AGT

Algebraic & Geometric Topology

Volume 25 (2025)

The cohomology of biquotients via a product on the two-sided bar construction

Jeffrey D Carlson

Appendix written jointly with Matthias Franz

msp

msp

Algebraic & Geometric Topology **25**:9 (2025) 5279–5317
DOI: 10.2140/agt.2025.25.5279
Published: 18 December 2025

The cohomology of biquotients via a product on the two-sided bar construction

JEFFREY D CARLSON
APPENDIX WRITTEN JOINTLY WITH MATTHIAS FRANZ

We compute the Borel equivariant cohomology ring of the left K-action on a homogeneous space G/H, where G is a connected Lie group, H and K are closed connected subgroups, and 2 as well as the torsion primes of the Lie groups are units of the coefficient ring. As a special case, this gives the singular cohomology rings of biquotients $H \setminus G/K$. This depends on a version of the Eilenberg–Moore theorem developed in the appendix, where a novel multiplicative structure on the two-sided bar construction $\mathbf{B}(A'', A, A')$ is defined, valid when $A'' \leftarrow A \rightarrow A$ is a pair of maps of homotopy Gerstenhaber algebras.

16U80, 55N91, 57T15, 57T30; 16E45, 55T20, 57T35

Homogeneous spaces, which can be realized as coset spaces G/H for G a transitively acting Lie group and H the stabilizer of a point, are arguably the most highly symmetric, most canonical, and most thoroughly investigated objects of study in differential geometry after Lie groups. A generalization of perennial interest, which offers many interesting examples in positive-curvature geometry, is the class of *biquotients*, the orbit spaces $K \backslash G/H$ of G under free left–right actions by products $K \times H$ of two closed subgroups.

The cohomology rings of a Lie group G over $\mathbb{Q}$ and those finite fields $\mathbb{F}_p$ for which $H^*(G;\mathbb{Z})$ lacks p-torsion have been known to be exterior algebras since the fundamental 1941 work of Hopf [17], and that of a homogeneous space G/H with connected stabilizer H has been known over $\mathbb{R}$ since work of Henri Cartan from 1950 [9]:

$$H^*(G/H) \cong \operatorname{Tor}_{H^*(BG)}(\mathbb{R}, H^*(BH)).$$

In his 1952 dissertation [6, Section 30], Borel used the Serre spectral sequence of the Borel fibration $G \to G/H \to BH$ to show the same ring isomorphism also holds when H is of maximal rank, with $\mathbb{F}_p$ coefficients if $H^*(G;\mathbb{Z})$ and $H^*(H;\mathbb{Z})$ lack p-torsion and with $\mathbb{Z}$ coefficients if they are torsion free. Beyond this, however, ring isomorphisms proved hard to find. Starting with Baum's 1962 dissertation, a program began to obtain a more general result, additively at least, using the then-new Eilenberg–Moore spectral sequence of the fibration $G/H \to BH \to BG$. Cartan's result implies this spectral sequence's collapse over $\mathbb{R}$, and Baum proved collapse under hypotheses covering the best-studied homogeneous spaces. Subsequent work of authors including Baum [2], Larry Smith [30], Gugenheim and May [15],

This paper is distilled from the preprint arXiv 2106.02986v1, which contains more generous exposition, much more historical context, and less-abbreviated proofs.

Munkholm [26], and Joel Wolf [34] in the 1960s–70s proved collapse under substantially more general hypotheses and with accordingly more involved proof techniques. Again, however, almost all of these proofs provided only the additive structure of $H^*(G/H)$, and those which in special cases gave the ring structure could be factored through Borel's theory. A collapse result guaranteeing a multiplicative isomorphism was provided by Franz only in 2019 [11].

Arguably the most general of the Eilenberg–Moore collapse results actually applies to a more general case than homogeneous spaces: Munkholm proves the collapse of the spectral sequence corresponding to the total space of a pullback bundle

$$\begin{array}{ccc} X \times_B E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & B \end{array}$$

when X, B, and E have polynomial cohomology on countably many generators and the fundamental group $\pi_1(B)$ is trivial [26].[1] A biquotient — and hence, in particular, a homogeneous space — fits into this setting when $X = BK$, $B = BG$, and $E = BH$ are models of classifying spaces chosen in such a way that $BH \to BG$ is a fiber bundle, and Singhof [29] used Munkholm's result in this case to determine that $H^*(K\backslash G/H)$ is additively isomorphic to $\operatorname{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH))$, and simply to Tor^0 in the special case where the rank of G is the sum of the ranks of K and H. Because the Eilenberg–Moore spectral sequence is concentrated in the 0^{th} column in this case, there is no extension problem, so this is in fact a ring isomorphism. One might hope Singhof's and Franz's theorems were instances of a more general result, and our initial motivation was to show this hope is justified.

Theorem 0.1 *Let G be a connected Lie group, H and K closed connected subgroups, and k a principal ideal domain in which 2 and the torsion primes of G, H, and K are units. Then the Borel equivariant cohomology ring of the left translation action of K on G/H, or, equivalently, of the two-sided action of $K \times H$ on G by $(x,h)\cdot g := xgh^{-1}$, is*

$$H^*_K(G/H;k) \cong H^*_{K\times H}(G;k) \cong \operatorname{Tor}^*_{H^*(BG;k)}(H^*(BK;k), H^*(BH;k)).$$

In particular, if the two-sided action of $K \times H$ on G is free, the cohomology ring of the biquotient $K\backslash G/H$ is given by

$$H^*(K\backslash G/H;k) \cong \operatorname{Tor}^*_{H^*(BG;k)}(H^*(BK;k), H^*(BH;k)).$$

See Example 6.1 for a sample computation. This result in fact extends to what could be called *generalized biquotients* by analogy with generalized homogeneous spaces; see Remark 4.3, where we recover a known result on the cohomology of certain free loop spaces.

In broadest outline, our proof of Theorem 0.1 uses the Eilenberg–Moore theorem to show the cohomology of the homotopy pullback of the diagram $BK \to BG \leftarrow BH$ is $\operatorname{Tor}_{C^*(BG)}(C^*(BK), C^*(BH))$ and constructs maps between $H^*(B\Gamma)$ and $C^*(B\Gamma)$ for $\Gamma \in \{G, K, H\}$ which then induce an isomorphism with

[1] And additionally, in characteristic 2, the cup-1 squares of polynomial generators of $H^*(X)$ and $H^*(E)$ vanish.

$\operatorname{Tor}_{H^*(BG)}(H^*(BK), H^*(BH))$. This much it has in common with many collapse results, but our result also shows that this map takes the classical product on the Tor of cochain algebras to that on the Tor of cohomology rings. The way this multiplicativity is established closely follows Franz's proof, but among the technical underpinnings necessary to extend his approach from a fibration to a general pullback is one substantial innovation.

A Tor of DGAs in full generality is not endowed with a product, and the product that is defined on $\operatorname{Tor}_{C^*(BG)}(C^*(BK), C^*(BH))$ is synthetic, arising from the homological external product rather than a multiplicative structure on the resolution itself. As a consequence, in previous collapse results, one could not say whether the isomorphisms shown were multiplicative. But there is a cochain complex $\mathbf{B}(A', A, A'')$, the two-sided bar construction, functorial in spans $A' \leftarrow A \rightarrow A''$ of DGA maps, whose cohomology is $\operatorname{Tor}_A(A', A'')$ under mild conditions. In Theorem A.1, assuming the maps in the span are actually maps of so-called *homotopy Gerstenhaber algebras*, a type of DGA with extra structure (of which cochain algebras are the main examples), we are able to define a product on $\mathbf{B}(A', A, A'')$ inducing a product on $\operatorname{Tor}_A(A', A'')$ which specializes to the known products when A', A, and A'' are cochain algebras or cohomology rings. This product may be the point of greatest interest in this paper; it is certainly the most difficult. Then the maps between $H^*(B\Gamma)$ and $C^*(B\Gamma)$ alluded to in the previous paragraph, chosen with sufficient care and assuming 2 is a unit of the coefficient ring k, will preserve this novel product in the transition from $\mathbf{B}(H^*(BK), H^*(BG), H^*(BH))$ to $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$.

Outline The layout of the paper is as follows:

(1) In Section 1 we recall algebraic conventions and the two-sided bar construction.

(2) In Section 2 we discuss extended homotopy Gerstenhaber algebras and recall results showing normalized cochain algebras and cohomology rings are examples.

(3) In Section 3 we recall the quasi-isomorphisms we need: A_∞-algebra maps λ from $H^*(B\Gamma)$ to $C^*(B\Gamma)$ for $\Gamma \in \{K, G, H\}$ and DGA maps $f\colon C^*(BT) \to H^*(BT)$ for $T \in \{T_K, T_H\}$ maximal tori in K and H. These f will be defined so as to annihilate the error terms distinguishing λ from a genuine DGA map, so that the composites $H^*(BK) \to C^*(BK) \to C^*(BT_K) \to H^*(BT_K)$ and $H^*(BH) \to H^*(BT_H)$ are just the functorially induced $H^*(B(T_K \hookrightarrow K))$ and $H^*(B(T_H \hookrightarrow H))$. The DGA quasi-isomorphisms f do not necessarily exist if T is replaced by a more general Lie group, and to construct them we need a simplicial model for BT, which necessitates the replacements discussed in Section 4.

It is an important technical point in showing the three A_∞-maps λ from $H^*(B\Gamma)$ to $C^*(B\Gamma)$ induce a map of Tors that they are essentially functorial up to homotopy. This fact, and our control over the error term annihilated by f, come from the existence of a certain auxiliary structure on extended homotopy Gerstenhaber algebras, a so-called *strongly homotopy commutative* algebra structure Φ whose existence was proven by Franz. The explicit formula for Φ plays a key role in the appendix. The fact that Φ, so defined, is a structure of the type sought follows from an extraordinarily involved cochain-level computation [10].

(4) In Section 4 we apply the enhanced version of the Eilenberg–Moore theorem proven in the appendix to the span $BK \to BG \leftarrow BH$ whose homotopy pullback $K\backslash G/H$ we are interested in, showing that $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$ carries a weak multiplicative structure inducing the cup product on $H^*(K\backslash G/H)$. We then show this product is preserved when we replace $BK \to BG \leftarrow BH$ with a simplicial model making available the DGA formality maps discussed in the previous paragraph.

(5) In Section 5 we weave these threads together to prove Theorem 0.1. We first construct a quasi-isomorphism

$$\Theta\colon \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \to \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)),$$

roughly to be thought of as $\mathbf{B}(\lambda_K, \lambda_G, \lambda_H)$, which is not necessarily multiplicative. Letting $T_K \le K$ and $T_H \le H$ be maximal tori, we then map the codomain into $\mathbf{B}(C^*(BT_K), C^*(BG), C^*(BT_H))$, using the maps $\rho\colon C^*(B\Gamma) \to C^*(BT_\Gamma)$ for $\Gamma \in \{K, H\}$ to define a strictly multiplicative map

$$\Psi = \mathbf{B}(f\rho, \mathrm{id}, f\rho)\colon \mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \to \mathbf{B}(H^*(BT_K), C^*(BG), H^*(BT_H)),$$

inducing an injection in cohomology. Because we have chosen f and λ to compose nicely, the composite $\Psi\Theta$ is multiplicative up to homotopy, and because the multiplicative map $H^*(\Psi)$ is injective, this shows $H^*(\Theta)$ is multiplicative as well, concluding the proof.

(6) Finally in the appendix, the technical core of the work, we construct a product on the two-sided bar construction and use it to prove an Eilenberg–Moore theorem (Theorem A.27) in which the product on the cohomology of the pullback of $X \to B \leftarrow E$ arises from the nonassociative product on $\mathbf{B}(C^*(X), C^*(B), C^*(E))$.

Remark 0.2 (added in proof) Since the submission of the present article, several new results have been obtained. Franz [12] has shown that the product on the two-sided bar construction mentioned above can indeed be extended to an A_∞-structure; this had been anticipated in Remark A.26. Moreover, when the domain is equipped with this A_∞-structure, the quasi-isomorphism ξ of Theorem A.27 is enhanced to a map of A_∞-algebras. Munkholm [26, Section 9] had already introduced a product on $\mathrm{Tor}_A(A', A'')$, for DGA maps $A \leftarrow A' \to A''$ meeting certain hypotheses. Carlson [8] has resurrected and studied the properties of this product and determined [7] that under mild flatness hypotheses, the product introduced in the present article induces Munkholm's product in cohomology.

Acknowledgments JDC would like to thank Omar Antolín Camarena for helpful discussions of and references on matters simplicial, Vincent Gélinas for fielding questions of varying lucidity regarding A_∞-algebras, and the organizers of the Fields Institute Spring 2020 Thematic Program on Toric Topology and Polyhedral Products for financial and moral support. MF was partially supported by an NSERC Discovery Grant.

1 Algebras, twisting cochains, and bar constructions

In this background section we establish notational conventions and recount some foundational lemmas. Nothing is original here, save possibly some of the lemmas on two-sided twisted tensor products and bar constructions, which if not published are still likely known.

Fix forever a commutative base ring k with unity over which we will consider cochain complexes, and with respect to which all tensor products and Hom-modules will be taken. We will take as understood the Koszul sign convention and the notions of derivation, coderivation, differential graded k-algebra (henceforth DGA), and differential graded k-coalgebra (DGC). A commutative DGA is a CDGA. All *algebras* we consider *are nonnegatively graded*, *associative*, *and connected* unless otherwise noted, and all *coalgebras nonnegatively graded*, *coassociative*, *and cocomplete*. All DGA *maps preserve unit and augmentation*, and all DGC *maps preserve counit and coaugmentation*. All *ideals* of DGAs *will be two-sided differential* ideals.

We also assume understood suspensions, desuspensions, tensor products, and Hom-modules of cochain complexes, homomorphisms of DGAs and DGCs, and the classical (reduced) bar construction of a DGA. We nevertheless need to establish notation:

Notation 1.1 (1) Given a cochain complex $B = \bigoplus B_n$, its differential is canonically written d_B. For the tensor differential on a tensor product $\bigotimes B^{(i)}$ of complexes $B^{(i)}$ we write $d_\otimes$. We write $|x|$ for the degree n of a homogeneous element $x \in B_n$. The *desuspension* $s^{-1}B = \{s^{-1}b : b \in B\}$ is the (re)graded k-module given by $(s^{-1}B)_n := B_{n+1}$, equipped with the differential $d_{s^{-1}B} \colon s^{-1}b \mapsto -s^{-1}db$. The desuspension map $s^{-1} \colon b \mapsto s^{-1}b$ is a degree-(-1) isomorphism of cochain complexes with inverse $s \colon s^{-1}B \xrightarrow{\sim} B$, the *suspension*.

(2) A DGA A is a list comprising, besides the underlying cochain complex (A, d_A), a canonically named multiplication $\mu_A \colon A \otimes A \to A$, unit $\eta_A \colon k \to A$, and augmentation $\epsilon_A \colon A \to k$, the clarifying decorations suppressed when practicable. The augmentation ideal is $\bar{A}$ and we write $\mu^{[n]} \colon A^{\otimes n} \to A$ for iterated multiplication.

(3) A DGC comprises a cochain complex (C, d_c), comultiplication $\Delta_C \colon C \to C \otimes C$, counit $\epsilon_C \colon C \to k$, and coaugmentation $\eta_C \colon k \to C$. *Cocompleteness* of C means that every element is annihilated by some iterate of the reduced comultiplication $c \mapsto \Delta c - 1 \otimes c - c \otimes 1$.

(4) We use $\bullet$ to abbreviate indices or exponents representing an indefinite number of tensor factors to be determined from context; for instance, if A is a graded algebra, $a_\bullet$ denotes a pure tensor $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$, where n is to be gleaned contextually. A repeated $\bullet$ implies summation unless explicitly stated otherwise, so that for example $d_\otimes = \mathrm{id}^\bullet \otimes d_A \otimes \mathrm{id}^\bullet \colon A^{\otimes n} \to A^{\otimes n}$ represents the sum of the n maps $\mathrm{id}^{\otimes \ell} \otimes d_A \otimes \mathrm{id}^{\otimes n-\ell-1}$ applying d_A to one tensor factor. Given a pure tensor $a_\bullet \in A^{\otimes n}$, an expression involving the string $a_\bullet$ multiple times represents a sum over order-preserving subdivisions of $a_\bullet$ into tensor factors.

(5) Our notation for tensor-factor permutations is typically cyclic. For example, $(12)\colon A\otimes B\to B\otimes A$ denotes the k-linear map given by $a\otimes b\mapsto(-1)^{|a||b|}b\otimes a$. We may at times also abusively use arguments to indicate permutations; for example we may instead of (123) write $\tau_{a\otimes b;c}\colon A\otimes B\otimes C\to C\otimes A\otimes B$ for the map taking $a\otimes b\otimes c$ to $(-1)^{|c||a|+|c||b|}c\otimes a\otimes b$. Amongst all permutations, *shuffles* will be of particular interest for us: a (p,q)-*shuffle* is a permutation π of $\{1,\dots,p+q\}$ such that $\pi(i)<\pi(j)$ whenever $i<j$ both lie in $\{1,\dots,p\}$ or both lie in $\{p+1,\dots,p+q\}$. Thus shuffles interleave two blocks, leaving the order within each block unaffected. To avoid writing Koszul signs, we generally manipulate such maps directly, without evaluating on elements.

Definition 1.2 The graded module underlying the *bar construction* $\mathbf{B}A$ of a DGA A is the direct sum of $\mathbf{B}_nA:=(s^{-1}\bar{A})^{\otimes n}$ for $n\geq 0$. The projection $\mathbf{B}A\to\mathbf{B}_nA$ is denoted by pr_n. Pure tensors are written as bar-words $[a_1|\cdots|a_n]\in\mathbf{B}_nA$, or $[a_\bullet]$ when the *length* $\ell([a_1|\cdots|a_n])=n$ is immaterial. The empty word $[]\in\mathbf{B}_0A$ is the image of $1\in k$ under the coaugmentation; the counit can be identified with pr_0. The comultiplication is the *deconcatenation* $\Delta_{\mathbf{B}A}\colon[a_\bullet]\mapsto\sum_{p=0}^{\ell(a_\bullet)}[a_1|\cdots|a_p]\otimes[a_{p+1}|\cdots|a_{\ell(a_\bullet)}]$. We will employ Einstein–Sweedler notation for the values of iterated deconcatenations: $\Delta_{\mathbf{B}A}^{[n]}[a_\bullet]=:[a_{(1)}]\otimes\cdots\otimes[a_{(n)}]$, implicitly summing over partitions of $[a_\bullet]$ into n bar-subwords.

The sum of the tensor differentials on the $\mathbf{B}_nA=(s^{-1}\bar{A})^{\otimes n}$ is the *internal differential* d_{int}; the *external differential* $d_{\mathrm{ext}}\colon\mathbf{B}_nA\to\mathbf{B}_{n-1}A$ is the bar-deletion operation $\mathrm{id}^\bullet\otimes s^{-1}\mu_As^{\otimes 2}\otimes\mathrm{id}^\bullet$ for $n\geq 2$ and is 0 on $\mathbf{B}_{\leq 1}A$. The total differential $d_{\mathbf{B}A}:=d_{\mathrm{int}}+d_{\mathrm{ext}}$ makes $\mathbf{B}A$ a DGC. This DGC structure is functorial in DGA maps $f\colon A\to B$, with the DGC map $\mathbf{B}f\colon\mathbf{B}A\to\mathbf{B}B$ given by $(s^{-1}fs)^{\otimes n}\colon[a_\bullet]\mapsto[fa_1|\cdots|fa_n]=:[fa_\bullet]$ on $\mathbf{B}_nA$.

Neglecting the DGC structure, there is a structurally irrelevant but sometimes notationally convenient *concatenation* map $\Delta_{\leftarrow}\colon[a_1|\cdots|a_p]\otimes[b_1|\cdots|b_q]\mapsto[a_1|\cdots|a_p|b_1|\cdots|b_q]$ which comes from viewing $\mathbf{B}A$ as the tensor algebra on $s^{-1}\bar{A}$.

Definition 1.3 Given a DGC C and DGA A, we write $\mathrm{Hom}_n(C,A)$ for the k-module of k-linear maps f sending each C_j to A_{j+n}, and set the degree $|f|$ to n for such a map. We write $\mathrm{Hom}(C,A)$ for the graded module $\bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}_n(C,A)$. This module becomes a cochain complex under the differential $D=d_{\mathrm{Hom}(C,A)}$ given by $Df:=d_Af-(-1)^{|f|}fd_C$ and a DGA with respect to the *cup product* $f\smile g:=\mu_A(f\otimes g)\Delta_C$, unity $\eta_A\epsilon_C\colon C\to k\to A$, and augmentation $f\mapsto\eta_Af\epsilon_C$. If $h\in\mathrm{Hom}_0(C,A)$ satisfies $h\eta_C=\eta_A$, then a cup inverse $h^{\smile-1}:=\sum_{\ell=0}^{\infty}(\eta_A\epsilon_C-h)^{\smile\ell}$ is well defined by cocompleteness of C. An element $t\in\mathrm{Hom}_1(C,A)$ satisfying the three conditions

$$\epsilon_At=0=t\eta_C,\quad Dt=t\smile t$$

is called a *twisting cochain*.

Example 1.4 For each connected DGA A, the composition

$$t_A\colon\mathbf{B}A\xrightarrow{\mathrm{pr}_1}s^{-1}\bar{A}\xrightarrow{s}\bar{A}\hookrightarrow A$$

of degree 1 is a twisting cochain called the *tautological twisting cochain*. The assignment $t_{(-)}$ is a natural transformation between the functors assigning a DGA A the underlying graded modules of $\mathbf{B}A$ and of A.

