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Given an orbifold, we construct a global spectrum representing its stable global homotopy type. Global spectra now represent orbifold cohomology theories which automatically satisfy certain properties such as additivity and the existence of Mayer–Vietoris sequences. Moreover, the value at a global quotient orbifold $M /\!\!/ G$ can be identified with the G-equivariant cohomology of the manifold M. Examples of orbifold cohomology theories which are represented by global spectra include Borel and Bredon cohomology theories and orbifold K-theory. This also implies that these cohomology groups are independent of the presentation of an orbifold as a global quotient orbifold.

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1 Introduction

The purpose of this paper is to use global homotopy theory for studying the homotopy theory of orbifolds as suggested by Schwede in [33, Preface].

Orbifolds are objects from geometric topology which model spaces that locally arise as quotients of smooth manifolds under smooth group actions. Special cases are *global quotient orbifolds* $M /\!\!/ G$ of a smooth manifold M under a smooth action of a compact Lie group G.

Orbifolds were first defined by Satake [31] who called them V-manifolds. The definition generalizes the definition of a manifold by endowing the euclidean charts with a finite group action. This notion was later used and popularized by Thurston [37], who started calling them orbifolds and used them to study three-manifolds. In a more general approach, due to Moerdijk and Pronk [25], one defines an orbifold as a special kind of Lie groupoid. Connections between groupoids and orbifolds were first studied by Haefliger [12].

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It is essential that an orbifold carries more information than just the underlying topological quotient space: It also keeps track of the group actions involved. The homotopy type of a global quotient orbifold $M /\!\!/ G$ should relate to the G-equivariant homotopy type of the underlying manifold M. For example, the cohomology of $M /\!\!/ G$ is usually defined as the G-equivariant cohomology of M. However, the group G may vary for different orbifolds and two quotients can represent the same orbifold, eg for H a subgroup of G, the quotients $M /\!\!/ H$ and $(G \times_H M) /\!\!/ G$ are equivalent as orbifolds. The presentation of an orbifold X as a global quotient orbifold $M /\!\!/ G$ hence is nonintrinsic data of the orbifold X. However, an orbifold cohomology theory should capture the G-equivariant behavior of the orbifold X. A conceptual treatment of *orbifold cohomology theories* must necessarily pay attention to the interplay between different equivariant cohomology theories. This leads us to global homotopy theory.

Global homotopy theory studies homotopy types with simultaneous, compatible group actions of all compact Lie groups. A particular framework for stable global homotopy theory was developed by Schwede in [33]. He uses the category of orthogonal spectra together with *global equivalences*. The notion of global equivalences is much more restrictive than the usual one of stable equivalences between orthogonal spectra. Two orthogonal spectra are globally equivalent if they represent the same *global homotopy type* and not just the same nonequivariant homotopy type. Localizing the category of orthogonal spectra at the global equivalences yields the *stable global homotopy category* $\mathcal{GH}$. There is a functor $\mathcal{GH} \to G\text{-}\mathcal{SH}$ from the stable global homotopy category to the G-equivariant stable homotopy category for any Lie group G which assigns the *underlying G-equivariant homotopy type* to a global homotopy type.

Similarly, the category of orthogonal spaces together with global equivalences of orthogonal spaces models unstable global homotopy types. Orthogonal spectra, ie stable global homotopy types, represent cohomology theories on these orthogonal spaces.

The goal of this paper is to construct a well-behaved functor from orbifolds to orthogonal spaces, so that we can use orthogonal spectra to represent cohomology theories on orbifolds.

Firstly, we use a different model for unstable global homotopy theory, the category $\mathbf{T}_{\mathbf{Orb}}$ of **Orb**-spaces, introduced by Gepner and Henriques in [13]. This category is defined to be the category of topological presheaves on the category **Orb**, which is the full subcategory of the topological category of topological groupoids **TGrpd** spanned by the groupoids with a single object and a compact Lie group as automorphisms. Hence, the restricted Yoneda embedding gives rise to a functor $Y : \mathbf{TGrpd} \to \mathbf{T}_{\mathbf{Orb}}$.

We verify that this functor is homotopical, ie it sends the so-called *essential equivalences* of topological groupoids to weak equivalences of **Orb**-spaces.

Theorem (Corollary 2.22) *The functor*

$$Y : \mathbf{TGrpd} \to \mathbf{T}_{\mathbf{Orb}}$$

sends essential equivalences of topological groupoids to weak equivalences of **Orb***-spaces.*

Precomposing the above functor with the forgetful functor **Orbfld** → **TGrpd** hence yields a functor sending essential equivalences of orbifolds to weak equivalences of **Orb**-spaces and therefore induces a functor

$$|-|_{\mathrm{gl}} : \mathbf{Orbfld}_\infty \to (\mathbf{T}_{\mathbf{Orb}})_\infty$$

on ∞-categorical localizations which we call the *global realization* functor.

Using a zigzag of Quillen equivalences established by Körschgen [17] and Schwede [34] between the category of **Orb**-spaces and the category of orthogonal spaces with global equivalences, there is a suspension functor $\Sigma^\infty_+ : (\mathbf{T}_{\mathbf{Orb}})_\infty \to \mathcal{GH}$ to the ∞-category of global spectra. We define the *global motiv* of an orbifold as

$$\mathbf{M}(\mathbb{G}) = \Sigma^\infty_+ |\mathbb{G}|_{\mathrm{gl}}\,.$$

Any other global spectrum E now represents a cohomology theory on orbifolds, by defining the k^{th} *orbifold cohomology group* represented by E of $\mathbb{G}$ as

$$E^k(\mathbb{G}) = \mathcal{GH}\,(\mathbf{M}(\mathbb{G}), E[k])\,.$$

It fulfills the following properties:

Theorem (Theorem 4.6) *Let $E \in \mathcal{GH}$ be a global spectrum.*

(i) *The orbifold cohomology functor $E^k : \mathbf{Orbfld}^{\mathrm{op}} \to \mathbf{Ab}$ takes essential equivalences of orbifolds to isomorphisms.*

(ii) *For every family $\{\mathbb{G}^i\}_{i\in I}$ of orbifolds, the natural map*

$$E^k\Big(\coprod_{i\in I} \mathbb{G}^i\Big) \to \prod_{i\in I} E^k(\mathbb{G}^i)$$

is an isomorphism.

(iii) *Let $\mathbb{G}$ be an orbifold covered by two open suborbifolds $\mathbb{U}$ and $\mathbb{V}$. There is a long exact sequence of orbifold cohomology groups*

$$\cdots \to E^k(\mathbb{G}) \xrightarrow{(i^*_{\mathbb{U}}, i^*_{\mathbb{V}})} E^k(\mathbb{U}) \oplus E^k(\mathbb{V}) \xrightarrow{j^*_{\mathbb{U}} - j^*_{\mathbb{V}}} E^k(\mathbb{U}\cap\mathbb{V}) \to E^{k+1}(\mathbb{G}) \to \cdots$$

for every orthogonal spectrum E. Here, $i_{\mathbb{U}} : \mathbb{U} \to \mathbb{G}$, $i_{\mathbb{V}} : \mathbb{V} \to \mathbb{G}$, $j_{\mathbb{U}} : \mathbb{U}\cap\mathbb{V} \to \mathbb{U}$ and $j_{\mathbb{V}} : \mathbb{U}\cap\mathbb{V} \to \mathbb{V}$ denote the inclusion morphisms.

(iv) *Let G be a compact Lie group acting almost freely on a manifold M. The orbifold cohomology of $G \ltimes M$ is isomorphic to the G-equivariant cohomology of M, ie*

$$E^k(G \ltimes M) \cong E^k_G(M).$$

Parts (i) and (iv) imply that orbifold cohomology groups are independent of the presentation of the orbifold as a global quotient orbifold.

Examples of equivariant cohomology theories which are represented by orthogonal spectra in the global sense include equivariant K-theory, Borel cohomology theories and Bredon cohomology theories. These cohomology theories hence are examples of *orbifold cohomology theories* in our sense. Particularly, we show that Bredon cohomology extends to a well-behaved cohomology theory on all orbifolds. These examples are discussed in greater detail in Section 4.

Relations to other work

Since this article was first posted, other papers have studied the relation between orbifolds and global homotopy theory, eg [6; 32]. The main difference is that those papers model orbifolds as differentiable stacks, whereas we work with the topological category of Lie groupoids and invert essential equivalences. There are several comparisons between these approaches, eg in [13; 18; 28], but the author is not aware of a precise comparison to the model used in this paper.

Organization of the paper

In Section 2.1 we recall the concept of an orbifold as a Lie groupoid. In Section 3, we discuss the zigzag of Quillen equivalences between models for orbispaces. The most important technical result is Corollary 3.13, saying that these equivalences preserve models for global quotient spaces. Finally, we associate an orthogonal space to an orbifold and define what an *orbifold cohomology theory* is in Section 4.

Conventions

Throughout this paper, $\mathbf{T}$ denotes the category of compactly generated, weak Hausdorff spaces together with continuous maps, as defined in [36]. This definition was first used by McCord in [22]. Other sources for details on this topic are [8, Section 7.9; 19, Appendix A; 33, Appendix A]. A *space* is always assumed to be an object of $\mathbf{T}$. Moreover, $\mathbf{T}_*$ denotes the category of based, compactly generated, weak Hausdorff spaces together with based continuous maps.

Let G be a topological group, ie a group object in $\mathbf{T}$. We write $G\text{-}\mathbf{T}$ for the category of (left) G-spaces together with G-equivariant continuous maps. Moreover, $G\text{-}\mathbf{T}_*$ denotes the category of based G-spaces together with based G-equivariant continuous maps.

A *manifold* is always assumed to be a smooth manifold which is second countable and Hausdorff. The category of smooth manifolds together with smooth maps is denoted by **Mfld**. We require all occurring group actions of Lie groups on manifolds to be smooth.

We will use the theory of ∞-categories as developed by Lurie in [20]. For $\mathcal{C}$ a topological category with a class of weak equivalences W, we write $\mathcal{C}_\infty = \mathrm{N}(\mathcal{C})[W^{-1}]$ for the ∞-categorical localization of the homotopy coherent nerve of $\mathcal{C}$ at W in the sense of [20, Definitions 1.1.5.5 and 1.3.4.1]. In the case where $\mathcal{C}$ admits the additional structure of a bicomplete compactly generated topological model category, this agrees with the ∞-categorical localization of the underlying 1-category, as can be shown using [21, Theorem 1.3.4.20; 30, Corollary 13.2.4].

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2 The global homotopy type of orbifolds

2.1 Orbifolds

We recall the approach to orbifolds via Lie groupoids. An excellent overview of this approach can be found in Moerdijk's paper [23]. A closely related approach using differentiable stacks can be found in [18].

Definition 2.1 (topological groupoid) A *topological groupoid* is a groupoid object in **T**, that is a tuple $\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, s, t, e, i, \circ)$ where $\mathbb{G}_0$ and $\mathbb{G}_1$ are spaces and $s\colon \mathbb{G}_1 \to \mathbb{G}_0$, $t\colon \mathbb{G}_1 \to \mathbb{G}_0$, $e\colon \mathbb{G}_0 \to \mathbb{G}_1$, $i\colon \mathbb{G}_1 \to \mathbb{G}_1$ and $\circ\colon \mathbb{G}_1 \times_{s,t} \mathbb{G}_1 \to \mathbb{G}_1$ are continuous maps such that the underlying sets of $\mathbb{G}_0$ (as objects) and $\mathbb{G}_1$ (as arrows) together with s as the source map, t as the target map, e as the identity map, i as the inverse map and $\circ$ as the composition map form a (small) groupoid.

