Omega i \times \text{id} & & \uparrow \Omega i \times \sigma & & \uparrow \eta \times \text{id} & & \\ \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B & & \end{array}$$

The triangle commutes, since η is a homotopy unit in ΩE . The upper right square commutes, since Ωp is a map of H-spaces. The bottom left square commutes, since $\eta_{\Omega E} = \sigma \circ \eta_{\Omega B} : * \rightarrow \Omega E$. Finally, the commutativity of bottom right square is equivalent to the existence of homotopies $\Omega p \circ \Omega i \sim \eta \circ \varepsilon$ and $\Omega p \circ \sigma \sim \text{id}$. The first homotopy exists, since $p \circ i$ is homotopy trivial; the second exists, since σ is a homotopy section for Ωp . Hence the whole diagram is commutative. The right side of the diagram is homotopic to $\text{id} : \Omega B \rightarrow \Omega B$, since η is a homotopy unit in ΩB . We obtain a commutative diagram

$$\begin{array}{ccccc} & & \Omega E & \xrightarrow{\Omega p} & \Omega B \\ & \nearrow \Omega i \times \text{id} & \uparrow f & & \uparrow \text{id} \\ \Omega F \times * & \xrightarrow{\text{id} \times \eta} & \Omega F \times \Omega B & \xrightarrow{\varepsilon \times \text{id}} & * \times \Omega B \end{array}$$

that is equivalent to the diagram from (ii). Now consider the diagram

$$\begin{array}{ccccc} \Omega F \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times \Omega i \times \sigma} & \Omega E \times \Omega E \times \Omega E & \xrightarrow{\text{id} \times \mu} & \Omega E \times \Omega E \\ \downarrow \mu \times \text{id} & & \downarrow \mu \times \text{id} & & \downarrow \mu \\ \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E \end{array}$$

The left square commutes, since $\Omega i : \Omega F \rightarrow \Omega E$ is a map of H-spaces; the right square commutes, since μ is homotopy associative. The top side of the diagram equals $\Omega i \times (\mu \circ (\Omega i \times \sigma)) = \Omega i \times f$, the bottom side equals f . Hence, it is the diagram from (iii). Finally, consider the diagram

$$\begin{array}{ccccc} \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma} & \Omega E \times \Omega E & \xrightarrow{\mu} & \Omega E \\ \downarrow D & & \downarrow D & & \downarrow \Delta \\ \Omega F \times \Omega B \times \Omega F \times \Omega B & \xrightarrow{\Omega i \times \sigma \times \Omega i \times \sigma} & \Omega E \times \Omega E \times \Omega E \times \Omega E & \xrightarrow{\mu \times \mu} & \Omega E \times \Omega E \\ \downarrow \text{pr}_{124} & \searrow \Omega i \times \sigma \times \varepsilon \times \text{id} & \downarrow \text{id} \times \text{id} \times \Omega p \times \Omega p & & \downarrow \text{id} \times \Omega p \\ \Omega F \times \Omega B \times \Omega B & \xrightarrow{\phi} & \Omega E \times \Omega E \times \Omega B \times \Omega B & \xrightarrow{\mu \times \mu} & \Omega E \times \Omega B \end{array}$$

where $\Delta(x) := (x, x)$, $D(x, y) := (x, y, x, y)$ and $\phi(f, b_1, b_2) := (\Omega i(f), \sigma(b_1), *, b_2)$. Clearly, the top two squares commute. The bottom right square commutes, since $\text{id} : \Omega E \rightarrow \Omega E$ and $\Omega p : \Omega E \rightarrow \Omega B$ are maps of H-spaces. The upper triangle commutes, since $\Omega p \circ \Omega i \sim \varepsilon$ and $\Omega p \circ \sigma \sim \text{id}$; the bottom triangle commutes by the definition of ϕ . The outer maps in the diagram give the required diagram (iv). $\square$

In the proof of next theorem we use the Künneth map $\kappa : H_*(X; k) \otimes H_*(Y; k) \rightarrow H_*(X \times Y; k)$. It is natural and associative. It is an isomorphism if $H_*(Y; k)$ is a free k -module.

If X is a simply connected space and $H_*(\Omega X; k)$ is free over k , this module is a connected k -Hopf algebra with the standard cup coproduct (see Section 2.4) and the Pontryagin product

$$m : H_*(\Omega X; k) \otimes H_*(\Omega X; k) \xrightarrow{\kappa} H_*(\Omega X \times \Omega X; k) \xrightarrow{\mu_*} H_*(\Omega X; k).$$

The unit and counit $k \xrightarrow{\eta} H_*(\Omega X; k) \xrightarrow{\varepsilon} k$ are induced by the H-space maps $* \xrightarrow{\eta} \Omega X \xrightarrow{\varepsilon} *$.

Theorem B.3 Let k be an associative ring with unit. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a homotopy fibration of simply connected spaces such that $H_*(\Omega B; k)$ and $H_*(\Omega F; k)$ are free k -modules, and the map Ωp admits a homotopy section $\sigma: \Omega B \rightarrow \Omega E$. Consider the composition

$$\Phi: H_*(\Omega F; k) \otimes H_*(\Omega B; k) \xrightarrow{(\Omega i)_* \otimes \sigma_*} H_*(\Omega E; k) \otimes H_*(\Omega E; k) \xrightarrow{m} H_*(\Omega E; k).$$

Then

- (i) Φ is an isomorphism of k -modules;
- (ii) $(\Omega i)_* = \Phi \circ (\text{id}_{H_*(\Omega F; k)} \otimes \eta_{H_*(\Omega B; k)})$;
- (iii) $(\Omega p)_* \circ \Phi = \varepsilon_{H_*(\Omega F; k)} \otimes \text{id}_{H_*(\Omega B; k)}$;
- (iv) Φ is a morphism of left $H_*(\Omega F; k)$ -modules and right $H_*(\Omega B; k)$ -comodules, where the (co)module structure on $H_*(\Omega E; k)$ is induced by the maps $(\Omega i)_*$ and $(\Omega p)_*$.

In particular, $k \rightarrow H_*(\Omega F; k) \xrightarrow{(\Omega i)_*} H_*(\Omega E; k) \xrightarrow{(\Omega p)_*} H_*(\Omega B; k) \rightarrow k$ is an extension of connected Hopf algebras over k .

Proof We write $H_*(\Omega X)$ instead of $H_*(\Omega X; k)$. Note that σ is continuous, and $\Omega i, \Omega p$ are maps of H-spaces. Hence σ_* is a map of coalgebras, and $(\Omega i)_*, (\Omega p)_*$ are maps of Hopf algebras. By the naturality of Künneth map, the following diagram commutes:

$$\begin{array}{ccccc} H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_* \otimes \sigma_*} & H_*(\Omega E) \otimes H_*(\Omega E) & \xrightarrow{m} & H_*(\Omega E) \\ \kappa \downarrow \simeq & & \kappa \downarrow & & \parallel \\ H_*(\Omega F \times \Omega B) & \xrightarrow{(\Omega i \times \sigma)_*} & H_*(\Omega E \times \Omega E) & \xrightarrow{\mu_*} & H_*(\Omega E) \end{array}$$

The top side of diagram equals Φ , and the bottom side equals f_* . Hence Φ is the composition $H_*(\Omega F) \otimes H_*(\Omega B) \xrightarrow{\kappa} H_*(\Omega F \times \Omega B) \xrightarrow{f_*} H_*(\Omega E)$. The left map is bijective by the assumption, the right map is bijective by Lemma B.2(i). Hence Φ is an isomorphism, so (i) is proved. Consider the diagram

$$\begin{array}{ccccc} & & H_*(\Omega E) & & \\ & \nearrow (\Omega i)_* & \uparrow f_* & \nwarrow (\Omega p)_* & \\ H_*(\Omega F \times *) & \xrightarrow{(\text{id} \times \eta)_*} & H_*(\Omega F \times \Omega B) & \xrightarrow{(\varepsilon \times \text{id})_*} & H_*(\Omega B) \\ \kappa \uparrow \simeq & & \kappa \uparrow & & \kappa \uparrow \simeq \\ H_*(\Omega F) \otimes k & \xrightarrow{\text{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \text{id}} & k \otimes H_*(\Omega B) \end{array}$$

The top half of the diagram commutes by Lemma B.2(ii), the bottom half commutes by naturality of κ . Since $f_* \circ \kappa = \Phi$, we have a commutative diagram

$$\begin{array}{ccccc} & & H_*(\Omega E) & & \\ & \nearrow (\Omega i)_* & \uparrow \Phi & \nwarrow (\Omega p)_* & \\ H_*(\Omega F) & \xrightarrow{\text{id} \otimes \eta} & H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\varepsilon \otimes \text{id}} & H_*(\Omega B) \end{array}$$

which proves (ii) and (iii). Now consider the diagram

$$\begin{array}{ccccc}
 H_*(\Omega F) \otimes H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\text{id} \otimes \kappa} & H_*(\Omega F) \otimes H_*(\Omega F \times \Omega B) & \xrightarrow{(\Omega i)_* \otimes f_*} & H_*(\Omega E) \otimes H_*(\Omega E) \\
 \downarrow \kappa \otimes \text{id} & & \downarrow \kappa & & \downarrow \kappa \\
 H_*(\Omega F \times \Omega F) \otimes H_*(\Omega B) & \xrightarrow{\kappa \otimes \text{id}} & H_*(\Omega F \times \Omega F \times \Omega B) & \xrightarrow{(\Omega i \times f)_*} & H_*(\Omega E \times \Omega E) \\
 \downarrow \mu_* \otimes \text{id} & & \downarrow (\mu \times \text{id})_* & & \downarrow \mu_* \\
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\kappa} & H_*(\Omega F \times \Omega B) & \xrightarrow{f_*} & H_*(\Omega E)
 \end{array}$$

The bottom right square commutes by Lemma B.2(iii), the other squares commute by naturality of κ . Since $\mu_* \circ \kappa = m: H_*(\Omega X) \otimes H_*(\Omega X) \rightarrow H_*(\Omega X)$, the outer maps in the diagram are

$$\begin{array}{ccc}
 H_*(\Omega F) \otimes H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{(\Omega i)_* \otimes \Phi} & H_*(\Omega E) \otimes H_*(\Omega E) \\
 \downarrow m \otimes \text{id} & & \downarrow m \\
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E)
 \end{array}$$

Hence Φ is a map of left $H_*(\Omega F)$ -modules. Similarly, by Lemma B.2(iv) and the Künneth isomorphisms we have the commutative diagram

$$\begin{array}{ccc}
 H_*(\Omega F) \otimes H_*(\Omega B) & \xrightarrow{\Phi} & H_*(\Omega E) \\
 \downarrow \text{id} \otimes \Delta & & \downarrow \Delta \\
 & & H_*(\Omega E) \otimes H_*(\Omega E) \\
 & & \downarrow \text{id} \otimes (\Omega p)_* \\
 H_*(\Omega F) \otimes H_*(\Omega B) \otimes H_*(\Omega B) & \xrightarrow{\Phi \otimes \text{id}} & H_*(\Omega E) \otimes H_*(\Omega B)
 \end{array}$$

hence Φ is a map of right $H_*(\Omega B)$ -comodules.

Since (i)–(iv) hold, the maps of Hopf algebras $(\Omega i)_*: H_*(\Omega F) \rightarrow H_*(\Omega E)$ and $(\Omega p)_*: H_*(\Omega E) \rightarrow H_*(\Omega B)$ form an extension of Hopf algebras by Proposition 2.4. $\square$

Recall that an element $x \in A$ of a Hopf algebra is *primitive* if $\Delta x = 1 \otimes x + x \otimes 1$. The set of primitive elements is a Lie subalgebra $PA \subset A$. Every map of Hopf algebras $f: A \rightarrow A'$ induces a map of Lie algebras $Pf := f|_{PA}: PA \rightarrow PA'$.

Corollary B.4 Suppose that the conditions of Theorem B.3 are met. Let $x \in H_*(\Omega E; \mathbf{k})$ be a primitive element such that $(\Omega p)_*(x) = 0$. Then $x = (\Omega i)_*(y)$ for some $y \in H_*(\Omega F; \mathbf{k})$.

Proof Since $\mathbf{k} \rightarrow H_*(\Omega F; \mathbf{k}) \rightarrow H_*(\Omega E; \mathbf{k}) \rightarrow H_*(\Omega B; \mathbf{k}) \rightarrow \mathbf{k}$ is a Hopf algebra extension, the sequence $0 \rightarrow PH_*(\Omega F; \mathbf{k}) \rightarrow PH_*(\Omega E; \mathbf{k}) \rightarrow PH_*(\Omega B; \mathbf{k})$ is exact; see [31, Proposition 4.10]. (This also easily follows from definitions). We have

$$x \in \text{Ker}(PH_*(\Omega E; \mathbf{k}) \rightarrow PH_*(\Omega B; \mathbf{k})) = \text{Im}(PH_*(\Omega F; \mathbf{k}) \rightarrow PH_*(\Omega E; \mathbf{k})). \quad \square$$

Appendix C Commutator identities

Fix elements $u_1, \dots, u_m$ of degree 1 in a graded associative algebra Γ . For a subset $I = \{i_1 < \dots < i_k\} \subset [m]$, we define

$$\hat{u}_I := u_{i_1} \cdots u_{i_k}, \quad c(I, x) := [u_{i_1}, [u_{i_2}, [\dots [u_{i_k}, x], \dots]]], \quad x \in \Gamma.$$

We write $A < B$ when $A, B \subset [m]$ and $\max(A) < \min(B)$. If $A < B$, we have $\hat{u}_{A \sqcup B} = \hat{u}_A \cdot \hat{u}_B$ and $c(A \sqcup B, x) = c(A, c(B, x))$. Also, $\hat{u}_\emptyset = 1$, $c(\emptyset, x) = x$.

Define the Koszul sign by $\theta(A, B) := |\{(a, b) \in A \times B : a > b\}|$. In a graded commutative algebra, we would have $\hat{u}_A \cdot \hat{u}_B = (-1)^{\theta(A, B)} \hat{u}_{A \sqcup B}$ if $A \cap B = \emptyset$. It has the following properties:

(i) $\theta(A, B) \equiv |A| \cdot |B| + \theta(B, A) \pmod{2}$.

(ii) If $A_1 \sqcup B_1 < A_2 \sqcup B_2$, then

$$\theta(A_1 \sqcup A_2, B_1 \sqcup B_2) \equiv \theta(A_1, B_1) + \theta(A_2, B_2) + |A_2| \cdot |B_1|.$$

For $I \subset [m]$, $j \in [m]$, we write $I_{<j} = \{i \in I : i < j\}$, $I_{>j} = \{i \in I : i > j\}$. We also use i as a shortened notation for $\{i\}$.

C.1 Regrouping of monomials

The following formulas can be used to express any monomial on $u_1, \dots, u_m$ as a linear combination of $c_1 \cdots c_s \cdot \hat{u}_B$, $c_i = c(A_i, u_{j_i})$, $A_i \neq \emptyset$.

Lemma C.1 *Let $I \subset [m]$, and let $x \in \Gamma$ be homogeneous. Then*

$$(13) \quad \hat{u}_I \cdot x = \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} c(A, x) \hat{u}_B.$$

Proof Define $d := \deg(x)$. We induct on $|I|$. The base case $I = \emptyset$ is clear. For the inductive step, let $i = \min(I)$, $I' = I \setminus i$. Then the right-hand side is equal to

$$\begin{aligned} \sum_{I'=A \sqcup B} (-1)^{\theta(i \sqcup A, B) + d \cdot |B|} c(i \sqcup A, x) \hat{u}_B &+ \sum_{I'=A \sqcup B} (-1)^{\theta(A, i \sqcup B) + d \cdot |i \sqcup B|} c(A, x) \hat{u}_{i \sqcup B} \\ &= \sum_{I'=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} ([u_i, c(A, x)] + (-1)^{|A|+d} c(A, x) u_i) \cdot \hat{u}_B \\ &= \sum_{I'=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} u_i c(A, x) \cdot \hat{u}_B. \end{aligned}$$

By the inductive hypothesis, this sum is equal to $u_i \cdot \hat{u}_{I'} x = \hat{u}_I \cdot x$. □

Proposition C.2 *Let $I \subset [m]$, $j \in [m]$. Then*

$$\hat{u}_I \cdot u_j = \sum_{\substack{I=A \sqcup B \\ \max(A) > j}} (-1)^{\theta(A, B) + |B|} c(A, u_j) \hat{u}_B + (-1)^{|I_{>j}|} \cdot \begin{cases} \hat{u}_{I \sqcup j}, & j \notin I \\ \hat{u}_{I_{<j}} \cdot u_j^2 \cdot \hat{u}_{I_{>j}}, & j \in I \end{cases}.$$

Proof Define $P = I_{\leq j}$, $Q = I_{> j}$. Then $P < Q$, therefore

$$\hat{u}_I = \hat{u}_P \hat{u}_Q, \quad r := \hat{u}_P u_j \hat{u}_Q = \begin{cases} \hat{u}_{I \sqcup \{j\}}, & j \notin I; \\ \hat{u}_{I_{< j}} \cdot u_j^2 \cdot \hat{u}_{I_{> j}}, & j \in I. \end{cases}$$

Apply (13) to $\hat{u}_Q \cdot u_j$, and consider the summand with $A_2 = \emptyset$ separately:

$$\begin{aligned} \hat{u}_I \cdot u_j &= \hat{u}_P \hat{u}_Q u_j = \sum_{Q=A_2 \sqcup B_2} (-1)^{\theta(A_2, B_2) + |B_2|} \hat{u}_P c(A_2, u_j) \hat{u}_{B_2} \\ &= (-1)^{|Q|} \hat{u}_P u_j \hat{u}_Q + \sum_{\substack{Q=A_2 \sqcup B_2 \\ A_2 \neq \emptyset}} (-1)^{\theta(A_2, B_2) + |B_2|} \hat{u}_P c(A_2, u_j) \hat{u}_{B_2}. \end{aligned}$$

Applying (13) to $\hat{u}_P \cdot c(A_2, u_j)$, we obtain the required identity:

$$\begin{aligned} \hat{u}_I \cdot u_j &= (-1)^{|Q|} r + \sum_{P=A_1 \sqcup B_1} \sum_{\substack{Q=A_2 \sqcup B_2 \\ A_2 \neq \emptyset}} (-1)^{\theta(A_1, B_1) + (|A_2|+1) \cdot |B_1| + \theta(A_2, B_2) + |B_2|} c(A_1, c(A_2, u_j)) \hat{u}_{B_1} \hat{u}_{B_2} \\ &= (-1)^{|Q|} r + \sum_{\substack{P \sqcup Q = A \sqcup B \\ A_{> j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} c(A, u_j) \hat{u}_B. \end{aligned} \quad \square$$

C.2 Identities for nested commutators

In this section Γ can be a Lie superalgebra.

Lemma C.3 For $I \subset [m]$ and homogeneous elements $x, y \in \Gamma$, we have

$$\begin{aligned} (14) \quad c(I, [x, y]) &= \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} [c(A, x), c(B, y)] \\ &= [c(I, x), y] + (-1)^{\deg(x) \cdot |I|} [x, c(I, y)] + \sum_{\substack{I=A \sqcup B \\ A, B \neq \emptyset}} (-1)^{\theta(A, B) + \deg(x) \cdot |B|} [c(A, x), c(B, y)]. \end{aligned}$$

Proof The second identity follows from $\theta(\emptyset, I) = \theta(I, \emptyset) = 0$ and $c(\emptyset, x) = x$. Let us prove the first identity by induction on $|I|$. The base case $I = \emptyset$ is clear. For the inductive step, define $i = \min(I)$, $I' = I \setminus i$, $d = \deg(x)$. Then, by the inductive hypothesis,

$$\begin{aligned} c(I, [x, y]) &= [u_i, c(I', [x, y])] = \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'|} [u_i, [c(A', x), c(B', y)]] \\ &= \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'|} [[u_i, c(A', x)], c(B', y)] \\ &\quad + \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', B') + d \cdot |B'| + d \cdot |A'|} [c(A', x), [u_i, c(B', y)]] \\ &= \sum_{I'=A' \sqcup B'} (-1)^{\theta(i \sqcup A', B') + d \cdot |B'|} [c(i \sqcup A', x), c(B', y)] \\ &\quad + \sum_{I'=A' \sqcup B'} (-1)^{\theta(A', i \sqcup B') + d \cdot |i \sqcup B'|} [c(A', x), c(i \sqcup B', y)] \\ &= \sum_{I=A \sqcup B} (-1)^{\theta(A, B) + d \cdot |B|} [c(A, x), c(B, y)]. \end{aligned} \quad \square$$

Corollary C.4 Let $I \subset [m]$, $I = I'' \sqcup I'$, $I'' < I'$. Let $x, y \in \Gamma$ be homogeneous, and let $\mathcal{A} \subset 2^{I'} \times 2^{I''}$ be a family of pairs of subsets. Then

$$(15) \quad \sum_{\substack{I' = A' \sqcup B' \\ (A', B') \in \mathcal{A}}} (-1)^{\theta(A', B') + |B'|} c(I'', [c(A', x), c(B', y)]) = \sum_{\substack{I = A \sqcup B \\ (A \cap I', B \cap I') \in \mathcal{A}}} (-1)^{\theta(A, B) + |B|} [c(A, x), c(B, y)].$$

Proof This follows from (14) and from identities

$$c(A'', c(A', x)) = c(A'' \sqcup A', x), \quad \theta(A'' \sqcup A', B'' \sqcup B') = \theta(A'', B'') + \theta(A', B') + |A'| \cdot |B''|$$

that are true for $A'', B'' < A', B'$. $\square$

Proposition C.5 Let $J \subset [m]$ and $i, j \in J$ such that $i < j$ and $J_{>j} \neq \emptyset$. Then

$$(16) \quad c(J \setminus ij, [u_i, u_j]) = (-1)^{|J_{>j}|} c(J \setminus i, u_i) - (-1)^{|J_{>i}|} c(J \setminus j, u_j) + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>i}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)].$$

Proof Define $P = J_{<j}$, $Q = J_{>i} \cap J_{<j}$, $R = J_{>j}$. Hence $P < i < Q < j < R$ and $R \neq \emptyset$. The left-hand side is equal to $x := c(P \sqcup Q, c(R, [u_i, u_j]))$. Define also $y := c(P \sqcup Q, [c(R, u_i), u_j])$, $z := c(P \sqcup Q, [u_i, c(R, u_j)])$. Then

$$\begin{aligned} x &= y + (-1)^{|R|} z + \sum_{\substack{R = A' \sqcup B' \\ A', B' \neq \emptyset}} (-1)^{\theta(A', B') + |B'|} c(P \sqcup Q, [c(A', u_i), c(B', u_j)]) \\ &\stackrel{(15)}{=} y + (-1)^{|R|} z + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>j}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)], \\ y &= (-1)^{|R|} c(P \sqcup Q, [u_j, c(R, u_i)]) = (-1)^{|R|} c(J \setminus i, u_i), \\ z &= c(P, c(Q, [u_i, c(R, u_j)])) \stackrel{(14)}{=} c(P, [c(Q, u_i), c(R, u_j)]) + (-1)^{|Q|} \underbrace{c(P, [u_i, c(Q \sqcup R, u_j)])}_{=c(J \setminus j, u_j)} \\ &\quad + \sum_{\substack{Q = A_2 \sqcup B_2 \\ A_2, B_2 \neq \emptyset}} (-1)^{\theta(A_2, B_2) + |B_2|} c(P, [c(A_2, u_i), c(B_2 \sqcup R, u_j)]) \\ &\stackrel{(15)}{=} (-1)^{|R|} \sum_{\substack{J \setminus ij = A \sqcup B \\ B \cap Q = \emptyset \\ A_{>j} = \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)] \\ &\quad + (-1)^{|Q|} c(J \setminus j, u_j) + (-1)^{|R|} \sum_{\substack{J \setminus ij = A \sqcup B \\ B \cap Q \neq \emptyset, Q \\ A_{>j} = \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)]. \end{aligned}$$

Therefore,

$$\begin{aligned} x &= (-1)^{|R|} c(J \setminus i, u_i) + (-1)^{|Q| + |R|} c(J \setminus j, u_j) \\ &\quad + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>i} \neq \emptyset \\ A_{>j} = \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)] + \sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>j}, B_{>j} \neq \emptyset}} (-1)^{\theta(A, B) + |B|} [c(A, u_i), c(B, u_j)]. \end{aligned}$$

In the first sum the condition $B_{>j} \neq \emptyset$ is always true, since $R = A_{>j} \sqcup B_{>j}$, $A_{>j} = \emptyset$ and $R \neq \emptyset$. In the second sum, $A_{>i} \neq \emptyset$ is always true. Hence the sums can be merged into

$$\sum_{\substack{J \setminus ij = A \sqcup B \\ A_{>i}, B_{>j} \neq \emptyset}} (-1)^{\theta(A,B) + |B|} [c(A, u_i), c(B, u_j)].$$

Using $|R| = |J_{>j}|$ and $|Q| + |R| = |J_{>i}| - 1$, we obtain (16). $\square$

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Algebraicity in monochromatic homotopy theory

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Using Patchkoria and Pstragowski’s version of Franke’s algebraicity theorem, we prove that the category of $K_p(n)$ -local spectra is exotically equivalent to the category of derived I_n -complete periodic comodules over the Adams Hopf algebroid (E_*, E_*E) for large primes. This gives a finite prime result analogous to the asymptotic algebraicity for $\mathrm{Sp}_{K_p(n)}$ of Barthel, Schlank, and Stapleton.

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1 Introduction

The central idea in chromatic homotopy theory is to study the symmetric monoidal stable ∞ -category of spectra, Sp , via its smaller building blocks. These are the categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ of $E_{n,p}$ -local and $K_p(n)$ -local spectra, where $E = E_{n,p}$ is Morava E -theory, and $K_p(n)$ is Morava K -theory; see, for example, [26]. These categories depend on a prime p and an integer n , called the height. For a fixed height n , increasing the prime p makes both categories behave more algebraically. This manifests itself, for example, in the E -Adams spectral sequence of signature

$$E_2^{s,t}(L_n\mathbb{S}) = \mathrm{Ext}_{E_*E}^{s,t}(E_*, E_*) \Rightarrow \pi_{t-s}L_n\mathbb{S}$$

computing the homotopy groups of the E -local sphere. By the smash product theorem of Ravenel, see [40, 7.5.6], this spectral sequence has a horizontal vanishing line at a finite page. If $p > n + 1$, this vanishing line appears already on the second page, where the information is completely described by the homological algebra of Comod_{E_*E} — the Grothendieck abelian category of comodules over the Hopf algebroid (E_*, E_*E) .

Increasing the prime p correspondingly increases the distance between objects appearing in the E -Adams spectral sequence. When $2p - 2$ exceeds $n^2 + n$, there is no longer room for any differentials, and the spectral sequence in fact collapses to an isomorphism

$$\pi_* L_n \mathbb{S} \cong \mathrm{Ext}_{E_* E}^{*,*}(E_*, E_*),$$

for degree reasons. In other words, the homotopy groups are completely algebraic in this range.

A natural question to ask is whether this collapse is a feature solely of the E -Adams spectral sequence or if it is a feature of the category $\mathrm{Sp}_{n,p}$. More precisely, is the entire category of E -local spectra algebraic, in the sense that it is equivalent to a derived category of an abelian category, whenever $2p - 2 > n^2 + n$? What about the category of $K_p(n)$ -local spectra?

At height $n = 0$, both categories $\mathrm{Sp}_{n,p}$ and $\mathrm{Sp}_{K_p(n)}$ are the category of rational spectra $\mathrm{Sp}_{\mathbb{Q}}$, which can be seen to be equivalent to the derived ∞ -category of rational vector spaces, but at positive heights $n > 0$, there can never be an equivalence of ∞ -categories $\mathrm{Sp}_{n,p} \simeq \mathrm{D}(\mathcal{A})$ or $\mathrm{Sp}_{K_p(n)} \simeq \mathrm{D}(\mathcal{A})$.

However, in [16] Bousfield showed that for $p > 2$ and $n = 1$, that there is an equivalence of homotopy categories

$$h\mathrm{Sp}_{1,p} \simeq h\mathrm{Fr}_{1,p},$$

where $\mathrm{Fr}_{n,p}$ is a certain derived ∞ -category of twisted comodules over $(E_*, E_* E)$. As this cannot be lifted to an equivalence of ∞ -categories, it is sometimes referred to as an *exotic* equivalence.

Franke expanded upon this in [19] by conjecturing — and attempting to prove — that for $2p - 2 > n^2 + n$ there should be an equivalence of homotopy categories

$$h\mathrm{Sp}_{n,p} \simeq h\mathrm{Fr}_{n,p}.$$

Unfortunately, a subtle error was discovered in the proof by Patchkoria in [35], but the result was recovered in [37] with a slightly worse bound: $2p - 2 > 2n^2 + 2n$. Pstrągowski also proved that this equivalence gets “stronger” the larger the prime, where we not only get an equivalence of categories but an equivalence of k -categories

$$h_k \mathrm{Sp}_{n,p} \simeq h_k \mathrm{Fr}_{n,p},$$

for $k = 2p - 2 - 2n^2 - 2n$. Here $h_k \mathcal{C}$ denotes taking the homotopy k -category, given by $(k-1)$ -truncating the mapping spaces in $\mathcal{C}$. At $k = 1$, this gives the classical situation of taking the homotopy category $h\mathcal{C}$. Using and developing a more general machinery, Pstrągowski and Patchkoria proved in [36] that the above equivalence holds in Franke’s conjectured bound, $2p - 2 > n^2 + n$.

The current belief is that these bounds are optimal. We know this to be true at the prime 2, as Roitzheim proved in [41] that the category $\mathrm{Sp}_{1,2}$ is *rigid*, in the sense that any equivalence of homotopy categories $h\mathrm{Sp}_{1,2} \simeq h\mathcal{C}$ lifts to an equivalence $\mathrm{Sp}_{1,2} \simeq \mathcal{C}$. The $K_p(n)$ -local analogue of Roitzheim’s result also

holds, as Ishak proved in [29] that $\mathrm{Sp}_{K_2(1)}$ is rigid as well. Hence, exotic equivalences for $\mathrm{Sp}_{n,p}$ or $\mathrm{Sp}_{K_p(n)}$ can only exist at primes that are large compared to the height.

The above results imply that increasing the prime p decreases how destructive the k -truncation of the mapping spaces needs to be. In the limit $p \rightarrow \infty$, we might expect that there is no need to truncate at all, giving an equivalence of ∞ -categories. But, there needs to be an appropriate notion of what “going to the infinite prime” should be. In [11], the authors use a notion of ultraproducts over a nonprincipal ultrafilter of primes, $\mathcal{F}$, to formalize this limiting process. They use this to prove the existence of a symmetric monoidal equivalence of ∞ -categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{n,p} \simeq \prod_{\mathcal{F}} \mathrm{Fr}_{n,p}.$$

Expanding on their work, Barthel, Schlank, and Stapleton proved in [12] a $K_p(n)$ -local version of the above result. More precisely, they show that there is a symmetric monoidal equivalence of ∞ -categories

$$\prod_{\mathcal{F}} \mathrm{Sp}_{K_p(n)} \simeq \prod_{\mathcal{F}} \widehat{\mathrm{Fr}}_{n,p},$$

where the right-hand side consists of derived complete twisted comodules for the naturally occurring Landweber ideal $I_n \subseteq E_*$.

Statement of results

We can summarize the most general of the above algebraicity results in the table

	$p < \infty$	$p \rightarrow \infty$
$\mathrm{Sp}_{n,p}$	[36]	[11]
$\mathrm{Sp}_{K_p(n)}$		[12]

A natural question arises: is there a finite prime exotic algebraicity for $\mathrm{Sp}_{K_p(n)}$? The goal of this paper is to give an affirmative answer. More precisely, we prove the following.

Theorem A (Theorem 4.15) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p - 2 - n^2 - n > 0$, then there is an equivalence of k -categories*

$$h_k \mathrm{Sp}_{K_p(n)} \simeq h_k \widehat{\mathrm{Fr}}_{n,p}.$$

In other words, $K_p(n)$ -local spectra are exotically algebraic at large primes.

The available tools for proving such a statement require an abelian category with enough injective objects admitting lifts to a stable ∞ -category. In lack of such a well-behaved abelian approximation for $\mathrm{Sp}_{K_p(n)}$, we take inspiration from [12] and instead use the dual category $\mathcal{M}_{n,p}$ of monochromatic spectra, which we show has the needed properties. Theorem A will then follow from the following result.

Theorem B (Theorem 4.13) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p - 2 - n^2 - n > 0$, then there is an equivalence of k -categories $h_k \mathcal{M}_{n,p} \simeq h_k \mathrm{Fr}_{n,p}^{I_{n-\mathrm{tors}}}$.*

In order to prove Theorem B, we first prove the analogous statement for monochromatic E -modules.

Theorem C (Theorem 4.5) *Let p be a prime and $n \in \mathbb{N}$. If $k = 2p - 2 - n > 0$, then there is an equivalence of k -categories $h_k \mathrm{Mod}_E^{I_{n-\mathrm{tors}}} \simeq h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_{n-\mathrm{tors}}})$.*

Overview of the paper

Section 1 introduces local duality, and the proposed exotic algebraic model using periodic chain complexes of torsion comodules. Section 3 focuses on Franke's algebraicity theorem. Most of the new results of the paper are presented in Sections 4.1 and 4.2, where we prove Theorems A, B and C. In Section 4.2 we prove that Barr–Beck adjunctions interact well with local duality, which is used to prove that periodization, torsion and taking the derived category all commute.

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2 The algebraic model

The goal of this section is to set up the necessary background material that will be used throughout the paper. We use these to construct convenient algebraic approximations of categories arising from chromatic homotopy theory.

Some conventions

We freely use the language of ∞ -categories, as developed by Joyal [31] and Lurie [32; 33]. Even though we are dealing with both classical 1-categories and ∞ -categories in this paper, we will sometimes refer to them both as *categories*, hoping that the prefix is clear from the context.

We denote by $\mathrm{Pr}_{\mathrm{st}}^L$ the ∞ -category of presentable stable ∞ -categories and colimit preserving functors. Together with the Lurie tensor product, it is a symmetric monoidal ∞ -category. The category of algebras $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ is then the category of presentable stable ∞ -categories with a symmetric monoidal structure commuting with colimits separately in each variable.

Let $\mathcal{C}, \mathcal{D} \in \text{CAlg}(\text{Pr}_{\text{st}}^L)$. A localization is a functor $f: \mathcal{C} \rightarrow \mathcal{D}$ with a fully faithful right adjoint i . We denote the composite by $L = i \circ f$. The adjoint i identifies $\mathcal{D}$ with a full subcategory of $\mathcal{C}$, which we denote by $\mathcal{C}_L$. We then view L as a functor $L: \mathcal{C} \rightarrow \mathcal{C}_L$, that is left adjoint to the inclusion, and by abuse of notation also call these localizations.

2.1 Local duality

The theory of abstract local duality, proved in [25] and generalized to the ∞ -categorical setting in [8], will be important for the entire paper. In particular, it will allow us to translate Theorem B into Theorem A.

Definition 2.1 A pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \in \text{CAlg}(\text{Pr}_{\text{st}}^L)$ is compactly generated by dualizable objects, and $\mathcal{K}$ is a subset of compact objects, is called a *local duality context*.

Construction 2.2 Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We define $\mathcal{C}^{\mathcal{K}\text{-tors}}$ to be the localizing tensor ideal generated by $\mathcal{K}$, denoted by $\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K})$. Further we define $\mathcal{C}^{\mathcal{K}\text{-loc}}$ to be the left orthogonal complement $(\mathcal{C}^{\mathcal{K}\text{-tors}})^{\perp}$, ie, the full subcategory consisting of objects $C \in \mathcal{C}$ such that $\text{Hom}_{\mathcal{C}}(T, C) \simeq 0$ for all $T \in \mathcal{C}^{\mathcal{K}\text{-tors}}$. Similarly, define $\mathcal{C}^{\mathcal{K}\text{-comp}}$ to be the left-orthogonal complement of $\mathcal{C}^{\mathcal{K}\text{-loc}}$, ie $\mathcal{C}^{\mathcal{K}\text{-comp}} = (\mathcal{C}^{\mathcal{K}\text{-loc}})^{\perp}$. These full subcategories are, respectively, called the $\mathcal{K}$ -torsion, $\mathcal{K}$ -local and $\mathcal{K}$ -complete objects in $\mathcal{C}$. We have inclusions into $\mathcal{C}$, denoted by $i_{\mathcal{K}\text{-tors}}$, $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{K}\text{-comp}}$, respectively.

By the adjoint functor theorem, [32, 5.5.2.9], the inclusions $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{K}\text{-comp}}$ have left adjoints $L_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$, respectively, while $i_{\mathcal{K}\text{-tors}}$ and $i_{\mathcal{K}\text{-loc}}$ have right adjoints $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$, respectively. These are then, by definition, localizations and colocalizations. Since the torsion, local and complete objects are ideals, these localizations and colocalizations are compatible with the symmetric monoidal structure of $\mathcal{C}$, in the sense of [33, 2.2.1.7]. In particular, by [33, 2.2.1.9] we get unique induced symmetric monoidal structures such that $L_{\mathcal{K}}$, $\Lambda_{\mathcal{K}}$, $\Gamma_{\mathcal{K}}$ and $V_{\mathcal{K}}$ are symmetric monoidal functors.