Any twisting cochain $C \to A$ extends uniquely to a DGC map $C \to \mathbf{B}A$, and inversely, given a DGC map $F\colon C \to \mathbf{B}A$, postcomposing $t_A\colon \mathbf{B}A \to A$ gives a twisting cochain $t_A F\colon C \to A$, thus prescribing a bijection between twisting cochains $C \to A$ and DGC maps $C \to \mathbf{B}A$.

Example 1.5 Given a DGA A, one can check that the map

$$t_\nabla = \mu_A(t_A \otimes \epsilon_A + \epsilon_A \otimes t_A)\colon \mathbf{B}A \otimes \mathbf{B}A \to A,$$

which takes $[a] \otimes 1$ and $1 \otimes [a]$ to a and annihilates all other pure tensors, is a twisting cochain if and only if A is a CDGA. In this case, the uniquely induced DGC map

$$\mu_\nabla\colon \mathbf{B}A \otimes \mathbf{B}A \to \mathbf{B}A,$$

called the *shuffle product*, is a product making $\mathbf{B}A$ a DG Hopf algebra. The shuffle product takes $[a_\bullet] \otimes [b_\bullet] \in \mathbf{B}_p A \otimes \mathbf{B}_q A$ to the sum of all (p,q)-shuffles (with Koszul sign) of $[a_\bullet|b_\bullet]$.

More generally, let B be another DGA. Then the *shuffle map*

$$\nabla\colon \mathbf{B}A \otimes \mathbf{B}B \to \mathbf{B}(A \otimes B)$$

is the direct sum of the maps $\mathbf{B}_p A \otimes \mathbf{B}_q B \to \mathbf{B}_{p+q}(A \otimes B)$ sending $[a_\bullet] \otimes [b_\bullet]$ to the sum of all tensor (p,q)-shuffles of $[a_1 \otimes 1|\cdots|a_p \otimes 1|1 \otimes b_1|\cdots|1 \otimes b_q]$. This is a DGC map, and if $A = B$ is a CDGA, then the composition $\mathbf{B}\mu \circ \nabla$ is the product μ_∇ of the previous paragraph.

The bijection between DGC maps and twisting cochains preserves suitable homotopy notions.

Definition 1.6 A DGC *homotopy* from one DGC map $F\colon C \to B$ to another such map G is a degree-(-1) map $H\colon C \to B$ such that

$$\epsilon_B H = 0, \quad H\eta_C = 0, \quad DH = G - F, \quad \text{and} \quad \Delta_B H = F \otimes H + H \otimes G.$$

We write $H\colon F \simeq G$ in this situation. Taking $B = \mathbf{B}A$ for an augmented DGA A, we may translate to an appropriate notion of homotopy for twisting cochains. Given two twisting cochains $t, u \in \mathrm{Hom}_1(C, A)$, a *twisting cochain homotopy* from the former to the latter is a map $h \in \mathrm{Hom}_0(C, A)$ satisfying the three conditions

$$\epsilon_C h = \epsilon_A, \quad h\eta_A = \eta_C, \quad \text{and} \quad Dh = t \smile h - h \smile u.$$

We again write $h\colon t \simeq u$.

Proposition 1.7 *Given a* DGC *homotopy $H\colon C \to \mathbf{B}A$, the map $h = \eta_A\epsilon_C + t_A H\colon C \to A$ is a twisting cochain homotopy, and the assignment $H \mapsto h$ is a bijection from the set of* DGC *homotopies $F \simeq G$ to the set of twisting cochain homotopies $t_A F \simeq t_A G$.*

Aside from providing a tractable encoding of DGC maps $C \to \mathbf{B}A$ and homotopies therebetween, twisting cochains $C \to A$ can also be harnessed to produce new differentials on $C \otimes A$ which we will use to define the two-sided bar construction.

Definition 1.8 Let C be a DGC, A' and A'' DGAs, M a differential right C-comodule, and N a differential left A''-module. One defines the *cap product* with an element $\phi \in \operatorname{Hom}(C, A'')$ by

$$\delta^{\mathrm{R}}_{\phi} := (\mathrm{id}_M \otimes \mu_N)(\mathrm{id}_M \otimes \phi \otimes \mathrm{id}_N)(\Delta_M \otimes \mathrm{id}_N)\colon M \otimes N \to M \otimes N, \quad x \otimes y \mapsto \pm x_{(1)} \otimes \phi(x_{(2)})y.$$

It is standard that if $\phi = t''$ is a twisting cochain, then $d_{\otimes} - \delta^{\mathrm{R}}_{t''}$ is a differential on $M \otimes N$, making it a cochain complex we denote by $M \otimes_{t''} N$ and call a *twisted tensor product*. When $M = C$ and $N = A''$, this prescription makes $C \otimes_{t''} A''$ a differential left C-module and a differential right A''-module.

A symmetric construction produces a cap product endomorphism $\delta^{\mathrm{L}}_{\phi}$ on $P \otimes Q$ from a graded linear $\phi\colon C \to A'$, a differential *left* C-comodule Q, and a differential *right* A'-module P, and if $\phi = t'$ is a twisting cochain then $\delta^{\mathrm{L}}_{t'} + d_{\otimes}$ is a differential on $P \otimes Q$; we again write $P \otimes_{t'} Q$ for the twisted tensor product. Applying this construction to $P = A'$ and $Q = C \otimes_{t''} A''$, we obtain a *two-sided twisted tensor product* [19, Remarks II.5.4]:

$$A' \otimes_{t'} C \otimes_{t''} A'' := A' \otimes_{t'} (C \otimes_{t''} A'').$$

This is an (A', A'')-bimodule, and given ideals $\mathfrak{a}' \trianglelefteq A'$ and $\mathfrak{a}'' \trianglelefteq A''$, we write $(\mathfrak{a}', \mathfrak{a}'')$ for the (A', A'')-subbimodule $\mathfrak{a}' \otimes C \otimes A'' + A' \otimes C \otimes \mathfrak{a}''$.

We will need some lemmas describing when (homotopy-)commutative squares of maps

$$\begin{array}{ccccc} A'_0 & \xleftarrow{\;t'_0\;} & C_0 & \xrightarrow{\;t''_0\;} & A''_0 \\ \downarrow & & \downarrow & & \downarrow \\ A'_1 & \xleftarrow[\;t'_1\;]{} & C_1 & \xrightarrow[\;t''_1\;]{} & A''_1 \end{array} \tag{1.9}$$

induce a map $A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \to A'_1 \otimes_{t'_1} C_1 \otimes_{t''_1} A''_1$ of twisted tensor products. These properties seem to be well known in the one-sided case (see e.g. Huebschmann [18, page 360] and Franz [11, Section 7]), so we can safely suppress the proofs of their easily guessed two-sided generalizations.

Lemma 1.10 *Let $G\colon C_0 \to C_1$ be a* DGC *map, $f'\colon A'_0 \to A'_1$ and $f''\colon A''_0 \to A''_1$* DGA *maps, and $t'_j\colon C_j \to A'_j$ and $t''_j\colon C_j \to A''_j$ twisting cochains for $j \in \{0, 1\}$. Then $f' \otimes G \otimes f''$ is a cochain map*

$$A'_0 \otimes_{t'_0} C_0 \otimes_{t''_0} A''_0 \to A'_1 \otimes_{t'_1} C_1 \otimes_{t''_1} A''_1$$

if $t'_1 G = f' t'_0$ and $t''_1 G = f'' t''_0$. If G is one such DGC *map and $\widetilde{G}$ another, and $H\colon C_0 \to C_1$ is a* DGC *homotopy $G \simeq \widetilde{G}$ such that $t'_1 H$ and $t''_1 H$ are 0, then $f' \otimes H \otimes f''$ is a cochain homotopy $f' \otimes G \otimes f'' \simeq f' \otimes \widetilde{G} \otimes f''$.*

Corollary 1.11 *Let C_0 and C_1 be* DGCs, *A' and A''* DGAs, *$t'\colon C_1 \to A'$ and $t''\colon C_1 \to A''$ twisting cochains, and $G\colon C_0 \to C_1$ a* DGC *homomorphism. Then $t' \circ G\colon C_0 \to A'$ and $t'' \circ G\colon C_0 \to A''$ are twisting cochains and*

$$\mathrm{id}_{A'} \otimes G \otimes \mathrm{id}_{A''}\colon A' \otimes_{t'G} C_0 \otimes_{t''G} A'' \to A' \otimes_{t'} C_1 \otimes_{t''} A''$$

a cochain map.

Lemma 1.12 *Let C be a* DGC, *A' and A''* DGAs, *$t'_j\colon C \to A'$ and $t''_j\colon C \to A''$ twisting cochains for $j \in \{0,1\}$, and $\mathfrak{a}' \trianglelefteq A'$ and $\mathfrak{a}'' \trianglelefteq A''$ ideals, and suppose there exist homotopies $h'\colon t'_0 \simeq t'_1$ and $h''\colon t''_1 \simeq t''_0$ with $h'\overline{C} \leq \mathfrak{a}'$ and $h''\overline{C} \leq \mathfrak{a}''$.*[2] *Then the composition*

$$(\delta^{\mathrm{L}}_{h'} \otimes \mathrm{id}_{A''})(\mathrm{id}_{A'} \otimes \delta^{\mathrm{R}}_{h''})\colon A' \otimes C \otimes A'' \to A' \otimes C \otimes A''$$

is a cochain isomorphism $A' \otimes_{t'_0} C \otimes_{t''_0} A'' \xrightarrow{\sim} A' \otimes_{t'_1} C \otimes_{t''_1} A''$ congruent to the identity modulo $(\mathfrak{a}', \mathfrak{a}'')$.

We now define the twisted tensor products of greatest interest to us:

Definition 1.13 Let A, A', and A'' be augmented DGAs and $F'\colon \mathbf{B}A \to \mathbf{B}A'$ and $F''\colon \mathbf{B}A \to \mathbf{B}A''$ DGC maps.[3] Associated to F' and F'' are twisting cochains $t' = t_{A'}F'$ and $t'' = t_{A''}F''$, and we define the *two-sided bar construction* as the two-sided twisted tensor product

$$\mathbf{B}(A', A, A'') := A' \otimes_{t_{A'}F'} \mathbf{B}A \otimes_{t_{A''}F''} A''.$$

We will write a pure tensor $a' \otimes [a_\bullet] \otimes a'' \in \mathbf{B}(A', A, A'')$ as $a'[a_\bullet]a''$. The differential of $\mathbf{B}(A', A, A'')$ preserves the length filtration $\mathbf{B}_{\leq \ell}(A', A, A'') := A' \otimes \mathbf{B}_{\leq \ell} A \otimes A''$.

Example 1.14 This two-sided bar construction, when we take $F' = \mathbf{B}f'$ and $F'' = \mathbf{B}f''$ for DGA maps $f'\colon A \to A'$ and $f''\colon A \to A''$, coincides with the familiar classical two-sided bar construction of differential graded algebras. In this case, the twisting cochains t' and t'' defining the twisted tensor product $\mathbf{B}(A', A, A'') = A' \otimes_{t'} \mathbf{B}A \otimes_{t''} A''$ are simply $t_{A'}\mathbf{B}f' = f't_A$ and $f''t_A$, respectively.

We see already from the preceding lemmas that DGA maps from A' and A'' and DGC maps from $\mathbf{B}A$ induce maps of two-sided bar constructions, and want to extend this technique to induce maps from commutative diagrams of DGC maps of the form

(1.15)
$$\begin{array}{ccccc} \mathbf{B}A'_0 & \xleftarrow{F'_0} & \mathbf{B}A_0 & \xrightarrow{F''_0} & \mathbf{B}A''_0 \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G''} \\ \mathbf{B}A'_1 & \xleftarrow[F'_1]{} & \mathbf{B}A_1 & \xrightarrow[F''_1]{} & \mathbf{B}A''_1 \end{array}$$

[2] Note this second homotopy goes the opposite direction to the one one might expect.

[3] These are A_∞*-algebra maps* from A to A' and A'' in the language to be introduced in Section 3.

We write $G'_{(1)}\colon \overline{A'_0} \to \overline{A'_1}$ for the composite $t_{A'_1} \circ G' \circ s^{-1}_{A'_0}$ and similarly for $G''_{(1)}$; later, in Definition 3.1, we will see $\eta\epsilon + G'_{(1)}\colon A'_0 \to A'_1$ and $\eta\epsilon + G''_{(1)}\colon A''_0 \to A''_1$ can be understood as a sort of approximate DGA map.

Proposition 1.16 (Wolf[4] [34, Theorem 7]; cf Gugenheim and Munkholm [16, Theorem 3.5$_*$]) *Suppose given a strictly commuting diagram of* DGC *maps as in (1.15). Then, in the notation of Definition 1.2, we may define a cochain map* $\mathbf{B}(A'_0, A_0, A''_0) \to \mathbf{B}(A'_1, A_1, A''_1)$ *by*

$$\mathbf{B}(G', G, G'') = (\Upsilon' \otimes G \otimes \Upsilon'')(\mathrm{id}_{A'_0} \otimes \Delta^{[3]}_{\mathbf{B}A_0} \otimes \mathrm{id}_{A''_0}), \tag{1.17}$$

where

$$\begin{aligned}
\Upsilon'|_{k\otimes \mathbf{B}A_0} &= \mathrm{id}_k \otimes \epsilon_{\mathbf{B}A_0}\colon && c'[a_{(1)}] \mapsto c'\epsilon[a_{(1)}],\\
\Upsilon'|_{\overline{A'_0}\otimes \mathbf{B}A_0} &= t_{A'_1} G' \Delta_{\leftarrow}(s^{-1}_{A'_0} \otimes F'_0)\colon && a'[a_{(1)}] \mapsto \pm t_{A'_1} G'\big([a'] \otimes F'_0[a_{(1)}]\big),\\
\Upsilon''|_{\mathbf{B}A_0\otimes k} &= \epsilon_{\mathbf{B}A_0} \otimes \mathrm{id}_k\colon && [a_{(3)}]c'' \mapsto \epsilon[a_{(3)}]c'',\\
\Upsilon''|_{\mathbf{B}A_0\otimes \overline{A''_0}} &= t_{A''_1} G'' \Delta_{\leftarrow}(F''_0 \otimes s^{-1}_{A''_0})\colon && [a_{(3)}]a'' \mapsto \pm t_{A''_1} G''\big(F''_0[a_{(3)}] \otimes [a'']\big).
\end{aligned}$$

If $t_{A'_1}G'$ *takes* $\mathbf{B}_{\geq 2}A'_0$ *into an ideal* $\mathfrak{a}' \trianglelefteq A'_1$ *and* $t_{A''_1}G''$ *takes* $\mathbf{B}_{\geq 2}A''_0$ *into* $\mathfrak{a}'' \trianglelefteq A''_1$, *then*

$$\mathbf{B}(G', g, G'') \equiv (\eta\epsilon + G'_{(1)}) \otimes G \otimes (G''_{(1)} + \eta\epsilon) \pmod{(\mathfrak{a}', \mathfrak{a}'')}.$$

Notation 1.18 In the event that the maps G, G', and G'' in (1.15), are respectively $\mathbf{B}g$, $\mathbf{B}g'$, and $\mathbf{B}g''$ for $g\colon A_0 \to A_1$, $g'\colon A'_0 \to A'_1$, and $g''\colon A''_0 \to A''_1$, we abuse notation by writing the map of Proposition 1.16 as $\mathbf{B}(g', g, g'') := \mathbf{B}(G', G, G'')$. Such a map is also a special case of that of Lemma 1.10.

We are interested in two-sided bar constructions because they provide functorial resolutions:

Proposition 1.19 (see Barthel, May, and Riehl [1, after Proposition 10.19]) *Suppose k is a principal ideal domain and $A \to A'$ a map of* DGAs *flat over k. Then given another* DGA *map $A \to A''$, the cohomology of the classical two-sided bar construction $\mathbf{B}(A', A, A'') = \mathbf{B}(A', A, A) \otimes_A A''$ of Example 1.14 is $\mathrm{Tor}_A(A', A'')$. If $H^*(A)$ and $H^*(A'')$ are flat over k, then the E_2 page of the associated algebraic Eilenberg–Moore spectral sequence is $\mathrm{Tor}_{H^*(A)}(H^*(A'), H^*(A''))$.*

Discussion 1.20 For the proof of Theorem A.13, we will need a more explicit expression for the differential on the classical two-sided bar construction $\mathbf{B}(A', A, A'')$ of Example 1.14. Note that the total differential is the sum of the tensor differential $d_\otimes$ on $A' \otimes \mathbf{B}A \otimes A''$ and two cap products. Separating the summand $\mathrm{id}_{A'} \otimes d_{\mathbf{B}A} \otimes \mathrm{id}_{A''}$ into its internal and external components, $d_{\mathbf{B}(A',A,A'')}$ decomposes as

[4] Wolf [34] cites his unpublished dissertation [33] for a proof of this result; however, this work was not publicly accessible at time of writing. (It is now available at the author's webpage.) For our proof, which turns out to be nontrivial, see the first version of this paper (arXiv 2106.02986v1). Proofs of the lemmas and many of the standard claims in this section can be found there as well.

the sum of tensor differentials and an "external" differential which is the sum of the two cap products and the bar-deletion differential. Explicitly, appealing to Definition 1.8, this external differential $d^{\mathrm{ext}}_{\mathbf{B}(A',A,A'')}$ is given on the summand $\mathbf{B}_\ell(A',A,A'')$ by

$$\begin{array}{rcccc} & \mu_{A'}(\mathrm{id}_{A'}\otimes f's) & \otimes\, \mathrm{id}^{\otimes \ell-1}_{s^{-1}\bar{A}} & \otimes & \mathrm{id}_{A''} \\ \textbf{(1.21)} & +\,\mathrm{id}_{A'} & \otimes \quad d_{\mathrm{ext}} & \otimes & \mathrm{id}_{A''} \\ & -\,\mathrm{id}_{A'} & \otimes\, \mathrm{id}^{\otimes \ell-1}_{s^{-1}\bar{A}} & \otimes & \mu_{A''}(f''s\otimes \mathrm{id}_{A''}). \end{array}$$

The minus sign before the third term of (1.21) is important and not consistently noted in the literature.

2 Homotopy Gerstenhaber algebras

We ultimately want to describe the multiplicative structure on $\mathrm{Tor}_{C^*(B)}(C^*(X),C^*(E))$ in terms of a product on the two-sided bar construction. The special property of a normalized singular cochain algebra A enabling us to do so will turn out to be the existence of a DG Hopf algebra structure on $\mathbf{B}A$, or in other words, a DGC map $\mathbf{B}A\otimes\mathbf{B}A\to\mathbf{B}A$ making it a DGA.[5]

Definition 2.1 Let A be a DGA such that $\mathbf{B}A$ admits a multiplication making it a DG Hopf algebra. If the twisting cochain $\mathbf{E}:=t_A\mu_{\mathbf{B}A}\colon \mathbf{B}A\otimes\mathbf{B}A\to A$ satisfies $\mathbf{E}_{j,\ell}:=\mathbf{E}|_{\mathbf{B}_jA\otimes\mathbf{B}_\ell A}=0$ for $j\geq 2$, we call A, equipped with $\mu_{\mathbf{B}A}$ (equivalently, with $\mathbf{E}$), a *homotopy Gerstenhaber algebra* (HGA).

Notation 2.2 We will on occasion write the values of the product $\mu_{\mathbf{B}A}$ in infix notation as $[a_\bullet]*[b_\bullet]$. It will be useful later to translate $\mathbf{E}_{1,\bullet}$ into a degree-zero map on $A\otimes\mathbf{B}A$ by taking

$$\mathfrak{E}:=\mathbf{E}_{1,\bullet}(s^{-1}\otimes\mathrm{id}_{\mathbf{B}A})\colon A\otimes\mathbf{B}A\to A,\qquad a[b_\bullet]\mapsto\mathbf{E}\big([a]\otimes[b_\bullet]\big).$$

We will also have occasion to use the operations $E_\ell=\mathbf{E}_{1,\ell}\circ(s^{-1})^{\otimes 1+\ell}\colon A\otimes A^{\otimes\ell}\to A$, whose values we write as $E_\ell(a;b_\bullet)=E_\ell(a\otimes b_1\otimes\cdots\otimes b_\ell)$. An HGA structure on A is more commonly phrased as a list of conditions [11, (6.2)–(6.4)] on the E_ℓ.

Remark 2.3 Unitality of $\mu_{\mathbf{B}A}$ implies $\mathbf{E}_{0,1}$ and $\mathbf{E}_{1,0}$ must both be $s\colon s^{-1}\bar{A}\xrightarrow{\sim}\bar{A}$ and $\mathbf{E}_{j,0}$ and $\mathbf{E}_{0,\ell}$ must be 0 for $j,\ell\geq 2$. The "$t\eta=0$" clause in the definition of a twisting cochain in Definition 1.3 implies $\mathbf{E}_{0,0}=\mathbf{E}\eta_{\mathbf{B}A}=0$. For notational convenience, we will extend $\mathbf{E}_{0,1}$ and $\mathbf{E}_{1,0}$ to both be $s\colon s^{-1}A\to A$ (thus respectively sending $[1]\otimes[]$ and $[]\otimes[1]$ to 1) and extend the operations $\mathbf{E}_{1,\ell}$ to $(s^{-1}A)^{\otimes 1+\ell}$ for $\ell\geq 1$ by setting them to 0 on pure tensors any of whose factors lies in $s^{-1}\operatorname{im}\eta_A$. That is, we will sometimes consider bar-words containing a letter c in the coefficient ring k, but such words are to be annihilated by $\mathbf{E}_{j,\ell}$ unless $(j,\ell)=(1,0)$ or $(0,1)$.

[5] Under our blanket assumption that A is graded and connected, a bialgebra structure on $\mathbf{B}A$ admits a unique antipode defined by a formula analogous to the one defining the cup inverse in Definition 1.3 [14, Propositions 1.4.14 and 1.4.24].

