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On the topology of character varieties of once-punctured torus bundles

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This paper presents, for the special case of once-punctured torus bundles, a natural method to study the character varieties of hyperbolic 3-manifolds that are bundles over the circle. The main strategy is to restrict characters to the fibre of the bundle, and to analyse the resulting branched covering map. This allows us to extend results of Steven Boyer, Erhard Luft and Xingru Zhang. Both $\mathrm{SL}(2,\mathbb{C})$-character varieties and $\mathrm{PSL}(2,\mathbb{C})$-character varieties are considered. As an explicit application of these methods, we build on work of Baker and Petersen to show that there is an infinite family of hyperbolic once-punctured bundles with canonical curves of $\mathrm{PSL}(2,\mathbb{C})$-characters of unbounded genus. A version of Thurston's dimension bound for character varieties is stated and proven for $\mathrm{PSL}(2,\mathbb{C})$-character varieties.

57K31, 57K32, 57M05; 14D20

1 Introduction

The first part of this paper extends results of Boyer, Luft and Zhang [4] concerning the $\mathrm{SL}(2,\mathbb{C})$-character variety $X(M_\varphi)$ of M_φ, a hyperbolic once-punctured torus bundle over S^1 with monodromy φ, and we determine its relationship with the $\mathrm{PSL}(2,\mathbb{C})$-character variety $\overline{X}(M_\varphi)$. Denote a once-punctured torus fibre by S and let λ be the associated longitude, ie the boundary of a compact core of S. The monodromy induces a polynomial automorphism $\overline{\varphi}\colon X(S)\to X(S)$ and we denote its *fixed point set* by $X_\varphi(S)$. This is the image of the *restriction map* $r\colon X(M_\varphi)\to X(S)$ (Lemma 11). We are interested in the topology of the Zariski components[1] of $X(M_\varphi)$ that contain the characters of irreducible representations. Each such component is one-dimensional (Proposition 14), and we let $X^{\mathrm{irr}}(M_\varphi)$ denote their union.

Boyer, Luft and Zhang [4] show that the binary dihedral characters in $X(M_\varphi(\lambda))\subset X(M_\varphi)$ are all simple points of $X(M_\varphi)$, and they provide an exact count of them. Namely their number is

$$\tfrac{1}{2}|2+\mathrm{tr}(\varphi_*)|-2^{b_1(\varphi)-2}, \tag{1-1}$$

where $b_1(\varphi)=b_1(M_\varphi;\mathbb{Z}_2)$ with $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$ and φ_* represents the automorphism induced by φ on first homology of S. We show that these are the only irreducible characters of M_φ that restrict to reducible characters of S (Scholion 12).

Let $\varepsilon\colon\pi_1(M_\varphi)\to\mathbb{Z}_2$ be the nontrivial homomorphism that is trivial on $\pi_1(S)$.

[0]Our *varieties* are affine algebraic sets, and to avoid using the word *irreducible* in two different ways, we refer to irreducible components of these sets as Zariski components.

Theorem 1 *Suppose M_φ is a hyperbolic once-punctured torus bundle. Then each Zariski component of $X^{\mathrm{irr}}(M_\varphi)$ is one-dimensional and $r\colon X^{\mathrm{irr}}(M_\varphi) \to X_\varphi(S)$ is a two-fold branched covering map. The ramification points of r are simple points, namely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$. The fibres of r correspond to the orbits of the action of $\langle \varepsilon \rangle \le H^1(M_\varphi, \mathbb{Z}_2)$. In particular,*

$$\chi(X^{\mathrm{irr}}(M_\varphi)) = 2\chi(X_\varphi(S)) - \tfrac{1}{2}|2 + \mathrm{tr}(\varphi_*)| + 2^{b_1(\varphi)-2},$$

where χ denotes the Euler characteristic in the above equation.

Implicit in the above statement is that the restriction map satisfies $r(X(M_\varphi)) = r(X^{\mathrm{irr}}(M_\varphi))$. This is shown as follows. Let C be a Zariski component containing only reducible representations. Then C is birationally equivalent to $\mathbb{C}$ and the restriction of r to C is constant. We apply a criterion due to Heusener and Porti [20] to show that $C \cap X^{\mathrm{irr}}(M_\varphi) \neq \varnothing$, and that each point of intersection is contained on a unique curve in $X^{\mathrm{irr}}(M_\varphi)$. Moreover, the intersection point is a smooth point of both curves and the intersection is transverse (Proposition 17).

In our applications, we either use genus bounds that follow from the above theorem (Corollary 20) or well-known results linking genus to the Newton polygon (see Section 5.2). The proof of Theorem 1 is organised as follows. We first show that no ramification point in $X^{\mathrm{irr}}(M_\varphi)$ is the character of a reducible representation (Section 3.4), and then characterise the fibres as orbits of ε and complete the proof with Proposition 19.

The natural map $X^{\mathrm{irr}}(M_\varphi) \to \overline{X}^{\mathrm{irr}}(M_\varphi)$ can be viewed as the quotient map

$$X^{\mathrm{irr}}(M_\varphi) \to X^{\mathrm{irr}}(M_\varphi)/H^1(M_\varphi, \mathbb{Z}_2) \subseteq \overline{X}^{\mathrm{irr}}(M_\varphi).$$

There is a natural splitting $H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H$, where H is the subgroup generated by all homomorphisms that are only nontrivial on $\pi_1(S)$. We may view $H \le H^1(S, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Theorem 1 and the analogous result for projective representations (Proposition 26) imply the following:

Corollary 2 *Suppose M_φ is a hyperbolic once-punctured torus bundle. We have the following commutative diagram of maps whose degrees are indicated in the diagram:*

$$\begin{array}{ccccccc}
X^{\mathrm{irr}}(M_\varphi) & \xrightarrow[2:1]{r} & X_\varphi(S) & \subset & X(S) & \cong & \mathbb{C}^3 \\
{\scriptstyle q_1}\Big\downarrow{\scriptstyle 2^{b_1(\varphi)}:1} & & {\scriptstyle q_2}\Big\downarrow{\scriptstyle 2^{b_1(\varphi)-1}:1} & & {\scriptstyle q_2}\Big\downarrow{\scriptstyle 4:1} & & \\
\overline{X}^{\mathrm{irr}}(M_\varphi) & \xrightarrow{\bar{r}} & \overline{X}_\varphi(S) & \subset & \overline{X}(S) & &
\end{array}$$

Here, each of the maps r, q_1, q_2 is a branched covering map. The map q_1 is the quotient map associated with the action of $H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H$, and the map q_2 is the quotient map associated with the action of $H^1(S, \mathbb{Z}_2)$ on $X(S)$. The map r is the quotient map associated with the action of $\langle \varepsilon \rangle$. The orbits of the action of $H^1(S, \mathbb{Z}_2)$ on $X_\varphi(S)$ are the same as the orbits of H except when $b_1(\varphi) = 2$ and one of the coordinate axes (with respect to the natural identification $X(S) = \mathbb{C}^3$) is contained in $X_\varphi(S)$.