For any $X \in \mathcal{C}$, these functors assemble into two cofiber sequences:

$$\Gamma_{\mathcal{K}}X \rightarrow X \rightarrow L_{\mathcal{K}}X \quad \text{and} \quad V_{\mathcal{K}}X \rightarrow X \rightarrow \Lambda_{\mathcal{K}}X.$$

Also, these functors only depend on the localizing subcategory $\mathcal{C}^{\mathcal{K}\text{-tors}}$, not on the particular choice of generators $\mathcal{K}$. Thus, when the set $\mathcal{K}$ is clear from the context, we often omit it as a subscript when writing the functors.

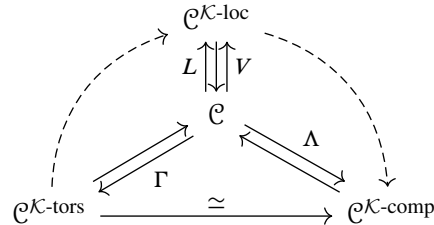
The next theorem is a slightly restricted version of the abstract local duality theorem of [8, 2.21; 25, 3.3.5].

Theorem 2.3 Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. Then

- (1) the functors Γ and L are **smashing**, meaning that there are natural equivalences $\Gamma X \simeq X \otimes \Gamma \mathbb{1}$ and $LX \simeq X \otimes L \mathbb{1}$,

- (2) the functors Λ and V are **cosmashing**, meaning there are natural equivalences $\Lambda X \simeq \underline{\mathrm{Hom}}(\Gamma \mathbb{1}, X)$ and $VX \simeq \underline{\mathrm{Hom}}(L \mathbb{1}, X)$, and
- (3) the functors $\Gamma: \mathcal{C}^{\mathcal{K}\text{-comp}} \rightarrow \mathcal{C}^{\mathcal{K}\text{-tors}}$ and $\Lambda: \mathcal{C}^{\mathcal{K}\text{-tors}} \rightarrow \mathcal{C}^{\mathcal{K}\text{-comp}}$ are mutually inverse symmetric monoidal equivalences of categories.

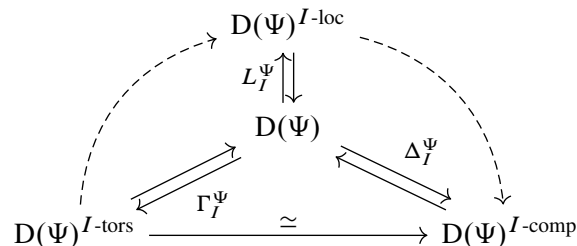
This can be summarized by the following diagram of adjoints:



Remark 2.4 Theorem 2.3 implies, in particular, that the symmetric monoidal structure induced by the localization L and the colocalization Γ is just the symmetric monoidal structure on $\mathcal{C}$ restricted to the full subcategories. This is not the case for $\mathcal{C}^{\mathcal{K}\text{-comp}}$, where the symmetric monoidal structure is given by $\Lambda(- \otimes_{\mathcal{C}} -)$. The functor V also induces a symmetric monoidal structure on $\mathcal{C}^{\mathcal{K}\text{-loc}}$, but this coincides with the one induced by L , due to their associated endofunctors on $\mathcal{C}$ defining an adjoint symmetric monoidal monad-comonad pair. We will not need or focus on the functor V , hence it will be omitted from the local duality diagrams for the rest of the paper.

We have two main examples of interest for this paper.

Example 2.5 Let (A, Ψ) be an Adams-type Hopf algebroid, for example, the Hopf algebroid $(R_*, R_* R)$ for an Adams-type ring spectrum R —see [24; 39, A.1] for details. Denote by $D(\Psi)$ the derived ∞ -category associated to the symmetric monoidal Grothendieck abelian category Comod_{Ψ} . This is defined using the model structure from [4]. If $I \subseteq A$ is a finitely generated invariant regular ideal, then $(D(\Psi), A/I)$ is a local duality context, with associated local duality diagram



In Section 2.2 we compare $D(\Psi)^{I\text{-tors}}$ to a more concrete category: the derived category of I -power torsion comodules.

The following example comes from chromatic homotopy theory. For a good introduction, see [5].

Example 2.6 Let E denote Morava E -theory at prime p and height n . If $F(n)$ is a finite type- n spectrum, then the pair $(\mathrm{Sp}_{n,p}, L_n F(n))$ is a local duality context. The corresponding diagram can be recognized as

$$\begin{array}{ccc}
 & \mathrm{Sp}_{n-1,p} & \\
 & \uparrow L_{n-1} \downarrow & \\
 & \mathrm{Sp}_{n,p} & \\
 \mathcal{M}_{n,p} & \xrightarrow{M_{n,p}} & \mathrm{Sp}_{K_p(n)} \\
 & \nwarrow \nearrow L_{K_p(n)} & \\
 & \mathrm{Sp}_{n,p} & \\
 & \downarrow L_{n-1} \uparrow & \\
 & \mathrm{Sp}_{n-1,p} &
 \end{array}$$

(Note: The diagram above is a schematic representation of the commutative diagram in the image. The image shows a diagram with $\mathcal{M}_{n,p}$ at the bottom left, $\mathrm{Sp}_{K_p(n)}$ at the bottom right, and $\mathrm{Sp}_{n,p}$ in the center. A horizontal arrow labeled $M_{n,p}$ points from $\mathcal{M}_{n,p}$ to $\mathrm{Sp}_{K_p(n)}$. A horizontal arrow labeled $\simeq$ points from $\mathcal{M}_{n,p}$ to $\mathrm{Sp}_{K_p(n)}$. A vertical arrow labeled L_{n-1} points from $\mathrm{Sp}_{n,p}$ to $\mathrm{Sp}_{n-1,p}$. A vertical arrow labeled L_{n-1} points from $\mathrm{Sp}_{n-1,p}$ to $\mathrm{Sp}_{n,p}$. A diagonal arrow labeled $L_{K_p(n)}$ points from $\mathrm{Sp}_{K_p(n)}$ to $\mathrm{Sp}_{n,p}$. A dashed curved arrow points from $\mathcal{M}_{n,p}$ to $\mathrm{Sp}_{n-1,p}$. A dashed curved arrow points from $\mathrm{Sp}_{n-1,p}$ to $\mathrm{Sp}_{K_p(n)}$.)

where $\mathcal{M}_{n,p}$ is the height- n monochromatic category and $\mathrm{Sp}_{K_p(n)}$ is the category of spectra localized at height- n Morava K -theory $K_p(n)$. The functor L_{n-1} is the Bousfield localization at E_{n-1} , while $L_{K_p(n)}$ is the Bousfield localization at $K_p(n)$; see [15]. The local duality then exhibits the classical equivalence $\mathcal{M}_{n,p} \simeq \mathrm{Sp}_{K_p(n)}$; see [26, 6.19].

Remark 2.7 There is also a version of this local duality diagram for modules over E , see [21, 4.2, 5.1], or alternatively [8, 3.7] for a version more similar to the above. This gives equivalences

$$\mathcal{M}_{n,p} \mathrm{Mod}_E \simeq \mathrm{Mod}_E^{I_n\text{-tors}} \simeq \mathrm{Mod}_E^{I_n\text{-comp}} \simeq L_{K_p(n)} \mathrm{Mod}_E,$$

where I_n is the Landweber ideal $(p, v_1, \dots, v_{n-1}) \subseteq E_*$.

2.2 The periodic derived torsion category

In this section we identify the category $\mathrm{D}(\Psi)^{I\text{-tors}}$ — as obtained in Example 2.5 — as the derived category of I -power torsion comodules. We also modify the category to exhibit some needed periodicity.

Definition 2.8 Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. The I -power torsion of a comodule M is defined as

$$T_I^\Psi M = \{x \in M \mid I^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

We say a comodule M is I -torsion if the natural map $T_I^\Psi M \rightarrow M$ is an equivalence.

Remark 2.9 One can similarly define I -power torsion A -modules. If (A, Ψ) is an Adams Hopf algebroid, a Ψ -comodule M is I -power torsion if and only if its underlying module is I -power torsion; see [8, 5.7].

Remark 2.10 By [8, 5.10] the full subcategory of I -torsion comodules, which we denote by $\mathrm{Comod}_\Psi^{I\text{-tors}}$, is a Grothendieck abelian category. It also inherits a symmetric monoidal structure from Comod_Ψ .

The following technical lemma will be needed later.

Lemma 2.11 *Let (A, Ψ) be an Adams Hopf algebroid, where A is noetherian and $I \subseteq A$ a regular invariant ideal. Then $\text{Comod}_{\Psi}^{I\text{-tors}}$ is generated under filtered colimits by the compact I -power torsion comodules.*

Proof By [9, 3.4] $\text{Comod}_{\Psi}^{I\text{-tors}}$ is generated by the set

$$\text{Tors}_{\Psi}^{\text{fp}} := \{G \otimes A/I^k \mid G \in \text{Comod}_{\Psi}^{\text{fp}}, k \geq 1\},$$

where $\text{Comod}_{\Psi}^{\text{fp}}$ is the full subcategory of dualizable Ψ -comodules. Since I is finitely generated and regular, A/I^k is finitely presented as an A -module, hence it is compact in Comod_{Ψ} by [24, 1.4.2], and in $\text{Comod}_{\Psi}^{I\text{-tors}}$ as colimits are computed in Comod_{Ψ} . As A is compact, being finitely generated is equivalent to being finitely presented. The tensor product of finitely generated modules is finitely generated, hence any element in $\text{Tors}_{\Psi}^{\text{fp}}$ is compact. $\square$

Remark 2.12 The assumption that the ring A is noetherian can most likely be removed, but it makes no difference to the results in this paper.

Notation 2.13 Since $\text{Comod}_{\Psi}^{I\text{-tors}}$ is Grothendieck abelian we have an associated derived stable ∞ -category $D(\text{Comod}_{\Psi}^{I\text{-tors}})$ which we denote simply by $D(\Psi^{I\text{-tors}})$.

We can now compare the torsion category obtained from local duality and the derived category of I -power torsion comodules.

Lemma 2.14 [9, 3.7(2)] *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a regular invariant ideal. There is an equivalence of categories*

$$D(\Psi)^{I\text{-tors}} \simeq D(\Psi^{I\text{-tors}}).$$

Furthermore, an object $M \in D(\Psi)$ is I -torsion if and only if the homology groups H_*M are I -power torsion Ψ -comodules.

In order to state both the general algebraicity machinery of [36] and our results, we need the respective derived categories to exhibit the periodic nature of the spectra we are interested in. This is done via the periodic derived category. There are several ways to constructing this, but we follow [19] in spirit, using periodic chain complexes.

Definition 2.15 Let $\mathcal{A}$ be an abelian category with a local grading, ie, an auto-equivalence $T: \mathcal{A} \rightarrow \mathcal{A}$, and denote by $[1]$ the shift functor on the category of chain complexes $\text{Ch}(\mathcal{A})$ in $\mathcal{A}$. A chain complex $C \in \text{Ch}(\mathcal{A})$ is called *periodic* if there is an isomorphism $\phi: C[1] \rightarrow TC$. The full subcategory of periodic chain complexes is denoted by $\text{Ch}^{\text{per}}(\mathcal{A})$.

Definition 2.16 The forgetful functor $\text{Ch}^{\text{per}}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{A})$ has a left adjoint P , called the *periodization*.

Definition 2.17 Let $\mathcal{A}$ be a locally graded abelian category. Then the *periodic derived category* of $\mathcal{A}$, denoted by $D^{\text{per}}(\mathcal{A})$, is the ∞ -category obtained by localizing $\text{Ch}^{\text{per}}(\mathcal{A})$ at the quasi-isomorphism. It is in fact stable by [36, 7.8].

Remark 2.18 If $\mathcal{A}$ is a symmetric monoidal category, then $P\mathbb{1}$ is a commutative ring object called the *periodic unit*. By [4, 2.3] the category of periodic chain complexes $\text{Ch}^{\text{per}}(\mathcal{A})$ is equivalent to $\text{Mod}_{P\mathbb{1}}(\text{Ch}(\mathcal{A}))$. This descends also to the derived categories, giving an equivalence

$$D^{\text{per}}(\mathcal{A}) \simeq \text{Mod}_{P\mathbb{1}}(D(\mathcal{A}));$$

see, for example, [37, 3.7].

We will also need local duality for the periodic derived category associated to a Hopf algebroid.

Construction 2.19 Let (A, Ψ) be an Adams-type (graded) Hopf algebroid. Then the shift functor $[1]: \text{Comod}_{\Psi} \rightarrow \text{Comod}_{\Psi}$ defined by $(TM)_k = M_{k-1}$ is a local grading on Comod_{Ψ} . Denote the corresponding periodic derived category by $D^{\text{per}}(\Psi)$. The pair $(D^{\text{per}}(\Psi), P(A/I))$ is a local duality context with associated local duality diagram

$$\begin{array}{ccc} & D^{\text{per}}(\Psi)^{I\text{-loc}} & \\ \text{---} \nearrow & \uparrow L_I^{\Psi} & \nwarrow \text{---} \\ & D^{\text{per}}(\Psi) & \\ \text{---} \nwarrow & \downarrow \Gamma_I^{\Psi} & \nearrow \text{---} \\ D^{\text{per}}(\Psi)^{I\text{-tors}} & \xrightarrow{\cong} & D^{\text{per}}(\Psi)^{I\text{-comp}} \end{array}$$

The functors in the diagram are induced by the functors from Example 2.5. In fact, there is a diagram

$$\begin{array}{ccccc} D(\Psi)^{I\text{-tors}} & \xrightleftharpoons[\Gamma_I^{\Psi}]{} & D(\Psi) & \xrightleftharpoons[L_I^{\Psi}]{} & D(\Psi)^{I\text{-loc}} \\ P \updownarrow & & P \updownarrow & & P \updownarrow \\ D^{\text{per}}(\Psi)^{I\text{-tors}} & \xrightleftharpoons[\Gamma_I^{\Psi}]{} & D^{\text{per}}(\Psi) & \xrightleftharpoons[L_I^{\Psi}]{} & D^{\text{per}}(\Psi)^{I\text{-loc}} \end{array}$$

that is commutative in all possible directions. Here the unmarked horizontal arrows are the respective fully faithful inclusions.

Remark 2.20 In the specific case of $(A, \Psi) = (E_0, E_0 E)$ and $I \subseteq E_0$ the Landweber ideal I_n , the above construction is [12, 3.12].

There is now some ambiguity to take care of for our category of interest $D^{\text{per}}(\Psi)^{I\text{-tors}}$. In the picture above, we do mean that we take I -torsion objects in $D^{\text{per}}(\Psi)$, ie, $[D^{\text{per}}(\Psi)]^{I\text{-tors}}$, but we could also take the periodization of the category $D(\Psi^{I\text{-tors}})$ as our model. Luckily, there is no choice, as they are equivalent. This can be thought of as the periodic version of Lemma 2.14.

Theorem 2.21 *Let (A, Ψ) be an Adams Hopf algebroid and $I \subseteq A$ a finitely generated invariant regular ideal. Then there is an equivalence of stable ∞ -categories*

$$[\mathbf{D}^{\text{per}}(\Psi)]^{I\text{-tors}} \simeq \mathbf{D}^{\text{per}}(\Psi^{I\text{-tors}}).$$

The proof of this uses the fact that Barr–Beck adjunctions commute with local duality. Proving this here disrupts the flow of the paper, so we defer it to Section 4.2.

Proof As Comod_{Ψ} is symmetric monoidal we have by Remark 2.18 an equivalence

$$\mathbf{D}^{\text{per}}(\Psi) \simeq \text{Mod}_{P\mathbb{1}}(\mathbf{D}(\Psi)),$$

coming from the periodicity Barr–Beck adjunction. By Theorem A.6 this induces a Barr–Beck adjunction on the torsion subcategories, which gives an equivalence

$$[\mathbf{D}^{\text{per}}(\Psi)]^{I\text{-tors}} \simeq \text{Mod}_{\Gamma_I^{\Psi}(P\mathbb{1})}(\mathbf{D}(\Psi)^{I\text{-tors}}).$$

Since Γ_I^{Ψ} is a smashing colocalization, and P is given by tensoring with $P(\mathbb{1})$, they do in fact commute. By Lemma 2.14 we have $\mathbf{D}(\Psi)^{I\text{-tors}} \simeq \mathbf{D}(\Psi^{I\text{-tors}})$, hence the above equivalence can be rewritten as

$$[\mathbf{D}^{\text{per}}(\Psi)]^{I\text{-tors}} \simeq \text{Mod}_{P(\Gamma_I^{\Psi}\mathbb{1})}(\mathbf{D}(\Psi^{I\text{-tors}})).$$

Now, also $\text{Comod}_{\Psi}^{I\text{-tors}}$ is symmetric monoidal, so Remark 2.18 gives an equivalence

$$\mathbf{D}^{\text{per}}(\Psi^{I\text{-tors}}) \simeq \text{Mod}_{P(\Gamma_I^{\Psi}\mathbb{1})}(\mathbf{D}(\Psi^{I\text{-tors}})).$$

□

3 Exotic algebraic models

We now have two sets of local duality diagrams, one coming from chromatic homotopy theory, see Example 2.6, and one from the homological algebra of Adams Hopf algebroids, see Example 2.5. We can also pass between these duality theories, by using homology theories. In particular, if we let $E = E_n$ be height- n Morava E -theory at a prime p , then we have the E -homology functor $E_*: \text{Sp}_{n,p} \rightarrow \text{Comod}_{E_*E}$ converting between homotopy theory and algebra. We can, in some sense, say that E_* approximates homotopical information by algebraic information.

The goal of this section is to set up an abstract framework for studying how good such approximations are. The version we recall below was developed in [36], taking inspiration from [19; 38].

3.1 Adapted homology theories

Adapted homology theories are particularly well-behaved homology theories that have associated Adams-type spectral sequences giving computational benefits over other homology theories.

Definition 3.1 Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞ -category and $\mathcal{A}$ an abelian category with a local grading [1]. A functor $H: \mathcal{C} \rightarrow \mathcal{A}$ is called a *conservative homology theory* if:

- (1) H is additive.
- (2) For a cofiber sequence $X \rightarrow Y \rightarrow Z$ in $\mathcal{C}$, the sequence $HX \rightarrow HY \rightarrow HZ$ is exact in $\mathcal{A}$.
- (3) There is a natural isomorphism $H(\Sigma X) \cong (HX)[1]$ for any $X \in \mathcal{C}$.
- (4) H reflects isomorphisms.

Remark 3.2 The first two axioms make H a homological functor, the third makes H into a locally graded functor, ie, a functor that preserves the local grading, and the last makes it a conservative functor.

Example 3.3 Let R be a ring spectrum. Then the functor $\pi_*: \text{Mod}_R \rightarrow \text{Mod}_{R_*}$ defined as $\pi_* M = [S, M]_*$ is a conservative homology theory.

Example 3.4 Let R be a ring spectrum. The functor $R_*(-): \text{Sp} \rightarrow \text{Mod}_R$, defined as the composition

$$\text{Sp} \xrightarrow{R \otimes (-)} \text{Mod}_R \xrightarrow{\pi_*} \text{Mod}_{R_*},$$

is a homology theory. If R is of Adams type, then $R_*(-)$ naturally lands in the subcategory Comod_{R_*R} . If we restrict the domain of R_* to the category of R -local spectra, then it is a conservative homology theory. For the rest of the paper we will use R_* to denote the restricted conservative homology theory $R_*: \text{Sp}_R \rightarrow \text{Comod}_{R_*R}$.

Remark 3.5 Recall that we are really interested in the category $\text{Sp}_{K_p(n)}$ of $K_p(n)$ -local spectra. The spectrum $K_p(n)$ is a field object in Sp , and its homotopy groups $\pi_* K_p(n)$ are graded fields. Hence the homology theory $K_p(n)_*: \text{Sp}_{K_p(n)} \rightarrow \text{Mod}_{K_p(n)_*}$ is too simple to exhibit the algebraicity properties that we want. As $K_p(n)$ is Adams type, $K_p(n)_*(-)$ factors through Comod_{K_*K} , but this category is very complicated. In particular, it does not have finite cohomological dimension, a feature we will need later. We learnt the argument for why this is the case from [10]. Having finite cohomological dimension would imply that the $K_p(n)$ -Adams spectral sequence has a horizontal vanishing line at a finite page. The groups in this spectral sequence are all torsion, hence this would imply that, for example, the homotopy groups of the $K_p(n)$ -local sphere is a finite filtration of torsion groups. In particular there could be no rational homotopy groups. But, by [13] the rational homotopy groups of the $K_p(n)$ -local sphere are highly nontrivial, meaning that the original assumption that Comod_{K_*K} has finite cohomological dimension must be wrong.

There is, however, a version of E_* -homology on $\text{Sp}_{K_p(n)}$, defined by sending a $K(n)$ -local spectrum X to $E_*^\vee(X) := \pi_* L_{K_p(n)}(E \otimes X)$. The functor does land in a category of comodules, specifically over the L -complete Hopf algebroid $(E_*, E_*^\vee E)$; see [1, 5.3]. However, the category $\text{Comod}_{E_*^\vee E}$ is not abelian. This is the reason for instead using the monochromatic category $\mathcal{M}_{n,p}$ and the category of I_n -power torsion comodules, as these inherit nicer homological properties we can exploit.

Definition 3.6 Let $H : \mathcal{C} \rightarrow \mathcal{A}$ be a homology theory and J an injective object in $\mathcal{A}$. An object $\bar{J} \in \mathcal{C}$ is said to be an *injective lift* of J if it represents the functor

$$\mathrm{Hom}_{\mathcal{A}}(H(-), J) : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{A}b$$

in the homotopy category $h\mathcal{C}$, ie $\mathrm{Hom}_{\mathcal{A}}(H(-), J) \cong [-, \bar{J}]$. We call $\bar{J}$ a *faithful lift* if the map $H(\bar{J}) \rightarrow J$ coming from the identity on $\bar{J}$ is an equivalence.

Definition 3.7 A homology theory $H : \mathcal{C} \rightarrow \mathcal{A}$ is said to be *adapted* if $\mathcal{A}$ has enough injective objects, and for any injective $J \in \mathcal{A}$ there is a faithful lift $\bar{J} \in \mathcal{C}$.

Example 3.8 We return to our two guiding examples $\pi_* : \mathrm{Mod}_R \rightarrow \mathrm{Mod}_{R_*}$ and $R_* : \mathrm{Sp}_R \rightarrow \mathrm{Comod}_{R_*R}$, where R is an Adams-type ring spectrum. Both functors are conservative adapted homology theories, with faithful lifts provided by Brown representability; see [36, 8.2] and [36, 8.13], respectively.

Remark 3.9 The definition of an adapted homology theory H states that for any injective $J \in \mathcal{A}$, there is some object $\bar{J} \in \mathcal{C}$ together with an equivalence $[X, \bar{J}] \simeq \mathrm{Hom}_{\mathcal{A}}(HX, J)$ induced by H . Because $\mathcal{A}$ has enough injective objects, we can use these equivalences to approximate homotopy classes of maps by repeatedly mapping into injectives. This gives precisely an associated Adams spectral sequence for the homology theory H . In fact, Patchkoria and Pstragowski proved that there is a bijection between adapted homology theories and Adams spectral sequences; see [36, 3.24, 3.25]. The construction of the Adams spectral sequence associated to an adapted homology theory $H : \mathcal{C} \rightarrow \mathcal{A}$ is given in [36, 2.24], or alternatively as a totalization spectral sequence in [36, 2.27].

In our particular interest $R = E_n$, the associated adapted homology theories π_* and E_* are even nicer than a general adapted homology theory. This is because the category of comodules is particularly simple.

Definition 3.10 Let $\mathcal{A}$ be a locally graded abelian category with enough injective objects. Then the *cohomological dimension* of $\mathcal{A}$ is the smallest integer d such that $\mathrm{Ext}_{\mathcal{A}}^{s,t}(-, -) \cong 0$ for all $s > d$.

Example 3.11 Let n be an integer, p a prime such that $p > n + 1$ and $E = E_n$ Morava E -theory at height n . Then by [37, 2.5] the category Comod_{E_*E} has cohomological dimension $n^2 + n$.

For certain Adams-type ring spectra R we get decompositions of the category Comod_{R_*R} into periodic families of subcategories. Such decompositions allows for the construction of partial inverses to the associated homology theories.

Construction 3.12 Let R be an Adams-type ring spectrum such that π_*R is concentrated in degrees divisible by some positive number $q + 1$, ie, $\pi_m R = 0$ for all $m \not\equiv 0 \pmod{q + 1}$. Any comodule M in the category Comod_{R_*R} splits uniquely into a direct sum of subcomodules $\bigoplus_{\phi \in \mathbb{Z}/q+1} M_{\phi}$ such that M_{ϕ} is

concentrated in degrees divisible by ϕ . Such a splitting induces a decomposition of the full subcategory of injective objects

$$\mathrm{Comod}_{R_*R}^{\mathrm{inj}} \simeq \mathrm{Comod}_{R_*R,0}^{\mathrm{inj}} \times \mathrm{Comod}_{R_*R,1}^{\mathrm{inj}} \times \cdots \times \mathrm{Comod}_{R_*R,q}^{\mathrm{inj}},$$

where the category $\mathrm{Comod}_{R_*R,\phi}^{\mathrm{inj}}$ denotes the full subcategory spanned by injective comodules concentrated in degrees divisible by ϕ .

Let $h_k\mathcal{C}$ denote the homotopy k -category of $\mathcal{C}$, obtained by $(k+1)$ -truncating all the mapping spaces in $\mathcal{C}$. The lift associated with each injective via the Adapted homology theory R_* allows us to construct a partial inverse to R_* , called the Bousfield functor β^{inj} in [36]. It is a functor $\beta^{\mathrm{inj}}: \mathrm{Comod}_{R_*R}^{\mathrm{inj}} \rightarrow h_{q+1}\mathrm{Sp}_R^{\mathrm{inj}}$, where the latter category is the homotopy $(q+1)$ -category of the full subcategory of Sp_R containing all spectra X such that R_*X is injective.

In order to mimic this behavior for a general adapted homology theory, Franke introduced the notion of a splitting of an abelian category.

Definition 3.13 [19] Let $\mathcal{A}$ be an abelian category with a local grading [1]. A *splitting* of $\mathcal{A}$ of order $q+1$ is a collection of Serre subcategories $\mathcal{A}_\phi \subseteq \mathcal{A}$ indexed by $\phi \in \mathbb{Z}/(q+1)$ satisfying

- (1) $[k]\mathcal{A}_n \subseteq \mathcal{A}_{n+k \bmod (q+1)}$ for any $k \in \mathbb{Z}$, and
- (2) the functor $\prod_\phi \mathcal{A}_\phi \rightarrow \mathcal{A}$, defined by $(a_\phi) \mapsto \bigoplus_\phi a_\phi$, is an equivalence of categories.

Example 3.14 As we saw above in Construction 3.12, the category of comodules over an Adams Hopf algebroid (R_*, R_*R) , where R_* is concentrated in degrees divisible by $q+1$, has a splitting of order $q+1$. This, then, also holds for the discrete Hopf algebroid (R_*, R_*) , giving the module category Mod_{R_*} a splitting of order $q+1$ as well.

Example 3.15 In the case $R = E(1)$ this has been written out in detail in [4, Section 4]. The Serre subcategories are all copies of the category of p -local abelian groups together with Adams operations ψ^k for $k \neq 0$ in $\mathbb{Z}_{(p)}$. The shift leaves the underlying module unchanged, but changes the Adams operation.

Definition 3.16 We will say that objects $A \in \mathcal{A}_\phi$ are of *pure weight* ϕ .

Remark 3.17 Just as for Comod_{R_*R} , a splitting of order $q+1$ of a locally graded abelian category $\mathcal{A}$ is enough to define, for any adapted homology theory $H: \mathcal{C} \rightarrow \mathcal{A}$, a partial inverse Bousfield functor β^{inj} ; see [36, Section 7.2].

3.2 Exotic homology theories

In order to make some statements about exotic equivalences a bit simpler, we introduce the concept of exotic adapted homology theories. This is not the way similar results are phrased in [36], but the notation serves as a shorthand for the criteria that they use.

Definition 3.18 Let $H : \mathcal{C} \rightarrow \mathcal{A}$ be a homology theory. We say H is k -exotic if H is adapted, conservative, $\mathcal{A}$ has finite cohomological dimension d and a splitting of order $q + 1$ such that $k = q + 1 - d > 0$.

The remarkable thing about a k -exotic homology theory $H : \mathcal{C} \rightarrow \mathcal{A}$ is that it forces the stable ∞ -category $\mathcal{C}$ to be approximately algebraic. Intuitively: as the order of the splitting is greater than the cohomological dimension, the H -Adams spectral sequence is very sparse and well behaved. There is a partial inverse of H via the Bousfield functor $\beta : \mathcal{A}^{\text{inj}} \rightarrow h_{q+1}\mathcal{C}^{\text{inj}}$, which forces a certain subcategory of a categorified deformation of H to be equivalent to both $h_k\mathcal{C}$ and $h_k\mathcal{D}^{\text{per}}(\mathcal{A})$. This is the contents of Franke's algebraicity theorem.

Theorem 3.19 [36, 7.56] Let $H : \mathcal{C} \rightarrow \mathcal{A}$ be a k -exotic homology theory. Then there is an equivalence of homotopy k -categories $h_k\mathcal{C} \simeq h_k\mathcal{D}^{\text{per}}(\mathcal{A})$.

There are several interesting examples of homology theories satisfying Theorem 3.19; see Section 8 in [36]. We highlight again our two guiding examples but focus specifically on certain Morava E -theories.

Example 3.20 [36, 8.7] Let p be a prime, n be a nonnegative integer, and E a height- n Morava E -theory concentrated in degrees divisible by $2p - 2$, for example, Johnson–Wilson theory $E(n)$. If $k = 2p - 2 - n > 0$, then the functor $\pi_* : \text{Mod}_E \rightarrow \text{Mod}_{E_*}$ is a k -exotic homology theory, giving an equivalence

$$h_k\text{Mod}_E \simeq h_k\mathcal{D}^{\text{per}}(\text{Mod}_{E_*}).$$

Notation 3.21 For the following example and the rest of the paper, we follow the notation of [3; 11; 12] and denote the category $\mathcal{D}^{\text{per}}(\text{Comod}_{E_*E})$ by $\text{Fr}_{n,p}$.

Example 3.22 [36, 8.13] Let p be a prime, n be a nonnegative integer, and E any height- n Morava E -theory. If $k = 2p - 2 - n^2 - n > 0$, then the functor $E_* : \text{Sp}_{n,p} \rightarrow \text{Comod}_{E_*E}$ is a k -exotic homology theory, giving an equivalence

$$h_k\text{Sp}_{n,p} \simeq h_k\text{Fr}_{n,p}.$$

Remark 3.23 As noted in [11, 5.29], this equivalence is strictly exotic for all $n \geq 1$ and primes p . In other words, it can never be made into an equivalence of stable ∞ -categories. In particular, the mapping spectra in $\text{Fr}_{n,p}$ are $H\mathbb{Z}$ -linear, while the mapping spectra in $\text{Sp}_{n,p}$ are only $H\mathbb{Z}$ -linear for $n = 0$.

Definition 3.24 Let $H : \mathcal{C} \rightarrow \mathcal{A}$ be a k -exotic homology theory. The category $D^{\text{per}}(\mathcal{A})$ is called an *exotic algebraic model* of $\mathcal{C}$ if the equivalence $h_k \mathcal{C} \simeq h_k D^{\text{per}}(\mathcal{A})$ can not be enhanced to an equivalence of ∞ -categories $\mathcal{C} \simeq D^{\text{per}}(\mathcal{A})$.

Remark 3.25 The notion of being exotically algebraic is part of a complex hierarchy of algebraicity levels; see [30] for a great exposé.

Remark 3.26 The existence of an exotic algebraic model for a stable ∞ -category $\mathcal{C}$ implies that the category is not rigid. This means, in particular, that there cannot exist a k -exotic homology theory with source Sp or $\text{Sp}_{(p)}$ as these are all rigid for all primes; see [42; 43; 44]. The same holds for $\text{Sp}_{1,2}$, as this is rigid by [41], and similarly for $\text{Sp}_{K_2(1)}$ by [29]. This shows that being k -exotic is quite a strong requirement.

4 Algebraicity for monochromatic categories

We are now ready to prove our main results. We start by proving Theorem C, which we will later use to prove Theorem B. The main result, Theorem A will then follow by using certain local duality arguments.

4.1 Monochromatic modules

For the rest of this section, we assume that E is the height- n Johnson–Wilson theory $E(n)$. This is an $\mathbb{E}_1$ -ring spectrum concentrated in degrees divisible by $2p - 2$, with coefficient ring

$$\pi_* E(n) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}],$$

where $|v_i| = 2p^i - 2$. The goal of this section is to prove Theorem C, which we do in three steps. First we show that the functor $\pi_* : \text{Mod}_E^{I_n\text{-tors}} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$ is a conservative adapted homology theory. We then show that $\text{Mod}_{E_*}^{I_n\text{-tors}}$ has finite cohomological dimension, and lastly that it admits a splitting.

The following lemma is the I_n -power torsion version [6, 3.14], and the proof is similar.

Lemma 4.1 *If M is an E -module, then $M \in \text{Mod}_E^{I_n\text{-tors}}$ if and only if $\pi_* M \in \text{Mod}_{E_*}^{I_n\text{-tors}}$.*

Proof Let $X \in \text{Mod}_E^{I_n\text{-tors}}$. By [8, 3.19] there is a strongly convergent spectral sequence of $E(n)_*$ -modules with signature

$$E_2^{s,t} = (H_{I_n}^{-s} \pi_* X)_t \Rightarrow \pi_{s+t} M_n X,$$

where $H_{I_n}^{-s}$ denotes local cohomology. By [17, 2.1.3(ii)] the E_2 -page consist of only I_n -power torsion modules. As $\text{Mod}_{E_*}^{I_n\text{-tors}}$ is abelian, it is closed under quotients and subobjects, as the higher pages are created from the E_2 -page using quotients and subobjects, they must also consist of only I_n -power torsion modules. In particular, the E_∞ -page is all I_n -power torsion. By Grothendieck’s vanishing theorem, see, for example, [17, 6.1.2], $H_{I_n}^s(-) \cong 0$ for $s > n$, hence the abutment of the spectral sequence $\pi_* M_n X$ is a finite

filtration of I_n -power torsion E_* -modules, and is therefore itself an I_n -power torsion module. Since X was assumed to be monochromatic, ie $X \in \text{Mod}_E^{I_n\text{-tors}}$, we have $\pi_* M_n X \cong \pi_* X$, and thus $\pi_* X \in \text{Mod}_{E_*}^{I_n\text{-tors}}$.

Assume now $X \in \text{Mod}_E$ such that its homotopy groups are I_n -power torsion. Monochromatization gives a map $\phi: M_n X \rightarrow X$, and as $\pi_* M_n X$ is I_n -power torsion this map factors on homotopy groups as

$$\pi_* M_n X \rightarrow H_{I_n}^0 \pi_* X \rightarrow \pi_* X,$$

where the first map is the edge morphism in the above-mentioned spectral sequence. As $\pi_* X$ was assumed to be I_n -power torsion we have $\pi_* X \cong H_{I_n}^0 \pi_* X$, and $H_{I_n}^s \pi_* X \cong 0$ for $s > 0$. Hence the spectral sequence collapses to give the isomorphism $\pi_* M_n X \cong H_{I_n}^0 \pi_* X$, which shows that $\pi_* \phi$ is an isomorphism. As π_* is conservative ϕ was already an isomorphism, hence $X \in \text{Mod}_E^{I_n\text{-tors}}$. $\square$

Lemma 4.2 *For any prime p and nonnegative integer n , the functor*

$$\pi_*: \text{Mod}_E^{I_n\text{-tors}} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$$

is a conservative adapted homology theory.

Proof The functor $\pi_*: \text{Mod}_E \rightarrow \text{Mod}_{E_*}$ is a conservative adapted homology theory. By Lemma 4.1 its restriction to $\text{Mod}_E^{I_n\text{-tors}}$ lands in $\text{Mod}_{E_*}^{I_n\text{-tors}}$, hence automatically $\pi_*: \text{Mod}_E^{I_n\text{-tors}} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$ is a conservative homology theory.

Let J be an injective I_n -power torsion E_* -module. We can embed $J \rightarrow Q$ into an injective E_* -module Q , as Mod_{E_*} has enough injectives. After applying the torsion functor $T_{I_n}^{E_*}$ this map has a section, as $J \cong T_{I_n}^{E_*} J$ is injective. In particular, any injective J is a retract of $T_{I_n}^{E_*} Q$ for some injective E_* -module Q , hence we can assume J to be of that form. By [17, 2.1.4] any such $J = T_{I_n}^{E_*} Q$ is injective as an object of Mod_{E_*} .