Example 2.4 If A is a commutative DGA, then the shuffle product μ_∇ of Example 1.5 makes the bar construction $\mathbf{B}A$ a DG Hopf algebra, so that A becomes an HGA. The corresponding twisting cochain $t_\nabla = \mathbf{E}$ satisfies $\mathbf{E} = \mathbf{E}_{0,1} + \mathbf{E}_{1,0}$, so the operations E_ℓ vanish for $\ell \geq 1$.

The Maurer–Cartan identity $D\mathbf{E} = \mathbf{E} \smile \mathbf{E}$ for the twisting cochain $\mathbf{E}\colon \mathbf{B}A \otimes \mathbf{B}A \to A$ defining an HGA structure on A yields, upon mild rearrangement, the useful formulae

$$\text{(2.5)}\quad \mathfrak{E}(\mu_A \otimes \mathrm{id}_{\mathbf{B}A}) = \sum_\pi \mu_A(\mathfrak{E} \otimes \mathfrak{E})\pi : \bar{A} \otimes \bar{A} \otimes \mathbf{B}A \to A, \quad \mathfrak{E}\big(a_1 a_2[b_\bullet]\big) = \sum \pm \mathfrak{E}\big(a_1[b_\bullet]\big)\mathfrak{E}\big(a_2[b_\bullet]\big),$$

where the sum is over shuffles $\pi : [a_1|a_2] \otimes [b_\bullet] \mapsto \pm[a_1|b_\bullet] \otimes [a_2|b_\bullet]$, and

$$\text{(2.6)}\quad \begin{aligned} D_\otimes \mathfrak{E} &= d_A \mathfrak{E} - \mathfrak{E} d_\otimes \mu_A(s \otimes \mathfrak{E})(12) + \mathfrak{E}(\mathrm{id}_A \otimes d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)}) - \mu_A(\mathfrak{E} \otimes s), \\ (D_\otimes \mathfrak{E})\big(a[b_\bullet]\big) &= \pm b_1 \mathfrak{E}\big(a[b_2|b_\bullet|b_\ell]\big) \pm \mathfrak{E}\big(a[b_1|b_\bullet|b_p b_{p+1}|b_\bullet|b_\ell]\big) \pm \mathfrak{E}\big(a[b_1|b_\bullet|b_{\ell-1}]\big) b_\ell \end{aligned}$$

for $\ell \geq 2$, with the middle term omitted if $\ell = 1$.[6] Here $d_\otimes$ refers to the "internal" differential $d_{\bar{A}} \otimes \mathrm{id}_{\mathbf{B}A} + \mathrm{id}_{\bar{A}} \otimes d_\otimes$ on $\bar{A} \otimes \mathbf{B}A$, omitting the external differential on the $\mathbf{B}A$ factor, and $D_\otimes$ is the derivation on $\mathrm{Hom}(\bar{A} \otimes \mathbf{B}A, A)$ defined with respect to this internal differential and d_A.

Critically, there is a natural way to make a variant of the cochain algebra into an HGA:

Definition 2.7 The *normalized cochain algebra* $C^*(X_\bullet; k)$ on a simplicial set $X_\bullet$ is the DG subalgebra containing all and only cochains vanishing on each degenerate simplex. It is augmented with respect to the map $C^*(X_\bullet; k) \twoheadrightarrow C^*(x_0; k) \xrightarrow{\sim} k$ induced by the inclusion of the simplicial subset associated to any chosen basepoint $x_0 \in X_0$.

The so-called *interval-cut operations*, among which the classical cup product and Steenrod cup-i products are the most famous examples, are known [24, Theorem 2.15] to define on $C^*(X_\bullet; k)$ the action of a symmetric DG-operad $\mathscr{X}$ called the *sequence operad* [24] or *surjection operad* [3]. This operad is known to be a quotient of the DG-operad $\mathscr{E}$ associated to the classical Barratt–Eccles simplicial operad [3, Theorem 1.3.2], an E_∞-operad filtered by an increasing sequence of suboperads $F_n\mathscr{E}$ whose geometric realizations are equivalent to the little n-cubes operads, and the sequence operad is accordingly filtered by quotients $F_n\mathscr{X}$ [3, Lemma 1.6.1]. As it turns out, an HGA structure on a DGA A is precisely an algebra structure over $F_2\mathscr{X}$ [23; 24, Theorem 4.1; 3, Section 1.6.6] (cf Franz [13, (3.13)] for the signs of the operations). Franz [10, Section 3.2] defines an *extended homotopy Gerstenhaber algebra* structure on a DGA A to be an algebra structure over a certain suboperad $F_3'\mathscr{X}$ of $F_3\mathscr{X}$ containing $F_2\mathscr{X}$. This structure comes in the form of operations $F_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A$ for $p, q \geq 1$, in addition to the E_ℓ, satisfying several compatibility conditions. An *extended* HGA *homomorphism* $f : A \to B$ is a DGA map which is simultaneously an $F_3'\mathscr{X}$-algebra homomorphism (which is to say it distributes over the E_ℓ and the $F_{p,q}$). Thus we have the following:

[6] Compare Franz [13, Lemma 3.1] for conditions for a map $C \to \mathbf{\Omega} C \otimes \mathbf{\Omega} C$ to be a twisting cochain inducing a DGC structure on the cobar construction $\mathbf{\Omega} C$ of a DGC C.

Corollary 2.8 *For any pointed simplicial set $X_\bullet$, its algebra $C^*(X_\bullet)$ of normalized cochains is naturally an extended* HGA*. Any* CDGA *A is naturally an extended* HGA*.*

Convention 2.9 *All* CDGAs *in this work, particularly cohomology rings, come equipped with this trivial extended* HGA *structure.* Consequently, if A is an extended HGA and B is a CDGA, then an extended HGA map $f\colon A \to B$ annihilates the values of $F_{p,q}$ and of E_ℓ for $\ell \geq 1$.

3 Strong homotopy commutativity

As adverted to in the introduction, we will require maps between DGAs that are not multiplicative, but still preserve the multiplication up to coherent homotopy.

Definition 3.1 Given two A_∞-algebras A and B, an *A_∞-map* from the former to the latter is defined to be a DGC homomorphism $F\colon \mathbf{B}A \to \mathbf{B}B$. Such a map F is determined by the sequence of compositions

$$F_{(n)}\colon A^{\otimes n} \xrightarrow[\sim]{(s^{-1})^{\otimes n}} \mathbf{B}_n A \hookrightarrow \mathbf{B}A \xrightarrow{F} \mathbf{B}B \xrightarrow{t_B} B.$$

If A and B are DGAs, then we write $F^+_{(1)} = F_{(1)} + \eta_B \epsilon_A\colon A \to B$. This is a cochain map, and if $H^*(F^+_{(1)})\colon H^*(A) \to H^*(B)$ is an isomorphism, we call F an *A_∞-quasi-isomorphism.*[7]

We will see it is possible to construct an A_∞-quasi-isomorphism from the CDGA $H^*(A)$ with trivial differential to A itself when A is a DGA with polynomial cohomology. To do so, we need two auxiliary notions:

Proposition 3.2 [26, Proposition 3.3] *There exists an **internal tensor product** of A_∞-maps which, given* DGAs *A, A', B, and B' and* DGC *maps $F\colon \mathbf{B}A \to \mathbf{B}A'$ and $G\colon \mathbf{B}B \to \mathbf{B}B'$, produces a* DGC *map*

$$F \mathbin{\underline{\otimes}} G\colon \mathbf{B}(A \otimes B) \to \mathbf{B}(A' \otimes B').$$

The operation $(F, G) \mapsto F \mathbin{\underline{\otimes}} G$

- *extends the tensor product of* DGA *maps in the sense that if $f\colon A \to A'$ and $g\colon B \to B'$ are* DGA *maps, then $\mathbf{B}f \mathbin{\underline{\otimes}} \mathbf{B}g = \mathbf{B}(f \otimes g)$,*
- *is functorial in each variable separately,*
- *is unital in the sense that $F \mathbin{\underline{\otimes}} \mathrm{id}_k$ and $\mathrm{id}_k \mathbin{\underline{\otimes}} F$ can be identified with F, and*
- *is associative in the sense that given a third pair C, C' of* DGAs *and a* DGC *map $H\colon \mathbf{B}C \to \mathbf{B}C'$, the iterated products $(F \mathbin{\underline{\otimes}} G) \mathbin{\underline{\otimes}} H$ and $F \mathbin{\underline{\otimes}} (G \mathbin{\underline{\otimes}} H)\colon \mathbf{B}(A \otimes B \otimes C) \to \mathbf{B}(A' \otimes B' \otimes C')$ agree.*

This notion of internal tensor product is connected to the ordinary external one by the shuffle map ∇ of Example 1.5.

[7] The map $F^+_{(1)}$ can be seen as a DGA map up to homotopy, and an ideal $\mathfrak{b} \trianglelefteq B$ containing the image of $t_B F(\mathbf{B}_{\geq 2} A)$ can be seen as a measure of its deviation from multiplicativity, for $F^+_{(1)}$ is a DGA map just if $t_B F(\mathbf{B}_{\geq 2} A) = 0$; and if so, then $F = \mathbf{B}F^+_{(1)}$.

Proposition 3.3 [11, Lemma 4.4] *Let* DGAs A'_0, A'_1, A''_0, *and* A''_1 *and* DGC *maps* $G' : \mathbf{B}A'_0 \to \mathbf{B}A'_1$ *and* $G'' : \mathbf{B}A''_0 \to \mathbf{B}A''_1$ *be given. Then the following square commutes*:

$$\begin{array}{ccc} \mathbf{B}A'_0 \otimes \mathbf{B}A''_0 & \xrightarrow{\nabla} & \mathbf{B}(A'_0 \otimes A''_0) \\ \big\downarrow{\scriptstyle G' \otimes G''} & & \big\downarrow{\scriptstyle G' \underline{\otimes} G''} \\ \mathbf{B}A'_1 \otimes \mathbf{B}A''_1 & \xrightarrow[\nabla]{} & \mathbf{B}(A'_1 \otimes A''_1) \end{array}$$

The internal tensor product allows us to define another notion of homotopy commutativity a priori unrelated to HGAs:

Definition 3.4 A *strongly homotopy commutative algebra* (henceforth SHC-*algebra*) is an augmented DGA A equipped with an A_∞-map from $A \otimes A$ to A (i.e. a DGC map $\Phi : \mathbf{B}(A \otimes A) \to \mathbf{B}A$), satisfying the following conditions:

1. Its 1-component $\Phi_{(1)} : \bar{A} \otimes \bar{A} \to \bar{A}$ is the restriction $\mu_A|_{\bar{A} \otimes \bar{A}}$ of the given product on A.
2. It is strictly unital in the sense that $\Phi \circ \mathbf{B}(\mathrm{id}_A \underline{\otimes}\, \eta_A) = \mathrm{id}_{\mathbf{B}A} = \Phi \circ \mathbf{B}(\eta_A \underline{\otimes}\, \mathrm{id}_A)$.
3. It is homotopy-associative: there is a homotopy between $\Phi(\Phi \underline{\otimes}\, \mathrm{id}_A) : \mathbf{B}(A \otimes A \otimes A) \to \mathbf{B}A$ and $\Phi(\mathrm{id}_A \underline{\otimes}\, \Phi)$.
4. It is homotopy-commutative: there is a homotopy between Φ and $\Phi \circ \mathbf{B}(12)$, where (12) is the tensor-factor transposition $A \otimes A \xrightarrow{\sim} A \otimes A$ of Notation 1.1(5).

We define the iterates $\Phi^{[n]} : \mathbf{B}(A^{\otimes n}) \to \mathbf{B}A$ of the structure map Φ by $\Phi^{[2]} := \Phi$ and $\Phi^{[n+1]} := \Phi(\Phi^{[n]} \underline{\otimes}\, \mathrm{id}_A)$.

An SHC-algebra structure Φ on a DGA A allows one to combine maps in a useful way: given sequences $(A_j)_{j=1}^n$ of DGAs and $(F_j : \mathbf{B}A_j \to \mathbf{B}A)_{j=1}^n$ of DGC maps, the composite

$$\mathbf{B}\Big(\bigotimes A_j\Big) \xrightarrow{\underline{\otimes} F_j} \mathbf{B}(A^{\otimes n}) \xrightarrow{\Phi^{[n]}} \mathbf{B}A \tag{3.5}$$

is guaranteed to be another DGC map. We will say this map is *compiled* from the F_j.

Associated to each homogeneous element $a \in A$ is a map from the free DGA on one generator of degree $|a|$ taking this generator to a. If a is a cocycle of even degree, this map factors through the map $\lambda_a : k[x] \to A$ taking x to a, where the differential on $k[x]$ is trivial. Thus, given a list $\vec{a} = (a_j)$ of even-degree cocycles of A and taking $A_j := k[x_j]$ for $|x_j| = |a_j|$ and $F_j := \mathbf{B}\lambda_{a_j}$, the compilation procedure (3.5) yields a DGC map

$$\lambda_{\vec{a}} : \mathbf{B}\Big(\bigotimes_{j=1}^{n} k[x_j]\Big) \xrightarrow{\mathbf{B}(\otimes \lambda_{a_j})} \mathbf{B}(A^{\otimes n}) \xrightarrow{\Phi^{[n]}} \mathbf{B}A. \tag{3.6}$$

Then $(\lambda_{\vec{a}})^{+}_{(1)}$ is easily seen to be the optimist's candidate for a ring homomorphism,

$$k[\vec{x}] := k[x_1, \dots, x_n] \to A, \quad x_1^{p_1} \cdots x_n^{p_n} \mapsto a_1^{p_1} \cdots a_n^{p_n}.$$

Though this map is in fact almost never multiplicative, it is at least a quasi-isomorphism.

Proposition 3.7 (Stasheff–Halperin [31, Theorem 9] and [26, 7.2]) *If A is an* SHC*-algebra whose cohomology ring $H^*(A)$ is polynomial on classes represented by even-degree elements $a_j \in A$ and $\lambda_{\vec{a}}$ is defined as in (3.6), then $H^*(\lambda_{\vec{a}})^+_{(1)}\colon k[\vec{x}] \to H^*(A)$ is an isomorphism.*

In the target application, A is the normalized cochain algebra on a classifying space BG. Munkholm [26, Proposition 4.7] (and stating a bit less, Gugenheim and Munkholm [16, Proposition 4.2]) showed that the cochain algebra of a simplicial set admits a natural SHC-algebra structure, but to define the product on the two-sided bar construction we will need later for our variant (Theorem A.27) of the Eilenberg–Moore theorem, we will use a result of Franz defining an SHC-algebra structure in terms of extended HGA operations.

Theorem 3.8 (Franz [10, Theorem 1.1 and (4.2)]) *An extended* HGA *A admits an* SHC*-algebra structure whose structure map Φ and associativity and commutativity homotopies are defined in terms of the extended* HGA *operations on A, and hence are natural in extended* HGA *maps. Moreover, the composite $\Phi \circ \nabla\colon \mathbf{B}A \otimes \mathbf{B}A \to \mathbf{B}(A \otimes A) \to \mathbf{B}A$ of the shuffle map of Example 1.5 and this structure map is the given product $\mu_{\mathbf{B}A}$ making A an* HGA.

We will require an explicit formula for $t_A \Phi_A$ in the proof of Theorem A.5, but hold off on stating it until then. From Corollary 2.8 and Theorem 3.8, we immediately have the following:

Corollary 3.9 *The normalized cochain algebra $C^*(X_\bullet; k)$ of a pointed simplicial set is naturally an* SHC*-algebra.*

Corollary 3.10 *For a pointed simplicial set $X_\bullet$ with polynomial cohomology and any list $\vec{a}$ in $C^*(X_\bullet; k)$ of representatives for k-algebra generators of $H^*(X_\bullet; k)$, the* DGC *map $\lambda_{\vec{a}}\colon \mathbf{B}H^*(X_\bullet; k) \to \mathbf{B}C^*(X_\bullet; k)$ given in (3.6) is such that $H^*(\lambda_{\vec{a}})^+_{(1)}$ can be identified with the identity map of $H^*(X_\bullet; k)$.*

We have observed that though it is a quasi-isomorphism in the cases that interest us, the extended 1-component $(\lambda_{\vec{a}})^+_{(1)}\colon H^*(X_\bullet) \to C^*(X_\bullet)$ is rarely a DGA homomorphism on the nose. However, the extended HGA structure and the resulting SHC structure on $C^*(X_\bullet)$ guarantee that, loosely speaking, it is a DGA map *up to an error term* contained in an ideal $\mathfrak{k}_{X_\bullet}$ of $C^*(X_\bullet)$ functorial in $X_\bullet$, independently of the choice of representatives $\vec{a}$ in $C^*(X_\bullet)$; thinking of $\mathfrak{k}_{X_\bullet}$ as a neighborhood of 0, we may consider it a sort of uniform bound on failure to be a DGA map. It will be an important point in the proof of our main result that the bounding ideal $\mathfrak{k}_{X_\bullet}$ lies in the kernel of the formality map f to be described in Theorem 3.16, and hence f annihilates the error term.

Definition 3.11 (Franz [11, (10.2)]) Given a pointed simplicial set $X_\bullet$, viewing its normalized cochain algebra $C^*(X_\bullet)$ as an extended HGA via Corollary 2.8 we denote by $\mathfrak{k} = \mathfrak{k}_{X_\bullet} \lhd C^*(X_\bullet)$ the ideal generated by the following elements, where a, b, $b_\bullet$, and $c_\bullet$ range over $C^*(X_\bullet)$:

(1) coboundaries,
(2) elements of odd degree,
(3) elements of the form $E_\ell(a; b_\bullet)$ for $\ell = \ell(b_\bullet) \geq 1$,
(4) elements of the form $F_{p,q}(b_\bullet; c_\bullet)$ with $(p,q) \neq (1,1)$,
(5) elements of the form $a \smile_2 E_\ell(b; c_\bullet)$ with $\ell \geq 2$,
(6) elements of the form $a \smile_2 \big(\cdots((b_0 \smile_1 b_1) \smile_1 b_2) \smile_1 \cdots\big)$ for cocycles a and $b_\bullet$.

From the naturality of the extended HGA structure on a cochain algebra, the following functoriality property of $\mathfrak{k}$ is immediate:

Lemma 3.12 (Franz [11, Proposition 10.1]) *If $\phi\colon Y_\bullet \to X_\bullet$ is a map of pointed simplicial sets, then the ideals of Definition 3.11 satisfy $\phi^*\mathfrak{k}_{X_\bullet} \leq \mathfrak{k}_{Y_\bullet}$.*

Morally speaking, then, our maps $\lambda_{\vec a}$ are functorial and multiplicative modulo $\mathfrak{k}$:

Theorem 3.13 (Franz [10, Proposition 7.2; 11, Proposition 11.5 and Theorem 11.6]) *Suppose 2 is a unit of k. Then the maps $\lambda_{\vec a}\colon k[\vec x] \to C^*(X_\bullet)$ of (3.6) are functorial modulo $\mathfrak{k}$ in the sense that, given a map $\phi\colon Y_\bullet \to X_\bullet$ of simplicial sets with polynomial cohomology and sequences $\vec a$ in $C^*(X_\bullet)$ and $\vec b$ in $C^*(Y_\bullet)$ representing generators of $H^*(X_\bullet)$ and $H^*(Y_\bullet)$, respectively, the left diagram of (3.14) commutes up to a DGC homotopy $H_\phi\colon \mathbf{B}H^*(X_\bullet) \to \mathbf{B}C^*(Y_\bullet)$ whose associated twisting cochain homotopy $\eta\epsilon + t_{C^*(Y_\bullet)}H_\phi$ sends $\mathbf{B}_{\geq 1}H^*(X_\bullet)$ into $\mathfrak{k}_{Y_\bullet}$.*

(3.14)
$$\begin{array}{ccc} \mathbf{B}H^*(X_\bullet) & \xrightarrow{\mathbf{B}H^*\phi} & \mathbf{B}H^*(Y_\bullet) \\ {\scriptstyle\lambda_{\vec a}}\downarrow & & \downarrow{\scriptstyle\lambda_{\vec b}} \\ \mathbf{B}C^*(X_\bullet) & \xrightarrow[\mathbf{B}C^*\phi]{} & \mathbf{B}C^*(Y_\bullet) \end{array} \qquad \begin{array}{ccc} \mathbf{B}\big(k[\vec x]\otimes k[\vec x]\big) & \xrightarrow{\mathbf{B}\mu} & \mathbf{B}\big(k[\vec x]\big) \\ {\scriptstyle\lambda_{\vec a}\otimes\lambda_{\vec a}}\downarrow & & \downarrow{\scriptstyle\lambda_{\vec a}} \\ \mathbf{B}(C^*(X_\bullet)\otimes C^*(X_\bullet)) & \xrightarrow[\Phi]{} & \mathbf{B}C^*(X_\bullet) \end{array}$$

Moreover $\lambda_{\vec a}$ is multiplicative modulo $\mathfrak{k}$ in the sense that $t_{C^*(X_\bullet)}\lambda_{\vec a}$ takes $\mathbf{B}_{\geq 2}\big(k[\vec x]\big)$ into $\mathfrak{k}_{X_\bullet}$ and the right diagram of (3.14) commutes up to a DGC homotopy $H_\mu\colon \mathbf{B}\big(k[\vec x]^{\otimes 2}\big) \to \mathbf{B}C^*(X_\bullet)$ whose associated twisting cochain homotopy $\eta\epsilon + t_{C^*(X_\bullet)}H_\mu$ sends $\mathbf{B}_{\geq 1}\big(k[\vec x]^{\otimes 2}\big)$ into $\mathfrak{k}_{X_\bullet}$.