The fibres of the map $\bar{r}$ may have size one, two or four, arising from irreducible representations with nontrivial centraliser. The restriction $\bar{r}\colon \overline{X} \to \overline{X}_\varphi(S)$ to a Zariski component $\overline{X} \subseteq \overline{X}^{\mathrm{irr}}(M_\varphi)$ is of degree one, except for a special situation when $b_1(\varphi) = 2$ and the degree is two (see Proposition 26).

Corollary 3 *If $b_1(\varphi) = 1$, then we have a birational isomorphism $\overline{X}^{\mathrm{irr}}(M_\varphi) \cong X_\varphi(S)$.*

Proof Since $H_1(M;\mathbb{Z}_2) = \mathbb{Z}_2$, we have $H^2(\pi_1(M);\mathbb{Z}_2) \cong 0$ and hence every representation into $\mathrm{PSL}(2,\mathbb{C})$ lifts to a representation into $\mathrm{SL}(2,\mathbb{C})$. See Section 2.1 for details. Hence we obtain the claimed birational isomorphism from the commutative diagram in Corollary 2 since the fibres of each of the surjective maps q_1 and r are the orbits of the action of $\langle \varepsilon \rangle$. □

The hypothesis of Corollary 3 is equivalent to φ_* mapping to an element of order three under the natural epimorphism $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}_2) \cong \mathrm{Sym}(3)$ (see Section 2.2).

Section 3 also contains a number of interesting facts about $X(M_\varphi)$ and $X_\varphi(S)$ that follow from our set-up. For instance, M_φ always admits an irreducible representation whose restriction to the fibre has image the quaternionic group. If $H_1(X,\mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then this character always lies on one (or three) special line(s) in $X_\varphi(S)$. Section 4 provides analogous results for the $\mathrm{PSL}(2,\mathbb{C})$-character variety $\overline{X}(M_\varphi)$ and the associated fixed point set $\overline{X}_\varphi(S) \subset \overline{X}(S)$. We use a dimension bound due to Thurston, for which we provide a proof (Proposition 8) and which is not implied by the dimension bounds in the setting of more general Lie groups due to Falbel and Guilloux [11] and Porti [24].

As pointed out by a referee, the combination of the restriction map to the fibre and the factorisation of the monodromy in terms of Dehn twists has previously been used by Porti [23, Section 4.5] to describe a method to compute the Reidemeister torsion of once-punctured torus bundles. In Section 5, we apply our refined analysis of the restriction map to compute the character varieties of infinite families of once-punctured torus bundles and a number of small examples. We follow the standard convention letting $\mathtt{A}, \mathtt{B} \in \mathrm{SL}(2,\mathbb{Z})$ correspond to right-handed Dehn twists (see Section 2.2).

First family Baker and Petersen [2] determined the genera of the character varieties of the family M_n of once-punctured torus bundles with monodromy $\varphi_n = \mathtt{AB}^{n+2}$. We note that $b_1(\varphi_n) = 1$ if n is odd and $b_1(\varphi_n) = 2$ if n is even. In the case where $n \geq 3$ is odd, we show that $X_{\varphi_n}(S)$ is a curve of genus zero, and hence Corollary 3 implies that the genus of $\overline{X}^{\mathrm{irr}}(M_n)$ is zero. This agrees with the computation in [2]. See Section 5.3.

Second family Let N_n be the once-punctured torus bundle with monodromy $\psi_n = \mathtt{AB}^{n+2}A$. We note that $b_1(\psi_n) = 2$ if n is odd and $b_1(\psi_n) = 3$ if n is even. In the case where $n = 2k+1 \geq 1$ is odd, we show that $X_{\psi_n}(S)$ is a curve of genus k and the genus of $\overline{X}^{\mathrm{irr}}(N_n)$ is zero. See Section 5.4.

Third family Let L_n be the once-punctured torus bundle with monodromy $\omega_n = \mathtt{A}^2\mathtt{B}^{n+2}\mathtt{A}$. We note that $b_1(\omega_n) = 1$ if n is odd and $b_1(\omega_n) = 2$ if n is even. In the case where $n = 2k+1 \geq 3$ is odd, we show that $X_{\omega_n}(S) \cong \overline{X}^{\text{irr}}(L_n)$ has $\frac{1}{2}(n+3)$ components, one of which is a curve of genus k and the others have genus 0. The curves of genus k are all canonical. This answers a question posed by Alan Reid of whether the canonical curves in the PSL$(2,\mathbb{C})$-character variety of once-punctured torus bundles always have genus zero. See Section 5.5.

Small examples We summarise computational results for monodromies that are short words in $\mathtt{A}$ and $\mathtt{B}$, or in $\mathtt{A}$ and $\mathtt{B}^{-1}$, in Section 5.6. These were executed with `Singular` [10].

We remark that in all examples produced in this paper, each fixed-point set $X_\varphi(S)$ has the property that there is at most one Zariski component of positive genus and each Zariski component is birationally equivalent to an elliptic or a hyperelliptic curve.

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2 Preliminaries

This section collects some basic facts about character varieties of 3-manifolds in Section 2.1 and about hyperbolic once-punctured torus bundles in Section 2.2. We also recall facts about mutation in Section 2.4 and the character variety of the free group of rank two in Section 2.5. None of the material in this section is new, and it mainly serves to set up notation. In Section 2.6 we also include a statement and proof of a dimension bound based on [25, Theorem 5.6] in Thurston's notes. For varieties of representations into SL$(2,\mathbb{C})$ this was given by Culler and Shalen [9, Proposition 3.2.1], but we could not find the statement of Proposition 8 for the case of representations into PSL$(2,\mathbb{C})$ in the literature.

