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Envelopes for algebraic patterns

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We generalize Lurie's construction of the symmetric monoidal envelope of an ∞-operad to the setting of algebraic patterns. This envelope becomes fully faithful when sliced over the envelope of the terminal object, and we characterize its essential image. Using this, we prove a comparison result that allows us to compare analogues of ∞-operads over various algebraic patterns. In particular, we show that the G-∞-operads of Nardin and Shah are equivalent to "fibrous patterns" over the $(2,1)$-category $\mathrm{Span}(\mathbb{F}_G)$ of spans of finite G-sets. When G is trivial this means that Lurie's ∞-operads can equivalently be defined over $\mathrm{Span}(\mathbb{F})$ instead of $\mathbb{F}_*$.

18N70; 18N60

1 Introduction

In Lurie's seminal work on homotopy-coherent algebra [23], the main objects used to encode algebraic structures are (symmetric) ∞-*operads*, which are defined as a certain type of functor of ∞-categories $\mathcal{O} \to \mathbb{F}_*$, where $\mathbb{F}_*$ is the category of finite pointed sets. However, as illustrated already in [23], it can sometimes be useful to consider variants of this notion, for instance because they give a combinatorially simpler description of some structure. For example, Lurie also considers *planar* (or nonsymmetric) ∞-operads, where the category $\mathbb{F}_*$ is replaced by the simplex category $\mathbf{\Delta}^{\mathrm{op}}$. As a special case of a general comparison theorem [23, Theorem 2.3.3.26] using the theory of *approximations* to ∞-operads, Lurie proves that there is an equivalence of ∞-categories between planar ∞-operads and ∞-operads over the (symmetric) associative operad Ass, given by pulling back along an explicit map $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ass}$.

Our main goal in this paper is to prove a more general version of such comparisons. Before we explain this result in more detail, let us motivate it by (informally) stating the two main new comparisons we will apply it to:

- In the definition of symmetric ∞-operads, we can equivalently replace the category $\mathbb{F}_*$ of finite pointed sets by the (2,1)-category $\mathsf{Span}(\mathbb{F})$ of *spans* of finite sets.
- For G a finite group, the G-equivariant ∞-operads of Nardin and Shah [26] can equivalently be described as ∞-operads over the (2,1)-category $\mathsf{Span}(\mathbb{F}_G)$ of spans of finite G-sets.

Fibrous patterns The general version of our main result is in the setting of *algebraic patterns* in the sense of Chu and Haugseng [7], which is a general framework for algebraic structures described by "Segal conditions". More precisely, an algebraic pattern is an ∞-category $\mathcal{O}$ equipped with a factorization system $(\mathcal{O}^{\mathrm{int}}, \mathcal{O}^{\mathrm{act}})$ of "inert" and "active" morphisms and a full subcategory $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$ of "elementary" objects. This data lets one define *Segal $\mathcal{O}$-objects* in a complete ∞-category $\mathcal{C}$ as functors $F\colon \mathcal{O} \to \mathcal{C}$ such that for any object $O \in \mathcal{O}$ the natural map

$$F(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)$$

is an equivalence, where $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{O/}$ consists of inert morphisms from O to elementary objects. We can then consider a version of ∞-operads where the category $\mathbb{F}_*$ is replaced by an arbitrary algebraic pattern $\mathcal{O}$; we will refer to them as *fibrous $\mathcal{O}$-patterns*.[1] Such a fibrous $\mathcal{O}$-pattern can be defined as a functor $\pi\colon \mathcal{P} \to \mathcal{O}$ such that:

(1) $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.

(2) For all $O \in \mathcal{O}$, the commutative square of ∞-categories

$$\begin{tikzcd}
\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] \arrow[d] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \arrow[d] \\
\mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}
\end{tikzcd}$$

is cartesian. This square is constructed in Definition 4.1.2 using the factorization system and the cocartesian lifts from (1).[2]

The ∞-category $\mathsf{Fbrs}(\mathcal{O})$ of fibrous $\mathcal{O}$-patterns is then defined as the subcategory of $\mathsf{Cat}_{\infty/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$-patterns and whose morphisms are required to preserve cocartesian morphisms over inert maps in $\mathcal{O}$.

---

[1] Under the mild technical assumption that $\mathcal{O}$ is *sound*, our definition of fibrous $\mathcal{O}$-patterns agrees with the definition of *weak Segal $\mathcal{O}$-fibrations* studied in [7]; see Proposition 4.1.7. However, we prove some results beyond this case, and here the notion of fibrous $\mathcal{O}$-pattern we introduce is better behaved for our purposes.

[2] The bottom horizontal functor is induced by the functors $\alpha_!\colon \mathcal{O}^{\mathrm{act}}_{/O} \to \mathcal{O}^{\mathrm{act}}_{/E}$ that are defined for an inert map $\alpha\colon O \rightarrowtail E$ by sending $\omega\colon X \to O$ to the active part of the factorization $\alpha \circ \omega\colon X \rightarrowtail \alpha_! X \rightsquigarrow E$. The top horizontal functor is defined similarly, by using the cocartesian lifts for inerts.

Let us mention a few examples of algebraic patterns where the corresponding notion of fibrous pattern has already been studied:

- If we take $\mathbb{F}_*$ with the classes of inert and active maps defined as in [23] (see Example 3.1.3) and $\langle 1\rangle := (\{0,1\},0)$ as the only elementary object, then a fibrous $\mathbb{F}_*$-pattern is a functor $\pi\colon \mathcal{P}\to\mathbb{F}_*$ that has cocartesian lifts for inerts and for which the functor
$$\mathcal{P}^{\mathrm{act}}\times_{\mathbb{F}}\mathbb{F}_{/\langle n\rangle}\simeq \mathcal{P}\times_{\mathbb{F}_*}(\mathbb{F}_*)^{\mathrm{act}}_{/\langle n\rangle}\to \prod_{\langle n\rangle\rightarrowtail\langle 1\rangle}\mathcal{P}\times_{\mathbb{F}_*}\mathbb{F}\simeq(\mathcal{P}^{\mathrm{act}})^n$$
is an equivalence. We will show in Proposition 4.1.7 that this is precisely equivalent to $\mathcal{P}$ being a (symmetric) ∞-operad in the sense of Lurie.

- If $\mathcal{O}\to\mathbb{F}_*$ is an ∞-operad in the sense of Lurie, then it has a canonical pattern structure for which a fibrous $\mathcal{O}$-pattern $\pi\colon\mathcal{P}\to\mathcal{O}$ is simply an ∞-operad over $\mathcal{O}$:
$$\mathsf{Fbrs}(\mathcal{O})\simeq\mathsf{Fbrs}(\mathbb{F}_*)_{/\mathcal{O}}=(\mathsf{Opd}_\infty)_{/\mathcal{O}}.$$

- Let $\mathbb{F}_*^\natural$ denote the algebraic pattern with underlying category $\mathbb{F}_*$ and the same factorization system as before, but with both $\langle 0\rangle$ and $\langle 1\rangle$ as elementary objects. Then a fibrous $\mathbb{F}_*^\natural$-pattern is a *generalized ∞-operad* in the sense of [23].

- If we equip $\mathbf{\Delta}^{\mathrm{op}}$ with the usual inert-active factorization system (see Example 3.1.4) and take $[1]$ as the only elementary object, then a fibrous $\mathbf{\Delta}^{\mathrm{op}}$-pattern is precisely a planar or nonsymmetric ∞-operad as in [23]. If we instead take both $[0]$ and $[1]$ as elementary we get *generalized nonsymmetric ∞-operads* as in [11].

- For a finite group G, the G-∞-operads of [26] are precisely fibrous $\underline{\mathbb{F}}_{G,*}$-patterns for a certain pattern $\underline{\mathbb{F}}_{G,*}$ (see Section 5.2).

Comparing fibrous patterns Our first main theorem lets us compare fibrous patterns over various bases:

Theorem A *Let $f\colon\mathcal{O}\to\mathcal{P}$ be a morphism of algebraic patterns (ie a functor that preserves active and inert morphisms and elementary objects). Suppose furthermore that:*

(i) *The induced functors $\mathcal{O}^{\mathrm{el}}_{O/}\to\mathcal{P}^{\mathrm{el}}_{f(O)/}$ are coinitial for all $O\in\mathcal{O}$.*

(ii) *The pattern $\mathcal{P}$ is **sound** in the sense of Definition 3.3.4.*

(iii) *The pattern $\mathcal{P}$ is **extendable**: for all $P\in\mathcal{P}$ the canonical functor*
$$\mathcal{P}^{\mathrm{act}}_{/P}\to\lim_{E\in\mathcal{P}^{\mathrm{el}}_{P/}}\mathcal{P}^{\mathrm{act}}_{/E},$$
is an equivalence.

(iv) *The restriction $f^{\mathrm{el}}\colon\mathcal{O}^{\mathrm{el}}\to\mathcal{P}^{\mathrm{el}}$ of f is an equivalence of ∞-categories,*

(v) *The functor $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}\to(\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ induced by f is an equivalence for all $O\in\mathcal{O}$.*

Then pullback along f gives an equivalence
$$f^*\colon\mathsf{Fbrs}(\mathcal{P})\xrightarrow{\sim}\mathsf{Fbrs}(\mathcal{O}).$$

Here the condition of *soundness* is a mild but rather technical assumption, which is satisfied in almost all examples of algebraic patterns we are aware of. We can now state the applications of Theorem A that we mentioned above more precisely:

Corollary B *Let G be a finite group and $\mathsf{Span}(\mathbb{F}_G)$ the $(2,1)$-category of spans of finite G-sets; we regard this as an algebraic pattern where the inert and active maps are the backwards and forwards maps, respectively, and the elementary objects are the orbits G/H for H a subgroup of G. There is a functor $\underline{\mathbb{F}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ such that pullback along it gives an equivalence*

$$\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \xrightarrow{\simeq} \mathsf{Fbrs}(\underline{\mathbb{F}}_{G*}) = \mathsf{Opd}_{G,\infty}.$$

If we restrict to those fibrous patterns that are also cocartesian fibrations (these are Segal fibrations, or equivalently Segal objects in Cat_∞) then we recover [26, Theorem 2.3.9] of Nardin and Shah, which says that the ∞-category of G-symmetric monoidal ∞-categories is equivalent to the ∞-category of product-preserving functors $\mathsf{Span}(\mathbb{F}_G) \to \mathsf{Cat}_\infty$.

In the case of the trivial group $G = \{e\}$, Corollary B yields an equivalence

$$\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F})) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*) = \mathsf{Opd}_\infty$$

between fibrous $\mathsf{Span}(\mathbb{F})$-patterns and ∞-operads in the sense of Lurie, given by pulling back along the inclusion of $\mathbb{F}_*$ in $\mathsf{Span}(\mathbb{F})$ as the wide subcategory containing the spans whose backwards map is injective.

Segal envelopes The crux of our strategy for proving Theorem A is a reduction to a comparison between Segal objects in Cat_∞ for the two patterns. For this purpose we need to develop an analogue of Lurie's *symmetric monoidal envelope* for ∞-operads over a general algebraic pattern.

A symmetric monoidal ∞-category can be viewed both as a commutative monoid in Cat_∞ (ie a Segal object for $\mathbb{F}_*$) and as an ∞-operad that is a cocartesian fibration; we thus have a (nonfull) subcategory inclusion $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$. In [23, Section 2.2.4], Lurie shows that this functor has a left adjoint, the symmetric monoidal envelope, which admits a very explicit description as a cocartesian fibration: the envelope of an ∞-operad $\mathcal{O}$ is simply the fiber product $\mathcal{O} \times_{\mathbb{F}_*} \mathsf{Ar}_{\mathrm{act}}(\mathbb{F}_*)$ where $\mathsf{Ar}_{\mathrm{act}}(\mathbb{F}_*)$ is the full subcategory of the arrow category of $\mathbb{F}_*$ on the active morphisms and the fiber product is over the source functor $\mathbb{F}^{\amalg} := \mathsf{Ar}_{\mathrm{act}}(\mathbb{F}_*) \to \mathbb{F}_*$, while the projection to $\mathbb{F}_*$ giving the symmetric monoidal ∞-category is by the target functor. Moreover, it was observed in [19] that if we instead regard the envelope as a functor to symmetric monoidal ∞-categories over $(\mathbb{F}, \amalg)$ (that is, finite sets with the disjoint union as symmetric monoidal structure) then it is fully faithful. We want to generalize these results to fibrous $\mathcal{O}$-patterns for a general algebraic pattern $\mathcal{O}$. To simplify exposition we assume here that $\mathcal{O}$ is both sound and extendable. For such $\mathcal{O}$, unstraightening restricts to give a functor $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \to \mathsf{Fbrs}(\mathcal{O})$ analogous to the inclusion $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$. Our second main result is a description of the left adjoint of this functor.

Theorem C *Let $\mathcal{O}$ be a sound and extendable pattern. Then*:

(1) *The unstraightening functor* $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \to \mathsf{Fbrs}(\mathcal{O})$ *has a left adjoint* $\mathsf{Env}_{\mathcal{O}}$ *whose value on a fibrous $\mathcal{O}$-pattern $\mathcal{P}$ is given by the functor* $O \mapsto \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$.

(2) *Slicing* $\mathsf{Env}_{\mathcal{O}}$ *over* $\mathcal{A}_{\mathcal{O}} := \mathsf{Env}_{\mathcal{O}}(\mathcal{O})$ *yields a fully faithful embedding*

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \colon \mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}},$$

which admits both a left and a right adjoint.

(3) *An object* $\mathcal{C} \to \mathcal{A}_{\mathcal{O}}$ *in* $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$ *lies in the essential image of* $\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$ *if and only if it is* $\mathbf{Ar}_{\mathbf{act}}(\mathcal{O})$***-equifibered****, ie for every active map* $O \rightsquigarrow O'$ *in* $\mathcal{O}$*, the square*

$$\begin{tikzcd}
\mathcal{C}(O) \arrow[r, "\mathcal{C}(\omega)"] \arrow[d] & \mathcal{C}(O') \arrow[d] \\
\mathcal{O}^{\mathrm{act}}_{/O} \arrow[r, "\omega_*"] & \mathcal{O}^{\mathrm{act}}_{/O'}
\end{tikzcd}$$

is cartesian.

In Section 4.2 we actually prove more general (but weaker) versions of this statement that do not require $\mathcal{O}$ to be sound or extendable. The comparison of Theorem A can now be shown by recalling a (simpler) comparison theorem for Segal objects from [2], passing to slices and then showing that the equivalence restricts to the essential image of the envelope.

In Section 4.3 we spell out Theorem C in several examples. In particular, for $\mathcal{O} = \mathbb{F}_*$, Theorem C recovers a result of [19], though with an alternative characterization of the image:[3]

Corollary D *The left adjoint to the forgetful functor* $\mathsf{CMon}(\mathsf{Cat}_\infty) \to \mathsf{Opd}_\infty$ *lifts to a fully faithful functor*:

$$\mathsf{Env} \colon \mathsf{Opd}_\infty \hookrightarrow \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}.$$

This functor has adjoints on both sides. A symmetric monoidal functor $\pi \colon (\mathcal{C}, \otimes) \to (\mathbb{F}, \amalg)$ *is in the essential image of* Env *if and only if the square*

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "\pi\times\pi"'] & \mathcal{C} \arrow[d, "\pi"] \\
\mathbb{F} \times \mathbb{F} \arrow[r, "\amalg"] & \mathbb{F}
\end{tikzcd}$$

is a pullback square in Cat_∞.

In Section 5.2 we also give a similar characterization of the essential image of the envelope for G-∞-operads, though in that case one has to require additional pullback squares involving the norm maps $\mathrm{Nm}_K^H \colon \mathcal{C}^K \to \mathcal{C}^H$.

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[3] See Observation 4.3.2 for a comparison.

Organization In Section 2 we prove a key part of Theorem C, which only depends on the factorization system on an algebraic pattern:

Theorem E *Let $\mathcal{B}$ be an ∞-category with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$.*

(1) *The forgetful functor $\mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}} \to \mathrm{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ has a left adjoint, which takes $\mathcal{E} \to \mathcal{B}$ to $\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B})$, where $\mathrm{Ar}_R(\mathcal{B})$ is the full subcategory of $\mathrm{Ar}(\mathcal{B}) := \mathrm{Fun}([1], \mathcal{B})$ spanned by the morphisms in $\mathcal{B}_R$, the fiber product is over evaluation at $0 \in [1]$, and the projection to $\mathcal{B}$ uses evaluation at 1.*

(2) *The induced functor $\mathrm{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \to (\mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathrm{Ar}_R(\mathcal{B})}$ is fully faithful, and a morphism $\mathcal{E} \to \mathrm{Ar}_R(\mathcal{B})$ in $\mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}}$ lies in the image of $\mathrm{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ if and only if it is $\mathrm{Ar}_R(\mathcal{B})$-**equifibered**, meaning that for every $\varphi\colon a \to b$ in $\mathrm{Ar}_R(\mathcal{B})$ the commutative square*

$$\begin{tikzcd}
\mathcal{E}_a \arrow[r, "\varphi_!"] \arrow[d] & \mathcal{E}_b \arrow[d] \\
(\mathcal{B}_R)_{/a} \arrow[r, "\varphi_!"] & (\mathcal{B}_R)_{/b}
\end{tikzcd}$$

is cartesian.

We emphasize that only the second point here is actually new — the first point has already been proved by both Ayala, Mazel-Gee, and Rozenblyum [1] and Shah [31].

We then review algebraic patterns in Section 3, where we also introduce the condition of soundness for patterns. In Section 4 we define fibrous patterns, specialize Theorem E to this context to prove Theorem C, and explore several examples. We are then ready to prove Theorem A in Section 5, where we also discuss the applications and an $(\infty, 2)$-categorical version of Theorem A.

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2 Envelopes for factorization systems

Our goal in this section is to prove Theorem E. We begin in Section 2.1 by explicitly describing the general procedure of freely adding cocartesian morphisms over a wide subcategory $\mathcal{B}_0$ of $\mathcal{B}$ to a functor $p\colon \mathcal{E} \to \mathcal{B}$, and then in Section 2.2 we specialize this to the situation where $\mathcal{B}_0$ is the right class of a factorization system and $\mathcal{E}$ already has p-cocartesian morphisms over the left class. As already mentioned, these results are not new, but we include complete proofs to make the paper more self-contained. In Section 2.3 we then prove the new part of Theorem E: we observe that for the induced adjunction on slices the left adjoint is fully faithful, and identify its image.

2.1 Adding cocartesian morphisms over a subcategory

Let $\mathcal{B}$ be an ∞-category equipped with a wide subcategory $\mathcal{B}_0$, and write $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ for the subcategory of $\mathsf{Cat}_{\infty/\mathcal{B}}$ whose objects have all cocartesian lifts of morphisms in $\mathcal{B}_0$ and whose morphisms preserve these. The aim of this subsection is to show that the forgetful functor

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$$

admits an (explicitly defined) left adjoint. Before explaining the construction of the left adjoint, let us first fix some notation: We let $\mathsf{Ar}(\mathcal{B}) := \mathsf{Fun}([1], \mathcal{B})$ denote the arrow ∞-category of $\mathcal{B}$, and write $\mathsf{Ar}_0(\mathcal{B})$ for the full subcategory of $\mathsf{Ar}(\mathcal{B})$ spanned by morphisms in $\mathcal{B}_0$. The left adjoint of the forgetful functor above is then given by

$$(\mathcal{E} \to \mathcal{B}) \mapsto (\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{B}),$$

where the fiber product is over $\mathrm{ev}_0\colon \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{B}$, and the map $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{B}$ is given by ev_1. We will prove this by showing that for any $\mathcal{E} \in \mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F} \in \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, restriction yields a natural equivalence

$$\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}),$$

where the left-hand side consists of functors that preserve cocartesian morphisms over $\mathcal{B}_0$. This result is by no means new, and has already appeared in [1; 31], but we include a proof for completeness, as this is the key input needed for our work in this paper.

Notation 2.1.1 Since $\mathcal{B}_0$ is a wide subcategory, the degeneracy map $s_0^*\colon \mathcal{B} \to \mathsf{Ar}(\mathcal{B})$ restricts to a functor $i\colon \mathcal{B} \to \mathsf{Ar}_0(\mathcal{B})$, taking an object of $\mathcal{B}$ to its identity map. We also have evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{B}$, and natural transformations $\sigma\colon i \circ \mathrm{ev}_0 \to \mathrm{id}$ and $\tau\colon \mathrm{id} \to i \circ \mathrm{ev}_1$, given for an object $x \xrightarrow{\varphi} y$ by the squares

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, equal] & x \arrow[d, "\varphi"] \\ x \arrow[r, "\varphi"] & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[r, "\varphi"] \arrow[d, "\varphi"] & y \arrow[d, equal] \\ y \arrow[r, equal] & y \end{tikzcd}$$

respectively. For any functor $p\colon \mathcal{E} \to \mathcal{B}$, the functor i induces a section $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})$ of the projection $\mathrm{pr}_{\mathcal{E}}\colon \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{E}$, and σ induces a natural transformation $\sigma_{\mathcal{E}}\colon i_{\mathcal{E}}\mathrm{pr}_{\mathcal{E}} \to \mathrm{id}$.

Observation 2.1.2 Suppose $p\colon \mathcal{E} \to \mathcal{B}$ is cocartesian over $\mathcal{B}_0$. Then $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})$ has a left adjoint $\pi_{\mathcal{E}}$, as we will show now (see [29, Theorem 5.2.8.(iii)] for the special case where $\mathcal{B}_0 = \mathcal{B}$): Such an adjoint exists if and only if, given an object $(x, \varphi\colon px \to b)$, there is an initial object in the ∞-category

$$\mathcal{E}_{(x,\varphi)/} := \mathcal{E} \times_{\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})} (\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}))_{(x,\varphi)/} \simeq \mathcal{E}_{x/} \times_{\mathcal{B}_{px/}} \mathcal{B}_{b/}$$

with the functor $\mathcal{B}_{b/} \to \mathcal{B}_{px/}$ given by composition with φ. A cocartesian morphism $x \to \varphi_! x$ is precisely an initial object in the right-hand side that maps to the identity in $\mathcal{B}_{b/}$. Thus $\pi_{\mathcal{E}}$ takes $(x, \varphi\colon px \to b)$ to the target $\varphi_! x$ of the cocartesian morphism over φ. Note that we have $\pi_{\mathcal{E}} i_{\mathcal{E}} \simeq \mathrm{id}$, and the unit transformation $\mathrm{id} \to i_{\mathcal{E}} \pi_{\mathcal{E}}$ is given at (x, φ) by

$$\left(\begin{tikzcd} x \arrow[d] \\ \varphi_! x \end{tikzcd}, \begin{tikzcd} px \arrow[r, "\varphi"] \arrow[d, "\varphi"] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd} \right).$$

Moreover, this is an adjunction over $\mathcal{B}$ in the sense that we have commutative squares

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "\iota_{\mathcal{E}}"] \arrow[d, "p"] & \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{B} \arrow[r, equal] & \mathcal{B} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \arrow[r, "\pi_{\mathcal{E}}"] \arrow[d, "\mathrm{ev}_1"] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, equal] & \mathcal{B}. \end{tikzcd}$$

For the left square this holds by construction and for the right square we have a transformation $p \circ \pi_{\mathcal{E}} = \mathrm{ev}_1 \circ \iota_{\mathcal{E}} \circ \pi_{\mathcal{E}} \to \mathrm{ev}_1$ coming from the counit of $\pi_{\mathcal{E}} \dashv \iota_{\mathcal{E}}$. This is an equivalence by the pointwise description of $\pi_{\mathcal{E}}$ given above.

Observation 2.1.3 Given $p\colon \mathcal{E} \to \mathcal{B}$, observe that $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})$ is cocartesian over $\mathcal{B}_0$, with cocartesian morphisms given by composition in $\mathsf{Ar}_0(\mathcal{B})$. (For instance, we can write $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})$ as a pullback $(\mathcal{E} \times \mathcal{B}) \times_{(\mathcal{B} \times \mathcal{B})} \mathsf{Ar}_0(\mathcal{B})$ over $\mathcal{B}$, where all three ∞-categories appearing are cocartesian over $\mathcal{B}_0$.)

Proposition 2.1.4 *If $q\colon \mathcal{F} \to \mathcal{B}$ is cocartesian over $\mathcal{B}_0$, composition with $i_{\mathcal{E}}$ gives a functor*

$$\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}), \mathcal{F}) \to \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}).$$

This is an equivalence, with inverse given by taking $F\colon \mathcal{E} \to \mathcal{F}$ to the composite

$$\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \xrightarrow{F \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})} \mathcal{F} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}.$$

Proof Given $G\colon \mathcal{E} \to \mathcal{F}$, the definitions of the sections $i_{\mathcal{E}}$ and $i_{\mathcal{F}}$ give

$$(G \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq i_{\mathcal{F}} \circ G,$$

and so we have

$$\pi_{\mathcal{F}} \circ (G \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})) \circ i_{\mathcal{E}} \simeq \pi_{\mathcal{F}} \circ i_{\mathcal{F}} \circ G \simeq G.$$

In the other direction, given $F\colon \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \to \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_0$, we have to show that F is naturally equivalent to $\pi_{\mathcal{F}} \circ (F i_{\mathcal{E}} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}))$. Here we can write $\mathrm{pr}_{\mathcal{F}} \circ (F i_{\mathcal{E}} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}))$ as the composite

$$\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \xrightarrow{\mathrm{pr}_{\mathcal{E}}} \mathcal{E} \xrightarrow{i_{\mathcal{E}}} \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B}) \xrightarrow{F} \mathcal{F},$$

so that $\sigma_{\mathcal{E}}$ induces a natural transformation

$$\alpha\colon \mathrm{pr}_{\mathcal{F}} \circ (F i_{\mathcal{E}} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})) \to F.$$

Note that this is given at $(e,\varphi\colon p(e)\to b)$ by the image $F(e,\mathrm{id}_{p(e)})\to F(e,\varphi)$ of a cocartesian morphism in $\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})$, and so is cocartesian in $\mathcal{F}$ since by assumption F preserves cocartesian morphisms over $\mathcal{B}_0$. Projecting to $\mathcal{B}$, we see that $q\alpha$ factors as the projection to $\mathsf{Ar}_0(\mathcal{B})$ followed by the evaluation map $\mathsf{Ar}_0(\mathcal{B})\times[1]\to\mathcal{B}$. We can therefore define a natural transformation

$$\beta\colon \mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})\times[1]\to\mathcal{F}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})$$

via the commutative diagram

$$\begin{tikzcd}
\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})\times[1] \arrow[r, "\alpha"] \arrow[d] & \mathcal{F} \arrow[dd, "q"] \\
\mathsf{Ar}_0(\mathcal{B})\times[1] \arrow[d, "\tau"] \arrow[dr, "\mathrm{ev}"] & \\
\mathsf{Ar}_0(\mathcal{B}) \arrow[r, "s"] & \mathcal{B}.
\end{tikzcd}$$

Here β is a natural transformation $(Fi_{\mathcal{E}}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B}))\to i_{\mathcal{F}}F$, and takes

$$(e,\varphi\colon p(e)\to b)\quad\text{to}\quad (F(e,\mathrm{id}_{p(e)}),\varphi)\to(F(e,\varphi),\mathrm{id}_c).$$

Composing with $\pi_{\mathcal{F}}$ we get a natural transformation $\pi_{\mathcal{F}}\beta\colon\pi_{\mathcal{F}}\circ(Fi_{\mathcal{E}}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B}))\to\pi_{\mathcal{F}}i_{\mathcal{F}}F\simeq F$. This is given at (e,φ) by the canonical morphism $\varphi_!F(e,\mathrm{id})\to F(e,\varphi)$. Since F preserves cocartesian morphisms over $\mathcal{B}_0$, this is an equivalence, and so we have obtained the natural equivalence we required. □

Corollary 2.1.5 *The forgetful functor*

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}\to\mathsf{Cat}_{\infty/\mathcal{B}}$$

has a left adjoint given by

$$(\mathcal{E}\to\mathcal{B})\mapsto\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})=s^*\mathcal{E}\to\mathsf{Ar}_0(\mathcal{B})\xrightarrow{t}\mathcal{B},$$

and unit given by $i_{\mathcal{E}}\colon\mathcal{E}\to\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B})$.

Proof By Proposition 2.1.4, for $\mathcal{E}\in\mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{F}\in\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ the composite

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B}),\mathcal{F})\to\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_0(\mathcal{B}),\mathcal{F})\xrightarrow{i_{\mathcal{E}}^*}\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{E},\mathcal{F})$$

is an equivalence, hence this natural transformation is indeed the unit of an adjunction. □

Observation 2.1.6 The forgetful functors $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}\to\mathsf{Cat}_{\infty/\mathcal{B}}\to\mathsf{Cat}_{\infty}$ detect pullbacks; in particular, the ∞-category $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ has all pullbacks. Indeed, given morphisms $\mathcal{E}_1\to\mathcal{E}_0\leftarrow\mathcal{E}_2$ in $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, it is easy to see that a morphism in the fiber product $\mathcal{E}_1\times_{\mathcal{E}_0}\mathcal{E}_2$ is cocartesian over $\mathcal{B}_0$ if and only if its images in $\mathcal{E}_1$ and $\mathcal{E}_2$ are cocartesian.

Observation 2.1.7 Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞-categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f\colon \mathcal{A} \to \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Pullback along f clearly gives a commutative diagram

$$\begin{tikzcd}
\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}} \arrow[d] \\
\mathsf{Cat}_{\infty/\mathcal{B}} \arrow[r, "f^*"] & \mathsf{Cat}_{\infty/\mathcal{A}}.
\end{tikzcd}$$

We then have an induced Beck–Chevalley transformation between the left adjoints of the vertical maps, given for $p\colon \mathcal{E} \to \mathcal{B}$ by the natural map

$$(\mathcal{E} \times_{\mathcal{B}} \mathcal{A}) \times_{\mathcal{A}} \mathsf{Ar}_0(\mathcal{A}) \to (\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_0(\mathcal{B})) \times_{\mathcal{B}} \mathcal{A},$$

which takes $(e \in \mathcal{E}, a \in \mathcal{A}, p(e) \simeq f(a), \varphi\colon a \to a' \in \mathsf{Ar}_0(\mathcal{A}))$ to $(e, f(a), f(\varphi), a)$. Note, however, that this is typically not an equivalence.

2.2 Free fibrations for factorization systems

In this subsection we specialize our previous results to the case of an ∞-category equipped with a factorization system. We again emphasize that this result already appears in [1; 31].

Notation 2.2.1 In this section we fix an ∞-category $\mathcal{B}$ with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$; we write $\mathsf{Ar}_L(\mathcal{B})$ and $\mathsf{Ar}_R(\mathcal{B})$ for the full subcategories of $\mathsf{Ar}(\mathcal{B})$ spanned by the morphisms in $\mathcal{B}_L$ and $\mathcal{B}_R$, respectively. We also abbreviate

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} := \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_L\text{-cocart}}$$

Proposition 2.2.2 [7, Proposition 7.3] *Let $(q\colon \mathcal{C} \to \mathcal{B}) \in \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$. Then:*

(1) *The functor $q'\colon \mathcal{C} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{B}$ given by evaluation at the target is a cocartesian fibration.*

(2) *A morphism $(\alpha, \beta)\colon (c_0, \varphi_0) \to (c_1, \varphi_1)$ in $\mathcal{C} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B})$ represented by the diagram*

$$\left(\begin{tikzcd}
c_0 \arrow[d, "\alpha"] & [-2em] , & q(c_0) \arrow[r, "\varphi_0"] \arrow[d, "q(\alpha)"] & b_0 \arrow[d, "\beta"] \\
c_1 & & q(c_1) \arrow[r, "\varphi_1"] & b_1
\end{tikzcd}\right)$$

is a q'-cocartesian lift of $\beta\colon b_0 \to b_1$ if and only if $q(\alpha)$ is in $\mathcal{B}_L$ and α is q-cocartesian.