We remark that the mere existence of such homotopies, without the bound in terms of $\mathfrak{k}$, lies at the heart of Munkholm's collapse proof [26]. To make precise our claim that there is a map $C^*(BT) \to H^*(BT)$ in the other direction annihilating $\mathfrak{k}$, we recall the classical construction of the simplicial classifying space:

Proposition 3.15 (see e.g. May [20, Section 21]) *Let $G_\bullet$ be a simplicial group. Then there exists a contractible simplicial $G_\bullet$-space $WG_\bullet$ functorial in $G_\bullet$. That is, if $\phi\colon G_\bullet \to H_\bullet$ is a homomorphism of simplicial groups, then $W\phi\colon WG_\bullet \to WH_\bullet$ is ϕ-equivariant in the sense that $(W\phi)(x\cdot g) = (W\phi)(x)\cdot\phi(g)$ for $x \in (WG_\bullet)_n$ and $g \in G_n$. The projection $WG_\bullet \to \overline{W}G_\bullet := WG_\bullet/G_\bullet$ is a principal $G_\bullet$-bundle, and the base $\overline{W}G_\bullet$ is the classifying simplicial set for simplicial principal $G_\bullet$-bundles.*

The promised map is then provided by the following result:

Theorem 3.16 (Franz [11, Theorem 9.6 and Proposition 9.7]) *Let $T_\bullet$ be a simplicial abelian group pointed at $1 \in T_0$ and such that the cohomology $H^*(T_\bullet; \mathbb{Z})$ of the normalized cochain complex $C^*(T_\bullet)$ is an exterior algebra on finitely many degree-1 generators. Then there exists a quasi-isomorphism $f = f_T : C^*(\overline{W}T_\bullet) \to H^*(\overline{W}T_\bullet)$ of extended* HGAs, *called the* **formality map**, *which annihilates all extended* HGA *operations $F_{p,q}$ except for $F_{1,1} = -\smile_2$. If 2 is a unit of k, the formality map can be chosen so as to annihilate all $\smile_2$-products of cocycles and such that its kernel contains the ideal $\mathfrak{k}_{\overline{W}T_\bullet}$ of Definition 3.11.*

4 Simplicial substitution

In this section we introduce a simplicial form of the homotopy pullback whose cohomology we will ultimately compute.

Discussion 4.1 We use the functorial Milgram model [25; 32; 28] of the universal principal bundle $EG \to BG$ associated to a Lie group G. It is well known that $K\backslash G/H$ is the homotopy pullback of the diagram $BK \to BG \leftarrow BH$ when $K \times H$ acts freely on G [29, Section 2]. More generally, writing $i: K \to G$ and $j: H \to G$ for the inclusions, the homotopy pullback can be identified as $EK \otimes_K G/H$,[8] which is the pullback of the diagram $BK \xrightarrow[Bi]{} BG \leftarrow EG/H$, the homeomorphism being given by $e \otimes gH \mapsto (eK, (Ei)(e)gH)$.

It is even better known that $Ej: EH \to EG$ is H-equivariant and induces a homotopy equivalence $BH \to EG/H$. The resulting map of triples induces a map of bar constructions

$$\mathbf{B}(C^*(BK), C^*(BG), C^*(EG/H)) \to \mathbf{B}(C^*(BK), C^*(BG), C^*(BH))$$

which the algebraic Eilenberg–Moore spectral sequence shows is a quasi-isomorphism by Proposition 1.19. Moreover, by Theorem A.27, this map is multiplicative with respect to the product of Theorem A.1, and hence induces a ring quasi-isomorphism.

If we write $\overline{G}$ for the singular complex $\operatorname{Sing} G$ of a connected Lie group, made a simplicial group by equipping each level $(\operatorname{Sing} G)_n$ with the valuewise multiplication of maps $\Delta^n \to G$, then the counit $|\overline{G}| \to G$ of the standard adjunction $|{-}| \dashv \operatorname{Sing}$ between the geometric realization and singular complex functors is a homomorphism and a homotopy equivalence, and is natural in continuous homomorphisms [20, Theorem 16.6(ii)]. For any simplicial group $G_\bullet$, there is a natural $|G_\bullet|$-equivariant homeomorphism $|WG_\bullet| \xrightarrow{\approx} E|G_\bullet|$ descending to a natural homeomorphism $|\overline{W}G_\bullet| \xrightarrow{\approx} B|G_\bullet|$ [4], so for $G_\bullet = \overline{G}$ we get a composite weak homotopy equivalence $|W\overline{G}| \to E|\overline{G}| \to EG$ equivariant with respect to the

[8] This is the quotient of $EK \times G/H$ under the equivalence relation $(ek, gH) \sim (e, kgH)$ for $k \in K$, more commonly denoted by $EK \times_K G/H$, the same notation as a pullback.

homomorphism $|\overline{G}| \to G$. Long exact homotopy sequences then show the map $\overline{W}\overline{G} = |W\overline{G}|/|\overline{G}| \to (EG)/G = BG$ is a natural weak homotopy equivalence. By naturality, these maps yield a multiplicative quasi-isomorphism $\mathbf{B}(C^*(BK), C^*(BG), C^*(BH)) \to \mathbf{B}(C^*|\overline{W}\overline{K}|, C^*|\overline{W}\overline{G}|, C^*|\overline{W}\overline{H}|)$.

Finally, the unit $X_\bullet \mapsto \operatorname{Sing}|X_\bullet|$ of the standard adjunction is a natural weak equivalence of simplicial sets whenever $X_\bullet$ is a Kan complex [20, Theorem 16.6(i)], which we know $\overline{W}\overline{K}$, $\overline{W}\overline{G}$, and $\overline{W}\overline{H}$ are [20, Lemma 21.3], so we have a multiplicative quasi-isomorphism to a more economical bar construction, $\mathbf{B}(C^*|\overline{W}\overline{K}|, C^*|\overline{W}\overline{G}|, C^*|\overline{W}\overline{H}|) \to \mathbf{B}(C^*(\overline{W}\overline{K}), C^*(\overline{W}\overline{G}), C^*(\overline{W}\overline{H}))$, although our proof does not strictly speaking require this simplification.

All told, we have the following:

Proposition 4.2 *Let G be a compact connected Lie group and H and K closed subgroups, and suppose the coefficient ring k is a principal ideal domain. Then there is an isomorphism of graded algebras*

$$H^*_K(G/H) \cong H^*\mathbf{B}(C^*(\overline{W}\overline{K}), C^*(\overline{W}\overline{G}), C^*(\overline{W}\overline{H})),$$

natural with respect to the diagram $K \hookrightarrow G \hookleftarrow H$.

Proof It remains only to check naturality, but this follows from the functoriality of W, Sing, $|-|$, and C^* and the naturality of the unit, the counit, and the Eilenberg–Moore isomorphism. □

Remark 4.3 May and Neumann observed [21] that the general result [15, Theorem 4.3] behind the Gugenheim–May computation of the cohomology groups of a homogeneous space G/H applies equally in many cases to a *generalized homogenous space*, meaning the homotopy fiber F of a map $B_H \to B_G$ of path-connected spaces. Here F is to be thought of as G/H for $G = \Omega B_G$ and $H = \Omega B_H$, and we assume a trivial $\pi_1(B_G)$-action on $H^*(F)$ and that $H^*(B_G)$ is a polynomial algebra over the principal ideal domain k, finitely generated in each degree. The key additional assumption is that there is a map $BT \to B_H$, where BT is a $K(\mathbb{Z}^n, 2)$, making $H^*(B_H)$ a free module of finite rank over $H^*(BT)$; then $T = \Omega BT$ is called a *maximal torus* of H.

Our result similarly generalizes. Given three path-connected spaces B_G, B_K, and B_H such that K and H admit maximal tori T_K and T_H and $H^*(B_G)$, $H^*(B_K)$, and $H^*(B_H)$ are polynomial algebras of finite type, then replacing $\operatorname{Sing} BT_K$ and $\operatorname{Sing} BT_H$ by $\overline{W}\overline{T}_K$ and $\overline{W}\overline{T}_K$, respectively, the same proof as in Section 5 shows that the cohomology of the homotopy pullback Y of the span $B_K \to B_G \leftarrow B_H$ is given as a graded group by $\operatorname{Tor}_{H^*(B_G)}(H^*(B_K), H^*(B_H))$, and as a ring if 2 is a unit of k.

We are not aware of great interest in such spaces Y in general, but some special cases have been studied. For instance, if B_K and B_H are points, we recover that $H^*(\Omega B_G)$ is an exterior algebra, and for the span $B_K \to B_K \times B_K \leftarrow B_K$, with both maps the diagonal, we find the cohomology ring of the free loop space LB_K is the tensor product of $H^*(B_K)$ and $H^*(K)$. This is an instance of a more general result of Saneblidze [27], derived using Hochschild homology and a model of the total space of the fibration $\Omega Y \to LY \to Y$ equipped with a cochain-level product related to the product described in Theorem A.5 and the HGA structure on $C^*(Y)$. His result, however, requires of $H^*(Y)$ only that the cup-1 squares of a set of polynomial generators be zero, without any maximal torus assumption, and of k only that it be a commutative ring.

5 The quasi-isomorphisms

We have finally assembled the necessary ingredients to prove Theorem 0.1.

Notation 5.1 In the calculation that follows, the base ring k *is now a principal ideal domain*, still usually suppressed in the notation. We will not require that 2 be a unit initially. Let G be a connected Lie group and H and K closed connected subgroups such that BG, BK, and BH have polynomial cohomology over k (equivalently, such that the torsion primes of G, H, and K are units). We will work with normalized cochains on the simplicial models $\overline{W}\,\overline{G}$ of Section 4, but in the notation for cohomology identify $H^*(BG)$ with $H^*(\overline{W}\,\overline{G})$ and so on, suppressing the natural isomorphisms induced by the simplicial weak equivalences $\overline{W}\,\overline{G} \to \operatorname{Sing}|\overline{W}\,\overline{G}| \to \operatorname{Sing} BG$.

From Proposition 4.2 we have a natural isomorphism

(5.2)
$$H^*_K(G/H) \cong \operatorname{Tor}_{C^*(\overline{W}\,\overline{G})}(C^*(\overline{W}\,\overline{K}), C^*(\overline{W}\,\overline{H}))$$

of graded algebras, and our goal is to use the maps λ of (3.6) and f of Theorem 3.16 to induce a CGA isomorphism

(5.3)
$$H^*_K(G/H) \cong \operatorname{Tor}_{H^*(BG)}(H^*(BK), H^*(BH)),$$

natural in inclusion diagrams $K \hookrightarrow G \hookleftarrow H$. On the level of graded modules, this isomorphism particularly means the Eilenberg–Moore spectral sequence of the homotopy pullback square of $BK \to BG \leftarrow BH$ collapses, a theorem of Munkholm [26] we will reprove as Proposition 5.7. To improve this result to a ring isomorphism, as both Tors arise as the cohomology of a two-sided bar construction, we will connect the argument DGAs of the two via maps preserving enough structure to guarantee our novel product on the two-sided bar construction is preserved up to homotopy. The structure of this section closely follows that of Franz [11, Section 12] and specializes to it in the case $K = 1$.

Discussion 5.4 We begin by constructing an additive cochain map between the two bar constructions. Selecting arbitrarily and fixing a list $\vec{a}$ of cocycles representing irredundant generators of $H^*(BG)$, and similarly for $H^*(BK)$ and $H^*(BH)$, we may use (3.6) to construct A_∞-quasi-isomorphisms $\lambda_G : \mathbf{B}H^*(BG) \to \mathbf{B}C^*(\overline{W}\,\overline{G})$, λ_K, and λ_H. Recall that these maps are selected so that if x_j denotes the cohomology class of $a_j \in C^*(\overline{W}\,\overline{G})$ and $\operatorname{rk} G = n$, then the extension $(\lambda_G)^+_{(1)} = \eta\epsilon + (\lambda_G)_{(1)}$ of the 1-component is the additive quasi-isomorphism

$$k[\vec{x}] \cong H^*(BG) \to C^*(\overline{W}\,\overline{G}), \qquad x_1^{p_1} \cdots x_n^{p_n} \mapsto a_1^{p_1} \cdots a_n^{p_n},$$

and similarly for λ_K and λ_H. We write the canonical twisting cochains $\mathbf{B}H^*(BG) \to H^*(BG)$ and $\mathbf{B}C^*(\overline{W}\,\overline{G}) \to C^*(\overline{W}\,\overline{G})$ respectively as t_{H} and t_{C}, and for any homomorphism $L' \to L$ of topological groups will write $\rho = \rho^L_{L'}$ for the functorially induced DGA maps $C^*(\overline{W}\,\overline{L}) \to C^*(\overline{W}\,\overline{L}')$ and $H^*(BL) \to H^*(BL')$. Then the candidate quasi-isomorphism

(5.5) $$\Theta\colon \mathbf{B}(H^*(BK), H^*(BG), H^*(BH)) \to \mathbf{B}(C^*(\overline{W}\,\overline{K}), C^*(\overline{W}\,\overline{G}), C^*(\overline{W}\,\overline{H}))$$

is defined as the composition of the cochain maps

$$\begin{array}{ccc}
H^*(BK) \otimes_{t_{\mathrm{H}}\mathbf{B}\rho} \mathbf{B}H^*(BG) \otimes_{t_{\mathrm{H}}\mathbf{B}\rho} H^*(BH) & & C^*(\overline{W}\,\overline{K}) \otimes_{t_{\mathrm{H}}\mathbf{B}\rho\lambda_G} \mathbf{B}H^*(BG) \otimes_{t_{\mathrm{H}}\mathbf{B}\rho\lambda_G} C^*(\overline{W}\,\overline{H}) \\
\downarrow{\scriptstyle \mathbf{B}(\lambda_K,\mathrm{id},\lambda_H)} & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle \mathbf{B}(\mathrm{id},\lambda_G,\mathrm{id})} \\
C^*(\overline{W}\,\overline{K}) \otimes_{t_{\mathrm{H}}\lambda_K\mathbf{B}\rho} \mathbf{B}H^*(BG) \otimes_{t_{\mathrm{H}}\lambda_H\mathbf{B}\rho} C^*(\overline{W}\,\overline{H}) & & C^*(\overline{W}\,\overline{K}) \otimes_{t_{\mathrm{C}}\mathbf{B}\rho} \mathbf{B}C^*(\overline{W}\,\overline{G}) \otimes_{t_{\mathrm{C}}\mathbf{B}\rho} C^*(\overline{W}\,\overline{H})
\end{array}$$

given from beginning to end by Lemmas 1.16, 1.12, and 1.11, where $\theta = (\delta^{\mathrm{L}}_{h^K} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta^{\mathrm{R}}_{h^H})$, the twisting cochain homotopies $h^K\colon \mathbf{B}H^*(BG) \to C^*(\overline{W}\,\overline{K})$ and $h^H\colon \mathbf{B}H^*(BG) \to C^*(\overline{W}\,\overline{H})$ coming from Theorem 3.13.

Lemma 5.6 *The map Θ defined in (5.5) satisfies*

$$\Theta \equiv (\lambda_K)^+_{(1)} \otimes \lambda_G \otimes (\lambda_H)^+_{(1)} \pmod{(\mathfrak{k}_{\overline{W}\,\overline{K}}, \mathfrak{k}_{\overline{W}\,\overline{H}})}.$$

Proof As $t_{C^*(\overline{W}\,\overline{K})}\lambda_K$ and $t_{C^*(\overline{W}\,\overline{H})}\lambda_H$ respectively take $\mathbf{B}_{\geq 2}H^*(BK)$ and $\mathbf{B}_{\geq 2}H^*(BH)$ into $\mathfrak{k}_{\overline{W}\,\overline{K}}$ and $\mathfrak{k}_{\overline{W}\,\overline{H}}$ by Theorem 3.13, the first factor $\mathbf{B}(\lambda_K, \mathrm{id}, \lambda_H)$ is congruent to $(\lambda_K)^+_{(1)} \otimes \mathrm{id} \otimes (\lambda_H)^+_{(1)}$ modulo $(\mathfrak{k}_{\overline{W}\,\overline{K}}, \mathfrak{k}_{\overline{W}\,\overline{H}})$ by Proposition 1.16. Next, by Theorem 3.13 again, the twisting cochain homotopies h^K and h^H respectively send $\mathbf{B}_{\geq 1}H^*(BG)$ into $\mathfrak{k}_{\overline{W}\,\overline{K}}$ and $\mathfrak{k}_{\overline{W}\,\overline{H}}$, so $(\delta^{\mathrm{L}}_{h^K} \otimes \mathrm{id})(\mathrm{id} \otimes \delta^{\mathrm{R}}_{h^H})$ is congruent to the identity modulo $(\mathfrak{k}_{\overline{W}\,\overline{K}}, \mathfrak{k}_{\overline{W}\,\overline{H}})$ by Lemma 1.12. Finally, $\mathbf{B}(\mathrm{id}, \lambda_G, \mathrm{id})$ is $\mathrm{id} \otimes \lambda_G \otimes \mathrm{id}$. □

Proposition 5.7 *We retain the notation G, K, H from Notation 5.1 and Θ from (5.5).*

(i) *The induced map*

$$H^*(\Theta)\colon \mathrm{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH)) \to \mathrm{Tor}^*_{C^*(\overline{W}\,\overline{G})}(C^*(\overline{W}\,\overline{K}), C^*(\overline{W}\,\overline{H}))$$

of graded k-modules is an isomorphism.

(ii) *The Eilenberg–Moore spectral sequence for the homotopy pullback of $BK \to BG \leftarrow BH$ collapses at the E_2 page.*

Proof (i) Because $H^*(BG)$ and $H^*(BK)$ are flat over k, under the length filtration of two-sided bar constructions discussed in Definition 1.13, the E_2 page of the target under the map of associated filtration spectral sequences induced by Θ is again $\operatorname{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH))$ as noted in Proposition 1.19. Since λ_G, $(\lambda_K)^+_{(1)}$, and $(\lambda_H)^+_{(1)}$ are each quasi-isomorphisms, by Lemma 5.6, the map of E_2 pages is the identity map. These are half-plane spectral sequences with exiting differentials and in the associated exact couples (E_1, A_1) one has $A_1^p = 0$ for $p > 0$, and particularly $\varprojlim_p A_1^p = 0$, so they are strongly convergent [5, Theorem 6.1(a)]; hence $H^*(\Theta)$ is a graded k-linear isomorphism [5, Theorem 5.3].

(ii) In the map of spectral sequences, the codomain is the Eilenberg–Moore spectral sequence of the homotopy pullback of the diagram $BK \leftarrow BG \rightarrow BH$. We have seen the spectral sequence map is a k-linear graded isomorphism from E_2 on, so it is enough to show the domain spectral sequence collapses. But E_2 of this domain sequence is already isomorphic to the sequence's target $H^*\mathbf{B}(H^*(BK), H^*(BG), H^*(BH))$ as a graded k-module. □

To show $H^*(\Theta)$ is multiplicative and natural will involve the formality map of Section 4, and particularly *from here on out, we will need* 2 *to be a unit in* k. We will soon specialize to maximal tori, but for now let arbitrary compact tori T_K and T_H and simplicial group homomorphisms $\alpha_K\colon \overline{T}_K \to \overline{K}$ and $\alpha_H\colon \overline{T}_H \to \overline{H}$ be given, and choose formality maps $f_{T_K}\colon C^*(\overline{W}\,\overline{T}_K) \to H^*(BT_K)$ and $f_{T_H}\colon C^*(\overline{W}\,\overline{T}_H) \to H^*(BT_H)$ as guaranteed by Theorem 3.16, recalling that these maps respectively annihilate the ideals $\mathfrak{k}_{\overline{W}\,\overline{T}_K}$ and $\mathfrak{k}_{\overline{W}\,\overline{T}_H}$ defined in Definition 3.11. As f and ρ are HGA and hence DGA maps, Lemma 1.10 provides a cochain map of two-sided bar constructions

(5.8)
$$\Psi_{\alpha_K,\alpha_H}\colon C^*(\overline{W}\,\overline{K}) \otimes_{\rho t_C} \mathbf{B}C^*(\overline{W}\,\overline{G}) \otimes_{\rho t_C} C^*(\overline{W}\,\overline{H}) \xrightarrow{f\rho\otimes \mathrm{id}\otimes f\rho} H^*(BT_K) \otimes_{f\rho t_C} \mathbf{B}C^*(\overline{W}\,\overline{G}) \otimes_{f\rho t_C} H^*(BT_H).$$

Lemma 5.9 *Let G, K, and H be as in Notation 5.1, Θ as in (5.5), and Ψ as in (5.8).*

(i) *The cochain map Ψ is multiplicative with respect to the product $\tilde{\mu}$ of Theorem A.1.*

(ii) *The composite $\Psi\Theta$ is equal to*

$$\rho^K_{T_K} \otimes \lambda_G \otimes \rho^H_{T_H}\colon H^*(BK) \otimes_{\rho t_{\mathrm{H}}} \mathbf{B}H^*(BG) \otimes_{\rho t_{\mathrm{H}}} H^*(BH) \to H^*(BT_K) \otimes_{f\rho t_{\mathrm{C}}} \mathbf{B}C^*(\overline{W}\,\overline{G}) \otimes_{f\rho t_{\mathrm{C}}} H^*(BT_H).$$

(iii) *If α_K and α_H are inclusions of maximal tori, the induced map $H^*(\Psi)$ in cohomology is injective.*

Proof (i) Multiplicativity of Ψ follows from naturality of $\tilde{\mu}$ since f and ρ are HGA maps.