A *morphism* of topological groupoids $f\colon \mathbb{G} \to \mathbb{H}$ is a pair of continuous maps $f_0\colon \mathbb{G}_0 \to \mathbb{H}_0$ and $f_1\colon \mathbb{G}_1 \to \mathbb{H}_1$ which is a functor between the underlying groupoids. A 2*-morphism* between two morphisms of topological groupoids $f, g\colon \mathbb{G} \to \mathbb{H}$ is a continuous map $\mathbb{G}_0 \to \mathbb{H}_1$ which is a natural transformation between f and g.

The category of topological groupoids is enriched over itself. We topologize the set $\mathbb{M}\mathrm{ap}(\mathbb{G}, \mathbb{H})_0$ of morphisms between $\mathbb{G}$ and $\mathbb{H}$ as a subspace of $\mathrm{map}(\mathbb{G}_1, \mathbb{H}_1)$. Moreover, $\mathbb{M}\mathrm{ap}(\mathbb{G}, \mathbb{H})_1$ is the space of 2-morphisms between $\mathbb{G}$ and $\mathbb{H}$ topologized as a subspace of $\mathrm{map}(\mathbb{G}_1, \mathbb{H}_1) \times \mathrm{map}(\mathbb{G}_1, \mathbb{H}_1) \times \mathrm{map}(\mathbb{G}_0, \mathbb{H}_1)$ where the inclusion into the first factor is given by the source functor, the one into the second factor by the target functor and the one into the third factor by the underlying map of the natural transformation itself.

The category of topological groupoids is denoted by **TGrpd**.

Definition 2.2 (Lie groupoid) Similarly, a *Lie groupoid* is a tuple $\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, s, t, e, i, \circ)$ where $\mathbb{G}_0$ and $\mathbb{G}_1$ are manifolds, $s\colon \mathbb{G}_1 \to \mathbb{G}_0$, $t\colon \mathbb{G}_1 \to \mathbb{G}_0$ are smooth submersions and $e\colon \mathbb{G}_0 \to \mathbb{G}_1$, $i\colon \mathbb{G}_1 \to \mathbb{G}_1$ and $\circ\colon \mathbb{G}_1 \times_{s,t} \mathbb{G}_1 \to \mathbb{G}_1$ are smooth maps such that the underlying sets form a groupoid as above.

Morphisms and 2-morphisms are defined just as for topological groupoids with smooth maps instead of continuous ones. The category of Lie groupoids is denoted by **LieGrpd**.

The category of Lie groupoids is not enriched over itself, but it is enriched over the category of topological groupoids by taking the C^∞-topology on mapping spaces. This topology is finer than the compact-open topology.

If s is a submersion, then t automatically is a submersion because $t = s \circ i$ and i is a self-inverse diffeomorphism. The requirement that $s, t\colon \mathbb{G}_1 \to \mathbb{G}_0$ are submersions ensures that the fibered product $\mathbb{G}_1 \times_{s,t} \mathbb{G}_1$ exists in **Mfld** and that the underlying space can be computed in **T**.

The forgetful functor **Mfld** $\to$ **T** induces a forgetful functor **LieGrpd** $\to$ **TGrpd**. In particular, every Lie groupoid can be regarded as a topological groupoid. We will not mention this when using this fact.

Example 2.3 We will give some examples without explicitly mentioning or defining all maps.

- Let G be a topological group (or Lie group, respectively). Let $\mathbb{B}G$ denote the topological groupoid (or Lie groupoid, respectively) with objects space $\mathbb{B}G_0 = *$ and arrow space $\mathbb{B}G_1 = G$. The identity map of $\mathbb{B}G$ sends $*$ to the identity of G. Composition and inverse maps are given by the corresponding maps of G.

- More generally, let G be a topological group and A be a G-space. Define the *action groupoid* $G \ltimes A$ with $(G \ltimes A)_0 = A$ and $(G \ltimes A)_1 = G \times A$. The map $s\colon G \times A \to A$ is the projection map and the map $t\colon G \times A \to A$ is the action map of A.

An arrow $g\colon x \to y$ in $G \ltimes A$ from x to y in A may be described as an element g of G such that $g.x = y$. The composition is defined by using the composition of G.

The topological groupoid $G \ltimes A$ carries the structure of a Lie groupoid if G is a Lie group acting smoothly on a manifold $M = A$.

Definition 2.4 (essential equivalence) An *essential equivalence* of topological groupoids (or Lie groupoids, respectively) is a morphism $f\colon \mathbb{G} \to \mathbb{H}$ of groupoids such that the following holds true:

- **(essentially surjective)** The map $\mathbb{H}_1 \times_{s,f_0} \mathbb{G}_0 \to \mathbb{H}_0$ induced by $t\colon \mathbb{H}_1 \to \mathbb{H}_0$ on the first factor is a surjection admitting local sections (or surjective submersion, respectively).

- **(fully faithful)** The square

$$\begin{array}{ccc} \mathbb{G}_1 & \xrightarrow{f_1} & \mathbb{H}_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ \mathbb{G}_0 \times \mathbb{G}_0 & \xrightarrow{(f_0,f_0)} & \mathbb{H}_0 \times \mathbb{H}_0 \end{array}$$

is a pullback in **T** (or in **Mfld**, respectively).

We say that two topological groupoids (or Lie groupoids, respectively) are *essentially equivalent* if there is a zigzag of essential equivalences between them. This is often also called *Morita equivalent*.

Definition 2.5 (stabilizer group) Let $\mathbb{G}$ be a topological groupoid and let x be a point in $\mathbb{G}_0$. We write
$$\mathbb{G}_x = (s,t)^{-1}(x,x)$$
for the *stabilizer group* of x. This becomes a topological group with the subspace topology.

Definition 2.6 (proper, foliation groupoid) Let $\mathbb{G}$ be a Lie groupoid. We say that $\mathbb{G}$ is *proper* if the map $(s,t)\colon \mathbb{G}_1 \to \mathbb{G}_0 \times \mathbb{G}_0$ is proper. We say that $\mathbb{G}$ is a *foliation groupoid* if $\mathbb{G}_x$ is discrete for all $x \in \mathbb{G}_0$.

Lemma 2.7 *Properness and being a foliation groupoid are invariant under essential equivalences.*

Proof It follows immediately from the fully faithfulness that essential equivalences induce isomorphism on stabilizer groups. This proves that being a foliation groupoid is invariant under essential equivalence. The statement for properness is proven in [24, Proposition 5.26]. □

Definition 2.8 (orbifold) An *orbifold* is a proper foliation groupoid. We write **Orbfld** for the full subcategory of orbifolds in **LieGrpd**.

Remark 2.9 Any essential equivalence between proper Lie groupoids is an essential equivalence of underlying topological groupoids. The essentially surjectiveness condition holds true because any submersions admits local sections. Let P denote the topological pullback of the diagram in the fully faithfulness condition. The induced map $\mathbb{G}_1 \to P$ is bijective and continuous, so we need to check that its inverse is continuous, too. We may check this locally at a point x in the codomain P. Let K be a compact neighborhood of the image of x under the structure map $P \to \mathbb{G}_0 \times \mathbb{G}_0$. The restriction of the map $\mathbb{G}_1 \to P$ to the preimage of K under $(s,t)\colon \mathbb{G}_1 \to \mathbb{G}_0 \times \mathbb{G}_0$ and the preimage of K under the structure map $P \to \mathbb{G}_0 \times \mathbb{G}_0$ is a bijective map from a compact space to a Hausdorff space and hence a homeomorphism.

Example 2.10 (global quotient orbifold) Let M be a manifold together with an almost free action of a compact Lie group G. The action groupoid $G \ltimes M$ is an orbifold and is called a *global quotient orbifold*.

In fact, every orbifold arises as such a global quotient orbifold, up to essential equivalence.

Theorem 2.11 [27, Corollary 1.3] *Every orbifold $\mathbb{G}$ is essentially equivalent to $G \ltimes M$ for some manifold M together with an almost free action of a compact Lie group G on M.*

For $\mathbb{G}$ a topological groupoid, we write $\|\mathbb{G}\|$ for the fat geometric realization of the nerve of $\mathbb{G}$ as defined in [35, Appendix A]. The fat geometric realization has a better homotopical behavior than the usual geometric realization, ie we have the following:

Proposition 2.12 *Let $f : \mathbb{G} \to \mathbb{H}$ be an essential equivalence of topological groupoids. Then the induced map $\|f\| : \|\mathbb{G}\| \to \|\mathbb{H}\|$ is a weak equivalence.*

Proof This is mentioned as a corollary of [26, Theorem 1.1]. □

However, in contrast to the usual geometric realization, the fat geometric realization does not commute with products on the nose, but it does up to weak equivalence:

Proposition 2.13 *Let $\mathbb{G}$ and $\mathbb{H}$ be two topological groupoids. The natural map*

$$\|\mathbb{G}\times\mathbb{H}\| \to \|\mathbb{G}\|\times\|\mathbb{H}\|$$

is a weak equivalence and admits a retract. These retracts together with the inclusion of the 0-simplex $* \to \|*\|$ assemble into a lax symmetric monoidal structure on the functor

$$\|-\|\colon \mathbf{TGrpd}\to\mathbf{T}$$

with respect to the cartesian monoidal structure on both sides.

Proof The above map is a weak equivalence by [35, Proposition A.1(iii)]. The nerve functor commutes with products and therefore is strong symmetric monoidal. The claim about the lax symmetric monoidality of the fat geometric realization is proven in [11, Appendix B]. □

Construction 2.14 Recall that the category of topological groupoids is enriched over itself. Applying the above lax symmetric monoidal functor to the topological groupoids $\mathbb{M}\mathrm{ap}(\mathbb{H},\mathbb{G})$, we obtain a topological enrichment on the category **TGrpd** with mapping spaces

$$\mathbf{TGrpd}(\mathbb{H},\mathbb{G}) = \|\mathbb{M}\mathrm{ap}(\mathbb{H},\mathbb{G})\|.$$

Similarly, we can change enrichment of the category of orbifolds from topological groupoids to topological spaces.

The category of orbifolds is usually defined as the categorical localization of the category of proper foliation groupoids at the essential equivalences. There are several explicit construction for this. A brief explanation using a calculus of fractions can be found in [23, Section 2]. An explanation how to invert those morphisms in a 2-categorical way can be found in [28]; see [18] for a comparison of different approaches. We will eventually consider the higher categorical version of this construction and define the *∞-category of orbifolds* $\mathbf{Orbfld}_\infty$ as the ∞-categorical localization of the homotopy coherent nerve of the category of orbifolds at the essential equivalences.

2.2 From orbifolds to Orb-spaces

We recall how an orbifold gives rise to an **Orb**-space, a particular model for the homotopy theory of orbispaces due to Gepner and Henriques [13, Section 4]. The main result of this section is Corollary 2.23, saying that the resulting functor $F\colon \mathbf{Orbfld}\to\mathbf{T}_{\mathbf{Orb}}$ sends essential equivalences of orbifolds to weak equivalences of **Orb**-spaces. An immediate corollary is that the functor F extends to the ∞-categorical localizations and yields a functor

$$|-|_{\mathrm{gl}}\colon \mathbf{Orbfld}_\infty\to(\mathbf{T}_{\mathbf{Orb}})_\infty$$

from the ∞-category of orbifolds to the ∞-category of **Orb**-spaces.

Definition 2.15 The topological category **Orb** is the full topological subcategory of **TGrpd** spanned by the topological groupoid $\mathbb{B}G$ for all compact Lie groups G. An **Orb**-*space* is a continuous functor $X\colon \mathbf{Orb}^{\mathrm{op}} \to \mathbf{T}$. A *morphism* of **Orb**-spaces is a continuous natural transformation. A map between **Orb**-spaces is called a weak equivalences if it is levelwise a weak equivalence. We write $\mathbf{T}_{\mathbf{Orb}}$ for the category of **Orb**-spaces.