Proof We first show that q' is a locally cocartesian fibration. A locally q'-cocartesian morphism over $\beta\colon b_0 \to b_1$ with source $(c_0, \varphi_0\colon q(c_0) \to b_0)$ is an initial object in the ∞-category

$$(\mathcal{C} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}.$$

We can identify this ∞-category as the fiber product

$$\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} (\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}),$$

where $\mathcal{B}^R_{/b_1}$ denotes the full subcategory of $\mathcal{B}_{/b_1}$ spanned by morphisms in $\mathcal{B}_R$.

We first observe that here $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$ has an initial object, given by

$$\begin{tikzcd} qc_0 \arrow[rr, "\lambda"] \arrow[dr, "\beta\varphi_0"'] & & b' \arrow[dl, "\rho"] \\ & b_1, & \end{tikzcd}$$

where (λ, ρ) is the (L, R)-factorization of $\beta\varphi_0$ — this follows from [22, Lemma 5.2.8.19].

The projection $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \to \mathcal{B}^R_{/b_1}$ is a left fibration, since it is a base change of the left fibration $(\mathcal{B}_{/b_1})_{\beta\varphi_0/} \to \mathcal{B}_{/b_1}$. The initial object of $\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/}$, which maps to ρ in $\mathcal{B}^R_{/b_1}$, therefore gives an equivalence

$$\mathcal{B}^R_{/b_1} \times_{\mathcal{B}_{/b_1}} (\mathcal{B}_{/b_1})_{\beta\varphi_0/} \simeq (\mathcal{B}^R_{/b_1})_{\rho/}$$

by [24, Tag 0199]. We can thus rewrite our expression for the ∞-category $(\mathcal{C} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/} \times_{\mathcal{B}_{b_0/}} \{\beta\}$ as

$$(\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}) \times_{\mathcal{B}_{b'/}} (\mathcal{B}^R_{/b_1})_{\rho/}.$$

A q-cocartesian morphism over λ with source c_0, which exists by assumption since λ is in $\mathcal{B}_L$, is precisely an initial object of $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ that maps to the initial object in $\mathcal{B}_{b'/}$. We thus have initial objects in $\mathcal{C}_{c_0/} \times_{\mathcal{B}_{qc_0/}} \mathcal{B}_{b'/}$ and $(\mathcal{B}^R_{/b_1})_{\rho/}$ that both map to the initial object in $\mathcal{B}_{b'/}$, and these thus give an initial object in the fiber product $(\mathcal{C} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}))_{(c_0,\varphi_0)/}$. This shows that if $\alpha\colon c_0 \to c_1$ is a q-cocartesian lift of λ, then

$$\left(\begin{tikzcd} c_0 \arrow[d, "\alpha"] & , & q(c_0) \arrow[r, "\varphi_0"] \arrow[d, "\lambda"] & b_0 \arrow[d, "\beta"] \\ c_1 & & b' \arrow[r, "\rho"] & b_1 \end{tikzcd}\right)$$

is a locally q-cocartesian lift of β with source (c_0, φ_0).

We have thus shown that q' is a locally cocartesian fibration, and the locally q'-cocartesian morphisms are precisely those in (2). To see that q' is a cocartesian fibration it then suffices by [22, Proposition 2.4.2.8] to check that the locally q'-cocartesian morphisms are closed under composition, which in our case is clear. □

Notation 2.2.3 It follows from Proposition 2.2.2 that the construction $\mathcal{E} \mapsto \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B})$ restricts to a well-defined functor

$$\mathrm{E}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \to \mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}}, \quad (\mathcal{E} \to \mathcal{B}) \mapsto (\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{B}).$$

Proposition 2.2.4 *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a functor admitting cocartesian lifts for all arrows in $\mathcal{B}_L$ and let $q\colon \mathcal{F} \to \mathcal{B}$ be a cocartesian fibration. Then the equivalence of Proposition 2.1.4 restricts to an equivalence*

$$\mathsf{Fun}^{\mathrm{cocart}}_{/\mathcal{B}}(\mathrm{E}(\mathcal{E}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}).$$

Proof We must show that these full subcategories are identified under the equivalence

$$\mathsf{Fun}^{R-\mathrm{cocart}}_{/\mathcal{B}}(\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}), \mathcal{F}) \xrightarrow{\sim} \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F})$$

of Proposition 2.1.4. Given a functor $F\colon \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{F}$ that preserves cocartesian morphisms over $\mathcal{B}_R$, we must thus check that F preserves all cocartesian morphisms if and only if $F \circ i_{\mathcal{E}}$ preserves cocartesian morphisms over $\mathcal{B}_L$. We write $p'\colon \mathcal{E} \times_{\mathcal{C}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{B}$ for the map induced by ev_1.

First, assume that $F\colon \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{F}$ preserves all cocartesian edges. For a p-cocartesian lift $\alpha\colon c_0 \to c_1$ of an edge $\beta\colon b_0 \to b_1$ in $\mathcal{B}_L$, its image under $i_{\mathcal{E}}$ is the edge

$$\left(\begin{tikzcd} c_0 \arrow[d, "\alpha"] \\ c_1 \end{tikzcd}, \begin{tikzcd} b_0 \arrow[r, "="] \arrow[d, "\beta"] & b_0 \arrow[d, "\beta"] \\ b_1 \arrow[r, "="] & b_1 \end{tikzcd}\right)$$

in $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B})$, which is p'-cocartesian by Proposition 2.2.2. In other words, $i_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B})$ preserves cocartesian lifts over $\mathcal{B}_L$, and hence so does $F \circ i_{\mathcal{E}}$.

For the converse assume that F preserves cocartesian lifts of edges in $\mathcal{B}_R$ and $F \circ i_{\mathcal{E}}$ preserves cocartesian lifts of edges in $\mathcal{B}_L$. We would like to show that a general p'-cocartesian morphism $(\alpha, \beta)\colon (c_0, \varphi_0) \to (c_1, \varphi_1)$ is sent to a q-cocartesian morphism in $\mathcal{F}$. According to Proposition 2.2.2, the morphism $p(\alpha)$ is in $\mathcal{B}_L$ and α is p-cocartesian. We can fit this morphism into the following diagram by applying the natural transformation $\sigma_{\mathcal{E}}\colon i_{\mathcal{E}}\mathrm{pr}_{\mathcal{E}} \to \mathrm{id}$:

$$\begin{tikzcd} (c_0, \mathrm{id}) \arrow[r, "{(\mathrm{id},\varphi_0)=(\sigma_{\mathcal{E}})_{(c_0,\varphi_0)}}"] \arrow[d, "{(\alpha,q(\alpha))}"'] & (c_0, \varphi_0) \arrow[d, "{(\alpha,\beta)}"] \\ (c_1, \mathrm{id}) \arrow[r, "{(\mathrm{id},\varphi_1)=(\sigma_{\mathcal{E}})_{(c_1,\varphi_1)}}"'] & (c_1, \varphi_1). \end{tikzcd}$$

Both horizontal morphisms are cocartesian edges over $\mathcal{B}_R$ (by Proposition 2.2.2) and the left-hand vertical morphism is the image under $i_{\mathcal{E}}$ of a p-cocartesian morphism over $\mathcal{B}_L$. Hence F sends three of the morphisms in the above square to cocartesian edges in $\mathcal{F}$ and it follows by composition and right-cancellation for cocartesian edges that $F(\alpha, \beta)$ is cocartesian too. □

Corollary 2.2.5 *The adjunction of Corollary 2.1.5 restricts to an adjunction*

$$\mathrm{E}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}} : \mathrm{forget}.$$

Observation 2.2.6 Suppose $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ are ∞-categories equipped with factorization systems, and that $f : \mathcal{A} \to \mathcal{B}$ is a functor that preserves both classes of maps in these. Pullback along f then gives a commutative diagram

$$\begin{tikzcd}
\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}_{\infty/\mathcal{A}}^{L\text{-cocart}} \arrow[d] \\
\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \arrow[r, "f^*"] & \mathsf{Cat}_{\infty/\mathcal{A}}^{\text{cocart}}.
\end{tikzcd}$$

As in Observation 2.1.7, this induces a Beck–Chevalley transformation, but this is typically not an equivalence.

2.3 Full faithfulness on slices

In this subsection we prove the main new result of this section: We observe that the adjunction of Corollary 2.2.5 induces an adjunction

$$\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})},$$

where the left adjoint is fully faithful, and characterize its image as in Theorem E.

To construct this adjunction, we recall the general construction of adjunctions on slices:

Observation 2.3.1 Given an adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R,$$

where $\mathcal{C}$ admits pullbacks, we have (by [22, Proposition 5.2.5.1]) for any c in $\mathcal{C}$ an induced adjunction

$$L_c : \mathcal{C}_{/c} \rightleftarrows \mathcal{D}_{/Lc} : R_c,$$

where L_c is simply given by applying L, while R_c is defined at $f : d \to Lc$ by the natural pullback square

$$\begin{tikzcd}
R_c d \arrow[r] \arrow[d] & Rd \arrow[d, "Rf"] \\
c \arrow[r, "\eta_c"] & RLc
\end{tikzcd}$$

over the unit map η_c. The unit for the new adjunction is then given at $c' \to c$ by the canonical map $c' \to R_c L_c c'$ obtained by factoring the square

$$\begin{tikzcd}
c' \arrow[r, "\eta_{c'}"] \arrow[d] & RLc' \arrow[d] \\
c \arrow[r, "\eta_c"] & RLc
\end{tikzcd}$$

through the pullback, while the counit $L_c R_c d \to d$ is given by the outer square in the diagram

$$\begin{tikzcd}
LR_c d \arrow[r] \arrow[d] & LRd \arrow[r, "\epsilon_d"] \arrow[d] & d \arrow[d] \\
Lc \arrow[r, "L\eta_c"] \arrow[rr, bend right, equal] & LRLc \arrow[r, "\epsilon_{Lc}"] & Lc,
\end{tikzcd}$$

where ϵ is the counit of the original adjunction.

Proposition 2.3.2 *By applying the construction of Observation 2.3.1 to the adjunction of Corollary 2.2.5 at the terminal object* $(\mathcal{B} \xrightarrow{=} \mathcal{B}) \in \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ *we obtain an adjunction*

$$\mathrm{E}\colon \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \rightleftarrows (\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} :\mathrm{Q}. \tag{1}$$

The left adjoint in this adjunction is fully faithful.

Proof Here E sends $\mathcal{E} \to \mathcal{B}$ to the cocartesian fibration $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_{R}(\mathcal{B}) \to \mathcal{B}$, equipped with the canonical projection to $\mathsf{Ar}_{R}(\mathcal{B}) \to \mathcal{B}$. The right adjoint Q is given by

$$\mathcal{E} \to \mathsf{Ar}_{R}(\mathcal{B}) \mapsto i^*\mathcal{E} = \mathcal{B} \times_{\mathsf{Ar}_{R}(\mathcal{B})} \mathcal{E} \to \mathcal{B},$$

where the pullback is taken along the inclusion of the identities $i : \mathcal{B} \to \mathsf{Ar}_{R}(\mathcal{B})$. The unit of this adjunction is then the map $\mathcal{E} \to \mathrm{Q}(\mathrm{E}(\mathcal{E}))$ obtained from the commutative square of units for the adjunction $\mathrm{E} \dashv \text{forget}$ (from Corollary 2.2.5) as the canonical map from $\mathcal{E}$ to the pullback. This square of units is the left-hand square in the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "i_{\mathcal{E}}"] \arrow[d] \arrow[rr, bend left, equal] & \mathrm{E}(\text{forget}(\mathcal{E})) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{B} \arrow[r, "i"] \arrow[rr, bend right, equal] & \mathsf{Ar}_{R}(\mathcal{B}) \arrow[r, "\mathrm{ev}_0"] & \mathcal{B},
\end{tikzcd}$$

where the right-hand square is cartesian by construction of $\mathrm{E}(\mathcal{E})$ in Notation 2.2.3. Hence the left-hand square is also cartesian and thus the unit $\mathcal{E} \to \mathrm{Q}(\mathrm{E}(\mathcal{E}))$ is an equivalence, and so E is indeed fully faithful. □

Now that we have the fully faithful envelope functor all that is left to do to prove Theorem E is to characterize its essential image:

Proposition 2.3.3 *A morphism* $\mathcal{D} \to \mathsf{Ar}_{R}(\mathcal{B})$ *of cocartesian fibrations over* $\mathcal{B}$ *is in the essential image of the left adjoint* E *from Proposition 2.3.2 if and only if it is* ***equifibered***, *meaning that for every object*

$\varphi\colon a \to b$ in $\mathrm{Ar}_R(\mathcal{B})$, the natural square

$$\begin{tikzcd}
\mathcal{D}_a \arrow[r, "\varphi_!"] \arrow[d] & \mathcal{D}_b \arrow[d] \\
\mathrm{Ar}_R(\mathcal{B})_a \arrow[r, "\varphi\circ(-)"] & \mathrm{Ar}_R(\mathcal{B})_b
\end{tikzcd}$$

is cartesian.

Proof We begin with the "only if" direction for $(\mathcal{E} \to \mathcal{B}) \in \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $(\varphi\colon a \to b) \in \mathrm{Ar}_R(\mathcal{B})$. We need to show that the left square of the following diagram is cartesian:

$$\begin{tikzcd}
(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_a \arrow[r, "\varphi_!"] \arrow[d] \arrow[rr, bend left, "\mathrm{pr}_{\mathcal{E}}"] & (\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))_b \arrow[r, "\mathrm{pr}_{\mathcal{E}}"] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathrm{Ar}_R(\mathcal{B})_a \arrow[r, "\varphi\circ(-)"] \arrow[rr, bend right, "s"'] & \mathrm{Ar}_R(\mathcal{B})_b \arrow[r, "s"] & \mathcal{B},
\end{tikzcd}$$

where the identification of the composite in the top row uses the description of cocartesian morphisms in $(\mathcal{E} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}))$ from Proposition 2.2.2. This follows since the right-hand square and the outer rectangle are both cartesian.

For the "if" direction we must show that the counit $\mathrm{E}(\mathrm{Q}(\mathcal{D})) \to \mathcal{D}$ is an equivalence if $\mathcal{D}$ is equifibered. By Observation 2.3.1 this counit can be factored as the composite of the top horizontal maps in the diagram

$$\begin{tikzcd}
\mathrm{E}(\mathrm{Q}(\mathcal{D})) \arrow[r] \arrow[d] & \mathrm{E}(\mathcal{D}) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathrm{Ar}_R(\mathcal{B}) \arrow[r] & \mathrm{E}(\mathrm{Ar}_R(\mathcal{B})) \arrow[r] & \mathrm{Ar}_R(\mathcal{B}).
\end{tikzcd} \tag{$\star$}$$

Here the right-hand horizontal maps come from the counit of the adjunction from Corollary 2.2.5. The bottom horizontal composite is an equivalence, so it will suffice to show that the composite rectangle is cartesian. Since the left-hand square is given by E applied to the cartesian square defining Q (as $\mathrm{Ar}_R(\mathcal{B})$ is $\mathrm{E}(\mathcal{B})$), and E preserves weakly contractible limits, it suffices to show that the right-hand square is cartesian.

By assumption, the functor $\mathcal{D} \to \mathcal{B}$ is a cocartesian fibration, and so the projection

$$\mathrm{E}(\mathcal{D}) \simeq \mathcal{D} \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \to \mathrm{Ar}_R(\mathcal{B})$$

is also a cocartesian fibration, with cocartesian morphisms exactly those that project to cocartesian morphisms in $\mathcal{D}$. Consider now the square

$$\begin{tikzcd}
\mathrm{E}(\mathcal{D}) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "\pi"] \\
\mathrm{Ar}_R(\mathcal{B}) \arrow[r, "t"] & \mathcal{B}
\end{tikzcd}$$

in which the top map is the counit for the adjunction of Corollary 2.2.5. The top map in the square takes cocartesian morphisms over $\mathsf{Ar}_R(\mathcal{B})$ to π-cocartesian morphisms in $\mathcal{D}$. To see this, note that a cocartesian morphism in $\mathrm{E}(\mathcal{D})$ over $\mathsf{Ar}_R(\mathcal{B})$ is of the form

$$\left(\begin{tikzcd} d \arrow[d] \\ \varphi_! d \end{tikzcd}, \begin{tikzcd} \pi(d) \arrow[r, "\alpha"] \arrow[d, "\varphi"] & b \arrow[d, "\beta"] \\ a \arrow[r, "\gamma"] & b' \end{tikzcd}\right),$$

and this is by construction sent to the canonical map $\alpha_! d \to \gamma_! \varphi_! d$, which is indeed cocartesian over β.

Consequently the top right square of $(\star)$ sits as the top face in the cube

$$\begin{tikzcd} \mathrm{E}(\mathcal{D}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{D} \arrow[dr] \arrow[dd] & \\ & \mathrm{E}(\mathsf{Ar}_R(\mathcal{B})) \arrow[rr] \arrow[dd] & & \mathsf{Ar}_R(\mathcal{B}) \arrow[dd] \\ \mathsf{Ar}_R(\mathcal{B}) \arrow[rr] \arrow[dr, equal] & & \mathcal{B} \arrow[dr, equal] & \\ & \mathsf{Ar}_R(\mathcal{B}) \arrow[rr] & & \mathcal{B} \end{tikzcd}$$

in which the vertical maps are cocartesian fibrations and the maps in the top square preserve cocartesian morphisms. Since the bottom square is obviously cartesian, to show that the top square is cartesian it suffices to check that taking fibers over any $\varphi \in \mathsf{Ar}_R(\mathcal{B})$ yields a cartesian square. We thus want to show that the following square is cartesian:

$$\begin{tikzcd} \mathrm{E}(\mathcal{D})_\varphi \arrow[r] \arrow[d] & \mathcal{D}_b \arrow[d] \\ \mathrm{E}(\mathsf{Ar}_R(\mathcal{B}))_\varphi \arrow[r] & \mathsf{Ar}_R(\mathcal{B})_b. \end{tikzcd}$$

Here there is a canonical equivalence $\mathrm{E}(\mathcal{D})_\varphi \simeq (\mathcal{D} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B})_\varphi) \simeq \mathcal{D}_a$ and similarly $\mathrm{E}(\mathsf{Ar}_R(\mathcal{B}))_\varphi \simeq (\mathsf{Ar}_R(\mathcal{B}))_a$. Via these equivalences the horizontal maps are identified with the cocartesian pushforward along φ. The resulting square is then precisely one of the squares that are cartesian by the assumption that $\mathcal{D}$ is equifibered. □

In Section 4.2 it will be notationally convenient to use a "straightened" version of the adjunction (1); to state this we first introduce some notation:

Notation 2.3.4 Let $\mathcal{B}$ be an ∞-category equipped with a factorization system $(\mathcal{B}_L, \mathcal{B}_R)$. Let $\mathcal{R}\colon \mathcal{B} \to \mathsf{Cat}_\infty$ be the straightening of the cocartesian fibration $\mathsf{Ar}_R(\mathcal{B}) \to \mathcal{B}$. We define the functor

$$\mathsf{St}^L_{\mathcal{B}}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \to \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}},$$

which we think of as a form of "straightening relative to the factorization system", as the composite

$$\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \xrightarrow{\mathrm{E}} (\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_R(\mathcal{B})} \xrightarrow{\mathrm{St}_{\mathcal{B}}} \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}},$$

sending $(p\colon \mathcal{E} \to \mathcal{B})$ to the straightening of $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathcal{B}$. Dually, we define

$$\mathrm{Un}^L_{\mathcal{B}}\colon \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} \to \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$$

as the composite

$$\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} \xrightarrow{\mathrm{Un}_{\mathcal{B}}} (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \xrightarrow{\mathrm{Q}} \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}.$$

For a functor $F\colon \mathcal{B} \to \mathsf{Cat}$ together with natural transformation $\alpha\colon F \to \mathcal{R}$ we then have that $\mathrm{Un}^L_{\mathcal{B}}(\alpha)$ is the pullback

$$\begin{tikzcd}
\mathrm{Un}^L_{\mathcal{B}}(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Un}_{\mathcal{B}}(F) \arrow[d, "\mathrm{Un}_{\mathcal{B}}(\alpha)"] \\
\mathcal{B} \arrow[r] & \mathsf{Ar}_R(\mathcal{B}).
\end{tikzcd}$$

This yields the following reformulation of Theorem E:

Theorem 2.3.5 *The functors* $\mathrm{St}^L_{\mathcal{B}}$ *and* $\mathrm{Un}^L_{\mathcal{B}}$ *give an adjunction*

$$\mathrm{St}^L_{\mathcal{B}}\colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}} :\mathrm{Un}^L_{\mathcal{B}}.$$

The functor $\mathrm{St}^L_{\mathcal{B}}$ *is fully faithful and a natural transformation* $F \to \mathcal{R}$ *is in the essential image of* $\mathrm{St}^L_{\mathcal{B}}$ *if and only if it is* ***equifibered****, meaning that for every object* $a \xrightarrow{\varphi} b$ *in* $\mathsf{Ar}_R(\mathcal{B})$*, the natural square*

$$\begin{tikzcd}
F(a) \arrow[r, "F(\varphi)"] \arrow[d] & F(b) \arrow[d] \\
\mathcal{R}(a) \arrow[r, "\mathcal{R}(\varphi)"] & \mathcal{R}(b)
\end{tikzcd}$$

is cartesian.

A pleasant consequence of Theorem 2.3.5 is that $\mathrm{St}^L_{\mathcal{B}}$ also has a *left* adjoint and that $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ is presentable. To see this, we use the following observation:

Observation 2.3.6 Let $\mathcal{C}$ be a presentable ∞-category, and S a set of morphisms in $\mathcal{C}$. Recall that a morphism $\varphi\colon X \to Y$ in $\mathcal{C}$ is *right orthogonal* to S if there exists a unique filler in every commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "\varphi"] \\
B \arrow[r] \arrow[ur, dotted] & Y,
\end{tikzcd}$$

where f is in S. Equivalently, φ is right orthogonal to S if and only if the commutative square

$$\begin{tikzcd}
\mathsf{Map}_{\mathcal{C}}(B, X) \arrow[r, "f^*"] \arrow[d, "\varphi_*"] & \mathsf{Map}_{\mathcal{C}}(A, X) \arrow[d, "\varphi_*"] \\
\mathsf{Map}_{\mathcal{C}}(B, Y) \arrow[r, "f^*"] & \mathsf{Map}_{\mathcal{C}}(A, Y)
\end{tikzcd}$$

is cartesian for all $f\colon A \to B$ in S. This square is in turn cartesian if and only if for all maps $B \to Y$, the map on fibers

$$\mathsf{Map}_{/Y}(B, X) \to \mathsf{Map}_{/Y}(A, X)$$

is an equivalence. Thus the map φ is right orthogonal to S if and only if as an object of $\mathcal{C}_{/Y}$ it is local with respect to the set of maps

$$\left\{ \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & Y & \end{tikzcd} : f \in S \right\}.$$

In particular, the full subcategory of $\mathcal{C}_{/Y}$ spanned by the objects that are right orthogonal to S is an accessible localization of $\mathcal{C}_{/Y}$, and so is also presentable.

Proposition 2.3.7 *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a small ∞-category equipped with a factorization system. The functor $\mathrm{St}^L_{\mathcal{B}}$ has a left adjoint, which exhibits $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ as an accessible localization of $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$. In particular, $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ is a presentable ∞-category.*

Proof The ∞-category $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ is clearly presentable, and we know that the functor $\mathrm{St}^L_{\mathcal{B}}$ is fully faithful, with its essential image given by functors equifibered over $\mathcal{R}$. It therefore suffices to show that this is the full subcategory of objects in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ that are local with respect to a set of morphisms. Let S be the collection of morphisms of the form

$$(y(\varphi) \times \mathrm{id})\colon y(b) \times [\epsilon] \to y(a) \times [\epsilon]$$

for $\epsilon \in \{0, 1\}$ and $(\varphi\colon a \to b) \in \mathsf{Ar}_R(\mathcal{B})$, where $y(a)(-) := \mathsf{Map}_{\mathcal{B}}(a, -)$ is the Yoneda embedding of $\mathcal{B}^{\mathrm{op}}$; this is a set since $\mathcal{B}$ is by assumption a small ∞-category. An object $\gamma\colon F \to \mathcal{R}$ in $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)_{/\mathcal{R}}$ is then equifibered if and only if it is right orthogonal to S: The latter means that the commutative squares

$$\begin{tikzcd} \mathsf{Map}(y(a) \times [\epsilon], F) \arrow[r] \arrow[d] & \mathsf{Map}(y(b) \times [\epsilon], F) \arrow[d] \\ \mathsf{Map}(y(a) \times [\epsilon], \mathcal{R}) \arrow[r] & \mathsf{Map}(y(b) \times [\epsilon], \mathcal{R}) \end{tikzcd}$$

are cartesian; by the Yoneda lemma this square can be identified with

$$\begin{tikzcd} \mathsf{Map}([\epsilon], F(a)) \arrow[r] \arrow[d] & \mathsf{Map}([\epsilon], F(b)) \arrow[d] \\ \mathsf{Map}([\epsilon], \mathcal{R}(a)) \arrow[r] & \mathsf{Map}([\epsilon], \mathcal{R}(b)), \end{tikzcd}$$

which is cartesian for $\epsilon = 0, 1$ if and only if the square

$$\begin{tikzcd} F(a) \arrow[r] \arrow[d] & F(b) \arrow[d] \\ \mathcal{R}(a) \arrow[r] & \mathcal{R}(b) \end{tikzcd}$$

is cartesian, since the objects $[0],[1]$ generate Cat_∞ under colimits. The result then follows from Observation 2.3.6. $\square$

Observation 2.3.8 It is easy to see (using the mapping space criterion for cocartesian morphisms) that the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ preserves limits and filtered colimits. Since both ∞-categories are presentable by Proposition 2.3.7, it follows by the adjoint functor theorem that this functor has a left adjoint.

Observation 2.3.9 Let $(\mathcal{A},\mathcal{A}_L,\mathcal{A}_R)$ and $(\mathcal{B},\mathcal{B}_L,\mathcal{B}_R)$ be ∞-categories equipped with factorization systems, and let $f\colon\mathcal{A}\to\mathcal{B}$ be a functor that preserves both classes of maps in these.

The functor f then induces a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i_{\mathcal{A}}"] \arrow[dd, "f"'] & \mathsf{Ar}_R(\mathcal{A}) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "q"] & \mathcal{A} \arrow[d, equal] \\
 & f^*\mathsf{Ar}_R(\mathcal{B}) \arrow[r, "\mathrm{ev}_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "f"] \\
\mathcal{B} \arrow[r, "i_{\mathcal{B}}"] & \mathsf{Ar}_R(\mathcal{B}) \arrow[r, "\mathrm{ev}_1"] & \mathcal{B}.
\end{tikzcd}$$

From this we get the following commutative diagram of ∞-categories:

$$\begin{tikzcd}
(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \arrow[r, "f^*"] \arrow[d] \arrow[dd, bend right=60, "\mathrm{Q}_{\mathcal{B}}"'] & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{A}})_{/f^*\mathsf{Ar}_R(\mathcal{B})} \arrow[r, "q^*"] \arrow[d] & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{A}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{A})} \arrow[d] \arrow[dd, bend left=60, "\mathrm{Q}_{\mathcal{A}}"] \\
(\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \arrow[r, "f^*"] \arrow[d, "\iota^*_{\mathcal{B}}"] & (\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}})_{/f^*\mathsf{Ar}_R(\mathcal{B})} \arrow[r, "q^*"] & (\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{A})} \arrow[d, "\iota^*_{\mathcal{A}}"] \\
\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \arrow[rr, "f^*"] & & \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}}.
\end{tikzcd} \tag{2}$$

Let us write $f^{\circledast}$ for the composite in the top row, which takes $\mathcal{E}\to\mathsf{Ar}_R(\mathcal{B})$ to the fiber product $\mathcal{E}\times_{\mathsf{Ar}_R(\mathcal{B})}\mathsf{Ar}_R(\mathcal{A})\to\mathsf{Ar}_R(\mathcal{A})$. Passing to vertical left adjoints now yields a Beck–Chevalley transformation

$$\mathrm{E}_{\mathcal{A}}f^* \to f^{\circledast}\mathrm{E}_{\mathcal{B}}.$$

Unwinding the definitions, this is given at $\mathcal{E}\to\mathcal{B}$ in $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ by the natural map

$$(\mathcal{E}\times_{\mathcal{B}}\mathcal{A})\times_{\mathcal{A}}\mathsf{Ar}_R(\mathcal{A}) \to (\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_R(\mathcal{B}))\times_{\mathsf{Ar}_R(\mathcal{B})}\mathsf{Ar}_R(\mathcal{A}).$$

This is an equivalence, so that we also have a commutative square

$$\begin{tikzcd}
\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \arrow[r, "f^*"] \arrow[d, "\mathrm{E}_{\mathcal{B}}"] & \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}} \arrow[d, "\mathrm{E}_{\mathcal{A}}"] \\
(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \arrow[r, "f^{\circledast}"] & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{A}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{A})}.
\end{tikzcd} \tag{3}$$

3 Algebraic patterns

In this section we will first review the basic definitions related to algebraic patterns and Segal objects in Section 3.1, and then look at some examples thereof in Section 3.2. We then introduce the condition of *soundness* for algebraic patterns in Section 3.3; this is somewhat technical, but turns out to be the key property needed for some of our results in the next section.

3.1 Algebraic patterns and Segal objects

In this subsection we review the definitions of algebraic patterns and Segal objects, and some related basic notions introduced in [7]. We also introduce a *relative* version of Segal objects, which will show up later.

Definition 3.1.1 An *algebraic pattern* is an ∞-category $\mathcal{O}$ equipped with a factorization system, whereby every morphism factors (uniquely up to equivalence) as an *inert* morphism followed by an *active* morphism, as well as a collection of *elementary objects*. We write $\mathcal{O}^{\mathrm{int}}$ and $\mathcal{O}^{\mathrm{act}}$ for the subcategories of $\mathcal{O}$ containing only the inert and active morphisms, respectively, and $\mathcal{O}^{\mathrm{el}}$ for the full subcategory of $\mathcal{O}^{\mathrm{int}}$ containing elementary objects and inert morphisms among them. For $X \in \mathcal{O}$, we also write

$$\mathcal{O}^{\mathrm{el}}_{X/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{int}}} \mathcal{O}^{\mathrm{int}}_{X/}$$

for the ∞-category of inert maps $X \to E$ with $E \in \mathcal{O}^{\mathrm{el}}$.

Notation 3.1.2 We often indicate inert maps as $X \rightarrowtail Y$ and active maps as $X \rightsquigarrow Y$. These arrows are not meant to indicate any particular intuition about inert or active morphisms.

Example 3.1.3 We write $\mathbb{F}_*$ for a skeleton of the category of pointed finite sets, with objects $\langle n\rangle := (\{0,1,\dots,n\},0)$, and we say a morphism $\varphi\colon \langle n\rangle \to \langle m\rangle$ is *inert* if φ restricts to an isomorphism $\langle n\rangle \setminus \varphi^{-1}(0) \to \langle m\rangle \setminus \{0\}$, and *active* if $\varphi^{-1}(0) = \{0\}$. Then the inert and active morphisms form a factorization system on $\mathbb{F}_*$, and we make this an algebraic pattern[4] by taking $\langle 1\rangle$ to be the single elementary object.