Now, as π_* is adapted on Mod_E we can chose a faithful injective lift $\bar{J}$ of J to Mod_E , and since $\bar{J}$ was assumed to have I_n -torsion homotopy groups we know by Lemma 4.1 that $\bar{J}$ is an object of $\text{Mod}_E^{I_n\text{-tors}}$. In particular, we have faithful lifts for any injective in $\text{Mod}_{E_*}^{I_n\text{-tors}}$, which means that $\pi_*: \text{Mod}_E^{I_n\text{-tors}} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$ is adapted. $\square$

Let C^{I_n} denote the I_n -adic completion functor on Mod_{E_*} . It is neither left nor right exact; see [26, Appendix A]. As E_* is an integral domain, the higher right derived functors vanish by [20, 5.1]. For $i \geq 0$ we denote the i^{th} left derived functor of C^{I_n} by $L_i^{I_n}$. For any $M \in \text{Mod}_E$ there is a natural map $L_0^{I_n} M \rightarrow C^{I_n} M$. It is always an epimorphism, but usually not an isomorphism.

Lemma 4.3 *For any prime p and nonnegative integer n , the category $\text{Mod}_{E_*}^{I_n\text{-tors}}$ has cohomological dimension n .*

Proof The category Mod_{E_*} has cohomological dimension n , and that Ext -groups in $\text{Mod}_{E_*}^{I_n\text{-tors}}$ are computed in Mod_{E_*} . By [17, 2.1.4], this implies that the cohomological dimension of $\text{Mod}_{E_*}^{I_n\text{-tors}}$ cannot be greater than n , so it remains to prove that it is exactly n . We prove this by computing an $\text{Ext}_{E_*}^n$ group that is nonzero.

By [26, A.2(d)] we have $L_0^{I_n} M \cong \text{Ext}_{E_*}^n(H_{I_n}^n(E_*), M)$ for any E_* -module M . In other words, the derived completion of an E_* -module is the n^{th} derived functor of maps from the I_n -local cohomology of E_* into M . Choosing $M = E_*/I_n$ we get

$$L_0^{I_n}(E_*/I_n) \cong \text{Ext}_{E_*}^n(H_{I_n}^n(E_*), E_*/I_n).$$

As any bounded I_n -torsion E_* -module is I_n -adically complete we have, as remarked in [7, 1.4], an isomorphism $L_0^{I_n}(E_*/I_n) \cong E_*/I_n$. The local cohomology of E_* is also I_n -torsion, in particular $H_{I_n}^n E_* = E_*/I_n^\infty$. Hence we have

$$\text{Ext}_{E_*}^n(E_*/I_n^\infty, E_*/I_n) \cong E_*/I_n \not\cong 0,$$

showing that there are two I_n -power torsion E_* -modules with nontrivial n^{th} Ext . $\square$

Lemma 4.4 *For any prime p and nonnegative integer n , the category $\text{Mod}_{E_*}^{I_n\text{-tors}}$ has a splitting of order $2p - 2$.*

Proof By [36, 8.1] the category Mod_{E_*} has a splitting of order $2p - 2$. We define the pure weight ϕ component of $\text{Mod}_{E_*}^{I_n\text{-tors}}$, denoted by $\text{Mod}_{E_*,\phi}^{I_n\text{-tors}}$, to be the essential image of $T_{I_n}^{E_*} : \text{Mod}_{E_*} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$ restricted to the pure weight ϕ component $\text{Mod}_{E_*,\phi}$. We claim that this defines a splitting of order $2p - 2$ on $\text{Mod}_{E_*}^{I_n\text{-tors}}$.

As $\text{Mod}_{E_*,\phi}$ is a Serre subcategory, and being I_n -power torsion is a property closed under subobjects, quotients, and extensions, also $\text{Mod}_{E_*,\phi}^{I_n\text{-tors}}$ is a Serre subcategory. As E_* is concentrated in degrees divisible by $2p - 2$ every I_n -power torsion module decomposes into its pure weight components. This also gives a decomposition of $\text{Mod}_{E_*}^{I_n\text{-tors}}$. The shift functor on I_n -power torsion modules simply shifts the underlying module, hence shift-invariance follows from the shift-invariance on Mod_{E_*} . $\square$

We can now summarize the above discussion with the first of our main results.

Theorem 4.5 (Theorem C) *Let p be a prime and n a nonnegative integer. If $k = 2p - 2 - n > 0$, then the functor*

$$\pi_* : \text{Mod}_E^{I_n\text{-tors}} \rightarrow \text{Mod}_{E_*}^{I_n\text{-tors}}$$

is a k -exotic homology theory, giving an equivalence

$$h_k \text{Mod}_E^{I_n\text{-tors}} \simeq h_k \text{D}^{\text{per}}(\text{Mod}_{E_*}^{I_n\text{-tors}}).$$

In particular, monochromatic E -modules are exotically algebraic at large primes.

Proof By Lemma 4.3 the cohomological dimension of the category $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ is n , and by Lemma 4.4 we have a splitting on $\mathrm{Mod}_{E_*}^{I_n\text{-tors}}$ of order $2p - 2$. Hence, by Lemma 4.2 the functor

$$\pi_* : \mathrm{Mod}_E^{I_n\text{-tors}} \rightarrow \mathrm{Mod}_{E_*}^{I_n\text{-tors}}$$

is a k -exotic homology theory for $k = 2p - 2 - n > 0$, which gives an equivalence

$$h_k \mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}})$$

by Theorem 3.19. □

We can also phrase this dually in terms of $K_p(n)$ -local E -modules.

Corollary 4.6 *Let p be a prime, n a positive integer and $K_p(n)$ be height- n Morava K -theory at the prime p . If $k = 2p - 2 - n > 0$, then we have a k -exotic algebraic equivalence*

$$h_k L_{K_p(n)} \mathrm{Mod}_E \simeq h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-comp}}.$$

In particular, $K_p(n)$ -local E -modules are exotically algebraic at large primes.

Proof The equivalence is constructed from the equivalences obtained from Remark 2.7, Theorems 4.5 and 2.21 and Construction 2.19. In particular, we have

$$\begin{aligned} h_k \mathrm{Mod}_E^{I_n\text{-comp}} &\stackrel{2.7}{\simeq} h_k \mathrm{Mod}_E^{I_n\text{-tors}} \\ &\stackrel{4.5}{\simeq} h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*}^{I_n\text{-tors}}) \\ &\stackrel{2.21}{\simeq} h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-tors}} \\ &\stackrel{2.19}{\simeq} h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Mod}_{E_*})^{I_n\text{-comp}}, \end{aligned}$$

where we have used that an equivalence of ∞ -categories induces an equivalence on homotopy k -categories. □

Now, let HE_* be the Eilenberg–MacLane spectrum of E_* . By Schwede and Shipley’s derived Morita theory, see [33, 7.1.1.16], there is a symmetric monoidal equivalence of categories $\mathrm{D}(E_*) \simeq \mathrm{Mod}_{HE_*}$, and we can form a local duality diagram for Mod_{HE_*} corresponding to Example 2.5 for the discrete Hopf algebroid (E_*, E_*) . By arguments similar to Lemmas 4.1 and 4.2 one can show that the homotopy groups functor $\pi_* : \mathrm{Mod}_{HE_*} \rightarrow \mathrm{Mod}_{E_*}$ restricts to a conservative adapted homology theory

$$\pi_* : \mathrm{Mod}_{HE_*}^{I_n\text{-tors}} \rightarrow \mathrm{Mod}_{E_*}^{I_n\text{-tors}}.$$

In the same range as Theorem 4.5 this is also k -exotic. We can then combine the algebraicity for $\mathrm{Mod}_E^{I_n\text{-tors}}$ and Mod_{HE_*} to get the following statement.

Corollary 4.7 *Let p be a prime and n a nonnegative integer. If $k = 2p - 2 - n > 0$, then there is an exotic equivalence $h_k \mathrm{Mod}_E^{I_n\text{-tors}} \simeq h_k \mathrm{Mod}_{HE_*}^{I_n\text{-tors}}$.*

4.2 Monochromatic spectra

Having proven that monochromatic E -modules are algebraic at large primes, we now turn to the larger category of all monochromatic spectra $\mathcal{M}_{n,p}$ with the same goal. The strategy is exactly the same as in Section 4.1: we first prove that the conservative adapted homology theory $E_*: \mathrm{Sp}_{n,p} \rightarrow \mathrm{Comod}_{E_*E}$ restricts to a conservative adapted homology theory on $\mathcal{M}_{n,p}$, before proving that $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ has a splitting and finite cohomological dimension. This will prove Theorem B, which we then convert into a proof of Theorem A, as in Corollary 4.6.

In this section the choice of v_n -periodic Landweber exact ring spectrum E does not matter, as the categories $\mathrm{Sp}_{n,p}$ and Comod_{E_*E} are equivalent for all such spectra—see [23, 1.12] and [27, 4.2], respectively. However, to make the interaction with Section 4.1 as simple as possible we will continue to use the height- n Johnson–Wilson spectrum $E(n)$.

Lemma 4.8 *If X is a E -local spectrum, then $X \in \mathcal{M}_{n,p}$ if and only if $E_*X \in \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$.*

Proof Assume first that $X \in \mathcal{M}_{n,p}$. We have $E \otimes X \in \mathrm{Mod}_E^{I_n\text{-tors}}$ as

$$E \otimes X \simeq E \otimes M_n X \simeq M_n(E \otimes X),$$

where the last equivalence follows from M_n being smashing. In particular, the restricted functor $E_*: \mathcal{M}_{n,p} \rightarrow \mathrm{Comod}_{E_*E}$ factors through $\mathrm{Mod}_E^{I_n\text{-tors}}$. By Lemma 4.1 and Remark 2.9 this means that E_*X is an I_n -power torsion E_*E -comodule.

For the converse, assume that we have $X \in \mathrm{Sp}_{n,p}$ such that $E_*X \in \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$. Using the monochromatization functor we obtain a comparison map $M_n X \rightarrow X$, which induces a map on E -modules $E \otimes M_n X \rightarrow E \otimes X$. This map is an isomorphism on homotopy groups, as E_*X was assumed to be I_n -power torsion. As E_* is conservative on $\mathrm{Sp}_{n,p}$, the original comparison map $M_n X \rightarrow X$ was an isomorphism, meaning that $X \in \mathcal{M}_{n,p}$. $\square$

Lemma 4.9 *For any prime p and nonnegative integer n , the functor*

$$E_*: \mathcal{M}_{n,p} \rightarrow \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$$

is a conservative adapted homology theory.

Proof First note that the image of the functor $E_*: \mathrm{Sp}_{n,p} \rightarrow \mathrm{Comod}_{E_*E}$ restricted to $\mathcal{M}_{n,p}$ is contained in $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ by Lemma 4.8. The functor $E_*: \mathcal{M}_{n,p} \rightarrow \mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ is then automatically a conservative homology theory. The category $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ has enough injectives as it is Grothendieck by Remark 2.10. Hence, it only remains to prove that we have faithful lifts for all injective objects.

Let J be an injective in $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$. As in the proof of Lemma 4.2 we can assume that J has the form $J = T_{I_n}^{E_*E} P$ for some injective E_*E -comodule P , as being torsion is a property of the underlying module. By [28, 2.1(c)] any injective E_*E -comodule is a retract of $E_*E \otimes_{E_*} Q$ for some injective E_* -module Q . Hence, we can further assume that J has the form $J = T_{I_n}^{E_*E} (E_*E \otimes_{E_*} Q)$.

From [8, 5.7] it follows that there is a commutative diagram of adjoint functors

$$\begin{array}{ccc} \mathrm{Comod}_{E_*E} & \xrightleftharpoons[\varepsilon^*]{\varepsilon_*} & \mathrm{Mod}_{E_*} \\ \updownarrow T_{I_n}^{E_*E} & & \updownarrow T_{I_n}^{E_*} \\ \mathrm{Comod}_{E_*E}^{I_n\text{-tors}} & \xrightleftharpoons[\varepsilon^*]{\varepsilon_*} & \mathrm{Mod}_{E_*}^{I_n\text{-tors}} \end{array}$$

where $\varepsilon_* \dashv \varepsilon^*$ is the forgetful-cofree adjunction. In particular, the functor ε^* is given by $E_*E \otimes_{E_*} (-)$. To justify the notation in the bottom row, let us prove that the cofree functor on I_n -power torsion modules is also given by $E_*E \otimes_{E_*} (-)$. In order to do this we prove that for an I_n -power torsion E_* -module M , that $T_{I_n}^{E_*E}(E_*E \otimes_{E_*} M) \cong E_*E \otimes_{E_*} M$.

By [8, 5.5] there is an isomorphism

$$T_{I_n}^{E_*E}(E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M),$$

which by [8, 4.4] gives

$$\mathrm{colim}_k \underline{\mathrm{Hom}}_{E_*E}(E_*/I_n^k, E_*E \otimes_{E_*} M) \cong \mathrm{colim}_k (E_*E \otimes_{E_*} \mathrm{Hom}_{E_*}(E_*/I_n^k, M)).$$

As the tensor product $- \otimes_{E_*} -$ commutes with filtered colimits separately in each variable, and M was assumed to be I_n -power torsion, the right hand side is $E_*E \otimes_{E_*} M$.

Now, choosing the injective Q in the top right corner and going through the square gives an isomorphism $T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q) \cong E_*E \otimes_{E_*} T_{I_n}^{E_*}Q$. By [17, 2.1.4] we know that $T_{I_n}^{E_*}Q$ is an injective E_* -module, and by [28, 2.1(a)] the cofree comodule $E_*E \otimes_{E_*} T_{I_n}^{E_*}Q$ is an injective E_*E -comodule. Hence, $J = T_{I_n}^{E_*E}(E_*E \otimes_{E_*} Q)$ is injective also as an object in Comod_{E_*E} .

Finally, as E_* has faithful injective lifts from Comod_{E_*E} to $\mathrm{Sp}_{n,p}$, there exists a lift $\bar{J}$ such that $[X, \bar{J}] \simeq \mathrm{Hom}_{E_*E}(E_*X, J)$ and $E_*\bar{J} \simeq J$. By Lemma 4.8 we know that $\bar{J} \in \mathcal{M}_{n,p}$, as J was assumed to be I_n -power torsion, hence we have found our faithful injective lift. $\square$

Lemma 4.10 *Let p be a prime and n a nonnegative integer. If $p-1 \nmid n$, then the category $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ has cohomological dimension $n^2 + n$.*

Proof The proof follows [37, 2.5] closely, which is itself a modern reformulation of [19, 3.4.3.9]. As in Lemma 4.3 we note that also Ext -groups in $\mathrm{Comod}_{E_*E}^{I_n\text{-tors}}$ are computed in Comod_{E_*E} . We start by defining *good targets* to be I_n -power torsion comodules N such that $\mathrm{Ext}_{E_*E}^{s,t}(E_*/I_n, N) = 0$ for all $s > n^2 + n$ and *good sources* to be I_n -power torsion comodules M such that $\mathrm{Ext}_{E_*E}^{s,t}(M, N) = 0$ for all $s > n^2 + n$ and I_n -torsion comodules N .

By the Landweber filtration theorem, see, for example, [27, 5.7], we know that any finitely presented comodule M has a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{s-1} \subseteq M_s = M,$$

where $M_r/M_{r-1} \cong E_*/I_{j_r}[t_r]$ and $j_r \leq n$. When M is I_n -power torsion we get $j_r = n$ for all r , as noted in [27, 4.3]. For primes p not dividing $n+1$ Morava's vanishing theorem, see, for example, [39, 6.2.10], gives us that $\text{Ext}_{E_*E}^{s,t}(E_*, E_*/I_n) = 0$ for all $s > n^2$. As the generators for the ideal I_i form a regular sequence, we get short exact sequences of the form

$$0 \rightarrow E_*/I_{i-1} \xrightarrow{v_i} E_*/I_{i-1} \rightarrow E_*/I_i \rightarrow 0$$

for $0 \leq i \leq n$. By the induced long exact sequence in Ext-groups, we get that

$$\text{Ext}_{E_*E}^{s,t}(E_*/I_n, E_*/I_n) = 0$$

for $s > n^2 + n$, which by the Landweber filtration implies that any finitely presented I_n -power torsion comodule is a good target.

The comodule E_*/I_n has a finite resolution of E_*E -comodules that are projective as modules over E_* . The Ext-functor out of these projectives can be computed using the cobar complex, see [39, A1.2.12], implying that the functor $\text{Ext}_{E_*E}^{s,t}(E_*/I_n, -)$ commutes with filtered colimits. By Lemma 2.11 any I_n -power torsion comodule is a filtered colimit of finitely presented ones, hence any I_n -power torsion comodule is a good target.

The above argument also proves that E_*/I_n is a good source, which by the Landweber filtration argument implies that any finitely presented I_n -torsion comodule is a good source. Again, by Lemma 2.11, the category $\text{Comod}_{E_*E}^{I_n\text{-tors}}$ is generated under filtered colimits by finitely presented comodules. Hence, we can apply [37, 2.4] to any injective resolution

$$0 \rightarrow M \rightarrow J_0 \rightarrow J_1 \rightarrow \cdots$$

to get that $J_{n^2+n} \rightarrow \text{Im}(J_{n^2+n} \rightarrow J_{n^2+n+1})$ is a split surjection, and that $\text{Im}(J_{n^2+n} \rightarrow J_{n^2+n+1})$ is injective. Hence, any injective resolution can be modified to have length $n^2 + n$. $\square$

Remark 4.11 In a previous version of this paper, we claimed that the cohomological dimension was n^2 . We want to thank Piotr Pstrągowski for pointing out the gap in the proof. This means that $\text{Comod}_{E_*E}^{I_n\text{-tors}}$ has the same cohomological dimension as the nontorsion category Comod_{E_*E} , as seen in Example 3.11. However, we do obtain something slightly stronger, as our result holds for all $p-1 \nmid n$, while the analogue in Comod_{E_*E} only holds when $p-1 > n$. In fact, Comod_{E_*E} does not have finite cohomological dimension when $p-1 \leq n$, as noted in [37, 2.6]. This difference happens because we only need the Ext^s -groups out of E_*/I_n to vanish for large s , which is given to us by Morava's vanishing theorem whenever $p-1 \nmid n$. For nontorsion comodules one has to have stronger vanishing results. These can be obtained by using the chromatic spectral sequence, which only gives the vanishing results for $p-1 > n$ instead of for $p-1 \nmid n$.

Lemma 4.12 For any prime p and nonnegative integer n , the category $\text{Comod}_{E_*E}^{I_n\text{-tors}}$ has a splitting of order $2p-2$.

Proof As E is concentrated in degrees divisible by $2p - 2$, [36, 8.13] shows that Comod_{E_*E} has a splitting of order $2p - 2$. The proof of the induced splitting on the I_n -torsion category is then identical to Lemma 4.4. $\square$

We can now summarize the above results with our second main result, which is the monochromatic analogue of Example 3.22.

Theorem 4.13 (Theorem B) *Let p be a prime and n a nonnegative integer. If $k = 2p - 2 - n^2 - n > 0$, then the restricted functor $E_*: \mathcal{M}_{n,p} \rightarrow \text{Comod}_{E_*E}^{I_n\text{-tors}}$ is k -exotic. In particular, there is an equivalence*

$$h_k \mathcal{M}_{n,p} \simeq h_k \mathbf{D}^{\text{per}}(E_* E^{I_n\text{-tors}}),$$

meaning that monochromatic homotopy theory is exotically algebraic at large primes.

Proof By Lemma 4.10, the cohomological dimension of $\text{Comod}_{E_*E}^{I_n\text{-tors}}$ is $n^2 + n$ and by Lemma 4.12 we have a splitting of order $2p - 2$. The restricted functor E_* is then by Lemma 4.9 k -exotic whenever $k = 2p - 2 - n^2 - n > 0$, which by Theorem 3.19 finishes the proof. $\square$

Remark 4.14 By Theorem 2.21 there is an equivalence

$$\mathbf{D}^{\text{per}}(E_* E^{I_n\text{-tors}}) \simeq \text{Fr}_{n,p}^{I_n\text{-tors}}$$

and by Example 2.6 there is an equivalence

$$\mathcal{M}_{n,p} \simeq \text{Sp}_{n,p}^{I_n\text{-tors}}.$$

This means that we can write the equivalence in Theorem 4.13 as

$$h_k \text{Sp}_{n,p}^{I_n\text{-tors}} \simeq h_k \text{Fr}_{n,p}^{I_n\text{-tors}}$$

for $k = 2p - 2 - n^2 - n > 0$. This is more in line with thinking about Theorem 4.13 as “coming from” the chromatic algebraicity of Example 3.22 on localizing ideals. This formulation is perhaps also easier to connect to the limiting case $p \rightarrow \infty$ as described using ultraproducts in [12], which can be stated informally as

$$\lim_{p \rightarrow \infty} \text{Sp}_{n,p}^{I_n\text{-tors}} \simeq \lim_{p \rightarrow \infty} \text{Fr}_{n,p}^{I_n\text{-tors}}.$$

Via Theorem 2.3 we can now obtain the associated exotic algebraicity statement for the category of $K_p(n)$ -local spectra.

Theorem 4.15 (Theorem A) *Let p be a prime, n a nonnegative integer and $K_p(n)$ be height- n Morava K -theory at the prime p . If $k = 2p - 2 - n^2 > 0$, then we have a k -exotic algebraic equivalence*

$$h_k \text{Sp}_{K_p(n)} \simeq h_k \text{Fr}_{n,p}^{I_n\text{-comp}}.$$

In other words, $K_p(n)$ -local homotopy theory is algebraic at large primes.

Proof As we did in Corollary 4.6, we construct the equivalence from a sequence of equivalences coming from Theorems 2.3 and 4.13. More precisely we use equivalences coming from Example 2.6, Theorems 4.13 and 2.21 and Construction 2.19, which give

$$\begin{aligned} h_k \mathrm{Sp}_{K_p(n)} &\stackrel{2.6}{\simeq} h_k \mathcal{M}_{n,p} \\ &\stackrel{4.13}{\simeq} h_k \mathrm{D}^{\mathrm{per}}(\mathrm{Comod}_{E_* E}^{I_n\text{-tors}}) \\ &\stackrel{2.21}{\simeq} h_k \mathrm{Fr}_{n,p}^{I_n\text{-tors}} \\ &\stackrel{2.19}{\simeq} h_k \mathrm{Fr}_{n,p}^{I_n\text{-comp}}, \end{aligned}$$

where we again have used that an equivalence of ∞ -categories induces an equivalence on homotopy k -categories. $\square$

Remark 4.16 As in Remark 4.14 we can phrase Theorem 4.15 as $h_k \mathrm{Sp}_{n,p}^{I_n\text{-comp}} \simeq h_k \mathrm{Fr}_{n,p}^{I_n\text{-comp}}$.

Appendix Barr–Beck for localizing ideals

In this appendix we prove that the monoidal Barr–Beck theorem — a monoidal version of Lurie’s ∞ -categorical version of the classical Barr–Beck monadicity theorem; see [33, Section 4.7] — interacts nicely with local duality.

Theorem A.1 [34, 5.29] *Let $\mathcal{C}, \mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^L)$ and $(F \dashv G): \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal adjunction. If in addition*

- (1) *G is conservative,*
- (2) *G preserves colimits, and*
- (3) *the projection formula holds,*

then (F, G) is a monoidally monadic adjunction and the monad GF is equivalent to the monad $G(\mathbb{1}_{\mathcal{D}}) \otimes (-)$. In particular this gives a symmetric monoidal equivalence $\mathcal{D} \simeq \mathrm{Mod}_{G(\mathbb{1}_{\mathcal{D}})}(\mathcal{C})$.

Proof By [33, 4.7.3.5] the adjunction is monadic by the first two criteria, giving an equivalence $\mathcal{D} \simeq \mathrm{Mod}_{GF}(\mathcal{C})$. The map of monads $G(\mathbb{1}_{\mathcal{D}}) \otimes (-) \rightarrow GF$ given by [18, 3.6] is seen to be an equivalence by applying the projection formula to the unit $\mathbb{1}_{\mathcal{D}}$. $\square$

Definition A.2 When the three criteria above hold for a given monoidal adjunction $(F \dashv G)$, we will say that the adjunction satisfies the monoidal Barr–Beck criteria or that it is a *monoidal Barr–Beck adjunction*. We will sometimes omit the prefix monoidal when it is clear from context.

Let $(\mathcal{C}, \mathcal{K})$ be a local duality context. We wish to prove that the associated local duality diagram is compatible with Theorem A.1. By modifying [14, 3.7] slightly, we know that any Barr–Beck adjunction induces a Barr–Beck adjunction on $\mathcal{K}$ -local and $\mathcal{K}$ -complete objects. Hence, it remains only to prove a similar statement for the $\mathcal{K}$ -torsion objects.

Definition A.3 Let $(\mathcal{C}, \mathcal{K})$ and $(\mathcal{D}, \mathcal{L})$ be local duality contexts. A *map of local duality contexts* is a symmetric monoidal colimit-preserving functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F(\mathcal{K}) \subseteq \mathcal{L}$. If, in addition $\text{Loc}_{\mathcal{D}}^{\otimes}(F(\mathcal{K})) = \text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$, then we say F is a *strict* map of local duality contexts. A monoidal adjunction $(F \dashv G): \mathcal{C} \rightarrow \mathcal{D}$ such that F is a strict map of local duality contexts is called a *local duality adjunction*, sometimes denoted by

$$(F \dashv G): (\mathcal{C}, \mathcal{K}) \rightarrow (\mathcal{D}, \mathcal{L}).$$

Given a local duality context and an appropriate functor, one can always extend the functor to a strict map of local duality context in the following way.

Construction A.4 Let $(\mathcal{C}, \mathcal{K})$ be a local duality context, $\mathcal{D} \in \text{CAlg}(\text{Pr}_{\text{st}}^L)$ and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a symmetric monoidal colimit-preserving functor. The image of $\mathcal{K}$ under F generates a localizing ideal $\text{Loc}_{\mathcal{D}}^{\otimes}(F(\mathcal{K}))$ in $\mathcal{D}$, which makes F a map of local duality contexts. We call this the local duality context on $\mathcal{D}$ induced by $\mathcal{C}$ via F .

The following lemma is essentially the “nongeometric” version of [2, 5.11]. The proof is also similar, but as we have phrased it in a different and slightly more general language, we present a full proof.

Lemma A.5 Let $(F \dashv G): (\mathcal{C}, \mathcal{K}) \rightarrow (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. Then, the adjunction induces a monoidal adjunction on localizing ideals

$$\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K}) \xrightleftharpoons[G']{F'} \text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L}).$$

Proof From Remark 2.4 we know that the symmetric monoidal structures on $\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K})$ and $\text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$ is simply the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, restricted to the full subcategories.

Since F is a map of local duality contexts, we have an inclusion $F(\mathcal{K}) \subseteq \mathcal{L}$, which gives inclusions

$$F(\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K})) \subseteq \text{Loc}_{\mathcal{D}}^{\otimes}(F(\mathcal{K})) \subseteq \text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L}),$$

meaning that the functor F restricts to the torsion objects. In particular we have for any object $X \in \mathcal{C}^{\mathcal{K}\text{-tors}}$ an equivalence $\Gamma_{\mathcal{L}} F(X) \simeq F(X)$. We let $F' = F|_{\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K})}$ and define G' to be the composition

$$\text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L}) \xrightarrow{i_{\mathcal{L}\text{-tors}}} \mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\Gamma_{\mathcal{K}}} \text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K}),$$

which is an adjoint to F' . We need to show that F is a symmetric monoidal functor, but, as the inclusions $i_{\mathcal{K}\text{-loc}}$ and $i_{\mathcal{L}\text{-loc}}$ are nonunitally monoidal all that remains to be proven is that F' sends the monoidal unit $\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}}$ to the monoidal unit $\Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}$.

The localizing ideals $\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K})$ and $\text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$ are equal to the localizing ideals generated by the respective units, ie

$$\text{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K}) = \text{Loc}_{\mathcal{C}}^{\otimes}(\Gamma_{\mathcal{K}} \mathbb{1}_{\mathcal{C}}) \quad \text{and} \quad \text{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L}) = \text{Loc}_{\mathcal{D}}^{\otimes}(\Gamma_{\mathcal{L}} \mathbb{1}_{\mathcal{D}}).$$

Since $(F \dashv G)$ is a local duality adjunction we also know that $\mathrm{Loc}_{\mathcal{D}}^{\otimes}(F(\mathcal{K})) = \mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$, which also means $\mathrm{Loc}_{\mathcal{D}}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) = \mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$.

Let $\mathcal{G}$ be the full subcategory of $\mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$ where $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ acts as a unit, in other words objects $M \in \mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$ such that $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M \simeq M$. In particular, $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$ is in $\mathcal{G}$. The category $\mathcal{G}$ is closed under retracts, suspension, and colimits, as well as tensoring with objects in $\mathcal{D}$, as we have

$$F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} (M \otimes_{\mathcal{D}} D) \simeq (F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} M) \otimes_{\mathcal{D}} D \simeq M \otimes_{\mathcal{D}} D$$

for any $M \in \mathcal{G}$ and $D \in \mathcal{D}$. Hence, it is a localizing tensor ideal of $\mathcal{D}$, with a symmetric monoidal structure where the unit is $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})$. In particular, $\mathcal{G} = \mathrm{Loc}_{\mathcal{D}}^{\otimes}(F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}))$, which we already know is equivalent to $\mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$.

Since the ideals are equivalent, and the unit is unique, we must have $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$. $\square$

The key feature for us is that such an induced adjunction inherits the property of being a Barr–Beck adjunction, ie, that the right adjoint is conservative, preserves colimits, and has a projection formula. An analogous, but not equivalent, statement was proven in [14, 4.5]. Another related, but not equivalent statement, is Greenlees and Shipley’s cellularization principle; see [22].

Theorem A.6 *Let $(F \dashv G): (\mathcal{C}, \mathcal{K}) \rightarrow (\mathcal{D}, \mathcal{L})$ be a local duality adjunction. If $(F \dashv G)$ satisfies the Barr–Beck criteria, then the induced monoidal adjunction on localizing ideals*

$$\mathrm{Loc}_{\mathcal{C}}^{\otimes}(\mathcal{K}) \xrightleftharpoons[G']{F'} \mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$$

constructed in Lemma A.5 also satisfies the Barr–Beck criteria.

Proof We need to prove that G' is conservative and colimit-preserving and that the projection formula holds. The first two will both follow from the following computation, showing that also G' is just the restriction of G to $\mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$.

Let $X \in \mathrm{Loc}_{\mathcal{D}}^{\otimes}(\mathcal{L})$. By definition we have $G'(X) = \Gamma_{\mathcal{K}}G(X)$, where we have omitted the inclusions from the notation for simplicity. Since $\Gamma_{\mathcal{K}}$ is smashing and $(F \dashv G)$ by assumption has a projection formula,

$$\Gamma_{\mathcal{K}}G(X) \simeq G(X) \otimes_{\mathcal{C}} \Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}} \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})).$$

By Lemma A.5 F' is symmetric monoidal, hence $F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}}) \simeq \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}$, which acts on X as the monoidal unit. Thus, we can summarize with

$$G'(X) \simeq G(X \otimes_{\mathcal{D}} F(\Gamma_{\mathcal{K}}\mathbb{1}_{\mathcal{C}})) \simeq G(X \otimes_{\mathcal{D}} \Gamma_{\mathcal{L}}\mathbb{1}_{\mathcal{D}}) \simeq G(X),$$

which shows that also G' is the restriction of G .

Now, as G is both conservative and preserves colimits, and colimits in the localizing ideals are computed in $\mathcal{C}$ and $\mathcal{D}$, respectively, then also G' is conservative and colimit-preserving. The projection formula for $(F' \dashv G')$ also automatically follows from the projection formula for $(F \dashv G)$. $\square$

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Twisted spectra revisited

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We recapture Douglas’s framework for twisted parametrized stable homotopy theory in the language of ∞ -categories. A twisted spectrum is essentially a section of a bundle of presentable stable ∞ -categories whose fiber is the ∞ -category of spectra, a perspective we refine. We recover some of Douglas’s results on classifications of such bundles, as well as further make precise the identification of categories of twisted spectra with module categories over Thom spectra in the pointed connected case. Furthermore, we examine subtle aspects of the functoriality which arise by virtue of being fibered over the Brauer space of the sphere spectrum. Extending beyond the scope of Douglas’s work, we introduce a total category of twisted spectra over a fixed space, where we allow the twists to vary, and show that these total categories themselves satisfy the essential features of a 6-functor formalism. We also introduce an $(\infty, 2)$ -category of twisted spectra, and use this framework to discuss duality theory for these types of objects.

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Introduction

Background and aim

The motivation behind this project is grounded in what is contemporaneously known as *Floer homotopy theory*. Without going into details, let us illustrate one way to think about the Floer homotopy theory

homotopy type of the manifold is build up using the critical points of the Morse function, and where one can compute the homology in many other ways, without even making use of the auxiliary data of a Morse function. In Floer theory, it is much harder to separate out what depends on the underlying homotopy types and what does not, and we do not yet know whether it is possible to define Floer homology in other ways that do not make use of a Morse–Floer function. It is possible that studying Floer homotopy types might give more insight into these structural questions.

In the aforementioned paper, Cohen, Jones, and Segal also sketch how to construct a (pro)spectrum from Floer data via flow categories in the situation in which our infinite-dimensional manifold is “trivially polarized”. Roughly, the reader should think of a trivialization of the manifold as a decompositions of its tangent space into two parts, one containing the positive eigenspace of the Hessian of the “Morse function” and another containing the negative eigenspace, allowing for some finite-dimensional ambiguity on where to place the nullspace [10; 12; 34]. Depending on the flavor of polarization, this corresponds to a class in either real or complex topological K-theory in cohomological degree 1. The manifold is trivially polarized when this class is zero. In particular, this implies that the flow category is framed, from which we can define a (pro)spectrum.

Question 0.2 What sort of objects are these “stable homotopy types” associated to Floer data where the polarization is *nontrivial*?

That some twisted version of stable homotopy theory is needed to deal with nontrivial polarizations was first pointed out by Furuta in the unfinished draft [15]. In his 2005 PhD thesis, Douglas proposed a more rigorous framework for these “twisted spectra” [12]. It is these twisted spectra that we are concerned with.

Our aim is to recast Douglas’s work in the modern framework of ∞ -categories, and to utilize this framework to prove structural results about these sort of objects. In particular, we are interested in functoriality properties for twisted spectra, akin to various 6-functor formalisms that have appeared in the literature in other contexts. We also want to get a better understanding of the duality theory of these types of objects.

Remark 0.3 Flow categories not being frameable is not the only issue when defining a prospectrum from Floer data. There is also the separate issue of “disc bubbling”, which complicates the picture, arguably even more than the issue of frameability. Roughly, disc bubbling leads to issues of compactness of our flow categories. In most flavors of Floer theory in which disc bubbling occurs, there is typically a natural way to compactify, but the precise nature of how a flow category relates to its compactification is a bit mysterious [10]. Two common ways of dealing with bubbling issues in Floer theory today are either the theory of Kuranishi structures [14] or the theory of polyfolds [19]. Here *we will not consider bubbling phenomena at all* and strictly focus on the issue of frameability, interpreted as the need to use twisted spectra rather than just ordinary spectra. A true model for Floer homotopy types would be some modified version of the twisted spectra which also takes into account bubbling issues.

Main results

Roughly, the reader should think of a twisted spectrum on a space B as “a section of a bundle of ∞ -categories over B whose fiber is the ∞ -category of spectra”. This is an obvious generalization of parametrized spectra: if we look at the trivial bundle $\pi_B: \mathrm{Sp} \times B \rightarrow B$, then sections of this are precisely functors $B \rightarrow \mathrm{Sp}$, that is, parametrized spectra. To rigorously make sense of twisted spectra, we will take two perspectives: invertible sheaves of categories or local systems of categories. For the first perspective one considers the constant sheaf of categories with value Sp on the space B :

$$\mathcal{O}_B: \mathrm{Open}(B)^{\mathrm{op}} \rightarrow \mathrm{Pr}_{\mathrm{St}}^L, \quad U \mapsto \mathrm{Fun}(\mathrm{Sing}(U), \mathrm{Sp}).$$

The ∞ -category of sheaves of presentable stable ∞ -categories on B is symmetric monoidal and the above sheaf is precisely the unit of this monoidal structure. A *haunt* over a space B is a locally free rank-1 module over $\mathcal{O}_B$, and a *twisted spectrum* over B is a section of a haunt. When our space B is locally contractible, then any haunt is a locally constant sheaf, and we can use the equivalence between locally constant sheaves and local systems to conclude the following theorem, which may be compared to [12, Proposition 2.8].

Theorem 0.4 *Let B be a locally contractible space. Then the space of haunts on B is equivalent to the space of map from $\mathrm{Sing}(B)$ to the classifying space of the Picard space of the sphere spectrum:*

$$\mathrm{Haunt}_B \simeq \mathrm{Map}(\mathrm{Sing}(B), B \mathrm{Pic}(\mathbb{S})).$$

This implies that we could have alternatively defined haunts and twisted spectra using local systems of categories. Indeed, from this point of view a haunt is simply a map² $\sigma: B \rightarrow B \mathrm{Pic}(\mathbb{S})$ and the ∞ -category of twisted spectra for this particular haunt is

$$\mathrm{TwSp}(B, \sigma) = \lim_B (B \xrightarrow{\sigma} B \mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Pr}_{\mathrm{St}}^L).$$

We summarize some of our results about these categories.