(ii) The restriction $\rho^K_{T_K}$ sends $\mathfrak{k}_{\overline{W}\,\overline{K}}$ to $\mathfrak{k}_{\overline{W}\,\overline{T}_K}$ by Lemma 3.12, while by Theorem 3.16 f_K annihilates $\mathfrak{k}_{\overline{W}\,\overline{T}_K}$, and similarly $f_H\rho^H_{T_H}$ annihilates $\mathfrak{k}_{\overline{W}\,\overline{H}}$. As Θ is congruent to $(\lambda_K)^+_{(1)} \otimes \lambda_G \otimes (\lambda_H)^+_{(1)}$ modulo $(\mathfrak{k}_{\overline{W}\,\overline{K}}, \mathfrak{k}_{\overline{W}\,\overline{H}})$ by Lemma 5.6, we then have

$$\Psi\Theta = (f\rho \otimes \mathrm{id} \otimes f\rho)\Theta = f\rho(\lambda_K)^+_{(1)} \otimes \lambda_G \otimes f\rho(\lambda_H)^+_{(1)}.$$

But taking cohomology of the cochain maps

$$H^*(BK) \xrightarrow{(\lambda_K)^+_{(1)}} C^*(\overline{W}\,\overline{K}) \xrightarrow{\rho} C^*(\overline{W}\,\overline{T}_K) \xrightarrow{f} H^*(BT_K)$$

yields

$$H^*(BK) \xrightarrow{\mathrm{id}} H^*(BK) \xrightarrow{\rho} C^*(BT_K) \xrightarrow{\mathrm{id}} H^*(BT_K),$$

and the differentials on $H^*(BK)$ and $H^*(BT_K)$ are zero, implying the cochain map $f_K \circ \rho^K_{T_K} \circ (\lambda_K)^+_{(1)}$ is itself $\rho^K_{T_K}$, and similarly for $\rho^H_{T_H}$.

(iii) We factor Ψ as $(f \otimes \mathrm{id} \otimes f)(\rho \otimes \mathrm{id} \otimes \rho)$ and show both factors induce injections in cohomology. As $f \otimes \mathrm{id} \otimes f : \mathbf{B}(C^*(\overline{W}\,\overline{T}_K), C^*(\overline{W}\,\overline{G}), C^*(\overline{W}\,\overline{T}_H)) \to \mathbf{B}(H^*(BT_K), C^*(\overline{W}\,\overline{G}), H^*(BT_H))$ induces the identity map between the E_2 pages of the associated filtration spectral sequences, it is a quasi-isomorphism. As for $\rho \otimes \mathrm{id} \otimes \rho$, consider the map of Serre spectral sequences induced by the map of (vertical) fibrations

$$\begin{array}{ccc} {}_{T_K}G_{T_H} & \longrightarrow & BT_K \times BT_H \\ \downarrow & & \downarrow \\ {}_KG_H & \longrightarrow & BK \times BH \end{array}$$

The homotopy fiber of both fibrations is $K/T_K \times H/T_H$. Thus the E_2 page of the right spectral sequence is concentrated in even degree, implying the sequence collapses, and the map of spectral sequences implies the left spectral sequence collapses as well; it follows that the left fibration induces an injection in cohomology. But by the naturality clause of Proposition 4.2 this injection is $H^*(\rho \otimes \mathrm{id} \otimes \rho)$. □

Theorem 5.10 *The isomorphism $H^*(\Theta)$ of Proposition 5.7 is multiplicative.*

Proof Since we know from Lemma 5.9 that $H^*(\Psi)$ is injective and multiplicative it will be enough to show the map $\Psi\Theta = \rho \otimes \lambda_G \otimes \rho$ is multiplicative up to homotopy. As $H^*(BK)$ and $H^*(BH)$ are CDGAs, the HGA operations E_k are zero for $k \geq 0$ by Convention 2.9, so the product $\tilde{\mu}$ of (A.6) reduces to $a'[a_\bullet]a'' \cdot b'[b_\bullet]b'' = \pm a'b' \otimes [a_\bullet] * [b_\bullet] \otimes a''b''$, which is just the tensor permutation Π of (A.4) rearranging the factors in the correct order followed by the coordinatewise product $\mu^{\otimes 3}$. Recalling from the proof of Theorem A.1 that Π gives a natural cochain isomorphism from the tensor-square of the two-sided bar construction to the two-sided twisted tensor product with respect to the twisting cochains t' and t'' given by $\rho^{\otimes 2}(t \otimes \eta\epsilon + \eta\epsilon \otimes t)$, we can transfer the desired multiplicativity of $\Psi\Theta$ up to homotopy to a question about maps

$$H^*(BK)^{\otimes 2} \otimes_{t'} (\mathbf{B}H^*(BG))^{\otimes 2} \otimes_{t''} H^*(BH)^{\otimes 2} \to H^*(BT_K) \otimes_{f\rho t_{\mathrm{C}}} \mathbf{B}C^*(\overline{W}\,\overline{G}) \otimes_{f\rho t_{\mathrm{C}}} H^*(BT_H).$$

We want to find a homotopy between the cochain maps

$$(\rho \otimes \lambda_G \otimes \rho)(\mu \otimes \mu_{\mathbf{B}H^*(BG)} \otimes \mu) = \rho\mu \otimes \lambda_G \mu_{\mathbf{B}H^*(BG)} \otimes \rho\mu$$

and

$$(\mu \otimes \mu_{\mathbf{B}C^*(\overline{W}\,\overline{G})} \otimes \mu)(\rho^{\otimes 2} \otimes \lambda_G^{\otimes 2} \otimes \rho^{\otimes 2}) = \mu\rho^{\otimes 2} \otimes \mu_{\mathbf{B}C^*(\overline{W}\,\overline{G})} \lambda_G^{\otimes 2} \otimes \mu\rho^{\otimes 2}.$$

On tensor factors we have $\rho\mu = \mu\rho^{\otimes 2}$ because ρ is a ring map, while Theorem 3.13 provides a coalgebra homotopy $H_\mu : \lambda_G \mathbf{B}\mu_{H^*(BG)} \simeq \Phi(\lambda_G \mathbin{\underline{\otimes}} \lambda_G)$ such that $t_{\mathrm{C}} H_\mu$ takes $\mathbf{B}_{\geq 1}(H^*(BG)^{\otimes 2})$ into $\mathfrak{k}_{\overline{W}\,\overline{G}}$, so that the twisting cochains $f\rho t_{\mathrm{C}}$ annihilate the image of H_μ. As the shuffle map ∇ is a natural transformation $\otimes \to \underline{\otimes}$ of bifunctors, we may append the square of Proposition 3.3:

$$\begin{array}{ccccc}
(\mathbf{B}H^*(BG))^{\otimes 2} & \xrightarrow[\nabla]{} & \mathbf{B}(H^*(BG)^{\otimes 2}) & \xrightarrow[\mathbf{B}\mu]{} & \mathbf{B}(H^*(BG)) \\
\downarrow{\scriptstyle \lambda_G\otimes\lambda_G} & & \downarrow{\scriptstyle \lambda_G\underline{\otimes}\lambda_G} & & \downarrow{\scriptstyle \lambda_G} \\
(\mathbf{B}C^*(\overline{W}\overline{G}))^{\otimes 2} & \xrightarrow[\nabla]{} & \mathbf{B}(C^*(\overline{W}\overline{G})^{\otimes 2}) & \xrightarrow[\Phi]{} & \mathbf{B}C^*(\overline{W}\overline{G})
\end{array}$$

Recalling $\Phi\circ\nabla = \mu_{\mathbf{B}C^*(\overline{W}\overline{G})}$ (Theorem 3.8) and $\mathbf{B}\mu\circ\nabla = \mu_{\mathbf{B}H^*(BG)}$ (Example 1.5), we then see $H_\mu\nabla$ is a DGC homotopy from $\lambda_G\mu_{\mathbf{B}H^*(BG)}$ to $\mu_{\mathbf{B}C^*(\overline{W}\overline{G})}\lambda_G^{\otimes 2}$. As $t_{\mathrm{C}}H_\mu\nabla$ takes the coaugmentation coideal of $(\mathbf{B}H^*(BG))^{\otimes 2}$ into $\mathfrak{k}_{\overline{W}\overline{G}}$, we may apply Lemma 1.10 to see that $f\rho\otimes H_\mu\nabla\otimes f\rho$ is a homotopy doing what we wanted. □

This establishes the ring isomorphism of Theorem 0.1. It remains to show this isomorphism is natural in inclusion diagrams.

Theorem 5.11 *Given a diagram*

$$\begin{array}{ccccc}
K_1 & \hookrightarrow & G_1 & \hookleftarrow & H_1 \\
\downarrow & & \downarrow & & \downarrow \\
K_0 & \hookrightarrow & G_0 & \hookleftarrow & H_0
\end{array} \tag{5.12}$$

of continuous group homomorphisms such that the cohomology of each of the groups' classifying spaces is a polynomial ring over k and given cocycle representatives for an irredundant set of polynomial generators of each cohomology ring, so as to define quasi-isomorphisms Θ_1 and Θ_0 as in (5.5), the following induced square commutes:

$$\begin{array}{ccc}
\mathrm{Tor}^*_{H^*(BG_0)}(H^*(BK_0), H^*(BH_0)) & \xrightarrow{\mathrm{Tor}^*_\rho(\rho,\rho)} & \mathrm{Tor}^*_{H^*(BG)}(H^*(BK_1), H^*(BH_1)) \\
{\scriptstyle H^*(\Theta_0)}\downarrow\wr & & \wr\downarrow{\scriptstyle H^*(\Theta_1)} \\
H^*_{K_0}(G_0/H_0) & \longrightarrow & H^*_{K_1}(G_1/H_1)
\end{array}$$

Proof Let $i_{K_1}: T_{K_1} \hookrightarrow K_1$ and $i_{H_1}: T_{H_1} \hookrightarrow H_1$ be inclusions of maximal tori. We expand the description of $\Psi\Theta$ in Lemma 5.9 to the diagram in Figure 1. Note carefully the asymmetry in the bottom square: on the left we have G_0 rather than G_1, but the tori T_{K_1} and T_{H_1} are subtori of G_1 rather than G_0. The composite $f\rho\otimes\mathrm{id}\otimes f\rho$ after Θ in the right column is the map $\Psi_{i_{K_1},i_{H_1}}$ of (5.8) and the composite after Θ the first column is Ψ_{j_K,j_H}, where j_K and j_H are respectively the compositions $T_{K_1} \hookrightarrow K_1 \to K_0$ and $T_{H_1} \hookrightarrow H_1 \to H_0$.

Our goal is to show the top square commutes in cohomology, but as $H^*(\Psi_{i_{K_1},i_{H_1}})$ is injective by Lemma 5.9(iii), it will suffice to postcompose $\Psi_{i_{K_1},i_{H_1}}$ to the two paths around the square and show the resulting maps induce the same map in cohomology. One confirms tensor factor by tensor factor that the

$$
\begin{array}{ccc}
H^*(BK_0)\otimes_{\rho t_{\mathrm{H}}}\mathbf{B}H^*(BG_0)\otimes_{\rho t_{\mathrm{H}}}H^*(BH_0) & \xrightarrow{\rho\otimes\mathbf{B}\rho\otimes\rho} & H^*(BK_1)\otimes_{\rho t_{\mathrm{H}}}\mathbf{B}H^*(BG_1)\otimes_{\rho t_{\mathrm{H}}}H^*(BH_1)\\
\downarrow{\scriptstyle\Theta_0} & & \downarrow{\scriptstyle\Theta_1}\\
C^*(\overline{W}\overline{K}_0)\otimes_{\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_0)\otimes_{\rho t_{\mathrm{C}}}C^*(\overline{W}\overline{H}_0) & \xrightarrow{\rho\otimes\mathbf{B}\rho\otimes\rho} & C^*(\overline{W}\overline{K}_1)\otimes_{\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_1)\otimes_{\rho t_{\mathrm{C}}}C^*(\overline{W}\overline{H}_1)\\
\downarrow{\scriptstyle\rho\otimes\mathrm{id}\otimes\rho} & & \downarrow{\scriptstyle\rho\otimes\mathrm{id}\otimes\rho}\\
C^*(\overline{W}\overline{T}_{K_1})\otimes_{\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_0)\otimes_{\rho t_{\mathrm{C}}}C^*(\overline{W}\overline{T}_{H_1}) & \xrightarrow[\mathrm{id}\otimes\mathbf{B}\rho\otimes\mathrm{id}]{} & C^*(\overline{W}\overline{T}_{K_1})\otimes_{\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_1)\otimes_{\rho t_{\mathrm{C}}}C^*(\overline{W}\overline{T}_{H_1})\\
\downarrow{\scriptstyle f\otimes\mathrm{id}\otimes f} & & \downarrow{\scriptstyle f\otimes\mathrm{id}\otimes f}\\
H^*(BT_{K_1})\otimes_{f\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_0)\otimes_{f\rho t_{\mathrm{C}}}H^*(BT_{H_1}) & \xrightarrow[\mathrm{id}\otimes\mathbf{B}\rho\otimes\mathrm{id}]{} & H^*(BT_{K_1})\otimes_{f\rho t_{\mathrm{C}}}\mathbf{B}C^*(\overline{W}\overline{G}_1)\otimes_{f\rho t_{\mathrm{C}}}H^*(BT_{H_1})
\end{array}
$$

Figure 1: A diagram of two-sided bar constructions.

bottom two squares commute, so it will be enough to find a cochain homotopy between the two paths along the outer rectangle. By Lemma 5.9(ii), the composite along the upper right is

$$(\rho^{K_1}_{T_{K_1}}\otimes\lambda_{G_1}\otimes\rho^{H_1}_{T_{H_1}})(\rho^{K_0}_{K_1}\otimes\mathbf{B}\rho^{G_0}_{G_1}\otimes\rho^{H_0}_{H_1})=\rho^{K_0}_{T_{K_1}}\otimes\lambda_{G_1}\mathbf{B}\rho^{G_0}_{G_1}\otimes\rho^{K_0}_{T_{K_1}} \tag{5.13}$$

and the composite along the lower left is

$$(\mathrm{id}\otimes\mathbf{B}\rho^{G_0}_{G_1}\otimes\mathrm{id})(\rho^{K_0}_{T_{K_1}}\otimes\lambda_{G_0}\otimes\rho^{H_0}_{T_{H_1}})=\rho^{K_0}_{T_{K_1}}\otimes\mathbf{B}\rho^{G_0}_{G_1}\lambda_{G_0}\otimes\rho^{H_0}_{T_{H_1}}. \tag{5.14}$$

Now Theorem 3.13 provides a DGC homotopy H_ρ between $\rho^{G_0}_{G_1}\lambda_{G_0}$ and $\lambda_G\rho^{G_0}_{G_1}$ such that $t_{\mathrm{C}}H_\rho$ takes $\mathbf{B}_{\geq 1}H^*(BG_0)$ into $\mathfrak{k}_{\overline{W}\overline{G}_1}$, so that the two twisting cochains $f\rho t_{\mathrm{C}}$ defining the common codomain of (5.13) and (5.14) annihilate the image of H_ρ. Applying Lemma 1.10, we see $f\rho\otimes H_\rho\otimes f\rho$ is the sought-after homotopy between (5.13) and (5.14). □

Corollary 5.15 *The isomorphism $H^*(\Theta)$ of Theorem 0.1 does not depend on the choice of representatives defining Θ in (5.5).*

Proof One can take the vertical maps to each be the identity in (5.12), defining Θ_0 in terms of one set of representatives and Θ_1 in terms of another. □

6 Example

In this section we give an example where Theorem 0.1 provides information not previously available from the standard techniques. This example is in a sense minimal in that in similar examples where the differentials form a regular sequence, the fact that an exterior algebra is a free CGA allows one to resolve the multiplicative extension problem, and in examples with $\mathrm{Tor}^{\leq -2}=0$ one can resolve the extension problem by noting that elements of Tor^{-1} annihilate one other and lie in odd degree, whereas Tor^0 is concentrated in even degree.

Example 6.1 Let S be the image of the embedding $z \mapsto \mathrm{diag}(z^{-4}, z, z, z)$ of $\mathrm{U}(1)$ in $\mathrm{SU}(4)$. Theorem 0.1 computes the ring $H_S^*(\mathrm{SU}(4)/S; \mathbb{Z}[\frac{1}{2}])$ as

$$\mathrm{Tor}^*_{H^*(B\mathrm{SU}(4);\mathbb{Z}[1/2])}\bigl(H^*(BS;\mathbb{Z}[\tfrac{1}{2}]), H^*(BS;\mathbb{Z}[\tfrac{1}{2}])\bigr)$$

(previously, the Sullivan model noted by Kapovich showed the same with $\mathbb{Z}[\frac{1}{2}]$ replaced by $\mathbb{Q}$). One finds explicitly that the induced map $\mathbb{Z}[\frac{1}{2}][c_2, c_3, c_4] \xrightarrow{\sim} H^*B\mathrm{SU}(4) \to H^*BS \xrightarrow{\sim} \mathbb{Z}[\frac{1}{2}][t_2]$ takes the universal Chern classes c_2, c_3, and c_4 respectively to $-6t^2$, $-8t^3$, and $-3t^4$, so one can compute the Tor using the associated Koszul complex, as the cohomology of the DGA $\mathbb{Z}[\frac{1}{2}][s_2, t_2] \otimes \bigwedge[v_3, w_5, x_7]$ with differential annihilating s and t and with

$$dv_3 = 6(s^2 - t^2), \quad dw_5 = 8(s^3 - t^3), \quad \text{and} \quad dx_7 = 3(s^4 - t^4).$$

Setting $p_{2j} = (s^j - t^j)/(s - t)$ for $2 \le j \le 4$ one finds that $\mathrm{Tor}^0 \cong \mathbb{Z}[\frac{1}{2}][s, t]/(s - t)(6p_2, 8p_4, 3p_6)$; that Tor^{-1} is the quotient of the $\mathbb{Z}[\frac{1}{2}][s, t]$-module generated by the cocycles

$$a = 3p_2w_5 - 4p_4v_3 = \frac{1}{2(s-t)}d(v_3w_5), \quad b = 2p_2x_7 - p_6v_3 = \frac{1}{3(s-t)}d(v_3x_7),$$
$$c = 8p_4x_7 - 3p_6w_5 = \frac{1}{s-t}d(w_5x_7),$$

which satisfy $p_6a - 4p_4b + p_2c = 0$, by the submodule generated by $d(v_3w_5)$, $d(v_3x_7)$, and $d(w_5x_7)$; and that Tor^{-2} is the quotient of the free $\mathbb{Z}[\frac{1}{2}][s, t]$-module generated by the class of

$$e = 3p_6v_3w_5 - 8p_4v_3x_7 + 6p_2w_5x_7 = \frac{1}{(s-t)}d(v_3w_5x_7)$$

by the submodule generated by $d(v_3w_5x_7)$. All told, one finds that $H_S^*(\mathrm{SU}(4)/S; \mathbb{Z}[\frac{1}{2}])$ is the quotient of $\mathbb{Z}[\frac{1}{2}][s_2, t_2, e_{14}] \otimes \bigwedge[a_7, b_9, c_{11}]$ by the ideal generated by

$$(s^3 + s^2t + st^2 + t)a - 4(s^2 + st + t^2)b + (s + t)c, \quad 2(s - t)a, \quad 3(s - t)b, \quad (s - t)c, \quad (s - t)e,$$
$$ab - (s + t)e, \quad ac - 4(s^2 + st + t^2)e, \quad bc - (s^3 + s^2t + st^2 + t^3)e, \quad ae, \quad be, \quad ce \quad \text{and} \quad e^2.$$

This computation resolves, for example, the multiplicative extension problem for products of the classes corresponding to a, b, c, and e on the E_∞ page of the Serre spectral sequence beginning with $E_2 = H^*(BS \times BS) \otimes H^*\mathrm{SU}(4)$ and converging to $H_S^*(\mathrm{SU}(4)/S; \mathbb{Z})$.

Appendix A product on the two-sided bar construction
joint with Matthias Franz

Given that an HGA structure defines a DGA structure on the bar construction of a DGA, one might hope HGA homomorphisms $A' \leftarrow A \to A''$ similarly induce a DGA structure on the two-sided bar construction. We cannot assert this, but we can at least define a nonassociative product sufficient to prove the variant Eilenberg–Moore theorem (Theorem A.27) needed for the proof of Theorem 0.1.

Theorem A.1 *Let $A' \xleftarrow{f'} A \xrightarrow{f''} A''$ be* HGA *homomorphisms. Then there exists a cochain map*

$$\widetilde{\mu}\colon \mathbf{B}(A', A, A'') \otimes \mathbf{B}(A', A, A'') \to \mathbf{B}(A', A, A''),$$

which we think of as a product, natural in the sense that given a commutative diagram

$$\begin{array}{ccccc} A' & \longleftarrow & A & \longrightarrow & A'' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} \\ B' & \longleftarrow & B & \longrightarrow & B'' \end{array}$$

of HGA *homomorphisms, the induced map $\mathbf{B}(g', g, g'')\colon \mathbf{B}(A', A, A'') \to \mathbf{B}(B', B, B'')$ of Notation 1.18 is multiplicative.*

Remark A.2 It requires the extended HGA operations $F_{p,q}$ to define the natural homotopy h^{c} witnessing the homotopy-commutativity axiom for Franz's natural SHC-algebra structure map Φ_A described in Theorem 3.8, but to define Φ_A itself and show that $\Phi_A \circ \nabla = \mu_{\mathbf{B}A}$, which is all we require here, one needs only that A be an HGA.