Example 3.1.4 Another basic example is $\mathbf{\Delta}^{\mathrm{op}}$, where $\mathbf{\Delta}$ is the simplex category. Recall that $\mathbf{\Delta}^{\mathrm{op}}$ admits an inert-active factorization system where inert maps are opposite to interval inclusions and active maps are opposite to maps preserving the maximal and minimal elements. To make $\mathbf{\Delta}^{\mathrm{op}}$ an algebraic pattern, we can take the elementary objects to be $[0]$ and $[1]$, in which case we denote the pattern by $\mathbf{\Delta}^{\mathrm{op},\natural}$, or alternatively just $[1]$, in which case the pattern is denoted by $\mathbf{\Delta}^{\mathrm{op},\flat}$.

The main reason for introducing algebraic patterns is that they describe algebraic structures via Segal conditions:

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[4] In [7] this pattern was denoted by $\mathbb{F}_*^{\flat}$ to distinguish it from the pattern $\mathbb{F}_*^{\natural}$, where the elementary objects are $\langle 0\rangle$ and $\langle 1\rangle$. However, in this paper $\mathbb{F}_* = \mathbb{F}_*^{\flat}$ is the key example, so we use a simplified notation for it.

Definition 3.1.5 A functor $F\colon \mathcal{O} \to \mathcal{C}$ is a *Segal $\mathcal{O}$-object* in the ∞-category $\mathcal{C}$ if for every object $X \in \mathcal{O}$ the induced functor

$$(\mathcal{O}^{\mathrm{el}}_{X/})^{\triangleleft} \to \mathcal{O} \xrightarrow{F} \mathcal{C}$$

is a limit diagram. If $\mathcal{C}$ has limits for diagrams indexed by $\mathcal{O}^{\mathrm{el}}_{X/}$ for all $X \in \mathcal{O}$, in which case we say that $\mathcal{C}$ is *$\mathcal{O}$-complete*, then this condition is equivalent to the canonical maps

$$F(X) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} F(E)$$

being equivalences. We refer to Segal $\mathcal{O}$-objects in the ∞-category $\mathcal{S}$ of spaces as *Segal $\mathcal{O}$-spaces* and Segal $\mathcal{O}$-objects in the ∞-category Cat_∞ of ∞-categories as *Segal $\mathcal{O}$-∞-categories*.

Example 3.1.6 We can identify $(\mathbb{F}_*)^{\mathrm{el}}_{\langle n\rangle/}$ with the set $\{\rho_i : i = 1, \dots, n\}$, where ρ_i is the inert morphism $\langle n\rangle \to \langle 1\rangle$ given by

$$\rho_i(j) = \begin{cases} 0, & j \neq i, \\ 1, & j = i. \end{cases}$$

A functor $F\colon \mathbb{F}_* \to \mathcal{C}$ is then a Segal $\mathbb{F}_*$-object if for every n the map

$$F(\langle n\rangle) \to \prod_{i=1}^{n} F(\langle 1\rangle),$$

induced by the maps ρ_i, is an equivalence. Thus Segal $\mathbb{F}_*$-objects are precisely commutative monoids in the sense of [23, Section 2.4.2]. For $\mathcal{C} = \mathcal{S}$, this gives the ∞-categorical analogue of *special Γ-spaces* in the sense of Segal [30].

Example 3.1.7 Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-spaces are precisely *Segal spaces* in the sense of [27], while Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-objects in $\mathcal{C}$ are associative monoids (or E_1-algebras).

Later on, we will also need to consider a *relative* version of Segal objects:

Definition 3.1.8 Let $\mathcal{O}$ be an algebraic pattern and $\mathcal{C}$ an $\mathcal{O}$-complete ∞-category. A *relative Segal $\mathcal{O}$-object* of $\mathcal{C}$ is a morphism $\pi\colon Y \to X$ in $\mathsf{Fun}(\mathcal{O}, \mathcal{C})$ such that for every $O \in \mathcal{O}$ the natural commutative square

$$\begin{tikzcd}
Y(O) \arrow[r] \arrow[d, "\pi(O)"'] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} Y(E) \arrow[d, "\lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \pi(E)"] \\
X(O) \arrow[r] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} X(E)
\end{tikzcd}$$

is cartesian. We denote by $\mathsf{Seg}^{/X}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{O}, \mathcal{C})_{/X}$ the full subcategory whose objects are the X-relative Segal $\mathcal{O}$-objects.

Observation 3.1.9 If $Y \to X$ is a relative Segal $\mathcal{O}$-object of $\mathcal{C}$, then the pasting lemma for cartesian squares implies that a morphism $Z \to Y$ is a relative Segal $\mathcal{O}$-object if and only if the composite $Z \to X$ is one. Moreover, a morphism $X \to *$ to the terminal object is a relative Segal $\mathcal{O}$-object if and only if X is a Segal $\mathcal{O}$-object in $\mathcal{C}$. Combining these two observations, we see that if X is a Segal $\mathcal{O}$-object of $\mathcal{C}$ then an X-relative Segal $\mathcal{O}$-object is just a Segal $\mathcal{O}$-object with a map to X, ie we have

$$\mathsf{Seg}_{\mathcal{O}}^{/X}(\mathcal{C}) = \mathsf{Seg}_{\mathcal{O}}(\mathcal{C})_{/X}$$

as full subcategories of $\mathsf{Fun}(\mathcal{C}, \mathcal{O})_{/X}$.

Lemma 3.1.10 *Suppose $X \to Y$ is a relative Segal $\mathcal{O}$-object in $\mathcal{C}$. Then for any map $\eta\colon Y' \to Y$, the pullback $X' := X \times_Y Y' \to Y'$ is also a relative Segal $\mathcal{O}$-object. In other words, pullback along η gives a functor $\eta^*\colon \mathsf{Seg}_{\mathcal{O}}^{/Y}(\mathcal{C}) \to \mathsf{Seg}_{\mathcal{O}}^{/Y'}(\mathcal{C})$.*

Proof For $O \in \mathcal{O}$, consider the commutative cube

$$\begin{tikzcd}
X'(O) \arrow[rr] \arrow[dr] \arrow[dd] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} X'(E) \arrow[dr] \arrow[dd] & \\
& X(O) \arrow[rr, crossing over] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} X(E) \arrow[dd] \\
Y'(O) \arrow[rr] \arrow[dr] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} Y'(E) \arrow[dr] & \\
& Y(O) \arrow[rr] \arrow[from=uu, crossing over] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} Y(E).
\end{tikzcd}$$

Here the left, right, and front faces are all cartesian, hence so is the back face. □

Lemma 3.1.11 *For every presentable ∞-category $\mathcal{C}$ the full subcategory*

$$\mathsf{Seg}_{\mathcal{O}}^{/X}(\mathcal{C}) \subseteq \mathsf{Fun}(\mathcal{O}, \mathcal{C})_{/X}$$

is an accessible localization. In particular, it is a presentable ∞-category.

Proof Consider the following collection of morphisms in $\mathsf{Fun}(\mathcal{O}, \mathcal{C})$:

$$\{\operatorname{colim}_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} y(E) \otimes C \to y(X) \otimes C\}_{X \in \mathcal{O}, C \in K},$$

where K is a set of κ-compact generators for $\mathcal{C}$, y is the Yoneda embedding for $\mathcal{O}^{\mathrm{op}}$, and $T \otimes C$ for $T \in \mathcal{S}$, $C \in \mathcal{C}$, is the canonical tensoring of $\mathcal{C}$ with $\mathcal{S}$, given by the colimit over T of the constant diagram with value C. A morphism $X \to Y$ in $\mathsf{Fun}(\mathcal{O}, \mathcal{C})$ is a relative Segal $\mathcal{O}$-object if and only if it is right orthogonal to this set of morphisms, hence the claim follows from Observation 2.3.6. □

Next, we take a brief look at morphisms between patterns:

Definition 3.1.12 If $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, a *morphism of algebraic patterns* is a functor $f\colon \mathcal{O} \to \mathcal{P}$ that preserves inert and active morphisms as well as elementary objects. We say that such a morphism is a *Segal morphism* if for every Segal $\mathcal{P}$-space F and every $X \in \mathcal{O}$ the functor $f^{\mathrm{el}}_{X/}\colon \mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{P}^{\mathrm{el}}_{f(X)/}$ arising from f induces an equivalence

$$\lim_{\mathcal{P}^{\mathrm{el}}_{f(X)/}} F \xrightarrow{\sim} \lim_{\mathcal{O}^{\mathrm{el}}_{X/}} F \circ f;$$

by [7, Lemma 4.5] this is equivalent to composition with f giving a functor

$$f^*\colon \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$$

for any $\mathcal{O}$-complete ∞-category $\mathcal{C}$. The Segal morphisms that occur in practice are those where the functor $f^{\mathrm{el}}_{X/}$ is coinitial for all $X \in \mathcal{O}$; if this is the case we say that f is a *strong Segal morphism.* In the special case where $f^{\mathrm{el}}_{X/}$ is an *equivalence* for every X, we say that f is an *iso-Segal morphism.*

Example 3.1.13 There is a morphism of algebraic patterns $\mathfrak{c}\colon \mathbf{\Delta}^{\mathrm{op},\flat} \to \mathbb{F}_*$, given on objects by $\mathfrak{c}([n]) = \langle n\rangle$, and with $\mathfrak{c}(\varphi)\colon \langle n\rangle \to \langle m\rangle$ for a morphism $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ given by

$$\mathfrak{c}(\varphi)(i) = \begin{cases} j & \text{if } \varphi(j-1) < i \le \varphi(j), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is an iso-Segal morphism.

Notation 3.1.14 We write $\mathrm{AlgPatt}$ for the ∞-category of algebraic patterns together with all morphisms of algebraic patterns.

Observation 3.1.15 Composition with a *strong* Segal morphism $f\colon \mathcal{O} \to \mathcal{P}$ also preserves *relative* Segal objects: If $X \to Y$ is a relative Segal $\mathcal{P}$-object in $\mathcal{C}$, then for $O \in \mathcal{O}$ we have a commutative diagram

$$\begin{tikzcd}
X(f(O)) \arrow[r] \arrow[d] & \lim_{E \in \mathcal{P}^{\mathrm{el}}_{f(O)/}} X(E) \arrow[r, "\sim"] \arrow[d] & \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{O/}} X(f(E')) \arrow[d] \\
Y(f(O)) \arrow[r] & \lim_{E \in \mathcal{P}^{\mathrm{el}}_{f(O)/}} Y(E) \arrow[r, "\sim"] & \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{O/}} Y(f(E'));
\end{tikzcd}$$

here both the left and right squares are cartesian, and hence so is the outer composite square. Composition with f thus gives a functor $f^*\colon \mathrm{Seg}^{/Y}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}^{/f^*Y}_{\mathcal{O}}(\mathcal{C})$.

We now recall a simple criterion for a Segal morphism to give an equivalence on Segal objects:

Proposition 3.1.16 [2, Corollary 2.64] *Suppose $\mathcal{O}$ and $\mathcal{P}$ are algebraic patterns, and $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism such that*

1. *$f^{\mathrm{el}}\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence of ∞-categories,*
2. *for every $O \in \mathcal{O}$, the functor $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ is an equivalence of ∞-groupoids.*

Then for any complete ∞-category $\mathcal{C}$ the functor $f^*\colon \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ is an equivalence, with inverse given by right Kan extension along f.

Proof We refer to [2, Section 2] for a detailed proof, but since this result will play an important role in this paper we recall the key steps for the reader's convenience.

By [7, Proposition 6.3], (2) implies that right Kan extension along f restricts to Segal objects, giving an adjunction

$$f^*\colon \mathrm{Seg}_{\mathcal{P}}(\mathcal{C}) \rightleftarrows \mathrm{Seg}_{\mathcal{O}}(\mathcal{C}) : f_*.$$

Moreover, the proof of [7, Proposition 6.3] shows that for $F \in \mathrm{Seg}_{\mathcal{O}}(\mathcal{C})$ we have $(f_*F)|_{\mathcal{P}^{\mathrm{el}}} \simeq f_*^{\mathrm{el}}(F|_{\mathcal{O}^{\mathrm{el}}})$.

Condition (1) therefore implies that the counit $f^*f_*F \to F$ is an equivalence for $F \in \mathrm{Seg}_{\mathcal{P}}(\mathcal{C})$, since it is an equivalence when evaluated on $\mathcal{P}^{\mathrm{el}}$. Moreover, (1) implies that f^* is conservative on Segal objects, again since equivalences are detected on elementary objects. To see that the unit map $G \to f_*f^*G$ is an equivalence, it then suffices to check this after applying f^*, but then the adjunction $f^* \dashv f_*$ implies that the composite

$$f^*G \to f^*f_*f^*G \xrightarrow{\sim} f^*G$$

is an equivalence, and here we already saw that the counit is an equivalence. Since both the unit and counit of the adjunction are natural equivalences, it must be an equivalence of ∞-categories. □

Remark 3.1.17 If $(\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}$ is a Segal $\mathcal{O}$-space and $(\mathcal{P}^{\mathrm{act}}_{/f(O)})^{\simeq}$ is a Segal $\mathcal{P}$-space in Proposition 3.1.16, then it suffices to check (2) for elementary objects in $\mathcal{O}$. This holds, for instance, if $\mathcal{O}$ and $\mathcal{P}$ are extendable (see Definition 3.3.16).

3.2 Examples of algebraic patterns

We now look at some examples of algebraic patterns. Our focus here will be on examples that will be relevant in the next sections; we refer the reader to [7, Section 3] for many other examples.

Example 3.2.1 We have patterns $\mathbf{\Delta}^{n,\mathrm{op},\natural}$ and $\mathbf{\Delta}^{n,\mathrm{op},\flat}$ with underlying category $\mathbf{\Delta}^{n,\mathrm{op}} := (\mathbf{\Delta}^{\mathrm{op}})^{\times n}$, equipped with the factorization system where the inert and active maps are those that are inert or active in $\mathbf{\Delta}^{\mathrm{op}}$ in each component. Here $(\mathbf{\Delta}^{n,\mathrm{op},\flat})^{\mathrm{el}} = \{([1],\dots,[1])\}$ while $(\mathbf{\Delta}^{n,\mathrm{op},\natural})^{\mathrm{el}}$ consists of all objects whose components are all either $[0]$ or $[1]$. Then Segal $\mathbf{\Delta}^{n,\mathrm{op},\natural}$-spaces are n-uple Segal spaces, which model n-fold ∞-categories, while Segal $\mathbf{\Delta}^{n,\mathrm{op},\flat}$-objects are $\mathbb{E}_n$-algebras (by the Dunn–Lurie additivity theorem).

Example 3.2.2 Let $\mathbf{\Theta}_n$ be the inductively defined wreath product $\mathbf{\Delta} \wr \mathbf{\Theta}_{n-1}$, starting with $\mathbf{\Theta}_0 = [0]$; see, for example, [5; 16] for more details. This has a factorization system where the active/inert maps are those whose components in $\mathbf{\Delta}$ and $\mathbf{\Theta}_{n-1}$ are both active or inert. There are two interesting pattern structures on $\mathbf{\Theta}_n^{\mathrm{op}}$: if we define the objects C_i in $\mathbf{\Theta}_n$ by $C_0 := [0]()$ and $C_i := [1](C_{i-1})$ for $i = 1,\dots,n$, then for $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ we take C_n to be the only elementary object, while for $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ we take all of $C_0,\dots,C_n$. Then Segal $\mathbf{\Theta}_n^{\mathrm{op},\natural}$-spaces are Rezk's model for (∞,n)-categories [28], while Segal $\mathbf{\Theta}_n^{\mathrm{op},\flat}$-objects are again $\mathbb{E}_n$-algebras (see [4]).

Example 3.2.3 Let $\mathbb{F}_*^{\leq k} \subseteq \mathbb{F}_*$ denote the full subcategory containing pointed finite sets of cardinality $\leq k$ (excluding the basepoint). Consider $\mathbb{F}_*^{\leq k}$ as an algebraic pattern by restricting the inert-active factorization system on $\mathbb{F}_*$ and choosing $\langle 1 \rangle$ to be the only elementary object. Segal objects for $\mathbb{F}_*^{\leq k}$ are *arity k-restricted commutative monoids* — a variant of commutative monoids in which the homotopy coherence data is only supplied up to arity k. More generally, if $\mathcal{O}$ is an ∞-operad then $\mathcal{O}^{\leq k} := \mathbb{F}_*^{\leq k} \times_{\mathbb{F}_*} \mathcal{O}$ has a natural structure of an algebraic pattern whose Segal objects are arity k-restricted $\mathcal{O}$-monoids. For more details see [2].

The remaining examples we want to discuss are all instances of a general class of algebraic patterns on ∞-categories of spans. For this purpose we briefly recall the construction of such ∞-categories — this is originally due to Barwick [3]; see also [18] for a more "model-independent" version.

Construction 3.2.4 Let $\mathfrak{X}$ be an ∞-category equipped with a pair of wide subcategories $\mathfrak{X}^b$ and $\mathfrak{X}^f$ (where "b" stands for *backwards* and "f" stands for *forwards*. Following Barwick, we say that the triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ is *adequate* if for every pair of morphisms $\beta\colon x \to y$ in $\mathfrak{X}^b$ and $\varphi\colon y' \to y$ in $\mathfrak{X}^f$, we have

(1) the pullback $x' := x \times_y y'$ exists in $\mathfrak{X}$,

(2) the projection $x' \to y'$ lies in $\mathfrak{X}^b$,

(3) the projection $x' \to x$ lies in $\mathfrak{X}^f$.

Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ Barwick defines an ∞-category $\mathsf{Span}_{b,f}(\mathfrak{X})$ (denoted in [3] by $\mathrm{A}^{\mathrm{eff}}(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$) such that the objects of $\mathsf{Span}_{b,f}(\mathfrak{X})$ are the objects of $\mathfrak{X}$ and the morphisms from x to y are spans (or correspondences)

$$\begin{tikzcd}
 & w \arrow[dl, "\beta"'] \arrow[dr, "\varphi"] & \\
x & & y
\end{tikzcd}$$

where the arrow β lies in $\mathfrak{X}^b$ and the arrow φ lies in $\mathfrak{X}^f$. The assumption that the triple is adequate allows for a composition law defined by taking pullbacks. If $\mathfrak{X}$ is an ∞-category with pullbacks, then we can take $\mathfrak{X}^b = \mathfrak{X}^f = \mathfrak{X}$, in which case we just write $\mathsf{Span}(\mathfrak{X})$ for the corresponding ∞-category of spans.

Observation 3.2.5 By the first part of [18, Proposition 4.9] the ∞-category $\mathsf{Span}_{b,f}(\mathfrak{X})$ always has a factorization system given by the classes of maps as above with φ or β required to be an equivalence (which we might call the "backwards" and "forwards" maps) and the subcategories of these maps are equivalent to $\mathfrak{X}^{b,\mathrm{op}}$ and $\mathfrak{X}^f$, respectively.

Definition 3.2.6 Given an adequate triple $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and a full subcategory $\mathfrak{X}_0 \subseteq \mathfrak{X}$, we denote by $\mathsf{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)$ the algebraic pattern given by $\mathsf{Span}_{b,f}(\mathfrak{X})$ with the factorization system whose inert and active maps are the backwards and forward maps, respectively, and with the objects of $\mathfrak{X}_0$ as the elementary objects.

Remark 3.2.7 The Segal condition for $\mathsf{Span}_{b,f}(\mathcal{X};\mathcal{X}_0)$ takes the following form for a functor F:

$$F(x) \simeq \lim_{e \to x \in (\mathcal{X}^b_{0/x})^{\mathrm{op}}} F(e),$$

where $\mathsf{Span}_{b,f}(\mathcal{X})^{\mathrm{el}}_{x/} \simeq (\mathcal{X}^b_{0/x})^{\mathrm{op}}$ with $\mathcal{X}^b_{0/x} := \mathcal{X}^b_0 \times_{\mathcal{X}^b} \mathcal{X}^b_{/x}$ and $\mathcal{X}^b_0$ is the full subcategory of $\mathcal{X}^b$ containing the objects of $\mathcal{X}_0$.

Example 3.2.8 Let $\mathbb{F}$ denote the category of finite sets. Since this has pullbacks, Construction 3.2.4 produces an ∞-category (in fact a (2,1)-category) $\mathsf{Span}(\mathbb{F})$ whose objects are finite sets, and whose morphisms are spans of the form

$$\begin{tikzcd} & \boldsymbol{m} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \boldsymbol{n} & & \boldsymbol{n}' \end{tikzcd}$$

for finite sets $\boldsymbol{n}$, $\boldsymbol{m}$, and $\boldsymbol{n}'$, with composition given by taking pullbacks. We consider $\mathsf{Span}(\mathbb{F}) = \mathsf{Span}(\mathbb{F};\{\mathbf{1}\})$ as an algebraic pattern by taking the backward maps as inerts, forward maps as actives and $\mathbf{1} \in \mathsf{Span}(\mathbb{F})$ as the only elementary object.

Observation 3.2.9 The category $\mathbb{F}_*$ may be thought of as the wide subcategory $\mathsf{Span}_{\mathrm{inj,all}}(\mathbb{F})$ of $\mathsf{Span}(\mathbb{F})$ containing only those morphisms where the backwards map is injective. The inert-active factorization system on $\mathbb{F}_*$ then coincides with the one obtained by restriction from $\mathsf{Span}(\mathbb{F})$, and the inclusion $\mathbb{F}_* \to \mathsf{Span}(\mathbb{F})$ is an iso-Segal morphism.

Example 3.2.10 Let G be a finite group and $\mathbb{F}_G$ the category of finite G-sets. Denote by $\mathrm{Orb}_G \subseteq \mathbb{F}_G$ the collection of G-orbits (ie transitive G-sets). Since $\mathbb{F}_G$ has pullbacks we have an ∞-category (really a (2,1)-category) $\mathsf{Span}(\mathbb{F}_G)$. Abusing notation slightly, we also denote the span pattern with the orbits as elementary objects by $\mathsf{Span}(\mathbb{F}_G) := \mathsf{Span}(\mathbb{F}_G;\mathrm{Orb}_G)$. Segal objects for this pattern are precisely G-commutative monoids in the sense of [25]; they also appear in [9] where they are called semi-Mackey functors. More generally, for any full subcategory $\mathcal{F} \subseteq \mathrm{Orb}_G$ we have a span pattern $\mathsf{Span}(\mathbb{F}_G;\mathcal{F})$ whose Segal objects may be thought of as G-commutative monoids that are Borel-$\mathcal{F}$-complete. Segal objects for $\mathsf{Span}(\mathbb{F}_G;\{G/e\})$ appear implicitly in [9], where they are called Borel-equivariant.

Example 3.2.11 As a variant of the previous example, we can consider subcategories $\mathbb{F}^f_G$ of $\mathbb{F}_G$ that are closed under base change; if $\mathbb{F}^f_G$ is moreover closed under finite coproducts, this data is equivalent to an *indexing system* in the sense of [6]. We can then define the span pattern $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G) := \mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G;\mathrm{Orb}_G)$, whose Segal objects we can think of as G-commutative monoids where only transfers that lie in $\mathbb{F}^f_G$ are allowed. As an illustrative example we may consider the extreme case where all forward maps are isomorphisms, ie $\mathbb{F}^f_G := \mathbb{F}^{\simeq}_G$. The corresponding span pattern $\mathsf{Span}_{\mathrm{all},\simeq}(\mathbb{F}_G;\mathrm{Orb}_G)$ has an underlying ∞-category equivalent to $\mathbb{F}^{\mathrm{op}}_G$ with all the maps inert and with $\mathrm{Orb}^{\mathrm{op}}_G$ as the subcategory of elementary objects. Segal objects for this pattern are thus equivalent to presheaves on Orb_G, and by Elemendorf's theorem this ∞-category is equivalent to that of G-spaces.

Example 3.2.12 A space $X \in \mathcal{S}$ is called *m-finite* if it is m-truncated and all of its homotopy groups are finite; we let $\mathcal{S}_m \subseteq \mathcal{S}$ denote the full subcategory of m-finite spaces. Since m-finite spaces are closed under finite limits we may consider the span pattern $\mathsf{Span}(\mathcal{S}_m) := \mathsf{Span}(\mathcal{S}_m; *)$. If we write $\mathcal{S}_m^{n\text{-tr}}$ for the wide subcategory of $\mathcal{S}_m$ whose maps are n-truncated, then $(\mathcal{S}_m, \mathcal{S}_m^{n\text{-tr}}, \mathcal{S}_m)$ is also an adequate triple, and we can likewise consider the pattern

$$\mathsf{Span}_{n\text{-tr},\mathrm{all}}(\mathcal{S}_m) := \mathsf{Span}_{n\text{-tr},\mathrm{all}}(\mathcal{S}_m; *)$$

for any n. For $n = m-1$, the Segal objects for $\mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m)$ are precisely the *m-commutative monoids* of Harpaz [14]. It also follows from [14, Proposition 5.14] that these are equivalent to Segal objects for $\mathsf{Span}(\mathcal{S}_m)$.

3.3 Sound patterns

In this subsection we define the notion of a *sound* pattern — a technical condition satisfied in almost all the usual examples. This requires first introducing some notation:

Notation 3.3.1 Fix a morphism $\omega\colon X \to Y$ in an algebraic pattern $\mathcal{O}$. For every elementary object $(\alpha\colon Y \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{Y/}$ we denote the inert-active factorization of $\alpha \circ \omega$ as follows:

$$\begin{tikzcd}
X \arrow[r, tail, "\omega_\alpha"] \arrow[d, "\omega"'] & \omega_{\alpha!}X \arrow[d, squiggly, "(\alpha\circ\omega)^{\mathrm{act}}"] \\
Y \arrow[r, tail, "\alpha"] & E.
\end{tikzcd}$$

Factorization defines a functor $\omega_{(-)}\colon \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\mathrm{int}}_{X/}$ by sending α to ω_α.

Definition 3.3.2 For $\omega\colon X \rightsquigarrow Y$ we define $\mathcal{O}^{\mathrm{el}}(\omega)$ as the pullback

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{el}}(\omega) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathcal{O}^{\mathrm{int}}_{X/}) \arrow[d, "{(s,t)}"] \\
\mathcal{O}^{\mathrm{el}}_{Y/} \times \mathcal{O}^{\mathrm{el}}_{X/} \arrow[r, "{(\omega_{(-)},\mathrm{id})}"] & \mathcal{O}^{\mathrm{int}}_{X/} \times \mathcal{O}^{\mathrm{int}}_{X/}.
\end{tikzcd}$$

An object in $\mathcal{O}^{\mathrm{el}}(\omega)$ can thus be represented by a diagram in $\mathcal{O}$ of the following shape:

$$\begin{tikzcd}
X \arrow[r, tail, "\omega_\alpha"] \arrow[d, squiggly, "\omega"'] & \omega_{\alpha!}X \arrow[r, tail] \arrow[d, squiggly] & E' \\
Y \arrow[r, tail, "\alpha"] & E,
\end{tikzcd}$$

where the arrows labeled by $\rightarrowtail$ and $\rightsquigarrow$ are required to be inert and active, respectively, E and E' are elementary, and ω is fixed. Morphisms in $\mathcal{O}^{\mathrm{el}}(\omega)$ are natural transformations of such diagrams that are constant at $\omega\colon X \rightsquigarrow Y$ and inert at all other objects.

Remark 3.3.3 By construction $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/} \times \mathcal{O}^{\mathrm{el}}_{X/}$ is the bifibration (see [22, Definition 2.4.7.2]) corresponding to the functor

$$(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \times \mathcal{O}^{\mathrm{el}}_{X/} \to \mathcal{S}, \quad (\alpha\colon Y \rightarrowtail E, \beta\colon X \rightarrowtail E') \mapsto \mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha, \beta).$$

Definition 3.3.4 We say that a pattern $\mathcal{O}$ is *sound* if for every active morphism $\omega\colon X \rightsquigarrow Y$ the functor $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/}$ is coinitial.

The point of introducing the condition of soundness is that it allows us to rewrite certain double limits, as described below in Lemma 3.3.8. Before we state this property we look at a first example, namely $\mathbb{F}_*$, where soundness is particularly easy to check; further examples will be given below.

Example 3.3.5 In the pattern $\mathbb{F}_*$ an active morphism $\omega\colon X_+ \rightsquigarrow Y_+$ is simply a map $\omega\colon X \to Y$ in $\mathbb{F}$. The inert undercategory $(\mathbb{F}_*)^{\mathrm{int}}_{Y_+/}$ may be identified with the poset $(\mathsf{Sub}(Y), \supseteq)$ of subsets of Y, by assigning to each $\gamma\colon Y_+ \rightarrowtail Z_+$ the subset $\gamma^{-1}(Z) \subset Y$. The category of elementary objects under Y_+ is given by the one-element subsets, and we may hence identify it with Y itself. For an elementary $\alpha\colon Y_+ \rightarrowtail E$ corresponding to $e \in Y$, the pushforward $\omega_{\alpha!}X_+$ can be identified with $\omega^{-1}(e)_+ \subset X_+$. Hence we have a cartesian square:

$$\begin{tikzcd}
\mathbb{F}^{\mathrm{el}}_*(\omega) \arrow[r] \arrow[d] & \mathsf{Ar}(\mathsf{Sub}(X)) \arrow[d, "{(t,s)}"] \\
X \times Y \arrow[r, "{(\mathrm{id},\omega^{-1})}"] & \mathsf{Sub}(X) \times \mathsf{Sub}(X)
\end{tikzcd}$$

and so $\mathbb{F}^{\mathrm{el}}_*(\omega)$ is the poset of pairs $(x, y) \in X \times Y$ such that $\{x\} \subset \omega^{-1}(y)$. In other words, $y = \omega(x)$ and hence the map $\mathbb{F}^{\mathrm{el}}_*(\omega) \to (\mathbb{F}_*)^{\mathrm{el}}_{X_+/} \simeq X$ is an equivalence. In particular it is coinitial and thus $\mathbb{F}_*$ is sound.

Observation 3.3.6 The composite $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/} \times \mathcal{O}^{\mathrm{el}}_{Y/} \to \mathcal{O}^{\mathrm{el}}_{Y/}$ is by construction a cartesian fibration. Its straightening is the functor

$$(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \xrightarrow{\omega_{(-)}} (\mathcal{O}^{\mathrm{int}}_{X/})^{\mathrm{op}} \xrightarrow{\mathcal{O}^{\mathrm{el}}_{-/}} \mathsf{Cat}$$

that sends $\alpha\colon Y \rightarrowtail E$ to the ∞-category $\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}X/}$ of elementaries under $\omega_{\alpha!}X$. Our definition of $\mathcal{O}^{\mathrm{el}}(\omega)$ therefore matches that given in [7, Remark 7.6]. Moreover, a limit over $\mathcal{O}^{\mathrm{el}}(\omega)$ can be rewritten as a double limit, that is for $F\colon \mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{C}$ we have

$$\lim_{\mathcal{O}^{\mathrm{el}}(\omega)} F \simeq \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{Y/}} \lim_{\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}X/}} F.$$

If $\mathcal{O}$ is sound, then we can use this to rewrite a limit over $\mathcal{O}^{\mathrm{el}}_{X/}$ as a double limit.