Theorem 0.5 *There exists a symmetric monoidal functor*

$$\mathrm{TwSp}: \mathcal{S}_{/B \mathrm{Pic}(\mathbb{S})} \rightarrow \mathrm{Pr}_{\mathrm{St}}^L.$$

This functor sends a map $f: (A, \tau) \rightarrow (B, \sigma)$ in $\mathcal{S}_{/B \mathrm{Pic}(\mathbb{S})}$ to the functor $f_!: \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{TwSp}(B, \sigma)$. Its right adjoint f^* is itself colimit-preserving, and thus further admits a right adjoint f_* . In particular, since the functor is symmetric monoidal, we have a natural equivalence

$$\mathrm{TwSp}(A, \tau) \otimes^L \mathrm{TwSp}(B, \sigma) \simeq \mathrm{TwSp}(A \times B, \tau \boxtimes \sigma),$$

which further implies an equivalence

$$\mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) \simeq \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma).$$

²In the setting on Floer homotopy theory, these maps always seem to factor through the J-homomorphism. Indeed, a polarization corresponds to a class in complex or real topological K -theory, and such classes correspond to maps $B \rightarrow U$ and $B \rightarrow U/O$, respectively. The relevant haunt is then obtained by using Bott periodicity and the J-homomorphism. A polarization being trivial (or the flow corresponding flow category being frameable) is equivalent to the map $B \rightarrow B \mathrm{Pic}(\mathbb{S})$ being nullhomotopic.

Remark 0.6 A key vehicle powering the theory forward at this stage, and for what comes next, is the notion of ambidexterity [20] for diagrams in $\mathrm{Pr}_{\mathrm{St}}^L$ indexed by Kan complexes. This more explicitly allows for one to control the homotopy coherences needed establish base-change and functoriality results. For more, see Section 1.5.

Under the assumption that B is connected, these categories can be identified with module categories, a fact that was established in [12, Proposition 3.13], and later rediscovered in a different context in [7, Theorem D]. We reprove this fact, establishing some extra naturality properties of identification.

Theorem 0.7 *Let B be a connected space and consider a haunt $\sigma : B \rightarrow B \mathrm{Pic}(\mathbb{S})$. Then there is a natural equivalence*

$$\mathrm{TwSp}(B, \sigma) \simeq \mathrm{Mod}_{\mathrm{Th}(\Omega\sigma)}$$

of ∞ -categories. Here $\mathrm{Th}(\Omega\sigma)$ denotes the Thom spectrum of the map $\Omega\sigma : \Omega B \rightarrow \mathrm{Pic}(\mathbb{S})$.

Note that the categories $\mathrm{TwSp}(A, \tau)$ are not themselves monoidal; the map $\Omega\sigma$ is a priori only $\mathbb{E}_1$. However, one can define a total category of twisted spectra over a fixed space, where we allow the twist to vary, and this total stable ∞ -category satisfies a 6-functor formalism. The following theorem is a summary of a number of different result found in Section 5, namely Propositions 5.8–5.10, 5.12, 5.15, 5.16, and 5.18.

Theorem 0.8 *We define the total ∞ -category of twisted spectra over a space A as*

$$\mathrm{TwSp}(A) = \mathrm{colim}(\mathrm{TwSp}(A, -) : \mathrm{Map}(A, B \mathrm{Pic}(\mathbb{S})) \rightarrow \mathrm{Pr}_{\mathrm{St}}^L).$$

These categories satisfy a 6-functor formalism where $f^! = f^*$. In particular:

- *The categories $\mathrm{TwSp}(A)$ are closed symmetric monoidal.*
- *For every map $f : A \rightarrow B$ of spaces there is an induced pullback functor*

$$f^* : \mathrm{TwSp}(B) \rightarrow \mathrm{TwSp}(A)$$

which is symmetric monoidal. This functor admits a right adjoint

$$f_* : \mathrm{TwSp}(A) \rightarrow \mathrm{TwSp}(B).$$

- *Given a map of spaces $f : A \rightarrow B$ such that the induced map $\mathrm{Map}(B, B \mathrm{Pic}(\mathbb{S})) \rightarrow \mathrm{Map}(A, B \mathrm{Pic}(\mathbb{S}))$ is an equivalence, there is a functor*

$$f_! : \mathrm{TwSp}(A) \rightarrow \mathrm{TwSp}(B).$$

This functor satisfies base change and projection formula isomorphisms. Moreover, the functor admits a right adjoint which coincides with f^* .

The naturality of the identification between categories of twisted spectra and module categories over Thom spectra guarantee that the pullback functor and its left and right adjoints on the level of categories of twisted spectra correspond to the standard restriction, extension, and coextension of scalars on the level of module categories, in the case that we restrict ourselves to connected spaces.

By using the natural equivalence

$$\mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) \simeq \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma)$$

already mentioned above, we can define and study dualizability of twisted spectra. In particular, we say that a (B, σ) -twisted spectrum X is *Costenoble–Waner dualizable* if its corresponding colimit-preserving functor

$$\Phi_X: \mathrm{Sp} \rightarrow \mathrm{TwSp}(B, \sigma)$$

is such that its right adjoint is itself colimit-preserving. This is a direct generalization of the notion of Costenoble–Waner dualizability of parametrized spectra. In the pointed connected case, under the natural equivalence between categories of twisted spectra and categories of modules over Thom spectra, we note that this type of dualizability corresponds to dualizability of bimodules.

Theorem 0.9 *Let $\alpha: G \rightarrow \mathrm{Pic}(\mathbb{S})$ and $\beta: H \rightarrow \mathrm{Pic}(\mathbb{S})$ be maps of groups over the Picard space of the sphere spectrum and suppose that we are given a $(\mathrm{Th}(\alpha) - \mathrm{Th}(\beta))$ -module M . Suppose that X is the twisted spectrum corresponding to M under the equivalence*

$$\mathrm{TwSp}(BG \times BH, -B\alpha \boxtimes B\beta) \simeq_{\mathrm{Th}(\alpha)} \mathrm{BiMod}_{\mathrm{Th}(\beta)}.$$

Then the diagram

$$\begin{array}{ccc} \mathrm{TwSp}(BG, B\alpha) & \xrightarrow{\Phi_X} & \mathrm{TwSp}(BH, B\beta) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Mod}_{\mathrm{Th}(\alpha)} & \xrightarrow{F_M} & \mathrm{Mod}_{\mathrm{Th}(\beta)} \end{array}$$

commutes, where

$$F_M: \mathrm{Mod}_{\mathrm{Th}(\alpha)} \rightarrow \mathrm{Mod}_{\mathrm{Th}(\beta)}, \quad N \mapsto N \otimes_{\mathrm{Th}(\alpha)} M.$$

We conclude in Section 7 by reinterpreting several of the notions of this paper in the context of spectral algebraic geometry. In particular, we identify twists as derived Azumaya algebras on the constant (or Betti) stack associated to a topological space. We then study this setup with a view towards the classical problem of Grothendieck regarding realizability of étale cohomology classes by Azumaya algebras.

Terminology and notation

Let us collect some common terminology and notation:

- As we will use $(\infty, 1)$ -categories extensively (which we will also refer to simply as ∞ -categories), we will follow common notation and terminology for these, following [26; 27]. In particular, all categorical notions should be interpreted in the ∞ -categorical sense. For example, colimits and limits will be what would classically have been referred to as homotopy colimits and homotopy limits.
- We will not differentiate between a 1-category and its corresponding ∞ -category obtained via the nerve construction.

- We denote by $\mathcal{S}$ and $\mathcal{Sp}$ the ∞ -categories of spaces³ and spectra.
- Given an ∞ -category $\mathcal{C}$, the mapping space between two objects will be denoted by $\mathrm{Map}_{\mathcal{C}}(x, y)$, or just $\mathrm{Map}(x, y)$ if $\mathcal{C}$ is implicit. If $\mathcal{C}$ is stable, we will denote the mapping spectra by $\mathrm{map}_{\mathcal{C}}(x, y)$. Recall that $\Omega^{\infty} \mathrm{map}(x, y) \simeq \mathrm{Map}(x, y)$. If the ∞ -category $\mathcal{C}$ has internal mapping objects, we will denote them by $\mathrm{hom}(x, y)$.
- To distinguish between $(\infty, 1)$ -categories and $(\infty, 2)$ -categories we will typically use bold font for the latter. For example, $\mathrm{Pr}_{\mathrm{St}}^L$ is the $(\infty, 1)$ -category of presentable stable ∞ -categories and left adjoints, while $\mathbf{Pr}_{\mathrm{St}}^L$ is the $(\infty, 2)$ -categorical version.
- Module categories are implicitly categories of right modules unless otherwise stated.

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1 Preliminaries

In this section, we collect a number of well-known structures that we will work with extensively in the paper. We recall aspects of the theory of (symmetric monoidal) stable presentable ∞ -categories, including Picard and Brauer spaces. We then discuss sheaves of presentable stable ∞ -categories, and review the connection between those sheaves that are locally constant and local systems of presentable stable ∞ -categories.

1.1 Stable presentable ∞ -categories

Let us start with some preliminaries on presentable ∞ -categories. We refer the reader in want of more details to [26, Section 5.5; 27, Section 4.8]. Let Pr^L denote the ∞ -category of presentable ∞ -categories and left adjoint functors. This admits a closed symmetric monoidal structure with tensor product denoted by $\otimes = \otimes^L$ and internal mapping objects denoted by $\mathrm{Fun}^L(-, -)$. The latter are just colimit-preserving functors from one presentable ∞ -category to another. We will typically restrict our attention to those presentable ∞ -categories that are stable, and let $\mathrm{Pr}_{\mathrm{St}}^L$ denote the full subcategory of Pr^L consisting of

³Or “ ∞ -groupoids”, “Kan complexes”, or “anima”, depending on your preferred terminology.

those. This inherits a closed symmetric monoidal structure which we denote by the same notation as before. The tensor product is sometimes referred to as the *Lurie tensor product*. The unit for the symmetric monoidal structure is Sp , the stable ∞ -category of spectra. Commutative algebra objects in $\mathrm{Pr}_{\mathrm{St}}^L$ are precisely symmetric monoidal presentable stable ∞ -categories whose tensor product preserves small colimits separately in each variable, also called *presentably symmetric monoidal ∞ -categories*.

Theorem 1.1 (universal property of the Lurie tensor product) *If $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞ -categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is the universal presentable ∞ -category receiving a functor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \otimes \mathcal{D}$ which preserves small colimits separately in each variable.*

If R is an $\mathbb{E}_\infty$ -ring, then Mod_R , being a presentably symmetric monoidal stable ∞ -category, is a commutative algebra object in $\mathrm{Pr}_{\mathrm{St}}^L$. Hence, we can make sense of modules over Mod_R using the Lurie tensor product.

Definition 1.2 Let R be an $\mathbb{E}_\infty$ -ring. The ∞ -category of *R -linear categories* is defined as

$$\mathrm{Cat}_R = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}_{\mathrm{St}}^L).$$

Note that Cat_R is itself a closed symmetric monoidal ∞ -category with unit Mod_R . By slight abuse of notation, we will denote the tensor product in Cat_R by $\otimes_R$. Note that $\mathrm{Cat}_{\mathbb{S}}$ is canonically identified with $\mathrm{Pr}_{\mathrm{St}}^L$ itself. Before we move on, here is a assortment of various remarks that we will use later on.

Remark 1.3 If R is an $\mathbb{E}_n$ -ring, then Mod_R is an $\mathbb{E}_{n-1}$ -algebra object in $\mathrm{Pr}_{\mathrm{St}}^L$, and Cat_R is an $\mathbb{E}_{n-2}$ -monoidal category. Hence, if we want Cat_R to be a monoidal category, then we need to assume that R is at least an $\mathbb{E}_3$ -ring. All of the examples of R that we have in mind in this paper will be $\mathbb{E}_\infty$ -rings though, so that Mod_R and Cat_R are themselves symmetric monoidal ∞ -categories.

Remark 1.4 By definition a presentable ∞ -category is κ -compactly generated for some infinite regular cardinal κ . For technical reasons, we will sometimes fix a sufficiently large uncountable regular cardinal κ , and work with presentable categories which are κ -compactly generated. One can form an ∞ -category Pr_{κ}^L of such categories. The point here is that while $\mathrm{Pr}_{\mathrm{St}}^L$ is not itself presentable, the ∞ -category $\mathrm{Pr}_{\mathrm{St},\kappa}^L$ is [27, Lemma 5.3.2.9]. Since $\mathrm{Pr}_{\mathrm{St}}^L$ is a filtered colimit of the presentable ∞ -categories $\mathrm{Pr}_{\mathrm{St},\kappa}^L$, this means that a lot of set-theoretical difficulties can be dealt with by using minor modifications. If we work over an $\mathbb{E}_\infty$ -ring R , as opposed to the sphere spectrum, we will use the notation

$$\mathrm{Cat}_{R,\kappa} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}_{\mathrm{St},\kappa}^L).$$

We denote by $\mathrm{Cat}_{R,\omega}$ the ∞ -category of *compactly generated R -linear categories*. If $R = \mathbb{S}$, we retain the use of the notation $\mathrm{Pr}_{\mathrm{St},\omega}^L$. The ∞ -category $\mathrm{Cat}_{R,\omega}$ inherits a symmetric monoidal structure from Cat_R .

Remark 1.5 There is also an ∞ -category Pr^R which consists of presentable ∞ -category and right adjoint functors. There is an equivalence

$$(\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathrm{Pr}^R$$

of ∞ -categories which is the identity on objects and sends a left adjoint to its right adjoint on the level of morphisms. We will write $\mathrm{Pr}_{\mathrm{St}}^R$ for the full subcategory of Pr^R consisting of those presentable ∞ -categories that are stable. We will also consider the ∞ -category $\mathrm{Pr}^{L,R}$ consisting of presentable ∞ -categories and functors that are *both* left and right adjoints. The full subcategory that consists of those presentable ∞ -categories that are stable will again be denoted $\mathrm{Pr}_{\mathrm{St}}^{L,R}$.

1.2 Picard and Brauer spaces

For a symmetric monoidal ∞ -category $\mathcal{C}$, we let $\mathrm{Pic}(\mathcal{C})$ denote the ∞ -groupoid of invertible objects in $\mathcal{C}$ and the equivalences between them. We refer to this as the *Picard space* of $\mathcal{C}$. Note that the Picard space is closed under the symmetric monoidal structure, so that it inherits the structure of an $\mathbb{E}_\infty$ -space. Moreover, this $\mathbb{E}_\infty$ -structure is grouplike.

Definition 1.6 If R is an $\mathbb{E}_\infty$ -ring, we will write

$$\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Mod}_R)$$

and refer to this as the *Picard space* of R .

We note that $\mathrm{Pic}(R)$ is essentially a delooping of the space of units $\mathrm{GL}_1(R)$. In particular, there is an equivalence

$$\mathrm{Pic}(R) \simeq \pi_0 \mathrm{Pic}(R) \times B \mathrm{GL}_1(R),$$

as spaces. As we remarked in the previous section, if R is an $\mathbb{E}_\infty$ -ring, then the ∞ -category of compactly generated R -linear categories is also symmetric monoidal, so we can make the following definition.

Definition 1.7 If R is an $\mathbb{E}_\infty$ -ring, we will write

$$\mathrm{Br}(R) = \mathrm{Pic}(\mathrm{Cat}_{R,\omega}),$$

and refer to this as the *Brauer space* of R .

The following result can be found in the literature; see for example [18, Proposition 5.7].

Proposition 1.8 *The inclusion of the identity component gives us a map $B \mathrm{Pic}(R) \rightarrow \mathrm{Br}(R)$ which induces an equivalence*

$$\mathrm{Pic}(R) \simeq \Omega B \mathrm{Pic}(R) \xrightarrow{\sim} \Omega \mathrm{Br}(R)$$

on applying the loop space functor.

$\pi_n \mathrm{Br}(R)$	0	1	2	3	4	5	6
$\mathbb{S}$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
$H\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	0	0	0
Hk	0	$\mathbb{Z}$	$k^\times$	0	0	0	0
MU	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
KU	??	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
KO	??	$\mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Table 1: Table of homotopy groups $\pi_n \mathrm{Br}(R)$ for various common $\mathbb{E}_\infty$ -rings R . Here k is an arbitrary field.

Remark 1.9 For $R = \mathbb{S}$, the inclusion $B \mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Br}(\mathbb{S})$ is already an equivalence.

The homotopy groups of the Brauer space $\mathrm{Br}(R)$ agree with the shifted homotopy groups of R in high enough dimensions and reduces to étale cohomology invariants in low dimensions.

Proposition 1.10 [4, Corollary 7.13] *Let R be a connective $\mathbb{E}_\infty$ -ring. Then we have*

$$\pi_n \mathrm{Br}(R) = \begin{cases} H_{\mathrm{ét}}^1(\mathrm{Spec} \pi_0 R, \mathbb{Z}) \times H_{\mathrm{ét}}^2(\mathrm{Spec} \pi_0 R, \mathbb{G}_m) & \text{if } n = 0, \\ H_{\mathrm{ét}}^0(\mathrm{Spec} \pi_0 R, \mathbb{Z}) \times H_{\mathrm{ét}}^1(\mathrm{Spec} \pi_0 R, \mathbb{G}_m) & \text{if } n = 1, \\ (\pi_0 R)^\times & \text{if } n = 2, \\ \pi_{n-2} R & \text{if } n \geq 3. \end{cases}$$

In particular, note that if R is a classical discrete commutative ring, then the Brauer space has at most three nontrivial homotopy groups.

For reference, we list some low-dimensional homotopy groups for various common $\mathbb{E}_\infty$ -rings in Table 1. We assembled this table from various sources, including [4, Theorem 7.16; 11; 12, Table 3].

1.3 The $(\infty, 2)$ -category of presentable stable ∞ -categories

The $(\infty, 1)$ -category of stable presentable ∞ -categories is simply a shadow of higher structure. Indeed, note that given any two presentable ∞ -categories $\mathcal{C}$ and $\mathcal{D}$, there exists an internal mapping object $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$, the ∞ -category of left adjoint functors between them. This suggests that there is a natural enrichment of $\mathrm{Pr}_{\mathrm{St}}^L$ in ∞ -categories, giving it the structure of an $(\infty, 2)$ -category [17]. A detailed description of the current status of the theory of $(\infty, 2)$ -categories would take us very far afield, and is not strictly necessary for results of this paper. There are several models for $(\infty, 2)$ -categories at this point, but we will typically think of these as enriched over ∞ -categories, the same way as ∞ -categories are enriched over ∞ -groupoids.

Definition 1.11 By [16, Chapter I.1], there exists a symmetric monoidal $(\infty, 2)$ -category consisting of stable cocomplete ∞ -categories. We let $\mathbf{Pr}_{\mathrm{St}}^L$ denote the full sub- $(\infty, 2)$ -category of the above, spanned by the presentable stable ∞ -categories.

By the discussion in [21, Section 4.4], for any $(\infty, 2)$ -category $\mathbf{C}$ and any $\mathbb{E}_\infty$ -algebra R in the core ∞ -category $\iota_1 \mathbf{C}$, there exists an $(\infty, 2)$ -category $\mathbf{Mod}_R$ for which

$$\iota_1 \mathbf{Mod}_R \simeq \mathbf{Mod}_R(\iota_1 \mathbf{C}).$$

Applying this to $\mathbf{C} = \mathbf{Pr}_{\mathrm{St}}^L$, we obtain an $(\infty, 2)$ -category organizing the totality of R -linear categories, which we denote by $\mathbf{Cat}_R$, again using bold notation.

Definition 1.12 As $\mathbf{Cat}_{R,\omega}$ is an full subcategory of $\mathbf{Cat}_R$, and since the latter has an $(\infty, 2)$ -categorical enhancement, we conclude that compactly generated R -linear categories form an $(\infty, 2)$ -category as well, which we denote by $\mathbf{Cat}_{R,\omega}$.

1.4 Sheaves of stable presentable ∞ -categories

Some of the main objects of this paper will arise via sheaves on a fixed topological space valued in presentable stable ∞ -categories. In this section, we go over what this means. First, we recall what a sheaf on an arbitrary (small) ∞ -category $\mathcal{C}$ is. We refer to [26, Chapter 5; 29, Section 1.3.1] for details.

Definition 1.13 Let $\mathcal{C}$ be a κ -small ∞ -category. Then we define the ∞ -category of *presheaves on $\mathcal{C}$* to be the functor category

$$\mathbf{PSh}(\mathcal{C}) = \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, S).$$

Suppose that $\mathcal{C}$ comes equipped with a Grothendieck topology. In this case, one might want to impose a descent condition with respect to the chosen topology. The language of localizations of ∞ -categories is well-suited for describing this passage.

Definition 1.14 We define the ∞ -category of *sheaves on $\mathcal{C}$* as the left-exact accessible localization

$$\mathbf{Shv}(\mathcal{C}) \hookrightarrow \mathbf{PSh}(\mathcal{C}).$$

This is the full subcategory of functors which satisfy descent with respect to covers given by the Grothendieck topology on $\mathcal{C}$.

Remark 1.15 As $\mathbf{PSh}(\mathcal{C})$ and $\mathbf{Shv}(\mathcal{C})$ are presentable ∞ -categories, we may stabilize by tensoring along

$$- \otimes \mathrm{Sp}: \mathbf{Pr}^L \rightarrow \mathbf{Pr}_{\mathrm{St}}^L$$

to obtain ∞ -categories of presheaves and sheaves on $\mathcal{C}$ valued in the ∞ -category of spectra. We use the notation $\mathbf{PSh}_{\mathrm{Sp}}(\mathcal{C})$ and $\mathbf{Shv}_{\mathrm{Sp}}(\mathcal{C})$ for these respective stable ∞ -categories.

Now we specialize, essentially once and for all, on the case $\mathcal{C} = \mathbf{Open}(B)$, the category of open subsets of a fixed topological space B , with morphisms given by inclusions of open sets. By identifying this with its nerve, we view this as an ∞ -site, with Grothendieck topology determined by the usual topology

on B . Recall that a continuous map $f: B \rightarrow A$ of topological spaces gives rise to a restriction morphism $\text{Open}(A) \rightarrow \text{Open}(B)$ of sites determined by the assignment $V \mapsto f^{-1}(V)$. We now introduce the ∞ -category of sheaves of κ -compactly generated R -linear categories on a fixed topological space B .

Definition 1.16 • Denote the ∞ -category of $\text{Cat}_{R,\kappa}$ -valued presheaves by

$$\text{PSh}_{\text{Cat}_{R,\kappa}}(B) = \text{Fun}(\text{Open}(B)^{\text{op}}, \text{Cat}_{R,\kappa}).$$

• Denote the full subcategory of $\text{Cat}_{R,\kappa}$ -valued sheaves with respect to the aforementioned Grothendieck topology by

$$\text{Shv}_{\text{Cat}_{R,\kappa}}(B) = \text{Fun}^{\tau}(\text{Open}(B)^{\text{op}}, \text{Cat}_{R,\kappa}).$$

The subscript τ on notation on the right denotes the accessible localization of the ∞ -category of presheaves corresponding to imposing descent along covers in the topology of B .

We shall refer to objects of $\text{Shv}_{\text{Cat}_{R,\kappa}}(B)$ as *R -linear categorical sheaves on B* . Recall that the fully faithful inclusion of sheaves into presheaves admits a left adjoint

$$L: \text{PSh}_{\text{Cat}_{R,\kappa}}(B) \rightarrow \text{Shv}_{\text{Cat}_{R,\kappa}}(B)$$

that is referred to as *sheafification*. This exhibits $\text{Shv}_{\text{Cat}_{R,\kappa}}(B)$ as an accessible localization.

Remark 1.17 Let $\mathcal{X}$ denote the ∞ -topos of sheaves of spaces on B . Then by [27, Proposition 4.8.1.17], one has the description

$$\text{Shv}_{\text{Cat}_{R,\kappa}}(B) \simeq \mathcal{X} \otimes \text{Cat}_{R,\kappa} \simeq \text{Fun}^R(\mathcal{X}^{\text{op}}, \text{Cat}_{R,\kappa})$$

where the middle and left terms denote the Lurie tensor product of $\mathcal{X}$ and $\text{Cat}_{R,\kappa}$ and the presentable ∞ -category of right adjoint functors between $\mathcal{X}$ and $\text{Cat}_{R,\kappa}$. This uses presentability of $\text{Cat}_{R,\kappa}$.

The category $\text{PSh}_{\text{Cat}_{R,\kappa}}(B)$ comes equipped with the pointwise symmetric monoidal structure. Note that the unit of this symmetric monoidal structure is the constant functor with value Mod_R . The sheafification L is compatible with the symmetric monoidal structure, with the result that $\text{Shv}_{\text{Cat}_{R,\kappa}}(B)$ itself inherits a symmetric monoidal structure, simply given as the sheafification of the pointwise tensor product [29, Proposition 1.3.4.6]. The sheafification functor is then a symmetric monoidal functor. We would like to understand the unit of this symmetric monoidal structure.

Definition 1.18 The *structure sheaf of parametrized R -modules* is the sheaf

$$\mathcal{O}_{B,R}: \text{Open}(B)^{\text{op}} \rightarrow \text{Cat}_R, \quad U \mapsto \text{Fun}(\text{Sing}(U), \text{Mod}_R).$$

We remark that the structure sheaf is manifestly a sheaf because if $\mathcal{U}_{\bullet} \rightarrow U$ is some open cover of U , then we have the equivalence

$$\mathcal{O}_{B,R}(U) = \text{Fun}(U, \text{Mod}_R) \simeq \text{Fun}(\text{colim}_i \mathcal{U}_i, \text{Mod}_R) \simeq \lim_i \text{Fun}(\mathcal{U}_i, \text{Mod}_R) = \lim_i \mathcal{O}(U_i).$$

We claim that this sheaf is equivalent to the sheafification of the constant functor.

Proposition 1.19 *There is an equivalence $L(\text{const}_{\text{Mod}_R}) \simeq \mathcal{O}_{B,R}$. Hence $\mathcal{O}_{B,R}$ is the unit for the symmetric monoidal structure on $\text{Shv}_{\text{Cat}_{R,\kappa}}(B)$.*

Proof We first claim that there exists a map of presheaves $\text{const}_{\text{Mod}_R} \rightarrow \mathcal{O}_{B,R}$, which for each open set U corresponds to a functor $\text{Mod}_R \rightarrow \text{Fun}(\text{Sing}(U), \text{Mod}_R)$. Upon taking adjoints, this gives rise to a map $L(\text{const}_{\text{Mod}_R}) \rightarrow \mathcal{O}_{B,R}$ in $\text{Shv}_{\text{Cat}_{R,\kappa}}(B)$ which we would like to show is an equivalence. As usual, it is enough to check this on stalks. Of course, the stalk of $L(\text{const}_{\text{Mod}_R})$ is exactly Mod_R , as it is determined by the stalks of the underlying presheaf. It is easy to see that the stalk of $\mathcal{O}_{B,R}$ at some point is Mod_R , as well. Thus the induced map $L(\text{const}_{\text{Mod}_R}) \rightarrow \mathcal{O}_{B,R}$ is an equivalence on stalks, and since taking stalks for all points is a conservative functor, this shows that the map is itself an equivalence. $\square$

1.5 Parametrized categories and ambidexterity

Before specializing to the situation at hand, we start with a basic treatment on parametrized stable ∞ -categories in general. In contrast to the last section, note that B will be an object in the ∞ -category of spaces, rather than a topological space, in what follows.

Definition 1.20 Let B be a space and let R denote a fixed $\mathbb{E}_\infty$ -ring. We define a *parametrized R -linear category* to be a functor $F: B \rightarrow \text{Cat}_R$.

Such gadgets naturally assemble to form an ∞ -category $\text{Fun}(B, \text{Cat}_R)$, which we refer to as the ∞ -category of parametrized R -linear categories on B . We remark that this inherits a pointwise symmetric monoidal structure from the symmetric monoidal structure on Cat_R . Note that if $F: B \rightarrow \text{Cat}_R$ is a parametrized R -linear category then, as Cat_R is both complete and cocomplete, we may form the limit and colimit of F . In fact, these two will agree, due to the phenomenon known as ambidexterity. This is not a new result: it appears for example in [20, Example 4.3.11]. However, for ease of reference, we state it below with a proof.

Proposition 1.21 (ambidexterity) *Given a parametrized R -linear category $F: B \rightarrow \text{Cat}_R$ there is a natural equivalence*

$$\text{colim}_B F \simeq \lim_B F.$$

Proof Without loss of generality, assume that $R = \mathbb{S}$. The limit of the parametrized R -linear category can be computed in Cat_∞ since the inclusion $\text{Pr}_{\text{St}}^L \rightarrow \text{Cat}_\infty$ preserves all small limits [26, Proposition 5.5.3.13]. Now, the colimit of the diagram F can be computed as the limit along the opposite diagram

$$F^{\text{op}}: B^{\text{op}} \rightarrow (\text{Pr}_{\text{St}}^L)^{\text{op}} \simeq \text{Pr}_{\text{St}}^R.$$

Again in this case, the inclusion $\text{Pr}_{\text{St}}^R \rightarrow \text{Cat}_\infty$ preserves all small limits [26, Theorem 5.5.3.18], so in order to compute the colimit of F , we find ourselves computing the limit of the diagram $B^{\text{op}} \rightarrow \text{Pr}_{\text{St}}^R \rightarrow \text{Cat}_\infty$.

Since B is a Kan complex, there is an equivalence $\alpha: B \simeq B^{\text{op}}$, identifying each path $\iota^{\text{op}}: x_1 \rightarrow x_0$ in B^{op} with $\iota^{-1}: x_1 \rightarrow x_0$ in B . Since the functor F factors through the core groupoid of Pr_{St}^L , and the natural diagram

$$\begin{array}{ccccccc} B & \longrightarrow & (\text{Pr}_{\text{St}}^L)^{\simeq} & \longrightarrow & \text{Pr}_{\text{St}}^L & \longrightarrow & \text{Cat}_{\infty} \\ \downarrow \simeq & & \downarrow \simeq & & & \nearrow & \\ B^{\text{op}} & \longrightarrow & ((\text{Pr}_{\text{St}}^L)^{\simeq})^{\text{op}} & \longrightarrow & \text{Pr}_{\text{St}}^R & & \end{array}$$

commutes, we are able to identify the diagrams F and F^{op} in Cat_{∞} . Putting all this together, we obtain the equivalence, as wanted. $\square$

As a consequence of Proposition 1.21, we will unambiguously refer to the colimit and the limit of a parametrized R -linear category as the *global sections*. If we have no reason to worry about whether we are referring to the colimit or the limit, we will denote the global sections functor by

$$\Gamma: \text{Fun}(B, \text{Cat}_R) \rightarrow \text{Cat}_R.$$

As an important consequence of the above ambidexterity result, we may identify the following adjoint pairs on global sections. For ease of notation, we will abusively denote objects in the ∞ -category ΓF by $(x_b)_{b \in B}$, where $x_b \in Fb$.

Lemma 1.22 *Let $F: B \rightarrow \text{Pr}_{\text{St}}^L$ be a parametrized category. The mapping space of ΓF is given by*

$$\text{Map}_{\Gamma F}((x_b)_{b \in B}, (y_b)_{b \in B}) \simeq \lim_{b \in B} \text{Map}_{Fb}(x_b, y_b).$$

Proof The mapping space in the category ΓF is described via the pullback diagram

$$\begin{array}{ccc} \text{Map}_{\Gamma F}((x_b)_{b \in B}, (y_b)_{b \in B}) & \longrightarrow & \text{Fun}(\Delta^1, \Gamma F) \\ \downarrow & & \downarrow (d_1, d_0) \\ \{*\} & \xrightarrow{((x_b)_{b \in B}, (y_b)_{b \in B})} & (\Gamma F)^{\text{op}} \times \Gamma F. \end{array}$$

By using the description of Γ as a limit and the appropriate commutativity of limits with pullbacks and products, we obtain the result. $\square$

Proposition 1.23 *Let B be a Kan complex and consider the functor*

$$\Gamma: \text{Fun}(B, \text{Pr}_{\text{St}}^L) \rightarrow \text{Pr}_{\text{St}}^L$$

and suppose we are given natural transformations $\lambda: F \rightarrow G$ and $\rho: G \rightarrow F$ of functors $F, G: B \rightarrow \text{Pr}_{\text{St}}^L$ such that $\lambda_b: Fb \rightarrow Gb$ is left adjoint to $\rho_b: Gb \rightarrow Fb$ for all $b \in B$. Then

$$\Gamma \lambda: \Gamma F \rightarrow \Gamma G \quad \dashv \quad \Gamma \rho: \Gamma G \rightarrow \Gamma F.$$

Proof Let us think of and denote objects in the categories ΓF and ΓG as families (x_b) where $x_b \in Fb$ and (y_b) where $y_b \in Gb$. Using this notation, we have that

$$\Gamma \lambda: \Gamma F \rightarrow \Gamma G, \quad (x_b)_{b \in B} \mapsto (\lambda_b(x_b))_{b \in B}.$$

Using the description of mapping spaces in the categories ΓF and ΓG , as well as the adjunction properties we obtain

$$\begin{aligned} \mathrm{Map}_{\Gamma G}((\Gamma\lambda)((x_b)_{b \in B}), (y_b)_{b \in B}) &\simeq \lim_B \mathrm{Map}_{Gb}(\lambda_b(x_b), y_b) \simeq \lim_B \mathrm{Map}_{Fb}(x_b, \rho_b(y_b)) \\ &\simeq \mathrm{Map}_{\Gamma F}((x_b)_{b \in B}, (\Gamma\rho)((y_b)_{b \in B})), \end{aligned}$$

which finishes the proof. $\square$

Proposition 1.24 *Let X be a Kan complex and consider the functor*

$$\mathrm{colim}_X \simeq \lim_X: \mathrm{Fun}(X, \mathrm{Pr}_{\mathrm{St}}^R) \rightarrow \mathrm{Pr}_{\mathrm{St}}^R.$$

Suppose we are given natural transformations $\lambda: F \rightarrow G$ and $\rho: G \rightarrow F$ of functors $F, G: X \rightarrow \mathrm{Pr}_{\mathrm{St}}^R$ such that $\lambda_x: Fx \rightarrow Gx$ is left adjoint to $\rho_x: Gx \rightarrow Fx$ for all $x \in X$. Then

$$\lim_X \lambda: \lim_X F \rightarrow \lim_X G \quad \dashv \quad \lim_X \rho: \lim_X G \rightarrow \lim_X F.$$

Proof This is similar to before. $\square$

As corollaries to ambidexterity, we also have the following results, that we will make heavy use of in the latter sections of this paper.

Proposition 1.25 *Let $f: A \rightarrow B$ be a map of spaces and let $F: B \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ be a parametrized category. The following holds:*

(1) *There is a functor*

$$f^*: \lim_B F \rightarrow \lim_A f^* F$$

which is a left adjoint.

(2) *There is a functor*

$$f_!: \mathrm{colim}_A f^* F \rightarrow \mathrm{colim}_B F$$

which is a left adjoint. Moreover, under ambidexterity its right adjoint can be identified with the functor f^* above.

Proof (1) Given a limiting cocone $B^\triangleleft \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ extending F , there is canonically an object of $\mathrm{Pr}_{\mathrm{St}/f^*F}^L$, so that we get a map $B^\triangleleft \rightarrow A^\triangleleft \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ by the universal property of $A^\triangleleft \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ as the terminal object in $\mathrm{Pr}_{\mathrm{St}/f^*F}^L$. Hence we get a map

$$f^*: \lim_B F \rightarrow \lim_A f^* F$$

in $\mathrm{Pr}_{\mathrm{St}}^L$, which is hence a left adjoint.

(2) The construction of $f_!$ is essentially dual to the construction of f^* above, using functoriality of the colimit construction. What is left to show is that $f_!$ is the left adjoint to f^* constructed above, under

ambidexterity. Since $f_!$ is a map in $\mathrm{Pr}_{\mathrm{St}}^L$ it must have a right adjoint. This right adjoint is obtained by passing to the opposite category $(\mathrm{Pr}_{\mathrm{St}}^L)^{\mathrm{op}} \simeq \mathrm{Pr}_{\mathrm{St}}^R$ where we obtain a map

$$\lim_B F \rightarrow \lim_A f^* F.$$

This map satisfies the same universal property as the map f^* constructed above, hence they must be naturally equivalent. $\square$

Note that the above theorem provides us with functors

$$\mathrm{colim}: \mathcal{S}_{/\mathrm{Pr}_{\mathrm{St}}^L} \rightarrow \mathrm{Pr}_{\mathrm{St}}^L \quad \text{and} \quad \lim: (\mathcal{S}_{/\mathrm{Pr}_{\mathrm{St}}^L})^{\mathrm{op}} \rightarrow \mathrm{Pr}_{\mathrm{St}}^L,$$

where the colimit functor sends the map $f: A \rightarrow B$ in $\mathcal{S}_{/\mathrm{Pr}_{\mathrm{St}}^L}$ to the functor $f_!$ and the limit functor sends it to f^* . For reference, let us state the following results regarding monoidality of the first functors.