2.1 Basic facts about character varieties

We start by summarising the discussion found in [22, pages 512–513]. Let M be a compact, connected and orientable 3-manifold. Let $R(M)$ be the variety of representations of $\pi_1(M)$ into SL$(2,\mathbb{C})$, and $X(M)$ be the associated character variety. Similarly, let $\overline{R}(M)$ and $\overline{X}(M)$ be the respective varieties of representations into PSL$(2,\mathbb{C})$ and their characters.

The quotient map $q\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ induces a regular map $q_\star\colon R(M) \to \overline{R}(M)$, and its image is a union of topological components of $\overline{R}(M)$. Indeed, $R(M) \to q_\star(R(M))$ is a regular covering map with group $H^1(\pi_1(M);\mathbb{Z}_2)$. Following [22], we let

$$Q(M) = R(M)/H^1(\pi_1(M);\mathbb{Z}_2) \cong q_\star(R(M)) \subseteq \overline{R}(M).$$

There also is a regular map $q_\star\colon X(M) \to \overline{X}(M)$, and its image is a union of topological components of $\overline{X}(M)$. The fibres over points in the image are again the orbits of the natural action of $H^1(\pi_1(M);\mathbb{Z}_2)$ on $X(M)$. Let $QX(M) = X(M)/H^1(\pi_1(M);\mathbb{Z}_2) \cong q_\star(X(M)) \subseteq \overline{X}(M)$. However, on the level of character varieties, the action may not be free. The points of $X(M)$ fixed by $h\colon \pi_1(M) \to \mathbb{Z}_2$ are characterised by $\chi(\gamma) = 0$ for all $\gamma \in \pi_1(M)$ with the property that $h(\gamma) = -1$. This defines an algebraic subset $S(h)$ of $X(M)$. Let $U(M)$ be the Zariski-open set $X(M) \setminus \bigcup_h S(h)$. Then on $U(M)$ the action of $H^1(\pi_1(M);\mathbb{Z}_2)$ is free and the quotient map $U(M) \to \overline{X}(M)$ onto its image is a regular covering map of its image.

To make these general notions explicit, we appeal to the following material from [17]. For each $\gamma \in \pi_1(M)$, there is a regular function $\tau_\gamma\colon R(M) \to \mathbb{C}$ defined by $\tau_\gamma(\rho) = \operatorname{tr}\rho(\gamma)$ and a regular function $I_\gamma\colon X(M) \to \mathbb{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$.

Suppose $\gamma_1, \dots, \gamma_n$ generate $\pi_1(M)$, and consider the $m = \frac{1}{6}n(n^2+5)$ functions τ_γ as γ ranges over the set

$$G = \{\gamma_i,\ \gamma_i\gamma_j,\ \gamma_i\gamma_j\gamma_k \mid 1 \le i \le n,\ 1 \le i < j \le n,\ 1 \le i < j < k \le n\}$$

Then we obtain a regular map $\tau\colon R(M) \to \mathbb{C}^m$ defined by

$$\rho \mapsto (\tau_\gamma(\rho))_{\gamma\in G}$$

and it is shown in [17] that there is a natural identification $\tau(R(M)) = X(M)$. We therefore use the notation $\tau\colon R(M) \to X(M)$ for the natural map.

Projective characters have a similar interpretation, using the fact that the *square* of the trace of an element in $\mathrm{PSL}(2,\mathbb{C})$ is well defined. Given a representation $\bar\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, its character is the map

$$\bar\chi_{\bar\rho}\colon \pi_1(M) \to \mathbb{C} \quad \text{defined by} \quad \gamma \mapsto (\operatorname{tr}\bar\rho(\gamma))^2$$

As above, let $\bar\tau\colon \overline{R}(M) \to \overline{X}(M)$ be the natural map from representations to characters.

Throughout this paper, we make use of the following trace identities. Let $A, B, C \in \mathrm{SL}(2,\mathbb{C})$. Then

(2-1) $$\operatorname{tr}A^{-1} = \operatorname{tr}A,$$

(2-2) $$\operatorname{tr}(B^{-1}AB) = \operatorname{tr}A,$$

(2-3) $$\operatorname{tr}A\operatorname{tr}B = \operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}),$$

(2-4) $$\operatorname{tr}(ABC) = \operatorname{tr}A\operatorname{tr}(BC) + \operatorname{tr}B\operatorname{tr}(AC) + \operatorname{tr}(C)\operatorname{tr}(AB) - \operatorname{tr}A\operatorname{tr}B\operatorname{tr}C - \operatorname{tr}(ACB).$$

Recall that a representation into $\mathrm{SL}(2,\mathbb{C})$ is *irreducible* if the only subspaces of $\mathbb{C}^2$ invariant under its image are $\{0\}$ and $\mathbb{C}^2$. Otherwise, a representation is *reducible*. If a representation is reducible, then it is conjugate into the upper triangular matrices, and hence there is an abelian representation with the same character. We make frequent use of the following results from [9]:

Lemma 4 *Suppose $\rho \in R(M)$.*

(1) *Then ρ is reducible if and only if $\chi_\rho(\gamma) = 2$ for each element γ of the commutator subgroup of $\pi_1(M)$.*

(2) *Suppose ρ is irreducible and $\sigma \in R(M)$ satisfies $\chi_\rho = \chi_\sigma$. Then ρ and σ are conjugate and, in particular, σ is irreducible.*

It follows from the above lemma that reducible representations form a closed subset of $R(M)$, and irreducible representations form an open subset of $R(M)$ with respect to the Euclidean topology. Moreover, the subset $X^{\text{red}}(M) \subseteq X(M)$ of all characters of reducible representations is an affine algebraic set, and it is the same as the subset of all characters of abelian representations.

Let $X^{\text{irr}}(M)$ be the union of those Zariski components of $X(M)$ that contain at least one (and hence a Zariski-open) subset of irreducible representations. The following is shown in [23, Lemma 3.9(iii)].