We now show that soundness is inherited along iso-Segal morphisms. Together with Example 3.3.5 this implies that all cartesian patterns of [8, Definition 2.6] are sound, and in particular any ∞-operad in the sense of Lurie is sound.

Lemma 3.3.7 *Let $f : \mathcal{O} \to \mathcal{P}$ be an iso-Segal morphism of algebraic patterns. Then $\mathcal{O}$ is sound, if $\mathcal{P}$ is sound. The converse implication holds if we further assume that*

$$\mathsf{Ar}^{\mathrm{act}}(f) \colon \mathsf{Ar}^{\mathrm{act}}(\mathcal{O}) \to \mathsf{Ar}^{\mathrm{act}}(\mathcal{P})$$

is essentially surjective.

Proof Being a morphism of algebraic patterns, f induces for each active $\omega \colon X \rightsquigarrow Y$ a morphism of cartesian fibrations

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{el}}(\omega) \arrow[r, "f"] \arrow[d] & \mathcal{P}^{\mathrm{el}}(f(\omega)) \arrow[d] \\
\mathcal{O}^{\mathrm{el}}_{Y/} \arrow[r, "f", "\simeq"'] & \mathcal{P}^{\mathrm{el}}_{f(Y)/},
\end{tikzcd}$$

where the bottom functor is an equivalence because we assumed that f is iso-Segal. On the fibers over some $(\alpha \colon Y \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{Y/}$ and $f(\alpha) \in \mathcal{P}^{\mathrm{el}}_{f(Y)/}$ we get an induced functor

$$f : \mathcal{O}^{\mathrm{el}}_{\alpha_! X/} \to \mathcal{P}^{\mathrm{el}}_{f(\alpha_! X)/},$$

which again is an equivalence because f is iso-Segal. Therefore it follows that $f : \mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{P}^{\mathrm{el}}(f(\omega))$ is an equivalence. This is also the top map in the square

$$\begin{tikzcd}
\mathcal{O}^{\mathrm{el}}(\omega) \arrow[r, "f", "\simeq"'] \arrow[d] & \mathcal{P}^{\mathrm{el}}(f(\omega)) \arrow[d] \\
\mathcal{O}^{\mathrm{el}}_{X/} \arrow[r, "f", "\simeq"'] & \mathcal{P}^{\mathrm{el}}_{f(X)/}.
\end{tikzcd}$$

Here the right functor is coinitial because $\mathcal{P}$ is sound, and hence the left functor is coinitial, which proves that $\mathcal{O}$ is sound.

For the converse implication, let $\rho \colon X' \rightsquigarrow Y'$ be some active morphism in $\mathcal{P}$. Because we assume that $\mathsf{Ar}^{\mathrm{act}}(f)$ is essentially surjective, we can write $\rho = f(\omega)$ for $\omega \colon X \rightsquigarrow Y$ active in $\mathcal{O}$ as before. Then the above argument shows that $\mathcal{P}^{\mathrm{el}}(\rho) \to \mathcal{P}^{\mathrm{el}}(Y')$ must be coinitial, as it is equivalent to $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}(Y)$. □

The crucial application of soundness for us will be through the following lemma: this will be used in the proof of Lemma 4.2.4, which is how the assumption of soundness enters our main theorem.

Lemma 3.3.8 *Let $\mathcal{O}$ be a sound pattern and $\mathcal{C}$ a sufficiently complete ∞-category. Consider a natural transformation $(\eta \colon F \Rightarrow G) \colon \mathcal{O}^{\mathrm{int}}_{X/} \to \mathcal{C}$ such that for all $X \rightarrowtail X' \in \mathcal{O}^{\mathrm{int}}_{X/}$ the square*

$$\begin{tikzcd}
F(X') \arrow[r] \arrow[d, "\eta_{X'}"'] & \lim_{X' \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{X'/}} F(E) \arrow[d, "\lim \eta_E"] \\
G(X') \arrow[r] & \lim_{X' \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{X'/}} G(E)
\end{tikzcd}$$

is cartesian. Then for every active morphism $\omega\colon X \rightsquigarrow Y$ *the square*

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "\eta_X"'] & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) \arrow[d, "\lim\eta_E"] \\
G(X) \arrow[r] & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X)
\end{tikzcd}$$

is cartesian.

Proof Consider the commutative cube

$$\begin{tikzcd}
F(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \lim_{\alpha\colon Y\rightarrowtail E'\in\mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) \arrow[dr] \arrow[dd] & \\
& \lim_{\beta\colon X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X/}} F(E) \arrow[rr, "\sim"] \arrow[dd] & & \lim_{(\alpha\colon Y\rightarrowtail E',\gamma\colon \omega_{\alpha!}X\rightarrowtail E)\in\mathcal{O}^{\mathrm{el}}(\omega)} F(E) \arrow[dd] \\
G(X) \arrow[rr] \arrow[dr] & & \lim_{\alpha\colon Y\rightarrowtail E'\in\mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X) \arrow[dr] & \\
& \lim_{\beta\colon X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X/}} G(E) \arrow[rr, "\sim"] & & \lim_{(\alpha\colon Y\rightarrowtail E',\gamma\colon \omega_{\alpha!}X\rightarrowtail E)\in\mathcal{O}^{\mathrm{el}}(\omega)} G(E).
\end{tikzcd}$$

The front horizontal maps are equivalences because $\mathcal{O}$ is assumed to be sound and hence $\mathcal{O}^{\mathrm{el}}(\omega)\to\mathcal{O}^{\mathrm{el}}_{X/}$ is coinitial. The left square is cartesian by applying the assumption. We would like to show that the back square is cartesian and by pullback pasting it will suffice to show that the right square is cartesian. We may write the limit over $\mathcal{O}^{\mathrm{el}}(\omega)$ as a double limit, by first right Kan extending along the cartesian fibration $\mathcal{O}^{\mathrm{el}}(\omega)\to\mathcal{O}^{\mathrm{el}}_{Y/}$, which is computed by taking limits over the fibers $\mathcal{O}^{\mathrm{el}}_{\omega_{\alpha!}Y/}$, and then taking the limit over $\mathcal{O}^{\mathrm{el}}_{Y/}$. Using this reformulation the right square can be written as an $\mathcal{O}^{\mathrm{el}}_{Y/}$-limit of diagrams of the form

$$\begin{tikzcd}
F(\omega_{\alpha!}X) \arrow[r] \arrow[d, "\eta_{\omega_{\alpha!}X}"'] & \lim_{\omega_{\alpha!}X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X'/}} F(E) \arrow[d, "\lim\eta_E"] \\
G(\omega_{\alpha!}X) \arrow[r] & \lim_{\omega_{\alpha!}X\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{X'/}} G(E).
\end{tikzcd}$$

Each of these diagrams is cartesian by assumption, and hence so is their limit. □

We will now check explicitly that the examples of patterns we discussed above are indeed sound. To do so, the following observation will be useful:

Lemma 3.3.9 *For an algebraic pattern* $\mathcal{O}$ *the following conditions are equivalent*:

1. (1) $\mathcal{O}$ *is sound.*
2. (2) *For every active morphism* $\omega\colon X\rightsquigarrow Y$ *and* $\beta\colon X\rightarrowtail E'\in\mathcal{O}^{\mathrm{el}}_{X/}$, *the* ∞*-category*
$$\mathcal{O}^{\mathrm{el}}_{\beta}(\omega) := \mathcal{O}^{\mathrm{el}}_{Y/}\times_{\mathcal{O}^{\mathrm{int}}_{X/}}(\mathcal{O}^{\mathrm{int}}_{X/})_{/\beta}$$
is weakly contractible.
3. (3) *For every* ω *and* β *as in* (2) *we have* $\operatorname{colim}_{\alpha\in(\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}}}\mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha,\beta)\simeq *$.

Proof ((1) $\Leftrightarrow$ (2)) The functor $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{X/}$ is a cocartesian fibration. By the dual of [22, Theorem 4.1.3.2.] it is coinitial if and only if its fibers are weakly contractible. Unwinding definitions yields the following description of the straightening:

$$\mathcal{O}^{\mathrm{el}}_{X/} \to \mathsf{Cat}, \quad (\beta\colon X \to E') \mapsto \mathcal{O}^{\mathrm{el}}_{Y/} \times_{\mathcal{O}^{\mathrm{int}}_{X/}} (\mathcal{O}^{\mathrm{int}}_{X/})_{/\beta}.$$

((2) $\Leftrightarrow$ (3)) Since $\mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/} \times \mathcal{O}^{\mathrm{el}}_{X/}$ is a bifibration, passing to the fiber over $\beta \in \mathcal{O}^{\mathrm{el}}_{X/}$ and taking opposites yields a left fibration $q\colon \mathcal{O}^{\mathrm{el}}_{\beta}(\omega)^{\mathrm{op}} \to (\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}}$. By [22, Corollary 3.3.4.6], the ∞-groupoid $|\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)| \simeq |\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)^{\mathrm{op}}|$ can be computed as the colimit of the straightening $\mathrm{St}(q)$, which is given by

$$\mathsf{St}(q)\colon (\mathcal{O}^{\mathrm{el}}_{Y/})^{\mathrm{op}} \to \mathcal{S}, \quad (\alpha\colon Y \to E) \mapsto \mathsf{Map}_{\mathcal{O}^{\mathrm{int}}_{X/}}(\omega_\alpha, \beta). \qquad \square$$

Observation 3.3.10 Suppose that $\mathcal{O}$ is a pattern such that for all $X \in \mathcal{O}$ the inert under-category $\mathcal{O}^{\mathrm{int}}_{X/}$ is a poset. In this case, spelling out the definition as in Example 3.3.5 we may identify $\mathcal{O}^{\mathrm{el}}_{\beta}(\omega)$ with the following subposet of $\mathcal{O}^{\mathrm{el}}_{Y/}$:

$$\mathcal{O}^{\mathrm{el}}_{\beta}(\omega) \simeq \{(\alpha\colon Y \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{Y/} \mid \beta = \gamma \circ \omega_\alpha\}.$$

Example 3.3.11 For the pattern $\mathbf{\Delta}^{\mathrm{op}}$ the inert under-category $(\mathbf{\Delta}^{\mathrm{op}})^{\mathrm{int}}_{[n]/}$ is equivalent to the poset of pairs $(a_0 \le a_1) \in [n]$. This is elementary in $\mathbf{\Delta}^{\mathrm{op},\flat}$ if and only if $a_1 - a_0 = 1$, and it is elementary in $\mathbf{\Delta}^{\mathrm{op},\natural}$ if and only if $a_1 - a_0 \le 1$. To check soundness we consider, for an active morphism $\omega\colon [m] \rightsquigarrow [n]$ in $\mathbf{\Delta}$ and elementary $(b_0 \le b_1) \in [n]$, the poset

$$(\mathbf{\Delta}^{\mathrm{op}})^{\mathrm{el}}_{\beta}(\omega) \simeq \{(a_0 \le a_1) \in (\mathbf{\Delta}^{\mathrm{op}})^{\mathrm{el}}_{[m]/} \mid \omega(a_0) \le b_0 \le b_1 \le \omega(a_1)\}.$$

In the case of $\mathbf{\Delta}^{\mathrm{op},\flat}$ this poset has a single element, namely that given by $a_0 = \max\{a \in [m] \mid \omega(a) \le b_0\}$ and $a_1 = a_0 + 1$, which satisfies $\omega(a_1) > b_0$ and hence $\omega(a_1) \ge b_1 = b_0 + 1$. For the pattern $\mathbf{\Delta}^{\mathrm{op},\natural}$ the poset still has a single element if $b_1 = b_0 + 1$ or if $b_1 = b_0$ with $b_i \notin \omega([m])$. But if $b_1 = b_0 = \omega(a)$ for some $a \in [m]$, then the poset is the category

$$(a - 1 \le a) \to (a \le a) \leftarrow (a \le a + 1),$$

which is not trivial, but still weakly contractible. This shows that $\mathbf{\Delta}^{\mathrm{op},\flat}$ and $\mathbf{\Delta}^{\mathrm{op},\natural}$ are both sound.

Example 3.3.12 The pattern $\mathbb{F}_*^\natural$ is sound. The inert under-category $(\mathbb{F}_*^\natural)_{A_+/}$ is the poset of subsets $U \subset A$. Given an active morphism $\omega\colon A_+ \to B_+$ and an elementary $E \subset A$ (ie $|E| \le 1$), we need to check that the poset of $E' \subset Y$ with $|E'| \le 1$ and $E \subset \omega^{-1}(E')$ is contractible. If $E = \{a\} \ne \varnothing$, then this poset has exactly one element $E' = \{\omega(a)\}$, and if $E = \varnothing$, then this poset has an initial element $E' = \varnothing$. So the poset is contractible in both cases, which proves that $\mathbb{F}_*^\natural$ is sound.

Lemma 3.3.13 *Products of sound patterns are sound: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are sound patterns, then $\mathcal{O}_1 \times \mathcal{O}_2$ is also a sound pattern.*

Proof Let $\omega = (\omega_1, \omega_2)\colon (X_1, X_2) \rightsquigarrow (Y_1, Y_2)$ be an active morphism in $\mathcal{O}_1 \times \mathcal{O}_2$. The projection $(\mathcal{O}_1 \times \mathcal{O}_2)^{\mathrm{el}}(\omega) \to (\mathcal{O}_1 \times \mathcal{O}_2)^{\mathrm{el}}_{(X_1,X_2)/}$ can be identified with the product of the projections

$$\mathcal{O}_1^{\mathrm{el}}(\omega_1) \times \mathcal{O}_2^{\mathrm{el}}(\omega_2) \to (\mathcal{O}_1)^{\mathrm{el}}_{X_1/} \times (\mathcal{O}_2)^{\mathrm{el}}_{X_2/},$$

which, by assumption, is a product of coinitial functors and hence again coinitial. □

Example 3.3.14 Applying Lemma 3.3.13 to Example 3.3.11, we see that the patterns $\mathbf{\Delta}^{n,\mathrm{op},\flat}$ and $\mathbf{\Delta}^{n,\mathrm{op},\natural}$ are both sound.

Next, we introduce a further condition for sound patterns; for this we first need some notation:

Notation 3.3.15 By Proposition 2.2.2, evaluation at the target $\mathrm{ev}_1\colon \mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathcal{O}$ is a cocartesian fibration. Its straightening, denoted by $\mathcal{A}_{\mathcal{O}}\colon \mathcal{O} \to \mathsf{Cat}_\infty$, takes $X \in \mathcal{O}$ to the ∞-category $\mathcal{A}_{\mathcal{O}}(X) \simeq \mathcal{O}^{\mathrm{act}}_{/X}$ of active morphisms to X. (Compare with [7, Corollary 7.4 and Remark 7.5].)

Definition 3.3.16 We say an algebraic pattern $\mathcal{O}$ is *soundly extendable* if it is sound and in addition the functor $\mathcal{A}_{\mathcal{O}}$ is a Segal $\mathcal{O}$-∞-category, ie for every $X \in \mathcal{O}$, the functor

$$\mathcal{O}^{\mathrm{act}}_{/X} \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{X/}} \mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence.

Remark 3.3.17 The notion of a soundly extendable pattern is a mild strengthening of the notion of extendable pattern from [7, Definition 8.5] (which uses a slightly weaker, but more complicated condition than what we are here calling "soundness"). It was shown in [7, Lemma 9.14] that every extendable pattern $\mathcal{O}$ satisfies the condition in Definition 3.3.16, so in particular a sound pattern is extendable if and only if it is soundly extendable. In principle, there could exist extendable patterns that are not sound, but we are not aware of any examples.

Example 3.3.18 The patterns $\mathbb{F}_*$, $\mathbf{\Delta}^{\mathrm{op},\natural}$, and $\mathbf{\Delta}^{\mathrm{op},\flat}$ are soundly extendable. Their soundness was verified in Examples 3.3.5 and 3.3.11. For extendability see [7, Examples 8.13 and 8.14]. The pattern $\mathbf{\Theta}_n^{\mathrm{op},\natural}$ is soundly extendable for all n (by [16, Proposition 2.7 and Lemma 3.5]), but note that $\mathbf{\Theta}_n^{\mathrm{op},\flat}$ fails to be extendable for $n > 1$. (See [7, Example 8.15].)

Example 3.3.19 Let $\mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. Then $\mathcal{O}$ is a soundly extendable pattern. This will follow by Example 4.1.5 and Lemma 4.1.15 in the next section.

Example 3.3.20 The patterns $\mathbb{F}_*^{\le k}$ are sound by Lemma 3.3.7, but not soundly extendable. Indeed, $\mathcal{A}_{\mathbb{F}_*^{\le k}}\colon \mathbb{F}_*^{\le k} \to \mathsf{Cat}_\infty$ does not satisfy the Segal condition: for any $n \le k$ the Segal map may be identified with the inclusion

$$(\mathbb{F}^{\times n})^{\le k} \simeq \mathcal{A}_{\mathbb{F}_*^{\le k}}(n) \to \mathcal{A}_{\mathbb{F}_*^{\le k}}(1)^{\times n} \simeq (\mathbb{F}^{\le k})^{\times n},$$

where $(\mathbb{F}^{\le k})^{\times n}$ is the category of n-tuples of sets such that each set has size $\le k$, and $(\mathbb{F}^{\times n})^{\le k}$ denotes the full subcategory on those n-tuples of total size $\le k$.

Lemma 3.3.21 *Let $\mathcal{O}$ and $\mathcal{P}$ be soundly extendable patterns such that $\mathcal{O}^{\mathrm{el}}_{O/}$ and $\mathcal{P}^{\mathrm{el}}_{P/}$ are weakly contractible for all $O \in \mathcal{O}$ and $P \in \mathcal{P}$. Then $\mathcal{O} \times \mathcal{P}$ is a soundly extendable pattern.*

Proof Soundness follows from Lemma 3.3.13. For extendability we have

$$\begin{aligned}\lim_{(\alpha,\beta)\colon (O,P)\rightarrowtail(E,E')\in(\mathcal{O}\times\mathcal{P})^{\mathrm{el}}_{(O,P)/}} (\mathcal{O}\times\mathcal{P})^{\mathrm{act}}_{/(E,E')} &\simeq \lim_{(\alpha\colon O\rightarrowtail E,\beta\colon P\rightarrowtail E')\in\mathcal{O}^{\mathrm{el}}_{O/}\times\mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{O}^{\mathrm{act}}_{/E}\times\mathcal{P}^{\mathrm{act}}_{/E'}\\ &\simeq \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}\times \lim_{\beta\colon P\rightarrowtail E'\in\mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{P}^{\mathrm{act}}_{/E'}\\ &\simeq \mathcal{O}^{\mathrm{act}}_{/O}\times\mathcal{P}^{\mathrm{act}}_{/P},\end{aligned}$$

where in the second line we used that in any ∞-category, products distribute over weakly contractible limits. □

Example 3.3.22 The pattern $\mathbf{\Delta}^{n,\mathrm{op},\natural}$ is soundly extendable. The case $n = 1$ appears in Example 3.3.18, and for $n > 1$ this follows from Lemma 3.3.21 by observing that $(\mathbf{\Delta}^{\mathrm{op},\natural})^{\mathrm{el}}_{[k]/}$ is weakly contractible for all k. (This argument fails for $\mathbf{\Delta}^{n,\mathrm{op},\flat}$ since $(\mathbf{\Delta}^{\mathrm{op},\flat})^{\mathrm{el}}_{[0]/} = \varnothing$, and indeed this pattern is *not* extendable for $n > 1$.)

Proposition 3.3.23 *The pattern $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)$, as defined in Definition 3.2.6, is*

1. *sound if $\mathfrak{X}^b_{/y} \to \mathfrak{X}_{/y}$ is fully faithful and the inclusion $\mathfrak{X}^b_{0/y} \hookrightarrow \mathfrak{X}_{0/y}$ is cofinal for every $y \in \mathfrak{X}$.*
2. *soundly extendable if and only if it is sound and the functor $\mathfrak{X}^f_{/-}\colon \mathfrak{X}^{b,\mathrm{op}} \to \mathsf{Cat}_\infty$ (defined on morphisms by pullback) is right Kan extended from $\mathfrak{X}^{b,\mathrm{op}}_0 \subseteq \mathfrak{X}^{b,\mathrm{op}}$.*

Proof (1) By Lemma 3.3.9(3) the pattern $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)$ is sound if and only if for every $\beta\colon e' \to x$ in $\mathfrak{X}^b$ and $\omega\colon x \to y$ in $\mathfrak{X}^f$ the following colimit indexed by $\alpha\colon e \to y \in \mathfrak{X}^b_{0/y}$ is contractible:

$$\begin{aligned}&\operatorname{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}^b_{/x}}(\beta\colon e'\to x,\ \omega^*\alpha\colon x\times_y e\to x)\\ &\qquad\simeq \operatorname{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{/x}}(\beta\colon e'\to x,\ \omega^*\alpha\colon x\times_y e\to x) \qquad (\mathfrak{X}^b_{/x}\subset\mathfrak{X}_{/x}\text{ full})\\ &\qquad\simeq \operatorname{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{/y}}(\omega\circ\beta\colon e'\to y,\ \alpha\colon e\to x) \qquad (\omega_!\dashv\omega^*)\\ &\qquad\simeq \operatorname{colim}_{\alpha\in\mathfrak{X}^b_{0/y}} \mathsf{Map}_{\mathfrak{X}_{0/y}}(\omega\circ\beta\colon e'\to y,\ \alpha\colon e\to x) \qquad (\mathfrak{X}_{0/y}\subset\mathfrak{X}_{/y}\text{ full})\\ &\qquad\simeq |\mathfrak{X}^b_{0/y}\times_{\mathfrak{X}_{0/y}}(\mathfrak{X}_{0/y})_{\omega\circ\beta/}|.\end{aligned}$$

By [22, Theorem 4.1.3.1] this ∞-category is weakly contractible if $\mathfrak{X}^b_{0/y} \to \mathfrak{X}_{0/y}$ is cofinal, so the claim follows.

(2) Since $\mathsf{Span}_{b,f}(\mathfrak{X};\mathfrak{X}_0)^{\mathrm{act}}_{/-} \simeq \mathfrak{X}^f_{/-}$, this is a consequence of the fact that a functor is Segal if and only if its restriction to the inert subcategory is right Kan extended from the elementaries by [7, Lemma 2.9]. □

As an important special case, we have:

Corollary 3.3.24 *If $\mathfrak{X}^b = \mathfrak{X}$ then $\mathsf{Span}_{\mathrm{all},f}(\mathfrak{X};\mathfrak{X}_0)$ is sound.*

Example 3.3.25 The pattern $\mathsf{Span}(\mathbb{F})$ is soundly extendable.

Example 3.3.26 Let $\mathbb{F}_G^f \subset \mathbb{F}_G$ be closed under base change and coproduct as in Example 3.2.11. The patterns $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)$ and $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)$ are soundly extendable. The slice $(\mathbb{F}_G)^f_{/A}$ decomposes as a product $\prod_{U\in A/G}(\mathbb{F}_G)^f_{/U}$ since the morphisms of $\mathbb{F}_G^f$ are closed under base change. This implies that $(\mathbb{F}_G)^f_{/-}$ is a $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)$-Segal category since the elementary slice category $\mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/} \simeq (\mathrm{Orb}_G)^{\mathrm{inj}}_{/A}$ is equivalent to the discrete set A/G over which we are taking the product. It also follows that $(\mathbb{F}_G)^f_{/-}$ is a Segal $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)$-category since

$$(\mathrm{Orb}_G)^{\mathrm{inj}}_{/A} \simeq \mathsf{Span}_{\mathrm{inj},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/} \hookrightarrow \mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G)^{\mathrm{el}}_{A/}$$

is coinitial.

Example 3.3.27 The pattern $\mathsf{Span}(\mathcal{S}_m)$ is soundly extendable. Soundness follows from Corollary 3.3.24. For extendability we need to show that the functor

$$(\mathcal{S}_m)_{/-}\colon \mathcal{S}_m^{\mathrm{op}} \to \mathsf{Cat}_\infty$$

is right Kan extended from its value at $* \in \mathcal{S}_m^{\mathrm{op}}$. Since being m-truncated can be checked fiberwise over $Y \in \mathcal{S}_m$, this functor is equivalent to $\mathsf{Fun}(-,\mathcal{S}_m)$ by straightening. This is now right Kan extended because $\mathsf{Fun}(X,\mathcal{S}_m) \simeq \lim_X \mathcal{S}_m$. One can show that $\mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m)$ is also soundly extendable; we will not need this, however.

Finally, we give an example of a pattern that is *not* sound:

Example 3.3.28 We expect that the pattern $\boldsymbol{U}^{\mathrm{op}}$ of undirected graphs of Hackney, Robertson, and Yau [13] is sound. However, this pattern does not include the nodeless loop S^1. In [12], Hackney gives a simpler description of $\boldsymbol{U}^{\mathrm{op}}$ and also defines a variant $\widetilde{\boldsymbol{U}}^{\mathrm{op}}$ that does include the nodeless loop. We will now show that this is an example of a nonsound pattern $\mathcal{O} = \widetilde{\boldsymbol{U}}^{\mathrm{op}}$. For the sake of brevity we shall not recall the definition, but rather the following facts:

- The category of elementaries under S^1 is trivial: $\mathcal{O}^{\mathrm{el}}_{S^1/} \simeq *$.
- There is an active morphism $\omega\colon S^1 \rightsquigarrow S^1_n$ to the n-vertex loop S^1_n ($n \geq 2$), for which $\mathcal{O}^{\mathrm{el}}_{S^1_n/}$ is the poset of simplices of S^1_n, which is weakly equivalent to S^1.

We can now use the characterization of soundness from Lemma 3.3.9(3) in the case of the active morphism $\omega\colon S^1 \rightsquigarrow S^1_n$ described above. Since $\mathcal{O}^{\mathrm{el}}_{S^1/}$ is trivial (and in this case $\omega_{\alpha !}S^1$ is always elementary), the colimit runs over the constant diagram on the point and hence evaluates to the classifying space of $\mathcal{O}^{\mathrm{el}}_{S^1_n/}$, which is not contractible. Note that this could be resolved by introducing a variant of $\widetilde{\boldsymbol{U}}^{\mathrm{op}}$ where $\mathsf{Map}_{\mathcal{O}}(S^1,S^1) \simeq \mathsf{Map}_{\mathcal{O}}(S^1,e) \simeq O(2)$, in which case $\mathcal{O}^{\mathrm{el}}_{S^1/}$ is equivalent to the ∞-groupoid S^1.

4 Fibrous patterns and Segal envelopes

We begin this section by introducing the notion of *fibrous $\mathcal{O}$-patterns* as a generalization of ∞-operads over an arbitrary base pattern $\mathcal{O}$ in Section 4.1. We then apply the results of Section 2 to fibrous patterns in Section 4.2, where we prove Theorem C. Finally, in Section 4.3 we give some examples of Segal envelopes.

4.1 Fibrous patterns

In this subsection we introduce the notion of a *fibrous $\mathcal{O}$-pattern* over a base algebraic pattern $\mathcal{O}$. (We borrow the adjective "fibrous" from [23, Section 2.3.3], where it is used for a somewhat related concept.) Fibrous patterns specialize to give, for example, Lurie's ∞-operads and generalized ∞-operads if we take the base pattern to be $\mathbb{F}_*$ or $\mathbb{F}_*^\natural$. The concept is also a variant of the definition of *weak Segal fibrations* given in [7]; as we will see in Proposition 4.1.7 the two notions coincide if the base pattern is sound, ie for almost all interesting examples of patterns, but the definition of fibrous patterns seems to be simpler and better behaved if we do not assume soundness.

Observation 4.1.1 Let $\mathcal{O}$ be an algebraic pattern. If $\pi: \mathcal{P} \to \mathcal{O}$ has cocartesian lifts of inert morphisms, then applying Proposition 2.2.2 to the inert-active factorization system on $\mathcal{O}$ furnishes a cocartesian fibration $\mathcal{P} \times_{\mathcal{O}} \mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathcal{O}$ (where this functor is given as $(P, \pi(P) \rightsquigarrow O) \mapsto O$). For a morphism $\omega\colon O_1 \to O_2$ in $\mathcal{O}$ the cocartesian transport functor $\omega_!\colon \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O_1} \to \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O_2}$ is given by

$$(P, \varphi\colon \pi(P) \rightsquigarrow O_1) \mapsto (\alpha_! P, \beta\colon O' \rightsquigarrow O_2),$$

where

$$\pi(P) \overset{\alpha}{\rightarrowtail} O' \overset{\beta}{\rightsquigarrow} O_2$$

is the inert-active factorization of the composite

$$\pi(P) \overset{\varphi}{\rightsquigarrow} O_1 \xrightarrow{\omega} O_2$$

and $P \to \alpha_! P$ is a cocartesian lift of α.

Definition 4.1.2 Let $\mathcal{O}$ be an algebraic pattern. Then a *fibrous $\mathcal{O}$-pattern* is a functor $\pi: \mathcal{P} \to \mathcal{O}$ such that:

(1) $\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.

(2) For all $O \in \mathcal{O}$, the commutative square of ∞-categories

$$\begin{tikzcd}
\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] \arrow[d] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \arrow[d] \\
\mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}
\end{tikzcd}$$

is cartesian. Here the horizontal functors are induced by cocartesian transport along the maps $O \rightarrowtail E$ in $\mathcal{O}^{\mathrm{el}}_{O/}$ for the cocartesian fibrations from Observation 4.1.1, applied to π and $\mathrm{id}_{\mathcal{O}}$.

Observation 4.1.3 Definition 4.1.2(2) says precisely that the straightening of the projection $\mathcal{P}\times_{\mathcal{O}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{O}) \to \mathrm{Ar}_{\mathrm{act}}(\mathcal{O})$ over $\mathcal{O}$, ie the natural transformation $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{P})\colon \mathrm{St}_{\mathcal{O}}(\mathcal{P}\times_{\mathcal{O}}\mathrm{Ar}_{\mathrm{act}}(\mathcal{O})) \to \mathcal{A}_{\mathcal{O}}$, is a relative Segal $\mathcal{O}$-∞-category.

Remark 4.1.4 For many patterns $\mathcal{O}$, the functor $\mathcal{O}^{\mathrm{act}}_{/-}$ is a Segal $\mathcal{O}$-∞-category; this is the case, for instance, if $\mathcal{O}$ is *extendable* in the sense of [7] by [7, Lemma 9.14]. In this case, Observation 3.1.9 implies that (2) is satisfied if and only if the functor $\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{P})$ is a Segal $\mathcal{O}$-∞-category, ie the functor

$$\mathcal{P}\times_{\mathcal{O}}\mathcal{O}^{\mathrm{act}}_{/O} \to \lim_{E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P}\times_{\mathcal{O}}\mathcal{O}^{\mathrm{act}}_{/E}$$

is an equivalence for all $O\in\mathcal{O}$.