Proof of Theorem A.1 By Theorem 3.8, there exists a DGC map $\Phi_A\colon \mathbf{B}(A \otimes A) \to \mathbf{B}A$ giving an SHC-algebra structure on A, and similarly for A' and A''. We set

$$\widetilde{\mu} := \underbrace{\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''}) \circ (\mathrm{id}_{A'\otimes A'} \otimes \nabla \otimes \mathrm{id}_{A''\otimes A''})}_{\mathbf{B}(\Phi_{A'}, \mu_{\mathbf{B}A}, \Phi_{A''})} \circ \Pi, \tag{A.3}$$

where the $\mathbf{B}(-,-,-)$ maps are as defined in Proposition 1.16 and the initial map

$$\begin{aligned} \Pi\colon \mathbf{B}(A', A, A'') \otimes \mathbf{B}(A', A, A'') &\to (A' \otimes A') \otimes_{t'} (\mathbf{B}A \otimes \mathbf{B}A) \otimes_{t''} (A'' \otimes A''), \\ a'[a_\bullet]a'' \otimes b'[b_\bullet]b'' &\mapsto \pm a' \otimes b' \otimes [a_\bullet] \otimes [b_\bullet] \otimes a'' \otimes b'', \end{aligned} \tag{A.4}$$

is the tensor permutation (2354). Here the two-sided twisted tensor product is determined by the twisting cochains

$$\begin{aligned} t' &= (f')^{\otimes 2} t_{A^{\otimes 2}} \nabla = (f')^{\otimes 2}(t_A \otimes \eta_A \epsilon_{\mathbf{B}A} + \eta_A \epsilon_{\mathbf{B}A} \otimes t_A)\colon (\mathbf{B}A)^{\otimes 2} \to (A')^{\otimes 2}, \\ t'' &= (f'')^{\otimes 2} t_{A^{\otimes 2}} \nabla = (f'')^{\otimes 2}(t_A \otimes \eta_A \epsilon_{\mathbf{B}A} + \eta_A \epsilon_{\mathbf{B}A} \otimes t_A)\colon (\mathbf{B}A)^{\otimes 2} \to (A'')^{\otimes 2}, \end{aligned}$$

for ∇ the shuffle map of Example 1.5, and it is not hard to check Π is a cochain map. To see

$$\mathrm{id} \otimes \nabla \otimes \mathrm{id}\colon (A' \otimes A') \otimes_{t'} (\mathbf{B}A \otimes \mathbf{B}A) \otimes_{t''} (A'' \otimes A'') \to (A' \otimes A') \otimes_{t'} \mathbf{B}(A \otimes A) \otimes_{t''} (A'' \otimes A'')$$

is a cochain map, it is enough by Lemma 1.10 to observe $t' = (f' \otimes f')t_{A\otimes A}\nabla = t_{A'\otimes A'}\mathbf{B}(f' \otimes f')\nabla$ and similarly $t'' = t_{A''\otimes A''}\mathbf{B}(f'' \otimes f'')\nabla$. To see

$$\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})\colon \mathbf{B}(A' \otimes A', A \otimes A, A'' \otimes A'') \to \mathbf{B}(A', A, A'')$$

is a well-defined cochain map, by Proposition 1.16 it is enough to note the diagram

$$\begin{array}{ccccc} \mathbf{B}(A'\otimes A') & \xleftarrow{\mathbf{B}(f')^{\otimes 2}} & \mathbf{B}(A\otimes A) & \xrightarrow{\mathbf{B}(f'')^{\otimes 2}} & \mathbf{B}(A''\otimes A'') \\ \downarrow{\scriptstyle \Phi_{A'}} & & \downarrow{\scriptstyle \Phi_{A}} & & \downarrow{\scriptstyle \Phi_{A''}} \\ \mathbf{B}A' & \xleftarrow[\mathbf{B}f']{} & \mathbf{B}A & \xrightarrow[\mathbf{B}f'']{} & \mathbf{B}A'' \end{array}$$

commutes by the naturality of Φ in HGAs.

Naturality follows because Π is natural in sextuples of cochain complexes, ∇ is natural in pairs of cochain complexes, Φ is natural in extended HGA maps, and $\mathbf{B}(-,-,-)$ is functorial in triples of DGA maps. $\square$

To make real use of this product, we will require a more explicit formula:

Theorem A.5 *The cochain map $\widetilde{\mu}$ of Theorem A.1 is given in terms of the notation set in Definition 2.1 and Notation 2.2 by*

$$\textbf{(A.6)}\qquad a'[a_\bullet]a''\otimes b'[b_\bullet]b'' \mapsto \pm a'b' \otimes [a_\bullet]*[b_{(2)}]\otimes \mathfrak{E}\big(a''[f''b_{(3)}]\big)b'' \\ \pm a'\mathfrak{E}\big(f'a_1[b'|f'b_{(1)}]\big)\otimes[a_{(2)}]*[b_{(2)}]\otimes\mathfrak{E}\big(a''[f''b_{(3)}]\big)b'',$$

where the first term is the sum over all decompositions $[b_\bullet]=[b_{(2)}]\otimes[b_{(3)}]$ into two tensor factors, and the second is the double sum over decompositions $[a_\bullet]=[a_1]\otimes[a_{(2)}]$ where the first tensor factor is of length 1 and decompositions $[b_\bullet]=[b_{(1)}]\otimes[b_{(2)}]\otimes[b_{(3)}]$ into three tensor factors, and where we recall from Remark 2.3 our convention that $\mathfrak{E}\big(f'a_1[b'|f'b_{(1)}]\big)=0$ when $b'=1$. The signs are those imposed by the Koszul convention. Explicitly, the first sum is

$$\textbf{(A.7)}\qquad \overbrace{\big(\mu_{A'}\otimes\mu_{\mathbf{B}A}\otimes\underbrace{\mu_{A''}(\mathfrak{E}(\mathrm{id}_{A''}\otimes\mathbf{B}f'')\otimes\mathrm{id}_{A''})\tau_{[b_{(3)}];a''}}_{T_{0,3}}\big)}^{T_0}\circ(\mathrm{id}_{A'\otimes A'}\otimes\mathrm{id}_{\mathbf{B}A}\otimes\Delta_{\mathbf{B}A}\otimes\mathrm{id}_{A''\otimes A''})\Pi$$

for $\tau_{[b_{(3)}];a''}$ the tensor shuffle taking the factors $[b_{(3)}]\otimes a'$ to $(-1)^{|[b_{(3)}]|\cdot|a'|}a'\otimes[b_{(3)}]$ and Π the permutation (2354) from (A.4), and the second is

$$\textbf{(A.8)}\qquad \overbrace{\big(\overbrace{\mu_{A'}(\mathrm{id}_{A'}\otimes\mathfrak{E}(f's\otimes s^{-1}\otimes\mathbf{B}f'))\tau_{b';[a_1]}}^{T_{1,1}}\otimes\mu_{\mathbf{B}A}\otimes\overbrace{\mu_{A''}(\mathfrak{E}(\mathrm{id}_{A''}\otimes\mathbf{B}f'')\otimes\mathrm{id}_{A''})\tau_{[b_{(3)}];a''}}^{T_{1,3}}\big)}^{T_1} \\ \circ\big(\mathrm{id}_{A'}\otimes(\mathrm{id}_{A'}-\epsilon_{A'})\otimes((\mathrm{pr}_1\otimes\mathrm{id}_{\mathbf{B}A})\otimes(\mathrm{id}_{\mathbf{B}A}\otimes\mathrm{id}_{\mathbf{B}A})\otimes(\epsilon_{\mathbf{B}A}\otimes\mathrm{id}_{\mathbf{B}A}))\Delta^{[3]}_{\mathbf{B}A\otimes\mathbf{B}A}\otimes\mathrm{id}_{A''\otimes A''}\big)\Pi,$$

where $\tau_{b';[a_1]}$ is again a tensor transposition.

Concatenations $\Delta_{\leftarrow}$ (as defined in Definition 1.2) have been omitted in the statement and proof to aid legibility (so far as possible).

Proof We first establish the formula modulo 2. The composition of the first two maps in (A.3) takes a pure tensor $a'[a_\bullet]a''\otimes b'[b_\bullet]b''$ to $(a'\otimes b')\otimes\nabla\big([a_\bullet]\otimes[b_\bullet]\big)\otimes(a''\otimes b'')$. Recall $\nabla\big([a_\bullet]\otimes[b_\bullet]\big)$ is the signed sum of terms $[\alpha\otimes\beta]_\bullet := [\alpha_1\otimes\beta_1|\cdots|\alpha_\ell\otimes\beta_\ell]$ where each $\alpha_j\otimes\beta_j$ is either $a_m\otimes 1$ or $1\otimes b_m$ for some m, and the indices of the a-arguments appear in increasing order as one encounters them from left to right, as do the indices of the b-arguments. The map $\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})$ first breaks each term $[\alpha \otimes \beta]_\bullet$ into three factors $[\alpha \otimes \beta]_{(1)} = [\alpha_1 \otimes \beta_1 | \cdots | \alpha_p \otimes \beta_p]$, $[\alpha \otimes \beta]_{(2)}$, and $[\alpha \otimes \beta]_{(3)}$ via $\Delta^{[3]}_{\mathbf{B}(A^{\otimes 2})}$, and applies f' to each α and β in the first block and f'' to each in the third block. The verification for each of the three tensor factors will run in parallel.

Now there is a case distinction to be made. Write $\alpha'_j = f'\alpha_j$ and so on. If $a' \otimes b' \in \overline{A' \otimes A'}$ and $a'' \otimes b'' \in \overline{A'' \otimes A''}$, then by (1.17), $\mathbf{B}(\Phi_{A'}, \Phi_A, \Phi_{A''})$ takes $(a' \otimes b') \otimes \nabla([a_\bullet] \otimes [b_\bullet]) \otimes (a'' \otimes b'')$ to

$$\textbf{(A.9)} \quad \pm t_{A'}\Phi_{A'}[a' \otimes b' | \alpha'_1 \otimes \beta'_1 | \cdots | \alpha'_p \otimes \beta'_p] \otimes [\alpha \otimes \beta]_{(2)} \otimes t_{A''}\Phi_{A''}[\alpha''_1 \otimes \beta''_1 | \cdots | \alpha''_q \otimes \beta''_q | a'' \otimes b''],$$

where $p + \ell([\alpha \otimes \beta]_{(2)}) + q = \ell(a_\bullet) + \ell(b_\bullet)$. If instead $a' \otimes b' \in \operatorname{im} \eta_{A' \otimes A'} = k$, then the associated clause $\Upsilon' = \mathrm{id}_k \otimes \epsilon$ from (1.17) applies instead, so the first tensor factor in (A.9) is replaced with $a'b'$, and $p = 0$; similarly, if $a'' \otimes b'' \in \operatorname{im} \eta_{A'' \otimes A''}$, then instead the third tensor factor in (A.9) is $a''b''$, and $q = 0$. In either case, these simpler factors agree with those given in (A.6), (A.7), and (A.8) by our observations and conventions from Remark 2.3 on the vanishing of $\mathbf{E}$, so for the remainder of the verification we may assume $a' \otimes b' \in \overline{A' \otimes A'}$ and $a'' \otimes b'' \in \overline{A'' \otimes A''}$.

At this point it becomes necessary to recall the unsigned formula [10, (4.2)] for $t_A\Phi_A$

$$\textbf{(A.10)} \qquad t_A\Phi_A[\alpha \otimes \beta]_\bullet = \sum_\sigma \pm \alpha_1 \mathbf{E}[\alpha_2 | \beta_\bullet] \cdots \mathbf{E}[\alpha_n | \beta_\bullet] \beta_n,$$

where the sum is over those permutations σ separately preserving the orders of the indices of the α-arguments and those of the β-arguments, and such that, moreover, at every point, reading left to right, one has encountered the symbol α more often than β. The key point in simplifying this formula in the present context is that $\mathbf{E}[\alpha_j | \beta_\bullet]$ vanishes whenever one of the β-letters other than β_n is 1, by the convention set in Remark 2.3.

Thus in order for $t_{A'}\Phi_{A'}([a' \otimes b'] \otimes [\alpha' \otimes \beta']_\bullet)$ to be nonzero, one of two things must happen. The first possibility is that $p = \ell[\alpha' \otimes \beta']_\bullet = 0$, so that $t_{A'}\Phi_{A'}[a' \otimes b'] = \pm a'b'$. The other possibility is that $b', \beta'_1, \ldots, \beta'_{p-1}$ are all non-1. Particularly, if $b' = 1$, then $\pm a'b'$ is again the only term, agreeing with (A.6), (A.7), and (A.8), so from now on we will assume $b' \in \overline{A'}$. Since for each j one of α'_j and β'_j is 1, it follows that $\alpha'_1 = \cdots = \alpha'_{p-1} = 1$. As $\mathbf{E}_{1,0}([1] \otimes []) = 1$ and otherwise $\mathbf{E}_{1,\bullet}([1] \otimes [\beta_\bullet]) = 0$ by the conventions of Remark 2.3, for a term in (A.10) to be nonzero one needs $\alpha'_p \neq 1$, so that $\beta'_p = 1$. Thus the only relevant terms have

$$[\alpha' \otimes \beta']_\bullet = [1 \otimes f'b_1 | \cdots | 1 \otimes f'b_{p-1} | f'a_1 \otimes 1],$$
$$t_{A'}\Phi_{A'}([a' \otimes b'] \otimes [\alpha' \otimes \beta']_\bullet) = \pm a'\mathbf{E}_{1,p}([f'a_1] \otimes [b' | f'b_{(1)}]),$$

where $[f'b_{(1)}] = [f'b_1 | \cdots | f'b_{p-1}]$.

Similarly, in order that $t_{A''}\Phi_{A''}[\alpha''_1 \otimes \beta''_1 | \cdots | \alpha''_q \otimes \beta''_q | a'' \otimes b'']$ be nonzero, the arguments β''_j must be non-1, so each α''_j is 1. Thus the relevant tensor factor is

$$t_{A''}\Phi_{A''}([1 \otimes f''b]_{(3)} \otimes [a'' \otimes b'']) = \pm \mathbf{E}_{1,q}([a''] \otimes [f''b_{(3)}])b'',$$

which vanishes by convention if $a''=1$. As for the middle tensor factor, we know from Theorem 3.8 that $\Phi\circ\nabla=\mu_{\mathbf{B}A}$. Combining our descriptions of the three tensor factors, we have established (A.6).

It remains to determine the signs. The sign for the permutation Π of (A.4) is by definition the Koszul permutation sign. For the composition of the second two maps, recall from (1.17) that before involving $\Phi_{A'}$ and $\Phi_{A''}$, the map $\mathbf{B}(\Phi_{A'},\mu_{\mathbf{B}A},\Phi_{A''})$ first breaks $[\alpha\otimes\beta]_\bullet=\nabla\big([a_\bullet]\otimes[b_\bullet]\big)$ into three chunks via $\Delta^{[3]}_{\mathbf{B}(A\otimes A)}$. As ∇ is a DGC map $\mathbf{B}A\otimes\mathbf{B}A\to\mathbf{B}(A\otimes A)$ by Example 1.5, we equally well have $\Delta^{[3]}_{\mathbf{B}(A\otimes A)}\nabla=\nabla^{\otimes 3}\Delta^{[3]}_{\mathbf{B}A\otimes\mathbf{B}A}$. Applying the definition (1.17), suppressing concatenations $\Delta_{\leftarrow}$, gives

$$\textbf{(A.11)}\quad \mathbf{B}(\Phi_{A'},\Phi_A,\Phi_{A''})(\mathrm{id}_{A'\otimes A'}\otimes\nabla\otimes\mathrm{id}_{A''\otimes A''})$$
$$=\underbrace{\big(\overbrace{t_{A'}\Phi_{A'}(s^{-1}_{A'\otimes A'}\otimes\mathbf{B}(f'\otimes f')\nabla)}^{Q_1}\otimes\Phi_A\nabla\otimes\overbrace{t_{A''}\Phi_{A''}(\mathbf{B}(f''\otimes f'')\nabla\otimes s^{-1}_{A''\otimes A''})}^{Q_3}\big)}_{Q}$$
$$\circ(\mathrm{id}_{A'\otimes A'}\otimes\Delta^{[3]}_{\mathbf{B}A\otimes\mathbf{B}A}\otimes\mathrm{id}_{A''\otimes A''}).$$

We will evaluate Q on terms

$$(a'\otimes b'\otimes[a_{(1)}]\otimes[b_{(1)}])\otimes([a_{(2)}]\otimes[b_{(2)}])\otimes([a_{(3)}]\otimes[b_{(3)}]\otimes a''\otimes b'')$$

and compare with the evaluations of T_0 and T_1 from (A.7) and (A.8), recalling from the discussion of nonzero values of $t_{A'}\Phi_{A'}$ and $t_{A''}\Phi_{A''}$ in the previous paragraph that the only nonzero terms in (A.11) arise when $[a_{(3)}]=[]$ is of length 0 and either we have $[a_{(1)}]=[b_{(1)}]=[]$ of length 0 as well or else $[a_{(1)}]=[a_1]$ is of length 1.

As each of the three tensor factors of Q, of T_0, and of T_1 is of degree zero, it will be enough to compare these factors separately. The middle tensor factor $\Phi_A\nabla$ of Q is $\mu_{\mathbf{B}A}$ by Theorem 3.8, as in T_0 and T_1. For the first and third factors, we will need to be explicit about the sign for Φ. The signed version of formula (A.10) for $(\Phi_A)_{(n)}=t_A\circ(\Phi_A)|_{\mathbf{B}_nA}\circ(s^{-1}_{A\otimes A})^{\otimes n}$ is

$$\textbf{(A.12)}\qquad (\Phi_A)_{(n)}:=(-1)^{n-1}\sum_\sigma\mu_A^{[\bullet]}\Big(\mathrm{id}_A\otimes\bigotimes_m E_{\ell_m}\otimes\mathrm{id}_A\Big)\sigma,$$

where the sum is over shuffles σ as in (A.10); thus, the sequence (ℓ_m) gives a partition of $n-1$, apportioning subsequences of $\beta_1,\dots,\beta_{n-1}$ as arguments of $\mathbf{E}[\alpha_2|-]$ through $\mathbf{E}[\alpha_n|-]$, and E_ℓ is $\mathbf{E}_{1,\ell}(s^{-1})^{\otimes 1+\ell}$ as in Notation 2.2; similar arguments hold for $\Phi_{A'}$ and $\Phi_{A''}$.

To interpret Q_1, note that $\ell[a_{(1)}]$ is either 0 or 1. In the case $[a_{(1)}]=[]$, the only term we need to worry about is $t_{A'}\Phi_{A'}s^{-1}_{A'\otimes A'}=(\Phi_{A'})_{(1)}$, which is $\mu_{A'}$ by Definition 3.4(1). This is the first factor of T_0 from (A.7). When $[a_{(1)}]=[a_1]$, we have seen from the discussion above that the only terms on which $t_{A'}\Phi_{A'}$ will potentially not vanish are those of the form $[a'\otimes b'|1\otimes f'b]_{(1)}\otimes[f'a_1\otimes 1]$. Here the notation $[a'\otimes b'|1\otimes f'b]_{(1)}$ means some initial subword $[a'\otimes b'|1\otimes f'b_1|\cdots|1\otimes f'b_\ell]$, which is $[a'\otimes b']$ if $\ell=0$.

A term of $\nabla\big([a_1][b_{(1)}]\big)$ leading to a term of this form is created by the operation

$$\upsilon\colon[a_1]\otimes[b_{(1)}]\mapsto[a_1\otimes 1|1\otimes b]_{(1)}$$

followed by the shuffle $\tau = \tau_{[a_1\otimes 1];[1\otimes b]_{(1)}}$ moving $[a_1\otimes 1]$ past each $[1\otimes b_j]$, resulting in

$$x := \pm[a'\otimes b']\otimes[1\otimes f'b]_{(1)}\otimes[f'a_1\otimes 1].$$

To determine the sign explicitly, (A.12) tells us we should rewrite things in terms of the operations E_ℓ. When we apply $\Phi_{A'}$ to x, we have seen above that the only nonzero term is $\pm a'E_p(a_1;b',b_{(1)})$, which comes from the summand in (A.12) corresponding to the shuffle

$$\sigma'\colon (a'\otimes b')\otimes\bigotimes_{j=1}^{p-1}(1\otimes f'b_j)\otimes(f'a_1\otimes 1)\mapsto(-1)^{|a_1|\cdot(|b'|+|b_{(1)}|)}a'\otimes 1^{\otimes p-1}\otimes f'a_1\otimes b'\otimes f'b_{(1)}\otimes 1.$$

If we write $\varkappa$ for the map omitting tensor factors 1, then this nonzero term can be written as

$$\mu'_A(\mathrm{id}_{A'}\otimes E_p)\tau_{b'\otimes b_{(1)};a_1}\varkappa(s^{-1}_{A'\otimes A'})^{\otimes p+1}x,$$

where $\tau_{b'\otimes b_{(1)};a_1}$ is the tensor shuffle moving a_1 past $b'\otimes b_{(1)}$. Since $(s^{-1})^{\otimes n}s^{\otimes n}$ is multiplication by $(-1)^{\binom{n}{2}}$, we conclude from (A.12) that on $(A'\otimes A')\otimes\mathbf{B}_1A\otimes\mathbf{B}_pA$, we have

$$\begin{aligned}
Q_1 &= t_{A'}\Phi_{A'}\circ(s^{-1}_{A'\otimes A'}\otimes\mathbf{B}(f')^{\otimes 2}\tau\upsilon)\\
&= (-1)^{\binom{p+1}{2}}(\Phi_{A'})_{(p+1)}s^{\otimes 1+p}_{A'\otimes A'}\circ(s^{-1}_{A'\otimes A'}\otimes\mathbf{B}(f')^{\otimes 2}\tau\upsilon)\\
&= (-1)^{\binom{p+1}{2}}\big((-1)^p\mu_{A'}(\mathrm{id}_{A'}\otimes\mathbf{E}_{1,p}(s^{-1}_{A'})^{\otimes 1+p})\tau_{b'\otimes b_{(1)};a_1}\varkappa\big)(-1)^p(\mathrm{id}_{A'\otimes A'}\otimes s^{\otimes p}_{A'\otimes A'}\mathbf{B}(f')^{\otimes 2}\tau\upsilon)\\
&= (-1)^{\binom{p+1}{2}}\mu_{A'}\big(\mathrm{id}_{A'}\otimes\mathfrak{E}(\mathrm{id}_{A'}\otimes(s^{-1}_{A'})^{\otimes p})\big)\tau_{b'\otimes b_{(1)};a_1}\varkappa(\mathrm{id}_{A'\otimes A'}\otimes s^{\otimes p}_{A'\otimes A'}\mathbf{B}(f')^{\otimes 2}\tau\upsilon),
\end{aligned}$$

where the accounting of signs is as follows: the $(-1)^{\binom{p+1}{2}}$ comes from $(\Phi_{A'})_{(1+p)} := t_{A'}\Phi_{A'}s^{\otimes 1+p}_{A'\otimes A'}$ in (A.12), the first $(-1)^p$ is the leftmost term of the right-hand side of (A.12), and the second $(-1)^p$ comes from moving $s^{\otimes p}_{A'\otimes A'}$ past $s^{-1}_{A'\otimes A'}$.