Lemma 5 *Suppose $\chi \in X^{\text{irr}}(M) \cap X^{\text{red}}(M)$. Then each representation with character χ is reducible and there is a nonabelian reducible representation with this character.*

The quotient $QX^{\text{irr}}(M) = X^{\text{irr}}(M)/H^1(\pi_1(M), \mathbb{Z}_2)$ is naturally identified with a union of topological components of the union $\overline{X}^{\text{irr}}(M)$ of Zariski-components of $\overline{X}(M)$ containing irreducible representations. The above lemma implies:

Corollary 6 *The map $X^{\text{irr}}(M) \to \overline{X}(M)$ is a branched covering map onto its image and its ramification points are characters of irreducible or nonabelian reducible representations in the set $\bigcup_h S(h)$.*

A point of a complex affine algebraic set X is a *simple point* if it is contained in a unique Zariski component X_0 of X and is a smooth point of X_0.

Suppose the boundary of M is a union of pairwise disjoint tori and the interior of M admits a complete hyperbolic structure of finite volume. Then the holonomy for the complete hyperbolic structure defines a discrete and faithful $\mathrm{PSL}(2,\mathbb{C})$-character, which is unique up to complex conjugation (the ambiguity arises from the two different orientations possible on M). Each Zariski component of $\overline{X}(M)$ containing such a character is called a *canonical component*. The discrete and faithful characters lift to $\mathrm{SL}(2,\mathbb{C})$, and again each Zariski component of $X(M)$ containing at least one of these lifts is called a *canonical component*. It is shown in [23, Corollary 3.28] that the discrete and faithful characters are simple points of the canonical components. (The proof in [23] is given in the setting of $\mathrm{SL}(2,\mathbb{C})$-characters, but descends to $\mathrm{PSL}(2,\mathbb{C})$-characters.)

We end this section with a general discussion of the obstruction for lifting characters from $\overline{X}(M)$ to $X(M)$. See [19, Section 4], [7], [16, Section 4.5], or [17, Section 2] for details. Suppose $\bar\rho \in \overline{R}(M)$. Then $\bar\rho$ determines a Stiefel–Whitney class $w_2(\bar\rho) \in H^2(\pi_1(M); \mathbb{Z}_2)$. This is zero if and only if $\bar\rho$ lifts to a representation $\rho \in R(M)$. Moreover, the class is constant on topological components of $R(M)$. In the case of interest in this paper, M is aspherical and has boundary a single torus. Hence it is a $K(\pi_1(M), 1)$ and so $H^2(\pi_1(M); \mathbb{Z}_2) \cong H^2(M; \mathbb{Z}_2)$. Moreover, Poincaré–Lefschetz duality implies $H^2(M; \mathbb{Z}_2) \cong H_1(M, \partial M; \mathbb{Z}_2)$. In particular, if $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2$, then $H^2(\pi_1(M); \mathbb{Z}_2) \cong 0$ and hence every representation lifts.

2.2 Once-punctured torus bundles

Throughout the remainder of this paper, M_φ denotes a compact core of the hyperbolic once-punctured torus bundle over S^1 with monodromy φ, and S denotes a fixed fibre of M_φ with the property that the monodromy is the identity on ∂S. For our fundamental groups, we assume that the base point lies in $\partial S \subset \partial M_\varphi$. (Note that this notation is a slight variation on the introduction, where we referred to the noncompact manifolds.)

We also denote by φ the automorphism $\pi_1(S) \to \pi_1(S)$ induced by the monodromy (this will cause no confusion in this paper). Let $\varphi_*\colon H_1(S) \to H_1(S)$ be the induced automorphism on homology. Since M_φ is hyperbolic, the associated monodromy is pseudo-Anosov and we have $|\operatorname{tr} \varphi_*| > 2$.

The fundamental group of M_φ admits the presentation

$$\pi_1(M_\varphi) = \langle t, a, b \mid t^{-1}at = \varphi(a),\ t^{-1}bt = \varphi(b) \rangle, \tag{2-5}$$

where t is a meridian of M_φ (that is, it corresponds to the trace of the base point of $\pi_1(M_\varphi)$ under the monodromy) and a, b are free generators of $\pi_1(S)$, chosen such that their commutator is the longitude of M_φ (that is, it corresponds to ∂S).

We choose the basis of $H_1(S)$ corresponding to the generators a, b of $\pi_1(S)$, and often implicitly identify φ_* with its corresponding matrix representation, $[\varphi_*] \in \mathrm{SL}(2, \mathbb{Z})$. We may write φ as a product of the automorphisms α and β induced by the right-handed Dehn-twists about the curves corresponding to a and b, since these generate the group of isotopy classes of orientation preserving homeomorphisms of the punctured torus. The automorphisms are defined as follows (where we also include β^{-1} for convenience):

$$\alpha := \begin{cases} a \mapsto a, \\ b \mapsto ba, \end{cases} \qquad \beta := \begin{cases} a \mapsto ab^{-1}, \\ b \mapsto b, \end{cases} \qquad \beta^{-1} = \begin{cases} a \mapsto ab, \\ b \mapsto b. \end{cases} \tag{2-6}$$

This also gives a factorisation of φ_* in terms of the induced maps α_* and β_* on homology. We have

$$\mathtt{A} := [\alpha_*] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathtt{B} := [\beta_*] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The matrices $\mathtt{A}$ and $\mathtt{B}$ generate $\mathrm{SL}(2, \mathbb{Z})$ and satisfy the relations

$$\mathtt{ABA} = \mathtt{BAB}, \quad \text{and} \quad (\mathtt{ABA})^4 = I.$$

Using the second relation, we note that any element of $\mathrm{SL}(2,\mathbb{Z})$ can be written as a word in positive powers of $\mathtt{A}$ and $\mathtt{B}$ by substituting

$$\mathtt{B}^{-1} = \mathtt{A}(\mathtt{ABA})^3\mathtt{A} = \mathtt{A}^2\mathtt{BA}^2\mathtt{BA}^2\mathtt{BA}^2 \quad \text{and} \quad \mathtt{A}^{-1} = \mathtt{B}(\mathtt{BAB})^3\mathtt{B} = \mathtt{B}^2\mathtt{AB}^2\mathtt{AB}^2\mathtt{AB}^2.$$

Hence any element of $\mathrm{SL}(2,\mathbb{Z})$ (including the identity and powers of $\mathtt{A}$ or $\mathtt{B}$) is conjugate to a word of the form

$$\mathtt{A}^{a_1}\mathtt{B}^{b_1}\mathtt{A}^{a_2}\mathtt{B}^{b_2}\cdots\mathtt{A}^{a_n}\mathtt{B}^{b_n}, \tag{2-7}$$

where $n > 0$ and the a_i and b_i are positive integers. Also note that $(\mathtt{ABA})^2 = -I$. The manifold M_φ is determined up to homeomorphism by $[\varphi_*] \in \mathrm{SL}(2,\mathbb{Z})$. It is a result of Murasugi, see [8, Proposition 1.3.3], that M_φ and M_ψ are homeomorphic if and only if $[\varphi_*]$ is conjugate to either $[\psi_*]$ or $[\psi_*^{-1}]$.