Remark 2.16 If G, H are compact Lie groups, then the morphism space $\mathbf{Orb}(H,G) = \|\mathbb{M}\mathrm{ap}(\mathbb{B}H, \mathbb{B}G)\|$ is homeomorphic to $\hom(H,G) \times_G EG$, the homotopy orbit space of the conjugation action of the group G on the space $\hom(H,G)$ of continuous homomorphisms; compare [13, Remark 4.5].

Now we recall how to associate an **Orb**-space to an orbifold. This definition is taken from [13, Section 4.2].

Construction 2.17 We define the topological functor

$$Y\colon \mathbf{TGrpd} \to \mathbf{T}_{\mathbf{Orb}}, \quad \mathbb{G} \mapsto (K \mapsto \|\mathrm{map}(\mathbb{B}K, \mathbb{G})\|),$$

as the restricted topological Yoneda embedding and

$$F\colon \mathbf{Orbfld} \to \mathbf{T}_{\mathbf{Orb}}$$

for the composite of the forgetful functor $\mathbf{Orbfld} \to \mathbf{TGrpd}$ and Y.

A key result for our purposes is that the functor $Y\colon \mathbf{TGrpd} \to \mathbf{T}_{\mathbf{Orb}}$ sends essential equivalences of topological groupoids to weak equivalences of **Orb**-spaces. This follows from the following:

Proposition 2.18 *Let K be a topological group. Let $f\colon \mathbb{G} \to \mathbb{H}$ be an essential equivalence of topological groupoids. Then the induced map of groupoids*

$$f_*\colon \mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{G}) \to \mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{H})$$

is again an essential equivalence.

The proof splits up into several lemmas. Let

$$p_{\mathbb{G}}\colon \mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{G}) \to \mathbb{M}\mathrm{ap}(\mathbb{B}e, \mathbb{G}) \cong \mathbb{G}$$

be induced by the unique morphism $\mathbb{B}e \to \mathbb{B}K$ where e denotes the trivial subgroup of $\mathcal{L}$.

Lemma 2.19 *Let K be a topological group and let $f\colon \mathbb{G} \to \mathbb{H}$ be an essential equivalence of topological groupoids. Then the diagram*

$$\begin{array}{ccc} \mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{G})_0 & \xrightarrow{(p_{\mathbb{G}})_0} & \mathbb{G}_0 \\ {\scriptstyle (f_*)_0}\downarrow & & \downarrow{\scriptstyle f_0} \\ \mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{H})_0 & \xrightarrow{(p_{\mathbb{H}})_0} & \mathbb{H}_0 \end{array}$$

is a pullback.

Proof Recall that $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0$ is topologized as a subset of $\mathrm{map}(K,\mathbb{H}_1)$. Evaluation at K hence induces a map $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times K\to\mathbb{H}_1$. One can check on elements that the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times K & \xrightarrow{(p_{\mathbb{G}})_0\circ\mathrm{pr}_1} & \mathbb{H}_0 & \xleftarrow{\;f_0\;} & \mathbb{G}_0 \\
\downarrow & & \downarrow{\scriptstyle\Delta_{\mathbb{H}_0}} & & \downarrow{\scriptstyle\Delta_{\mathbb{G}_0}} \\
\mathbb{H}_1 & \xrightarrow{(s_{\mathbb{H}},t_{\mathbb{H}})} & \mathbb{H}_0\times\mathbb{H}_0 & \xleftarrow{(f_0,f_0)} & \mathbb{G}_0\times\mathbb{G}_0
\end{array}$$

Using that

$$(\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times K)\times_{\mathbb{H}_0}\mathbb{G}_0\cong(\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times_{\mathbb{H}_0}\mathbb{G}_0)\times K,$$

we obtain an induced map on pullbacks

$$(\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times_{\mathbb{H}_0}\mathbb{G}_0)\times K\to\mathbb{H}_1\times_{\mathbb{H}_0\times\mathbb{H}_0}(\mathbb{G}_0\times\mathbb{G}_0)\cong\mathbb{G}_1,$$

where the last isomorphism uses that f is an essential equivalence. One can now check on elements that the adjoint map

$$\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times_{\mathbb{H}_0}\mathbb{G}_0\to\mathrm{map}(K,\mathbb{G}_1)$$

takes values in $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0$ and that this map is inverse to the map

$$\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0\to\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0\times_{\mathbb{H}_0}\mathbb{G}_0$$

induced by $(f_*)_0$ and $(p_{\mathbb{G}})_0$. □

We write $s_{\mathbb{B}K,\mathbb{G}}$ and $t_{\mathbb{B}K,\mathbb{G}}$ for the source and target map of the topological groupoid $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})$.

Lemma 2.20 *Let K be a topological group and let $\mathbb{G}$ be a topological groupoid. The diagram*

$$\begin{array}{ccc}
\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_1 & \xrightarrow{s_{\mathbb{B}K,\mathbb{G}}} & \mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0 \\
{\scriptstyle(p_{\mathbb{G}})_1}\downarrow & & \downarrow{\scriptstyle(p_{\mathbb{G}})_0} \\
\mathbb{G}_1 & \xrightarrow{s_{\mathbb{G}}} & \mathbb{G}_0
\end{array}$$

is a pullback.

Proof Similarly as in the proof of Lemma 2.19, evaluation at an element of K and diagonal maps induce a map

$$\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0\times_{\mathbb{G}_0}\mathbb{G}_1\times K\to\mathbb{G}_1\times_{\mathbb{G}_0\times\mathbb{G}_0}(\mathbb{G}_1\times\mathbb{G}_1).$$

There is a homeomorphism

$$\mathbb{G}_1\times_{\mathbb{G}_0\times\mathbb{G}_0}(\mathbb{G}_1\times\mathbb{G}_1)\cong\mathbb{G}_1\times_{\mathbb{G}_0}\mathbb{G}_1\times_{\mathbb{G}_0}\mathbb{G}_1$$

sending $(g_1,(g_2,g_3))$ to (g_3,g_1,g_2^{-1}). Composition in $\mathbb{G}$ hence induces a map

$$\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0\times_{\mathbb{G}_0}\mathbb{G}_1\times K\to\mathbb{G}_1.$$

Consider the map

$$\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1 \to \mathrm{map}(K,\mathbb{G}_1) \times \mathrm{map}(K,\mathbb{G}_1) \times \mathrm{map}(*,\mathbb{G}_1),$$

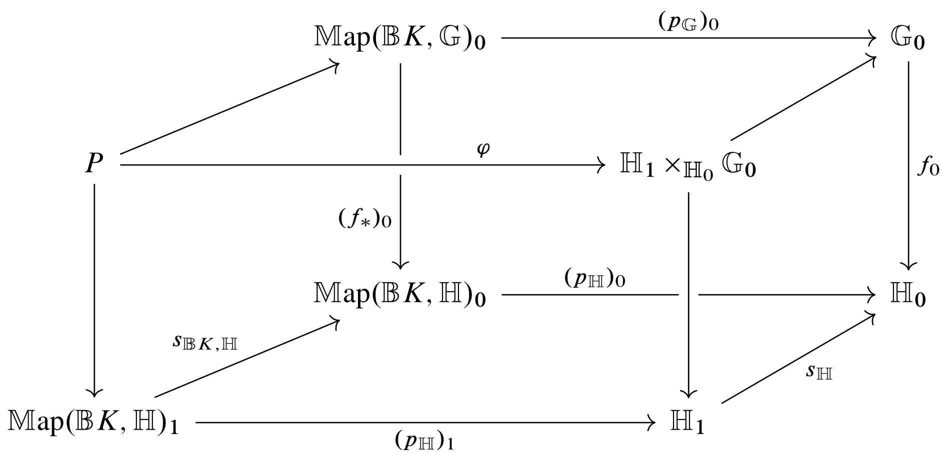
which is induced by the inclusion $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0 \hookrightarrow \mathrm{map}(K,\mathbb{G}_1)$, the adjoint of the above map and the homeomorphism $\mathbb{G}_1 \cong \mathrm{map}(*,\mathbb{G}_1)$. One can check on elements that this map takes values in $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_1$ and that it is inverse to the map from $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_1$ to the pullback $\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0 \times_{\mathbb{G}_0} \mathbb{G}_1$ induced by $s_{\mathbb{B}K,\mathbb{G}}$ and $(p_{\mathbb{G}})_1$. □

Remark 2.21 Similarly one shows that the diagram in Lemma 2.20 is a pullback when replacing the source maps with target maps.

Proof of Proposition 2.18 We first prove that f_* is essentially surjective as in Definition 2.4 of essential equivalences. Let

$$P = \mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_1 \times_{\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0} \mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})_0$$

denote the pullback along $s_{\mathbb{B}K,\mathbb{H}}$ and $(f_*)_0$. Consider the commutative diagram



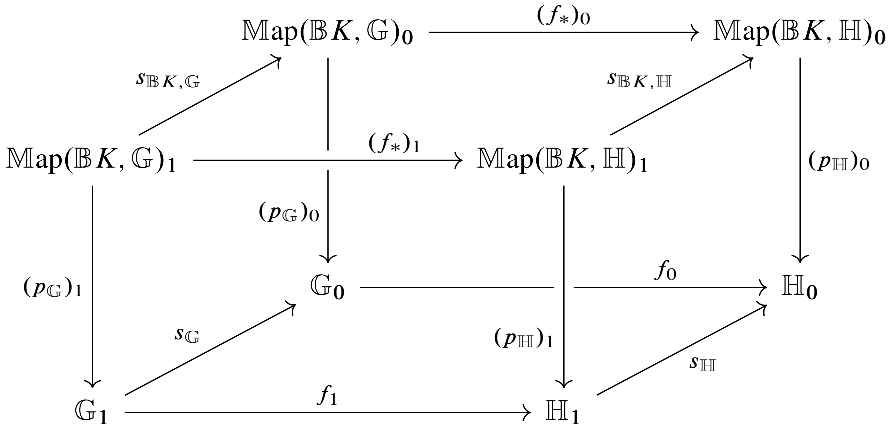
where the unlabeled maps out of the pullbacks are the structure maps and φ is the map induced by the functoriality of pullbacks.

The squares on the left and on the right are pullbacks by definition. The square in the back also is a pullback by Lemma 2.19. Hence, the square in the front is a pullback. Consider the following commutative diagram:

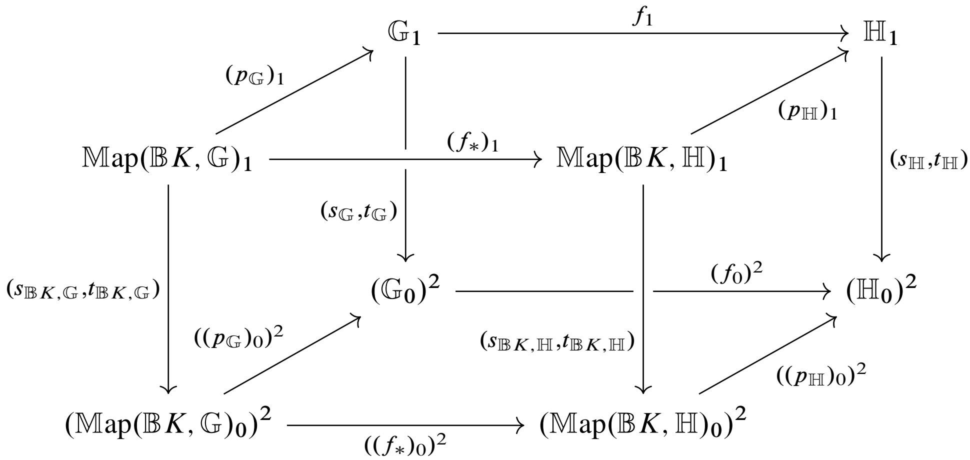
$$\begin{array}{ccccc}
P & \longrightarrow & \mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_1 & \xrightarrow{t_{\mathbb{B}K,\mathbb{H}}} & \mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{H})_0 \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle (p_{\mathbb{H}})_1} & & \downarrow{\scriptstyle (p_{\mathbb{H}})_0} \\
\mathbb{H}_1 \times_{\mathbb{H}_0} \mathbb{G}_0 & \longrightarrow & \mathbb{H}_1 & \xrightarrow{t_{\mathbb{H}}} & \mathbb{H}_0
\end{array}$$

We have just shown that the left square is a pullback and the right square is a pullback by Remark 2.21. Therefore, the outer square also is a pullback. The composition of the lower maps is a surjection admitting local sections by assumption on f. Now we can conclude that the composition of the maps in the top row also is a surjection that admits local sections.