We must compare the sign of the value of Q_1 on $\big([a'\otimes b']\otimes[a_1|b_{(1)}]\big)$ with that of the first factor $T_{1,1} = \mu_{A'}(\mathrm{id}_{A'}\otimes\mathfrak{E}(f's\otimes s^{-1}\otimes\mathbf{B}f'))\tau_{b';[a_1]}$ from (A.8). This is done symbolically by determining which tensor factors are moved past which others in the computation. For these purposes, the factors $\mu_{A'}$, $\mathrm{id}_{A'}$, $\mathfrak{E}$, $\varkappa$, f', $\mathbf{B}(f'\otimes f')$, and υ of degree 0 are invisible, as are the tensor factors 1 of degree 0 produced by υ and deleted by $\varkappa$. In Figure 2 below, Q_1 is in the left column and $T_{1,1}$ in the right, and in each row of each column, operations are listed on the left and arguments on the right, the result of applying the operations appearing on the right in the following row. As usual, moving two symbols y and z past each other incurs the sign $(-1)^{|y|\cdot|z|}$, and the rule goes equally for $y = s^{\pm 1}$.

In determining whether the signs agree, we can remove cancelling signs within one diagram or pairs of matching signs in both diagrams, referring to this as "cancellation" either way. Then two crossings cancel if both involve transposing the same symbols y and z, or if one crossing transposes s and z while the other transposes s^{-1} and z. After cancellation, the only signs remaining are $(-1)^{|s^{-1}|\cdot|s^{-1}|} = -1$, arising from the crossing marked with a dot on the right, and the underlined

$$(-1)^{\binom{p+1}{2}}\cdot|(s^{-1})^{\otimes p-1}s^{\otimes p-1}| = (-1)^{\binom{p+1}{2}+\binom{p-1}{2}} = (-1)^{p^2-p+1} = -1$$

at the bottom on the left, so the operations in question are indeed the same.

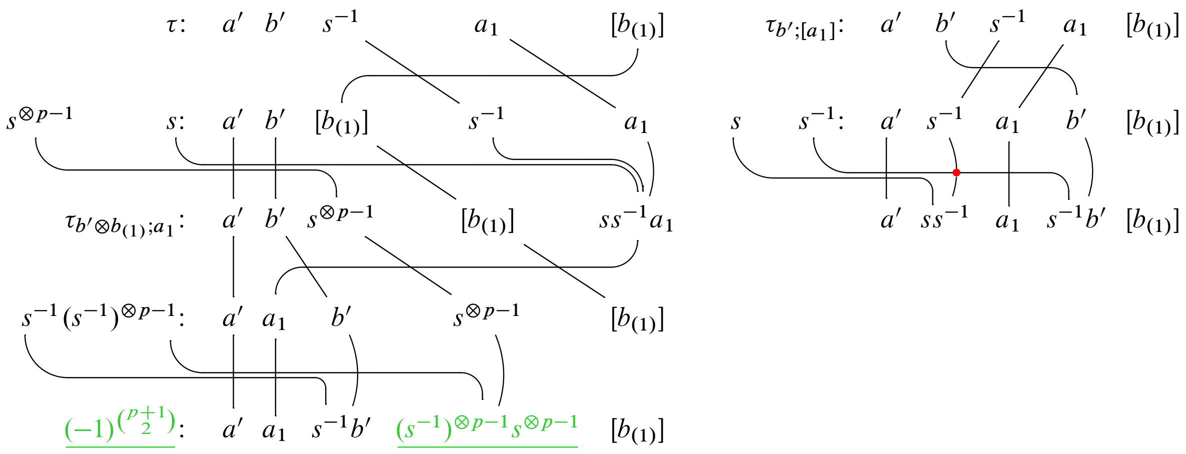


Figure 2

As for Q_3 and $T_{0,3} = T_{1,3}$ (from (A.11), (A.7), and (A.8)), we have seen above that the terms on which the $\Phi_{A''}$ in Q_3 is potentially nonzero are those of the form $[1 \otimes f''b]_{(3)} \otimes [a'' \otimes b'']$. In particular, $[a_{(3)}] = []$. Recall that we write $q = \ell[b_{(3)}]$. We see the many 1 factors arising because $\nabla\big([] \otimes [b_{(3)}]\big) = [1 \otimes b]_{(3)}$ contribute nothing to the iterated $\mu_{A''}$-product, so we may omit them with the map $\varkappa$. When they are omitted, the shuffle involved in the nonvanishing summand of (A.12) simplifies to the shuffle $\tau_{f''b_{(3)};a''}$ switching the A'' tensor factors containing a'' and $f''b_{(3)}$ in $f''b_{(3)} \otimes a'' \otimes b''$. Thus on $\mathbf{B}_q A \otimes A'' \otimes A''$, we can expand Q_3 as

$$\begin{aligned} Q_3 &= t_{A''}\Phi_{A''} \circ (\mathbf{B}(f'')^{\otimes 2} \otimes s^{-1}_{A''\otimes A''}) \\ &= (-1)^{\binom{q+1}{2}} (\Phi_{A''})_{(1+q)} \circ s^{\otimes 1+q}_{A''\otimes A''} \circ (\mathbf{B}(f'')^{\otimes 2}\nabla \otimes s^{-1}_{A''\otimes A''}) \\ &= (-1)^{\binom{q+1}{2}} (-1)^q \mu_{A''}(\mathbf{E}_{1,q}(s^{-1}_{A''})^{\otimes 1+q} \otimes \mathrm{id}_{A''})\tau_{f''b_{(3)};a''}\varkappa \circ (s^{\otimes q}_{A''\otimes A''}\mathbf{B}(f'')^{\otimes 2}\nabla \otimes \mathrm{id}_{A''\otimes A''}) \\ &= (-1)^{\binom{q+1}{2}-q} \mu_{A''}\big(\mathfrak{E}(\mathrm{id}_{A''} \otimes (s^{-1}_{A''})^{\otimes q}) \otimes \mathrm{id}_{A''}\big)\tau_{f''b_{(3)};a''}\varkappa(s^{\otimes q}_{A''\otimes A''}\mathbf{B}(f'')^{\otimes 2}\nabla \otimes \mathrm{id}_{A''\otimes A''}), \end{aligned}$$

where, as in the analysis of Q_1, the $(-1)^{\binom{q+1}{2}}$ comes from $(\Phi_{A''})_{q+1} = t_{A''}\Phi_{A''}s^{\otimes q+1}_{A''\otimes A''}$ and the $(-1)^q$ is the leftmost factor of the right-hand side of (A.12). We may simplify the exponent of -1 by observing that

$$\binom{q+1}{2} - q = \binom{q+1}{2} - \binom{q}{1} = \binom{q}{2}.$$

In Figure 3 the crossings for Q_3 appear on the left and that for

$$T_{0,3} = T_{1,3} = \mu_{A''}(\mathfrak{E}(\mathrm{id}_{A''} \otimes \mathbf{B}f'') \otimes \mathrm{id}_{A''})\tau_{[b_{(3)}];a''}$$

appears on the right.

When matching crossings are cancelled, the only signs remaining are the underlined $(-1)^{\binom{q}{2}}$ and $|(s^{-1})^{\otimes q}s^{\otimes q}| = (-1)^{\binom{q}{2}}$ on the left, so the operations are equal, concluding the proof. □

With this more explicit formulation of $\tilde{\mu}$, we are able to relate it to the product on an HGA:

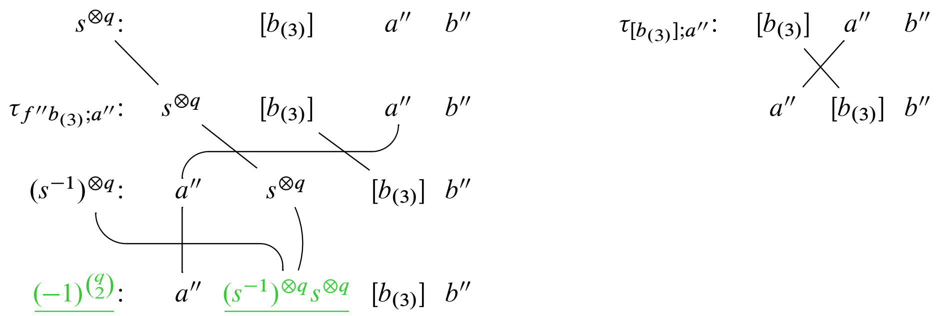
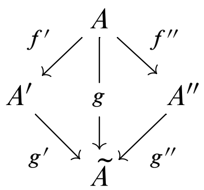


Figure 3

Theorem A.13 *Given a commutative diagram*



of HGA *homomorphisms, the natural cochain map*

$$\textbf{(A.14)}\qquad \xi := \mu^{[3]}_{\widetilde{A}}(g'\otimes \eta_A\epsilon_{\mathbf{B}A}\otimes g'')\colon \mathbf{B}(A',A,A'')\to \widetilde{A},\quad a'[a_\bullet]a''\mapsto g'(a')\eta_{\widetilde{A}}\epsilon_{\mathbf{B}A}[a_\bullet]g''(a''),$$

induces a map in cohomology multiplicative with respect to the product on $H^*\mathbf{B}(A',A,A'')$ induced by the product defined by Theorem A.1 and the expected product on $H^*(\widetilde{A})$.

Proof By Theorem A.1, $\mathbf{B}(g',g,g'')$ is a multiplicative cochain map $\mathbf{B}(A',A,A'')\to\mathbf{B}(\widetilde{A},\widetilde{A},\widetilde{A})$, so it will be enough to show $\xi\colon\mathbf{B}(\widetilde{A},\widetilde{A},\widetilde{A})\to\widetilde{A}$ is a cochain map inducing a multiplicative map in cohomology. In other words, we may as well start by assuming $A'=A=A''=\widetilde{A}$ and $f'=f''=g'=g''=\mathrm{id}_A$. Let us then agree to write $\mu=\mu_A$ and $\mathrm{id}=\mathrm{id}_A$ and $\epsilon=\eta_A\epsilon_{\mathbf{B}A}$, so we can restate (A.14) as

$$\textbf{(A.15)}\qquad \xi=\mu^{[3]}(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\colon c'[c_\bullet]c''\mapsto c'\epsilon[c_\bullet]c''.$$

It is trivial to check this is a cochain map.

To show the induced map $H^*(\xi)$ in cohomology is multiplicative, we show that

$$\textbf{(A.16)}\qquad Dh=\xi\widetilde{\mu}-\mu\xi^{\otimes 2}\colon\mathbf{B}(A,A,A)^{\otimes 2}\to A$$

for a certain homotopy $h\colon\mathbf{B}(A,A,A)^{\otimes 2}\to A$ to be produced momentarily. It will help to adapt (A.6), (A.7), and (A.8) to the present setup:

$$\textbf{(A.17)}\qquad \begin{aligned}\widetilde{\mu}=\Big(&\underset{a'b'}{\mu}\otimes\underset{\otimes[a_\bullet]*[b_{(2)}]}{\mu_{\mathbf{B}A}}\otimes\underset{\otimes\,\mathfrak{E}(a''\ \ [b_{(3)}])\ \ b''}{\mu(\mathfrak{E}(\mathrm{id}\otimes\mathrm{id}_{\mathbf{B}A})\otimes\mathrm{id})}\Big)\circ\Pi'\\ &+\Big(\underset{a'\otimes\mathfrak{E}(a_1\ \ [b'\ \mid b_{(1)}])}{\mu(\mathrm{id}\otimes\mathfrak{E}(s\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}A}))}\otimes\underset{\otimes[a_{(2)}]*[b_{(2)}]}{\mu_{\mathbf{B}A}}\otimes\underset{\otimes\,\mathfrak{E}(a''\ \ [b_{(3)}])\ \ b''}{\mu(\mathfrak{E}(\mathrm{id}\otimes\mathrm{id}_{\mathbf{B}A})\otimes\mathrm{id})}\Big)\circ\Pi'',\end{aligned}$$

where Π' and Π'' are preparatory tensor permutations. The permutations τ in particular have been absorbed into these without sign change because the operators between τ and Π in (A.7) and (A.8) are of degree zero. Here the value, up to sign (determined by the Koszul convention, but not written out), on the standard pure tensor $a'[a_\bullet]a'' \otimes b'[b_\bullet]b''$ is displayed below the function, to make the argument easier to follow, though in principle these values are optional. We will continue with this notation throughout the proof. Recall that if b' is in the image of the unit map η_A, then the $\mathfrak{C}$ factor in the second term is defined to vanish on bar-words containing it by Remark 2.3.

It is evident from (A.15) that $-\mu\xi^{\otimes 2}$ vanishes unless $\ell = \ell(a_\bullet)$ and $r = \ell(b_\bullet)$ are both zero, in which case it is

$$-\mu^{[6]}(\underset{a'}{\mathrm{id}} \otimes \eta_A \otimes \underset{a''}{\mathrm{id}} \otimes \underset{b'}{\mathrm{id}} \otimes \eta_A \otimes \underset{b''}{\mathrm{id}}) = -\underset{a'a''b'b''.}{\mu^{[4]}}$$

As $\mu_{\mathbf{B}A}$ is of degree 0 and $\epsilon_{\mathbf{B}A}$ annihilates $\mathbf{B}_{\geq 1}A$, the first term of $\xi\widetilde{\mu}$ vanishes according to (A.15) and (A.17) unless $\ell(a_\bullet) = \ell(b_{(2)}) = 0$, in which case it reduces to

$$\mu^{[3]}\big(\underset{a'b'}{\mu} \otimes \eta_A \otimes \mu(\mathfrak{C}(\underset{\mathfrak{C}(a''}{\mathrm{id}} \otimes \underset{[b_{(3)}])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \underset{b''.}{\mathrm{id}})\big) \circ \tau_{a'';b'}$$

If additionally $\ell(b_{(3)}) = 0 = r$, this further reduces to

$$\mu^{[3]}(\underset{a'b'}{\mu} \otimes \eta_A \otimes \mu(\underset{a''}{\mathrm{id}} \otimes \underset{b''.}{\mathrm{id}})) \circ \tau_{a'';b'} = \mu^{[4]} \circ \tau_{a'';b'}$$

The second term of $\xi\widetilde{\mu}$ vanishes unless $[a_\bullet] = [a]$ is of length 1 and $[b_{(2)}]$ is of length 0, in which case it contributes

$$\mu^{[4]}(\underset{a'}{\mathrm{id}} \otimes \mathfrak{C}(\underset{\mathfrak{C}(a_1}{s} \otimes \underset{[b'}{s^{-1}} \otimes \underset{|\, b_{(1)}])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \mathfrak{C}(\underset{\mathfrak{C}(a''}{\mathrm{id}} \otimes \underset{[b_{(3)}])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \underset{b''}{\mathrm{id}}) \circ \pi''$$

for π'' running over tensor permutations $a'[a]a'' \otimes b'[b_\bullet]b'' \mapsto a' \otimes [a]b'[b_{(1)}] \otimes a''[b_{(3)}] \otimes b''$. These π'' are the specialization of Π'' from (A.17) to the case $\ell(a_{(2)}) = 0$.

All told, on $\mathbf{B}_\ell(A, A, A) \otimes \mathbf{B}_r(A, A, A)$ one has

$$\textbf{(A.18)}\quad \xi\widetilde{\mu} - \mu_A\xi^{\otimes 2} = \begin{cases} \underset{a'b'a''b'}{\mu^{[4]}}\tau_{a'';b'} - \underset{a'a''b'b''}{\mu^{[4]}} & \text{if } \ell = 0 = r, \\ \mu^{[4]}(\underset{a'}{\mathrm{id}} \otimes \underset{b'}{\mathrm{id}} \otimes \underset{\mathfrak{C}(a''[b_\bullet])}{\mathfrak{C}} \otimes \underset{b''}{\mathrm{id}})\tau_{a'';b'} & \text{if } \ell = 0 < r, \\ \mu^{[4]}(\underset{a'}{\mathrm{id}} \otimes \mathfrak{C}(\underset{\mathfrak{C}(a_1}{s} \otimes \underset{[b'}{s^{-1}} \otimes \underset{|\, b_{(1)}])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \mathfrak{C}(\underset{\mathfrak{C}(a''}{\mathrm{id}} \otimes \underset{[b_{(3)}])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \underset{b''}{\mathrm{id}}) \circ \pi'' & \text{if } \ell = 1, \\ 0 & \text{if } \ell \geq 2. \end{cases}$$

The promised homotopy is given as

$$\textbf{(A.19)}\quad h := \mu^{[4]}(\underset{a'}{\mathrm{id}} \otimes \underset{\epsilon[a_\bullet]}{\epsilon} \otimes \mathfrak{C}(\underset{\mathfrak{C}(a''}{\mathrm{id}} \otimes \underset{[b'}{s^{-1}} \otimes \underset{|\, b_\bullet])}{\mathrm{id}_{\mathbf{B}A}}) \otimes \underset{b''.}{\mathrm{id}})$$

To show Dh agrees with $\xi\widetilde{\mu} - \mu_A\xi^{\otimes 2}$, we follow the same case distinctions.

If $\ell = \ell(a_\bullet) \geq 2$, then h and hence dh vanish since $\epsilon[a_\bullet] = 0$, and since the differential on $\mathbf{B}(A, A, A)$ takes $\mathbf{B}_{\geq 2}(A, A, A)$ into $\mathbf{B}_{\geq 1}(A, A, A)$, so does $hd_{\mathbf{B}(A,A,A)^{\otimes 2}}$.

If $\ell = 1$, then h and hence dh vanish as before, but hd need not, by the formula for the "external" differential $d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)} = d_{\mathbf{B}(A,A,A)} - d_\otimes$ given in (1.21), since the outer two operators reduce the length of the bar-word in $a'[a]a''$ to zero, taking it respectively to $\pm a'a[]a''$ and $\pm a'[]aa''$. Thus $hd_\otimes$ and $h(\mathrm{id}_{\mathbf{B}(A,A,A)} \otimes d_{\mathbf{B}(A,A,A)})$ vanish, but for $h(d_{\mathbf{B}(A,A,A)} \otimes \mathrm{id}_{\mathbf{B}(A,A,A)})$, plugging (1.21) into (A.19), we get

(A.20)
$$
\begin{aligned}
&\mu^{[4]}(\mathrm{id} \otimes \epsilon \otimes \mathfrak{E}(\mathrm{id} \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) \\
&\qquad \circ \big((\mu(\mathrm{id} \otimes s) \otimes \eta_{\mathbf{B}A} \otimes \mathrm{id}) \otimes \mathrm{id}_{\mathbf{B}(A,A,A)} - \mathrm{id} \otimes \eta_{\mathbf{B}A} \otimes \mu(s \otimes \mathrm{id}) \otimes \mathrm{id}_{\mathbf{B}(A,A,A)}\big) \\
&\qquad\quad\;\; a' \quad a \qquad [] \qquad a'' \qquad b'[b_\bullet]b'' \qquad a' \qquad [] \qquad a \quad a'' \qquad b'[b_\bullet]b'' \\
&= -\mu^{[4]}(\mathrm{id} \otimes s \otimes \mathfrak{E}(\mathrm{id} \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) + \mu^{[3]}\big(\mathrm{id} \otimes \mathfrak{E}(\mu(s \otimes \mathrm{id}) \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}\big) \\
&\qquad a' \quad a \quad \mathfrak{E}\big(a'' \quad [b' \;\; | \; b_{(1)}]\big) \quad b'' \qquad\qquad a' \quad \mathfrak{E}\big(\; a \quad a'' \quad [b' \;\; | \; b_{(1)}]\big) \quad b'',
\end{aligned}
$$

the change in signs coming in both terms from the factor s of degree 1 coming from $d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)}$ being moved past the factor s^{-1} of degree -1 in the argument of $\mathfrak{E}$. By the Cartanesque formula (2.5), the second term on the right-hand side of (A.20) can be replaced by

$$
\begin{aligned}
&\mu^{[4]}(\mathrm{id} \otimes (\mathfrak{E} \otimes \mathfrak{E})\pi(s \otimes \mathrm{id} \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) \\
&\qquad a' \quad (\mathfrak{E} \otimes \mathfrak{E})\pi\big(a \quad a'' \quad [b' \; | \; b_\bullet]\big) \quad b'',
\end{aligned}
$$

where π runs over shuffles $a \otimes a'' \otimes [b'|b_\bullet] \mapsto a[b'|b_\bullet]_{(1)} \otimes a''[b'|b_\bullet]_{(2)}$ and $[b'|b_\bullet]_{(1)} \otimes [b'|b_\bullet]_{(2)}$ is a deconcatenation of $[b'|b_\bullet]$. There is a case distinction here: The deconcatenation yielding $[] \otimes [b'|b_\bullet]$ recovers

$$
\begin{aligned}
&\mu^{[4]}(\mathrm{id} \otimes s \otimes \mathfrak{E}(\mathrm{id} \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) \\
&\qquad a' \quad a \quad \mathfrak{E}\big(a'' \quad [b' \; | \; b_\bullet]\big) \quad b'',
\end{aligned}
$$

cancelling the first term of (A.20). Otherwise, b' is assigned to the first word in the deconcatenation, so the value is $[b'|b_{(1)}] \otimes [b_{(3)}]$, and one gets operations

$$
\begin{aligned}
&(\mathrm{id} \otimes \mathfrak{E}(s \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathfrak{E}(\mathrm{id} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) \circ \pi'' \\
&\;\; a' \quad \mathfrak{E}\big(a_1 \quad [b' \; | \; b_{(1)}]\big) \quad \mathfrak{E}\big(a'' \quad [b_{(3)}]\big) \quad b''
\end{aligned}
$$

agreeing with the case $\ell = 1$ of (A.18). Because π preserves the relative positions of s and s^{-1}, it does not incur a sign change in moving past them to become π''.