Lemma 1.26 *The functor*

$$\mathrm{colim}: \mathcal{S}_{/\mathrm{Pr}_{\mathrm{St}}^L} \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$$

is symmetric monoidal. Here, the symmetric monoidal structure on the source category is obtained by intertwining the cartesian product on $\mathcal{S}$ with the Lurie tensor product. In particular, if $F: A \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ and $G: B \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$ are two parametrized categories, then their tensor product is the parametrized category

$$F \boxtimes G: A \times B \xrightarrow{F \otimes G} \mathrm{Pr}_{\mathrm{St}}^L \times \mathrm{Pr}_{\mathrm{St}}^L \xrightarrow{\otimes} \mathrm{Pr}_{\mathrm{St}}^L.$$

Proof This follows from the Lurie tensor product preserving small colimits. $\square$

1.6 Locally constant sheaves and parametrized objects

Let $\mathcal{C}$ be a presentable ∞ -category, which for the sake of specificity, the reader may take to be $\mathrm{Cat}_{R,\kappa}$. We now invoke a recurring theme in the story, namely that the homotopy theory of *locally constant sheaves* of objects in $\mathcal{C}$ agrees with the homotopy theory of B -parametrized objects of $\mathcal{C}$. After recalling the relevant notions, we review this correspondence.

Definition 1.27 Let B be a topological space and let $\mathcal{C}$ be a presentable ∞ -category.

- We say that a sheaf $\mathcal{F}: \mathrm{Open}(B)^{\mathrm{op}} \rightarrow \mathcal{C}$ is *constant* if it lies in the image of the canonical morphism $\mathcal{C} \rightarrow \mathrm{Shv}_{\mathcal{C}}(B)$ induced by the terminal geometric morphism $\mathcal{S} \rightarrow \mathrm{Shv}_{\mathcal{C}}(B)$ of ∞ -topoi.
- We say $\mathcal{F}$ is *locally constant* if there is a jointly surjective open cover $\coprod \phi_\alpha: \coprod U_\alpha \rightarrow B$, such that for every α , $\phi_\alpha^*(\mathcal{F})$ is constant in $\mathrm{Shv}_{\mathcal{C}}(U_\alpha)$.

We denote the full subcategory of $\mathrm{Shv}_{\mathcal{C}}(B)$ consisting of those sheaves that are locally constant by $\mathrm{Shv}_{\mathcal{C}}(B)^{\mathrm{lc}}$.

Theorem 1.28 *Let B be a locally contractible topological space.⁴ Then there is an equivalence*

$$\mathrm{Shv}_{\mathrm{Cat}_{R,\kappa}}(B)^{\mathrm{lc}} \simeq \mathrm{Fun}(\mathrm{Sing}(B), \mathrm{Cat}_{R,\kappa}).$$

Proof A proof of this fact in the context of sheaves of spaces may be found in [27, Theorem A.1.15]. In particular, it is there shown that there is an equivalence

$$\mathcal{S}_{/B} \simeq \mathrm{Shv}(B)^{\mathrm{lc}}$$

of parametrized spaces over B with locally constant sheaves of spaces on B . We can tensor this equivalence with the presentable ∞ -category $\mathcal{C}$ using the symmetric monoidal structure of Pr^L . Using the fact that there are equivalences

$$\mathcal{S}_{/B} \otimes^L \mathcal{C} \simeq \mathrm{Fun}(B, \mathcal{C}), \quad \mathrm{Shv}(B)^{\mathrm{lc}} \otimes^L \mathcal{C} \simeq \mathrm{Fun}^R(\mathrm{Shv}(B)^{\mathrm{lc}, \mathrm{op}}, \mathcal{C}) \simeq \mathrm{Shv}_{\mathrm{Cat}_{R,\kappa}}(B)^{\mathrm{lc}},$$

we obtain the desired equivalence by setting $\mathcal{C} = \mathrm{Cat}_{R,\kappa}$. $\square$

Remark 1.29 The above theorem holds true even if we replace $\mathrm{Cat}_{R,\kappa}$ with any ∞ -category $\mathcal{C}$ admitting small limits, with a different proof.

Remark 1.30 We remind the reader that $\mathcal{C}$ -valued sheaves are not homotopy invariant. That is to say that $\mathrm{Shv}_{\mathcal{C}}(B) \not\simeq \mathrm{Shv}_{\mathcal{C}}(A)$ even though $B \simeq A$. Specializing to our context where $\mathcal{C} = \mathrm{Cat}_{R,\kappa}$, and noting that $\mathrm{Fun}(B, \mathrm{Cat}_R)$ is manifestly homotopy invariant, we interpret the above as saying that ∞ -categories of locally constant sheaves of categories are themselves homotopy invariant.

2 Haunts

In this section, we introduce the notion of a haunt. These were first introduced by Douglas [12]. Roughly, a haunt over a space B is to be thought of as a bundle of ∞ -categories over B whose fiber is the ∞ -category of spectra. We will consider haunts mainly from two perspectives: sheaves of categories, and local systems of categories. Having the two perspectives on haunts in mind is usually quite useful. Thinking about haunts in terms of sheaves is well-suited for constructions and geometric intuition, while thinking about haunts in terms of local systems comes in handy for abstract properties and proving theorems. In the last subsection, we look at explicit examples of haunts over a few different spaces, as well as examples of haunts that naturally appear in the context of various flavors of Floer theory.

2.1 PoV I: invertible sheaves of categories

Let us fix a topological space B and an $\mathbb{E}_{\infty}$ -ring R . Recall that the structure sheaf of R -modules parametrized over B is the sheaf of categories

$$\mathcal{O}_{B,R}: \mathrm{Open}(B)^{\mathrm{op}} \rightarrow \mathrm{Cat}_R, \quad U \mapsto \mathrm{Fun}(\mathrm{Sing}(U), \mathrm{Mod}_R).$$

⁴All CW-complexes and paracompact manifolds are locally contractible, so this is a very large class of topological spaces, which encompasses any applications that we have in mind.

This is also the sheafification of the constant functor with value Mod_R , and is hence the unit of the symmetric monoidal structure on $\text{Shv}_B(\text{Cat}_{R,\kappa})$.

Definition 2.1 Let $\mathcal{F}$ be sheaf of R -linear categories over B . We say $\mathcal{F}$ is *locally free of rank 1*, if for every point in B there is an open set U containing the point, for which $\mathcal{F}(U) \simeq \mathcal{O}_{B,R}(U)$.

Definition 2.2 We will refer to a locally free rank 1 modules over $\mathcal{O}_{B,R}$ as an R -*haunt* over B . Let us set the notation $\text{Haunt}_{B,R}$ to denote the space of R -haunts on B . If $R = \mathbb{S}$, we will drop the $\mathbb{S}$ and simply call these *haunts*.

Remark 2.3 We could have more generally defined haunts as invertible sheaves of categories, namely, as objects in the Picard ∞ -groupoid

$$\text{Pic}(\text{Mod}_{\mathcal{O}_{B,R}}(\text{Shv}_{\text{Cat}_{R,\kappa}}(B))).$$

Every locally free rank 1 module is invertible with respect to the tensor product, since invertibility can be checked on stalks. However, it is not always true that invertible objects are locally free of rank 1. We will return to the discussion of this subtle point.

To visualize haunts, we can think of them as categorified versions of line bundles. In particular, one may construct haunts in the same way that line bundles on a space B are constructed, by picking an open cover of B and gluing together trivial bundles with transition maps over intersections. We will make this precise after introducing parametrized category perspective on haunts.

2.2 PoV II: parametrized categories

We now switch to the second point of view of haunts, by the following direct analogue of Theorem 1.28 in this situation.

Theorem 2.4 *If B is locally contractible, then we have*

$$\text{Haunt}_{B,R} \simeq \text{Map}(B, B \text{Pic}(R)).$$

Proof Let $\mathcal{H}$ be a haunt. By assumption, we can find a cover $\{U_i\}$ of B by contractible spaces. Since U is contractible, we have $\text{Fun}(U, \text{Sp}) \simeq \text{Sp}$. Hence, in this case any haunt is a locally constant sheaf of categories, taking the constant value Sp . By Theorem 1.28, there is an equivalence between locally constant sheaves on B valued in Pr_{St}^L and local systems $\text{Fun}(\text{Sing}(B), \text{Pr}_{\text{St}}^L)$. As our sheaf is constant with value Sp , it takes values in the connected component of the core ∞ -groupoid containing Sp , and the local system necessarily factors through $B \text{Aut}(\text{Sp}) \simeq B \text{Pic}(\mathbb{S})$. $\square$

Remark 2.5 If we want to consider invertible $\mathcal{O}_{B,R}$ -modules, rather than locally free rank-1 ones, the corresponding space will be equivalent to $\text{Map}(B, \text{Br}(R))$ instead. Keeping in mind that our main

motivation comes from Floer homotopy theory, where the map corresponding to the relevant haunt always seems to factor through the real or complex J-homomorphism

$$B(\mathbb{Z} \times BO) \rightarrow B \operatorname{Pic}(\mathbb{S}) \quad \text{or} \quad B(\mathbb{Z} \times BU) \rightarrow B \operatorname{Pic}(\mathbb{S}),$$

we will exclusively deal with the locally free rank-1 case. It is possible that applications of twisted spectra to settings where we have to look at invertible modules, rather than locally free modules, exists within the mathematical wilderness, but as of the time of writing, the authors of the paper know of no such examples. As a potential application, we mention the introduction of [37], where the author muses, with reference to hints in [8], that the Brauer space may be the appropriate target for an elliptic J-homomorphism. Hence, it is possible that one needs to consider invertible modules in applications to elliptic cohomology.

Due to the above theorem, we will sometimes abusively refer to maps $B \rightarrow B \operatorname{Pic}(R)$ also as haunts, but we will try to stick to the terminology *twists on the space B* if we want to make the distinction clear. Given a haunt $\mathcal{H}$, we will refer to its corresponding twist as $\tau_{\mathcal{H}}$. Given a twist $\tau: B \rightarrow B \operatorname{Pic}(R)$, we will refer to its corresponding haunt as $\mathcal{H}_{(B, \tau)}$. In particular, note that the identity map $1: B \operatorname{Pic}(R) \rightarrow B \operatorname{Pic}(R)$ provides us with a haunt over the classifying space of the Picard space. We refer to this as the *universal R -haunt* and denote it by $\mathcal{U}_R = \mathcal{H}_{(B \operatorname{Pic}(R), 1)}$.

2.3 Haunts as categorified line bundles

Let us discuss in detail how one may construct haunts “by hand”, so that the reader may visualize these. Let B be a topological space and let $\{U_i\}$ denote an open cover of B . We let

$$p: \coprod_i U_i \rightarrow B$$

denote the jointly surjective map and let $N_{\bullet}(p)$ be the nerve of this epimorphism. This is a simplicial object in $\mathcal{S}$ for which there is an equivalence

$$\operatorname{colim}_{\Delta^{\operatorname{op}}} N_{\bullet}(p) \simeq B,$$

where the topological space is now regarded as an object in $\mathcal{S}$.

Remark 2.6 We remark that $B \operatorname{Pic}(\mathbb{S})$ may itself be written as the geometric realization of the colimit diagram $B \operatorname{Pic}_{\bullet}$ which is nerve of the map $* \rightarrow B \operatorname{Pic}(\mathbb{S})$. Thus, the map $B \rightarrow B \operatorname{Pic}(\mathbb{S})$ is equivalent to a map of colimit diagrams $N_{\bullet}(p) \rightarrow B \operatorname{Pic}_{\bullet}$.

Since the functor $\operatorname{Map}(-, B \operatorname{Pic})$ sends colimits of spaces to limits, we may express a haunt $\mathcal{H}_{(B, \tau)}$ corresponding to a twist $\tau: B \rightarrow B \operatorname{Pic}(\mathbb{S})$ as the totalization of an object in the cosimplicial space

$$(1) \quad \operatorname{Map}(N_{\bullet}(p), B \operatorname{Pic}) := \operatorname{Map}(N_0(p), B \operatorname{Pic}) \rightrightarrows \operatorname{Map}(N_1(p), B \operatorname{Pic}) \rightrightarrows \operatorname{Map}(N_2(p), B \operatorname{Pic}) \cdots.$$

Composing levelwise with the inclusion $B \operatorname{Pic}(\mathbb{S}) \rightarrow \operatorname{Pr}_{\operatorname{St}}^L$, we may express the haunt $\mathcal{H}_{(B, \tau)}$ itself as an object in the cosimplicial ∞ -category

$$\operatorname{Fun}(N_{\bullet}(p), \operatorname{Pr}_{\operatorname{St}}^L) := (\operatorname{Fun}(N_0(p), \operatorname{Pr}_{\operatorname{St}}^L) \rightrightarrows \operatorname{Fun}(N_1(p), \operatorname{Pr}_{\operatorname{St}}^L) \rightrightarrows \operatorname{Fun}(N_2(p), \operatorname{Pr}_{\operatorname{St}}^L) \rightrightarrows \cdots).$$

Let us write

$$\mathcal{H} \simeq \lim_{\Delta} \mathcal{H}_{(N_i(p), \tau_i)}.$$

Note that since B is assumed to be locally contractible, we can always choose the cover so that it trivializes the twist, so without loss of generality we will assume that we have such a cover. In this case, each $\tau_i: N_i(p) \rightarrow B \operatorname{Pic}$ is nullhomotopic, and therefore gives rise to the trivial haunt on this restriction. Thus, every $\mathcal{H}_{N_i(p), \tau_i}$ will be equivalent to the trivial haunt $\mathcal{O}_{(N_i(p))}$. The cosimplicial structure abstractly determines equivalences on the restrictions of these haunts on n -fold intersections of the cover. Unpacking what this means, and using Remark 2.6 above, we see that in simplicial degree 1, we obtain maps

$$\coprod_{i,j} U_i \times_B U_j \rightarrow \operatorname{Pic}(\mathbb{S})$$

encoding coherent choices of parametrized invertible $\mathbb{S}$ -modules on intersections. Similarly, in degree 2, we obtain maps

$$\coprod_{i,j,k} U_i \times_B U_j \times_B U_k \rightarrow \operatorname{Pic}(\mathbb{S}) \times \operatorname{Pic}(\mathbb{S});$$

a moment's reflection reveals that this is exactly the cocycle condition on triple intersections.

2.4 Examples of haunts

In this section, we will construct some explicit examples of haunts. Recall that in order to construct examples of haunts on a space B , it is often enough to give gluing data on intersections $U_i \cap U_j$ of a trivializing open cover $\{U_i\}_{i \in I}$ in the form of a parametrized invertible spectrum.

Example 2.7 (haunts over the circle) Pick a open cover of the circle consisting of two open sets U_0 and U_1 with intersection consisting of two contractible connected components V_0 and V_1 . The gluing data then amounts to an equivalence v_i of the ∞ -category of spectra on each connected component V_i . Let us pick

$$v_0 = \operatorname{Id}: \operatorname{Sp} \rightarrow \operatorname{Sp} \quad \text{and} \quad v_1 = \Sigma^n: \operatorname{Sp} \rightarrow \operatorname{Sp}.$$

We refer to this haunt as the n^{th} *suspension haunt*. Since equivalence classes of haunts over S^1 are classified by homotopy classes of maps $S^1 \rightarrow B \operatorname{Pic}(\mathbb{S})$, that is, by $\pi_1 B \operatorname{Pic}(\mathbb{S}) \cong \mathbb{Z}$, there should be essentially one haunt for each integer n . Indeed, this is the n^{th} suspension haunt that we constructed above.

Example 2.8 (haunts over the 2-sphere) We cover the 2-sphere by two open disks U_0 and U_1 containing the northern hemisphere and the southern hemisphere, respectively. The intersection is then an equatorial band which is homotopy equivalent to S^1 , so we need to give an equivalence

$$r_{01}: \operatorname{Sp}^{S^1} \rightarrow \operatorname{Sp}^{S^1}.$$

There is one nontrivial S^0 -bundle over S^1 , namely the real Hopf fibration

$$S^0 \rightarrow S^1 \rightarrow S^1.$$

This gives us an invertible S^1 -spectrum that we denote by $\mathbb{S}^\eta$. The equivalence r_{01} is given by taking the smash product with $\mathbb{S}^\eta$. This is the only nontrivial haunt over S^2 there is, since

$$\pi_0 \operatorname{Map}(S^2, B \operatorname{Pic}(\mathbb{S})) = \pi_2 B \operatorname{Pic}(\mathbb{S}) \cong \mathbb{Z}/2.$$

Example 2.9 (haunts over the 3-sphere) We cover the 3-sphere by two open 3-disks U_0 and U_1 . The intersection $U_0 \cap U_1$ is then homotopy equivalent to a 2-sphere. Note that there is one nontrivial S^2 -bundle over S^2 , namely the bundle

$$S^2 \rightarrow S^3 \times_{S^1} S^2 \rightarrow S^2$$

where S^1 acts on S^2 by rotation and on S^3 via the Hopf action. This bundle gives us an invertible S^2 -spectrum T . The gluing equivalence

$$r_{01} : \operatorname{Sp}^{S^2} \rightarrow \operatorname{Sp}^{S^2}$$

is given by taking the smash product with this spectrum. This is the only nontrivial haunt over S^3 there is, since

$$\pi_0 \operatorname{Map}(S^3, B \operatorname{Pic}(\mathbb{S})) = \pi_3 B \operatorname{Pic}(\mathbb{S}) \cong \mathbb{Z}/2.$$

We will now give brief introduction of haunts, and phenomena related to haunts, that arise naturally in the setting of two different flavors of Floer theory: monopole Floer theory and symplectic Floer theory.

Example 2.10 (haunts in Seiberg–Witten Floer theory) Let us give a brief sketch on how haunts are constructed from Seiberg–Witten Floer theory. This is a condensed version of some of the results that will become available in the forthcoming article [6]. Some of the basic statements on 3-manifolds equipped with complex spin structures can be found in [25, Chapter 1]. One considers pairs $(Y, \mathfrak{s})$ where Y is an oriented riemannian 3-manifold and $\mathfrak{s}$ is a complex spin structure on Y . From this data one may canonically associate a class $[(Y, \mathfrak{s})] \in K^1(P)$ where

$$P = H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$$

is the so-called *Picard torus* of Y and $K^1(P)$ denotes complex topological K-theory of the Picard torus in cohomological degree 1. There are a couple of different ways of constructing this class, but to keep it simple let us give a construction of the class completely contained within complex K-theory. One auxiliary bundle which is convenient to use is the so-called *Poincaré bundle* $\mathcal{L} \rightarrow P \times Y$. This is the bundle whose total space is given as

$$\mathcal{L} = P \times \tilde{Y} \times \mathbb{C} / \pi_1(Y)$$

where the second term denotes the universal cover of Y , and where the fundamental group acts as

$$\gamma \cdot (a, x, z) = (a, \gamma \cdot x, \exp(ia(\gamma))z).$$

Here the notation in the second factor just refers to the ordinary action the fundamental group on the universal cover and the notation $a(\gamma)$ in the third factor comes from remembering that $a \in H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$

and that $H^1(Y; \mathbb{R}) \cong \text{Hom}(\pi_1(Y), \mathbb{R})$. We remark that the Poincaré bundle should be viewed as a family of flat line bundles over Y parametrized by P . Indeed, the following statements can be used to characterize it:

- From the point of view of the Riemann–Hilbert correspondence these line bundles with flat connection are coming from the family of unitary representations

$$\rho_a: \pi_1(Y) \rightarrow U(1), \quad \gamma \mapsto \exp(ia(\gamma)).$$

- One can check that the first Chern character of the Poincaré bundle is precisely the element in $H^2(P \times Y; \mathbb{Z})$ corresponding to the identity morphism on $\pi_1(P)$ under the isomorphism

$$\text{Hom}(\pi_1(P), \pi_1(P)) \cong H^1(P; \mathbb{Z}) \otimes H^1(Y; \mathbb{Z}).$$

With this bundle in the back of our head, let us construct the wanted K-theory class. The data of the complex spin structure $\mathfrak{s}$ can alternatively be phrased in terms of a hermitian rank 2 bundle $S \rightarrow Y$, the spinor bundle, which comes equipped with Clifford multiplication. Note that we can consider the isomorphism class $[S]$ of the spinor bundle as an object of $K^0(Y)$ and that we can consider the isomorphism class $[\mathcal{L}]$ of the Poincaré bundle as an object of $K^0(P \times Y)$. Let us consider the projections $p: P \times Y \rightarrow P$ and $q: P \times Y \rightarrow Y$. Note that we have a pullback map

$$q^*: K^0(Y) \rightarrow K^0(P \times Y).$$

Moreover, since Y is equipped with a complex spin structure, which is equivalent to an orientation with respect to complex topological K-theory by [5], we are provided with a umkehr map

$$p_!: K^0(P \times Y) \rightarrow K^{-3}(P) \cong K^1(P).$$

We define

$$[(Y, \mathfrak{s})] = p_!(q^*([S]) \otimes [\mathcal{L}]) \in K^1(P) = [P, U].$$

We note that the same class in K-theory can be obtained through other methods, like for example applying index theory to the family of Dirac operators associated to $(Y, \mathfrak{s})$. To get a well-defined haunt on the Picard torus, we use Bott periodicity and the complex J-homomorphism to end up with a twist

$$\tau_{(Y, \mathfrak{s})}: P \rightarrow U \simeq B(\mathbb{Z} \times BU) \rightarrow B \text{Pic}(\mathbb{S}).$$

In particular, this map is not trivializable when Y is the 3-torus, and one needs to take this into account when constructing stable homotopy types in that situation.

Example 2.11 (haunts in Lagrangian Floer theory) For a lagrangian submanifold L in some symplectic manifold, we can construct the so-called *stable Gauss map*

$$\gamma: L \rightarrow U/O.$$

This gives rise to a twist on L via real Bott periodicity and the real J-homomorphism

$$\tau: L \rightarrow U/O \simeq B(\mathbb{Z} \times BO) \rightarrow B \text{Pic}(\mathbb{S}).$$

We remark that for a commutative ring k , the composition

$$\tau_{Hk}: L \rightarrow B \operatorname{Pic}(\mathbb{S}) \rightarrow B \operatorname{Pic}(Hk)$$

picks up the Maslov class on π_1 and the second relative Stiefel–Whitney class on π_2 . The possible nontriviality of this map explains why certain Floer homology theories are not integrally graded. It is known that the map τ_{Hk} is zero on homotopy groups when dealing with exact lagrangians in cotangent bundles; see for example [2] and [22] for proofs using two different methods.

3 Twisted spectra

In this section, we introduce the focal notion of study in this paper, namely twisted spectra. Again, twisted spectra are global sections of haunts, and since we have two main perspectives on haunts, this means that a priori we also have two perspectives on twisted spectra. Moreover, it turns out that there is a third perspective: twisted spectra are simply modules over Thom spectra. While this is certainly a fruitful perspective, much is known about modules categories and Thom spectra, viewing twisted spectra solely from this perspective tends to obscure some important structural insights on how twisted spectra behave. In this section, we will discuss such structural results, such as symmetric monoidal structures, functoriality with respect to maps of spaces, and more.

3.1 Twisted spectra

We now define the main objects of this paper.

Definition 3.1 A global section of an R -haunt is called a *twisted R -module*. If $R = \mathbb{S}$, we will simply call these objects *twisted spectra*.

In the original paper by Douglas, twisted spectra were referred to as “specters”, which also explains the terminology “haunt”. We will use both terminologies, but stick to twisted spectra for most of the time. The category of twisted R -modules associated to the R -haunt $\mathcal{H}$ is the global sections of the haunt. We will denote this category by

$$\operatorname{TwMod}_R(\mathcal{H}) = \Gamma \mathcal{H}.$$

Recall that taking global sections of a haunt over a space B corresponds to taking the limit of the corresponding parametrized category. Hence, we may alternatively define the ∞ -category of twisted spectra corresponding to a haunt $\mathcal{H}$ over B as

$$\operatorname{TwSp}(B, \tau_{\mathcal{H}}) := \lim_B (B \xrightarrow{\tau_{\mathcal{H}}} B \operatorname{Pic}(\mathbb{S}) \rightarrow \operatorname{Pr}_{\operatorname{St}}^L).$$

Similarly, the ∞ -category of twisted R -modules corresponding to an R -haunt $\tau: B \rightarrow B \operatorname{Pic}(R)$ is the limit

$$\operatorname{TwMod}(B, \tau) = \lim_B (B \xrightarrow{\tau} B \operatorname{Pic}(R) \rightarrow \operatorname{Cat}_R).$$

We will typically adopt this perspective for the remainder of this work. Note also, and this will be important, that ambidexterity tells us that we might as well have replaced the limit by the colimit in the above formula. Many results on twisted spectra arises from interpolating between thinking of $\mathrm{TwSp}(B, \tau)$ as a colimit and as a limit.

Remark 3.2 (visualizing twisted spectra) Let B be a space with a twist $\tau: B \rightarrow B \operatorname{Pic}(\mathbb{S})$. Following the discussion of Section 2.4, we pull back the twist τ along a trivializing open cover $p: \coprod_{i \in I} U_i \rightarrow B$ to obtain the following limit diagram in $\operatorname{Pr}_{\operatorname{St}}^L$:

$$(2) \quad \mathrm{TwSp}(B, \tau) \rightarrow (\mathrm{Fun}(N_0(p), \operatorname{Sp}) \rightrightarrows \mathrm{Fun}(N_1(p), \operatorname{Sp}) \rightrightarrows \mathrm{Fun}(N_2(p), \operatorname{Sp}) \cdots).$$

Thus, to specify an object in $\mathrm{TwSp}(B, \tau)$, it is enough to specify compatible choices of parametrized spectra on the open cover. Hence, a twisted spectrum X determines (untwisted) parametrized spectra $X_i \in \mathrm{Fun}(U_i, \operatorname{Sp})$, together with equivalences

$$(3) \quad X_i|_{U_i \cap U_j} \simeq \Phi_{ij}(X_j|_{U_i \cap U_j})$$

in $\mathrm{Fun}(U_i \cap U_j, \operatorname{Sp})$, where Φ_{ij} is an autoequivalence of $\mathrm{Fun}(U_i \cap U_j, \operatorname{Sp})$. Furthermore, these autoequivalences are determined by invertible spectra satisfying a cocycle condition, so that in fact

$$X_i|_{U_i \cap U_j} \simeq \mathcal{L}_{ij} \otimes X_j|_{U_i \cap U_j}$$

for $\mathcal{L}_{ij} \in \operatorname{Map}(U_i \cap U_j, \operatorname{Pic}(\mathbb{S}))$.

Remark 3.3 We expect that twisted spectra should be related to the notion of twisted vector bundles in a natural way. Roughly, a twisted vector bundle on a space is obtained by gluing vector bundles $\{V_i\}$ along on open cover, $\{U_i\}_{i \in I}$ where the standard equivalence on intersections is replaced by an equivalence

$$V_i|_{U_i \cap U_j} \simeq V_j|_{U_i \cap U_j} \otimes \mathcal{L}_{ij}.$$

Here $\mathcal{L}_{ij}$ is an auxiliary line bundle; the collection of line bundles on intersections should satisfy a cocycle condition. See [13, Section 1] for a more precise discussion. We will see that the descent data needed to glue together a twisted spectrum from parametrized spectra is exactly of this form. This leads us to speculate about the existence of a “twisted J-homomorphism”. Roughly, fixing the space B , we expect this to be a functor

$$\mathrm{TwVect}(B) \rightarrow \operatorname{Pic}(\mathrm{TwSp}(B)),$$

where $\mathrm{TwVect}(B)$ is a suitably chosen category of twisted vector bundles on B , and $\mathrm{TwSp}(B)$ is a total category of twisted spectra which we introduce in Section 5. We will not, however, pursue this line of investigation any further herein.

3.2 Examples of twisted spectra

Before proving structural results about twisted spectra, let us give some explicit examples to guide the reader’s intuition. The reader should keep in mind the various examples of haunts that we gave in Section 2.4. We will typically make reference to and also use the same notation used in those examples.

Example 3.4 (twisted spectra over the circle) By the above discussion, we have that

$$\mathrm{TwSp}(S^1, n) \xrightarrow{\simeq} \mathrm{eq}(\mathrm{Fun}(U_0, \mathrm{Sp}) \times \mathrm{Fun}(U_1, \mathrm{Sp}) \rightrightarrows \mathrm{Fun}(V_0, \mathrm{Sp}) \times \mathrm{Fun}(V_1, \mathrm{Sp}))$$

for the ∞ -category of twisted spectra corresponding to the n^{th} suspension haunt. In particular, we may think of $\mathrm{TwSp}(S^1, n)$ as sitting in a pullback diagram

$$\begin{array}{ccc} \mathrm{TwSp}(S^1, n) & \longrightarrow & \mathrm{Sp} \\ \downarrow & & \downarrow \\ \mathrm{Sp} & \longrightarrow & \mathrm{Sp} \times \mathrm{Sp} \end{array}$$

where the bottom horizontal map sends a spectrum X to (X, X) and the right-hand vertical map sends a spectrum Y to $(Y, \Sigma^n Y)$. Hence, a twisted spectrum for the n^{th} suspension haunt is then essentially a spectrum X together with an equivalence $\phi: \Sigma^n X \simeq X$.

Example 3.5 (twisted spectra over the 2-sphere) There is only one nontrivial haunt over S^2 , and by the above, its category of twisted spectra can be described as

$$\mathrm{TwSp}(S^2, \tau) \simeq \mathrm{eq}(\mathrm{Fun}(U_0, \mathrm{Sp}) \times \mathrm{Fun}(U_1, \mathrm{Sp}) \rightrightarrows \mathrm{Fun}(U_0 \cap U_1, \mathrm{Sp})).$$

In particular, we may think of $\mathrm{TwSp}(S^2, \tau)$ as sitting in a pullback diagram

$$\begin{array}{ccc} \mathrm{TwSp}(S^2, \tau) & \longrightarrow & \mathrm{Sp} \\ \downarrow & & \downarrow X \mapsto \underline{X} \otimes S^n \\ \mathrm{Sp} & \xrightarrow{Y \mapsto \underline{Y}} & \mathrm{Sp}^{S^1} \end{array}$$

where the bottom map takes a spectrum Y and constructs the trivially parametrized S^1 -spectrum $\underline{Y}$, and the right vertical map sends X to $\underline{X} \otimes S^n$. Recall now the equivalence

$$\mathrm{Sp}^{S^1} \simeq \mathrm{RMod}_{S[\mathbb{Z}]};$$

the latter ∞ -category can be thought of as a spectrum equipped with an automorphism. Viewed in this way, we can reinterpret the bottom map as the one assigning the identity id_Y to the spectrum Y , and the right vertical map as assigning the automorphism $-\mathrm{id}_X$ to X . From this description, we may think of an object of $\mathrm{TwSp}(S^2, \tau)$ as a pair (X, ι) where X is a spectrum and where ι is a homotopy $\iota: \mathrm{id}_X \simeq -\mathrm{id}_X$.

Example 3.6 (twisted spectra over the 3-sphere) There is only one nontrivial haunt over S^3 , and by the above, its category of twisted spectra can be described as

$$\mathrm{TwSp}(S^3, \tau) \simeq \mathrm{eq}(\mathrm{Sp} \rightrightarrows \mathrm{Fun}(S^2, \mathrm{Sp})).$$

In particular, we can think of $\mathrm{TwSp}(S^3, \tau)$ as sitting in the pullback diagram

$$\begin{array}{ccc} \mathrm{TwSp}(S^3, \tau) & \longrightarrow & \mathrm{Sp} \\ \downarrow & & \downarrow \\ \mathrm{Sp} & \longrightarrow & \mathrm{Sp}^{S^2} \end{array}$$

where the bottom map takes a spectrum X and constructs the trivially parametrized S^2 -spectrum $\underline{X}$ and the second map takes a spectrum Y and sends it to $\underline{Y} \otimes T$. A twisted spectrum over the 3-sphere is then a spectrum X together with an equivalence $\underline{X} \simeq \underline{X} \otimes T$. Let us analyze this a bit further. Recall that we have an equivalence

$$\mathrm{Sp}^{S^2} \simeq \mathrm{RMod}_{\mathbb{S}[\Omega S^2]},$$

of ∞ -categories. The map $\eta: S^1 \rightarrow \mathrm{Aut}(\mathbb{S}) \simeq \mathrm{GL}_1(\mathbb{S})$ classifying $\eta \in \pi_1(\mathrm{GL}_1(\mathbb{S}))$, factors as a map

$$S^1 \rightarrow \Omega S^2 \rightarrow \mathrm{Aut}(\mathbb{S}) \simeq \mathrm{GL}_1(\mathbb{S}),$$

where $\Omega S^2 \simeq \Omega \Sigma S^1$ is the free $\mathbb{E}_1$ -algebra on S^1 , since $\mathrm{GL}_1(\mathbb{S})$ is an $\mathbb{E}_1$ algebra. Now, given a spectrum X , we may view it as a $\mathbb{S}[\Omega S^2]$ -module via restriction of scalars along $\mathbb{S}[\Omega S^2] \rightarrow \mathbb{S}$. Since an automorphism of $\mathbb{S}$ induces an automorphism of $\mathbb{S} \otimes X \simeq X$, this induces a map

$$\eta_X: S^1 \rightarrow \Omega S^2 \rightarrow \mathrm{Aut}(X) \simeq \mathrm{GL}_1(X),$$

for a general spectrum X . The equivalence $\underline{X} \simeq \underline{X} \otimes T$ precisely corresponds to a nullhomotopy of η_X . Hence, we can think of an object in $\mathrm{TwSp}(S^3, \tau)$ as a pair (X, α) where X is a spectrum and α is a nullhomotopy of η_X .

Example 3.7 (twisted spectra in Seiberg–Witten Floer theory) Let us discuss briefly how twisted spectra are constructed from Seiberg–Witten Floer data. We already explained how the twist was constructed completely within K-theory. This is perhaps not the best way to construct the twist if one wants to build a spectrum-like object from geometric data, though. In practice, one builds an explicit cocycle description of the class by studying a “total Dirac operator” on a certain Hilbert bundle over the corresponding Picard torus:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathbb{D}} & \mathcal{H} \\ & \searrow & \swarrow \\ & P & \end{array}$$

One then studies the distribution of eigenvalues of the operator $\mathbb{D}$ and picks an open cover $\{U_i\}$ of P in a suitable way with respect to this eigenvalue distribution. Once such an open cover is chosen, one constructs the spectra X_i on U_i following the constructions of [31], [24], and [36], using finite-dimensional approximation and Conley indices. See also [23]. The resulting twisted spectrum associated to the pair $(Y, \mathfrak{s})$ will be denoted by $\mathrm{SWF}(Y, \mathfrak{s})$. Again, more details will appear in the forthcoming article [6].

3.3 Pullbacks along maps of spaces

We now move to some important functoriality of the categories of twisted spectra with respect to maps of spaces. We first assert that we have the standard pullback, and its left and right adjoints.

Proposition 3.8 Given a map $f: A \rightarrow B$ of spaces we have a pullback functor

$$f^*: \mathrm{TwSp}(B, \sigma) \rightarrow \mathrm{TwSp}(A, f^*\sigma).$$

This functor has left and right adjoints that we will denote by

$$f_!: \mathrm{TwSp}(A, f^*\sigma) \rightarrow \mathrm{TwSp}(B, \sigma) \quad \text{and} \quad f_*: \mathrm{TwSp}(A, f^*\sigma) \rightarrow \mathrm{TwSp}(B, \sigma),$$

respectively.

Proof This follows from applying Proposition 1.25 to the composite functor $B \rightarrow B \mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Pr}_{\mathrm{St}}^L$. $\square$

Note that the existence of the left adjoint $f_!$ crucially uses ambidexterity, while the right adjoint f_* more or less follows from our construction of f^* as a left adjoint. Note that the above provides us with functors

$$\begin{aligned} \mathrm{TwSp}_! : \mathcal{S}/_{B \mathrm{Pic}(\mathbb{S})} &\rightarrow \mathrm{Pr}_{\mathrm{St}}^L, & f &\mapsto f_!, \\ \mathrm{TwSp}^* : (\mathcal{S}/_{B \mathrm{Pic}(\mathbb{S})})^{\mathrm{op}} &\rightarrow \mathrm{Pr}_{\mathrm{St}}^{L,R}, & f &\mapsto f^*, \\ \mathrm{TwSp}_* : \mathcal{S}/_{B \mathrm{Pic}(\mathbb{S})} &\rightarrow \mathrm{Pr}_{\mathrm{St}}^R, & f &\mapsto f_*, \end{aligned}$$

all of which have the same assignment on objects and where the assignment on a morphism $f: (A, f^*\sigma) \rightarrow (B, \sigma)$ is shown on the right. Later on, we will want to show that the functors described above have the expected compatibility with the tensor product of twisted spectra, which will be defined in the next subsection. An important result that will help with this is the Beck–Chevalley condition, which tells us how the functors above behave with respect to change of base.

Proposition 3.9 (Beck and Chevalley) Suppose that

$$\begin{array}{ccc} (A, \tau) & \xrightarrow{j} & (B, \sigma) \\ i \downarrow & & \downarrow f \\ (C, \rho) & \xrightarrow{g} & (D, \pi) \end{array}$$

is a pullback square in $\mathcal{S}/_{B \mathrm{Pic}(\mathbb{S})}$. Then the canonical transformation

$$i_! j^* \rightarrow g^* f_!$$

is an equivalence of functors $\mathrm{TwSp}(B, \sigma) \rightarrow \mathrm{TwSp}(C, \rho)$.