To see that f_* is fully faithful as in Definition 2.4, consider the following commutative diagram:



The square in the back is a pullback by Lemma 2.19. The squares on the left and on the right are pullbacks by Lemma 2.20. Therefore, the square in the front is also a pullback. Consider the following commutative diagram:



The square in the back is a pullback by assumption on f. The lower square is a pullback by Lemma 2.19. We have just shown that the square on the top is also a pullback. All in all, we can conclude that the square in the front also is a pullback. □

Propositions 2.18 and 2.12 imply that the induced map

$$\|\mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{G})\| \to \|\mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{H})\|$$

is a weak equivalence for every essential equivalence $f : \mathbb{G} \to \mathbb{H}$. We hence proved the following:

Corollary 2.22 *The functor*

$$Y : \mathbf{TGrpd} \to \mathbf{T}_{\mathbf{Orb}}$$

from Construction 2.17 sends essential equivalences of topological groupoids to weak equivalences of **Orb***-spaces.*

Corollary 2.23 *The functor*

$$F : \mathbf{Orbfld} \to \mathbf{T}_{\mathbf{Orb}}$$

from Construction 2.17 sends essential equivalences of orbifolds to weak equivalences of **Orb***-spaces.*

Definition 2.24 Because the functor $F : \mathbf{Orbfld} \to \mathbf{T}_{\mathbf{Orb}}$ takes essential equivalences of orbifolds to weak equivalences of **Orb**-spaces, it extends in an essentially unique way to a functor of the ∞-categorical realizations and yields a functor

$$|-|_{\mathrm{gl}} : \mathbf{Orbfld}_{\infty} \to (\mathbf{T}_{\mathbf{Orb}})_{\infty}, \tag{2-1}$$

we refer to $|M|_{\mathrm{gl}}$ as the *global realization* of the orbifold M.

The following theorem summarizes the main properties of the global realization functor. For $\widetilde{U}$ an open subset of the object manifold $\mathbb{G}_0$ of a Lie groupoid $\mathbb{G}$, we define the *restricted groupoid* $\mathbb{G}|_{\widetilde{U}}$ of $\mathbb{G}$ to $\widetilde{U}$ as the Lie groupoid with object manifold $\widetilde{U}$ and arrow manifold $(s,t)^{-1}(\widetilde{U})$. The structure maps are given by restricting the corresponding maps of $\mathbb{G}$. We moreover write

$$\mathbb{G}_0/\mathbb{G}_1 = \mathrm{coeq}(s,t : \mathbb{G}_1 \rightrightarrows \mathbb{G}_0)$$

for the *orbit space* of $\mathbb{G}$.

Theorem 2.25 (i) *The global realization functor (2-1) takes countable disjoint unions of orbifolds to coproducts of* **Orb***-spaces.*

(ii) *Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\widetilde{U}$ and $\widetilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \to \mathbb{G}_0/\mathbb{G}_1$. Then the square*

$$\begin{array}{ccc} \big|\mathbb{G}|_{\widetilde{U}\cap\widetilde{V}}\big|_{\mathrm{gl}} & \longrightarrow & \big|\mathbb{G}|_{\widetilde{U}}\big|_{\mathrm{gl}} \\ \downarrow & & \downarrow \\ \big|\mathbb{G}|_{\widetilde{V}}\big|_{\mathrm{gl}} & \longrightarrow & |\mathbb{G}|_{\mathrm{gl}} \end{array} \tag{2-2}$$

is a pushout in the ∞-category of **Orb***-spaces, where all morphisms are induced by the respective inclusions.*

(iii) *For every orbifold $\mathbb{G}$, the underlying space of the global realization $|\mathbb{G}|_{\mathrm{gl}}$ is weakly equivalent to $\|\mathbb{G}\|$.*

Proof (i) The functor $F\colon \mathbf{Orbfld} \to \mathbf{T_{Orb}}$ preserves coproducts because they can be computed levelwise and $\|\mathbb{M}\mathrm{ap}(\mathbb{B}K, -)\|\colon \mathbf{Orbfld} \to \mathbf{T}$ preserves coproducts for any universal subgroup K of $\mathcal{L}$ by a similar reasoning as for pushouts in the next part of the proof. Disjoint union of **Orb**-spaces preserves weak equivalences, so coproducts in the 1-category $\mathbf{T_{Orb}}$ represent coproducts in the ∞-category **Orb**-spaces.

(ii) We first check that for every compact Lie group K, the commutative diagram of orbifolds

$$\begin{array}{ccc} \mathbb{G}|_{\widetilde{U}\cap\widetilde{V}} & \xrightarrow{j_U} & \mathbb{G}|_{\widetilde{U}} \\ {\scriptstyle j_V}\downarrow & & \downarrow{\scriptstyle i_U} \\ \mathbb{G}|_{\widetilde{V}} & \xrightarrow{i_V} & \mathbb{G} \end{array} \tag{2-3}$$

is sent to a pushout square in $\mathbf{T}$ under the functor $\|\mathbb{M}\mathrm{ap}(\mathbb{B}K, -)\|\colon \mathbf{Orbfld} \to \mathbf{T}$. Fat geometric realization is a left adjoint and therefore commutes with pushouts. We can hence also check that the functor which assigns the n^{th} space of the nerve of $\mathbb{M}\mathrm{ap}(\mathbb{B}K, \mathbb{G})$ to a topological groupoid $\mathbb{G}$ sends (2-3) to a pushout. It can be checked on elements that this is true for the underlying sets. It is also a pushout in $\mathbf{T}$ because all maps are sent to inclusions of open subsets.

The natural map from the homotopy pushout to a pushout along open inclusions is a weak equivalence; see [9, Corollary 1.6]. This applies to the above situation. The functor F from Construction 2.17 hence takes (2-3) to a levelwise homotopy pushout in $\mathbf{T_{Orb}}$.

We are now going to prove that a levelwise homotopy pushout in $\mathbf{T_{Orb}}$ is a homotopy pushout. Let P denote the diagram $(\cdot \leftarrow \cdot \rightarrow \cdot)$. Pushouts can be computed levelwise. The homotopy pushout functor in $\mathbf{T_{Orb}}$ therefore is the left derived functor of

$$\big(\operatorname*{colim}_P\big)^{\mathbf{Orb}}\colon [P, \mathbf{T_{Orb}}] \cong [\mathbf{Orb}, [P, \mathbf{T}]] \to [\mathbf{Orb}, \mathbf{T}] = \mathbf{T_{Orb}}.$$

It follows from general model category theory that the homotopy pushout can be computed as the pushout of a cofibrant replacement in the projective model structure on $[P, \mathbf{T_{Orb}}]$; compare [14, Corollary 5.1.6]. We claim that a cofibrant diagram in $[P, \mathbf{T_{Orb}}] \cong [\mathbf{Orb}, [P, \mathbf{T}]]$ levelwise consists of diagrams in $[P, \mathbf{T}]$ for which the natural map from the homotopy pushout to the pushout is a weak equivalence. This is the case for diagrams in $[P, \mathbf{T}]$ where both arrows of P are mapped to Hurewicz cofibrations; see eg [9, Theorem A.7]. As can be seen from the characterization given in [14, Theorem 5.1.3], a cofibrant object in $[P, \mathbf{T_{Orb}}]$ maps the two arrows of P to cofibrations between **Orb**-spaces and these cofibrations are levelwise Hurewicz cofibrations by [16, Proposition 2.2(iv)].

(iii) Let e denote the trivial subgroup of $\mathcal{L}$. Then the groupoid $\mathbb{M}\mathrm{ap}(\mathbb{B}e, \mathbb{G})$ is isomorphic to $\mathbb{G}$, and so the value of the **Orb**-space $|\mathbb{G}|_{\mathrm{gl}}$ at the trivial group is isomorphic to $\|\mathbb{G}\|$. $\square$

3 A comparison of global quotient constructions

There are different models for the ∞-category of orbispaces. In this section we recall a result by Körschgen [17] establishing a Quillen equivalence between the **Orb**-space model of Gepner and Henriques [13] and the $\mathcal{L}$-space model of Schwede [34] as well as a result by Schwede [34] establishing a Quillen equivalence between the latter and the orthogonal space model from [33].

Each of these models comes with a specific *global quotient functor* with source the category of G-spaces. The goal of this section is to prove that the above Quillen equivalences are compatible with these global quotient constructions.

3.1 Orb-spaces versus $\mathbf{O}_{\mathrm{gl}}$-spaces

We recall the definition of the topological monoid $\mathcal{L}$, which contains every compact Lie group as a subgroup. A space with an action of $\mathcal{L}$ then gives rise to a G-space for an arbitrary compact Lie group G by restricting the action of $\mathcal{L}$ to G.

The following is mainly taken from [34]. We require all representations to be orthogonal representations on inner product spaces.

Definition 3.1 ($\mathcal{L}$-space) Let $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ be the space of linear isometric self-embeddings of $\mathbb{R}^\infty$ together with the standard inner product that carries the compactly generated function space topology as described in [34, Section 1]. The space $\mathcal{L}$ together with the composition of self-embeddings is a topological monoid.

An *$\mathcal{L}$-space A* is a space together with a continuous action of the topological monoid $\mathcal{L}$. A *morphism* of $\mathcal{L}$-spaces is an $\mathcal{L}$-equivariant continuous map of the underlying spaces. We write $\mathcal{L}\text{-}\mathbf{T}$ for the category of $\mathcal{L}$-spaces.

We let G be a compact Lie group. We recall that a *complete G-universe* is a G-representation of countably infinite dimension into which every finite-dimensional G-representation embeds by an equivariant linear isometry.

Definition 3.2 (universal subgroup) A subgroup G of $\mathcal{L}$ is *universal* if it is compact, admits the structure of a Lie group, and the tautological G-action through the inclusion $G \hookrightarrow \mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ makes $\mathbb{R}^\infty$ a complete G-universe.

By [34, Proposition 1.5], every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$. Moreover, every isomorphism between universal subgroups of $\mathcal{L}$ is given by conjugation with an invertible element of $\mathcal{L}$.

Definition 3.3 ($\mathbf{O}_{\mathrm{gl}}$-spaces and $\mathbf{Orb}'$-spaces) We recall two topological categories $\mathbf{O}_{\mathrm{gl}}$ and $\mathbf{Orb}'$ with the same objects, namely the universal subgroups of the linear isometries monoid $\mathcal{L}$. In **Orb**, the space of morphisms between two universal subgroups G and H of $\mathcal{L}$ is given by

$$\mathbf{O}_{\mathrm{gl}}(H,G) = \mathrm{map}^{\mathcal{L}}(\mathcal{L}/H, \mathcal{L}/G).$$

The composition in $\mathbf{O}_{\mathrm{gl}}$ is composition of $\mathcal{L}$-equivariant maps.