The choice of a monodromy as a positive word in $\mathtt{A}$ and $\mathtt{B}$ is not unique, and there also is no simple criterion to ensure that the absolute value of the trace is greater than two. It is well-known, see [18, Proposition 2.1], that if $|\operatorname{tr}\varphi_*| > 2$, then the conjugacy class of φ_* has a representative of the form

$$\pm\mathtt{A}^{a_1}\mathtt{B}^{-b_1}\mathtt{A}^{a_2}\mathtt{B}^{-b_2}\cdots\mathtt{A}^{a_n}\mathtt{B}^{-b_n}, \tag{2-8}$$

where $n > 0$, the a_i and b_i are positive integers, and the sign equals the sign of the trace of φ_*. This representative is unique up to cyclic permutation of the factors $\mathtt{A}^{a_i}\mathtt{B}^{-b_i}$.

If $\operatorname{tr}\varphi_* \neq 2$, then $H_1(M_\varphi,\mathbb{Z}) \cong \mathbb{Z} \oplus \operatorname{Tor}(M_\varphi,\mathbb{Z})$. The *parity* of the trace contains information on the rank of homology with coefficients in $\mathbb{Z}_2$. We now describe this in more detail.

2.3 Order and rank

The natural homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ gives an epimorphism from $H_1(S) \to H_1(S;\mathbb{Z}_2)$, and takes φ_* to a map $\varphi_2\colon H_1(S;\mathbb{Z}_2) \to H_1(S;\mathbb{Z}_2)$. On the level of matrices, $[\varphi_*] \mapsto [\varphi_2]$ under the natural epimorphism $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}_2)$. Since $\mathrm{SL}(2,\mathbb{Z}_2) \cong \mathrm{Sym}(3)$, the order $o(\varphi_2)$ of φ_2 is either one, two or three. If φ_2 is not the identity, then it has order two if $\operatorname{tr}(\varphi_2) = 0$ and order three if $\operatorname{tr}(\varphi_2) = 1$. In particular, $o(\alpha_2) = o(\beta_2) = 2$ and the word in (2-7) maps to an element of order two if and only if the sum of all exponents is odd.

A direct calculation shows that $b_1(M_\varphi;\mathbb{Z}_2) = \operatorname{rank} H_1(M_\varphi,\mathbb{Z}_2) = 4 - o(\varphi_2) \in \{1,2,3\}$. To simplify notation, we write $b_1(\varphi) = b_1(M_\varphi;\mathbb{Z}_2)$. The maximal rank $b_1(M_\varphi;\mathbb{Z}_2) = 3$ is attained if φ_2 is the identity, and hence for at least one of φ, φ^2 or φ^3. Hence each once-punctured torus bundle is either 2-fold or 3-fold covered by a once-punctured torus bundle with homology with coefficients in $\mathbb{Z}_2$ of maximal rank three.

We identify $H_1(M_\varphi,\mathbb{Z}_2) = \operatorname{Hom}(\pi_1(M_\varphi),\mathbb{Z}_2) = H^1(M_\varphi,\mathbb{Z}_2)$, and denote $\varepsilon\colon \pi_1(M_\varphi) \to \mathbb{Z}_2$ the homomorphism defined by

$$\varepsilon(t) = -1, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 1. \tag{2-9}$$

Let $H \leq \mathrm{Hom}(\pi_1(M_\varphi), \mathbb{Z}_2)$ be the subgroup consisting of all homomorphisms

$$h\colon \pi_1(M_\varphi) \to \mathbb{Z}_2$$

satisfying $h(t) = 1$. Note that H has rank zero if $\mathrm{tr}(\varphi_*)$ is odd. In the following, we will make use of the splitting

$$H^1(M_\varphi, \mathbb{Z}_2) = \langle \varepsilon \rangle \oplus H. \tag{2-10}$$

2.4 Mutation

The once-punctured torus S admits an orientation preserving involution $\iota\colon S \to S$ defined by

$$\iota = \begin{cases} a \mapsto ba^{-1}b^{-1}, \\ b \mapsto bab^{-1}a^{-1}b^{-1}. \end{cases}$$

We have $\iota_* = -I$. If one cuts M_φ along S and changes the monodromy by ι, one obtains $M_{\iota\varphi}$. This process is called *mutation*.

It follows from [26, Proposition 1] that there is a birational equivalence between Zariski components of $\overline{X}(M_\varphi)$ and $\overline{X}(M_{\iota\varphi})$ that contain the character of a representation $\bar{\rho}$ whose restriction to S is irreducible and has trivial centraliser. In particular, if one is interested in the topology of generic components of $\overline{X}(M_\varphi)$, then one may restrict attention to bundles where the trace of φ_* is positive. Moreover, components where all irreducible representations have nontrivial centraliser can be understood directly (see Section 4.2).

In the case of $\mathrm{SL}(2,\mathbb{C})$-character varieties, using the HNN-splitting for the fundamental group as in the proof of [26, Proposition 1], one sees that the restriction maps to the fibre result in the same fixed-point set; that is: $X_\varphi(S) = X_{\iota\varphi}(S)$. However, the topology of the $\mathrm{SL}(2,\mathbb{C})$-character varieties may be different. For instance, mutation of the figure eight knot complement (with $\varphi_* = \mathtt{AB}^{-1}$) along the fibre results in the associated sister manifold, and the smooth projective models of their canonical components are a torus and a sphere respectively [26, Section 3.4].