Example 4.1.5 Since $\mathbb{F}_*$ is extendable, a fibrous $\mathbb{F}_*$-pattern is a functor $\pi\colon\mathcal{P}\to\mathbb{F}_*$ such that $\mathcal{P}$ has π-cocartesian lifts for inerts, and for all n the functor

$$\mathcal{P}^{\mathrm{act}}\times_{\mathbb{F}}\mathbb{F}_{/\langle n\rangle} \simeq \mathcal{P}\times_{\mathbb{F}_*}(\mathbb{F}_*)^{\mathrm{act}}_{/\langle n\rangle} \to \prod_{\langle n\rangle\rightarrowtail\langle 1\rangle} \mathcal{P}\times_{\mathbb{F}_*}\mathbb{F} \simeq (\mathcal{P}^{\mathrm{act}})^n$$

is an equivalence. This functor takes an object $P\in\mathcal{P}^{\mathrm{act}}$ over $\langle m\rangle$ in $\mathbb{F}_*$ together with an active map $\omega\colon\langle m\rangle\rightsquigarrow\langle n\rangle$ to the list of objects $(P_1,\dots,P_n)$ where $P\rightarrowtail P_j$ is the cocartesian lift of the inert map $\omega_j := (\rho_j\circ\omega)^{\mathrm{int}}\colon\langle m\rangle\rightarrowtail\langle m\rangle_j$ where $\rho_j\colon\langle n\rangle\rightarrowtail\langle 1\rangle$ is as in Example 3.1.6. We will see later (Proposition 4.1.7) that this condition is equivalent to $\mathcal{P}\to\mathbb{F}_*$ being an ∞-operad in the sense of Lurie.

We can rewrite the second condition in Definition 4.1.2 to obtain the following equivalent characterization of fibrous patterns:

Proposition 4.1.6 *For any algebraic pattern $\mathcal{O}$, a functor $\pi\colon\mathcal{P}\to\mathcal{O}$ is a fibrous $\mathcal{O}$-pattern if and only if*:

(1) *$\mathcal{P}$ has π-cocartesian morphisms over inert morphisms in $\mathcal{O}$.*

(2) *For every active morphism $\omega\colon O_1\rightsquigarrow O_2$ in $\mathcal{O}$, and all objects $X_0\in\mathcal{P}_{O_0}$, $X_1\in\mathcal{P}_{O_1}$, the commutative square*

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{P}}(X_0,X_1) \arrow[r] \arrow[d] & \lim_{\alpha\colon O_2\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_2/}} \mathrm{Map}_{\mathcal{P}}(X_0,\omega_{\alpha,!}X_1) \arrow[d] \\
\mathrm{Map}_{\mathcal{O}}(O_0,O_1) \arrow[r] & \lim_{\alpha\colon O_2\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_2/}} \mathrm{Map}_{\mathcal{O}}(O_0,\omega_{\alpha,!}O_1)
\end{tikzcd} \tag{4}$$

is cartesian. Here the horizontal maps are defined using the functor $\omega_{(-)}\colon\mathcal{O}^{\mathrm{el}}_{O_2/}\to\mathcal{O}^{\mathrm{int}}_{O_1/}$, which was introduced in Notation 3.3.1.

(3) *For every active morphism $\omega\colon O_1\rightsquigarrow O_2$ in $\mathcal{O}$, the functor*

$$\mathcal{P}^{\simeq}_{O_1} \to \lim_{\alpha\colon O_2\to E\in\mathcal{O}^{\mathrm{el}}_{O_2/}} \mathcal{P}^{\simeq}_{\omega_{\alpha!}O_1},$$

induced by cocartesian transport along the inert morphisms $\omega_\alpha\colon O_1\rightarrowtail\omega_{\alpha!}O_1$ in $\mathcal{O}^{\mathrm{int}}_{O_1/}$, is an equivalence.

Proof A square of ∞-categories is cartesian if and only if the underlying square of ∞-groupoids as well as all induced squares of mapping spaces are cartesian. For the square in the definition of a fibrous pattern the underlying square of ∞-groupoids is

$$\begin{tikzcd}
\mathcal{P}^{\simeq} \times_{\mathcal{O}^{\simeq}} (\mathcal{O}_{/O})^{\simeq} \arrow[r] \arrow[d] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P}^{\simeq} \times_{\mathcal{O}^{\simeq}} (\mathcal{O}_{/E})^{\simeq} \arrow[d] \\
(\mathcal{O}_{/O})^{\simeq} \arrow[r] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} (\mathcal{O}_{/E})^{\simeq}.
\end{tikzcd}$$

This square is cartesian if and only if the map on fibers over each $\omega \colon O' \rightsquigarrow O$ is an equivalence. This map takes the form

$$\mathcal{P}^{\simeq}_{O'} \to \lim_{\alpha \colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{P}^{\simeq}_{\omega_{\alpha,!} O'} \tag{5}$$

and is induced by the cocartesian transport along the inert morphisms $\omega_\alpha \colon O' \rightarrowtail \omega_{\alpha !} O'$ as in Notation 3.3.1. This is exactly the map from (3), so the square of ∞-groupoids is cartesian if and only if (3) holds.

Now consider, for two objects $(P, \varphi \colon \pi(P) \rightsquigarrow O)$ and $(P', \varphi' \colon \pi(P') \rightsquigarrow O) \in \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$, the square of mapping spaces

$$\begin{tikzcd}
\mathsf{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}}((P, \varphi), (P', \varphi')) \arrow[r] \arrow[d] & \lim_{\alpha \colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E}}((P, \varphi), (\varphi'_{\alpha,!} P', (\alpha \circ \varphi')^{\mathrm{act}})) \arrow[d] \\
\mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi, \varphi') \arrow[r] & \lim_{\alpha \colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/E}}(\varphi, (\alpha \circ \varphi')^{\mathrm{act}}).
\end{tikzcd} \tag{6}$$

A point in $\mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi, \varphi')$ is a (necessarily active) morphism $f \colon \pi(P) \rightsquigarrow \pi(P')$ together with a homotopy $\varphi \simeq \varphi' \circ f$. To compute the fiber of the vertical maps at this point, note that the mapping space in $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}$ can be computed as

$$\mathsf{Map}_{\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O}}((P, \varphi), (P', \varphi')) \simeq \mathsf{Map}_{\mathcal{P}}(P, P') \times_{\mathsf{Map}_{\mathcal{O}}(\pi(P), \pi(P'))} \mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/O}}(\varphi, \varphi').$$

Hence the map on the vertical fibers of the square is given by

$$\mathsf{Map}^{f}_{\mathcal{P}}(P, P') \to \lim_{\alpha \colon O \rightarrowtail E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha \circ f}_{\mathcal{P}}(P, \varphi'_{\alpha,!} P'), \tag{7}$$

where the superscripts indicate fibers over maps in $\mathcal{O}$. This agrees with the map on fibers over f of the square in (2). Therefore (2) implies that the square of mapping spaces is a pullback.

However, we have not shown the converse yet, because we have only considered the fibers in (4) over morphisms $f \in \mathsf{Map}_{\mathcal{O}}(O_0, O_1)$ that are active. Let us now assume that the square of mapping spaces (6) is cartesian. For a general morphism $O_0 \to O_1$ we can find an inert-active factorization $O_0 \overset{j}{\rightarrowtail} Q \overset{g}{\rightsquigarrow} O_1$. Since j is inert we can find a cocartesian lift $\tilde{j} \colon P_0 \to j_! P_0$ and by virtue of this being cocartesian, precomposition with $\tilde{j}$ induces the vertical equivalences in the diagram

$$\begin{tikzcd}
\mathsf{Map}^{g}_{\mathcal{P}}(j_!P, P') \arrow[r] \arrow[d, "(-)\circ\tilde{j}"', "\simeq"] & \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha\circ g}_{\mathcal{P}}(j_!P, \varphi'_{\alpha,!}P') \arrow[d, "\simeq"', "(-)\circ\tilde{j}"] \\
\mathsf{Map}^{g\circ j}_{\mathcal{P}}(P, P') \arrow[r] & \lim_{\alpha\colon O\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O/}} \mathsf{Map}^{\varphi'_\alpha\circ g\circ j}_{\mathcal{P}}(P, \varphi'_{\alpha,!}P').
\end{tikzcd}$$

Since g is active, the previous argument shows that the top map is an equivalence. Hence the bottom map is an equivalence and as $f = g\circ j$ was arbitrary this shows that (2) is implied. □

The conditions in Proposition 4.1.6 are reminiscent of Lurie's definition of an ∞-operad [23]. Note, however, that in conditions (2) and (3) we need to consider *all* active maps in $\mathcal{O}$, while Lurie's definition of ∞-operads, or the definition of *weak Segal fibrations* in [7], only involve the conditions corresponding to identity maps. If the base pattern is *sound*, however, the conditions for all active maps are implied by this special case:

Proposition 4.1.7 *Suppose $\mathcal{O}$ is a sound pattern. Then a functor $\pi\colon\mathcal{P}\to\mathcal{O}$ is a fibrous $\mathcal{O}$-pattern if and only if it is a weak Segal $\mathcal{O}$-fibration in the sense of* [7, Definition 9.6], *ie the conditions of Proposition 4.1.6 hold whenever ω is an identity morphism. Concretely*:

(1) *$\mathcal{P}$ has all π-cocartesian lifts of inert morphisms in $\mathcal{O}$.*

(2) *For every $O_1\in\mathcal{O}$, the functor*

$$\mathcal{P}^{\simeq}_{O_1}\to\lim_{\alpha\colon O_1\to E\in\mathcal{O}^{\mathrm{el}}_{O_1/}}\mathcal{P}^{\simeq}_{E},$$

induced by cocartesian transport along $\alpha\colon O_1\rightarrowtail E$, is an equivalence.

(3) *For all $O_0, O_1\in\mathcal{O}$, and all objects $X_0\in\mathcal{P}_{O_0}$, $X_1\in\mathcal{P}_{O_1}$, the commutative square*

$$\begin{tikzcd}
\mathsf{Map}_{\mathcal{P}}(X_0, X_1) \arrow[r] \arrow[d] & \lim_{\alpha\colon O_1\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathsf{Map}_{\mathcal{P}}(X_0, \alpha_!X_1) \arrow[d] \\
\mathsf{Map}_{\mathcal{O}}(O_0, O_1) \arrow[r] & \lim_{\alpha\colon O_1\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{O_1/}} \mathsf{Map}_{\mathcal{O}}(O_0, E)
\end{tikzcd}$$

is cartesian.

Remark 4.1.8 In [7] (and [23]), the analogue of (2) says that the functor

$$\mathcal{P}_{O_1}\to\lim_{\alpha\colon O_1\to E\in\mathcal{O}^{\mathrm{el}}_{O_1/}}\mathcal{P}_{\alpha_!O'_2}$$

is an equivalence, rather than that the underlying map of ∞-groupoids is one. However, it follows from (3) that this functor gives an equivalence on mapping spaces, ie it is already fully faithful, and so it is an equivalence if and only if it is an equivalence on underlying ∞-groupoids. In fact, it would suffice in (2) to assume that the map is merely surjective on π_0.

Proof of Proposition 4.1.7 Suppose $\pi\colon \mathcal{P} \to \mathcal{O}$ is a weak Segal fibration. Consider the functor $F\colon \mathcal{O}^{\mathrm{int}}_{O_1/} \to \mathcal{O}^{\mathrm{int}} \to \mathcal{S}$ defined by $F(O_1 \rightarrowtail O_2) := \mathcal{P}^{\simeq}_{O_2}$ and cocartesian transport along inerts. The natural transformation $\eta\colon F \Rightarrow *$ to the terminal functor satisfies the conditions of Lemma 3.3.8. The conclusion of the lemma tells us that (2) holds for all $\omega\colon O_1 \rightsquigarrow O_2$.

For property (3), fix $X_0, X_1 \in \mathcal{P}$ with $\pi(X_0) = O_0$ and $\pi(X_1) = O_1$. Then cocartesian transport along inerts defines a functor

$$F\colon \mathcal{O}^{\mathrm{int}}_{O_1/} \to \mathcal{S}, \quad (\varphi\colon O_1 \rightarrowtail O_2) \mapsto \mathrm{Map}(X_0, \varphi_! X_1),$$

and this admits a canonical natural transformation to the functor $G(\varphi\colon O_1 \rightarrowtail O_2) := \mathrm{Map}(O_0, O_2)$. Applying Lemma 3.3.8 to $\eta\colon F \Rightarrow G$ shows that (3) holds for all $\omega\colon O_1 \rightsquigarrow O_2$. □

Example 4.1.9 Fibrous $\mathbb{F}_*$-patterns are precisely (symmetric) ∞-operads as defined in [23], while fibrous $\mathbb{F}_*^{\natural}$- patterns are generalized (symmetric) ∞-operads. Similarly, fibrous $\mathbf{\Delta}^{\mathrm{op},\flat}$- and $\mathbf{\Delta}^{\mathrm{op},\natural}$-patterns are nonsymmetric (or planar) ∞-operads and generalized nonsymmetric ∞-operads, respectively.

Observation 4.1.10 For a sound pattern $\mathcal{O}$ we can also describe the fibrous $\mathcal{O}$-patterns that are cocartesian fibrations as the unstraightenings of Segal $\mathcal{O}$-∞-categories, ie as the *Segal $\mathcal{O}$-fibrations* of [7, Definition 9.1]. This is easy to check directly, but it is also a special case of Lemma 4.2.4 (taking $Y = *$), which we will prove below.

Fibrous $\mathcal{O}$-patterns admit a canonical pattern structure, which we now introduce:

Definition 4.1.11 Suppose $\pi: \mathcal{P} \to \mathcal{O}$ is a fibrous $\mathcal{O}$-pattern. We say a morphism in $\mathcal{P}$ is *inert* if it is π-cocartesian and lies over an inert morphism in $\mathcal{O}$, and *active* if it just lies over an active morphism in $\mathcal{O}$. The inert and active morphisms then form a factorization system on $\mathcal{P}$ by [23, Proposition 2.1.2.5], and we give $\mathcal{P}$ an algebraic pattern structure with this factorization system by taking the elementary objects to be all those that lie over elementary objects in $\mathcal{O}$.

Definition 4.1.12 A *morphism of fibrous $\mathcal{O}$-patterns* is a commutative triangle

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathcal{P}' \arrow[dl, "\pi'"] \\
& \mathcal{O}, &
\end{tikzcd}$$

where π and π' are fibrous $\mathcal{O}$-patterns and f is a morphism of algebraic patterns. It is immediate from the definition of the pattern structures that for this it suffices to require that f preserves inert morphisms. We write $\mathrm{Fbrs}(\mathcal{O})$ for the full subcategory of $\mathrm{AlgPatt}_{/\mathcal{O}}$ whose objects are the fibrous $\mathcal{O}$-patterns; this is equivalently a full subcategory of $\mathrm{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$.

Lemma 4.1.13 *The inclusion* $\mathrm{Fbrs}(\mathcal{O}) \hookrightarrow \mathrm{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ *preserves limits and κ-filtered colimits where κ is a regular cardinal such that $\mathcal{O}^{\mathrm{el}}_{O/}$ is κ-small for all $O \in \mathcal{O}$. Limits and κ-filtered colimits of $\mathcal{O}$-fibrous patterns can therefore be computed in* $\mathrm{Cat}_{\infty/\mathcal{O}}$.

Proof By Observation 2.3.8 the forgetful functor $\mathrm{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \to \mathrm{Cat}_{\infty/\mathcal{O}}$ preserves limits and κ-filtered colimits, and is also conservative. It therefore suffices to observe that the commutative square that is required to be cartesian for an object of $\mathrm{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ to be a fibrous $\mathcal{O}$-pattern commutes with limits and κ-filtered colimits of ∞-categories. Since a limit or filtered colimit of cartesian squares in Cat_∞ is again cartesian, this implies the result. □

Observation 4.1.14 If $\pi : \mathcal{P} \to \mathcal{O}$ is a fibrous $\mathcal{O}$-pattern, then for every object $\overline{X} \in \mathcal{P}$ over X in $\mathcal{O}$, the functor

$$\mathcal{P}^{\mathrm{el}}_{\overline{X}/} \to \mathcal{O}^{\mathrm{el}}_{X/}$$

is an equivalence. Indeed, since $\mathcal{P}^{\mathrm{int}} \to \mathcal{O}^{\mathrm{int}}$ is a cocartesian fibration the functor $\mathcal{P}^{\mathrm{int}}_{\overline{X}/} \to \mathcal{O}^{\mathrm{int}}_{X/}$ is an equivalence, and the above functor is obtained by restricting to the full subcategories of elementary objects. In particular, π is an iso-Segal morphism. More generally, if $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of fibrous $\mathcal{O}$-patterns, then f induces an equivalence

$$\mathcal{P}^{\mathrm{el}}_{\overline{X}/} \xrightarrow{\simeq} \mathcal{Q}^{\mathrm{el}}_{f(\overline{X})/}$$

for the same reason, so that f is also an iso-Segal morphism.

Lemma 4.1.15 *Suppose $\mathcal{O}$ is a sound pattern and $\pi : \mathcal{P} \to \mathcal{O}$ is $\mathcal{O}$-fibrous. Then $\mathcal{P}$ is also a sound pattern. Moreover, if $\mathcal{O}$ is soundly extendable, then so is $\mathcal{P}$.*

Proof As we just observed that π is iso-Segal, soundness follows from Lemma 3.3.7.

Now assume $\mathcal{O}$ is soundly extendable. Then, by Remark 4.1.4, the functor

$$\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/Y} \to \lim_{E' \in \mathcal{O}^{\mathrm{el}}_{Y/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E'}$$

is an equivalence. Since any morphism in $\mathcal{P}$ that is mapped to an active morphism in $\mathcal{O}$ is active by definition and active morphisms satisfy cancellation, we have that $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/Y} = \mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/Y}$. Consider the case where $Y = \pi(X)$ for $X \in \mathcal{P}$. Since $\mathcal{P} \to \mathcal{O}$ is an equivalence on elementary slices, we can rewrite the limit on the right-hand side as a limit over $E \in \mathcal{P}^{\mathrm{el}}_{X/}$ and set $E' := \pi(E)$:

$$\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(X)} \xrightarrow{\simeq} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} \mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(E)}.$$

Now, passing to the over-category of $(X, \mathrm{id}_{\pi(X)})$ we obtain an equivalence:

$$\mathcal{P}^{\mathrm{act}}_{/X} \simeq (\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(X)})_{/(X,\mathrm{id}_{\pi(X)})} \xrightarrow{\simeq} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} (\mathcal{P}^{\mathrm{act}} \times_{\mathcal{O}^{\mathrm{act}}} \mathcal{O}^{\mathrm{act}}_{/\pi(E)})_{/(E,\mathrm{id}_{\pi(E)})} \simeq \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} \mathcal{P}^{\mathrm{act}}_{/E},$$

which shows that $\mathcal{P}$ is soundly extendable. □

Proposition 4.1.16 *Suppose we have a commutative triangle of algebraic patterns*

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{P} \arrow[dl, "p"] \\ & \mathcal{O}, & \end{tikzcd}$$

where $\mathcal{P}$ is $\mathcal{O}$-fibrous. Assume further that $\mathcal{O}$ is sound. Then $\mathcal{Q}$ is $\mathcal{O}$-fibrous if and only if it is $\mathcal{P}$-fibrous.

Proof By Lemma 4.1.15 $\mathcal{P}$ is also sound, so we may use the characterization from Proposition 4.1.7.

Any inert morphism $\pi\colon P \to P'$ in $\mathcal{P}$ is cocartesian over an inert morphism $\omega\colon O \to O'$ in $\mathcal{O}$; if $\varphi\colon Q \to Q'$ is an inert morphism over ω in $\mathcal{Q}$ such that $F(Q) \simeq P$, then we have $F(\varphi) \simeq \pi$ since F preserves inert morphisms and π is the unique inert morphism over ω with source P. It now follows from [22, Proposition 2.4.1.3] that $\varphi\colon Q \to Q'$ is F-cocartesian if and only if it is q-cocartesian. Thus (1) in Proposition 4.1.7 holds for F if and only if it holds for q.

Assuming this holds, then for $Q, Q' \in \mathcal{Q}$, $P = F(Q)$, $P' = F(Q')$ and $O = q(Q)$, $O' = q(Q')$, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Map}_{\mathcal{Q}}(Q, Q') \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{Q}}(Q, \alpha_! Q') \arrow[d] \\
\mathsf{Map}_{\mathcal{P}}(P, P') \arrow[r] \arrow[d] & \lim_{\alpha \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{P}}(P, \alpha_! P') \arrow[d] \\
\mathsf{Map}_{\mathcal{O}}(O, O') \arrow[r] & \lim_{(\alpha\colon O' \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{O'/}} \mathsf{Map}_{\mathcal{O}}(O, E).
\end{tikzcd}$$

Here the bottom square is cartesian since $\mathcal{P}$ is $\mathcal{O}$-fibrous, so the top square is cartesian if and only if the outer square is cartesian. But since p is an iso-Segal morphism (by Observation 4.1.14) we can rewrite the top square as

$$\begin{tikzcd}
\mathsf{Map}_{\mathcal{Q}}(Q, Q') \arrow[r] \arrow[d] & \lim_{\beta \in \mathcal{P}^{\mathrm{el}}_{P'/}} \mathsf{Map}_{\mathcal{Q}}(Q, \beta_! Q') \arrow[d] \\
\mathsf{Map}_{\mathcal{P}}(P, P') \arrow[r] & \lim_{(\beta\colon P' \to E') \in \mathcal{P}^{\mathrm{el}}_{P'/}} \mathsf{Map}_{\mathcal{P}}(P, E'),
\end{tikzcd}$$

and so (3) in Proposition 4.1.7 holds for F if and only if it holds for q. The proof for (2) is similar. □

Corollary 4.1.17 *If $\mathcal{O}$ is sound and $\pi : \mathcal{P} \to \mathcal{O}$ exhibits $\mathcal{P}$ as an $\mathcal{O}$-fibrous pattern, then composition with π gives a functor*

$$\pi_!\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O}),$$

and this induces an equivalence

$$\mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})_{/\mathcal{P}}.$$

Example 4.1.18 Let $(\pi : \mathcal{O} \to \mathbb{F}_*) \in \mathsf{Opd}_\infty$ be an ∞-operad in the sense of Lurie, ie a fibrous $\mathbb{F}_*$-pattern. Applying Corollary 4.1.17 we obtain an equivalence:

$$\mathsf{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*)_{/\mathcal{O}} = \mathsf{Opd}_{\infty/\mathcal{O}}.$$

So fibrous $\mathcal{O}$-patterns are simply ∞-operads over $\mathcal{O}$.

Lemma 4.1.19 *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism. Then pullback along f restricts to a functor*

$$f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O}), \quad (\pi\colon \mathcal{F} \to \mathcal{P}) \mapsto (f^*\pi\colon \mathcal{F} \times_{\mathcal{P}} \mathcal{O} \to \mathcal{O}).$$

Proof Suppose $\pi\colon \mathcal{F} \to \mathcal{P}$ is a $\mathcal{P}$-fibrous pattern. Definition 4.1.2(1) for $f^*\mathcal{F}$ follows from the usual description of cocartesian morphisms in a pullback, since f preserves inert morphisms. To prove (2), we observe that $f^*\mathcal{F} \times_{\mathcal{O}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) \simeq \mathcal{F} \times_{\mathcal{P}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{O})$, so that we have a cartesian square

$$\begin{tikzcd}
f^*\mathcal{F} \times_{\mathcal{O}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) \arrow[r] \arrow[d] & \mathcal{F} \times_{\mathcal{P}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{P}) \times_{\mathcal{P}} \mathcal{O} \arrow[d] \\
\mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) \arrow[r] & \mathsf{Ar}_{\mathrm{act}}(\mathcal{P}) \times_{\mathcal{P}} \mathcal{O}
\end{tikzcd}$$

of cocartesian fibrations over $\mathcal{O}$. Straightening yields the cartesian square

$$\begin{tikzcd}
\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(f^*\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{St}^{\mathrm{int}}_{\mathcal{P}}(\mathcal{F}) \circ f \arrow[d] \\
\mathcal{A}_{\mathcal{O}} \arrow[r] & \mathcal{A}_{\mathcal{P}} \circ f
\end{tikzcd}$$

of functors $\mathcal{O} \to \mathsf{Cat}_\infty$. By Observation 4.1.3 the natural transformation $\mathrm{St}^{\mathrm{int}}_{\mathcal{P}}(\mathcal{F}) \to \mathcal{A}_{\mathcal{P}}$ is a relative $\mathcal{P}$-Segal ∞-category. This remains true after precomposing with f (by Observation 3.1.15, since f is strong Segal). Hence the right vertical map in the square is a relative $\mathcal{O}$-Segal ∞-category and by Lemma 3.1.10 so is the left vertical arrow. Using Observation 4.1.3 again we see that $f^*\mathcal{F}$ is fibrous. □

Example 4.1.20 The morphism $\mathfrak{c}\colon \mathbf{\Delta}^{\mathrm{op},\flat} \to \mathbb{F}_*$ from Example 3.1.13 is iso-Segal and hence Lemma 4.1.19 shows that pulling back along it defines a functor:

$$\mathfrak{c}^*\colon \mathsf{Fbrs}(\mathbb{F}_*) \to \mathsf{Fbrs}(\mathbf{\Delta}^{\mathrm{op},\flat}).$$

Under the identifications of Example 4.1.9 this is exactly the forgetful functor from (symmetric) ∞-operads to nonsymmetric ∞-operads.

Finally, let us note that we can lift the comparison of Proposition 3.1.16 to fibrous patterns:

Proposition 4.1.21 *Suppose $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and let $\pi\colon \mathcal{Q} \to \mathcal{P}$ be a fibrous pattern. Then $\bar{f}\colon f^*\mathcal{Q} \to \mathcal{Q}$ is also a strong Segal morphism that satisfies the conditions of Proposition 3.1.16 and thus induces an equivalence*

$$\bar{f}^*\colon \mathsf{Seg}_{\mathcal{Q}}(\mathcal{S}) \xrightarrow{\sim} \mathsf{Seg}_{f^*\mathcal{Q}}(\mathcal{S}).$$

Proof Denote by $\pi'\colon \mathcal{Q}' := f^*\mathcal{Q} \to \mathcal{O}$ the projection map. Since $\mathcal{Q}$ is fibrous and f is strong Segal, it follows from Lemma 4.1.19 that $\mathcal{Q}'$ is also fibrous. By Observation 4.1.14 we have $\mathcal{Q}^{\mathrm{el}}_{Q/} \simeq \mathcal{P}^{\mathrm{el}}_{\pi(Q)/}$

and similarly for $\mathcal{Q}'$ and $\mathcal{O}$. The map $(\mathcal{Q}')^{\mathrm{el}}_{Q/} \to \mathcal{Q}^{\mathrm{el}}_{\bar{f}(Q)/}$ therefore identifies with $\mathcal{O}^{\mathrm{el}}_{\pi'(Q)/} \to \mathcal{P}^{\mathrm{el}}_{f(\pi'(Q))/}$ which is coinitial by the assumption that f is strong Segal. We conclude that $\bar{f}$ is strong Segal. We proceed by verifying the conditions. Condition (1) of Proposition 3.1.16 is visibly stable under base change so it remains to check (2). Observe that for every object $Q \in \mathcal{Q}'$ that lies over $O \in \mathcal{O}$ we have by [22, Lemma 5.4.5.4] a pullback square of slice ∞-categories

$$\begin{tikzcd}
\mathcal{Q}'_{/Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Q}_{/\bar{f}(Q)} \arrow[d] \\
\mathcal{O}_{/O} \arrow[r] & \mathcal{P}_{/f(O)}.
\end{tikzcd}$$

By assumption the bottom map induces an equivalence on the underlying spaces of active maps and since the square is cartesian the same holds for the top map. $\square$

4.2 Segal envelopes

In this section we will specialize our results from Section 2 to fibrous $\mathcal{O}$-patterns over an algebraic pattern $\mathcal{O}$. Recall that we have shown that from the inert-active factorization system on $\mathcal{O}$ we obtain an adjunction

$$(-) \times_{\mathcal{O}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) \colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \rightleftarrows (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}})_{/\mathsf{Ar}_{\mathrm{act}}(\mathcal{O})},$$

where the right adjoint is given by pulling back along the map $\mathcal{O} \to \mathsf{Ar}_{\mathrm{act}}(\mathcal{O})$ given by the degeneracy $[1] \to [0]$. This can equivalently be interpreted as a "straightening-unstraightening" adjunction

$$\mathsf{St}^{\mathrm{int}}_{\mathcal{O}} \colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \rightleftarrows \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}} : \mathsf{Un}^{\mathrm{int}}_{\mathcal{O}}$$

in which the left adjoint is fully faithful with image the $\mathcal{A}_{\mathcal{O}}$-equifibered functors.

We can immediately identify the image of the full subcategory $\mathsf{Fbrs}(\mathcal{O})$ under this fully faithful functor:

Proposition 4.2.1 *For any algebraic pattern $\mathcal{O}$, the fully faithful functor $\mathsf{St}^{\mathrm{int}}_{\mathcal{O}}$ identifies $\mathsf{Fbrs}(\mathcal{O})$ with the full subcategory of $\mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$ spanned by the equifibered maps that are also relative Segal objects. In other words, the functor $\mathsf{St}^{\mathrm{int}}_{\mathcal{O}}$ restricts to a fully faithful functor*

$$\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}} := \mathsf{St}^{\mathrm{int}}_{\mathcal{O}}|_{\mathsf{Fbrs}(\mathcal{O})} \colon \mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$$

with image the equifibered objects. Moreover, for any strong Segal morphism $f \colon \mathcal{O} \to \mathcal{P}$, we have a commutative square

$$\begin{tikzcd}
\mathsf{Fbrs}(\mathcal{P}) \arrow[r, "f^*"] \arrow[d, hook, "\mathsf{Env}^{/\mathcal{A}_{\mathcal{P}}}_{\mathcal{P}}"] & \mathsf{Fbrs}(\mathcal{O}) \arrow[d, hook, "\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}"] \\
\mathsf{Seg}^{/\mathcal{A}_{\mathcal{P}}}_{\mathcal{P}}(\mathsf{Cat}_\infty) \arrow[r, "f^{\circledast}"] & \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty),
\end{tikzcd} \tag{8}$$

where the functor $f^{\circledast}$ *is given by the composite*

$$\mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \xrightarrow{f^*} \mathsf{Seg}_{\mathcal{O}}^{/f^*\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \to \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$$

of restriction along f *and pullback along the natural map* $\mathcal{A}_{\mathcal{O}} \to f^*\mathcal{A}_{\mathcal{P}}$ *(see Observation 3.1.15 and Lemma 3.1.10).*

Proof From Observation 4.1.3 we know that an object $\mathcal{P}$ of $\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is a fibrous $\mathcal{O}$-pattern if and only if $\mathrm{St}_{\mathcal{O}}^{\text{int}}(\mathcal{P})$ is a relative Segal $\mathcal{O}$-object in Cat_∞. The commutative square (8) likewise follows by restricting the square (3) in Observation 2.3.9 to full subcategories. □

From this observation we can deduce some pleasant properties of the ∞-categories of fibrous patterns:

Corollary 4.2.2 *For any algebraic pattern* $\mathcal{O}$, *the* ∞*-category* $\mathsf{Fbrs}(\mathcal{O})$ *is presentable, and fits in a cartesian square of fully faithful right adjoints*

$$\begin{tikzcd}
\mathsf{Fbrs}(\mathcal{O}) \arrow[r, hook, "\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}"] \arrow[d, hook] & \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty) \arrow[d, hook] \\
\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \arrow[r, hook, "\mathrm{St}_{\mathcal{O}}^{\text{int}}"] & \mathsf{Fun}(\mathcal{O}^{\mathrm{op}}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}.
\end{tikzcd}$$

Proof We know from Proposition 4.2.1 that we have the given cartesian square of fully faithful functors; it remains to show that this is a square in Pr^R. For the bottom horizontal and right vertical functor we have shown this in Proposition 2.3.7 and Lemma 3.1.11, respectively. It now follows that the rest of the diagram also lies in Pr^R, since the diagram is cartesian and by [22, Theorem 5.5.3.18] Pr^R admits pullbacks and the inclusion $\mathsf{Pr}^R \subset \mathsf{Cat}_\infty$ preserves them. □

Corollary 4.2.3 (1) *For any algebraic pattern* $\mathcal{O}$, *the following functors admit left adjoints*:

$$\mathsf{Fbrs}(\mathcal{O}) \hookrightarrow \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{O}}.$$

(2) *For any strong Segal morphism* $f\colon \mathcal{O} \to \mathcal{P}$, *the functor* $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ *admits a left adjoint.*

Proof The first claim was shown in Corollary 4.2.2 and Observation 2.3.8. In particular limits and κ-filtered colimits in $\mathsf{Fbrs}(\mathcal{O})$ for appropriate κ are computed in $\mathsf{Cat}_{\infty/\mathcal{O}}$. This implies that $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{P})$ preserves limits and κ-filtered colimits, since we know pullback along f preserves limits and filtered colimits as a functor $\mathsf{Cat}_{\infty/\mathcal{P}} \to \mathsf{Cat}_{\infty/\mathcal{O}}$. Hence the claim follows from the adjoint functor theorem. □

In Proposition 4.2.1 we only showed that the left adjoint $\mathrm{St}_{\mathcal{O}}^{\text{int}}$ restricts to a functor from fibrous patterns to relative Segal objects — in general the right adjoint $\mathrm{Un}_{\mathcal{O}}^{\text{int}}$ does not necessarily take relative Segal $\mathcal{O}$-∞-categories over $\mathcal{A}_{\mathcal{O}}$ to fibrous $\mathcal{O}$-patterns. However, this is the case if $\mathcal{O}$ is sound; to see this, we first need a technical lemma:

Lemma 4.2.4 *Let $\mathcal{O}$ be a sound algebraic pattern and let $\gamma\colon X \to Y$ be a morphism in $\mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)$, with $\Gamma\colon \mathcal{X} \to \mathcal{Y}$ denoting its unstraightening. Then the following are equivalent*:

(1) *$\Gamma\colon X \to Y$ is a relative Segal object.*

(2) *$\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\Gamma)\colon \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{X}) \to \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\mathcal{Y})$ is a relative Segal object*, *ie the commutative square*

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] \arrow[d] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \arrow[d] \\
\mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[r] & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E}
\end{tikzcd}$$

is cartesian for all $O \in \mathcal{O}$.