To define the morphism spaces in $\mathbf{Orb}'$ we consider the space

$$\widetilde{E}(H,G) = \{(\alpha,\varphi) \in \hom(H,G) \times \mathcal{L} \mid h\varphi = \varphi\alpha(h) \text{ for all } h \in H\}.$$

The group G acts on $\widetilde{E}(H,G)$ via $g.(\alpha,\varphi) = (\alpha^g, \varphi g^{-1})$ where $(\alpha^g)(h) = g\alpha(h)g^{-1}$ for $h \in H$. The morphism space is then given by the homotopy orbit space

$$\mathbf{Orb}'(H,G) = \widetilde{E}(H,G) \times_G EG.$$

The composition is defined in [17, Section 3.1].

In much the same way as for the indexing category **Orb**, and $\mathbf{O}_{\mathrm{gl}}$-*space* is a contravariant continuous functor from $\mathbf{O}_{\mathrm{gl}}$ to the category **T** of spaces; and an $\mathbf{Orb}'$-*space* is a contravariant continuous functor from $\mathbf{Orb}'$ to **T**. Morphisms of $\mathbf{O}_{\mathrm{gl}}$-spaces and $\mathbf{Orb}'$-spaces are natural transformations. We write $\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ and $\mathbf{T}_{\mathbf{Orb}'}$ for the categories of $\mathbf{O}_{\mathrm{gl}}$-spaces and $\mathbf{Orb}'$-spaces, respectively.

Construction 3.4 Körschgen [17] provided a zigzag of Quillen equivalences relating the category $\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ and $\mathbf{T}_{\mathbf{Orb}}$ through the category $\mathbf{T}_{\mathbf{Orb}'}$, each equipped with the projective model structures. The right Quillen functors are restriction functors along continuous functors $f_1\colon \mathbf{Orb}' \to \mathbf{Orb}$ and $f_2\colon \mathbf{Orb}' \to \mathbf{O}_{\mathrm{gl}}$ between the indexing categories. Both functors are the identity on objects. On morphism space, the first functor

$$f_1(H,G)\colon \mathbf{Orb}'(H,G) = \widetilde{E}(H,G) \times_G EG \to \mathrm{map}(H,G) \times_G EG \cong \mathbf{Orb}(H,G)$$

arises from the projection $\widetilde{E}(H,G) \to \mathrm{map}(H,G)$ by applying the homotopy orbit construction. The behavior of the second functor on morphism spaces is

$$f_2(H,G)\colon \widetilde{E}(H,G) \times_G EG \to \mathrm{map}^{\mathcal{L}}(\mathcal{L}/H, \mathcal{L}/G) = \mathbf{O}_{\mathrm{gl}}(H,G), \quad [\alpha,\varphi] \mapsto \varphi^\flat,$$

where $\varphi^\flat\colon \mathcal{L}/H \to \mathcal{L}/G$ is defined by $\varphi^\flat(fH) = f\varphi G$.

Körschgen proved in [17, Propositions 2.18 and 2.22] that all the maps $f_1(H,G)$ and $f_2(H,G)$ are weak equivalences. So in particular, both functors $f_1\colon \mathbf{Orb}' \to \mathbf{Orb}$ and $f_2\colon \mathbf{Orb}' \to \mathbf{O}_{\mathrm{gl}}$ are Dwyer–Kan equivalences of topologically enriched categories; see [17, Corollary 3.13]. Consequently, the restriction functors

$$f_1^*\colon \mathbf{T}_{\mathbf{Orb}} \to \mathbf{T}_{\mathbf{Orb}'} \quad \text{and} \quad f_2^*\colon \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}} \to \mathbf{T}_{\mathbf{Orb}'}$$

are fully homotopical and right Quillen functors for the projective model structure; see [13, Lemma A.6] or [16, Theorem 3.5]. So the induced functors of localizations are equivalences of ∞-categories.

3.2 $\mathbf{O}_{\mathrm{gl}}$-spaces versus orthogonal spaces

In this section we recall the ∞-categorical essence of Schwede's paper [34], in the form of an equivalence between the ∞-categories of $\mathbf{O}_{\mathrm{gl}}$-spaces and global spaces.

Global spaces are the ∞-category underlying the global model structure on the category of orthogonal spaces [33].

Definition 3.5 (orthogonal spaces) The topological category $\mathbf{L}$ has all finite-dimensional real inner product spaces as its object. The morphism space $\mathbf{L}(V, W)$ between two inner product spaces is the space of linear isometric maps between V and W topologized as a Stiefel manifold. An *orthogonal space* is a continuous functor from $\mathbf{L}$ to the category $\mathbf{T}$ of spaces; *morphisms* of orthogonal spaces are the natural transformations. We write **Spc** for the category of orthogonal spaces and morphisms between them.

In [33, Definition 1.1.2], Schwede introduced the notion of *global equivalence* for morphisms between orthogonal spaces. This notion is much finer than the nonequivariant notion of equivalence; orthogonal spaces encode compatible G-equivariant homotopy types for all compact Lie groups that are invariant under global equivalences. The global equivalences are part of a model structure on the category of orthogonal spaces, the *global model structure* of [33, Proposition 1.2.21].

Now we recall that and how the ∞-category of global spaces is equivalent to the ∞-category of $\mathbf{O}_{\mathrm{gl}}$-spaces. We define the value of an orthogonal space X at $\mathbb{R}^\infty$ by

$$X(\mathbb{R}^\infty) = \operatorname*{colim}_{V \in s(\mathbb{R}^\infty)} X(V),$$

where the colimit is taken over the poset $s(\mathbb{R}^\infty)$ of finite-dimensional subspaces of $\mathbb{R}^\infty$. The functoriality of X extends by colimits to a continuous action of $\mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) = \mathcal{L}$ on $X(\mathbb{R}^\infty)$; see [34, Conjecture 3.2].

Construction 3.6 We introduce a functor $\Psi\colon \mathbf{Spc} \to \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ from the category of orthogonal spaces to the category of $\mathbf{O}_{\mathrm{gl}}$-spaces. We let X be an orthogonal space. The value of the $\mathbf{O}_{\mathrm{gl}}$-space $\Psi(X)$ at a universal subgroup G of the linear isometries monoid $\mathcal{L}$ is

$$\Psi(X)(G) \;=\; X(\mathbb{R}^\infty)^G.$$

An orthogonal space is *closed* if all of its structure maps are closed embeddings. It is shown in [33, Proposition 1.1.17] that the functor $\Psi\colon \mathbf{Spc} \to \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ takes global equivalences between closed orthogonal spaces to weak equivalences of $\mathbf{O}_{\mathrm{gl}}$-spaces. The following result is the ∞-categorical essence of [34].

Theorem 3.7 *The functor $\Psi\colon \mathbf{Spc} \to \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ admits a left derived functor*

$$\Psi_\infty\colon \mathbf{Spc}_\infty \to (\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}})_\infty$$

that is an equivalence of ∞-categories.

Proof In [34, Theorem 3.9] Schwede constructed a left Quillen equivalence $L\colon \mathbf{Spc} \to \mathcal{L}\text{-}\mathbf{T}$ between the global model structure on $\mathcal{L}\text{-}\mathbf{T}$ and the positive global model structure on $\mathcal{L}\text{-}\mathbf{T}$. This functor hence derives to an equivalence of ∞-categories

$$\mathbf{Spc}_\infty \simeq (\mathcal{L}\text{-}\mathbf{T})_\infty.$$

The result [34, Theorem 2.5] exhibits another Quillen equivalence between the global model structure on $\mathcal{L}\text{-}\mathbf{T}$ and the projective model structure on the category of $\mathbf{O}_{\mathrm{gl}}$-spaces. The right adjoint $\Phi\colon \mathcal{L}\text{-}\mathbf{T} \to \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$ of this Quillen equivalence is "of Elmendorf type" in that it records the fixed points for all universal subgroups of $\mathcal{L}$ and the natural maps between these. The right adjoint preserves all equivalences, so it extends to an equivalence of ∞-categories

$$(\mathcal{L}\text{-}\mathbf{T})_\infty \;\simeq\; (\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}})_\infty.$$

In combination we obtain an equivalence of ∞-categories from $\mathbf{Spc}_\infty$ to $(\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}})_\infty$ that is left derived from the functor $\Phi \circ L\colon \mathbf{Spc} \to \mathbf{T}_{\mathbf{O}_{\mathrm{gl}}}$. Finally, [34, Proposition 3.7] provides a natural morphism of $\mathcal{L}$-spaces

$$\xi(X)\colon L(X) \to X(\mathbb{R}^\infty),$$

which is a global equivalence if X is cofibrant in the global model structure on orthogonal spaces. So

$$\Phi(\xi(X))\colon \Phi(L(X)) \to \Phi(X(\mathbb{R}^\infty)) = \Psi(X)$$

is natural morphism of $\mathbf{O}_{\mathrm{gl}}$-spaces that is a weak equivalence whenever X is cofibrant. Since the left derived functor of $\Phi \circ L$ is an equivalence of ∞-categories, so is the left derived functor of Ψ. $\square$

Summarizing the above discussion, there is a chain of equivalences of ∞-categories:

$$(\mathbf{T}_{\mathbf{Orb}})_\infty \xrightarrow{f_1^*} (\mathbf{T}_{\mathbf{Orb}'})_\infty \xleftarrow{f_2^*} (\mathbf{T}_{\mathbf{O}_{\mathrm{gl}}})_\infty \xleftarrow{\Psi} \mathbf{Spc}_\infty.$$

We denote the resulting composite of those equivalences or their inverses, respectively, by

$$\omega\colon (\mathbf{T}_{\mathbf{Orb}})_\infty \to \mathbf{Spc}_\infty. \tag{3-1}$$

3.3 Comparing orbispace quotients

The **Orb**-space model and the orthogonal space model for global spaces both come with specific constructions that implement the "orbispace quotient" $A/\!\!/G$ of a compact Lie group G acting on a space A. We need to know that these two kinds of orbispace quotients match up under the equivalence (3-1). For orthogonal spaces, the orbispace quotient $A/\!\!/G$ is modeled by the following semifree orthogonal space. The construction depends on a choice of faithful G-representation, but it is of course independent up to global equivalence of this choice; see [33, Proposition 1.1.26].

Definition 3.8 (semifree orthogonal spaces) Let V be a finite-dimensional faithful representation of a compact Lie group G. We define the *semifree orthogonal space* functor

$$\mathbf{L}_{G,V}\colon G\text{-}\mathbf{T}\to\mathbf{Spc}$$

as follows: Let A be a G-space. For an inner product space W, we set

$$(\mathbf{L}_{G,V}A)(W)=\mathbf{L}(V,W)\times_G A,$$

where G acts on $\mathbf{L}(V,W)$ by precomposing the action on V. The structure maps act by postcomposition.

We write $\overline{F}\colon\mathbf{TGrpd}\to\mathbf{T}_{\mathbf{Orb}}$ for the composite of the Yoneda embedding and restriction along the full embedding $\mathbf{T}_{\mathbf{Orb}}^{\mathrm{op}}\to\mathbf{TGrpd}^{\mathrm{op}}$; this functor is one ingredient in the functor Construction 2.17 that induces the global realization (2-1). So the **Orb**-space $\overline{F}(\mathbb{G})$ associated to a topological groupoid $\mathbb{G}$ is given on objects by $\overline{F}(\mathbb{G})(K)=\|\mathbb{M}\mathrm{ap}(\mathbb{B}K,\mathbb{G})\|$.