For the $\ell = 0$ cases, we write the differential on $\mathbf{B}(A, A, A)^{\otimes 2}$ as the sum of the "internal" tensor differential $d_\otimes$ and the "external" differentials $\mathrm{id}_{\mathbf{B}(A,A,A)} \otimes d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)}$ and $d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)} \otimes \mathrm{id}_{\mathbf{B}(A,A,A)}$. We may omit the second external differential in consideration of $hd_{\mathbf{B}(A,A,A)}$ since $d_{\mathbf{B}(A,A,A)}$ vanishes on $\mathbf{B}_0(A, A, A)$, and when $r = 0$ we may omit the first external differential for the same reason. Write $D_\otimes$ for the differential on $\mathrm{Hom}((A \otimes \mathbf{B}A \otimes A)^{\otimes 2}, A)$, where the domain is $\mathbf{B}(A, A, A)^{\otimes 2}$ equipped with the tensor differential $d_\otimes$. Thus $D_\otimes h = d_A h + hd_\otimes$. Because composition is a cochain map, as are μ_A, id_A, $\epsilon_{\mathbf{B}A}$, $\mathrm{id}_{\mathbf{B}A}$, and s^{-1}, we have

(A.21)
$$
\begin{aligned}
D_\otimes h &= \mu^{[4]}(\mathrm{id} \otimes \eta_{\mathbf{B}A} \otimes (D_\otimes \mathfrak{E})(\mathrm{id} \otimes s^{-1} \otimes \mathrm{id}_{\mathbf{B}A}) \otimes \mathrm{id}) \\
&\qquad\quad a' \qquad\qquad (D_\otimes \mathfrak{E})\big(a'' \quad [b' \; | \; b_\bullet]\big) \quad b''.
\end{aligned}
$$

Equation (2.6) shows that for fixed r, the function $D_\otimes\mathfrak{E}\colon \bar{A}\otimes \mathbf{B}_{1+r}A\to A$ is given by

(A.22)
$$D_\otimes\mathfrak{E} = \underset{b'\quad \mathfrak{E}(a''[b_\bullet])}{\mu(s\otimes\mathfrak{E})(12)} + \underset{\mathfrak{E}(a''d_{\mathrm{ext}}[b'|b_\bullet])}{\mathfrak{E}(\mathrm{id}\otimes d_{\mathrm{ext}})} - \underset{\begin{cases}\mathfrak{E}(a''[])b' & r=0,\\ \mathfrak{E}(a''[b'|b_1|b_\bullet|b_{r-1}])b_r & r>0.\end{cases}}{\mu(\mathfrak{E}\otimes s)},$$

The second term also vanishes when $r=0$, and then plugging (A.22) into (A.21) yields

(A.23)
$$\begin{aligned} D_\otimes h &= \mu^{[4]}(\underset{a'}{\mathrm{id}}\otimes(\underset{b'}{s}\otimes\underset{\mathfrak{E}(a''[])}{\mathfrak{E}})\,\tau_{a'';[b']}(\mathrm{id}\otimes s^{-1}\otimes\eta_{\mathbf{B}A})\otimes\underset{b''}{\mathrm{id}})\\ &= \underset{a'b'a''b''}{\mu^{[4]}}\circ\tau_{a'';b'} - \mu^{[4]}(\underset{a'}{\mathrm{id}}\otimes(\underset{\mathfrak{E}(a''[])}{\mathfrak{E}}\otimes\underset{b'}{s})(\mathrm{id}\otimes s^{-1}\otimes\eta_{\mathbf{B}A})\otimes\underset{b''}{\mathrm{id}}) - \underset{a'a''b'b''}{\mu^{[4]}}. \end{aligned}$$

Here we use that $\mathfrak{E}=\mathrm{id}$ on $A\otimes\mathbf{B}_0A\cong A$, and the further simplification in the first term comes from the calculation

$$(s\otimes\mathrm{id})\tau_{a'';[b']}(\mathrm{id}\otimes s^{-1}) = \tau_{a'';b'}(\mathrm{id}\otimes s)(\mathrm{id}\otimes s^{-1}) = \tau_{a'';b'}.$$

The $\ell=0=r$ case of (A.18) agrees with (A.23), concluding that case.

For $\ell=0<r$, substituting (A.22) into (A.21) now gives instead

(A.24)
$$\begin{aligned} D_\otimes h = {}& \mu^{[4]}(\underset{a'}{\mathrm{id}}\otimes(\underset{b'}{s}\otimes\underset{\mathfrak{E}(a''[b_\bullet])}{\mathfrak{E}})\tau_{a'';[b']}(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}_rA})\otimes\underset{b''}{\mathrm{id}})\\ &+\mu^{[3]}(\underset{a'}{\mathrm{id}}\otimes\underset{\mathfrak{E}(a''[b'b_1|b_2|\cdots|b_r])}{\mathfrak{E}(\mathrm{id}\otimes s^{-1}\mu s^{\otimes 2}\otimes\mathrm{id}_{\mathbf{B}_{r-1}A})}(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{s^{-1}\bar{A}}\otimes\mathrm{id}_{\mathbf{B}_{r-1}A})\otimes\underset{b''}{\mathrm{id}})\\ &+\mu^{[3]}(\underset{a'}{\mathrm{id}}\otimes\underset{\mathfrak{E}(a''\ [b']\otimes d_{\mathrm{ext}}[b_\bullet])}{\mathfrak{E}(\mathrm{id}\otimes\mathrm{id}\otimes d_{\mathrm{ext}}|_{\mathbf{B}_rA})}(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}_rA})\otimes\underset{b''}{\mathrm{id}})\\ &-\mu^{[4]}(\underset{a'}{\mathrm{id}}\otimes(\underset{\mathfrak{E}(a''[b'|b_1|\cdots|b_{r-1}])}{\mathfrak{E}}\otimes\underset{b_r}{s})(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}_{r-1}A}\otimes\mathrm{id}_{s^{-1}\bar{A}})\otimes\underset{b''}{\mathrm{id}}). \end{aligned}$$

As for $h(\mathrm{id}_{\mathbf{B}(A,A,A)}\otimes d^{\mathrm{ext}}_{\mathbf{B}(A,A,A)})$, composing its definition from (1.21) with the expression for h from (A.19) yields

(A.25)
$$\begin{gathered} \mu^{[3]}(\underset{a'}{\mathrm{id}}\otimes\underset{\mathfrak{E}(a''\ [b'b_1|\cdots|b_r])}{\mathfrak{E}(\mathrm{id}\otimes s^{-1}\mu(\mathrm{id}\otimes s)\otimes\mathrm{id}_{\mathbf{B}_{r-1}A})}\otimes\underset{b''}{\mathrm{id}})\\ +\mu^{[3]}(\underset{a'}{\mathrm{id}}\otimes\underset{\mathfrak{E}(a''\ [b']\otimes d_{\mathrm{ext}}[b_\bullet])}{\mathfrak{E}(\mathrm{id}\otimes s^{-1}\otimes d_{\mathrm{ext}}|_{\mathbf{B}_rA})}\otimes\underset{b''}{\mathrm{id}})\\ -\mu^{[3]}(\underset{a'}{\mathrm{id}}\otimes\underset{\mathfrak{E}(a''\ [b'\ |\ b_1|\cdots|b_{r-1}])}{\mathfrak{E}(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}_{r-1}A})}\otimes\mu(\underset{b_r}{s}\otimes\underset{b''}{\mathrm{id}})). \end{gathered}$$

These three terms are the respective opposites of the second through fourth terms of (A.24), as transitioning from one ordering of symbols to the other involves moving s^{-1} past s in the second and fourth terms and past d_{ext} in the third, incurring each time a sign change of $(-1)^{-1\cdot 1}$. Thus these three pairs of terms cancel. It remains only to check that the uncancelled first term of (A.24) agrees with the $\ell=0<r$ clause

in (A.18). For this, note that id has degree 0, so we have $\tau_{a'';[b']}(\mathrm{id}\otimes s^{-1}) = (s^{-1}\otimes\mathrm{id})\tau_{a'';b'}$, and $\mathfrak{E}$ has degree zero as well, so

$$\begin{aligned}\mu^{[4]}(\mathrm{id}\otimes(s\otimes\mathfrak{E})\tau_{a'';[b']}(\mathrm{id}\otimes s^{-1}\otimes\mathrm{id}_{\mathbf{B}_r A})\otimes\mathrm{id}) &= \mu^{[4]}(\mathrm{id}\otimes(s\otimes\mathfrak{E})(s^{-1}\otimes\mathrm{id}\otimes\mathrm{id}_{\mathbf{B}_r A})\otimes\mathrm{id})\tau_{a'';b'}\\ &= \mu^{[4]}(\mathrm{id}\otimes\mathrm{id}\otimes\mathfrak{E}\otimes\mathrm{id})\tau_{a'';b'}.\end{aligned}$$ □

Remark A.26 We expect that if A', A, and A'' are extended HGAs, the product $\widetilde{\mu}$ of Theorem A.1 is the 2-component of a differential on $\mathbf{BB}(A', A, A'')$ making $\mathbf{B}(A', A, A'')$ an A_∞-algebra and making the map ξ of Theorem A.13 the extended 1-component $\Xi^{+}_{(1)}$ of an A_∞-map Ξ from $\mathbf{B}(A', A, A'')$ to $\widetilde{A}$, but we will leave the exploration of this possibility for another occasion.[9]

We now can use the new product to establish the particular version of the Eilenberg–Moore theorem we will need:

Theorem A.27 (Eilenberg–Moore with induced product) *Suppose the coefficient ring k is a principal ideal domain, and suppose given a pullback square*

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & E \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

of pointed topological spaces in which $E \to B$ is a Serre fibration, B and X are path connected, the action of $\pi_1(B)$ on the cohomology of the fiber F is trivial, and

(1) *each $H^n(F)$ is a finitely generated k-module, or*

(2) *each $H^n(B)$ and each $H^n(X)$ is a finitely generated k-module.*

Then there is a natural quasi-isomorphism

$$\xi\colon \mathbf{B}(C^*(X), C^*(B), C^*(E)) \to C^*(Y), \quad x[b_\bullet]e \mapsto \alpha^*(x) \smile \eta_{C^*(Y)}\epsilon_{\mathbf{B}C^*(E)}[b_\bullet] \smile \beta^*(e),$$

inducing a ring isomorphism

$$\mathrm{Tor}^*_{C^*(B)}(C^*(X), C^*(E)) \xrightarrow{\sim} H^*(Y)$$

with respect to the product on the domain induced by the product given in Theorem A.1.

We rely on the presentation of Gugenheim and May [15, Theorem 3.3] with some modification of hypotheses.

Proof We assume B and X are path connected to guarantee that the homotopy type of the fiber F is well defined and we only have to discuss one group $\pi_1(B)$; one otherwise needs a separate argument for each path component. This assumption then also implies $\pi_1(X)$ acts trivially on $H^*(F)$. Given

[9] Added in proof: this expectation has now been confirmed; see Franz [12] and Remark 0.2.

any proper projective resolution $P^\bullet$ of $C^*(X)$ as a $C^*(B)$-module (or resolution in the more general sense of Gugenheim and May [15, Definitions 1.1]), Gugenheim and May show the expected composite filtered map $\vartheta\colon P^\bullet \otimes_{C^*(B)} C^*(E) \to C^*(X) \otimes_{C^*(B)} C^*(E) \to C^*(Y)$ is a quasi-isomorphism [15, Theorem 3.3 and Corollary 3.5] under slightly different finiteness hypotheses, namely that k is Noetherian and the groups $H_n(X;\mathbb{Z})$ and $H_n(B;\mathbb{Z})$ are all finitely generated.

These hypotheses are used only to see the canonical maps $C^*(B) \otimes H^*(F) \to C^*(B; H^*(F))$ and $C^*(X) \otimes H^*(F) \to C^*(X; H^*(F))$ are quasi-isomorphisms, using a lemma [15, Lemma 3.2] asserting that if k is a commutative Noetherian ring, G an k-module, and C a chain complex over $\mathbb{Z}$ with each $H_n(C)$ finite, then $\mathrm{Hom}_{\mathbb{Z}}(C,k) \otimes_k G \to \mathrm{Hom}_{\mathbb{Z}}(C,G)$ is a quasi-isomorphism. The proof of this lemma uses only that $\mathbb{Z}$ is a principal ideal domain, and hence the same argument shows that if k is a principal ideal domain, C now a differential graded k-module, and G a k-module, then $\mathrm{Hom}_k(C,k) \otimes_k G \to \mathrm{Hom}_k(C,G)$ is a quasi-isomorphism. Thus, assuming k is a principal ideal domain, we can replace Gugenheim and May's hypothesis that each integral homology group $H_n(B;\mathbb{Z})$ and $H_n(X;\mathbb{Z})$ is finitely generated with the weaker hypothesis (2) that the (co)homology groups with coefficients in k are. If, alternatively, we assume (1) that the $H^n(F)$ is finitely generated over k, then again assuming k is a principal ideal domain, the decomposition of each $H^n(F)$ as a finite product of cyclic k-modules shows $C^*(B) \otimes H^n(F) \to C^*(B; H^n(F))$ and $C^*(X) \otimes H^n(F) \to C^*(X; H^n(F))$ are isomorphisms of differential graded k-modules.

To see the suppressed details in the proof of the multiplicativity of $H^*(\vartheta)$ with respect to the classical product on Tor [15, page 26], subdivide the vertical cohomological Eilenberg–Zilber map featuring along the upper-right of McCleary's diagram gesturing at such a proof [22, page 255] as

$$H^*(Y) \otimes H^*(Y) \xrightarrow{H^*(\Phi)} H^*(C^*(Y) \otimes C^*(Y)) \xrightarrow{i} H^*\big((C_*(Y) \otimes C_*(Y))^*\big) \xleftarrow{\mathrm{EZ}^*} H^*(Y \times Y).$$

Then there are evident horizontal maps subdividing the region into three rectangles that can be seen to commute on choosing maps between resolutions [15, Theorem 1.7] and expanding out the definition of the external product.

As k is a principal ideal domain, Proposition 1.19 implies that $\widetilde{C} = \mathbf{B}(C^*(X), C^*(B), C^*(B))$ gives a resolution $\widetilde{C} \to C^*(X)$ of $C^*(X)$ as a differential $C^*(B)$-module and $H^*(\widetilde{C})$ is the desired Tor. The map ϑ then specializes to our ξ. As C^* is a functor valued in HGAs, by Theorems A.1 and A.13 there is a natural cochain map $\widetilde{\mu}\colon \widetilde{C}^{\otimes 2} \to \widetilde{C}$ such that the quasi-isomorphism ξ is multiplicative up to homotopy in the sense that if μ is the cup product on $C^*(Y)$, then there exists $h\colon \widetilde{C} \to C^*(Y)$ with $Dh = \xi\widetilde{\mu} - \mu\xi^{\otimes 2}$. Thus the k-module isomorphism $H^*(\xi)\colon H^*(\widetilde{C}) \to H^*(Y)$ takes the induced product

$$H^*(\widetilde{C}) \otimes H^*(\widetilde{C}) \xrightarrow{\times} H^*(\widetilde{C} \otimes \widetilde{C}) \xrightarrow{H^*(\widetilde{\mu})} H^*(\widetilde{C})$$

on Tor to the cup product on $H^*(Y)$. Moreover, since the classical proof shows $H^*(\xi)$ is multiplicative with respect the standard product on Tor and the cup product on $H^*(Y)$, we conclude that $H^*(\widetilde{\mu}) \circ \times$ is the standard product. □

Remark A.28 The classical Eilenberg–Moore theorem obtains the ring structure on the domain through the external product and the Eilenberg–Zilber theorem, without reference to a cochain-level product on any complex computing Tor. We went to all this trouble because in Section 5, we need the fact that the product on Tor is induced by such a product in order to show certain maps of Tors are multiplicative and finally obtain Theorem 0.1.

References

[1] **T Barthel**, **J P May**, **E Riehl**, *Six model structures for DG-modules over DGAs: model category theory in homological action*, New York J. Math. 20 (2014) 1077–1159 MR

[2] **P F Baum**, *On the cohomology of homogeneous spaces*, Topology 7 (1968) 15–38 MR

[3] **C Berger**, **B Fresse**, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 135–174 MR

[4] **C Berger**, **J Huebschmann**, *Comparison of the geometric bar and W-constructions*, J. Pure Appl. Algebra 131 (1998) 109–123 MR

[5] **J M Boardman**, *Conditionally convergent spectral sequences*, from "Homotopy invariant algebraic structures" (J-P Meyer, J Morava, W S Wilson, editors), Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 49–84 MR

[6] **A Borel**, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. 57 (1953) 115–207 MR

[7] **J D Carlson**, *Products on Tor*, preprint (2023) arXiv 2311.16007

[8] **J D Carlson**, *A ring structure on Tor*, preprint (2023) arXiv 2306.04860 To appear in Forum Math. Sigma

[9] **H Cartan**, *La transgression dans un groupe de Lie et dans un espace fibré principal*, from "Colloque de topologie (espaces fibrés)", Thone, Liège (1951) 57–71 MR

[10] **M Franz**, *Homotopy Gerstenhaber algebras are strongly homotopy commutative*, J. Homotopy Relat. Struct. 15 (2020) 557–595 MR

[11] **M Franz**, *The cohomology rings of homogeneous spaces*, J. Topol. 14 (2021) 1396–1447 MR

[12] **M Franz**, *An A_∞-version of the Eilenberg–Moore theorem*, preprint (2023) arXiv 2311.16947

[13] **M Franz**, *Szczarba's twisting cochain is comultiplicative*, Homology Homotopy Appl. 26 (2024) 287–317 MR

[14] **D Grinberg**, **V Reiner**, *Hopf algebras in combinatorics*, preprint (2014) Available at `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb.pdf`

[15] **V K A M Gugenheim**, **J P May**, *On the theory and applications of differential torsion products*, Mem. Amer. Math. Soc. 142, Amer. Math. Soc., Providence, RI (1974) MR

[16] **V K A M Gugenheim**, **H J Munkholm**, *On the extended functoriality of Tor and Cotor*, J. Pure Appl. Algebra 4 (1974) 9–29 MR

[17] **H Hopf**, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. 42 (1941) 22–52 MR

[18] **J Huebschmann**, *Perturbation theory and free resolutions for nilpotent groups of class 2*, J. Algebra 126 (1989) 348–399 MR

[19] **D Husemoller**, **J C Moore**, **J Stasheff**, *Differential homological algebra and homogeneous spaces*, J. Pure Appl. Algebra 5 (1974) 113–185 MR

[20] **J P May**, *Simplicial objects in algebraic topology*, Van Nostrand Math. Stud. 11, Van Nostrand, Princeton, NJ (1967) MR

[21] **J P May**, **F Neumann**, *On the cohomology of generalized homogeneous spaces*, Proc. Amer. Math. Soc. 130 (2002) 267–270 MR

[22] **J McCleary**, *A user's guide to spectral sequences*, 2nd edition, Cambridge Stud. Adv. Math. 58, Cambridge Univ. Press (2001) MR

[23] **J E McClure**, **J H Smith**, *A solution of Deligne's Hochschild cohomology conjecture*, from "Recent progress in homotopy theory" (D M Davis, J Morava, G Nishida, W S Wilson, N Yagita, editors), Contemp. Math. 293, Amer. Math. Soc., Providence, RI (2002) 153–193 MR

[24] **J E McClure**, **J H Smith**, *Multivariable cochain operations and little n-cubes*, J. Amer. Math. Soc. 16 (2003) 681–704 MR

[25] **R J Milgram**, *The bar construction and abelian H-spaces*, Illinois J. Math. 11 (1967) 242–250 MR

[26] **H J Munkholm**, *The Eilenberg–Moore spectral sequence and strongly homotopy multiplicative maps*, J. Pure Appl. Algebra 5 (1974) 1–50 MR

[27] **S Saneblidze**, *The bitwisted Cartesian model for the free loop fibration*, Topology Appl. 156 (2009) 897–910 MR

[28] **G Segal**, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 105–112 MR

[29] **W Singhof**, *On the topology of double coset manifolds*, Math. Ann. 297 (1993) 133–146 MR

[30] **L Smith**, *Homological algebra and the Eilenberg–Moore spectral sequence*, Trans. Amer. Math. Soc. 129 (1967) 58–93 MR

[31] **J Stasheff**, **S Halperin**, *Differential algebra in its own rite*, from "Proceedings of the Advanced Study Institute on algebraic topology, III", Var. Publ. Series 13, Aarhus Univ. (1970) 567–577 MR

[32] **N E Steenrod**, *Milgram's classifying space of a topological group*, Topology 7 (1968) 349–368 MR

[33] **J L Wolf**, *The cohomology of homogeneous spaces and related topics*, PhD thesis, Brown University (1973)

[34] **J Wolf**, *The cohomology of homogeneous spaces*, Amer. J. Math. 99 (1977) 312–340 MR

Department of Mathematics, Tufts University
Medford, MA, United States

Department of Mathematics, University of Western Ontario
London ON, Canada

jeffrey.carlson@tufts.edu, mfranz@uwo.ca

Received: 25 February 2022 Revised: 13 March 2023

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2025 is US $760/year for the electronic version, and $1110/year (+$75, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 9 (pages 5175–5754) 2025