Proof For $O \in \mathcal{O}$, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[rr] \arrow[dr] \arrow[dd] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{X} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \arrow[dr] \arrow[dd] & \\
& \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \arrow[rr] \arrow[dl] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{Y} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/E} \arrow[dl] \\
\mathcal{O}^{\mathrm{act}}_{/O} \arrow[rr] & & \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}. &
\end{tikzcd}$$

Here all four functors to the bottom row are cocartesian fibrations, and the morphisms in the top square preserve cocartesian morphisms. We therefore see that condition (2), which asks for the top square to be cartesian, is equivalent to all squares of fibers over $\omega\colon O' \rightsquigarrow O$ in $\mathcal{O}^{\mathrm{act}}_{/O}$ being cartesian. The relevant square of fibers is

$$\begin{tikzcd}
X(O') \arrow[r] \arrow[d] & \lim_{(\alpha\colon O \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{O/}} X(\omega_{\alpha !} O') \arrow[d] \\
Y(O') \arrow[r] & \lim_{(\alpha\colon O \rightarrowtail E) \in \mathcal{O}^{\mathrm{el}}_{O/}} Y(\omega_{\alpha !} O').
\end{tikzcd}$$

Considering the special case $\omega = \mathrm{id}_O$ we see that (2) implies (1), while to see that the converse holds when $\mathcal{O}$ is sound we apply Lemma 3.3.8 with $F = X$ and $G = Y$. ☐

Proposition 4.2.5 *If the pattern $\mathcal{O}$ is sound*, *then the adjunction of Notation 2.3.4 restricts to an adjunction*

$$\mathsf{Env}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}\colon \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty) : \mathsf{Un}^{\mathrm{int}}_{\mathcal{O}}.$$

Moreover, if $f : \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism between sound patterns, then in addition to the square (8) we have a commutative square

$$\begin{tikzcd}
\mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \arrow[r, "f^{\circledast}"] \arrow[d, "\mathrm{Un}_{\mathcal{P}}^{\mathrm{int}}"'] & \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty) \arrow[d, "\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}"] \\
\mathsf{Fbrs}(\mathcal{P}) \arrow[r, "f^*"] & \mathsf{Fbrs}(\mathcal{O}).
\end{tikzcd} \tag{9}$$

Proof We need to show that $\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}} : \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}} \to \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ sends $\mathcal{A}_{\mathcal{O}}$-relative Segal objects to fibrous $\mathcal{O}$-patterns. Since we know an object of $\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is fibrous if and only if its image under $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ is a relative Segal object, it suffices to show that $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}} \circ \mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}$ preserves relative Segal objects.

Let $X \to \mathcal{A}_{\mathcal{O}}$ be a relative Segal object; then $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X))$ fits into a cartesian square

$$\begin{tikzcd}
\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X)) \arrow[r] \arrow[d] & \mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}(X)) \arrow[d] \\
\mathcal{A}_{\mathcal{O}} \arrow[r] & \mathrm{St}_{\mathcal{O}}^{\mathrm{int}}(\mathrm{Un}_{\mathcal{O}}(\mathcal{A}_{\mathcal{O}}))
\end{tikzcd}$$

obtained from applying $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ to the cartesian square defining $\mathrm{Un}_{\mathcal{O}}^{\mathrm{int}}(X)$. Since relative Segal objects are stable under base change by Lemma 3.1.10, it suffices to show the right vertical map is a relative Segal object, which follows from Lemma 4.2.4. The commutative square (9) follows by restricting the square (2) in Observation 2.3.9 to full subcategories. □

For soundly extendable patterns $\mathcal{O}$ we can furthermore think of this adjunction as being induced by one between fibrous patterns and Segal $\mathcal{O}$-∞-categories:

Theorem 4.2.6 *Let $\mathcal{O}$ be a soundly extendable pattern. Then there is an adjunction*

$$\mathsf{Env}_{\mathcal{O}} : \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty),$$

where $\mathsf{Env}_{\mathcal{O}}(\mathcal{P})(X) := \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/X}$ and the right adjoint is given by unstraightening. This induces an adjunction

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} : \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}},$$

where the left adjoint is fully faithful and the image consists of the Segal $\mathcal{O}$-∞-categories that are equifibered over $\mathcal{A}_{\mathcal{O}}$.

Proof It remains to show that the adjunction

$$(-) \times_{\mathcal{O}} \mathsf{Ar}_{\mathrm{act}}(\mathcal{O}) : \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \rightleftarrows \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{cocart}} \simeq \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)$$

from Corollary 2.2.5 restricts to an adjunction between $\mathsf{Fbrs}(\mathcal{O})$ and $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$. Since $\mathcal{A}_{\mathcal{O}}$ is a Segal $\mathcal{O}$-∞-category, we have by Observation 3.1.9 and Proposition 4.2.1 that the left adjoint takes fibrous patterns to Segal $\mathcal{O}$-∞-categories. On the other hand, the right adjoint takes the latter to fibrous patterns by Observation 4.1.10. □

Remark 4.2.7 In the context of Theorem 4.2.6 the right adjoint of $\mathsf{Env}_{\mathcal{O}}$ is faithful and replete. It induces an equivalence between $\mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ and the subcategory of $\mathsf{Fbrs}(\mathcal{O})$ whose objects are cocartesian fibrous patterns and whose morphisms preserve all cocartesian edges.

Remark 4.2.8 If $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism between soundly extendable patterns, then pullback/restriction along f gives a commutative square

$$\begin{tikzcd}
\mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty) \arrow[r, "f^*"] \arrow[d] & \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \arrow[d] \\
\mathsf{Fbrs}(\mathcal{P}) \arrow[r, "f^*"] & \mathsf{Fbrs}(\mathcal{O}).
\end{tikzcd}$$

Note, however, that the corresponding Beck–Chevalley transformation is usually not invertible, so we have to slice over $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{O}}$ to get a commutative square of envelopes

$$\begin{tikzcd}
\mathsf{Fbrs}(\mathcal{P}) \arrow[r, "f^*"] \arrow[d, "\mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}"'] & \mathsf{Fbrs}(\mathcal{O}) \arrow[d, "\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}"] \\
\mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} \arrow[r, "f^{\circledast}"] & \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}
\end{tikzcd} \tag{10}$$

as a special case of (8).

4.3 Examples of Segal envelopes

Example 4.3.1 For the soundly extendable pattern $\mathbb{F}_*$, we know that fibrous patterns are exactly ∞-operads, while Segal $\mathbb{F}_*$-∞-categories are symmetric monoidal ∞-categories; here $\mathcal{A}_{\mathbb{F}_*}$ is the symmetric monoidal category $\mathbb{F}^{\amalg}$ of finite sets under disjoint union. Hence Theorem 4.2.6 yields an adjunction

$$\mathsf{Env}_{\mathbb{F}_*}^{/\mathbb{F}^{\amalg}}\colon \mathsf{Opd}_\infty = \mathsf{Fbrs}(\mathbb{F}_*) \rightleftarrows \mathsf{Seg}_{\mathbb{F}_*}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathbb{F}_*}} = \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}.$$

The left adjoint is fully faithful and a symmetric monoidal functor $\pi\colon (\mathcal{C},\otimes) \to (\mathbb{F},\amalg)$ lies in the essential image if and only if it is equifibered. This means that the following square is cartesian for all maps $\omega\colon X \to Y$ in $\mathbb{F}$:

$$\begin{tikzcd}
\mathcal{C}^X \arrow[r, "\omega_\otimes"] \arrow[d, "\pi^X"'] & \mathcal{C}^Y \arrow[d, "\pi^Y"] \\
\mathbb{F}^X \arrow[r, "\omega_\amalg"] & \mathbb{F}^Y .
\end{tikzcd}$$

Here the horizontal functors tensor over fibers of ω. In fact, it follows by taking products and pasting pullback diagrams[5] that it suffices to check the case of $\omega\colon \{1,2\} \to \{1\}$.

Observation 4.3.2 The essential image of the sliced envelope functor

$$\mathsf{Env}_{\mathbb{F}_*}^{/\mathbb{F}^{\amalg}}\colon \mathsf{Opd}_\infty \hookrightarrow \mathsf{CMon}(\mathsf{Cat}_\infty)_{/(\mathbb{F},\amalg)}$$

was first described in [19], but the characterization there looks at first glance quite different from ours. Let us therefore compare these two descriptions:

---

[5] See Lemma 5.2.16 for an elaboration of this argument.

For a symmetric monoidal functor $\pi\colon \mathcal{C} \to \mathbb{F}$, let us write $\mathcal{C}_{(1)} \subset \mathcal{C}$ for the full subcategory of those $x \in \mathcal{C}$ with $|\pi(x)| = 1$. Then the characterization of [19] is that the essential image consists of those π that satisfy the following pair of conditions:

(1) Every object $x \in \mathcal{C}$ is equivalent to $x_1 \otimes \cdots \otimes x_n$ for some $x_i \in \mathcal{C}_{(1)}$.

(2) For every $n, m \geq 0$ and any two tuples $x_1, \ldots, x_m \in \mathcal{C}_{(1)}$ and $y_1, \ldots, y_n \in \mathcal{C}_{(1)}$, the canonical map

$$\coprod_{\varphi\colon m \to n} \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{C}}\Bigl(\bigotimes_{j \in \varphi^{-1}(i)} x_j, y_i\Bigr) \to \mathrm{Map}\Bigl(\bigotimes_{j=1}^{m} x_j, \bigotimes_{i=1}^{n} y_i\Bigr)$$

is an equivalence.

These conditions must be equivalent to our equifiberedness condition since they describe the same full subcategory. To check this more explicitly, we consider the functor

$$D_n\colon \mathcal{C}^n \to \mathbb{F}^n \times_{\mathbb{F}} \mathcal{C},$$

which is an equivalence for all n if and only if $p\colon \mathcal{C} \to \mathbb{F}$ is equifibered. The functor D_n is essentially surjective if and only if for any $x \in \mathcal{C}$ and a decomposition $\pi(x) = A_1 \amalg \cdots \amalg A_n$ there is a decomposition $x = x_1 \otimes \cdots \otimes x_n$ such that $\pi(x_i) \cong A_i$ compatibly with the decomposition. By choosing the trivial decomposition with $|A_i| = 1$ this recovers condition (1). Conversely, given condition (1) we can decompose x as $\bigotimes_{a \in \pi(x)} y_a$ and then find the desired x_i as $x_i = \bigotimes_{a \in A_i} y_a$.

To see that the full faithfulness of the D_n's corresponds to condition (2), we first observe that in the presence of condition (1) we can replace condition (2) with the following:

(2′) For every $n \geq 0$ and any two tuples $z_1, \ldots, z_n \in \mathcal{C}$ and $y_1, \ldots, y_n \in \mathcal{C}$, the canonical map

$$\prod_{i=1}^{n} \mathrm{Map}_{\mathcal{C}}(z_i, y_i) \to \coprod_{(\varphi_i\colon \pi(z_i) \to \pi(y_i))} \mathrm{Map}^{\coprod_{i=1}^{n} \varphi_i}\Bigl(\bigotimes_{i=1}^{n} z_i, \bigotimes_{i=1}^{n} y_i\Bigr)$$

is an equivalence. Here we write $\mathrm{Map}_{\mathcal{C}}^{\varphi}(a, b)$ for the fiber of $\mathrm{Map}_{\mathcal{C}}(a, b)$ over some $\varphi\colon \pi(a) \to \pi(b)$.

To relate this to condition (2), first decompose y_i using condition (1) and use two-out-of-three to reduce to the case where $|\pi(y_i)| = 1$. Then write $z_i = \bigotimes_{j \in \varphi^{-1}(i)} x_j$ and argue as in [19, Remark 2.4.8].

Now we can observe that D_n is fully faithful if and only if condition (2′) holds: indeed, the mapping space in $\mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}$ can be described as

$$\begin{aligned}\mathrm{Map}_{\mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}}\bigl((x, \pi(x) = A_1 \amalg \cdots \amalg A_n), (y, \pi(y) = B_1 \amalg \cdots \amalg B_n)\bigr)& \\ \simeq \mathrm{Map}_{\mathbb{F}^n}((A_i), (B_i)) \times_{\mathrm{Map}_{\mathbb{F}}(\pi(x), \pi(y))} \mathrm{Map}_{\mathcal{C}}(x, y)& \\ \simeq \coprod_{(\varphi_i\colon A_i \to B_i)} \mathrm{Map}^{\coprod \varphi_i}(x, y).&\end{aligned}$$

Applying this to the images of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ under $\mathcal{C}^n \to \mathbb{F}^n \times_{\mathbb{F}} \mathcal{C}$ yields the desired form.

It is interesting that while in condition (2) we need to quantify over all $n, m \geq 0$, in condition (2′) it suffices to consider only the case $n = 2$ as all other cases can be obtained inductively. This works because the objects z_i and y_i in condition (2′) are themselves allowed to be composite.

Example 4.3.3 For the soundly extendable pattern $\mathbf{\Delta}^{\mathrm{op},\flat}$ fibrous patterns are nonsymmetric ∞-operads, while Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-$\infty$-categories are monoidal ∞-categories. We therefore define $\mathsf{Opd}_\infty^{\mathrm{ns}} := \mathsf{Fbrs}(\mathbf{\Delta}^{\mathrm{op},\flat})$ and $\mathsf{Mon}(\mathsf{Cat}_\infty) := \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op},\flat}}(\mathsf{Cat}_\infty)$. The Segal $\mathbf{\Delta}^{\mathrm{op},\flat}$-category $\mathcal{A}_{\mathbf{\Delta}^{\mathrm{op},\flat}}$ is equivalent to the category $\mathbf{\Delta}_+$ of finite (possibly empty) linearly ordered sets, with the monoidal structure given by concatenation. The envelope functor $\mathsf{Env}_{\mathbf{\Delta}^{\mathrm{op},\flat}}^{/\mathbf{\Delta}_+}$ can then be interpreted as a fully faithful embedding:

$$\mathsf{Env}_{\mathbf{\Delta}^{\mathrm{op},\flat}}^{/\mathbf{\Delta}_+} : \mathsf{Opd}_\infty^{\mathrm{ns}} \hookrightarrow \mathsf{Mon}(\mathsf{Cat}_\infty)_{/\mathbf{\Delta}_+}.$$

Similarly to Example 4.3.1 we can describe the essential image as those monoidal functors $\pi : \mathcal{V} \to \mathbf{\Delta}_+$ for which the following natural square is cartesian:

$$\begin{tikzcd}
\mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"] \arrow[d, "\pi"'] & \mathcal{V} \arrow[d, "\pi"] \\
\mathbf{\Delta}_+ \times \mathbf{\Delta}_+ \arrow[r, "\otimes"] & \mathbf{\Delta}_+.
\end{tikzcd}$$

Example 4.3.4 For the soundly extendable pattern $\mathbf{\Delta}^{\mathrm{op},\natural}$, fibrous patterns are generalized nonsymmetric ∞-operads as defined in [11], while Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-$\infty$-categories are category objects in Cat_∞, ie double ∞-categories. We thus write $\mathsf{Opd}_\infty^{\mathrm{gen,ns}} := \mathsf{Fbrs}(\mathbf{\Delta}^{\mathrm{op},\natural})$ and $\mathsf{DblCat}_\infty := \mathsf{Seg}_{\mathbf{\Delta}^{\mathrm{op},\natural}}(\mathsf{Cat}_\infty)$. We may regard $(\infty,2)$-categories (in the form of complete 2-fold Segal spaces) as those double ∞-categories $\mathcal{X}_\bullet$ such that $\mathcal{X}_0$ is an ∞-groupoid and which satisfy a completeness condition. In particular, the Segal $\mathbf{\Delta}^{\mathrm{op},\natural}$-$\infty$-category $\mathcal{A}_{\mathbf{\Delta}^{\mathrm{op},\natural}} \simeq \mathcal{A}_{\mathbf{\Delta}^{\mathrm{op},\flat}}$ may be thought of as the one-object $(\infty,2)$-category $\mathfrak{B}\mathbf{\Delta}_+$ where the endomorphisms of the single object are $\mathbf{\Delta}_+$, with the monoidal structure corresponding to composition. The envelope functor $\mathsf{Env}_{\mathbf{\Delta}^{\mathrm{op},\natural}}^{/\mathfrak{B}\mathbf{\Delta}_+}$ can then be interpreted as giving a fully faithful embedding:

$$\mathsf{Env}_{\mathbf{\Delta}^{\mathrm{op},\natural}}^{/\mathfrak{B}\mathbf{\Delta}_+} : \mathsf{Opd}_\infty^{\mathrm{gen,ns}} \hookrightarrow \mathsf{DblCat}_{\infty/\mathfrak{B}\mathbf{\Delta}_+}.$$

The essential image is characterized by a pullback square analogous to the one from Example 4.3.3. The morphisms in $\mathsf{Opd}_\infty^{\mathrm{gen,ns}}$ among the cocartesian fibrations that correspond to $(\infty,2)$-categories are precisely *lax functors* as defined for instance in [10], so we obtain a description of these in terms of $\mathsf{DblCat}_{\infty/\mathfrak{B}\mathbf{\Delta}_+}$. (More generally, we can also consider the envelope for $\mathbf{\Delta}^{n,\mathrm{op},\natural}$, which was briefly discussed in [15].)

Example 4.3.5 Let $\mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. Fibrous $\mathcal{O}$-patterns are, by Example 4.1.18, exactly ∞-operads over $\mathcal{O}$, while Segal $\mathcal{O}$-∞-categories are precisely $\mathcal{O}$-monoidal ∞-categories which we denote by $\mathsf{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty) := \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$. By Example 3.3.19, $\mathcal{O}$ is soundly extendable and our construction recovers the $\mathcal{O}$-monoidal envelope of [23, Section 2.2.4]. In particular, this gives a fully faithful embedding

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} : \mathsf{Opd}_{\infty/\mathcal{O}} \to \mathsf{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}.$$

In the case $\mathcal{O} = \mathbb{E}_n$, the ∞-category $\mathcal{A}_{\mathbb{E}_n}$ admits an alternative description as the $\mathbb{E}_n$-monoidal ∞-category of embedded n-disks in $\mathbb{R}^n$.

5 The comparison theorem

In Section 5.1 we use the Segal envelopes to prove the comparison result, Theorem A. We then discuss the application of this to equivariant ∞-operads, Corollary B, in Section 5.2. Finally, we explain how to upgrade the envelope and comparison equivalences to equivalences of $(\infty,2)$-categories in Section 5.3.

5.1 Comparing fibrous patterns

In this subsection we will use Segal envelopes to obtain a criterion for a morphism of patterns $f\colon \mathcal{O} \to \mathcal{P}$ to induce via pullback an equivalence

$$f^*\colon \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})$$

between the corresponding ∞-categories of fibrous patterns. We specialize this to recover some comparison results from [23] without using the technical results on approximations to ∞-operads from [23, Section 2.3.3]. As new applications, we show that (symmetric) ∞-operads can also be described as fibrous patterns over $\mathsf{Span}(\mathbb{F})$, and that fibrous patterns over $\mathsf{Span}(\mathcal{S}_m)$ and $\mathsf{Span}_{(m-1)\text{-tr},\text{all}}(\mathcal{S}_m)$ are equivalent.

Theorem 5.1.1 *Suppose $\mathcal{O}$ is a pattern, $\mathcal{P}$ is a soundly extendable pattern, and $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism such that the following conditions hold:*

(i) $f^{\mathrm{el}}\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ *is an equivalence of ∞-categories.*

(ii) $(\mathcal{O}^{\mathrm{act}}_{/X})^{\simeq} \to (\mathcal{P}^{\mathrm{act}}_{/f(X)})^{\simeq}$ *is an equivalence for all $X \in \mathcal{O}$.*

Then pullback along f gives an equivalence

$$f^*\colon \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O}).$$

Remark 5.1.2 If we also assume that $\mathcal{A}^{\simeq}_{\mathcal{O}} = (\mathcal{O}^{\mathrm{act}}_{/-})^{\simeq}$ is an $\mathcal{O}$-Segal space, for example if $\mathcal{O}$ is soundly extendable, then it suffices to check condition (ii) when X is elementary.

Example 5.1.3 Let $\mathcal{P}$ be a soundly extendable pattern, and define $\mathcal{O} \subset \mathcal{P}$ as the full subpattern on the "necessary objects" in the sense of [7, Definition 14.7]. This means that $\mathcal{O}$ contains those $X \in \mathcal{P}$ for which there exists an active morphism $X \rightsquigarrow E$ with E elementary. Then Theorem 5.1.1 applies to the full inclusion $\mathcal{O} \subset \mathcal{P}$ and hence restriction yields an equivalence $\mathsf{Fbrs}(\mathcal{P}) \simeq \mathsf{Fbrs}(\mathcal{O})$.

First we show that condition (ii) can always be strengthened as follows.

Lemma 5.1.4 *In the situation of Theorem 5.1.1 the induced natural transformation*

$$\alpha\colon \mathcal{A}_{\mathcal{O}} \to f^*\mathcal{A}_{\mathcal{P}}$$

of functors $\mathcal{O} \to \mathsf{Cat}_\infty$ is an equivalence. In particular $\mathcal{A}_{\mathcal{O}}$ is $\mathcal{O}$-Segal.

Proof By assumption, the functor $\mathcal{A}_{\mathcal{O}}(X)\to\mathcal{A}_{\mathcal{P}}(f(X))$ is an equivalence on underlying ∞-groupoids, so it remains to show that it is fully faithful. To see this, observe that given active maps $\varphi\colon X\rightsquigarrow Y$ and $\varphi'\colon X'\rightsquigarrow Y$, the mapping space $\mathsf{Map}_{\mathcal{O}^{\mathrm{act}}_{/Y}}(\varphi',\varphi)$ is the fiber at φ' of the map $(\varphi\circ -)\colon\mathcal{A}^{\simeq}_{\mathcal{O}}(X)\to\mathcal{A}^{\simeq}_{\mathcal{O}}(Y)$. This map fits into a square

$$\begin{tikzcd}
\mathcal{A}^{\simeq}_{\mathcal{O}}(X) \arrow[r, "\sim"] \arrow[d] & \mathcal{A}^{\simeq}_{\mathcal{P}}(fX) \arrow[d] \\
\mathcal{A}^{\simeq}_{\mathcal{O}}(Y) \arrow[r, "\sim"] & \mathcal{A}^{\simeq}_{\mathcal{P}}(fY),
\end{tikzcd}$$

where the horizontal maps are equivalences. Then we also have equivalences on fibers, which gives the desired full faithfulness. Finally we note that $\mathcal{A}_{\mathcal{O}}\simeq f^*\mathcal{A}_{\mathcal{P}}$ implies that $\mathcal{A}_{\mathcal{O}}$ is Segal since $\mathcal{A}_{\mathcal{P}}$ was assumed to be Segal and f^* preserves Segal objects. □

The following lemma tells us that for sound patterns it suffices to check $\mathsf{Ar}_{\mathrm{act}}(\mathcal{O})$-equifiberedness on active morphisms that end in elementary objects.

Lemma 5.1.5 *Let $\mathcal{O}$ be a sound pattern and let $(\eta\colon F\Rightarrow G)$ be a relative Segal object over $\mathcal{O}$ in a sufficiently complete ∞-category $\mathcal{C}$. Suppose that the naturality squares*

$$\begin{tikzcd}
F(X) \arrow[r, "F(\omega)"] \arrow[d, "\eta_X"] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(\omega)"] & G(Y)
\end{tikzcd}$$

are cartesian for active morphisms $\omega\colon X\rightsquigarrow Y$ where Y is elementary. Then they are also cartesian for arbitrary Y, ie η is $\mathsf{Ar}_{\mathrm{act}}(\mathcal{O})$-equifibered.

Proof For an arbitrary active morphism $\omega\colon X\rightsquigarrow Y$ consider the commutative cube

$$\begin{tikzcd}
 & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} F(\omega_{\alpha!}X) \arrow[rr, "\lim F(\omega_\alpha)"] \arrow[dd, "\lim \eta_{\omega_{\alpha!}X}"] & & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} F(E) \arrow[dd, "\lim \eta_E"] \\
F(X) \arrow[ur] \arrow[rr, "F(\omega)"] \arrow[dd, "\eta_X"] & & F(Y) \arrow[ur] \arrow[dd, "\eta_Y"] & \\
 & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} G(\omega_{\alpha!}X) \arrow[rr, "\lim G(\omega_\alpha)"] & & \lim_{\alpha\colon Y\rightarrowtail E\in\mathcal{O}^{\mathrm{el}}_{Y/}} G(E) \\
G(X) \arrow[ur] \arrow[rr, "G(\omega)"] & & G(Y). \arrow[ur] &
\end{tikzcd}$$

The back square is cartesian as it is a limit over squares that we have assumed to be cartesian. (Note that $\omega_\alpha\colon\omega_{\alpha!}X\rightsquigarrow E$ is an active morphism with elementary target.) The right face is cartesian because η is a relative Segal object, and so is the left face from this and Lemma 3.3.8. Therefore the front face is cartesian by the pullback pasting lemma. □

Proof of Theorem 5.1.1 It follows from Proposition 3.1.16 that the functor

$$f^*\colon \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty) \to \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$$

is an equivalence. From Lemma 5.1.4 we have $\mathcal{A}_{\mathcal{O}} \simeq f^*\mathcal{A}_{\mathcal{P}}$ and that $\mathcal{A}_{\mathcal{O}}$ is Segal. Hence the induced functor

$$f^{\circledast}\colon \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} \to \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$$

is also an equivalence. This means in the commutative square

$$\begin{tikzcd}
\mathsf{Fbrs}(\mathcal{P}) \arrow[r, "f^*"] \arrow[d, hook, "\mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}"] & \mathsf{Fbrs}(\mathcal{O}) \arrow[d, hook, "\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}"] \\
\mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}} \arrow[r, "f^{\circledast}"] & \mathsf{Seg}_{\mathcal{O}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}
\end{tikzcd}$$

from Proposition 4.2.1, the bottom horizontal functor $f^{\circledast}$ is an equivalence, while the vertical functors are fully faithful. It follows that the top horizontal functor f^* is also fully faithful. To prove that it is also essentially surjective, it suffices to show that an object of $\mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}}$ is in the image of $\mathsf{Env}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}$ if its image under the equivalence $f^{\circledast}$ is in the image of $\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}$.

Suppose we are given some $(\eta\colon F \Rightarrow \mathcal{A}_{\mathcal{P}}) \in \mathsf{Seg}_{\mathcal{P}}(\mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{P}}}$ such that $f^{\circledast}F \Rightarrow \mathcal{A}_{\mathcal{O}}$ is equifibered. Equivalently, $\eta_{\circ f}\colon (F \circ f) \Rightarrow (\mathcal{A}_{\mathcal{P}} \circ f)$ is equifibered. By Lemma 5.1.5 it suffices to check that the naturality squares are cartesian for active morphisms $\omega\colon X \rightsquigarrow E \in \mathcal{P}$ ending in an elementary. Since $f\colon \mathcal{O}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ is an equivalence, we may write $E \simeq f(E')$ for $E' \in \mathcal{O}$. Moreover, since $f\colon \mathcal{O}^{\mathrm{act}}_{/E'} \to \mathcal{P}^{\mathrm{act}}_{/f(E')}$ is an equivalence, we can find $\rho\colon Y \rightsquigarrow E' \in \mathcal{O}$ such that $f(\rho) \simeq \omega$ as objects of $\mathsf{Ar}_{\mathrm{act}}(\mathcal{P})$. Now it follows that the naturality square of η at ω is cartesian since we assumed that the naturality square of $\eta_{\circ f}$ at ρ is equifibered. This shows that η is $\mathsf{Ar}_{\mathrm{act}}(\mathcal{P})$-equifibered, and hence that $f^*\colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ is essentially surjective. □

As a variant of Theorem 5.1.1, we get a useful criterion for identifying the effect of the pushforward functor $f_!\colon \mathsf{Fbrs}(\mathcal{O}) \to \mathsf{Fbrs}(\mathcal{P})$ for a map of patterns $f\colon \mathcal{O} \to \mathcal{P}$:

Corollary 5.1.6 *Suppose we have a commutative diagram of patterns*

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "g"] \arrow[d, "p"] & \mathcal{R} \arrow[d, "q"] \\
\mathcal{O} \arrow[r, "f"] & \mathcal{P}
\end{tikzcd}$$

such that

(i) *$\mathcal{O}$ is sound and $\mathcal{Q}$ is a fibrous $\mathcal{O}$-pattern,*
(ii) *$\mathcal{P}$ is soundly extendable and $\mathcal{R}$ is a fibrous $\mathcal{P}$-pattern,*
(iii) *f is a strong Segal morphism,*
(iv) *g satisfies the assumptions of Theorem 5.1.1.*

Then the induced map of fibrous $\mathcal{O}$-patterns $\mathcal{Q} \to f^*\mathcal{R}$ is adjoint to an equivalence $f_!\mathcal{Q} \xrightarrow{\sim} \mathcal{R}$.