For the rest of this section, we let G be a universal subgroup of the linear isometries monoid $\mathcal{L}$, we let V be a finite-dimensional faithful G-subrepresentation of the complete G-universe $\mathbb{R}^\infty$, and we let A be a G-space. By adapting constructions from Körschgen's paper [17], we will now construct a functor $Q\colon G\,\mathbf{T}\to\mathbf{T}_{\mathbf{Orb}'}$ and two natural transformations of $\mathbf{Orb}'$-spaces

$$f_1^*(\overline{F}(G\ltimes A))\xleftarrow{\ l\ }Q(A)\xrightarrow{\ \nu\ }f_2^*(\Psi(\mathbf{L}_{G,V}A)).$$

We will then show in Theorem 3.12 that the two morphisms are weak equivalences of $\mathbf{Orb}'$-spaces whenever A is Hausdorff. So in this case the equivalences witness that the two orbispace quotients $A/\!\!/G$ in the worlds of **Orb**-spaces and orthogonal spaces correspond under the equivalence ω from (3-1).

Construction 3.9 Let H be another universal subgroup of $\mathcal{L}$. We define

$$\widetilde{E}(H,G,A)=\{(\alpha,\varphi,a)\in\mathrm{hom}(H,G)\times\mathcal{L}\times A\mid h\varphi=\varphi\alpha(h)\text{ and }\alpha(h)a=a\text{ for all }h\in H\}.$$

The group G acts on $\widetilde{E}(H,G,A)$ by

$$g.(\alpha,\varphi,a)=(\alpha^g,\varphi g^{-1},g.a).$$

We define the $\mathbf{Orb}'$-space $Q(A)$ at the group H by

$$Q(A)(H)=\widetilde{E}(H,G,A)\times_G EG\cong\|G\ltimes\widetilde{E}(H,G,A)\|.$$

Functoriality is defined similarly as is [17, Definition 3.5]: We define a composition law of topological groupoids

$$(H\ltimes\widetilde{E}(K,H))\times(G\ltimes\widetilde{E}(H,G,A))\to G\ltimes\widetilde{E}(K,G,A),$$

which is given by

$$H\times\widetilde{E}(K,H)\times G\times\widetilde{E}(H,G,A)\to G\times\widetilde{E}(K,G,A),\quad (h,\alpha,\varphi),(g,\beta,\psi,x)\mapsto(g\beta(h),\beta\circ\alpha,\varphi\psi,x),$$

on arrows. The verification that this indeed gives rise to a functor between topological groupoids is analogous to the one in [17, Proposition 3.7]. Using Proposition 2.13, we obtain a map

$$\mathbf{Orb}'(K,H)\times Q(A)(H)\to Q(A)(K)$$

by applying fat geometric realizations on both sides. In the case where $A=*$, then $Q(*)(H)=\mathbf{Orb}'(H,G)$, and this construction reduces to the composition maps in the category $\mathbf{Orb}'$ as defined in [17, Proposition 3.9]. The verification of the associativity and unitality with respect to the composition in $\mathbf{Orb}'$ are a computation on the level of groupoids similar to the one for $A=*$ in [17, Proposition 3.8]. This concludes the definition of the $\mathbf{Orb}'$-space $Q(A)$.

Next we define the weak equivalence of $\mathbf{Orb}'$-spaces $l\colon Q(A)\to f_1^*(\bar{F}(G\ltimes A))$.

Construction 3.10 We let H be another universal subgroup of $\mathcal{L}$. The projection

$$\rho_A^H\colon \widetilde{E}(H,G,A)\to \mathbb{M}\mathrm{ap}(\mathbb{B}H,\mathbb{G}\ltimes A)_0=\{(\alpha,x)\in \hom(H,G)\times A\mid x\in A^{\alpha(H)}\},\quad \rho_A^H(\alpha,\varphi,x)=(\alpha,x),$$

participates in a pullback square

$$\begin{array}{ccc} \widetilde{E}(H,G,A) & \longrightarrow & \widetilde{E}(H,G) \\ \downarrow{\scriptstyle \rho_A^H} & & \downarrow{\scriptstyle \rho^H} \\ \mathbb{M}\mathrm{ap}(\mathbb{B}H,G\ltimes A)_0 & \longrightarrow & \hom(H,G) \end{array}$$

where the horizontal maps are also projections (forgetting the factor A) and the map $\rho^H=\rho_*^H$ is also defined in [17, Definition 2.12]. The map ρ_A^H gives rise to the map

$$\rho_A^H\times_G EG\colon \widetilde{E}(H,G,A)\times_G EG\to \mathbb{M}\mathrm{ap}(\mathbb{B}H,G\ltimes A)_0\times_G EG.$$

Note that

$$\begin{aligned}\big((f_1)^*\circ F\big)(G\ltimes A)(H) &= \|\mathbb{M}\mathrm{ap}(\mathbb{B}H,G\ltimes A)\| \\ &\cong \|G\ltimes \mathbb{M}\mathrm{ap}(\mathbb{B}H,G\ltimes A)_0\|\cong \mathbb{M}\mathrm{ap}(\mathbb{B}H,G\ltimes A)_0\times_G EG.\end{aligned}$$

Using this homeomorphism, the map $\rho_A^H\times_G EG$ rewrites as a map

$$l(H)\colon Q(A)(H)\to \big((f_1)^*\circ F\big)(G\ltimes A)(H)$$

and a computation on the level of groupoids similar to [17, Proposition 3.8] shows that these maps are compatible with the action maps. We obtain a morphism of $\mathbf{Orb}'$-spaces $l\colon Q(A)\to ((f_1)^*\circ F)(G\ltimes A)$.

Finally, we define the weak equivalence of $\mathbf{Orb}'$-spaces $\nu\colon Q(A)\to ((f_2)^*\circ\Psi)(\mathbf{L}_{G,V}A)$.

Construction 3.11 We let V be a finite-dimensional faithful G-subrepresentation of the complete G-universe $\mathbb{R}^\infty$. Then restriction of linear isometric embeddings from $\mathbb{R}^\infty$ to V is map of $\mathcal{L}$-spaces $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \to \mathbf{L}(V, \mathbb{R}^\infty)$ that is also G-equivariant for precomposition. If H is another universal subgroup of $\mathcal{L}$, we define a continuous map

$$\nu(H)\colon Q(A)(H) = \widetilde{E}(H, G, A) \times_G EG \to (\mathbf{L}(V, \mathbb{R}^\infty) \times_G A)^H = \big(f_2^*(\Psi(\mathbf{L}_{G,V} A))\big)(H)$$

by

$$\nu(H)[\alpha, \varphi, a; x] = [\varphi|_V, a].$$

For varying universal subgroups H, these maps form a morphism of $\mathbf{Orb}'$-spaces

$$\nu\colon Q(A) \to ((f_2)^* \circ \Psi)(\mathbf{L}_{G,V} A).$$

This can be proven similarly as in [17, Proposition 3.12] because functoriality on both sides does not interact with the elements of A.

The following is the main result of this section.

Theorem 3.12 *Let G be a universal subgroup of the linear isometries monoid $\mathcal{L}$, let V be a finite-dimensional faithful G-subrepresentation of the complete G-universe $\mathbb{R}^\infty$, and let A be a Hausdorff G-space. Then the morphisms of $\mathbf{Orb}'$-spaces*

$$f_1^*(\overline{F}(G \ltimes A)) \xleftarrow{\ l\ } Q(A) \xrightarrow{\ \nu\ } f_2^*(\Psi(\mathbf{L}_{G,V} A))$$

are weak equivalences.

Proof Let H be a universal subgroup of $\mathcal{L}$. The projection $\rho^H : \widetilde{E}(H, G) \to \hom(H, G)$ is a fiber bundle with contractible fiber by [17, Proposition 2.15] and so is the pullback

$$\rho_A^H : \widetilde{E}(H, G, A) \to \mathbb{M}\mathrm{ap}(\mathbb{B}H, \mathbb{G} \ltimes A)_0.$$

The proof that $l(H) = \rho_A^H \times_G EG$ is a weak equivalence is completely analogous to the one in [17, Proposition 2.18], exploiting that the homotopy quotient construction $- \times_G EG$ is homotopical.

We now turn to the proof that the second map is an equivalence. The space A is Hausdorff by assumption, the space $\hom(H, G)$ is Hausdorff because it is metrizable, the space $\mathcal{L}$ is Hausdorff by [34, Proposition A.1]. The product of Hausdorff spaces is Hausdorff and we conclude that the space $\widetilde{E}(H, G, A)$ is Hausdorff because it is a subspace of a Hausdorff space. The natural map

$$\widetilde{E}(H, G, A) \times_G EG \to \widetilde{E}(H, G, A)/G$$

from the homotopy quotient to the orbit space is a weak equivalence for all universal subgroups H of $\mathcal{L}$ by [17, Theorem A.7] because $\tilde{E}(H, G, A)$ is a free G-space and Hausdorff.

Now we exploit that the map

$$\widetilde{E}(H, G, A)/G \to (\mathcal{L} \times_G A)^H, \quad [\alpha, \varphi, x] \mapsto [\varphi, x],$$

is well defined and a homeomorphism. We can adapt the proof of [17, Proposition 2.20]. More in detail, the proof does not use any properties of $\mathcal{L}$ other than being a $(H \times G)$-space such that the restricted G-action is free. These assumption also hold for the space $\mathcal{L} \times A$ where $H \times G$ acts via

$$(k,g).(\varphi,x) = (k\varphi g^{-1}, g.x)$$

for $k \in H$, $g \in G$, $\varphi \in \mathcal{L}$ and $x \in A$. We may replace $\mathcal{L}$ with $\mathcal{L} \times A$. The space $\widetilde{E}(H,G)$ then becomes $\widetilde{E}(H,G,A)$ and $(\mathcal{L}/G)^H$ becomes $(\mathcal{L} \times_G A)^H$.

The final ingredient is [34, Proposition A.10], which implies that the restriction map

$$\rho_V \colon \mathcal{L} \times_G A = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty) \times_G A \to \mathbf{L}(V, R^\infty) \times_G A = (\mathbf{L}_{G,V} A)(\mathbb{R}^\infty)$$

induces a weak equivalence on H-fixed point spaces. □

We now apply the previous theorem to the main case of interest for us, namely global quotient orbifolds.

Corollary 3.13 *Let M be a manifold together with an almost free action of a universal subgroup G of $\mathcal{L}$ on M. Then the image of the global realization $|G \ltimes M|_{\mathrm{gl}}$ under the equivalence $\omega \colon (\mathbf{T}_{\mathbf{Orb}})_\infty \to \mathbf{Spc}_\infty$ defined in (3-1) is equivalent to the semifree global space $\mathbf{L}_{G,V} M$, where V is any faithful G-representation.*

Proof This follows from Theorem 3.12, also using that f_1^* and f_2^* are fully homotopical as all objects are fibrant in the projective model structure and that the semifree global space $\mathbf{L}_{G,V}$ is closed by [33, Example 1.1.14], so that we can directly apply the functor Ψ from Construction 3.6 without replacing the object cofibrantly. □

4 Orbifold cohomology theories from global spectra

In this section we use global spectra in the sense of Schwede [33] to define orbifold cohomology theories. We then discuss various properties and examples of our construction.

In [33, Chapter 4], Schwede developed a setting for global stable homotopy theory based on orthogonal spectra; see [33, Definition 3.1.1]. For a compact Lie group G, every orthogonal spectrum X can be made an orthogonal G-spectrum via the trivial G-action; thus, the G-equivariant homotopy groups $\pi_*^G(X)$ are defined, compare [33, (3.1.11)]. A morphism $f \colon X \to Y$ of orthogonal spectra is a *global equivalence* if the induced homomorphism $\pi_k^G(f) \colon \pi_k^G(X) \to \pi_k^G(Y)$ is an isomorphism for every compact Lie group G and all integers k.