2.5 The character variety of the free group of rank two

The fundamental group of the once-punctured torus fibre S is the free group $\langle a, b\rangle$, and the character variety $X(S)$ admits the affine coordinates $(x, y, z) \in \mathbb{C}^3$, where

$$x = \tau_a(\rho), \quad y = \tau_b(\rho), \quad z = \tau_{ab}(\rho)$$

as ρ ranges over $R(S)$. An elementary calculation, which goes back to classical work of Fricke and Klein [14; 15] and Vogt [28], shows that $X(S) = \mathbb{C}^3$.

Let $\rho \in R(S)$ be an irreducible representation. Up to conjugation, ρ is of the form

$$\rho(a) = \begin{pmatrix} s & 0 \\ 1 & s^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}, \tag{2-11}$$

where $u \neq 0$.

If $\rho \in R(S)$ is a reducible representation, then ρ is conjugate to one of the following:

$$\rho(a) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}, \tag{2-12}$$

$$\rho(a) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & u \\ 0 & p^{-1} \end{pmatrix}. \tag{2-13}$$

Every point in $X^{\mathrm{red}}(S)$ is the character of a reducible character as in (2-12) with $u = 0$. A representation $\rho \in R(S)$ is reducible if and only if its character satisfies the equation

$$x^2 + y^2 + z^2 - xyz - 4 = 0.$$

This is a consequence of the first part of Lemma 4. See [9] for details.

A parametrisation of the $\mathrm{PSL}(2,\mathbb{C})$-character variety $\overline{X}(S)$ is described in [19, Section 4.2] as follows. Since the fundamental group of S is free in two generators, every character in $\overline{X}(S)$ lifts to $X(S)$. By studying the invariant functions on $X(S)$, one obtains an identification

$$\overline{X}(S) \cong \{(X, Y, Z, W) \in \mathbb{C}^4 \mid W^2 = XYZ\}, \tag{2-14}$$

where $X = \tau_a^2$, $Y = \tau_b^2$, $Z = \tau_{ab}^2$ and $W = \tau_a \tau_b \tau_{ab} = \frac{1}{2}(\tau_a^2 \tau_b^2 + \tau_{ab}^2 - \tau_{ab^{-1}}^2)$. The four-fold branched covering map $X(S) \to \overline{X}(S)$ thus has the following description in these coordinates:

$$\overline{X}(S) \ni (x, y, z) \mapsto (x^2, y^2, z^2, xyz) \in \overline{X}(S). \tag{2-15}$$

2.6 Thurston's dimension bound

Based on Thurston [25, Theorem 5.6], Culler and Shalen [9, Proposition 3.2.1] showed:

Proposition 7 (Thurston) *Let N be a compact orientable 3-manifold. Suppose $\rho\colon \pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$ is an irreducible representation such that for each torus component T of ∂N,*

$$\rho\big(\mathrm{im}(\pi_1(T) \to \pi_1(N))\big) \not\subseteq \{\pm 1\}.$$

Let R_0 be an irreducible component of $R(N)$ containing ρ. Then $X_0 = \tau(R_0)$ has dimension $\geq s - 3\chi(N)$, where s is the number of torus components of ∂N.

An analogous statement for representations into $\mathrm{PSL}(2,\mathbb{C})$ is the following:

Proposition 8 (Thurston) *Let N be a compact orientable 3-manifold. Suppose $\bar{\rho}_0\colon \pi_1(N) \to \mathrm{PSL}(2,\mathbb{C})$ is an irreducible representation such that for each torus component T of ∂N, $\bar{\rho}_0\big(\mathrm{im}(\pi_1(T) \to \pi_1(N))\big)$ is nontrivial and not isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\overline{R}_0$ be an irreducible component of $\overline{R}(N)$ containing $\bar{\rho}_0$. Then $\overline{X}_0 = \bar{\tau}(\overline{R}_0)$ has dimension $\geq s - 3\chi(N)$, where s is the number of torus components of ∂N.*

We first interpret the statement of the proposition before giving its proof.

The group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not occur as an exclusion in the statement of [25, Theorem 5.6] because Thurston was concerned with *holonomies of hyperbolic structures* and not arbitrary irreducible representations. Its appearance is rather natural in light of the following observations.

First, abelian subgroups of $\mathrm{PSL}(2,\mathbb{C})$ either have a global fixed point in $P^1(\mathbb{C})$, or they are isomorphic to the Klein four group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (and are irreducible but elementary). Up to conjugacy, the latter has the nontrivial elements[2]

$$\kappa_1 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \kappa_2 = \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \kappa_3 = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2-16}$$

In particular, for representations of $\langle \mu, \lambda \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, the two cases are distinguished by $\operatorname{tr} \rho[\mu,\lambda] = 2$ and $\operatorname{tr} \rho[\mu,\lambda] = -2$, with the first equation determining a positive dimensional component in the $\mathrm{PSL}(2,\mathbb{C})$-character variety of $\mathbb{Z} \oplus \mathbb{Z}$ and the second an isolated point. It also follows that an equivalent requirement on $\bar{\rho}_0$ in the statement of Proposition 8 is that

the image of each peripheral torus subgroup is nontrivial and reducible.

Second, each lift of the Klein four group to $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to the quaternionic group and hence if a peripheral subgroup corresponding to a torus boundary component has this image under a projective representation, then it does not lift. This explains why this case does not appear in Proposition 7.

Proof of Proposition 8 We follow the wording of the proof of [9, Proposition 3.2.1] closely, and combine this with observations from Thurston's proof and results stated in [19].

By [19, Corollary 3.3.5], the conclusion is equivalent to the assertion that $\dim \overline{R}_0 \geq s - 3\chi(N) + 3$. We shall prove this by induction on s. First suppose that $s = 0$. We may assume that $\partial N \neq \varnothing$ since otherwise $\chi(N) = 0$ and there is nothing to prove. Now N has the homotopy type of a finite 2-dimensional CW-complex K with one 0-cell, and, say, m 1-cells and n 2-cells. Thus $\pi_1(N)$ has a presentation $\langle g_1, \dots, g_m \mid r_1 = \dots = r_n = 1 \rangle$. This presentation gives a natural identification of $\overline{R}(N)$ with an algebraic subset of $(\mathrm{PSL}(2,\mathbb{C}))^m$.