Proof For a fibrous $\mathcal{P}$-pattern $\mathcal{T}$, we have natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathsf{Fbrs}(\mathcal{O})}(\mathcal{Q}, f^*\mathcal{T}) &\simeq \mathrm{Map}_{\mathsf{Fbrs}(\mathcal{O})_{/\mathcal{Q}}}(\mathcal{Q}, p^* f^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathsf{Fbrs}(\mathcal{Q})}(g^*\mathcal{R}, g^*q^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathsf{Fbrs}(\mathcal{R})}(\mathcal{R}, q^*\mathcal{T}) \\ &\simeq \mathrm{Map}_{\mathsf{Fbrs}(\mathcal{P})}(\mathcal{R}, \mathcal{T}),\end{aligned}$$

where we have used Theorem 5.1.1 and Corollary 4.1.17. □

Corollary 5.1.7 *Suppose $\mathcal{O}$ is a sound pattern, $q\colon \mathcal{P} \to \mathbb{F}_*$ is a symmetric ∞-operad, and $f\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism that satisfies the assumptions of Theorem 5.1.1. Then f exhibits $\mathcal{P}$ as the* ***symmetrization*** *of $\mathcal{O}$, in the sense that the induced map $(qf)_!\mathcal{O} \to \mathcal{P}$ is an equivalence.* □

Example 5.1.8 Let Ass be the (symmetric) associative ∞-operad as defined in [23, Definition 4.1.1.1.], and let Cut: $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ass}$ denote the functor defined in [23, Construction 4.1.2.9.]. Then pullback along Cut gives an equivalence

$$\mathsf{Fbrs}(\mathbf{\Delta}^{\mathrm{op},\flat}) \xleftarrow{\sim} \mathsf{Fbrs}(\mathrm{Ass}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*)_{/\mathrm{Ass}}$$

between nonsymmetric ∞-operads and symmetric ∞-operads over Ass, where the second equivalence is that of Corollary 4.1.17. In other words, nonsymmetric ∞-operads are equivalent to symmetric ∞-operads over the associative ∞-operad. Moreover, Ass is the symmetrization of $\mathbf{\Delta}^{\mathrm{op},\flat}$.

The equivalence of Example 5.1.8 is also proved by Lurie as [23, Theorem 4.1.3.14], which is a special case of [23, Theorem 2.3.3.26]. This more general statement can also be proved by our methods; to see this, we first need to recall some definitions:

Definition 5.1.9 Let $\pi\colon \mathcal{O} \to \mathbb{F}_*$ be an ∞-operad. We say a functor $f\colon \mathcal{C} \to \mathcal{O}$ is an *approximation* if the following conditions hold:

(1) For $C \in \mathcal{C}$ over $\langle n \rangle$ in $\mathbb{F}_*$, there exists for $i = 1, \dots, n$ a locally cocartesian morphism $\rho_i^C\colon C \to C_i$ in $\mathcal{C}$ over $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$. Moreover, the image of ρ_i^C in $\mathcal{O}$ is inert.

(2) $\mathcal{C}$ has all f-cartesian lifts of active morphisms in $\mathcal{O}$.

Following [21], we say that f is a *strong approximation* if we additionally have:

(3) The functor $\mathcal{C}_{\langle 1 \rangle} \to \mathcal{O}_{\langle 1 \rangle}$ is an equivalence.

Observation 5.1.10 Suppose $\mathcal{O}$ is an ∞-operad and $f\colon \mathcal{C} \to \mathcal{O}$ is an approximation. We say a morphism in $\mathcal{C}$ is *inert* if its image in $\mathcal{O}$ is inert, and *active* if it is f-cartesian and its image in $\mathcal{O}$ is active. Then the inert and active morphisms in $\mathcal{C}$ give a factorization system. We think of $\mathcal{C}$ as an algebraic pattern using this factorization system, with the elementary objects being those that map to $\langle 1 \rangle$ in $\mathbb{F}_*$; then f is a morphism of algebraic patterns.

Proposition 5.1.11 *Suppose $\mathcal{O}$ is an ∞-operad and $f\colon \mathcal{C} \to \mathcal{O}$ is a strong approximation.*

(i) $\mathcal{C}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ *is an equivalence.*

(ii) $\mathcal{C}^{\mathrm{el}}_{C/} \to \mathcal{O}^{\mathrm{el}}_{f(C)/}$ *is an equivalence for all $C \in \mathcal{C}$, ie f is an iso-Segal morphism.*

(iii) $\mathcal{C}^{\mathrm{act}}_{/C} \to \mathcal{O}^{\mathrm{act}}_{/f(C)}$ *is an equivalence for all $C \in \mathcal{C}$.*

Proof For (i), observe that from the equivalence $\mathcal{C}_{\langle 1\rangle} \xrightarrow{\sim} \mathcal{O}_{\langle 1\rangle}$ it follows that a morphism in $\mathcal{C}$ over $\langle 1\rangle$ is inert if and only if it is an equivalence (since the equivalences are precisely the inert morphisms in $\mathcal{O}_{\langle 1\rangle}$). Hence $\mathcal{C}^{\mathrm{el}} = \mathcal{C}^{\simeq}_{\langle 1\rangle}$, so the functor $\mathcal{C}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{el}}$ is just the underlying morphism of ∞-groupoids of the functor between fibers over $\langle 1\rangle$ that is an equivalence by assumption.

To show (ii), we first observe that $\mathcal{C}^{\mathrm{el}}_{C/}$ is an ∞-groupoid, since morphisms are given by inert maps over $\langle 1\rangle$ and these are invertible. Moreover, if C lies over $\langle n\rangle$ then the fiber of $\mathcal{C}^{\mathrm{el}}_{C/}$ over ρ_i is contractible, since there by assumption exists a locally cocartesian morphism over ρ_i — this is then initial in the ∞-category $(\mathcal{C}_{C/})_{\rho_i}$ and so in particular has no automorphisms.

We thus have a commutative triangle

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{el}}_{C/} \arrow[rr] \arrow[dr, "\sim"'] & & \mathcal{O}^{\mathrm{el}}_{f(C)/} \arrow[dl, "\sim"] \\
& (\mathbb{F}^{\mathrm{el}}_{*})_{\langle n\rangle}, &
\end{tikzcd}$$

where both maps to $(\mathbb{F}^{\mathrm{el}}_{*})_{\langle n\rangle}$ are equivalences, hence so is the top horizontal map.

To prove (iii), observe that by assumption $\mathcal{C}^{\mathrm{act}} \to \mathcal{O}^{\mathrm{act}}$ is the underlying right fibration of the cartesian fibration $\mathcal{C} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}} \to \mathcal{O}^{\mathrm{act}}$. This gives the required equivalence of slices by [24, Tag `00TE`]. □

Corollary 5.1.12 *Suppose $f\colon \mathcal{C} \to \mathcal{O}$ is a strong approximation to an ∞-operad $q\colon \mathcal{O} \to \mathbb{F}_*$.*

(1) *If $\mathcal{X}$ is an ∞-category with finite products, then restriction along f gives an equivalence*

$$f^*\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Seg}_{\mathcal{C}}(\mathcal{X}).$$

(2) *Pullback along f gives an equivalence*

$$f^*\colon \mathrm{Fbrs}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Fbrs}(\mathcal{C}).$$

(3) *The map f exhibits $\mathcal{O}$ as the symmetrization of $\mathcal{C}$, ie $(qf)_!\mathcal{C} \simeq \mathcal{O}$.*

Proof Combine Proposition 5.1.11 with Proposition 3.1.16, Theorem 5.1.1, and Corollary 5.1.7. □

Remark 5.1.13 Lurie's proof of [23, Theorem 2.3.3.26] uses envelopes for approximations to ∞-operads, just as our proof of Theorem 5.1.1, and we do not claim that our proof is different in any essential way.

We end this section with a couple of examples that do not follow from Corollary 5.1.12 or [23, Theorem 2.3.3.26]. These involve patterns defined using spans, so we start with a general observation about comparisons of these:

Observation 5.1.14 Consider two adequate triples $(\mathfrak{X}, \mathfrak{X}^b, \mathfrak{X}^f)$ and $(\mathfrak{Y}, \mathfrak{Y}^b, \mathfrak{Y}^f)$ and a functor $F\colon \mathfrak{X} \to \mathfrak{Y}$ that preserves the two subcategories and also preserves pullbacks of backwards maps along forwards maps. Suppose further that we have full subcategories $\mathfrak{X}_0 \subset \mathfrak{X}$ and $\mathfrak{Y}_0 \subset \mathfrak{Y}$ such that $F(\mathfrak{X}_0) \subset \mathfrak{Y}_0$. Then F induces a morphism of patterns:

$$F\colon \mathsf{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0) \to \mathsf{Span}_{b,f}(\mathfrak{Y}; \mathfrak{Y}_0).$$

We may apply Theorem 5.1.1 to this if the following conditions hold:

1. $\mathsf{Span}_{b,f}(\mathfrak{Y}; \mathfrak{Y}_0)$ is soundly extendable. (See Proposition 3.3.23.)
2. For all $x \in \mathfrak{X}$, the map $\mathfrak{X}_0^b \times_{\mathfrak{X}^b} \mathfrak{X}^b_{/x} \to \mathfrak{Y}_0^b \times_{\mathfrak{Y}^b} \mathfrak{Y}^b_{/F(x)}$ is cofinal.
3. $F\colon \mathfrak{X}_0^b \to \mathfrak{Y}_0^b$ is an equivalence of ∞-categories.
4. $F\colon \mathfrak{X}^f_{/x} \to \mathfrak{Y}^f_{/F(x)}$ induces an equivalence on maximal subgroupoids for all $x \in \mathfrak{X}$.

Point (2) ensures that F is a strong Segal morphism since $\mathsf{Span}_{b,f}(\mathfrak{X}; \mathfrak{X}_0)^{\mathrm{int}} \simeq (\mathfrak{X}^b)^{\mathrm{op}}$ with the elementaries being $(\mathfrak{X}_0^b)^{\mathrm{op}}$.

Corollary 5.1.15 *Pullback along the inclusion* $\mathfrak{i}\colon \mathbb{F}_* \simeq \mathsf{Span}_{\mathrm{inj,all}}(\mathbb{F}) \to \mathsf{Span}(\mathbb{F})$ *gives an equivalence*

$$\mathfrak{i}^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F})) \xrightarrow{\sim} \mathsf{Fbrs}(\mathbb{F}_*) \simeq \mathsf{Opd}_\infty.$$

Proof We check the conditions of Theorem 5.1.1 in the form stated in Observation 5.1.14:

1. The pattern is soundly extendable by Example 3.3.25.
2. For $A \in \mathbb{F}$ the relevant functor is the restriction of $\mathbb{F}^{\mathrm{inj}}_{/A} \to \mathbb{F}_{/A}$ to elementaries. But every map out of a one-point set is injective, so this is an equivalence.
3. Similarly, the functor on backwards morphisms $\mathbb{F}^{\mathrm{inj}} \to \mathbb{F}$ restricts to an equivalence on elementaries.
4. Both categories have the same forward morphisms. □

More generally, we have:

Corollary 5.1.16 *Pullback along the inclusion* $\mathfrak{i}_m\colon \mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m) \to \mathsf{Span}(\mathcal{S}_m)$ *induces an equivalence*

$$\mathfrak{i}_m^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathcal{S}_m)) \to \mathsf{Fbrs}(\mathsf{Span}_{(m-1)\text{-tr},\mathrm{all}}(\mathcal{S}_m)).$$

Proof We can apply Theorem 5.1.1: $\mathsf{Span}(\mathcal{S}_m)$ is soundly extendable by Example 3.3.27 and in this example we also note that $\mathfrak{i}_m$ is an iso-Segal morphism. Condition (i) of Theorem 5.1.1 holds because in both cases the elementary ∞-category is the terminal ∞-category. Condition (ii) holds because both span ∞-categories have the same forward morphisms. □

5.2 G-equivariant ∞-operads

In this section we apply the theory of fibrous patterns and envelopes in the setting of G-equivariant ∞-operads developed by Nardin and Shah in [26]. While their paper works in the generality of T-parametrized ∞-operads, we will restrict to the special case of the orbit category $T = \mathrm{Orb}_G$ for simplicity. Our main result is that the G-∞-operads of [26] are equivalent to fibrous $\mathsf{Span}(\mathbb{F}_G)$-patterns; we will also show that the sliced envelope for G-∞-operads is fully faithful and characterize the image, giving a third description of these objects.

First, we recall some constructions in equivariant higher algebra, which were pioneered in [3] and further developed in [25; 26]. Fix a finite group G throughout.

Definition 5.2.1 Let $\mathbb{F}_G$ be the category of finite G-sets, $\mathbb{F}_{G,*}$ the category of finite pointed G-sets, and $\mathrm{Orb}_G \subset \mathbb{F}_G$ the full subcategory of G-orbits.

Definition 5.2.2 A *G-∞-category* is a functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ and a *G-symmetric monoidal ∞-category* is a $\mathsf{Span}(\mathbb{F}_G)$-Segal object in Cat_∞. We write

$$\mathsf{Cat}_{G,\infty} := \mathsf{Fun}(\mathrm{Orb}_G^{\mathrm{op}}, \mathsf{Cat}_\infty) \quad \text{and} \quad \mathsf{Cat}_{G,\infty}^{\otimes} := \mathsf{Seg}_{\mathsf{Span}(\mathbb{F}_G)}(\mathsf{Cat}_\infty)$$

and define the forgetful functor $\mathsf{Cat}_{G,\infty}^{\otimes} \to \mathsf{Cat}_{G,\infty}$ by restricting to the elementaries $\mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Span}(\mathbb{F}_G)$.

Notation 5.2.3 For a G-∞-category $\mathcal{C}\colon \mathrm{Orb}_G^{\mathrm{op}} \to \mathsf{Cat}_\infty$ we denote its value at G/H by $\mathcal{C}^H$ and refer to it as the H-fixed point ∞-category of $\mathcal{C}$. There are restriction maps $\mathcal{C}^H \to \mathcal{C}^K$ for $K \subset H \subset G$. Given a G-symmetric monoidal ∞-category $\mathcal{D}\colon \mathsf{Span}(\mathbb{F}_G) \to \mathsf{Cat}_\infty$ we further have tensor products $\otimes\colon \mathcal{D}^H \times \mathcal{D}^H \to \mathcal{D}^H$ and so-called norm maps $\mathrm{Nm}_K^H\colon \mathcal{D}^K \to \mathcal{D}^H$ for all $K \subset H \subset G$ coming from the span $(G/K \xleftarrow{=} G/K \to G/H)$.

Example 5.2.4 Since $\mathsf{Span}(\mathbb{F}_G)$ is an extendable pattern (Example 3.3.26) $\mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)}$ is a Segal object in Cat_∞. We denote this G-symmetric monoidal ∞-category by

$$\underline{\mathbb{F}}_G := \mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)}(-) = \mathsf{Span}(\mathbb{F}_G)^{\mathrm{act}}_{/-}\colon \mathsf{Span}(\mathbb{F}_G) \to \mathsf{Cat}_\infty.$$

The H-fixed point category is the category of finite H-sets:

$$(\underline{\mathbb{F}}_G)^H = \mathsf{Span}(\mathbb{F}_G)^{\mathrm{act}}_{/(G/H)} \simeq (\mathbb{F}_G)_{/(G/H)} \simeq \mathbb{F}_H.$$

The restriction maps are given by restriction, the tensor product by disjoint union, and the norm maps are $(- \times H)_{/K}\colon \mathbb{F}_K \to \mathbb{F}_H$. In summary, $\underline{\mathbb{F}}_G$ is $\mathbb{F}_G$ with its natural structure as a G-symmetric monoidal ∞-category.

Below we will see that fibrous $\mathsf{Span}(\mathbb{F}_G)$-patterns model G-∞-operads. We now explain how $\mathcal{N}_\infty$-operads fit into this framework:

Example 5.2.5 Let $\mathbb{F}_G^f \subset \mathbb{F}_G$ be a wide subcategory closed under base change and disjoint union. Then the inclusion functor $\mathsf{Span}_{\mathrm{all},f}(\mathbb{F}_G) \to \mathsf{Span}(\mathbb{F}_G)$ defines a fibrous $\mathsf{Span}(\mathbb{F}_G)$-pattern. To see that it has cocartesian lifts for inerts, note that any functor of the form $\mathsf{Span}_{b,f}(\mathcal{C}) \to \mathsf{Span}_{b,\mathrm{all}}(\mathcal{C})$ has cocartesian lifts for backwards maps. For the second condition we need to show that

$$(\mathbb{F}_G^f)_{/A} \to \lim_{U \in (\mathrm{Orb}_G)_{/A}} (\mathbb{F}_G^f)_{/U}$$

is an equivalence. The limit may be rewritten as a product over the set of orbits of A and then the equivalence follows because $\mathbb{F}_G^f$ is closed under base change and disjoint union.

Categories $\mathbb{F}_G^f$ that in addition to the above also contain all fold maps $\nabla\colon G/H \amalg G/H \to G/H$, are in bijection with the indexing systems of [6], see [26, Remark 2.4.12]. Under the equivalence $\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \simeq \mathsf{Opd}_{G,\infty}$ proved below the fibrous $\mathsf{Span}(\mathbb{F}_G)$-patters described above are the "commutative G-∞-operads" from [26, Definition 2.4.10], which correspond to the $\mathcal{N}_\infty$-operads of [6] by [26, Remark 2.4.12].

We now quickly recall the necessary notation from [26] to state their definition of G-∞-operads, but we refer the reader there for details.

Definition 5.2.6 Define $\underline{\mathbb{F}}_G^\vee \subset \mathsf{Ar}(\mathbb{F}_G)$ as the full subcategory of those morphisms $(f : U \to V)$ where V is an orbit: $\underline{\mathbb{F}}_G^\vee := \mathsf{Ar}(\mathbb{F}_G) \times_{\mathbb{F}_G} \mathrm{Orb}_G$. We say that a morphism $f \to g$ given by

$$\begin{tikzcd}
U \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d, "g"] \\
V \arrow[r, "k"] & Y
\end{tikzcd}$$

- lies in $(\underline{\mathbb{F}}_G^\vee)^{\mathrm{si}}$ if it is a summand inclusion, ie $U \to X \times_Y V$ is injective,
- lies in $(\underline{\mathbb{F}}_G^\vee)^{\mathrm{tdeg}}$ if it is target degenerate, ie $k : V \to Y$ is an equivalence.

Definition 5.2.7 Define $\underline{\mathbb{F}}_{G,*}$ as the algebraic pattern

$$\underline{\mathbb{F}}_{G,*} := \mathsf{Span}_{\mathrm{si},\mathrm{tdeg}}(\underline{\mathbb{F}}_G^\vee; \mathrm{Orb}_G),$$

where the elementary objects are those in the essential image of the identity inclusion $\mathrm{Orb}_G \to \mathsf{Ar}(\mathrm{Orb}_G) \subset \underline{\mathbb{F}}_G^\vee$.

Remark 5.2.8 The functor $\mathrm{ev}_1 : \underline{\mathbb{F}}_G^\vee \to \mathrm{Orb}_G$ induces a cocartesian fibration

$$\underline{\mathbb{F}}_{G,*} = \mathsf{Span}_{\mathrm{si},\mathrm{tdeg}}(\underline{\mathbb{F}}_G^\vee) \xrightarrow{\mathrm{ev}_1} \mathsf{Span}_{\mathrm{all},\mathrm{iso}}(\mathrm{Orb}_G) \simeq \mathrm{Orb}_G^{\mathrm{op}}.$$

Straightening this yields a G-∞-category whose H-fixed point category is $(\underline{\mathbb{F}}_{G,*})^H \simeq \mathbb{F}_{H,*}$, similarly to Example 5.2.4.

Observation 5.2.9 For $(U \to V) \in \underline{\mathbb{F}}_{G,*}$ the category of elementaries under $(U \to V)$ is equivalent to the opposite of the category of orbits over U (as in Remark 3.2.7):

$$(\underline{\mathbb{F}}_{G,*})^{\mathrm{el}}_{(U\to V)/} \simeq (\mathrm{Orb}_G \times_{(\underline{\mathbb{F}}_G^\vee)} (\underline{\mathbb{F}}_G^{\vee,\mathrm{si}})_{/(U\to V)})^{\mathrm{op}} \simeq (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\mathrm{op}}.$$

Here we used that any morphism $(Q \xrightarrow{=} Q) \to (U \to V)$ (where Q is an orbit) is automatically in $(\underline{\mathbb{F}}_G^{\vee})^{\mathrm{si}}$ since $Q \to Q \times_V U$ is injective. Now consider the full subcategory on those $(Q \to U)$ that are injective. This subcategory is equivalent to the discrete set of orbits U/G and moreover the inclusion of the subcategory is a left adjoint:

$$U/G \hookrightarrow (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\mathbb{F}_G)_{/U})^{\mathrm{op}} \simeq (\underline{\mathbb{F}}_{G,*})^{\mathrm{el}}_{(U \to V)/},$$

with right adjoint given by sending $(f : Q \to U)$ to $(f(Q) \hookrightarrow U)$. In particular, the inclusion of U/G is a coinitial functor. This means that for any kind of (weak) Segal condition over $\underline{\mathbb{F}}_{G,*}$ the limit involved can be rewritten as a product indexed by the finite set U/G.

Corollary 5.2.10 *The pattern* $\underline{\mathbb{F}}_{G,*}$ *is sound.*[6]

Proof We check the conditions of Proposition 3.3.23. First we show that the backwards maps satisfy cancellation. Consider two morphisms in $\underline{\mathbb{F}}_G^{\vee}$:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "e"] & U \arrow[r, "h"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "b"] & V \arrow[r, "k"] & Y
\end{tikzcd}$$

such that $A \to B \times_Y X$ is injective. We can write this map as a composite $A \to B \times_V U \to B \times_Y X$, the first map of which then has to be injective. In other words $(a, b)\colon e \to f$ is in $\underline{\mathbb{F}}_G^{\vee,\mathrm{si}}$ as claimed.

We also need to show that the inclusion $\mathfrak{X}^b_{0/y} \hookrightarrow \mathfrak{X}_{0/y}$ is cofinal. In the case at hand this inclusion is $\mathrm{Orb}_G \times_{\underline{\mathbb{F}}_G^{\vee}} (\underline{\mathbb{F}}_G^{\vee,\mathrm{si}})_{/(U \to V)} \to \mathrm{Orb}_G \times_{\underline{\mathbb{F}}_G^{\vee}} (\underline{\mathbb{F}}_G^{\vee})_{/(U \to V)}$, which is an equivalence by the argument from Observation 5.2.9. ☐

Definition 5.2.11 [26] A *G-∞-operad* is a weak Segal fibration over $\underline{\mathbb{F}}_{G,*}$ in the sense of [7, Definition 9.6]; see also Proposition 4.1.7. Let $\mathsf{Opd}_{G,\infty}$ denote the full subcategory of $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\underline{\mathbb{F}}_{G,*}}$ on the G-∞-operads.

Observation 5.2.12 This agrees with the definition of [26]. First we note that given $p\colon \mathcal{P} \to \underline{\mathbb{F}}_{G,*}$ with cocartesian lifts for inerts, the composite $\mathrm{ev}_1 \circ p\colon \mathcal{P} \to \mathrm{Orb}_G^{\mathrm{op}}$ exhibits $\mathcal{P}$ as a cocartesian fibration over $\mathrm{Orb}_G^{\mathrm{op}}$, ie an Orb_G-∞-category, and p as an Orb_G-functor. This holds because the inert morphisms in $\underline{\mathbb{F}}_{G,*}$ contain all the cocartesian lifts of $\mathrm{ev}_1\colon \underline{\mathbb{F}}_{G,*} \to \mathrm{Orb}_G^{\mathrm{op}}$. We hence have an identification:

$$\mathsf{Cat}^{\text{int-cocart}}_{\infty/\underline{\mathbb{F}}_{G,*}} = (\mathsf{Cat}_{G,\infty})^{\text{int-cocart}}_{/\underline{\mathbb{F}}_{G,*}}.$$

It remains to see that their conditions (2) and (3) exactly amount to the weak Segal conditions (2) and (3) in [7, Definition 9.6]. Indeed, this follows by inspection using Observation 5.2.9 and [7, Remark 9.7].

---

[6] In fact this pattern is soundly extendable. This follows because the functor $\underline{\mathbb{F}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ discussed in Proposition 5.2.14 is iso-Segal and induces an equivalence on forward maps. However, the extendability of $\underline{\mathbb{F}}_{G,*}$ will not be needed here.

Corollary 5.2.13 *We have* $\mathsf{Opd}_{G,\infty} = \mathsf{Fbrs}(\underline{\mathbb{F}}_{G,*})$.

Proof The pattern $\underline{\mathbb{F}}_{G,*}$ is sound by Corollary 5.2.10 and hence weak Segal fibrations and fibrous patterns are the same by Proposition 4.1.7. □

Proposition 5.2.14 *Restriction along the morphism of patterns* $\underline{\mathbb{F}}_{G,*} \xrightarrow{s} \mathsf{Span}(\mathbb{F}_G)$ *induced by the functor* $\underline{\mathbb{F}}_G^{\vee} \to \mathbb{F}_G$ *given by evaluation at* 0 *yields an equivalence*

$$s^*\colon \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G)) \xrightarrow{\simeq} \mathsf{Fbrs}(\underline{\mathbb{F}}_{G,*}) = \mathsf{Opd}_{G,\infty}.$$

Proof We need to show that the morphism of patterns

$$s\colon \underline{\mathbb{F}}_{G,*} = \mathsf{Span}_{\mathrm{si,tdeg}}(\underline{\mathbb{F}}_G^{\vee}; \mathrm{Orb}_G) \to \mathsf{Span}(\mathbb{F}_G; \mathrm{Orb}_G)$$

satisfies the conditions of Theorem 5.1.1. Since this comes from a morphism of adequate triples, we can use the formulation in Observation 5.1.14. We check each of the conditions there in turn:

1. It was checked in Example 3.3.25 that $\mathsf{Span}(\mathbb{F}_G)$ is soundly extendable.
2. We need to show that

   $$(\mathrm{Orb}_G \times_{(\underline{\mathbb{F}}_G^{\vee})} (\underline{\mathbb{F}}_G^{\vee,\mathrm{si}})_{/(U\to V)})^{\mathrm{op}} \to (\mathrm{Orb}_G \times_{\mathbb{F}_G} (\underline{\mathbb{F}}_G)_{/U})^{\mathrm{op}}$$

   is cofinal. But we have already noted in Observation 5.2.9 that it is an equivalence.
3. This holds since the functor induces the identity on Orb_G.
4. For all $U \in \mathbb{F}_G$ the functor

   $$(\underline{\mathbb{F}}_G^{\vee,\mathrm{tdeg}})_{/(U\to V)} \to (\mathbb{F}_G)_{/U}$$

   is an equivalence by inspection of the definition of $(\underline{\mathbb{F}}_G^{\vee})^{\mathrm{tdeg}}$. □

As a consequence we obtain a fully faithful envelope into the ∞-category of G-symmetric monoidal ∞-categories over $\underline{\mathbb{F}}_G$ and a characterization of the image.

Corollary 5.2.15 *There is an adjunction*

$$\mathsf{Env}_G\colon \mathsf{Opd}_{G,\infty} \rightleftarrows \mathsf{Cat}^{\otimes}_{G,\infty} :\!\text{forget},$$

where the left adjoint may be lifted to a fully faithful functor

$$\mathsf{Env}_G\colon \mathsf{Opd}_{G,\infty} \hookrightarrow (\mathsf{Cat}^{\otimes}_{G,\infty})_{/\underline{\mathbb{F}}_G}.$$

This functor has both adjoints and its essential image consists of those G-symmetric monoidal functors $p\colon \mathcal{C} \to \underline{\mathbb{F}}_G$ *that are* $\mathsf{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$*-equifibered.*

Proof Using that $\mathsf{Opd}_{G,\infty} \simeq \mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G))$ by Proposition 5.2.14, this is an instance of Theorem 4.2.6. The envelope of the terminal G-∞-operad is $\mathsf{Env}_{\mathsf{Span}(\mathbb{F}_G)}(*) = \mathcal{A}_{\mathsf{Span}(\mathbb{F}_G)} = \underline{\mathbb{F}}_G$ by Example 5.2.4. □

We elaborate further on the characterization of the image:

Lemma 5.2.16 *A G-symmetric monoidal functor $F:\mathcal{C}\to\mathcal{D}$ is $\mathsf{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$-equifibered if and only if*

$$\begin{tikzcd} \mathcal{C}^H\times\mathcal{C}^H \arrow[r, "\otimes"] \arrow[d] & \mathcal{C}^H \arrow[d] \\ \mathcal{D}^H\times\mathcal{D}^H \arrow[r, "\otimes"] & \mathcal{D}^H \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}^K \arrow[r, "\mathrm{Nm}_K^H"] \arrow[d] & \mathcal{C}^H \arrow[d] \\ \mathcal{D}^K \arrow[r, "\mathrm{Nm}_K^H"] & \mathcal{D}^H \end{tikzcd}$$

are pullback squares of ∞-categories for all subgroups $K\subset H\subset G$.

Proof F induces a natural transformation of functors $\mathbb{F}_G\to\mathsf{Cat}$, defined by restricting to forwards maps in $\mathsf{Span}(\mathbb{F}_G)$. Let $\mathcal{K}\subset\mathbb{F}_G$ denote the maximal subcategory such that the restriction of F to $\mathcal{K}$ is a cartesian natural transformation. Then F is $\mathsf{Ar}_{\mathrm{act}}(\mathsf{Span}(\mathbb{F}_G))$-equifibered if and only if $\mathcal{K}=\mathbb{F}_G$. Note that $\mathcal{K}$ is closed under composition and right-cancellation, since pullback squares are, and contains all equivalences. Moreover, $\mathcal{K}$ is closed under disjoint union since both functors $\mathcal{C},\mathcal{D}\colon\mathbb{F}_G\to\mathsf{Cat}$ send disjoint unions to products. Using this one can see that to show $\mathcal{K}=\mathbb{F}_G$, it suffices to check that $\mathcal{K}$ contains the morphisms

$$\nabla\colon G/H\amalg G/H\to G/H \quad\text{and}\quad G/K\to G/H$$

for all subgroups $K\subset H\subset G$. This is exactly the condition stated in the lemma. □

Remark 5.2.17 One might hope that G-∞-operads are also equivalent to fibrous $\mathbb{F}_{G,*}$-patterns, in analogy with what we showed in Corollary 5.1.15 for $G=\{e\}$, but this is false for nontrivial groups. Note that the orbit functor $(-)_G:\mathbb{F}_{G,*}\to\mathbb{F}_*$ exhibits $\mathbb{F}_{G,*}$ as a fibrous $\mathbb{F}_*$-pattern, ie an ∞-operad in the sense of Lurie. Therefore there is an equivalence $\mathsf{Fbrs}(\mathbb{F}_{G,*})\simeq(\mathsf{Opd}_\infty)_{/\mathbb{F}_{G,*}}$. We refer to this as the ∞-category of *naive G-∞-operads*. There is an inclusion of patterns $\mathbb{F}_{G,*}\to\mathsf{Span}(\mathbb{F}_G)$ similar to the one used in Corollary 5.1.15, and this is a strong Segal morphism by an argument as in Observation 5.2.9. Therefore there is a restriction functor:

$$\mathsf{Opd}_{G,\infty}\simeq\mathsf{Fbrs}(\mathsf{Span}(\mathbb{F}_G))\to\mathsf{Fbrs}(\mathbb{F}_{G,*})\simeq(\mathsf{Opd}_\infty)_{/\mathbb{F}_{G,*}},$$

which forgets from (genuine) G-∞-operads to naive G-∞-operads. However, we cannot apply the comparison Theorem 5.1.1 since $(\mathsf{Orb}_G^{\mathrm{op}})^{\simeq}\simeq\mathbb{F}_{G,*}^{\mathrm{el}}\to\mathsf{Span}(\mathbb{F}_G)^{\mathrm{el}}\simeq\mathsf{Orb}_G^{\mathrm{op}}$ is not an equivalence.