Definition 4.1 The ∞*-category of global spectra* $\mathcal{GH}$ is the ∞-categorical localization of the homotopy coherent nerve of the topological category **Sp** of orthogonal spectra at the class of global equivalences.

Remark 4.2 Usually, one would define $\mathcal{GH}$ as the ∞-categorical localization of the 1-category of orthogonal spectra at the global equivalences. However, one can first regard orthogonal spectra as a simplicial model category and then use [30, Theorem 13.2.1] to deduce that there exists simplicial (co)fibrant replacement functors. Combining this with [21, Theorem 1.3.4.20] implies that the two definitions agree.

By [33, Theorem 4.3.18], the global equivalences of orthogonal spectra are part of a stable, cofibrantly generated, proper, topological model structure. The ∞-category of global spectra is thus complete and cocomplete; compare [3, Theorem 2.5.9]. Since the model structure is stable, the suspension functor on its homotopy category is invertible, so the ∞-category $\mathcal{GH}$ is stable; compare [21, Corollary 1.4.2.27].

The *suspension spectrum* functor

$$\Sigma^\infty_+ \colon \mathbf{Spc} \to \mathbf{Sp}$$

is defined in [33, Conjecture 4.1.7]. Given an orthogonal space X, the value of $\Sigma^\infty_+ X$ at an inner product space V is $(\Sigma^\infty_+ X)(V) = S^V \wedge X(V)_+$. The functor Σ^∞_+ takes global equivalences of orthogonal spaces to global equivalences of orthogonal spectra [33, Corollary 4.1.9], and it is a left Quillen functor with respect to the two global model structures, see [33, Theorem 4.3.17(v)]. So it descends to a colimit preserving functor

$$\Sigma^\infty_+ \colon \mathbf{Spc}_\infty \to \mathcal{GH}$$

of underlying ∞-categories, for which we use the same name.

Definition 4.3 (global motiv) We write $\mathbf{M} \colon \mathbf{Orbfld}_\infty \to \mathcal{GH}$ for the composite

$$\mathbf{Orbfld}_\infty \xrightarrow{|-|_{\mathrm{gl}}} (\mathbf{T}_{\mathbf{Orb}})_\infty \xrightarrow[\sim]{\omega} \mathbf{Spc}_\infty \xrightarrow{\Sigma^\infty_+} \mathcal{GH}$$

of the global homotopy type, the equivalence ω from (3-1), and the suspension spectrum functor. We call $\mathbf{M}(\mathbb{G})$ the *global motiv* of the orbifold $\mathbb{G}$.

The following theorem summarizes the main properties of the global motiv functor. All four parts are consequences of the corresponding properties of the global realization, see Theorem 2.25, and the fact that the functor Σ^∞_+ preserves colimits. The last item also uses Corollary 3.13.

Theorem 4.4 (i) *The global motiv functor takes essential equivalences of orbifolds to global equivalences of global spectra.*

(ii) *The global motiv functor takes disjoint unions of orbifolds to coproducts of global spectra.*

(iii) *Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\widetilde{U}$ and $\widetilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \to \mathbb{G}_0/\mathbb{G}_1$. Then the square*

(4-1)
$$\begin{array}{ccc} \mathbf{M}(\mathbb{G}|_{\widetilde{U}\cap\widetilde{V}}) & \longrightarrow & \mathbf{M}(\mathbb{G}|_{\widetilde{U}}) \\ \downarrow & & \downarrow \\ \mathbf{M}(\mathbb{G}|_{\widetilde{V}}) & \longrightarrow & \mathbf{M}(\mathbb{G}) \end{array}$$

is a bicartesian square in the ∞-category of global spectra, where all morphisms are induced by the respective inclusions.

(iv) *Let M be a manifold together with an almost free action of a universal subgroup G of $\mathcal{L}$ on M. Then the global motiv $\mathbf{M}(G \ltimes M)$ is equivalent to the suspension spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} M$.*

Definition 4.5 (orbifold cohomology) Let $\mathbb{G}$ be an orbifold and let E be an orthogonal spectrum and let k be an integer. We define the k^{th} *orbifold cohomology group* represented by E of $\mathbb{G}$ by

$$E^k(\mathbb{G}) = \mathcal{GH}\left(\mathbf{M}(\mathbb{G}), E[k]\right).$$

This extends to a functor $E^k\colon \mathbf{Orbfld}^{\mathrm{op}} \to \mathbf{Ab}$ and morphisms between orthogonal spectra induce natural transformations between the associated orbifold cohomology theories.

The first three items of Theorem 4.4 immediate imply the following properties of the orbifold cohomology groups. Part (iv) follows from the equivalence between the global motiv $\mathbf{M}(G \ltimes M)$ and the suspension spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V} M$, and the adjunction isomorphism

$$E^k_G(M) = G\text{-}\mathcal{SH}(\Sigma^\infty_{G,+} M, E[k]) \cong \mathcal{GH}(\Sigma^\infty_+ \mathbf{L}_{G,V} M, E[k])$$

from [33, Remark 4.5.25].

Theorem 4.6 *Let E be an orthogonal spectrum.*

(i) *The orbifold cohomology functor $E^k\colon \mathbf{Orbfld}^{\mathrm{op}} \to \mathbf{Ab}$ takes essential equivalences of orbifolds to isomorphisms.*

(ii) *For every family $\{\mathbb{G}^i\}_{i\in I}$ of orbifolds, the natural map*

$$E^k\Big(\coprod_{i\in I} \mathbb{G}^i\Big) \to \prod_{i\in I} E^k(\mathbb{G}^i)$$

is an isomorphism.

(iii) *Let $\mathbb{G}$ be an orbifold and let U, V be two open subsets covering the orbit space $\mathbb{G}_0/\mathbb{G}_1$. Let $\widetilde{U}$ and $\widetilde{V}$ be the preimages of U and V under the quotient map $\mathbb{G}_0 \to \mathbb{G}_0/\mathbb{G}_1$. There is a long exact sequence of orbifold cohomology groups*

$$\cdots \to E^k(\mathbb{G}) \xrightarrow{(i_U^*, i_V^*)} E^k\left(\mathbb{G}|_{\widetilde{U}}\right) \oplus E^k\left(\mathbb{G}|_{\widetilde{V}}\right) \xrightarrow{j_U^* - j_V^*} E^k\left(\mathbb{G}|_{\widetilde{U}\cap\widetilde{V}}\right) \to E^{k+1}(\mathbb{G}) \to \cdots$$

for every orthogonal spectrum E. Here, $i_U\colon \mathbb{G}|_{\widetilde{U}} \to \mathbb{G}$, $i_V\colon \mathbb{G}|_{\widetilde{V}} \to \mathbb{G}$, $j_U\colon \mathbb{G}|_{\widetilde{U}\cap\widetilde{V}} \to G|_{\widetilde{U}}$ and $j_V\colon \mathbb{G}|_{\widetilde{U}\cap\widetilde{V}} \to \mathbb{G}|_{\widetilde{V}}$ denote the inclusion morphisms.

(iv) *Let G be a compact Lie group acting almost freely on a manifold M. The orbifold cohomology of $G \ltimes M$ is isomorphic to the G-equivariant cohomology of M, ie*

$$E^k(G \ltimes M) \cong E^k_G(M).$$

Example 4.7 (cohomology of the classifying space, Borel cohomology) Let E be a nonequivariant generalized cohomology theory. By Brown's representability theorem, there is an orthogonal spectrum representing E which we also denote by E. The forgetful functor $U\colon \mathcal{GH} \to \mathcal{SH}$ admits a right adjoint $b\colon \mathcal{SH} \to \mathcal{GH}$; see [33, Theorem 4.5.1]. This right adjoint admits an explicit lift to the category of orthogonal spectra described in [33, Conjecture 4.5.21].

Let $\mathbb{G}$ be an orbifold. Since the underlying space of the global realization $|\mathbb{G}|_{\mathrm{gl}}$ is the nonequivariant realization $|\mathbb{G}|$, the underlying nonequivariant homotopy type of the global motiv $\mathbf{M}(\mathbb{G})$ is the unreduced suspension spectrum of $|\mathbb{G}|$. So we obtain a chain of natural isomorphisms

$$\begin{aligned}(bE)^k(\mathbb{G}) = \mathcal{GH}(\mathbf{M}(\mathbb{G}), bE[k]) &\cong \mathcal{SH}\big(U(\mathbf{M}(\mathbb{G})), E[k]\big)\\ &\cong \mathcal{SH}(\Sigma^\infty_+|\mathbb{G}|, E[k]) = E^k(|\mathbb{G}|).\end{aligned}$$

We used that the functor b commutes with the respective shift functors in $\mathcal{SH}$ and $\mathcal{GH}$ because its adjoint U does so. We just showed that the orbifold cohomology theory represented by bE can be identified with the cohomology represented by E of the classifying space. For $G \ltimes M$ a global quotient orbifold, we have $|G \ltimes M| \cong M \times_G EG$. Therefore, bE represents the equivariant Borel cohomology represented by E for all global quotients orbifolds.

Example 4.8 (Bredon cohomology) Let F be a global Mackey functor as defined in [33, Section 4.2]. By [33, Theorem 4.4.9], there is an *Eilenberg–MacLane spectrum HF*, ie an object of the global stable homotopy category with equivariant homotopy groups concentrated in degree 0, and such that $\pi_0^G(HF) \cong F(G)$ for all compact Lie groups G with corresponding transfer and restriction maps. The underlying orthogonal G-spectrum $(HF)_G$ represents Bredon cohomology for the underlying G-coefficient system of $F|_F$, the underlying contravariant functor of the restriction of F to a G-Mackey functor; see [7, Example 3.2.16]. For $G \ltimes M$ a global quotient orbifold, we obtain

$$HF^k(G \ltimes M) \cong HF^k_G(M) \cong H^k_G(M, F|_G)$$

by Theorem 4.6(iv). We conclude that the spectrum HF represents Bredon cohomology for global quotient orbifolds. This in particular proves that the Bredon cohomology of an orbifold is independent of the choice of the presentation as a global quotient orbifold. Adem and Ruan [2, Section 3, Remark 5.11] proved this for the special case where $F = \mathbf{RU}_{\mathbb{Q}}$ is the rationalized representation functor, Pronk and Scull [29, Proposition 5.11] proved this for what they call an *orbifold coefficient system*. Moreover, Bredon cohomology extends to an actual cohomology theory on all orbifolds.

Construction 4.9 (Atiyah–Hirzebruch spectral sequence) The full subcategories of globally connective spectra and globally coconnective spectra form a t-structure on $\mathcal{GH}$; see [33, Theorem 4.4.9]. The heart of this t-structure consists of those spectra whose homotopy groups are concentrated in degree 0, ie Eilenberg–MacLane spectra. The functor $\underline{\pi}_0$ induces an equivalence between this heart and the category of global Mackey functors. This t-structure is induced by a t-model structure on **Sp** in the sense of Fausk and Isaksen [10, Definition 4.1]. They give a short argument why this is true for model categories of spectra which also applies in our case.

Their paper [10, Section 10] explains how to construct an Atiyah–Hirzebruch spectral sequence from a t-model structure on a stable model category. We briefly recall the most important ideas and adjust them to our situation: There is a $(q-1)$-connective cover $Y_{\geq q}$ for any spectrum Y and any integer q and natural transformations $Y_{\geq q+1} \to Y_{\geq q}$ in $\mathcal{GH}$. The homotopy cofiber of this map is a shifted Eilenberg–MacLane spectrum of type $\underline{\pi}_q(Y)$.