Suppose $\bar{\rho}_0(g_i) = \pm G_i$. Then for each r_j there exists $\epsilon_j \in \{\pm 1\}$ such that

$$r_j(G_1, \dots, G_m) = \epsilon_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that if $\epsilon_j = 1$ for each j, then $\bar{\rho}_0$ lifts to a representation to $\mathrm{SL}(2,\mathbb{C})$. Now consider the coordinate ring $\mathbb{C}[a_1, b_1, c_1, d_1, \dots, a_m, b_m, c_m, d_m]$. Let $\overline{R}_0$ be the variety with ideal defined by the m equations $a_i c_i - b_i d_i = 1$ and the $3n$ equations arising from the $(1,1), (1,2)$ and $(2,1)$ entries of the m matrix equations

$$r_j\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}\right) = \epsilon_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[2] Following standard convention, we write elements of $\mathrm{PSL}(2,\mathbb{C})$ not as sets $\{\pm C\}$ but rather in the form $\pm C$.

The equations given by the $(2,2)$ entries of the above matrix equations are a consequence of the other equations and the multiplicative property of determinants. Our set-up gives a natural algebraic embedding $\overline{R}_0 \to (\mathrm{PSL}(2,\mathbb{C}))^m$ with image in $\overline{R}(N)$. Hence we may identify $\overline{R}_0$ with a subvariety of $\overline{R}(N)$ containing $\bar\rho_0$. Now

$$\dim \overline{R}_0 \ge 4m - m - 3n = 3m - 3n = -3\chi(K) + 3 = -3\chi(N) + 3$$

gives the desired conclusion.

Now suppose $s > 0$. Let T be a torus component of ∂N. Since $\bar\rho_0\big(\mathrm{im}(\pi_1(T) \to \pi_1(N))\big) \not\subseteq \{1\}$, there is $\alpha \in \pi_1(N)$, represented by a simple closed curve on T, such that $\bar\rho_0(\alpha) \neq 1 \in \mathrm{PSL}(2,\mathbb{C})$. Since $\bar\rho_0$ is irreducible, there is $\gamma \in \pi_1(N)$ such that $\bar\rho_0$ restricted to the subgroup generated by α and γ is irreducible.

Following Thurston, we show that γ can be chosen such that $\mathrm{tr}^2\,\bar\rho_0(\gamma) \neq 4$. Hence suppose that $\mathrm{tr}^2\,\bar\rho_0(\gamma) = 4$. Since $\bar\rho_0(\alpha)$ and $\bar\rho_0(\gamma)$ have no common fixed point on $P^1(\mathbb{C})$, it follows that $\bar\rho_0(\gamma)$ is parabolic and has a unique fixed point on $P^1(\mathbb{C})$. We may conjugate $\bar\rho_0$ such that

$$\bar\rho_0(\gamma) = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar\rho_0(\alpha) = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $c \neq 0$. Then

$$\bar\rho_0(\alpha\gamma^{2n}) = \pm\begin{pmatrix} a & b+2na \\ c & d+2nc \end{pmatrix}.$$

Hence $\mathrm{tr}^2\,\bar\rho_0(\alpha\gamma^{2n}) = (a+d+2nc)^2$ and since $c \neq 0$ we may choose n such that $\mathrm{tr}^2\,\bar\rho_0(\alpha\gamma^{2n})$ does not equal 4. Now the parabolic $\bar\rho_0(\gamma^{2n})$ has the same fixed point on $P^1(\mathbb{C})$ as $\bar\rho(\gamma)$. Hence $\bar\rho_0(\alpha)$ and $\bar\rho_0(\gamma^{2n})$ have no common fixed point on $P^1(\mathbb{C})$, and so the representation restricted to $\langle \alpha, \gamma^{2n}\rangle = \langle \alpha, \alpha\gamma^{2n}\rangle$ is irreducible. This completes the argument that we may choose γ such that $\mathrm{tr}^2\,\bar\rho_0(\gamma) \neq 4$.

Choose a base point for the fundamental group of N based on T and thus represent γ by a simple closed curve based at this point. Drilling out a regular open neighbourhood of γ gives a submanifold M of N such that N is obtained by adding a 2-handle to a genus two boundary component S of M. We may choose a standard basis $\alpha', \beta', \gamma', \delta'$ of $\pi_1(S)$ such that α' and γ' are mapped to α and γ under the natural surjection $i_*\colon \pi_1(M) \to \pi_1(N)$; the 2-handle is attached to S along a simple closed curve that represents the conjugacy class of δ; and the relation for $\pi_1(S)$ is given by $[\alpha', \beta'] = [\gamma', \delta']$.

The representation $\bar\rho'_0 = \bar\rho_0 \circ i_*\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ is irreducible since i_* is a surjection. By the induction hypothesis, there is an irreducible component $\overline{R}'_0$ of $\overline{R}(M)$ that contains $\bar\rho'_0$ and has dimension

$$\dim \overline{R}'_0 \ge (s-1) - 3\chi(M) + 3 = s - 3\chi(N) + 5$$

since $\chi(M) = \chi(N) - 1$. Let W be the intersection of $\overline{R}'_0$ with the subvariety defined by the two equations

$$\mathrm{tr}^2\,\bar\rho(\delta') = 4, \quad \mathrm{tr}\,\bar\rho[\gamma', \delta'] = 2.$$

By [19, Section 2.4], these are well defined invariant functions on $\overline{R}'_0$. Then the dimension of each component of W is at least $s - 3\chi(N) + 3$.

We first show that $\bar{\rho}_0'$ is contained in W. The kernel of i_* is the normal closure of δ', and hence $\bar{\rho}_0'(\delta') = 1$. This implies $\operatorname{tr}^2 \bar{\rho}_0'(\delta') = 4$ and $\operatorname{tr} \bar{\rho}_0'[\gamma', \delta'] = 2$. Hence there is an irreducible component $\bar{R}_0$ of W that contains $\bar{\rho}_0'$. To obtain the desired conclusion that we can identify $\bar{R}_0$ with a subvariety of $\bar{R}(N)$ it remains to show that all representations $\bar{\rho} \in \bar{R}_0$ near $\bar{\rho}_0$ satisfy $\bar{\rho}(\delta') = 1$.