Proof In order to prove the result, we temporarily adopt the perspective of twisted spectra as the global sections of a sheaf of categories. Let us pick a trivializing open cover for the haunt $\mathcal{H}_{(D, \pi)}$. By pullback along g and h , this will trivialize the haunts $\mathcal{H}_{(C, \rho)}$ and $\mathcal{H}_{(B, \sigma)}$ respectively. Taking the Čech nerve of the covers, and using the description of $\mathrm{TwSp}(-, -)$ as the totalization of the cosimplicial object in categories (cf (2)), with each term being of the form $\mathrm{Fun}(U_i, \mathrm{Sp})$, we can reexpress the natural transformation $i_! j^* \rightarrow g^* f_!$ as a natural transformation of functors

$$\mathrm{Fun}(U_\bullet, \mathrm{Sp}) \rightarrow \mathrm{Fun}(V_\bullet, \mathrm{Sp}),$$

where $U_\bullet$ is the Čech nerve of the cover of B and $V_\bullet$ is the Čech nerve of the cover of C . Note that this decomposes, in each cosimplicial degree i , to the ordinary Beck–Chevalley maps

$$i_! j^* \rightarrow g^* f_! : \text{Fun}(V_i, \text{Sp}) \rightarrow \text{Fun}(U_i, \text{Sp}),$$

which are well known to be equivalences. Thus, we have displayed the natural transformation in question as natural transformation of functors of cosimplicial objects in categories, with each component being an equivalence. It follows that the natural transformation on totalizations will itself be an equivalence, as we wanted to show. $\square$

3.4 Tensor products of twisted spectra

Let us start by considering some monoidal structures on the ∞ -category of spaces over $B \text{Pic}(\mathbb{S})$. To simplify terminology, we will refer to the overcategory $\mathcal{S}_{/B \text{Pic}(\mathbb{S})}$ as the ∞ -category of *twisted spaces*. Note that this ∞ -category is symmetric monoidal by intertwining the cartesian product on $\mathcal{S}$ with the $\mathbb{E}_\infty$ -structure on $B \text{Pic}(\mathbb{S})$. More specifically, the symmetric monoidal structure, which will be denoted by $\boxtimes$ and referred to as the *exterior product*, is given as follows: if (A, τ) and (B, σ) are twisted spaces, we have

$$(A, \tau) \boxtimes (B, \sigma) = (A \times B, \tau \boxtimes \sigma),$$

where $\tau \boxtimes \sigma$ denotes the composition

$$\tau \boxtimes \sigma : A \times B \xrightarrow{\tau \times \sigma} B \text{Pic}(\mathbb{S}) \times B \text{Pic}(\mathbb{S}) \rightarrow B \text{Pic}(\mathbb{S}).$$

Compare this to the *interior product* on the mapping space $\text{Map}(A, B \text{Pic}(\mathbb{S}))$ for a fixed space A which is given by using the diagonal on A together with the $\mathbb{E}_\infty$ -structure on the Picard space:

$$+ : \text{Map}(A, B \text{Pic}(\mathbb{S})) \times \text{Map}(A, B \text{Pic}(\mathbb{S})) \rightarrow \text{Map}(A \times A, B \text{Pic}(\mathbb{S}) \times B \text{Pic}(\mathbb{S})) \rightarrow \text{Map}(A, B \text{Pic}(\mathbb{S})).$$

In particular, note that the interior product $\tau + \sigma$ is obtained by pulling back the exterior product $\tau \boxtimes \sigma$ along the diagonal map $\Delta_A : A \rightarrow A \times A$. Similar to the above situation, twisted spectra will also have an exterior and an interior product. We obtain the first one by considering tensor products of categories of twisted spectra.

Proposition 3.10 *The functor $\text{TwSp}_! : \mathcal{S}_{/B \text{Pic}(\mathbb{S})} \rightarrow \text{Pr}_{\text{St}}^L$ is symmetric monoidal with respect to the exterior product on the source and the Lurie tensor product on the target. In particular, we have*

$$\text{TwSp}(A, \tau) \otimes \text{TwSp}(B, \sigma) \simeq \text{TwSp}(A \times B, \tau \boxtimes \sigma),$$

and the following diagram commutes:

$$\begin{array}{ccc} \text{TwSp}(A, \tau) \otimes \text{TwSp}(B, \sigma) & \xrightarrow{\simeq} & \text{TwSp}(A \times B, \tau \boxtimes \sigma) \\ \downarrow f_! \otimes g_! & & \downarrow (f \times g)_! \\ \text{TwSp}(C, \rho) \otimes \text{TwSp}(D, \pi) & \xrightarrow{\simeq} & \text{TwSp}(C \times D, \rho \boxtimes \pi) \end{array}$$

Proof Note that the functor TwSp that we are referring to in the proposition is the composition

$$\mathcal{S}/B\mathrm{Pic}(\mathcal{S}) \rightarrow \mathcal{S}/\mathrm{Pr}_{\mathrm{St}}^L \xrightarrow{\mathrm{colim}} \mathrm{Pr}_{\mathrm{St}}^L.$$

Note that these are all symmetric monoidal functors. That the diagram is commutative follows from the fact that the coherence maps of a symmetric monoidal functor is supposed to be natural. $\square$

Corollary 3.11 *Given maps $f: A \rightarrow C$ and $g: B \rightarrow D$ of spaces, the following hold:*

(1) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{TwSp}(C, \tau) \otimes \mathrm{TwSp}(D, \sigma) & \xrightarrow{\cong} & \mathrm{TwSp}(C \times D, \tau \boxtimes \sigma) \\ \downarrow f^* \otimes g^* & & \downarrow (f \times g)^* \\ \mathrm{TwSp}(A, f^* \tau) \otimes \mathrm{TwSp}(B, f^* \sigma) & \xrightarrow{\cong} & \mathrm{TwSp}(A \times B, f^*(\tau \boxtimes \sigma)) \end{array}$$

(2) *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{TwSp}(A, f^* \tau) \otimes \mathrm{TwSp}(B, f^* \sigma) & \xrightarrow{\cong} & \mathrm{TwSp}(A \times B, f^*(\tau \boxtimes \sigma)) \\ \downarrow f_* \otimes g_* & & \downarrow (f \times g)_* \\ \mathrm{TwSp}(C, \tau) \otimes \mathrm{TwSp}(D, \sigma) & \xrightarrow{\cong} & \mathrm{TwSp}(C \times D, \tau \boxtimes \sigma) \end{array}$$

Proof The first statement follows from the functors f^* , g^* , and $(f \times g)^*$ being right adjoints to $f_!$, $g_!$, and $(f \times g)_!$, respectively. The second statement then follows from the first statement since f_* , g_* , and $(f \times g)_*$ are right adjoints to f^* , g^* , and $(f \times g)^*$, respectively. $\square$

We can use this equivalence to define the exterior tensor product of twisted spectra, simply by appealing to the universal property of tensor products of presentable stable ∞ -categories.

Definition 3.12 The *exterior product of twisted spectra* is the functor

$$- \boxtimes -: \mathrm{TwSp}(A, \tau) \times \mathrm{TwSp}(B, \sigma) \rightarrow \mathrm{TwSp}(A \times B, \tau \boxtimes \sigma)$$

corresponding to the equivalence of Proposition 3.10 under the universal property of tensor products of presentable stable ∞ -categories.

Note that the diagonal map on a space A induces a pullback functor

$$\Delta_A^*: \mathrm{TwSp}(A \times A, \tau \boxtimes \sigma) \rightarrow \mathrm{TwSp}(A, \tau + \sigma)$$

which can be used to define the interior product of twisted spectra.

Definition 3.13 The *interior product of twisted spectra* is the functor

$$- \otimes -: \mathrm{TwSp}(A, \tau) \times \mathrm{TwSp}(A, \sigma) \rightarrow \mathrm{TwSp}(A, \tau + \sigma)$$

obtained by pulling the exterior product back along the diagonal map on the space A .

The following lemma alternatively describes the external product in terms of the internal product.

Lemma 3.14 *Let $X \in \text{TwSp}(A, \tau)$ and let $Y \in \text{TwSp}(B, \sigma)$. Then*

$$X \boxtimes Y \simeq p_A^* X \otimes p_B^* Y$$

where $p_A: A \times B \rightarrow A$ and $p_B: A \times B \rightarrow B$ denotes the projections. Here, the tensor product is taken in twisted spectra over $A \times B$.

Proof We have

$$p_A^* X \otimes p_B^* Y = \Delta_{A \times B}^*(p_A^* X \boxtimes p_B^* Y) \simeq \Delta_{A \times B}^*((p_A \times p_B)^*(X \boxtimes Y)) \simeq X \boxtimes Y,$$

where we are using Proposition 3.15; see below. □

If we want to be explicit about the fact that we work with twisted spectra over A , we will sometimes denote the interior product by $\otimes_A$, but in most cases, we will be implicit about this and simply write $\otimes$. Note that the fact that the twists add up under the interior tensor product suggests that there is a symmetric monoidal structure on some category of twisted spectra over A where we let the twists vary. We will look at this category in Section 5.

3.5 Projection isomorphism

In this section, we will prove a preliminary version of the projection isomorphism for twisted spectra. We begin with some compatibilities between pullbacks and the symmetric monoidal structure on twisted spectra, giving us a rudimentary version of pullbacks being symmetric monoidal with respect to the interior tensor product of twisted spectra.

Proposition 3.15 *Let $f: A \rightarrow C$ and $g: B \rightarrow D$. Let $X \in \text{TwSp}(C, \rho)$ and $Y \in \text{TwSp}(D, \pi)$. Then we have*

$$(f \times g)^*(X \boxtimes Y) \simeq f^* X \boxtimes g^* Y.$$

Proof The statement of the theorem essentially follows from the commutativity of the diagram

$$\begin{array}{ccc} \text{TwSp}(C, \rho) \otimes \text{TwSp}(D, \pi) & \xrightarrow{\simeq} & \text{TwSp}(C \times D, \rho \boxtimes \pi) \\ \downarrow f^* \otimes g^* & & \downarrow (f \times g)^* \\ \text{TwSp}(A, f^* \rho) \otimes \text{TwSp}(B, g^* \pi) & \xrightarrow{\simeq} & \text{TwSp}(A \times B, f^* \rho \boxtimes g^* \pi) \end{array}$$

in Proposition 3.10. □

Corollary 3.16 *Let $f: A \rightarrow B$ be a map of spaces, $X \in \text{TwSp}(B, \tau)$ and $Y \in \text{TwSp}(B, \sigma)$. Then we have*

$$f^*(X \otimes Y) \simeq f^* X \otimes f^* Y.$$

Proof This follows from the computation

$$f^*(X \otimes Y) \simeq f^*((\Delta_B)^*(X \boxtimes Y)) \simeq (\Delta_A)^*((f \times f)^*(X \boxtimes Y)) \simeq f^*X \otimes f^*Y. \quad \square$$

Proposition 3.17 *Let $f: A \rightarrow B$ be a map of spaces, $Y \in \text{TwSp}(B, \sigma)$, and $X \in \text{TwSp}(A, f^*\rho)$ for some twists $\sigma, \rho: B \rightarrow B \text{ Pic}(\mathbb{S})$. Then we have an equivalence*

$$f_!(f^*Y \otimes X) \simeq Y \otimes f_!X$$

of $(A, \sigma + \rho)$ -twisted spectra.

Proof Using the definition of the internal tensor product, the first part of Corollary 3.11, and commutativity of some maps we get:

$$\begin{aligned} f_!(f^* \otimes X) &\simeq f_!(\Delta_A^*(f^*Y \boxtimes X)) \simeq f_!(\Delta_A^*((f \times 1)^*(Y \boxtimes X))) \simeq f_!(((f \times 1) \circ \Delta_A)^*(Y \boxtimes X)) \\ &\simeq f_!((f, 1)^*(Y \boxtimes X)). \end{aligned}$$

We finish the proof by using Proposition 3.9 on the pullback diagram

$$\begin{array}{ccc} (A, f^*\sigma + f^*\rho) & \xrightarrow{(f, 1)} & (B \times A, \sigma \boxtimes f^*\rho) \\ f \downarrow & & \downarrow 1 \times f \\ (B, \sigma + \rho) & \xrightarrow{\Delta_B} & (B \times B, \sigma \boxtimes \rho) \end{array}$$

so that we have equivalences

$$f_!((f, 1)^*(Y \boxtimes X)) \simeq \Delta_B^*((1 \times f)_!(Y \boxtimes X)) \simeq \Delta_B^*(Y \boxtimes f_!X) \simeq Y \otimes f_!X. \quad \square$$

3.6 Closed structure

Twisted spectra also have a closed structure. Indeed, the exterior and internal tensor product on twisted spectra admit right adjoints, which we will refer to as the exterior and internal hom-twisted spectra. Let us start with the exterior tensor product. Let Y be a (B, σ) -twisted spectrum and consider the functor $- \boxtimes Y: \text{TwSp}(A, \tau) \rightarrow \text{TwSp}(A \times B, \tau \boxtimes \sigma)$. We claim that this has a right adjoint that we will denote by

$$\text{hom}^{\boxtimes}(Y, -): \text{TwSp}(A \times B, \tau \boxtimes \sigma) \rightarrow \text{TwSp}(A, \tau)$$

This follows formally from the fact that $- \boxtimes Y$ preserves colimits, and that we are dealing with presentable stable ∞ -categories.

Definition 3.18 *The external hom object of twisted spectra is the functor*

$$\text{hom}^{\boxtimes}(-, -): \text{TwSp}(B, \sigma)^{\text{op}} \times \text{TwSp}(A \times B, \tau \boxtimes \sigma) \rightarrow \text{TwSp}(A, \tau).$$

Moreover, we have a right adjoint to the functor $- \otimes Y: \text{TwSp}(B, \tau) \rightarrow \text{TwSp}(B, \tau + \sigma)$, which we denote by

$$\text{hom}(Y, -): \text{TwSp}(A, \tau + \sigma) \rightarrow \text{TwSp}(A, \tau).$$

Since the interior product is obtained by pulling back the exterior product along the diagonal map, it follows that the internal hom twisted spectra functor is the composition

$$\mathrm{TwSp}(A, \tau + \sigma) \xrightarrow{(\Delta_A)^*} \mathrm{TwSp}(A \times A, \tau \boxtimes \sigma) \xrightarrow{\mathrm{hom}^{\boxtimes}(Y, -)} \mathrm{TwSp}(A, \tau).$$

Definition 3.19 The internal hom object of twisted spectra is the functor

$$\mathrm{hom}(-, -) : \mathrm{TwSp}(A, \sigma)^{\mathrm{op}} \times \mathrm{TwSp}(A, \tau + \sigma) \rightarrow \mathrm{TwSp}(A, \tau).$$

Let us prove a preliminary version of pullbacks being closed monoidal with respect to the internal hom objects.

Proposition 3.20 Let $f : A \rightarrow B$ be a map of spaces, $X \in \mathrm{TwSp}(B, \tau)$, and $Y \in \mathrm{TwSp}(B, \sigma)$. Then we have an equivalence

$$f^* \mathrm{hom}(X, Y) \simeq \mathrm{hom}(f^* X, f^* Y)$$

of $(A, -f^* \tau + f^* \sigma)$ -twisted spectra.

Proof Let Z be an arbitrary $(A, -f^* \tau + f^* \sigma)$ -twisted spectrum. By using Proposition 3.17 and Corollary 3.16 as well as adjunctions, we obtain

$$\begin{aligned} \mathrm{Map}(Z, f^* \mathrm{hom}(X, Y)) &\simeq \mathrm{Map}(f_! Z, \mathrm{hom}(X, Y)) \simeq \mathrm{Map}(f_! Z \otimes X, Y) \simeq \mathrm{Map}(f_!(Z \otimes f^* X), Y) \\ &\simeq \mathrm{Map}(Z \otimes f^* X, f^* Y) \simeq \mathrm{Map}(Z, \mathrm{hom}(f^* X, f^* Y)), \end{aligned}$$

from which the result follows by a Yoneda lemma argument. $\square$

3.7 Thom spectrum perspective

Before we move on, let us also explain to the reader how to think about the formalism that we have just covered from the perspective of thinking of twisted spectra as modules over Thom spectra. Doing this, we will follow some of the results in [7], and extend them slightly to account for functoriality in spaces. Let us consider the adjunction

$$\mathrm{Mod}_{(-)} : \mathrm{Alg}(R) \rightleftarrows (\mathrm{Cat}_R)_{\mathrm{Mod}_R} : \mathrm{End}_R(1_{(-)}).$$

If A is a connected space, let us pick a base point a and consider the inclusion $a \rightarrow A$, which we by abuse of notation will simply denote by a . If $\tau : A \rightarrow B \mathrm{Pic}$ is a twist, then note that we have an induced functor $a_!^{\tau} : \mathrm{Sp} \rightarrow \mathrm{TwSp}(A, \tau)$ which makes $\mathrm{TwSp}(A, \tau)$ an object of $(\mathrm{Cat}_{\mathbb{S}})_{\mathrm{Sp}/}$. The counit of the adjunction above provides us with a functor

$$\epsilon_{\mathrm{TwSp}(A, \tau)} : \mathrm{Mod}_{\mathrm{End}(a_! \mathbb{S})} \rightarrow \mathrm{TwSp}(A, \tau)$$

which is natural in $(\mathrm{Cat}_{\mathbb{S}})_{\mathrm{Sp}/}$. In [7, Theorem 7.13], it is essentially shown that this counit map is an equivalence. We want to extend this result slightly to be a natural equivalence, so that we know that the functors f^* , $f_!$, and f_* on categories of twisted spectra correspond to the standard restriction, extension,

and coextension of scalars on the level of module categories. In particular, we want to exhibit a *natural* identification of $\text{End}(a_! \mathbb{S})$ with $\text{Th}(\Omega\tau)$, so that this holds true.

Remark 3.21 In particular, given a pointed map $f: (A, a) \rightarrow (B, b)$ the diagram

$$\begin{array}{ccc} & \text{Sp} & \\ a_! \swarrow & & \searrow b_! \\ \text{TwSp}(A, f^* \sigma) & \xrightarrow{f_!} & \text{TwSp}(B, \sigma) \end{array}$$

provides us with a morphism in $(\text{Cat}_R)_{\text{Mod}_R}/.$ It then follows that the diagram

$$\begin{array}{ccc} \text{Mod}_{\text{End}(a_! \mathbb{S})} & \longrightarrow & \text{TwSp}(A, f^* \sigma) \\ \downarrow \text{End}(f_! \mathbb{S})_! & & \downarrow f_! \\ \text{Mod}_{\text{End}(b_! \mathbb{S})} & \longrightarrow & \text{TwSp}(B, \sigma) \end{array}$$

commutes. Note that the left hand vertical map is precisely extension of scalars along the $\mathbb{E}_1$ -ring map $\text{End}(f_! \mathbb{S}): \text{End}(a_! \mathbb{S}) \rightarrow \text{End}(b_! \mathbb{S})$.

Let us state and prove this desired natural equivalence. Note that ΩA is an $\mathbb{E}_1$ -group and that we can think of the map $\Omega\tau: \Omega A \rightarrow \text{Pic}(R)$ as an object of $\text{Grp}/_{\text{Pic}(R)}$. In particular, the inclusion $a \rightarrow A$ is corresponds to the inclusion of the unit in ΩA . For some convenient notation, let us denote the inclusion of the unit of an arbitrary group G by ι_G . Another way of writing the endomorphism ring $\text{End}(a_! \mathbb{S})$ is then $\text{End}((B\iota_{\Omega A})_! \mathbb{S})$.

Definition 3.22 We let $\mathcal{S}[G]$ denote the free presentable ∞ -category of the grouplike $\mathbb{E}_n$ -space G . We recall that there is an adjunction

$$\mathcal{S}[-]: \text{Grp}_{\mathbb{E}_n} \rightleftarrows \text{Alg}_{\mathbb{E}_n}(\text{Pr}^L): \text{Pic},$$

which might be found for example in [3, Theorem 7.7]. Recall that the underlying ∞ -category of $\mathcal{S}[G]$ is $\text{Fun}(G, \mathcal{S})$, but with symmetric monoidal structure given by the Day convolution product, rather than the pointwise one.

Lemma 3.23 The inclusion $B \text{Pic}(\mathbb{S}) \rightarrow \text{Pr}_{\text{St}}^L$ factors as

$$\begin{array}{ccc} B \text{Pic}(\mathbb{S}) & \xrightarrow{\quad} & \text{Pr}_{\text{St}}^L \\ & \searrow & \uparrow (\epsilon_{\text{Sp}})_! \\ & \text{Mod}_{\mathcal{S}[\text{Pic}(\mathbb{S})]}(\text{Pr}^L) & \end{array}$$

Proof First, note that there is a natural factorization $\text{Pic} \rightarrow \mathcal{S}[\text{Pic}(\mathbb{S})] \rightarrow \text{Sp}$ of the inclusion $\text{Pic}(\mathbb{S}) \rightarrow \text{Sp}$, which follows by the universal property of $\mathcal{S}[\text{Pic}(\mathbb{S})]$ as the initial presentably symmetric monoidal ∞ -category receiving an $\mathbb{E}_\infty$ -map from $\text{Pic}(\mathbb{S})$. Then there is a map $B \text{Pic}(\mathbb{S}) \rightarrow \text{Mod}_{\mathcal{S}[\text{Pic}(\mathbb{S})]}(\text{Pr}_{\text{St}}^L)$, since there is a group map $\text{Pic}(\mathbb{S}) \rightarrow \text{Aut}(\mathcal{S}[\text{Pic}(\mathbb{S})])$ by functoriality of $\text{Pic}(\mathbb{S}) \mapsto \mathcal{S}[\text{Pic}(\mathbb{S})]$. $\square$

Proposition 3.24 *The functors*

$$\mathrm{Th}: \mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Alg} \quad \text{and} \quad \mathrm{End}((B\iota(-))_!\mathbb{S}): \mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Alg}$$

are naturally equivalent.

Proof We will divide up the proof in three steps.

Step 1 We first show that the two functors both factor through functors of the type

$$\mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Alg}(\mathcal{S}[\mathrm{Pic}(\mathbb{S})]).$$

The counit of the $(\mathcal{S}[-] \vdash \mathrm{Pic})$ -adjunction provides us with a natural symmetric monoidal functor

$$\epsilon_{\mathrm{Sp}}: \mathcal{S}[\mathrm{Pic}(\mathbb{S})] \rightarrow \mathrm{Sp}.$$

For the Thom spectrum functor, note that there is a factorization

$$\mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \xrightarrow{\mathrm{Th}_{\mathrm{Pic}}} \mathrm{Alg}(\mathcal{S}[\mathrm{Pic}(\mathbb{S})]) \xrightarrow{\mathrm{Alg}(\epsilon_{\mathrm{Sp}})} \mathrm{Alg},$$

since $\mathcal{S}[\mathrm{Pic}(\mathbb{S})]$ is the universal symmetric monoidal category admitting an $\mathbb{E}_{\infty}$ -map from $\mathrm{Pic}(\mathbb{S})$. The functor $\mathrm{Th}_{\mathrm{Pic}}$ is then just the functor that sends $\alpha \in \mathrm{Grp}/\mathrm{Pic}(\mathbb{S})$ to the colimit

$$\mathrm{Th}_{\mathrm{Pic}}(\alpha) = \mathrm{colim}_G (G \xrightarrow{\alpha} \mathrm{Pic}(\mathbb{S}) \rightarrow \mathcal{S}[\mathrm{Pic}(\mathbb{S})]).$$

For the second functor in the statement of the theorem, it is clear, essentially by definition, that we have a factorization

$$\mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \rightarrow (\mathrm{Pr}_{\mathrm{St}}^L)_{\mathrm{Sp}} / \xrightarrow{\mathrm{End}(1(-))} \mathrm{Alg},$$

where the first functor sends $\alpha: G \rightarrow \mathrm{Pic}(\mathbb{S})$ to the left adjoint functor $(B\iota_G)_!: \mathrm{Sp} \rightarrow \mathrm{TwSp}(BG, B\alpha)$. Moreover, by Lemma 3.23 the first functor in the composition above factors as

$$\mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \simeq \mathcal{S}_*^{\geq 1} / B\mathrm{Pic}(\mathbb{S}) \xrightarrow{F_{\mathrm{Pic}}} (\mathrm{Mod}_{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]}(\mathrm{Pr}^L))_{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]} / \xrightarrow{(\epsilon_{\mathrm{Sp}})_!} (\mathrm{Pr}_{\mathrm{St}}^L)_{\mathrm{Sp}} /.$$

Here, the functor F_{Pic} is the functor

$$F_{\mathrm{Pic}}(\alpha) = \mathrm{colim}_{BG} (BG \rightarrow B\mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Mod}_{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]}(\mathrm{Pr}^L)).$$

Next, we claim that after composing with the functor $\mathrm{End}(1(-))$, the above argument implies a factorization of the assignment $\mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) \rightarrow \mathrm{Alg}(\mathcal{S}[\mathrm{Pic}(\mathbb{S})]) \rightarrow \mathrm{Alg}(\mathrm{Sp})$. This can be seen from the commutativity of the diagram

$$(4) \quad \begin{array}{ccccc} \mathrm{Grp}/\mathrm{Pic}(\mathbb{S}) & \xrightarrow{\quad} & (\mathrm{Pr}_{\mathrm{St}}^L)_{\mathrm{Sp}} / & \xrightarrow{\quad \mathrm{End}(1(-)) \quad} & \mathrm{Alg} \\ & \searrow & \uparrow (\epsilon_{\mathrm{Sp}})_! & & \uparrow \mathrm{Alg}(\epsilon_{\mathrm{Sp}}) \\ & & \mathrm{Mod}_{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]}(\mathrm{Pr}^L)_{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]} / & \xrightarrow{\quad \mathrm{End}^{\mathcal{S}[\mathrm{Pic}(\mathbb{S})]}(1(-)) \quad} & \mathrm{Alg}(\mathcal{S}[\mathrm{Pic}(\mathbb{S})]) \end{array}$$

where the right hand square commutes due to the naturality properties of the endomorphism functor established in [27, Sections 4.8.3–4.8.5].

Step 2 So far, we concluded that there are morphisms

$$\mathrm{Th}_G: \mathrm{Grp}/_G \rightarrow \mathrm{Alg}(\mathcal{S}[G]) \quad \text{and} \quad \mathrm{End}_G: \mathrm{Grp}/_G \rightarrow \mathrm{Alg}(\mathcal{S}[G])$$

for $G = \mathrm{Pic}(\mathbb{S})$ factoring the functors in the statement. In fact, such functors exist for any group G . We now show that these are natural in the group G . That is, we would like to show that if $f: G \rightarrow H$ is a map of grouplike $\mathbb{E}_\infty$ -spaces, then there will be an induced commutative diagrams

$$\begin{array}{ccc} \mathrm{Grp}/_G & \xrightarrow{\mathrm{Th}_G} & \mathrm{Alg}(\mathcal{S}[G]) \\ \downarrow & & \downarrow f_! \\ \mathrm{Grp}/_H & \xrightarrow{\mathrm{Th}_{G'}} & \mathrm{Alg}(\mathcal{S}[H]) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Grp}/_G & \xrightarrow{\mathrm{End}_G} & \mathrm{Alg}(\mathcal{S}[G]) \\ \downarrow & & \downarrow f_! \\ \mathrm{Grp}/_H & \xrightarrow{\mathrm{End}_{G'}} & \mathrm{Alg}(\mathcal{S}[H]) \end{array}$$

For the first diagram, note that the horizontal arrows are induced by the straightening equivalences $\mathcal{S}/_G \simeq \mathcal{S}[G]$ and $\mathcal{S}/_H \simeq \mathcal{S}[H]$, which are monoidal by [35, Corollary D] and are natural by definition. The commutativity of the second diagram reduces again to commutativity of the diagram (4)).

Step 3 Once, this is understood, we argue as in Proposition 7.8 of [7]. We will show that the identity map $G \rightarrow G$ is sent to the terminal object of $\mathrm{Alg}(\mathcal{S}[G])$. We already know from [7, Theorem 7.13] that these two functors, Th_G and $\mathrm{End}((B\iota_{(-)})_! \mathbb{S})$ agree pointwise. Thus, it is enough to show that the functor $\mathrm{Th}_G: \mathrm{Grp}/_G \rightarrow \mathrm{Alg}(\mathcal{S}[G])$ sends the identity map $G \rightarrow G$ to the terminal object. Note that this is true because we can identify Th_G with the straightening equivalence, so terminal objects get sent to terminal objects, under the equivalence. We now defer to the lemma below — we set $\mathcal{C} = \mathrm{Grp}(\mathcal{S})$ and $f: \mathrm{Grp}(\mathcal{S}) \rightarrow \mathrm{Cat}_\infty$ to be the functor $G \mapsto \mathrm{Alg}(\mathcal{S}[G])$. Since we now know that the two functors $\mathrm{Th}_G(-)$ and $\mathrm{End}(B(-)_!(*))$ satisfy the criteria, they are identified. $\square$

Lemma 3.25 *Let $\mathcal{C}$ be any category and f any functor $\mathcal{C} \rightarrow \mathrm{Cat}_\infty$. Then the category of natural transformations*

$$\mathcal{C}/_{(-)} \rightarrow f$$

sending $i d_x$ to terminal objects of $f(x)$ is contractible. Moreover, any such natural transformation is terminal in this category.

Proof Natural transformations $\mathcal{C}/_{(-)} \rightarrow f$ can be identified with (non-Cartesian) sections of the cocartesian fibration F classifying f . Informally, the section σ sends some element y to a pair $(y, \beta(y) \in f(y))$, from which the natural transformation $\tau: \mathcal{C}/_{(-)} \rightarrow f$ corresponding to σ is such that

$$\tau_x: \mathcal{C}/_x \rightarrow f(x), \quad (g: y \rightarrow x) \mapsto g_*(\beta(y)),$$

where $g_*: f(y) \rightarrow f(x)$ is the functor induced by evaluating f on g . Since we are assuming that we are dealing with natural transformations that send the identity to terminal objects, we must then have that $\beta(y)$ is terminal in $f(y)$ for every y . Since terminal objects in the ∞ -category of sections $\Gamma(\mathcal{C}, F)$ may be understood pointwise, it follows that the section σ itself is terminal in this ∞ -category. Hence it is determined uniquely up to contractible choice. $\square$

Theorem 3.26 *Let A be a connected space, and consider a twist $\tau: A \rightarrow B \text{ Pic}(\mathbb{S})$. There is a natural equivalence*

$$\text{TwSp}(A, \tau) \simeq \text{Mod}_{\text{Th}(\Omega\tau)}.$$

Remark 3.27 The above result gives a conceptual reason for why the 2-periodic sphere spectrum, and modules over this ring, seem to be central to work in symplectic Floer theory, see for example [1]. Indeed, the 2-periodic sphere spectrum is precisely the Thom spectrum of the loop of the map

$$S^1 \simeq U(1) \rightarrow U \simeq B(\mathbb{Z} \times BU) \rightarrow B \text{ Pic}(\mathbb{S})$$

where we are using the inclusion of $U(1)$ into U , Bott periodicity, and the complex J-homomorphism [28]. Interestingly, the loop of this map is actually $\mathbb{E}_2$. Hence, the ∞ -category of modules over the 2-periodic sphere spectrum is monoidal in its own right. This raises questions about how this monoidal structure interacts with the monoidal structure

$$\otimes: \text{TwSp}(S^1, \tau) \otimes \text{TwSp}(S^1, \sigma) \rightarrow \text{TwSp}(S^1, \tau + \sigma),$$

on twisted spectra over the circle that we introduced in Section 3.4.

As corollaries, we can conclude that the pullback functors, and their left and right adjoints, on categories of twisted spectra correspond to the standard restriction, extension, and coextension of scalars on module categories.

Corollary 3.28 *Assume that the spaces A and B are pointed and connected and let $\sigma: B \rightarrow B \text{ Pic}(\mathbb{S})$ be a twist. Let $f: A \rightarrow B$ be a map of based spaces. Then the following diagram commutes:*

$$\begin{array}{ccc} \text{Mod}_{\text{Th}(\Omega(f^*\sigma))} & \xrightarrow{\text{Th}(\Omega f)!} & \text{Mod}_{\text{Th}(\Omega\sigma)} \\ \downarrow \simeq & & \downarrow \simeq \\ \text{TwSp}(A, f^*\sigma) & \xrightarrow{f!} & \text{TwSp}(B, \sigma) \end{array}$$

Proof Let $f: A \rightarrow B$ be a based map of connected spaces. Then the diagram

$$\begin{array}{ccc} \Omega A & \xrightarrow{\Omega f} & \Omega B \\ & \searrow \Omega(f^*\sigma) & \swarrow \Omega\sigma \\ & \text{Pic}(\mathbb{S}) & \end{array}$$

provides us with a morphism in $\text{Grp}/\text{Pic}(\mathbb{S})$. Because of the naturality of the functors, the diagram

$$\begin{array}{ccc} \text{Th}(\Omega(f^*\sigma)) & \xrightarrow{\text{Th}(\Omega f)} & \text{Th}(\Omega\sigma) \\ \downarrow & & \downarrow \\ \text{End}(a! \mathbb{S}) & \xrightarrow{\text{End}(f! \mathbb{S})} & \text{End}(b! \mathbb{S}) \end{array}$$

commutes in Alg . The result then follows by using Remark 3.21. □

Corollary 3.29 Assume that the spaces A and B are pointed and connected and let $\sigma: B \rightarrow B \operatorname{Pic}(\mathbb{S})$ be a twist. Let $f: A \rightarrow B$ be a map of based spaces. Then:

(1) The following diagram commutes:

$$\begin{array}{ccc} \operatorname{Mod}_{\operatorname{Th}(\Omega\sigma)} & \xrightarrow{\operatorname{Th}(\Omega f)^*} & \operatorname{Mod}_{\operatorname{Th}(\Omega(f^*\sigma))} \\ \downarrow \simeq & & \downarrow \simeq \\ \operatorname{TwSp}(B, \sigma) & \xrightarrow{f^*} & \operatorname{TwSp}(A, f^*\sigma) \end{array}$$

(2) The following diagram commutes:

$$\begin{array}{ccc} \operatorname{Mod}_{\operatorname{Th}(\Omega(f^*\sigma))} & \xrightarrow{\operatorname{Th}(\Omega f)^*} & \operatorname{Mod}_{\operatorname{Th}(\Omega\sigma)} \\ \downarrow \simeq & & \downarrow \simeq \\ \operatorname{TwSp}(A, f^*\sigma) & \xrightarrow{f_*} & \operatorname{TwSp}(B, \sigma) \end{array}$$

4 Homology and cohomology invariants of twisted spectra

It is natural to ask what sorts of invariants can be extracted from a twisted spectrum. As these should often arise from Floer homotopical data, it would be useful to extract honest abelian groups which can say something about the context at hand. In this section, we investigate what sort of invariants one can attach to a twisted spectrum. As we will see, it will not be so straightforward to define global invariants as it is in the untwisted case.

4.1 Nonexistence of global homotopy invariants

Let B be a fixed space. To a parametrized spectrum $X: B \rightarrow \operatorname{Sp}$, one may associate its *homology type* and *cohomology type*

$$p_!(X) \simeq \operatorname{colim}_B X \in \operatorname{Sp} \quad \text{and} \quad p_*(X) \simeq \lim_B X \in \operatorname{Sp},$$

respectively. Now let $\tau: B \rightarrow B \operatorname{Pic}$ be the data of a haunt over B , and let $X \in \operatorname{TwSp}(B, \tau)$ be a twisted parametrized spectrum. One is led to ask the following natural question.

Question 4.1 Do globally defined homology and cohomology invariants exist in general for X as they do in the untwisted case?

The answer, as we shall see, is a resounding no. Namely, there is generally *no canonical way* to associate homology and cohomology invariants to a twisted spectrum, in stark contrast to ordinary parametrized spectra. In this section, we propose a simple conceptual explanation for this phenomenon: that the ∞ -category of twisted parametrized spectra is fibered over the ∞ -category $\mathcal{S}_{/B \operatorname{Pic}(\mathbb{S})}$ of twisted spaces, whose structure adds constraints to the resulting functorialities that may arise in the total space.

Porism 4.2 As a consequence of the discussion in Proposition 3.8, the assignment $(B, \tau) \mapsto \mathrm{TwSp}(B, \tau)$ may be packaged into a symmetric monoidal functor

$$\mathrm{TwSp}^*: \mathcal{S}_{/B \mathrm{Pic}(\mathbb{S})}^{\mathrm{op}} \rightarrow \mathrm{Pr}_{\mathrm{St}}^{L,R},$$

where the target denotes the ∞ -category of stable presentable ∞ -categories with morphisms which are both left and right adjoints.