5.3 Upgrading to $(\infty,2)$-categories

In this subsection we will upgrade our main results from ∞-categories to $(\infty,2)$-categories: we will see that the comparison equivalence of Theorem 5.1.1 is an equivalence of $(\infty,2)$-categories and the fully faithful envelope functor of Proposition 4.2.1 is a fully faithful functor of $(\infty,2)$-categories. More precisely, we will show that these functors are compatible with natural Cat_∞-module structures on the ∞-categories involved. It then follows from results of Hinich [21] and Heine [20] that these ∞-categories can be upgraded to (∞,2)-categories and the functors to functors of (∞,2)-categories. We will not comment further on this, however, as our primary interest is in showing that our equivalences are compatible with the natural ∞-categories of maps, which is an immediate consequence of compatibility with the Cat_∞-module structures. We begin by defining such module structures on the ∞-categories and functors we studied in Section 2:

Construction 5.3.1 Let $\mathcal{B}$ be an ∞-category equipped with a wide subcategory $\mathcal{B}_0$. The forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_\infty$ has a right adjoint, taking $\mathcal{C} \in \mathsf{Cat}_\infty$ to the projection $\mathcal{C} \times \mathcal{B} \to \mathcal{B}$; this factors through the subcategory $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and thus gives symmetric monoidal functors

$$\mathsf{Cat}_\infty \to \mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$$

with respect to the cartesian products. It follows that both $\mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ are Cat_∞-modules, with the tensoring in both cases simply given by cartesian product, ie

$$(\mathcal{C}, \mathcal{E} \to \mathcal{B}) \mapsto \mathcal{E} \times \mathcal{C} \to \mathcal{B},$$

and that the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ is a Cat_∞-module functor. Moreover, both Cat_∞-module structures are adjoint to an enrichment in Cat_∞, given, respectively, by $\mathsf{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-,-)$ and $\mathsf{Fun}_{/\mathcal{B}}(-,-)$. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the ∞-categories $\mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ are Cat_∞-modules, with the tensoring given by the cartesian product, and the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ is a Cat_∞-module functor; it is easy to see that this Cat_∞-module structure on $\mathsf{Cat}_{\infty/\mathcal{B}}^{\text{cocart}}$ corresponds under the equivalence with $\mathsf{Fun}(\mathcal{B}, \mathsf{Cat}_\infty)$ to that given by taking products with constant functors.

Proposition 5.3.2 (i) *For any ∞-category $\mathcal{B}$, the tensoring of $\mathsf{Cat}_{\infty/\mathcal{B}}$ over Cat_∞ in Construction 5.3.1 is adjoint to a cotensoring, with the cotensor of $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ given by the pullback*

$$\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}} := \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \times_{\mathsf{Fun}(\mathcal{C},\mathcal{B})} \mathcal{B}$$

along the constant diagram functor $\mathcal{B} \to \mathsf{Fun}(\mathcal{C}, \mathcal{B})$.

(ii) *If $\mathcal{B}_0$ is a wide subcategory of $\mathcal{B}$, then $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ is also cotensored over Cat_∞, with the cotensor of $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ again given by $\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$. In particular, the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ preserves the cotensoring.*

Proof Part (i) follows from the natural equivalences

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{C} \times \mathcal{F}, \mathcal{E}) \simeq \left\{ \begin{tikzcd} \mathcal{C} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{C} \times \mathcal{B} \arrow[r, "\text{proj}"] & \mathcal{B} \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \arrow[d] \\ \mathcal{B} \arrow[r, "\text{const}"] & \mathsf{Fun}(\mathcal{C}, \mathcal{B}) \end{tikzcd} \right\} \simeq \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{B}}}(\mathcal{F}, \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}).$$

To prove (ii), we observe that if $\mathcal{E} \to \mathcal{B}$ is in $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$, then so is $\mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ by [22, Proposition 3.1.2.3], and a morphism $[1] \to \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ is cocartesian if and only if the corresponding map $[1] \times \mathcal{C} \to \mathcal{E}$ has cocartesian components at every $c \in \mathcal{C}$. Thus a morphism $\mathcal{F} \to \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}$ over $\mathcal{B}$ preserves cocartesian morphisms over $\mathcal{B}_0$ if and only if the corresponding map $\mathcal{F} \times \mathcal{C} \to \mathcal{E}$ preserves cocartesian morphisms over $\mathcal{B}_0$, so that the previous equivalence of mapping spaces restricts on subspaces to an equivalence

$$\mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{C} \times \mathcal{F}, \mathcal{E}) \simeq \mathrm{Map}_{\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}}(\mathcal{F}, \mathcal{E}_{/\mathcal{B}}^{\mathcal{C}}). \qquad \square$$

Observation 5.3.3 If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the ∞-categories $\mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ and $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}}$ are similarly cotensored over Cat_∞, with the same cotensors as in Proposition 5.3.2, and the forgetful functor $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}} \to \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}}$ preserves the cotensoring.

Proposition 5.3.4 (i) *Let $\mathcal{B}$ be an ∞-category with a wide subcategory $\mathcal{B}_0$. Then the left adjoint*

$$(-) \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \colon \mathrm{Cat}_{\infty/\mathcal{B}} \to \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$$

of the forgetful functor from Corollary 2.1.5 is a Cat_∞-module functor, with the adjunction being an adjunction of Cat_∞-modules.

(ii) *If $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, then the left adjoint*

$$(-) \times_{\mathcal{B}} \mathrm{Ar}_R(\mathcal{B}) \colon \mathrm{Cat}_{\infty/\mathcal{B}}^{L\text{-cocart}} \to \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathrm{cocart}}$$

of the forgetful functor from Corollary 2.2.5 is a Cat_∞-module functor, with the adjunction being an adjunction of Cat_∞-modules.

Proof The forgetful functor $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathrm{Cat}_{\infty/\mathcal{B}}$ is a Cat_∞-module functor by Construction 5.3.1. By [17, Theorem 3.4.7], the left adjoint then has a canonical oplax Cat_∞-module structure, given for $\mathcal{C} \in \mathrm{Cat}_\infty$ and $\mathcal{E} \to \mathcal{B}$ in $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ by the natural map

$$(\mathcal{C} \times \mathcal{B}) \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}) \to \mathcal{C} \times (\mathcal{B} \times_{\mathcal{B}} \mathrm{Ar}_0(\mathcal{B}));$$

this is clearly an equivalence, so the adjunction of Corollary 2.1.5 lifts to an adjunction of Cat_∞-modules. This proves (i), and the proof of (ii) is the same. $\square$

Remark 5.3.5 The Cat_∞-module structures on $\mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}$ and $\mathrm{Cat}_{\infty/\mathcal{B}}$ are adjoint to enrichments in Cat_∞, given, respectively, by $\mathrm{Fun}_{/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}}(-,-)$ and $\mathrm{Fun}_{/\mathcal{B}}(-,-)$; the equivalence of Proposition 2.1.4 is then precisely that induced by the Cat_∞-module adjunction from Proposition 5.3.4. Similarly, if $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ is an ∞-category equipped with a factorization system, the equivalence of Proposition 2.2.4 is also induced by the Cat_∞-module adjunction above.

Lemma 5.3.6 (i) *For any functor of ∞-categories $f : \mathcal{A} \to \mathcal{B}$ the functor $f^* \colon \mathrm{Cat}_{\infty/\mathcal{B}} \to \mathrm{Cat}_{\infty/\mathcal{A}}$ given by pullback along f is a Cat_∞-module functor and also preserves the cotensoring with Cat_∞.*

(ii) *Suppose $\mathcal{A}$ and $\mathcal{B}$ are ∞-categories equipped with wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, respectively, and that $f : \mathcal{A} \to \mathcal{B}$ is a functor that takes $\mathcal{A}_0$ into $\mathcal{B}_0$. Then the functor $f^* \colon \mathrm{Cat}_{\infty/\mathcal{B}}^{\mathcal{B}_0\text{-cocart}} \to \mathrm{Cat}_{\infty/\mathcal{A}}^{\mathcal{A}_0\text{-cocart}}$ given by pullback along f is a Cat_∞-module functor and also preserves the cotensoring with Cat_∞.*

Proof We prove (i); the proof of (ii) is the same. The functor f^* fits in a commutative triangle

$$\begin{tikzcd}
 & \mathsf{Cat}_\infty \arrow[dl] \arrow[dr] & \\
\mathsf{Cat}_{\infty/\mathcal{B}} \arrow[rr, "f^*"] & & \mathsf{Cat}_{\infty/\mathcal{A}},
\end{tikzcd}$$

where all three functors preserve finite products, and so are symmetric monoidal with respect to the cartesian products. Hence $f^*\colon \mathsf{Cat}_{\infty/\mathcal{B}} \to \mathsf{Cat}_{\infty/\mathcal{A}}$ is a Cat_∞-module functor. To see that f^* also preserves the cotensoring, observe that for $\mathcal{E} \to \mathcal{B}$ in $\mathsf{Cat}^{\mathcal{B}_0\text{-cocart}}_{\infty/\mathcal{B}}$ or $\mathsf{Cat}_{\infty/\mathcal{B}}$ and $\mathcal{C} \in \mathsf{Cat}_\infty$ we have a natural commutative cube

$$\begin{tikzcd}
(f^*\mathcal{E})^{\mathcal{C}}_{/\mathcal{A}} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Fun}(\mathcal{C}, f^*\mathcal{E}) \arrow[dr] \arrow[dd] & \\
 & \mathcal{E}^{\mathcal{C}}_{/\mathcal{B}} \arrow[rr] \arrow[dd] & & \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \arrow[dd] \\
\mathcal{A} \arrow[rr] \arrow[dr] & & \mathsf{Fun}(\mathcal{C}, \mathcal{A}) \arrow[dr] & \\
 & \mathcal{B} \arrow[rr] & & \mathsf{Fun}(\mathcal{C}, \mathcal{B}),
\end{tikzcd}$$

where the front, back and right faces are cartesian. The left vertical square is therefore also cartesian, giving an equivalence

$$(f^*\mathcal{E})^{\mathcal{C}}_{/\mathcal{A}} \xrightarrow{\sim} f^*(\mathcal{E}^{\mathcal{C}}_{/\mathcal{B}}). \qquad \square$$

Observation 5.3.7 For $f\colon \mathcal{A} \to \mathcal{B}$ a functor that preserves wide subcategories $\mathcal{A}_0$ and $\mathcal{B}_0$, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Cat}_\infty \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \mathsf{Cat}^{\mathcal{B}_0\text{-cocart}}_{\infty/\mathcal{B}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}^{\mathcal{A}_0\text{-cocart}}_{\infty/\mathcal{A}} \arrow[d] \\
 & \mathsf{Cat}_{\infty/\mathcal{B}} \arrow[r, "f^*"] & \mathsf{Cat}_{\infty/\mathcal{A}}
\end{tikzcd}$$

of symmetric monoidal functors (with the cartesian monoidal structures). It follows that the commutative square on the bottom right (as in Observation 2.1.7) is a square of Cat_∞-modules. Similarly, if f is compatible with factorization systems $(\mathcal{A}, \mathcal{A}_L, \mathcal{A}_R)$ and $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$, then the commutative square

$$\begin{tikzcd}
\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{B}} \arrow[r, "f^*"] \arrow[d] & \mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{A}} \arrow[d] \\
\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \arrow[r, "f^*"] & \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{A}}
\end{tikzcd}$$

is a square of Cat_∞-modules. It follows that for both squares the Beck–Chevalley map is a natural transformation of Cat_∞-modules.

Proposition 5.3.8 *Let $(\mathcal{B}, \mathcal{B}_L, \mathcal{B}_R)$ be a factorization system. Then there is a natural Cat_∞-module structure on the ∞-category $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$, with the tensoring given by cartesian products, and the adjunction*

$$\mathrm{E} \colon \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \rightleftarrows (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} : \mathrm{Q}$$

is compatible with the Cat_∞-module structures. Moreover, $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ is also cotensored over Cat_∞, with the cotensor of $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \to \mathsf{Ar}_R(\mathcal{B})$ in $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ being $\mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$.

Proof The forgetful functor $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \to \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}}$ has a right adjoint, which takes a cocartesian fibration $\mathcal{E} \to \mathcal{B}$ to the projection $\mathcal{E} \times_{\mathcal{B}} \mathsf{Ar}_R(\mathcal{B}) \to \mathsf{Ar}_R(\mathcal{B})$. We thus have a commutative diagram

$$\begin{tikzcd}
 & \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}} \arrow[dr, "\text{forget}"] \arrow[dd, "(-)\times_{\mathcal{B}}\mathsf{Ar}_R(\mathcal{B})"] & \\
\mathsf{Cat}_\infty \arrow[ur, "(-)\times\mathcal{B}"] \arrow[dr, swap, "(-)\times\mathsf{Ar}_R(\mathcal{B})"] & & \mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}} \\
 & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \arrow[ur, "\mathrm{Q}"] &
\end{tikzcd}$$

of right adjoints, which are then symmetric monoidal functors with respect to cartesian products. This in particular shows that $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ is a Cat_∞-module, with the tensoring given by taking cartesian products, and the functor Q is compatible with the Cat_∞-module structures. As in Construction 5.3.1, it follows that the left adjoint E is an oplax Cat_∞-module functor, and that the oplax structure maps are equivalences; thus we have a Cat_∞-module adjunction.

To identify the cotensor, we first observe that $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ can be described as a subcategory of $\mathsf{Cat}_{\infty/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$; the Cat_∞-module structures on both are clearly compatible, and the latter has a cotensoring given by $(\mathcal{C}, \mathcal{E}) \mapsto \mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ by Proposition 5.3.2. It thus suffices to show that $\mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ is an object of $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$, ie that the composite to $\mathcal{B}$ is a cocartesian fibration, that the morphism to $\mathsf{Ar}_R(\mathcal{B})$ preserves cocartesian morphisms over $\mathcal{B}$, and that a morphism $\mathcal{F} \to \mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$ preserves cocartesian morphisms over $\mathcal{B}$ if and only if the adjoint map $\mathcal{F} \times \mathcal{C} \to \mathcal{E}$ does so. To see this, consider the commutative cube

$$\begin{tikzcd}
\mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{E}^{\mathcal{C}} \arrow[dr] \arrow[dd] & \\
 & \mathsf{Ar}_R(\mathcal{B}) \arrow[rr] \arrow[dd] & & \mathsf{Ar}_R(\mathcal{B})^{\mathcal{C}} \arrow[dd] \\
\mathcal{B} \arrow[rr] \arrow[dr, equal] & & \mathcal{B}^{\mathcal{C}} \arrow[dr, equal] & \\
 & \mathcal{B} \arrow[rr] & & \mathcal{B}^{\mathcal{C}}.
\end{tikzcd} \tag{11}$$

Here the top and bottom squares are cartesian, the vertical maps are cocartesian fibrations, and both maps to $\mathsf{Ar}_R(\mathcal{B})^{\mathcal{C}}$ preserve cocartesian morphisms. It follows that $\mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})} \to \mathcal{B}$ is a cocartesian fibration, and a morphism here is cocartesian if and only if its images in $\mathsf{Ar}_R(\mathcal{B})$ and $\mathcal{E}^{\mathcal{C}}$ are both cocartesian. Combining this with the description of cocartesian morphisms in $\mathcal{E}^{\mathcal{C}}$ from [22, Proposition 3.1.2.1] gives the required description of cocartesian morphisms in $\mathcal{E}^{\mathcal{C}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$. □

Observation 5.3.9 Let us write $\mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}(-,-)$ for the enrichment adjoint to the Cat_∞-module structure on $(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}$; this satisfies

$$\mathsf{Map}_{\mathsf{Cat}_\infty}(\mathcal{C}, \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}(-,-)) \simeq \mathsf{Map}_{(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{B}})_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}}(\mathcal{C}\times -,-);$$

identifying the right-hand side as a fiber product we see that for $\alpha\colon \mathcal{E}\to \mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B}), \beta\colon \mathcal{F}\to \mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B})$ we have a natural cartesian square

$$\begin{tikzcd}
\mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}((\mathcal{E},\alpha),(\mathcal{F},\beta)) \arrow[r] \arrow[d] & \mathsf{Fun}^{\mathrm{cocart}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) \arrow[d] \\
\{\alpha\} \arrow[r] & \mathsf{Fun}^{\mathrm{cocart}}_{/\mathcal{B}}(\mathcal{E},\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B})).
\end{tikzcd}$$

Since the functor E is fully faithful and compatible with the Cat_∞-module structures we conclude that it gives a natural equivalence

$$\mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{B}}(-,-) \xrightarrow{\sim} \mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}(\mathrm{E}(-),\mathrm{E}(-)).$$

Observation 5.3.10 Suppose $f:\mathcal{A}\to\mathcal{B}$ is a functor compatible with specified factorization systems. Passing to vertical left adjoints in the commutative square of Observation 2.3.9 yields a Beck–Chevalley transformation

$$\mathrm{E}_{\mathcal{A}}f^* \to f^{\circledast}\mathrm{E}_{\mathcal{B}}.$$

Unwinding the definitions, this is given at $\mathcal{E}\to\mathcal{B}$ in $\mathsf{Cat}^{L\text{-cocart}}_{\infty/\mathcal{B}}$ by the natural map

$$(\mathcal{E}\times_{\mathcal{B}}\mathcal{A})\times_{\mathcal{A}}\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{A}) \to (\mathcal{E}\times_{\mathcal{B}}\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B}))\times_{\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B})}\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{A}),$$

which is an equivalence. The functors and transformations here are also compatible with the Cat_∞-module structures, by the same argument as in Observation 5.3.7 , so for $\mathcal{E},\mathcal{F}\to\mathsf{Ar}_{\boldsymbol{R}}(\mathcal{B})$ we have a natural commutative square in which the vertical maps are equivalences:

$$\begin{tikzcd}
\mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) \arrow[r] \arrow[d, "\sim"] & \mathsf{Fun}^{L\text{-cocart}}_{/\mathcal{A}}(f^*\mathcal{E},f^*\mathcal{E}) \arrow[d, "\sim"] \\
\mathsf{Fun}^{\mathcal{B}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{B})}(\mathrm{E}_{\mathcal{B}}\mathcal{E},\mathrm{E}_{\mathcal{B}}\mathcal{F}) \arrow[r] & \mathsf{Fun}^{\mathcal{A}\text{-cocart}}_{/\mathsf{Ar}_{\mathrm{R}}(\mathcal{A})}(\mathrm{E}_{\mathcal{A}}f^*\mathcal{E},\mathrm{E}_{\mathcal{A}}f^*\mathcal{F}).
\end{tikzcd} \tag{12}$$

After these preliminaries we are finally ready to consider fibrous patterns and their envelopes. First, we want to show that the ∞-categories $\mathsf{Fbrs}(\mathcal{O})$ and $\mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ have Cat_∞-module structures inherited from those we have already considered. This is slightly complicated by the fact that $\mathsf{Fbrs}(\mathcal{O})$ may not be closed under tensors in $\mathsf{Cat}^{\mathrm{int}\text{-cocart}}_{\infty/\mathcal{O}}$, and similarly for the relative Segal objects. (For example, for $\mathcal{O}\in\mathsf{Fbrs}(\mathbb{F}_*)$ and $\mathcal{C}\in\mathsf{Cat}_\infty$, the ∞-category $\mathcal{C}\times\mathcal{O}$ is not an object of $\mathsf{Fbrs}(\mathbb{F}_*)$ since its fiber over $\langle 0\rangle$ is $\mathcal{C}$, not $*$; on the other hand, $\mathsf{Fbrs}(\mathbb{F}_*^{\natural})$ *is* closed under tensoring with Cat_∞.) Luckily, cotensors are better behaved:

Proposition 5.3.11 *Let $\mathcal{O}$ be an algebraic pattern.*

(i) *For $\mathcal{P} \in \mathsf{Fbrs}(\mathcal{O})$ and $\mathcal{C} \in \mathsf{Cat}$, the cotensor $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ in $\mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}}$ is again fibrous.*

(ii) *For $X \in \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ corresponding to $\mathcal{X} \in (\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{O}})_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$ and $\mathcal{C} \in \mathsf{Cat}$, the cotensor $\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$ in $(\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{O}})_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$ again straightens to a relative Segal object.*

Proof To prove (i), first observe that we can identify $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/O}$ as the fiber product

$$\mathsf{Fun}(\mathcal{C}, \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/O}) \times_{\mathsf{Fun}(\mathcal{C}, \mathcal{O}^{\text{act}}_{/O})} \mathcal{O}^{\text{act}}_{/O},$$

so that we have a commutative cube

$$\begin{tikzcd}
\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/O} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Fun}(\mathcal{C}, \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/O}) \arrow[dr] \arrow[dd] & \\
& \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/E} \arrow[rr, crossing over] & & \mathsf{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\text{act}}_{/E}) \arrow[dd] \\
\mathcal{O}^{\text{act}}_{/O} \arrow[rr] \arrow[dr] & & \mathsf{Fun}(\mathcal{C}, \mathcal{O}^{\text{act}}_{/O}) \arrow[dr] & \\
& \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{O}^{\text{act}}_{/E} \arrow[rr] \arrow[from=uu, crossing over] & & \mathsf{Fun}(\mathcal{C}, \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathcal{O}^{\text{act}}_{/E}),
\end{tikzcd}$$

where the front and back faces are cartesian. Here the right vertical face is also cartesian since $\mathcal{P}$ is $\mathcal{O}$-fibrous. It then follows that the left vertical face is also cartesian, ie $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ is also $\mathcal{O}$-fibrous.

For (ii), we extract the following commutative diagram from the cube (11) that describes $\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$:

$$\begin{tikzcd}
(\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})})_O \arrow[rr] \arrow[dr] \arrow[dd] & & (\mathcal{X}^{\mathcal{C}})_O \arrow[dr] \arrow[dd] & \\
& \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})})_E \arrow[rr, crossing over] & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathcal{X}^{\mathcal{C}})_E \arrow[dd] \\
\mathsf{Ar}_{\text{act}}(\mathcal{O})_O \arrow[rr] \arrow[dr] & & (\mathsf{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_O \arrow[dr] & \\
& \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} \mathsf{Ar}_{\text{act}}(\mathcal{O})_E \arrow[rr] \arrow[from=uu, crossing over] & & \lim_{E \in \mathcal{O}^{\text{el}}_{O/}} (\mathsf{Ar}_{\text{act}}(\mathcal{O})^{\mathcal{C}})_E .
\end{tikzcd}$$

(Here we have also used O for the constant functor $\mathcal{C} \to \mathcal{O}$ with this value.) The front and back vertical faces in this cube are cartesian by the definition of $\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$, while the right vertical face is cartesian since $\mathcal{X}$ by assumption straightens to a relative Segal object (and we can identify $\mathcal{X}^{\mathcal{C}}_O$ as $\mathsf{Fun}(\mathcal{C}, \mathcal{X}_O)$ etc). Hence the left vertical face is also cartesian, and this is precisely the relative Segal condition for $\mathcal{X}^{\mathcal{C}}_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})}$. □

Corollary 5.3.12 *Let $\mathcal{O}$ be an algebraic pattern.*

(i) *The localization $L_{\text{fbrs}} \colon \mathsf{Cat}^{\text{int-cocart}}_{\infty/\mathcal{O}} \to \mathsf{Fbrs}(\mathcal{O})$ is a localization of Cat_∞-modules.*

(ii) *The localization $L_{\text{rseg}} \colon (\mathsf{Cat}^{\text{cocart}}_{\infty/\mathcal{O}})_{/\mathsf{Ar}_{\text{act}}(\mathcal{O})} \to \mathsf{Seg}^{/\mathcal{A}_{\mathcal{O}}}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ is a localization of Cat_∞-modules.*

Proof We prove the first claim; the proof of the second is the same — in particular, both follow from [23, Proposition 2.2.1.9]. In order to apply this to L_{fbrs}, we must verify the required hypothesis, which amounts to checking that for $\mathcal{C} \in \mathsf{Cat}_\infty$ and $\mathcal{E} \in \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$, the canonical map $\mathcal{C} \times \mathcal{E} \to \mathcal{C} \times L_{\mathrm{fbrs}}(\mathcal{E})$ is taken to an equivalence by L_{fbrs}. Equivalently, we must show that for $\mathcal{P} \in \mathsf{Fbrs}(\mathcal{O})$, the induced map

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{C} \times L_{\mathrm{fbrs}}(\mathcal{E}), \mathcal{P}) \to \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{C} \times \mathcal{E}, \mathcal{P})$$

is an equivalence. Using the cotensoring, this is the same as the map

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(L_{\mathrm{fbrs}}(\mathcal{E}), \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}) \to \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}}(\mathcal{E}, \mathcal{P}^{\mathcal{C}}_{/\mathcal{O}})$$

given by composition with the localization map $\mathcal{E} \to L_{\mathrm{fbrs}}(\mathcal{E})$. This map is indeed an equivalence, since $\mathcal{P}^{\mathcal{C}}_{/\mathcal{O}}$ is fibrous by Proposition 5.3.11. □

Corollary 5.3.13 *Let $\mathcal{O}$ be a sound pattern. Then we have a commutative square*

$$\begin{tikzcd}
\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \arrow[r, "L_{\mathrm{fbrs}}"] \arrow[d, "\mathrm{E}"] & \mathsf{Fbrs}(\mathcal{O}) \arrow[d, "\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}"] \\
(\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\mathsf{Ar}_{\mathrm{act}}(\mathcal{O})} \arrow[r, "L_{\mathrm{rseg}}"] & \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)
\end{tikzcd}$$

of Cat_∞-module functors. Moreover, the adjunction

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} : \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty) : \mathsf{Un}_{\mathcal{O}}^{\mathrm{int}}$$

of Proposition 4.2.5 is an adjunction of Cat_∞-modules, with the right adjoint being a **lax** *Cat_∞-module functor.*

Proof Let us use the universal property of $\mathsf{Fbrs}(\mathcal{O})$ as a Cat_∞-module localization to verify that the composite

$$\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \xrightarrow{\mathrm{E}} (\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{cocart}})_{/\mathsf{Ar}_{\mathrm{act}}(\mathcal{O})} \xrightarrow{L_{\mathrm{rseg}}} \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$$

factors through L_{fbrs}, as a functor of Cat_∞-modules. Thus we need to verify that if a morphism $\mathcal{E} \to \mathcal{F}$ in $\mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}}$ is taken to an equivalence by L_{fbrs}, then $\mathrm{E}\mathcal{E} \to \mathrm{E}\mathcal{F}$ is taken to an equivalence by L_{rseg}. The latter condition is equivalent to the induced morphism

$$\mathsf{Map}(\mathrm{E}\mathcal{F}, \mathcal{X}) \to \mathsf{Map}(\mathrm{E}\mathcal{E}, \mathcal{X})$$

being an equivalence provided $\mathcal{X}$ is the unstraightening of an object in $\mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$. By adjunction this holds if and only if the map

$$\mathsf{Map}(\mathcal{F}, \mathrm{Q}\mathcal{X}) \to \mathsf{Map}(\mathcal{E}, \mathrm{Q}\mathcal{X})$$

is an equivalence for all such $\mathcal{X}$, but since $\mathcal{O}$ is sound the object $\mathrm{Q}\mathcal{X}$ is fibrous, and hence this is indeed an equivalence as by assumption $\mathcal{E} \to \mathcal{F}$ is taken to an equivalence by L_{fbrs}. It follows that the right adjoint inherits a lax Cat_∞-module structure. □

Remark 5.3.14 For any pattern $\mathcal{O}$ the Segal envelope

$$\mathsf{Env}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}} \colon \mathsf{Fbrs}(\mathcal{O}) \to \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$$

is a *lax* Cat_∞-module functor, since it can be defined by restricting $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}}$ to these full subcategories, the inclusions of which are lax Cat_∞-module functors. This suffices to upgrade the envelope to a functor of $(\infty,2)$-categories, and we can see that it is fully faithful since it is obtained by restricting the functor $\mathrm{St}_{\mathcal{O}}^{\mathrm{int}} \colon \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \to \mathsf{Fun}(\mathcal{O}, \mathsf{Cat}_\infty)_{/\mathcal{A}_{\mathcal{O}}}$, which is a fully faithful functor of $(\infty,2)$-categories by Observation 5.3.9.

Proposition 5.3.15 *Let $\mathcal{O}$ and $\mathcal{P}$ be algebraic patterns and $f \colon \mathcal{O} \to \mathcal{P}$ a strong Segal morphism.*

(i) *The functor $f^* \colon \mathsf{Fbrs}(\mathcal{P}) \to \mathsf{Fbrs}(\mathcal{O})$ is a lax Cat_∞-module functor and its left adjoint $f_!$ is a Cat_∞-module functor.*

(ii) *The functor $f^{\circledast} \colon \mathsf{Seg}_{\mathcal{P}}^{/\mathcal{A}_{\mathcal{P}}}(\mathsf{Cat}_\infty) \to \mathsf{Seg}_{\mathcal{O}}^{/\mathcal{A}_{\mathcal{O}}}(\mathsf{Cat}_\infty)$ is a lax Cat_∞-module functor and its left adjoint $f_!$ is a Cat_∞-module functor.*

Proof To prove (i), we observe that f^* is obtained by restricting $f^* \colon \mathsf{Cat}_{\infty/\mathcal{O}}^{\text{int-cocart}} \to \mathsf{Cat}_{\infty/\mathcal{P}}^{\text{int-cocart}}$, which is a Cat_∞-module functor by Observation 5.3.10, to full subcategories; it is therefore a lax Cat_∞-module functor. The left adjoint $f_!$ is then automatically an oplax Cat_∞-module functor, and the oplax structure map is an equivalence if and only if the right adjoint f^* preserves Cat_∞-cotensors, which we know from Lemma 5.3.6 and Proposition 5.3.11. The proof of (ii) is the same. □

Remark 5.3.16 It follows that for $\mathcal{Q} \in \mathsf{Fbrs}(\mathcal{O})$ and $\mathcal{R} \in \mathsf{Fbrs}(\mathcal{P})$ we have a natural equivalence

$$\mathsf{Fun}_{/\mathcal{P}}^{\text{int-cocart}}(f_!\mathcal{Q}, \mathcal{R}) \simeq \mathsf{Fun}_{/\mathcal{O}}^{\text{int-cocart}}(\mathcal{Q}, f^*\mathcal{R}).$$

Corollary 5.3.17 *Let $f \colon \mathcal{O} \to \mathcal{P}$ be a strong Segal morphism between soundly extendable patterns that satisfies the hypotheses of Theorem 5.1.1. Then pullback along f gives an equivalence*

$$f^* \colon \mathsf{Fbrs}(\mathcal{P}) \xrightarrow{\sim} \mathsf{Fbrs}(\mathcal{O})$$

of Cat_∞-modules. In particular, for any $\mathcal{Q}, \mathcal{Q}'$ in $\mathsf{Fbrs}(\mathcal{P})$, the induced functor

$$\mathsf{Fun}_{/\mathcal{P}}^{\text{int-cocart}}(\mathcal{Q}, \mathcal{Q}') \to \mathsf{Fun}_{/\mathcal{O}}^{\text{int-cocart}}(f^*\mathcal{Q}, f^*\mathcal{Q}')$$

is an equivalence. □

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