For X another orthogonal spectrum, we apply $\mathcal{GH}(X, -[-p-q])$ to the distinguished triangle

$$Y_{\geq q+1} \to Y_{\geq q} \to (H\underline{\pi}_q(Y))[q] \to Y_{\geq q+1}[1]$$

to get an exact couple with $D^2_{p,q} = \mathcal{GH}(X, Y_{\geq q}[-p-q])$ and $E^2_{p,q} = \mathcal{GH}\big(X, (H\underline{\pi}_q(Y))[-p]\big)$.

For $X = \big(\Sigma^\infty_+ \circ R \circ \mathrm{R}_{\mathrm{fib}} \circ \mathcal{L}_{\mathrm{Lie}}\big)(\mathbb{G})$, we obtain the following:

Proposition 4.10 *Let Y be an orthogonal spectrum and let $\mathbb{G}$ be an orbifold. There is a spectral sequence with $E^2_{p,q} = (H\underline{\pi}_q(Y))^{-p}(\mathbb{G})$ — Bredon cohomology of $\mathbb{G}$ with coefficient system $\underline{\pi}_q(Y)$ — which conditionally converges to $Y^{p+q}(\mathbb{G})$.*

Here, conditional convergences of spectral sequences is defined in Boardman's paper [5, Definition 5.10]. It also provides several criteria for deducing strong convergence from conditional convergence.

Proof The statement follows from Fausk and Isaksen's theorem [10, Theorem 10.1]. We need to check that $\big(\Sigma^\infty_+ \circ R \circ \mathrm{R}_{\mathrm{fib}} \circ \mathcal{L}_{\mathrm{Lie}}\big)(\mathbb{G})$ is bounded below and that $\mathrm{holim}_{n\to\infty} Y_{\geq n}$ is contractible.

The first condition holds because suspension spectra are globally connective by [33, Proposition 4.1.11].

Let G be a compact Lie group and k be an integer. The functor $\pi_0^G \colon \mathcal{GH} \to \mathbf{Ab}$ is corepresentable by [33, Theorem 4.4.3(i)]. We hence obtain a Milnor short exact sequence; see eg [4, Lemma 0919]:

$$0 \to \lim^1_{n\to\infty} \pi^G_{k+1}(Y_{\geq n}) \to \pi^G_k\big(\operatorname*{holim}_{n\to\infty} Y_{\geq n}\big) \to \lim_{n\to\infty} \pi^G_k(Y_{\geq n}) \to 0$$

The homotopy groups $\pi^G_k(Y_{\geq n})$ eventually vanish. The left and the right term of the sequence therefore vanish and so does the middle term. □

Example 4.11 (orbifold K-theory) Firstly, we briefly discuss the geometric definition of orbifold K-theory via orbifold vector bundles. A full definition and proofs of the following statements can be found in [1, Section 3.3]. Let $\mathbb{G}$ be a compact orbifold, that is a proper foliation groupoid where $\mathbb{G}_0$ is compact. A $\mathbb{G}$-*space* is a smooth manifold M together with a smooth *anchor map* $\alpha \colon M \to \mathbb{G}_0$ and a smooth *action map* $\mu \colon \mathbb{G}_1 \times_{\mathbb{G}_0} M \to M$ which satisfy the usual unitality and associativity conditions. A $\mathbb{G}$-vector bundle is a $\mathbb{G}$-space M where the anchor map $\alpha \colon M \to \mathbb{G}_0$ is a complex vector bundle such that the action of $\mathbb{G}$ is fiberwise linear. Similarly as for manifolds, one can define Whitney sums and tensor products of $\mathbb{G}$-vector bundles.

The *orbifold K-theory* $\mathbf{K}_{\mathbf{Orbfld}}(\mathbb{G})$ of $\mathbb{G}$ is defined as the Grothendieck ring of isomorphism classes of $\mathbb{G}$-vector bundles. This is a functorial construction (using pullbacks of vector bundles) and one can prove that every essential equivalence between orbifolds induces an isomorphism on these rings; compare [2, Proposition 4.2].

Moreover, the K-theory of a global quotient orbifold $G \ltimes M$ may be identified with the G-equivariant K-theory of M, ie there is an isomorphism

$$\mathbf{K}_{\mathbf{Orbfld}}(G \ltimes M) \cong \mathbf{K}_G(M),$$

which is natural in M; see [1, Proposition 3.6].

On the other hand, there is the *periodic global K-theory spectrum* **KU** which represents K-theory in the following sense: for every finite G-CW complex A, there is a natural isomorphism

$$\mathbf{K}_G(A) \cong \mathbf{KU}^0_G(A).$$

This applies when $A = M$ is a compact manifold together with a G-action on M. The definition of **KU** is due to Joachim [15]. See [33, Section 6.4] for a self-contained exposition from a global perspective.

By also using Theorem 4.6(iv), we finally obtain an isomorphism

$$\mathbf{K}_{\mathbf{Orbfld}}(G \ltimes M) \cong \mathbf{KU}^0(G \ltimes M).$$

We conclude that the ring spectrum **KU** represents orbifold K-theory for compact global quotient orbifolds. Moreover, both sides are invariant under essential equivalences of orbifolds. We conclude hat **KU** represents orbifold cohomology for all compact orbifolds by Theorem 2.11.

It is however not clear that this isomorphism is natural with respect to all maps of orbifolds.

Example 4.12 (rationalized K-theory) The rationalized K-theory is represented by the rationalized periodic global K-theory spectrum $\mathbf{KU}_{\mathbb{Q}}$. By [38, Theorem 1.6], this spectrum decomposes up to $\mathcal{F}$in-global equivalence into a sum of shifted Eilenberg–MacLane spectra

$$\mathbf{KU}_{\mathbb{Q}} \simeq \bigvee_{n \in \mathbb{Z}} H(\mathbf{RU}_{\mathbb{Q}})[2n],$$

which is enough for our purposes. This implies that there is an isomorphism

$$\mathbf{KU}^0(\mathbb{G}) \otimes \mathbb{Q} \cong \bigoplus_{n \in \mathbb{Z}} (H(\mathbf{RU}_{\mathbb{Q}}))^{2n}(\mathbb{G})$$

between the rationalized K-theory and Bredon cohomology with coefficients system $\mathbf{RU}_{\mathbb{Q}}$ of an orbifold $\mathbb{G}$. For $\mathbb{G} = G \ltimes M$ a global quotient orbifold of a compact manifold M under the action of a compact Lie group G, this rewrites as

$$\mathbf{K}_{\mathbf{Orbfld}}(G \ltimes M) \otimes \mathbb{Q} \cong \bigoplus_{n \in \mathbb{Z}} H^{2n}_G(M, \mathbf{RU}_{\mathbb{Q}})$$

by Examples 4.11 and 4.8. This has been noted before by Adem and Ruan [2, Remark 5.11].

References

[1] **A Adem**, **J Leida**, **Y Ruan**, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics 171, Cambridge Univ. Press (2007) MR

[2] **A Adem**, **Y Ruan**, *Twisted orbifold K-theory*, Comm. Math. Phys. 237 (2003) 533–556 MR

[3] **I Barnea**, **Y Harpaz**, **G Horel**, *Pro-categories in homotopy theory*, Algebr. Geom. Topol. 17 (2017) 567–643 MR

[4] **P Belmans**, **A J de Jong**, et al., *The Stacks project*, electronic reference (2005–) Available at `http://stacks.math.columbia.edu`

[5] **J M Boardman**, *Conditionally convergent spectral sequences*, from "Homotopy invariant algebraic structures", Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 49–84 MR

[6] **A Clough**, **B Cnossen**, **S Linskens**, *Global spaces and the homotopy theory of stacks*, preprint (2024) arXiv 2407.06877

[7] **D Degrijse**, **M Hausmann**, **W Luck**, **I Patchkoria**, **S Schwede**, *Proper equivariant stable homotopy theory*, Mem. Amer. Math. Soc. 1432, Amer. Math. Soc., Providence, RI (2023) MR

[8] **T tom Dieck**, *Algebraic topology*, European Mathematical Society (EMS), Zürich (2008) MR

[9] **D Dugger**, **D C Isaksen**, *Topological hypercovers and $\mathbb{A}^1$-realizations*, Math. Z. 246 (2004) 667–689 MR

[10] **H Fausk**, **D C Isaksen**, *t-model structures*, Homology Homotopy Appl. 9 (2007) 399–438 MR

[11] **D Gepner**, **L Meier**, *On equivariant topological modular forms*, Compos. Math. 159 (2023) 2638–2693

[12] **A Haefliger**, *Groupoïdes d'holonomie et classifiants*, from "Transversal structure of foliations", Astérisque 116, Société Mathématique de France, Paris (1984) 70–97 MR

[13] **A G Henriques**, *Orbispaces*, PhD thesis, Massachusetts Institute of Technology (2005) MR Available at `https://www.proquest.com/docview/305368406`

[14] **M Hovey**, *Model categories*, Mathematical Surveys and Monographs 63, Amer. Math. Soc., Providence, RI (1999) MR

[15] **M Joachim**, *Higher coherences for equivariant K-theory*, from "Structured ring spectra", London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 87–114 MR

[16] **A Körschgen**, *Dwyer–Kan equivalences induce equivalences on topologically enriched presheaves*, preprint (2017) arXiv 1704.07472

[17] **A Körschgen**, *A comparison of two models of orbispaces*, Homology Homotopy Appl. 20 (2018) 329–358 MR

[18] **E Lerman**, *Orbifolds as stacks?*, Enseign. Math. 56 (2010) 315–363 MR

[19] **L G J Lewis**, *The stable category and generalized Thom spectra*, PhD thesis, The University of Chicago (1978) MR Available at `https://www.proquest.com/docview/251798616`

[20] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR

[21] **J Lurie**, *Higher algebra*, preprint (2017) Available at `https://url.msp.org/Lurie-HA`

[22] **M C McCord**, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. 146 (1969) 273–298 MR

[23] **I Moerdijk**, *Orbifolds as groupoids: an introduction*, from "Orbifolds in mathematics and physics", Contemp. Math. 310, Amer. Math. Soc., Providence, RI (2002) 205–222 MR

[24] **I Moerdijk**, **J Mrčun**, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics 91, Cambridge Univ. Press (2003) MR

[25] **I Moerdijk**, **D A Pronk**, *Orbifolds, sheaves and groupoids*, K-Theory 12 (1997) 3–21 MR

[26] **B Noohi**, *Homotopy types of topological stacks*, Adv. Math. 230 (2012) 2014–2047 MR

[27] **J Pardon**, *Enough vector bundles on orbispaces*, Compos. Math. 158 (2022) 2046–2081 MR

[28] **D A Pronk**, *Etendues and stacks as bicategories of fractions*, Compositio Math. 102 (1996) 243–303 MR

[29] **D Pronk**, **L Scull**, *Translation groupoids and orbifold cohomology*, Canad. J. Math. 62 (2010) 614–645 MR

[30] **E Riehl**, *Categorical homotopy theory*, New Mathematical Monographs 24, Cambridge Univ. Press (2014) MR

[31] **I Satake**, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 359–363 MR

[32] **H Sati**, **U Schreiber**, *Proper orbifold cohomology*, preprint (2020) arXiv 2008.01101

[33] **S Schwede**, *Global homotopy theory*, New Mathematical Monographs 34, Cambridge Univ. Press (2018) MR

[34] **S Schwede**, *Orbispaces, orthogonal spaces, and the universal compact Lie group*, Math. Z. 294 (2020) 71–107 MR

[35] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR

[36] **N P Strickland**, *The category of CGWH spaces*, preprint (2009) Available at `https://ncatlab.org/nlab/files/StricklandCGHWSpaces.pdf`

[37] **W T Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) Available at `https://url.msp.org/gt3m`

[38] **C Wimmer**, *Rational extensions of the representation ring global functor and a splitting of global equivariant K-theory*, Bull. Lond. Math. Soc. 50 (2018) 863–873 MR

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