In particular, let $\tau: A \rightarrow B \mathrm{Pic}(\mathbb{S})$ be a twist, and let us try to define a functor $\mathrm{TwSp}(A, \tau) \rightarrow \mathrm{Sp}$. If we try to define a functor $p_!: \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{Sp}$ or $p_*: \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{Sp}$ via the functoriality of Porism 4.2, we immediately see that we need the collapse map $p: A \rightarrow *$ to sit in the diagram

$$\begin{array}{ccc} A & \xrightarrow{p} & * \\ & \searrow \tau \quad \swarrow 0 & \\ & B \mathrm{Pic}(\mathbb{S}) & \end{array}$$

In particular, the equivalence $0 \simeq p^*(0) \simeq \tau$ is forced upon us. This may not appear so interesting at first glance. Something slightly interesting is hidden in the notation though. Indeed, while the above only make sense when the twist is nullhomotopic, the invariants that we get depend on how we trivialize the twist. Note that the trivialization corresponds to a particular equivalence $\mathrm{TwSp}(A, \tau) \simeq \mathrm{Sp}^A$ with the category of spectra parametrized over A .

4.2 Homology and cohomology invariants

Now, it could of course happen that we are dealing with a twisted spectrum with a nontrivializable twisting τ , in which case our twisted spectrum does not have a globally defined homotopy invariant. We are not completely out of luck, though. Indeed, we might still have (co)homology invariants with respect to other generalized cohomology theories. Let R be an $\mathbb{E}_\infty$ -ring, so that we have an induced map $B \mathrm{Pic}(\mathbb{S}) \rightarrow B \mathrm{Pic}(R)$ of Picard spaces. Now, if the composition $\tau_R: A \rightarrow B \mathrm{Pic}(\mathbb{S}) \rightarrow B \mathrm{Pic}(R)$ is nullhomotopic, then it makes sense to talk about R -homology and R -cohomology of X , completely analogous to before. Again the (co)homology will depend on our choice of trivialization of τ_R , though.

Proposition 4.3 *The space of trivializations of $\tau_R: A \rightarrow B \mathrm{Pic}(R)$ is a torsor for $\mathrm{Map}(A, \mathrm{Pic}(R))$.*

Proof Let $\tau \in \pi_0(\mathrm{Map}(A, B \mathrm{Pic}))$ be an equivalence class of twists, such that there is a nullhomotopy $\iota: \tau \simeq 0$. Note first that every $\alpha: A \rightarrow \mathrm{Pic}$ classifies an automorphism of the trivial twist. In particular

$$\mathrm{Aut}_{\mathrm{Map}(A, B \mathrm{Pic})}(0) \simeq \mathrm{Map}(A, \mathrm{Pic}(R)).$$

Thus, we may postcompose ι with any such automorphism to obtain another trivialization $\iota \circ \alpha$. We see therefore that there is a transitive action of $\mathrm{Map}(A, \mathrm{Pic}(R))$ on the space of trivializations. Now, since the group $\mathrm{Map}(A, \mathrm{Pic})$, being the automorphism group of the trivial twist, acts transitively on itself, we conclude that the action on the space of trivializations is itself transitive. $\square$

Each one of these trivializations α corresponds to an equivalence

$$\mathrm{TwMod}_R(A, \tau_R) \xrightarrow{\alpha} \mathrm{Fun}(A, \mathrm{Mod}_R)$$

that we by abuse of notation will also denote by α .

Definition 4.4 Let $X \in \mathrm{TwSp}(A, \tau)$ be a twisted spectra for a haunt satisfying that $\tau_R: A \rightarrow B \mathrm{Pic}(R)$ is nullhomotopic. Given a choice of trivialization α of τ_R , we write

$$R_i^\alpha = \pi_i(\mathrm{colim}_A \alpha(X)) \quad \text{and} \quad R_\alpha^i = \pi_i(\lim_A \alpha(X)),$$

and refer to these as the (R, α) -homology and (R, α) -cohomology of X , respectively.

Let us look at some examples, where the cohomology theories that we keep in mind are those that appear in Table 1.

Example 4.5 (invariants of twisted spectra over the circle) Twisted spectra over the circle have KU-invariants whenever we are dealing with a haunt corresponding to an $n \in \mathbb{Z} \cong \pi_1(B \mathrm{Pic}(\mathbb{S}))$ which is zero mod 2. This follows from the maps

$$\pi_1 B \mathrm{Pic}(\mathbb{S}) \rightarrow \pi_1 B \mathrm{Pic}(\mathrm{KU})$$

being reduction mod 2. In the cases that the map $\tau_{\mathrm{KU}}: S^1 \rightarrow B \mathrm{Pic}(\mathrm{KU})$ is trivializable, note that there are essentially two different trivializations, since $\pi_1 \mathrm{Pic}(\mathrm{KU}) \cong \mathbb{Z}/2$.

Example 4.6 (invariants of twisted spectra over the 3-sphere) Consider the nontrivial haunt

$$\tau: S^3 \rightarrow B \mathrm{Pic}(\mathbb{S}).$$

Since $\pi_3 B \mathrm{Pic}(H\mathbb{Z})$, the map $\tau_{H\mathbb{Z}}: S^3 \rightarrow B \mathrm{Pic}(\mathbb{S})$ is trivializable, and since $\pi_0 \mathrm{Map}(S^3, \mathrm{Pic}(H\mathbb{Z})) \cong \pi_3 \mathrm{Pic}(H\mathbb{Z})$, there is an essentially unique way to trivialize the twist. The $H\mathbb{Z}$ -invariants for a twisted spectrum over S^3 is hence uniquely determined.

Example 4.7 (invariants of twisted spectra from Seiberg–Witten Floer theory) A haunt $\tau: P \rightarrow B \mathrm{Pic}(\mathbb{S})$ obtained from a complex spin manifold $(Y, \mathfrak{s})$ always admits KU-invariants. Note that the induced map

$$\pi_i \tau_{\mathrm{KU}}: \pi_i P \rightarrow \pi_i U \rightarrow \pi_i B \mathrm{Pic}(\mathbb{S}) \rightarrow \pi_i B \mathrm{Pic}(\mathrm{KU})$$

is zero for all i . Indeed, it is clear that the map is zero for even $i \geq 2$ since $\pi_i U \cong 0$ for all even i , and that it is zero for odd $i \geq 3$ since $\pi_i B \mathrm{Pic}(\mathrm{KU}) \cong 0$ for all odd i in this range. We also claim that the map induced for $i = 1$ is zero. The rough argument uses that isomorphism classes of complex spin structures on Y is in one-to-one correspondence with complex line bundles over Y [25, Proposition 1.1.1]. Indeed, given one complex spin structure on Y with spinor bundle S , all other complex spin structures can be obtained by tensoring $S \otimes L$ with a hermitian line bundle over Y . The first Chern character of $S \otimes L$ is known to be

$$c_1(S \otimes L) = c_1(S) + 2c_1(L).$$

Since the tangent bundle of an oriented 3-manifold is always trivial, we can always find a complex spin structure on Y , namely the one for which the spinor bundle is trivial. Hence, the first Chern character of any complex spin structure is necessarily even, and this will imply that the corresponding haunt is such that the induced map

$$\pi_1 \tau : \pi_1 P \rightarrow \pi_1 B \operatorname{Pic}(\mathbb{S}) \cong \mathbb{Z}$$

takes values in $2\mathbb{Z}$. Since the map $\pi_1 B \operatorname{Pic}(\mathbb{S}) \rightarrow \pi_1 B \operatorname{Pic}(\operatorname{KU})$ is mod 2, we conclude that $\pi_1 \tau_{\operatorname{KU}}$ is zero.

Moreover, we want to point out the following result of Maegawa which tells us that twisted spectra coming from haunts that factor through the complex J-homomorphism, which are common in Floer homotopy theory, always admit R -(co)homology group with respect to any even $\mathbb{E}_2$ -ring R .

Proposition 4.8 [30, Theorem 3.1] *Let B be a CW-complex and let $\tau : B \rightarrow B(\mathbb{Z} \times \operatorname{BU}) \rightarrow B \operatorname{Pic}(\mathbb{S})$ be a haunt that factor through the complex J-homomorphism. Then $\tau_R : B \rightarrow B \operatorname{Pic}(R)$ is nullhomotopic for any even periodic $\mathbb{E}_2$ -ring spectrum. Moreover, the nullhomotopy is determined by essentially uniquely by the choice of an $\mathbb{E}_1$ -complex orientation on R .*

5 The total category of twisted spectra

The discussion on tensor products of twisted spectra suggests that there is a symmetric monoidal structure on some category of twisted spectra on a fixed space, where we let the twists vary. In this section, we introduce this total category of twisted spectra over a space, show that it has the wanted closed symmetric monoidal structure, and discuss pullbacks and their adjoints on these sorts of categories. For all intents and purposes, we will essentially show that the total categories admit a 6-functor formalism.

5.1 The total category of twisted spectra

We start by simply introducing the wanted total category of twisted spectra over a fixed space.

Definition 5.1 The *total category of twisted spectra* over A is defined as

$$\operatorname{TwSp}(A) = \operatorname{colim}(\operatorname{TwSp}(A, -) : \operatorname{Map}(A, B \operatorname{Pic}(\mathbb{S})) \rightarrow \operatorname{Pr}_{\operatorname{St}}^L).$$

Note that by ambidexterity, the total category is also obtained as the limit of the diagram.

Remark 5.2 There is another potential definition of a total category for twisted spectra that we can form by looking at the total space of the fibration classified by the global sections functor

$$\operatorname{colim} \circ \operatorname{Map}(A, B \operatorname{Pic}(\mathbb{S})) \rightarrow \operatorname{Cat}_\infty, \quad \tau \mapsto \operatorname{TwSp}(A, \tau).$$

Heuristically, we think of objects in this category as a pair (τ, X) where $X \in \operatorname{TwSp}(A, \tau)$. We will not study this category in this paper, mostly because it does not have the wanted properties that are looking for. In particular, note that the lax version of the total category is not stable.

Let us look at a possible example of an object in one of these total categories that appear in nature.

Example 5.3 (Seiberg–Witten Floer theory) Recall Examples 2.10 and 3.7, where we sketched the haunt and twisted spectrum associated to an oriented riemannian 3-manifold equipped with a complex spin structure $(Y, \mathfrak{s})$. It is well-understood how the haunt $\tau_{(Y, \mathfrak{s})}: P \rightarrow B \operatorname{Pic}(\mathbb{S})$ changes when we vary the complex spin structure $\mathfrak{s}$. We expect that, given an oriented riemannian 3-manifold Y , it should be possible to construct an object $\operatorname{SWF}(Y)$ in the total category $\operatorname{TwSp}(P)$ of twisted spectra over the Picard torus $P = H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$ in such a way that

$$\operatorname{SWF}(Y)_\tau = \begin{cases} \operatorname{SWF}(Y, \mathfrak{s}) & \text{if there is a (necessarily unique) } \mathfrak{s} \text{ such that } \tau_{(Y, \mathfrak{s})} = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

That such a “total twisted spectrum” might be interesting to study is hinted at by the fact that one in monopole Floer homology tend to consider all possible isomorphism classes of complex spin structures on a 3-manifold [25, Chapter 3].

5.2 The total pullback functors and their adjoints

In this section, we show that the pullback functors we introduced in Section 3.3 assemble into a pullback functor on the level of total categories, and that this pullback functor admits a right adjoint, but not necessarily a left adjoint. Now, there are a few maps that we could denote by f^* for a map $f: A \rightarrow B$ of spaces, so to avoid overlap we will start this section by introducing the following notation.

Notation 5.4 Given a map $f: A \rightarrow B$ of spaces, the precomposition map induced by applying the functor $\operatorname{Map}(-, B \operatorname{Pic}(\mathbb{S}))$ will be denoted by

$$B \operatorname{Pic}(f): \operatorname{Map}(B, B \operatorname{Pic}(\mathbb{S})) \rightarrow \operatorname{Map}(A, B \operatorname{Pic}(\mathbb{S})).$$

We also introduce the following definition of an auxiliary category that will be of interest.

Definition 5.5 Let $f: A \rightarrow B$ be a map of spaces. We write

$$\operatorname{TwSp}(A, f^*) = \operatorname{colim}(\operatorname{Map}(B, B \operatorname{Pic}(\mathbb{S})) \xrightarrow{B \operatorname{Pic}(f)} \operatorname{Map}(A, B \operatorname{Pic}(\mathbb{S})) \xrightarrow{\operatorname{TwSp}(A, -)} \operatorname{Pr}_{\operatorname{St}}^L).$$

This is essentially the subcategory of the total category $\operatorname{TwSp}(A)$ generated by twists that are in the image of $B \operatorname{Pic}(f)$. Note that the inclusion $\operatorname{TwSp}(A, f^*) \rightarrow \operatorname{TwSp}(A)$ can be identified with the functor $(B \operatorname{Pic}(f))_!$ of the second part of Proposition 1.25.

Note that we can think of all the pullback functors $f^*: \operatorname{TwSp}(B, \tau) \rightarrow \operatorname{TwSp}(A, f^*\tau)$ for various twists τ as assembling into a natural transformation

$$\operatorname{Nat}(f^*): \operatorname{TwSp}(B, -) \rightarrow \operatorname{TwSp}(A, -) \circ B \operatorname{Pic}(f)$$

of functors $\operatorname{Map}(B, B \operatorname{Pic}(\mathbb{S})) \rightarrow \operatorname{Pr}_{\operatorname{St}}^L$. Let us consider the following construction.

Construction 5.6 Let $f: A \rightarrow B$ be a map of spaces. We can construct an induced *total pullback functor* as a composition

$$f^*: \text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*) \rightarrow \text{TwSp}(A)$$

as follows:

- The first functor $\text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*)$ is the one obtained by feeding the natural transformation $\text{Nat}(f^*)$ through the colimit functor

$$\text{colim}: \text{Fun}(\text{Map}(B, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^L) \rightarrow \text{Pr}_{\text{St}}^L.$$

- The second functor $\text{TwSp}(A, f^*) \rightarrow \text{TwSp}(A)$ is the inclusion of the full subcategory.

We now arrive at the question to the existence of adjoints to this total pullback functors. Similarly to before, note that the different functors $f_!$ on the categories of twisted spectra for various twists assemble into a natural transformation

$$\text{Nat}(f_!): \text{TwSp}(A, -) \circ B \text{ Pic}(f) \rightarrow \text{TwSp}(B, -)$$

of functors $\text{Map}(B, B \text{ Pic}(\mathbb{S})) \rightarrow \text{Pr}_{\text{St}}^L$. Of course, the different functors f_* on categories of twisted spectra for various twists also assemble into a natural transformation

$$\text{Nat}(f_*): \text{TwSp}(A, -) \circ B \text{ Pic}(f) \rightarrow \text{TwSp}(B, -)$$

of functors $\text{Map}(B, B \text{ Pic}(\mathbb{S}), \text{Pr}_{\text{St}}^R) \rightarrow \text{Pr}_{\text{St}}^R$. Note that we need use Pr_{St}^R here since the functors f_* are right adjoints.

Proposition 5.7 *The following hold:*

- (1) *The functor we get from feeding $\text{Nat}(f^*)$ into the limit functor*

$$\lim: \text{Fun}(\text{Map}(B, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^R) \rightarrow \text{Pr}_{\text{St}}^R$$

is equivalent to the functor $\text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*)$ appearing above.

- (2) *The functor we get from feeding $\text{Nat}(f_!)$ into the colimit functor*

$$\text{colim}: \text{Fun}(\text{Map}(B, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^L) \rightarrow \text{Pr}_{\text{St}}^L$$

is left adjoint to $\text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*)$.

- (3) *The functor we get from feeding $\text{Nat}(f_*)$ into the limit functor*

$$\lim: \text{Fun}(\text{Map}(B, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^R) \rightarrow \text{Pr}_{\text{St}}^R$$

is right adjoint to $\text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*)$.

Proof The first statement is due to ambidexterity in Pr_{St}^L and Pr_{St}^R . The second and third statements then follow from Propositions 1.25 and 1.26 □

Because of the above result, the existence of a left and right adjoint to the total pullback functor essentially comes down to the existence of adjoints to the inclusion $\mathrm{TwSp}(A, f^*) \rightarrow \mathrm{TwSp}(A)$. It turns out that the wanted right adjoint to the pullback on total categories always exists, but that the left adjoint only exists under some very strict finiteness assumptions.

Proposition 5.8 *The total pullback functor $f^*: \mathrm{TwSp}(B) \rightarrow \mathrm{TwSp}(A)$ has a right adjoint.*

Proof Recall that the inclusion $\mathrm{TwSp}(A, f^*) \rightarrow \mathrm{TwSp}(A)$ is simply the functor $(B \mathrm{Pic}(f))_!$ of the second part of Proposition 1.25. Hence, it necessarily has a right adjoint. $\square$

Note that the same argument cannot be run for the left adjoint; the functor $\mathrm{TwSp}(A, f^*) \rightarrow \mathrm{TwSp}(A)$ does not in general admit a left adjoint, unless we have some pretty strict finiteness conditions on the fiber of f .

Proposition 5.9 *Let $f: A \rightarrow B$ be a map of spaces such that the induced map $B \mathrm{Pic}(f)$ is an equivalence, then we have an induced left adjoint functor*

$$f_!: \mathrm{TwSp}(A) \rightarrow \mathrm{TwSp}(B).$$

Moreover, the right adjoint to this functor is f^* .

Proof Note that if $B \mathrm{Pic}(f)$ is an equivalence, then the functor $B \mathrm{Pic}(f)_!$ is an equivalence, so it certainly has a left adjoint, namely its inverse. $\square$

We also prove Beck–Chevalley in this setting.

Proposition 5.10 (Beck–Chevalley) *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ \downarrow i & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

is a pullback square of spaces in which i and f are such that $B \mathrm{Pic}(i)$ and $B \mathrm{Pic}(f)$ are equivalences. Then, the canonical natural transformation

$$i_! j^* \rightarrow g^* f_!$$

is an equivalence of functors $\mathrm{TwSp}(B) \rightarrow \mathrm{TwSp}(C)$.

Proof It is enough to check that the functors $\mathrm{TwSp}(B, \sigma) \rightarrow \mathrm{TwSp}(C)$ are naturally equivalent for all choices of twist σ . Fixing a twist σ , the upper composition corresponds to the functor

$$\mathrm{TwSp}(B, \sigma) \xrightarrow{j^*} \mathrm{TwSp}(A, j^* \sigma) \simeq \mathrm{TwSp}(A, i^* \rho) \xrightarrow{i_!} \mathrm{TwSp}(C, \rho),$$

where the middle equivalence is coming from the fact that $B \operatorname{Pic}(i)$ is an equivalence, so that there is an essentially unique $\rho \in \operatorname{Map}(C, B \operatorname{Pic}(\mathbb{S}))$ such that $i^* \rho \simeq j^* \sigma$. Similarly, the lower composition corresponds to the functor

$$\operatorname{TwSp}(B, \sigma) \simeq \operatorname{TwSp}(B, f^* \pi) \xrightarrow{f_!} \operatorname{TwSp}(D, \pi) \xrightarrow{g^*} \operatorname{TwSp}(C, g^* \pi),$$

where the equivalence at the start is coming from the fact that $B \operatorname{Pic}(f)$ is an equivalence, so that there is an essentially unique $\pi \in \operatorname{Map}(D, B \operatorname{Pic}(\mathbb{S}))$ such that $f^* \pi \simeq \sigma$. Next, we note that $g^* \pi \simeq \rho$. Indeed, since $B \operatorname{Pic}(i)$ is an equivalence, this follows from

$$i^* g^* \pi \simeq j^* f^* \pi \simeq j^* \sigma \simeq i^* \rho.$$

The wanted result now follows from applying Proposition 3.9 to the pullback diagram

$$\begin{array}{ccc} (A, \tau) & \xrightarrow{j} & (B, \sigma) \\ \downarrow i & & \downarrow f \\ (C, \rho) & \xrightarrow{g} & (D, \pi) \end{array}$$

in the category $\mathcal{S}_{/B \operatorname{Pic}(\mathbb{S})}$. Here, τ is the necessarily unique twist on A such that $\tau \simeq j^* f^* \pi \simeq i^* g^* \pi$. $\square$

Remark 5.11 If $f: A \rightarrow B$ is a map of spaces such that the induced map $B \operatorname{Pic}(f)$ is an equivalence, then the map f is itself very close to being an equivalence itself. Indeed, such a map must necessarily induce isomorphisms of homotopy groups in degrees ≥ 2 , since $\Omega \operatorname{Pic}(\mathbb{S}) \simeq \Omega^\infty \mathbb{S}$. In fact, the authors of this paper have not yet come up with an example of a map $f: A \rightarrow B$ for which $B \operatorname{Pic}(f)$ is an equivalence, but which is itself *not* an equivalence.

5.3 Symmetric monoidal structure on the total category

We want to assemble the interior product to a symmetric monoidal structure on the total category of twisted spectra over the space A . Let us recall some important facts. For every space A , the mapping space $\operatorname{Map}(A, B \operatorname{Pic}(\mathbb{S}))$ becomes an $\mathbb{E}_\infty$ -space by using the $\mathbb{E}_\infty$ -structure on $\operatorname{Pic}(\mathbb{S})$ and the cartesian symmetric monoidal structure on $\mathcal{S}$, which makes every space into a co- $\mathbb{E}_\infty$ -spaces via its diagonal map. Note that the map $B \operatorname{Pic}(f)$ is always an $\mathbb{E}_\infty$ -map.

Proposition 5.12 *The category $\operatorname{TwSp}(A)$ is endowed with a symmetric monoidal structure that we will denote by $\otimes_A$. This symmetric monoidal structure preserves small colimits in each variable separately. Moreover, this symmetric monoidal structure can be identified with the interior product described in Definition 3.13 in the sense that the following diagram commutes:*

$$\begin{array}{ccc} \operatorname{TwSp}(A, \tau) \otimes \operatorname{TwSp}(A, \sigma) & \xrightarrow{\otimes} & \operatorname{TwSp}(A, \tau + \sigma) \\ \downarrow & & \downarrow \\ \operatorname{TwSp}(A) \otimes \operatorname{TwSp}(A) & \xrightarrow{\otimes_A} & \operatorname{TwSp}(A) \end{array}$$

Proof Note that $\text{Map}(A, B \text{ Pic}(\mathbb{S}))$ is an $\mathbb{E}_\infty$ -space by the interior product and the fact that the functor

$$\text{TwSp}(A, -): \text{Map}(A, B \text{ Pic}(\mathbb{S})) \rightarrow \text{Fun}(A, \text{Pr}_{\text{St}}^L) \rightarrow \text{Pr}_{\text{St}}^L$$

is lax symmetric monoidal. In other words, it is an $\mathbb{E}_\infty$ -algebra in $\text{Fun}(\text{Map}(A, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^L)$. Since the functor

$$\text{colim}: \text{Fun}(\text{Map}(A, B \text{ Pic}(\mathbb{S})), \text{Pr}_{\text{St}}^L) \rightarrow \text{Pr}_{\text{St}}^L$$

is symmetric monoidal, this tells us that the colimit $\text{TwSp}(A)$ is an $\mathbb{E}_\infty$ -algebra in Pr_{St}^L , that is, a presentably symmetric monoidal ∞ -category. All that is left now is identifying this with the interior product on twisted spectra already defined in Definition 3.13. Recall that the upper map is defined as the composition $\text{TwSp}(A, \tau) \times \text{TwSp}(A, \sigma) \rightarrow \text{TwSp}(A, \tau) \otimes \text{TwSp}(A, \sigma) \simeq \text{TwSp}(A \times A, \tau \boxtimes \sigma) \xrightarrow{\Delta^*} \text{TwSp}(A, \tau + \sigma)$, the identifications done in the proof of Proposition 3.10, and the fact that commuting colimits is natural. $\square$

For reference, we also include the following lemma.

Lemma 5.13 *Let $f: A \rightarrow B$ be a map of spaces. The category $\text{TwSp}(A, f^*)$ is symmetric monoidal and the inclusion $\text{TwSp}(A, f^*) \rightarrow \text{TwSp}(A)$ is symmetric monoidal.*

Proof The map $B \text{ Pic}(f)$ is a map of $\mathbb{E}_\infty$ -spaces, so the symmetric monoidality of $\text{TwSp}(A, f^*)$ follows from the same line of argument as the proof of Proposition 5.12. That the inclusion is symmetric monoidal is clear since $\text{TwSp}(A, f^*)$ is a subcategory of $\text{TwSp}(A)$ which is closed under the tensor product. $\square$

Definition 5.14 If $X \in \text{TwSp}(A, \tau)$ is dualizable in the symmetric monoidal ∞ -category $\text{TwSp}(A)$, we will say that X is *Spanier–Whitehead dualizable*. We will refer to its dual $D^{SW} X$, which is necessarily a $\text{TwSp}(A, -\tau)$ -twisted spectrum, as the *Spanier–Whitehead dual* of X .

We note that, just as Spanier–Whitehead duality for parametrized spectra is not the “correct” notion of dualizability in that setting (see [32, Part IV]), there is another form of dualizability of twisted spectra that is more appropriate. We will look at this in Section 6.2.

Proposition 5.15 *The functor $f^*: \text{TwSp}(B) \rightarrow \text{TwSp}(A)$ is symmetric monoidal with respect to the interior product $\otimes$. That is, if X and Y are twisted spectra over B , then*

$$f^*(X \otimes Y) \simeq f^* X \otimes f^* Y.$$

Proof Recall that the functor $f^*: \text{TwSp}(B) \rightarrow \text{TwSp}(A)$ factors as

$$\text{TwSp}(B) \rightarrow \text{TwSp}(A, f^*) \rightarrow \text{TwSp}(A).$$

So it is enough to show that both functors in the composition are symmetric monoidal. That the first functor is symmetric monoidal follows from the fact that the natural transformation $\text{Nat}(f^*)$ is a symmetric monoidal transformation, which follows from Corollary 3.16, and that the colimit functor is symmetric monoidal. That the second functor is symmetric monoidal follows from Lemma 5.13. $\square$

5.4 Projection isomorphism on total category

In this section, we prove the projection isomorphism in the total category of twisted spectra over a fixed space. This essentially follows directly from the Beck–Chevalley result that we proved in Section 5.2.

Proposition 5.16 (Projection isomorphism) *Let $f : A \rightarrow B$ be a map of spaces such that $B \operatorname{Pic}(f)$ is an equivalence and let $X \in \operatorname{TwSp}(A)$ and $Y \in \operatorname{TwSp}(B)$. Then we have an equivalence*

$$f_!(f^*Y \otimes X) \simeq Y \otimes f_!X$$

of twisted spectra over B .

Proof Formally, we have that

$$\begin{aligned} f_!(f^*Y \otimes X) &\simeq f_!(\Delta_A^*(f^*Y \boxtimes X)) \simeq f_!(\Delta_A^*((f \times 1)^*(Y \boxtimes X))) \simeq f_!(((f \times 1) \circ \Delta_A)^*(Y \boxtimes X)) \\ &\simeq f_!((f, 1)^*(Y \boxtimes X)), \end{aligned}$$

while

$$Y \otimes f_!X \simeq \Delta_B^*(Y \boxtimes f_!X) \simeq \Delta_B^*((1 \times f)_!(Y \boxtimes X)).$$

To finish the proof we need to use Proposition 5.10 for the diagram

$$\begin{array}{ccc} A & \xrightarrow{(f, 1)} & B \times A \\ f \downarrow & & \downarrow 1 \times f \\ B & \xrightarrow{\Delta_B} & B \times B \end{array}$$

so that we get an equivalence

$$f_! \circ (f, 1)^* \simeq \Delta_B^* \circ (1 \times f)_!.$$

Note that since f satisfies the necessary assumption for applying Proposition 5.10, so does $1 \times f$ by observing that $\operatorname{cofib}(f) \simeq \operatorname{cofib}(1 \times f)$ and using the fiber sequence

$$\operatorname{Map}(\operatorname{cofib}(1 \times f), B \operatorname{Pic}(\mathbb{S})) \rightarrow \operatorname{Map}(B \times B, B \operatorname{Pic}(\mathbb{S})) \xrightarrow{B \operatorname{Pic}(1 \times f)} \operatorname{Map}(B \times A, B \operatorname{Pic}(\mathbb{S})). \quad \square$$

5.5 Closed structure on the total category

In this section, we show that the symmetric monoidal structure on $\operatorname{TwSp}(A)$ is closed, so that there exist functorial internal hom twisted spectra, which are right adjoint to the tensor product. Moreover, we identify this with the internal hom twisted spectra functors that we considered in Section 3.6. First we record the following lemma, which is immediate from the adjoint functor theorem.

Proposition 5.17 *Let $\mathcal{C}^\otimes$ be a presentably symmetric monoidal presentable ∞ -category. Then $\mathcal{C}^\otimes$ is closed in the sense that for each object $Y \in \mathcal{C}$, the functor $- \otimes Y : \mathcal{C} \rightarrow \mathcal{C}$ admits a right adjoint.*

Turning to the case of twisted spectra at hand, we set $Y \in \operatorname{TwSp}(A)$. Then the functor $- \otimes Y$ has right adjoint, which we denote by $\operatorname{hom}_{\operatorname{TwSp}(A)}(Y, -)$, or simply $\operatorname{hom}(Y, -)$ when the context is clear. Being an adjoint, we have the natural equivalence

$$(5) \quad \operatorname{map}(X \otimes Y, Z) \simeq \operatorname{map}(X, \operatorname{hom}(Y, Z)).$$

We would like to show that this constructed internal hom is compatible with internal homs that we constructed in Section 3.6. Morally this should just follow by uniqueness of adjoints and the fact that the symmetric monoidal structure on the total category $\mathrm{TwSp}(A)$ is determined by the various pairings

$$\mathrm{TwSp}(A, \tau) \otimes \mathrm{TwSp}(A, \sigma) \rightarrow \mathrm{TwSp}(A, \tau + \sigma),$$

as per Proposition 5.12. However, let us be a bit more precise about this.

Proposition 5.18 *The symmetric monoidal structure of $\mathrm{TwSp}(A)$ is closed, and this closed structure can be identified by the definition of the interior homs of Section 3.6. Indeed, let $Y \in \mathrm{TwSp}(A, \sigma)$ be a twisted spectrum viewed as an object in $\mathrm{TwSp}(A)$ via the canonical inclusion $i_\sigma: \mathrm{TwSp}(A, \sigma) \rightarrow \mathrm{TwSp}(A)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{TwSp}(A) & \xrightarrow{\mathrm{hom}(X, -)} & \mathrm{TwSp}(A) \\ \downarrow & & \downarrow \\ \mathrm{TwSp}(A, \tau + \sigma) & \xrightarrow{\mathrm{hom}(X, -)} & \mathrm{TwSp}(A, \tau) \end{array}$$

for all twists τ .

Proof Let $B\tau$ denote the connected component of τ in the space $\mathrm{Map}(A, B\mathrm{Pic}(\mathbb{S}))$. Note that the inclusion $\iota_\tau: B\tau \hookrightarrow \mathrm{Map}(A, B\mathrm{Pic}(\mathbb{S}))$ induces a map $(\iota_\tau)_!: \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{TwSp}(A)$; we can identify this with the “inclusion map” mentioned in the statement. Moreover, we can identify the (internal) right adjoint of this functor, $(\iota_\tau)^*$, with the projection map $\pi_\alpha: \mathrm{TwSp}(A) \rightarrow \mathrm{TwSp}(A, \alpha)$, obtained appealing to ambidexterity and the universal property of $\mathrm{TwSp}(A)$ as a limit. We now turn our attention to the commutative diagram of functors

$$\begin{array}{ccc} \mathrm{TwSp}(A, \tau) & \xrightarrow{-\otimes Y} & \mathrm{TwSp}(A, \tau + \sigma) \\ \downarrow (\iota_\tau)_! & & \downarrow (\iota_{\tau+\sigma})_! \\ \mathrm{TwSp}(A) & \xrightarrow{-\otimes_A Y} & \mathrm{TwSp}(A) \end{array}$$

obtained by restricting the diagram from Proposition 5.12. Taking right adjoints, we obtain the desired diagram. $\square$

6 The $(\infty, 2)$ -category of twisted spectra

In this section, we will construct an $(\infty, 2)$ -category of twisted spectra, much like how May–Sigurdsson construct a bicategory of parametrized spectra [32, Chapter 17]. We will use the $(\infty, 2)$ -categorical structure to define a corresponding notion of Costenoble–Waner duality for twisted spectra. Lastly, we return to the perspective on twisted spectra as modules over Thom spectra from Section 3.7, and tie the $(\infty, 2)$ -category of twisted spectra in with categories of bimodules and the $(\infty, 2)$ -categorical structure of these.

6.1 The $(\infty, 2)$ -category of twisted parametrized spectra

Twisted spectra, over all spaces and twists, can be viewed as an $(\infty, 2)$ -category, along the lines of Section 2 of [9].

Definition 6.1 The $(\infty, 2)$ -category of twisted spectra $\mathbf{TwSp}$ is the full subcategory of $\mathbf{Pr}_{\mathcal{S}\mathcal{I}}^L$ spanned by those stable presentable ∞ -categories of the form $\mathbf{TwSp}(A, \tau)$.

Notation 6.2 Fix a twist $\tau: A \rightarrow B \operatorname{Pic}(\mathcal{S})$ and consider the diagonal map $\Delta: A \rightarrow A \times A$. We have an induced functor

$$\Delta_!: \operatorname{Sp}^A \simeq \mathbf{TwSp}(A, -\tau + \tau) \rightarrow \mathbf{TwSp}(A, -\tau \boxtimes \tau).$$

Evaluating this functor at the trivially parametrized sphere spectrum over $\mathcal{S}$, we obtain

$$\mathcal{S}[\Omega A]^\tau := \Delta_! \mathcal{S} \in \mathbf{TwSp}(A \times A, -\tau \boxtimes \tau),$$

which will refer to as the τ -twisted spherical group ring.

Indeed, this works because of the following result.

Proposition 6.3 *There is an equivalence of categories*

$$\mathbf{TwSp}(A \times B, -\tau \boxtimes \sigma) \simeq \operatorname{Fun}^L(\mathbf{TwSp}(A, \tau), \mathbf{TwSp}(B, \sigma)).$$

Explicitly, it should work as follows:

- It sends $X \in \mathbf{TwSp}(A \times B, -\tau \boxtimes \sigma)$ to the colimit-preserving functor

$$\Phi_X = (p_B)_!((p_A)^*(-) \otimes X): \mathbf{TwSp}(A, \tau) \rightarrow \mathbf{TwSp}(B, \sigma),$$

where $p_A: A \times B \rightarrow A$ and $p_B: A \times B \rightarrow B$ denotes the projections.

- It sends the left adjoint functor $F: \mathbf{TwSp}(A, \tau) \rightarrow \mathbf{TwSp}(B, \sigma)$ to the evaluation

$$(1 \otimes F)(\mathcal{S}[\Omega A]^\tau) \in \mathbf{TwSp}(B, \sigma),$$

where we are implicitly using Proposition 3.10.

Proof First, let us note the following: if $\mathcal{C}$ is an invertible presentable stable ∞ -category, that is if $\mathcal{C} \in B \operatorname{Pic}(\mathcal{S})$, then its dual $\mathcal{C}^\vee = \operatorname{Fun}^L(\mathcal{C}, \operatorname{Sp})$, which is also equivalent to $\mathcal{C}^{\operatorname{op}}$, simply corresponds to the inverse in the group structure in $B \operatorname{Pic}(\mathcal{S})$. From this, we can conclude that

$$\begin{aligned} \mathbf{TwSp}(A, \tau)^\vee &= \operatorname{Fun}^L(\mathbf{TwSp}(A, \tau), \operatorname{Sp}) \simeq \operatorname{Fun}^L(\operatorname{colim}_{a \in A} \tau(a), \operatorname{Sp}) \simeq \lim_{a \in A} \operatorname{Fun}^L(\tau(a), \operatorname{Sp}) \simeq \lim_{a \in A} -\tau(a) \\ &= \mathbf{TwSp}(A, -\tau), \end{aligned}$$

using that $\operatorname{Fun}^L(-, \operatorname{Sp})$ turns colimits into limits and that $\tau(a) \in B \operatorname{Pic}(\mathcal{S})$ for all $a \in A$.

Now we use the fact that $\mathrm{TwSp}(A, \tau)$ is dualizable with dual $\mathrm{TwSp}(A, -\tau)$ to conclude the general case. We have

$$\begin{aligned} \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma) &\simeq \mathrm{TwSp}(A, -\tau) \otimes^L \mathrm{TwSp}(B, \sigma) \simeq \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) \otimes^L \mathrm{TwSp}(B, \sigma) \\ &\simeq \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)), \end{aligned}$$

where the first equivalence is Proposition 3.10, and the last equivalence follows by dualizability of $\mathrm{TwSp}(A, \tau)$.

The functor $\Phi: \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma) \rightarrow \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma))$ sends a twisted spectrum X to its Fourier Mukai transform Φ_X . The inverse is given by the composition

$$\begin{aligned} \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) &\rightarrow \mathrm{Fun}^L(\mathrm{TwSp}(A \times A, -\tau \boxtimes \tau), \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma)) \\ &\xrightarrow{\mathrm{ev}_{u_{A, \tau}}} \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma). \quad \square \end{aligned}$$

Definition 6.4 Given a $(A \times B, -\tau \boxtimes \sigma)$ -twisted spectrum X , we write

$$\Phi_X = (p_B)_! \circ (- \otimes X) \circ (p_A)^*: \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{TwSp}(B, \sigma)$$

and call this the *Fourier–Mukai transform* of X .

Remark 6.5 Let us be a bit more explicit about how Proposition 6.3 implies the above expression. Indeed first recall we have the standard evaluation pairing

$$\mathrm{ev}: \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) \otimes \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{TwSp}(B, \sigma)$$

given by

$$(X, F) \mapsto F(X) \in \mathrm{TwSp}(B, \sigma).$$

The proof of Proposition 6.3 shows that there is a natural equivalence of pairings

$$\begin{array}{ccc} \mathrm{Fun}^L(\mathrm{TwSp}(A, \tau), \mathrm{TwSp}(B, \sigma)) \otimes \mathrm{TwSp}(A, \tau) & \xrightarrow{\mathrm{ev}} & \mathrm{TwSp}(B, \sigma) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{TwSp}(A, -\tau) \otimes \mathrm{TwSp}(B, \sigma) \otimes \mathrm{TwSp}(A, -\tau) & \xrightarrow{\mathrm{ev}} & \mathrm{TwSp}(B, \sigma) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{TwSp}(A \times B, -\tau \boxtimes \sigma) \otimes \mathrm{TwSp}(B, \sigma) & \xrightarrow{\mathrm{ev}} & \mathrm{TwSp}(B, \sigma) \end{array}$$

where the bottom pairing is given by

$$(X, M) \mapsto (p_B)_!(p_A^*(M) \otimes X) \in \mathrm{TwSp}(B, \sigma).$$

Note that the middle pairing above is none other than evaluation pairing $\mathrm{TwSp}(A, -\tau) \otimes \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{Sp}$ corresponding to the duality datum on the dualizable category $\mathrm{TwSp}(A, -\tau)$, tensored with $\mathrm{TwSp}(B, \sigma)$.

The above means that we can heuristically think of the $(\infty, 2)$ -category $\mathbf{TwSp}$, as shown in Table 2.

	TwSp	Pr_{St}^L
objects	twist $\tau: A \rightarrow B \text{ Pic}(\mathbb{S})$	presentable stable ∞ -category $\text{TwSp}(A, \tau)$
1-morphism	$X \in \text{TwSp}(A \times B, -\tau \boxtimes \sigma)$	colimit-preserving functor $\Phi_X: \text{TwSp}(A, \tau) \rightarrow \text{TwSp}(B, \sigma)$
2-morphism	map $X \rightarrow Y$ of twisted spectra	natural transformation $\eta: \Phi_X \rightarrow \Phi_Y$
identity 1-morphism	τ -twisted spherical group ring $\Delta_1^\tau(\mathbb{S}) \in \text{TwSp}(A \times A, -\tau \boxtimes \tau)$	identity functor $1: \text{TwSp}(A, \tau) \rightarrow \text{TwSp}(A, \tau)$
vertical composition	composition of maps of twisted spectra	vertical composition of natural transformations
horizontal composition of 1-morphisms	$Y \odot X \in \text{TwSp}(A \times C, -\tau \boxtimes \rho)$	composition of functors

Table 2: How to think of **TwSp** as a full subcategory of **Pr_{St}^L**.

Definition 6.6 Let $X \in \text{TwSp}(A \times B, -\tau \boxtimes \sigma)$ and $Y \in \text{TwSp}(B \times C, -\sigma \boxtimes \rho)$. The *horizontal composition* of X and Y is defined as

$$Y \odot X = (p_{A \times C})_!(p_{B \times C}^* Y \otimes p_{A \times B}^* X) \in \text{TwSp}(A \times C, -\tau \boxtimes \rho),$$

where $p_{A \times B}: A \times B \times C \rightarrow A \times B$ and $p_{B \times C}: A \times B \times C \rightarrow B \times C$ are the projections. Here, the tensor product is taken in twisted spectra over the space $A \times B \times C$.

Let us prove that this actually corresponds to the horizontal composition in **Pr_{St}^L**, which is simply composition of functors.

Proposition 6.7 Let $X \in \text{TwSp}(A \times B, -\tau \boxtimes \sigma)$ and let $Y \in \text{TwSp}(B \times C, -\sigma \boxtimes \rho)$. Then

$$\Phi_{Y \odot X} \simeq \Phi_Y \circ \Phi_X \quad \text{and} \quad \Phi_{\Delta_1^\tau \mathbb{S}} \simeq 1_{\text{TwSp}(A, \tau)}.$$

Proof The assertions essentially follows from using the projection isomorphism and Beck–Chevalley a couple of times for various maps. To keep track of maps let us use the notation

$$p_A: A \times B \rightarrow A, \quad q_B: B \times C \rightarrow B, \quad r_A: A \times C \rightarrow C,$$

$$p_B: A \times B \rightarrow B, \quad q_C: B \times C \rightarrow C, \quad r_C: A \times C \rightarrow A,$$

for the various projections. We will also write

$$p_{A \times B}: A \times B \times C \rightarrow A \times B, \quad p_{B \times C}: A \times B \times C \rightarrow B \times C, \quad p_{A \times C}: A \times B \times C \rightarrow A \times C,$$

and

$$\pi_A: A \times B \times C \rightarrow A, \quad \pi_B: A \times B \times C \rightarrow B, \quad \pi_C: A \times B \times C \rightarrow C.$$

Per definition, we have that

$$\Phi_Y \circ \Phi_X = (q_C)_! \circ (- \otimes Y) \circ (q_B)^* \circ (p_B)_! \circ (- \otimes X) \circ (p_A)^*,$$

and that

$$\Phi_{Y \odot X} = (r_C)_! \circ (- \otimes (Y \odot X)) \circ (r_A)^* = (r_C)_! \circ (- \otimes ((p_{A \times C})_!(p_{B \times C}^* Y \otimes p_{A \times B}^* X))) \circ (r_A)^*.$$

These are the compositions that we are trying to identify. We start by using Beck–Chevalley on the pullback diagram

$$\begin{array}{ccc} A \times B \times C & \xrightarrow{p_{A \times B}} & A \times B \\ p_{B \times C} \downarrow & & \downarrow p_B \\ B \times C & \xrightarrow{q_B} & B \end{array}$$

so that we have that $(q_B)^* \circ (p_B)_! \simeq (p_{B \times C})_! \circ (p_{A \times B})^*$. By combining this with the fact that the pullback $(p_{A \times B})^*$ is symmetric monoidal we obtain

$$\Phi_Y \circ \Phi_X \simeq (q_C)_! \circ (- \otimes Y) \circ (p_{B \times C})_! \circ (- \otimes (p_{A \times B})^*(X)) \circ (\pi_A)^*.$$

By using the projection isomorphism for the map $p_{B \times C}$ we can further write

$$\Phi_Y \circ \Phi_X \simeq (\pi_C)_! \circ (- \otimes (p_{B \times C})^*(Y)) \circ (- \otimes (p_{A \times B})^*(X)) \circ (\pi_A)^*.$$

Let us now look closer at the functor $\Phi_{Y \odot X}$. By using the projection isomorphism for the map $p_{A \times C}$ to make the identification

$$\Phi_{Y \odot X} \simeq (\pi_C)_! \circ (- \otimes ((p_{B \times C})^* Y \otimes (p_{A \times B})^* X)) \circ (\pi_A)^*$$

which by inspection agrees with the identification we made for $\Phi_Y \circ \Phi_X$. This finishes the proof of the first assertion. For the second assertion, we will use the notation $p_1, p_2: A \times A \rightarrow A$ for the projection onto the first and second factor, respectively. Using the projection isomorphism for the diagonal map $\Delta: A \rightarrow A \times A$ we obtain

$$(p_2)_! \circ (- \otimes \Delta_! \mathbb{S}) \circ (p_1)^* \simeq (p_2)_! \circ \Delta_! \circ (- \otimes \mathbb{S}) \circ \Delta^* (p_1)^* \simeq (-),$$

where we have used that $p_i \circ \Delta \simeq 1$ for $i = 1, 2$ and that tensoring with $\mathbb{S}$ is the identity. $\square$

Remark 6.8 The above describes the composition of Fourier–Mukai functor in terms of some operation on twisted spectra. There should also be a horizontal composition on 2-morphisms that, interpreted in the language of twisted spectra, should allow us to take a map $f: X \rightarrow Y$ in $\text{TwSp}(A \times B, -\tau \boxtimes \sigma)$ and a map $g: Z \rightarrow W$ in $\text{TwSp}(B \times C, -\sigma \boxtimes \rho)$, and construct a map

$$g \odot f: Z \odot X \rightarrow W \odot Y$$

in $\text{TwSp}(A \times C, -\tau \boxtimes \rho)$. Due to the functoriality, this should be the map

$$g \odot f = (p_{A \times C})_! (p_{B \times C}^* g \otimes p_{A \times B}^* f): Z \odot X \rightarrow W \odot Y.$$

We will now consider the functors

$$f_!: \text{TwSp}(A, f^* \sigma) \rightarrow \text{TwSp}(B, \sigma) \quad \text{and} \quad f^*: \text{TwSp}(B, \sigma) \rightarrow \text{TwSp}(A, f^* \sigma)$$

associated to a map $f: A \rightarrow B$ of spaces. Since both functors are colimit-preserving, Proposition 6.3 tells us there has to be a $(A \times B, -f^* \sigma \boxtimes \sigma)$ -twisted spectrum and $(B \times A, -\sigma \boxtimes f^* \sigma)$ -twisted spectra corresponding to the functors $f_!$ and f^* , respectively. Let us figure out what these are.

Proposition 6.9 *Let $f: A \rightarrow B$ be a map of spaces and $\sigma: B \rightarrow B \text{ Pic}(\mathbb{S})$ classify a haunt over B . Then the following hold:*

- (1) *The pullback of $-f^*\sigma \boxtimes \sigma: A \times B \rightarrow B \text{ Pic}(\mathbb{S})$ along the map $(1, f): A \rightarrow A \times B$ is trivial. Hence it induces a functor*

$$(1, f)_!: \text{Sp}^A \rightarrow \text{TwSp}(A \times B, -\tau \boxtimes \sigma).$$

Similarly, the pullback of $-\sigma \boxtimes f^*\sigma: B \times A \rightarrow B \text{ Pic}(\mathbb{S})$ along the map $(f, 1): A \rightarrow B \times A$ is trivial and hence induces a functor

$$(f, 1)_!: \text{Sp}^A \rightarrow \text{TwSp}(B \times A, -\sigma \boxtimes f^*\sigma).$$

- (2) *The functor $f_!: \text{TwSp}(A, f^*\sigma) \rightarrow \text{TwSp}(B, \sigma)$ corresponds to the $(A \times B, -\tau \boxtimes \sigma)$ -twisted spectrum $(1, f)_! \mathbb{S}$ under the Fourier–Mukai transform. That is:*

$$\Phi_{(1, f)_! \mathbb{S}} \simeq f_!.$$

- (3) *The functor $f^*: \text{TwSp}(B, \sigma) \rightarrow \text{TwSp}(A, f^*\sigma)$ corresponds to the $(B \times A, -\sigma \boxtimes f^*\sigma)$ -twisted spectrum $(f, 1)_! \mathbb{S}$. That is:*

$$\Phi_{(f, 1)_! \mathbb{S}} \simeq f^*.$$

Proof The first part of the theorem is straight-forward. The second and third statement are proved in the same way, so we only prove the second statement, leaving the third for the reader. By using the projection isomorphism for the map $(1, f)$, we obtain

$$\Phi_{(1, f)_! \mathbb{S}} = (p_B)_! \circ (- \otimes (1, f)_! \mathbb{S}) \circ (p_A)^* \simeq (p_B)_! \circ (1, f)_! \circ (- \otimes \mathbb{S}) \circ (1, f)^* \circ (p_A)^*$$

from which the result follows by observing that $p_B \circ (1, f)$ is just f and that $p_A \circ (1, f)$ is the identity, and that $\mathbb{S}$ is the unit for the symmetric monoidal structure. $\square$

6.2 Dualizability of twisted spectra

Note that, per construction, the Fourier–Mukai transform of a twisted spectrum $X \in \text{TwSp}(A \times B, -\tau \boxtimes \sigma)$ is colimit-preserving, so it must admit a right adjoint, namely the composition

$$\text{TwSp}(B, \sigma) \xrightarrow{(p_B)^*} \text{TwSp}(A \times B, (p_B)^*\sigma) \xrightarrow{\text{hom}(X, -)} \text{TwSp}(A \times B, (p_B)^*\sigma + \tau \boxtimes \sigma) \xrightarrow{(p_A)^*} \text{TwSp}(A, \tau).$$

Dualizability of a twisted spectrum is going to be phrased in terms of whether this functor is itself colimit-preserving.

Definition 6.10 The $(A \times B, -\tau \boxtimes \sigma)$ -twisted spectrum X is *dualizable* if the functor $\Phi_X: \text{TwSp}(A, \tau) \rightarrow \text{TwSp}(B, \sigma)$ is an internal left adjoint in $\text{Pr}_{\mathbb{S}}^L$. In other words, it is dualizable if the right adjoint to its Fourier–Mukai transform is itself colimit-preserving.

In particular, notice that if X is dualizable, then by Proposition 6.3 the right adjoint must be on the form $\Phi_{X^\vee}$ for some essentially unique $(B \times A, -\sigma \boxtimes \tau)$ -twisted spectrum $X^\vee$. We refer to this $X^\vee$ as the *dual* of X .

Example 6.11 Since the right adjoint of $f_!$ is f^* , which itself admits a left adjoint, we can use Proposition 6.9 to conclude that the dual of the $\text{TwSp}(A \times B, -f^*\sigma \boxtimes \sigma)$ -twisted spectrum $X = (1, f)_! S$ is the $\text{TwSp}(B \times A, -\sigma \boxtimes f^*\sigma)$ -twisted spectrum $X^\vee = (f, 1)_! S$.

We will now recast some of the above discussion in the extreme cases that one of the spaces A or B is a point. We will abusively denote the projections $A \rightarrow *$ and $B \rightarrow *$ simply by A and B , respectively. We first note the following:

- If A is a point, then the (B, σ) -twisted spectrum X corresponds to the colimit-preserving functor

$$\Phi_X: \text{Sp} \rightarrow \text{TwSp}(B, \sigma), \quad Y \mapsto B^*(Y) \otimes X.$$

- If B is a point, then the $(A, -\tau)$ -twisted spectrum X corresponds to the colimit-preserving functor

$$\Phi_X: \text{TwSp}(A, \tau) \rightarrow \text{Sp}, \quad Y \mapsto A_!(Y \otimes X).$$

Per the above, a twisted spectrum can thus be viewed as a colimit-preserving functor in essentially two ways: as a functor with either target or source Sp . This gives rise to a lot of interesting structures. To be able to be slightly more explicit in our writing, let us set the following notation in this section.

Notation 6.12 Let $X \in \text{TwSp}(A, \tau)$. We retain the notation

$$\Phi_X: \text{Sp} \rightarrow \text{TwSp}(A, \tau), \quad Y \mapsto A^*(Y) \otimes X$$

for the Fourier–Mukai transform thought of as a functor $\text{Sp} \rightarrow \text{TwSp}(A, \tau)$, but write

$$\Psi_X: \text{TwSp}(A, -\tau) \rightarrow \text{Sp}, \quad Y \mapsto A_!(Y \otimes X)$$

for the Fourier–Mukai transform thought of as a functor $\text{TwSp}(A, -\tau) \rightarrow \text{Sp}$.

In particular, note that the Φ - and Ψ -notation are exchanged under duality: if we think of $X \in \text{TwSp}(A, \tau)$ as the functor Φ_X , then we must think of its dual $X^\vee \in \text{TwSp}(A, -\tau)$ as the functor $\Psi_{X^\vee}$, and vice versa. The following preliminary lemma might be useful.

Lemma 6.13 Let $X \in \text{TwSp}(A, -\tau)$ and let $Y \in \text{TwSp}(A, \tau)$. Then

$$\Psi_X \circ \Phi_Y \simeq \Phi_{A_!(X \otimes Y)} \quad \text{and} \quad \Phi_Y \circ \Psi_X \simeq \Phi_{X \boxtimes Y}$$

where the right hand sides refer to the Fourier–Mukai transforms associated to $A_!(Y \otimes X) \in \text{Sp}$ and $X \boxtimes Y \in \text{TwSp}(A \times A, -\tau \boxtimes \tau)$, respectively.

Proof The first statement follows from using the projection isomorphism for the map $A \rightarrow *$ to obtain

$$\Psi_X \circ \Phi_Y \simeq A_! \circ (- \otimes Y) \circ (- \otimes X) \circ A^* \simeq (- \otimes A_!(X \otimes Y)) \circ A^* \simeq \Phi_{A_!(X \otimes Y)}.$$

For the second statement, let us try to try to understand $\Phi_{X \boxtimes Y}$. We will use the notation $p_1, p_2: A \times A \rightarrow A$ refers to the first and second projection, respectively. Then we have that

$$\Phi_{X \boxtimes Y} = (p_2)_! \circ (- \otimes (X \boxtimes Y)) \circ (p_1)^* \simeq (p_2)_! \circ (- \otimes ((p_1)^* X \otimes (p_2)^* Y)) \circ (p_1)^*$$

by using Lemma 3.14. By using that $(p_1)^*$ is symmetric monoidal and the projection isomorphism for the map p_2 we obtain

$$\Phi_{X \boxtimes Y} \simeq (- \otimes Y) \circ (p_2)_! \circ (p_1)^* \circ (- \otimes X).$$

We finish the proof by using Beck–Chevalley for the pullback diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{p_1} & A \\ p_2 \downarrow & & \downarrow \\ A & \longrightarrow & * \end{array}$$

so that we can write

$$\Phi_{X \boxtimes Y} \simeq (- \otimes Y) \circ A^* \circ A_! \circ (- \otimes X) = \Phi_Y \circ \Psi_X. \quad \square$$

The reason for the auxiliary notation Ψ_X is that duality of X in the sense of the functor Ψ_X being an internal left adjoint is not very interesting from a structural point of view; it just reduces to Spanier–Whitehead dualizability.

Proposition 6.14 *Consider a twisted spectrum $X \in \text{TwSp}(A, -\tau)$, which we view as a colimit-preserving functor*

$$\Psi_X : \text{TwSp}(A, \tau) \rightarrow \text{Sp}.$$

The dualizability of X in the sense of Definition 6.10 then reduces to dualizability of X in the symmetric monoidal ∞ -category $\text{TwSp}(A)$.

Proof Let us write DX for the supposed dual of X . Dualizability in the sense of Definition 6.10, then just says that $\Phi_{DX} : \text{Sp} \rightarrow \text{TwSp}(A, \tau)$ is right adjoint to Ψ_X . In particular, this means that we have a unit and counit

$$\eta : 1 \rightarrow \Phi_{DX} \circ \Psi_X \quad \text{and} \quad \epsilon : \Psi_X \circ \Phi_{DX} \rightarrow 1,$$

which are natural transformations of functors $\text{TwSp}(A, \tau) \rightarrow \text{TwSp}(A, \tau)$ and $\text{Sp} \rightarrow \text{Sp}$ respectively, satisfying the triangle equalities. We recall that these triangle equality simply says that the compositions

$$\Psi_X \xrightarrow{\Psi_X \eta} \Psi_X \Phi_{DX} \Psi_X \xrightarrow{\epsilon \Psi_X} \Psi_X \quad \text{and} \quad \Phi_{DX} \xrightarrow{\eta \Phi_{DX}} \Phi_{DX} \Psi_X \Phi_{DX} \xrightarrow{\Phi_{DX} \epsilon} \Phi_{DX}$$

are the identity natural transformations on Ψ_X and Φ_{DX} , respectively. In terms of $(A \times A, -\tau \boxtimes \tau)$ -twisted spectra and ordinary spectra, the unit and counit just correspond to maps

$$\eta : \Delta_!^{\tau} \mathbb{S} \rightarrow DX \boxtimes X \quad \text{and} \quad \epsilon : A_!(X \otimes DX) \rightarrow \mathbb{S},$$

which under the $(\Delta_! \vdash \Delta^*)$ -adjunction and the $(A_! \vdash A^*)$ -adjunction correspond to maps

$$\eta : \mathbb{S} \rightarrow DX \otimes X \quad \text{and} \quad \epsilon : X \otimes DX \rightarrow \mathbb{S}$$

of parametrized spectra over A . Under these identifications, the triangle equalities translate to the condition that the compositions

$$X \xrightarrow{1 \otimes \eta} X \otimes DX \otimes X \xrightarrow{\epsilon \otimes 1} X \quad \text{and} \quad DX \xrightarrow{\eta \otimes 1} DX \otimes X \otimes DX \xrightarrow{1 \otimes \epsilon} DX$$

are the identity maps on X and DX , respectively. This is precisely the condition for dualizability in the symmetric monoidal category $\mathrm{TwSp}(A)$. $\square$

The other kind of dualizability does give us something new, though.

Definition 6.15 Suppose that A is a point and consider a twisted spectrum $X \in \mathrm{TwSp}(B, \sigma)$, which we view as a colimit-preserving functor

$$\Phi_X : \mathrm{Sp} \rightarrow \mathrm{TwSp}(B, \sigma)$$

via the Fourier–Mukai transform. The dualizability of X in the sense of Definition 6.10 will then be referred as *Costenoble–Waner dualizability* and the dual $D^{CW} X \in \mathrm{TwSp}(B, -\sigma)$ will be referred as the *Costenoble–Waner dual*.

Example 6.16 Suppose that A is connected and consider the inclusion of a point $a \in A$. By abuse of notation, we will also denote the inclusion by a . For any twist $\tau : A \rightarrow B \mathrm{Pic}(\mathbb{S})$ we have an induced functor $a_!^\tau : \mathrm{Sp} \rightarrow \mathrm{TwSp}(A, \tau)$, where we have decorated the superscript of the functor to remember what twist we are dealing with. Consider the object $X = a_!^\tau \mathbb{S}$. We claim that this is Costenoble–Waner dual to $a_!^{-\tau} \mathbb{S}$. Indeed, one can easily check, by using the projection isomorphism, that

$$\Phi_{a_!^\tau}(\mathbb{S}) \simeq a_!^\tau : \mathrm{Sp} \rightarrow \mathrm{TwSp}(A, \tau).$$

The right adjoint to $\Phi_{a_!^\tau} \mathbb{S}$ is hence the functor $a^{*,\tau} : \mathrm{TwSp}(A, \tau) \rightarrow \mathrm{Sp}$.

6.3 Bimodule perspective on the $(\infty, 2)$ -category of twisted spectra

Let us restrict ourselves to spaces that are connected. We have already seen that we have natural equivalences

$$\mathrm{TwSp}(A, \tau) \simeq \mathrm{Mod}_{\mathrm{Th}(\Omega A)}$$

of ∞ -categories whenever A is pointed and connected. In this section, we will identify the subcategory of the $(\infty, 2)$ -category of twisted spectra with a subcategory of the Morita $(\infty, 2)$ -category. That is, we will recast some of the discussion on $(\infty, 2)$ -categories in terms of bimodules of Thom spectra. We start with the following definition.

Definition 6.17 Let $\mathbf{ThMod}$ denote the full subcategory of $\mathbf{Pr}_{\mathrm{St}}^L$ spanned by those stable presentable ∞ -categories of the form $\mathrm{Mod}_{\mathrm{Th}(\alpha)}$ where $\alpha : G \rightarrow \mathrm{Pic}(\mathbb{S})$ is some object in $\mathrm{Grp}/\mathrm{Pic}(\mathbb{S})$.

	ThMod	Pr_{St}^L
objects	$\mathbb{E}_1$ -algebra $\mathrm{Th}(\alpha)$	presentable stable ∞ -category $\mathrm{Mod}_{\mathrm{Th}(\alpha)}$
1-morphism	$(\mathrm{Th}(\alpha) - \mathrm{Th}(\beta))$ -bimodule M	colimit-preserving functor $F_M : \mathrm{Mod}_{\mathrm{Th}(\alpha)} \rightarrow \mathrm{Mod}_{\mathrm{Th}(\beta)}$
2-morphism	bimodule map $M \rightarrow N$	natural transformation $\eta : F_M \rightarrow F_N$
identity 1-morphism	diagonal bimodule $\mathrm{Th}(\alpha)$	identity functor $1 : \mathrm{Mod}_{\mathrm{Th}(\alpha)} \rightarrow \mathrm{Mod}_{\mathrm{Th}(\alpha)}$
vertical composition	vertical composition of natural transformations	composition of bimodule maps
horizontal composition on 1-morphisms	composition of functors	tensoring over the joint common action

Table 3: How to think of **ThMod** as a full subcategory of **Pr_{St}^L**.

This definition makes sense since for every pair of $\mathbb{E}_1$ -rings R and S , we have an equivalence

$${}_R\mathrm{BiMod}_S \simeq \mathrm{Fun}^L(\mathrm{Mod}_R, \mathrm{Mod}_S)$$

of ∞ -categories; see for example [27, Section 4.8]. In particular, the colimit-preserving functor associated to the $(R-S)$ -bimodule M is

$$F_M = - \otimes_R M : \mathrm{Mod}_R \rightarrow \mathrm{Mod}_S.$$

It is well-known that composition of such functors corresponds to tensoring bimodules over their joint action: if M is an $(R-S)$ -bimodule and N is an $(S-T)$ -bimodule, then the composition $F_N \circ F_M$ corresponds to the $(R-T)$ -bimodule $M \otimes_S N$. This means that we can heuristically think of the $(\infty, 2)$ -category **ThMod** as shown in Table 3.

Essentially, we would like to know that, under the equivalence of twisted spectra and modules over Thom spectra, that the Fourier–Mukai transform precisely corresponds to the functor associated to a bimodule. This is the content of the following result.

Proposition 6.18 *Let $\alpha : G \rightarrow \mathrm{Pic}(\mathbb{S})$ and $\beta : H \rightarrow \mathrm{Pic}(\mathbb{S})$ and suppose we have a $(\mathrm{Th}(\alpha) - \mathrm{Th}(\beta))$ -module M . Suppose that X is the twisted spectra corresponding to M under the equivalence*

$$\mathrm{TwSp}(BG \times BH, -B\alpha \boxtimes B\beta) \simeq_{\mathrm{Th}(\alpha)} \mathrm{BiMod}_{\mathrm{Th}(\beta)}.$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{TwSp}(BG, B\alpha) & \xrightarrow{\Phi_X} & \mathrm{TwSp}(BH, B\beta) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Mod}_{\mathrm{Th}(\alpha)} & \xrightarrow{F_M} & \mathrm{Mod}_{\mathrm{Th}(\beta)} \end{array}$$

Proof Let us study the adjunctions

$$(- \otimes \mathrm{TwSp}(BG, B\alpha)) \vdash \mathrm{Fun}^L(\mathrm{TwSp}(BG, B\alpha), -) \quad \text{and} \quad (- \otimes \mathrm{Mod}_{\mathrm{Th}(\alpha)}) \vdash \mathrm{Fun}^L(\mathrm{Mod}_{\mathrm{Th}(\alpha)}, -)$$

of functors $\mathbf{Pr}_{\mathrm{St}}^L \rightarrow \mathbf{Pr}_{\mathrm{St}}^L$. Due to Theorem 3.26, we know that these adjunctions are equivalent to one another, so for ease of notation let us not make a distinction between them, and simply write $- \otimes \mathcal{C} \vdash \mathbf{Fun}^L(\mathcal{C}, -)$. Consider the counit of the adjunction, the evaluation

$$\mathrm{ev}_{\mathcal{D}}: \mathbf{Fun}^L(\mathcal{C}, \mathcal{D}) \otimes \mathcal{C} \rightarrow \mathcal{D}.$$

Since this is a natural transformation we know that the diagram

$$\begin{array}{ccc} \mathbf{Fun}^L(\mathcal{C}, \mathrm{TwSp}(BH, B\beta)) \otimes \mathcal{C} & \xrightarrow{\mathrm{ev}_{\mathrm{TwSp}(BH, B\beta)}} & \mathrm{TwSp}(BH, B\beta) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{Fun}^L(\mathcal{C}, \mathrm{Mod}_{\mathrm{Th}(\beta)}) \otimes \mathcal{C} & \xrightarrow{\mathrm{ev}_{\mathrm{Mod}_{\mathrm{Th}(\beta)}}} & \mathrm{Mod}_{\mathrm{Th}(\beta)} \end{array}$$

commutes. The result follows by noting that the functors Φ_X and F_M are precisely $\mathrm{ev}_{\mathrm{TwSp}(BH, B\beta)}(X \otimes -)$ and $\mathrm{ev}_{\mathrm{Mod}_{\mathrm{Th}(\beta)}}(M \otimes -)$ under the equivalences

$$\mathbf{Fun}^L(\mathrm{TwSp}(BG, B\alpha), \mathrm{TwSp}(BH, B\beta)) \simeq \mathrm{TwSp}(BG \times BH, -B\alpha \boxtimes B\beta)$$

and

$$\mathbf{Fun}^L(\mathrm{Mod}_{\mathrm{Th}(\alpha)}, \mathrm{Mod}_{\mathrm{Th}(\beta)}) \simeq {}_{\mathrm{Th}(\alpha)}\mathrm{BiMod}_{\mathrm{Th}(\beta)},$$

respectively. □

Putting this all together, we deduce the following statement:

Theorem 6.19 *Let $\mathbf{TwSp}^{\mathrm{conn}}$ denote the full subcategory of $\mathbf{Pr}_{\mathrm{St}}^L$ spanned by those stable ∞ -categories of the form $\mathrm{TwSp}(BG, B\alpha)$ for some object $\alpha \in \mathrm{Grp}/\mathrm{Pic}(\mathbb{S})$. Then there is an equivalence*

$$\mathbf{TwSp}^{\mathrm{conn}} \simeq \mathbf{ThMod}$$

of $(\infty, 2)$ -categories.

7 Derived algebraic geometry perspective

The notion of a twisted spectrum on a topological space B has a natural interpretation within the setting of spectral algebraic geometry. Roughly, this is a setting of (derived) algebraic geometry where one takes the basic affine objects to be connective $\mathbb{E}_{\infty}$ -ring spectra. Gluing these together in the appropriate homotopical context, one obtains spectral schemes, and more generally, spectral stacks. The goal of this section is to identify a twist on a space B with a *Brauer class*, in the sense of derived algebraic geometry, on the associated constant stack B .

7.1 Derived stacks

Let $\mathbf{CAlg}^{\mathrm{cn}}$ denote the ∞ -category of connective $\mathbb{E}_{\infty}$ -rings. One endows its opposite category with the ∞ -categorical variant of the étale topology; cf [39, Chapter 2.4]. Our treatment here will follow the functor of points approach to defining stacks.

Definition 7.1 The ∞ -category of *spectral stacks* is the full subcategory $\mathrm{dStk} := \mathrm{Fun}^\tau(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S}) \subset \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ of space-valued functors which satisfy descent with respect to the étale topology.

Remark 7.2 The definition of a derived stack is manifestly very general; one typically focuses on stacks which have certain properties, for example spectral schemes or spectral Deligne–Mumford stacks.

Since dStk is a geometric localization of a presheaf category, it naturally forms an ∞ -topos. As such, there is a geometric morphism adjunction

$$c^* : \mathcal{S} \rightleftarrows \mathrm{dStk} : \Gamma,$$

where c^* denotes the *constant*, or *Betti* stack, and Γ denotes the global sections morphism.

Example 7.3 For the purposes of this work, an important class of examples of spectral stacks will be the constant stacks, ie the essential image of the functor $c^*(-)$ above.

As in classical algebraic geometry, one probes the geometry of a spectral stack by studying its category of quasicoherent sheaves. Given a fixed spectral stack F , the symmetric monoidal ∞ -category of quasicoherent sheaves as a limit

$$\mathrm{QCoh}(F) \simeq \lim_{\mathrm{Spec} A \rightarrow F} \mathrm{Mod}(A)$$

of module categories. If F is a constant stack associated to a topological space one has the following identification.

Proposition 7.4 *There is an equivalence*

$$\mathrm{QCoh}(c^*(B)) \simeq \mathrm{Fun}(B, \mathrm{Sp})$$

between quasicoherent sheaves on $c^*(B)$ and parametrized categories on B .

7.2 The Brauer stack and derived Azumaya algebras

As we saw in Section 1, the Brauer space of a $\mathbb{E}_\infty$ ring R captures the invertible objects with respect to the Lurie tensor product on compactly generated R -linear categories. Such objects in this affine case correspond precisely to derived Azumaya algebras over R , whose definition we now recall.

Definition 7.5 An R -algebra A is an *Azumaya R -algebra* if A is a compact generator of Mod_R and if the natural map

$$A \otimes_R A^{\mathrm{op}} \rightarrow \mathrm{End}_R(A)$$

is an equivalence of R -algebras.

The assignment $R \mapsto \mathbf{Br}(R)$ satisfies étale descent and hence can be extended to the ∞ -category of spectral stacks. In fact, there is a spectral stack $\mathbf{Br}$ [4, Section 6.1] such that for an arbitrary $F \in \mathbf{dStk}$, one has an equivalence

$$\mathrm{Map}_{\mathbf{dStk}}(F, \mathbf{Br}) \simeq \mathbf{Br}(F).$$

The notion of an Azumaya algebra globalizes, as well. Namely, one obtains a spectral stack $\mathbf{Az}$ whose value on a connective $\mathbb{E}_\infty$ ring R is exactly the space of Azumaya algebras. A derived Azumaya algebra on a stack F will then just be a map $F \rightarrow \mathbf{Az}$. There is a natural map

$$\mathbf{Az} \rightarrow \mathbf{Br}$$

of functors which on R -points is simply given by $A \mapsto \mathrm{Mod}_A$. A motivating question, going back to Grothendieck's study of Azumaya algebras and étale cohomology, is whether or not every Brauer class of a stack F arises from a derived Azumaya algebra over F . We may phrase this in this language as whether or not the map $F \rightarrow \mathbf{Br}$ classifying a Brauer class lifts across the map $\mathbf{Az} \rightarrow \mathbf{Br}$ to a map $F \rightarrow \mathbf{Az}$.

Now, by a result of Toën in the simplicial case (cf [38, Theorem 3.7]), and Antieau and Gepner in the spectral setting at hand (cf [4, Theorem 6.1]), for a qcqs spectral scheme the above is true. We remark that this was extended by Lurie in [29] to the case of spectral algebraic spaces using his theory of scallop decompositions.

7.3 Betti stacks and twisted spectra

We now tie the ensuing discussion with our study of twisted spectra. Let B be a fixed topological space whose associated Betti stack we denote by $c^*(B)$

Proposition 7.6 *There exists an equivalence*

$$\mathrm{Map}_{\mathcal{S}}(B, \mathbf{Br}(\mathbb{S})) \simeq \mathbf{Br}(c^*(B)).$$

Proof This is simply a consequence of the constant stack/global sections adjunction described above. In particular

$$\mathbf{Br}(c^*(B)) \simeq \mathrm{Map}_{\mathbf{dStk}}(c^*(F), \mathbf{Br}) \simeq \mathrm{Map}_{\mathbf{dStk}}(F, \Gamma(\mathbf{Br})).$$

We now identify $\Gamma(\mathbf{Br})$, the global sections of the stack $\mathbf{Br}$. This is computed by its value on the final object, $\mathrm{Spec} \mathbb{S}$; thus $\Gamma(\mathbf{Br}) \simeq \mathbf{Br}(\mathbb{S})$. $\square$

Thus we may identify a twist on a topological space B with a Brauer class on the associated Betti stack. We now rephrase the realizability question of Grothendieck in the following relevant form.

Question 7.7 Let $c^*(B)$ be the Betti stack associated to a topological space, and let $\alpha: B \rightarrow B\mathrm{Pic}(\mathbb{S})$ be a twist corresponding to a Brauer class on $c^*(B)$. Does there exist an algebra object $\mathcal{A}$ in $\mathrm{QCoh}(c^*(B))$ for which $\mathrm{TwSp}(B, \alpha) \simeq \mathrm{Mod}_{\mathcal{A}}$ as $\mathrm{QCoh}(c^*(B))$ -linear categories?

In fact this question is easily seen to have a positive answer when the space B is pointed connected. Recall from Section 3.7 that there is a natural equivalence

$$\mathrm{TwSp}(B, \alpha) \simeq \mathrm{RMod}_{\mathrm{Th}(\Omega\alpha)}.$$

Via the equivalence $\mathrm{Sp}^B \simeq \mathrm{RMod}_{\mathbb{S}[\Omega B]}$ (eg by [7, Lemma 4.46]), where the symmetric monoidal structure on the right hand side is transported over from the pointwise monoidal structure on Sp^B , we may view the $\mathbb{E}_1$ -algebra $\mathrm{Th}(\Omega\alpha)$ as corresponding to an $\mathbb{E}_1$ -algebra object in Sp^B . In particular, this lies in the essential image of the functor

$$\Theta: \mathrm{Alg}_{\mathrm{Sp}^B} \rightarrow \mathrm{Mod}_{\mathrm{Sp}^B}(\mathrm{Pr}_{\mathrm{St}}^L),$$

given by $A \mapsto \mathrm{RMod}_A$; cf [27, Remark 4.8.5.17]. Thus, we may conclude that Brauer classes on constant stacks arise from algebra objects over constant stacks c^*B for B pointed and connected. Hence we may think of $\mathrm{Th}(\Omega\alpha)$ as a type of (derived) Azumaya algebra over c^*B .

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