Hence suppose that for $\bar{\rho} \in \bar{R}_0$ near $\bar{\rho}_0$, we have $\bar{\rho}(\delta') \neq 1$. Then $\bar{\rho}(\delta')$ is a nontrivial parabolic. Since $\operatorname{tr} \bar{\rho}[\gamma', \delta'] = 2$, it follows that $\bar{\rho}(\delta')$ and $\bar{\rho}(\gamma')$ share a fixed point on $P^1(\mathbb{C})$. Since $\bar{\rho}$ is near $\bar{\rho}_0$, $\bar{\rho}(\gamma')$ is not a parabolic. Hence $\bar{\rho}[\gamma', \delta']$ is a nontrivial parabolic that has the same fixed point as $\bar{\rho}(\delta')$ and shares a fixed point with $\bar{\rho}(\gamma')$.

Since $[\gamma', \delta'] = [\alpha', \beta']$ in $\pi_1(M)$, we have $\operatorname{tr} \bar{\rho}[\alpha', \beta'] = \pm 2$. Since $\bar{\rho}$ is near $\bar{\rho}_0$, it follows from our hypothesis on the peripheral subgroups that $\operatorname{tr} \bar{\rho}[\alpha', \beta'] = 2$. Hence $\bar{\rho}(\alpha')$ and $\bar{\rho}(\beta')$ have a common fixed point on $P^1(\mathbb{C})$. Hence the unique fixed point of the nontrivial parabolic $\bar{\rho}[\gamma', \delta'] = \bar{\rho}[\alpha', \beta']$ is in common with both $\bar{\rho}(\gamma')$ and $\bar{\rho}(\alpha')$ and thus $\operatorname{tr} \bar{\rho}[\alpha', \gamma'] = 2$. But this is not possible since $\bar{\rho}$ is near $\bar{\rho}_0'$, and $\bar{\rho}_0'$ restricted to the subgroup generated by α' and γ' is irreducible. This is the final contradiction to our hypothesis that $\bar{\rho}(\delta') \neq 1$. □

Example 9 The $\operatorname{PSL}(2, \mathbb{C})$-character variety of the Whitehead link complement contains 0-dimensional components that satisfy the condition that all peripheral subgroups have image isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. See [27, Appendix C].

Example 10 Baker and Petersen [2, Theorem 7.6] give zero-dimensional components in character varieties of once-punctured torus bundles that appear to contradict Proposition 8. A less sophisticated approach than given in [2] shows that these characters indeed lie on one-dimensional components. The presentation for the fundamental group of M_n with monodromy $\varphi_n = \mathtt{AB}^{n+2}$ is given in [2, Section 2.1] as

$$\Gamma_n = \langle \alpha, \beta \mid \beta^{-n} = \alpha^{-1}\beta\alpha^2\beta\alpha^{-1} \rangle,$$

where a meridian is the element $\mu = \beta\alpha$ and a corresponding longitude is $\lambda = (\alpha\beta\alpha^{-1})\beta(\alpha\beta\alpha^{-1})^{-1}\beta^{-1}$; see [2, Lemma 2.3]. Suppose $|n| \geq 2$ is even. Then

$$\bar{\rho}(\alpha) = \pm \begin{pmatrix} -yx^n & -1 \\ \dfrac{1+x^{2+n}}{(1-x^2)(1-x^n)} & y \end{pmatrix} \quad \text{and} \quad \bar{\rho}(\beta) = \pm \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

gives a representation of Γ_n into $\operatorname{PSL}(2, \mathbb{C})$ that does not lift to $\operatorname{SL}(2, \mathbb{C})$ if

$$x^2 + x^n = y^2 x^n (1 - x^2)(1 - x^n).$$

Indeed, the above equation is equivalent to the determinant condition for $\bar{\rho}(\alpha)$, and it is also a sufficient condition for $\bar{\rho}(\alpha)\bar{\rho}(\beta)^{-n}\bar{\rho}(\alpha) + \bar{\rho}(\beta)\bar{\rho}(\alpha)^2\bar{\rho}(\beta) = 0$ (noting that both summands are independent of the choice of sign).

The alleged isolated points in [2, Theorem 7.6] correspond to the points (x_0, y_0) satisfying $x_0^n = -1$ and $y_0 = \pm\frac{1}{\sqrt{2}}$ on this curve. At each of these points, the image of the commutator of meridian and longitude has trace equal to two (hence the representation restricted to the peripheral subgroup is reducible) and the image of the meridian has trace $\pm\frac{1}{\sqrt{2}}(x_0 + x_0^{-1})$ (hence the representation restricted to the peripheral subgroup is nontrivial since the absolute value of the trace is bounded from above by $\sqrt{2}$). The triple of squares of traces is indeed as stated in [2, Theorem 7.6]

$$(\operatorname{tr}^2 \bar{\rho}(\alpha), \operatorname{tr}^2 \bar{\rho}(\beta), \operatorname{tr}^2 \bar{\rho}(\alpha\beta)) = \bigl(2, (x_0 + x_0^{-1})^2, \tfrac{1}{2}(x_0 + x_0^{-1})^2\bigr).$$

We have intentionally used the notation from [2] in this example. With respect to the notation of (2-5), one has $t = \mu^{-1}$, $a = \beta^{-1}\mu\beta\mu^{-1}$ and $b = \beta$ and

$$\pi_1(M_n) = \langle t, a, b \mid t^{-1}at = a(a^{-1}b^{-1})^{n+2}, t^{-1}bt = ba\rangle.$$

3 Character varieties of once-punctured torus bundles

We continue with the notation introduced in Section 2 and analyse the $\mathrm{SL}(2, \mathbb{C})$-character varieties of once-punctured torus bundles using the restriction to the fibre.

3.1 The restriction map and the fixed point set

The fundamental group of a once-punctured torus bundle M_φ with monodromy φ is

$$\pi_1(M_\varphi) = \langle t, a, b \mid t^{-1}at = \varphi(a), t^{-1}bt = \varphi(b)\rangle.$$

Throughout, we assume

$$[\varphi_*] = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Since we restrict our attention to hyperbolic once-punctured torus bundles, we have $|\operatorname{tr}\varphi_*| = |k_1 + k_4| > 2$.

As in Section 2.1, $X(M_\varphi)$ admits the affine coordinates (x, y, z, u, v, w, q) in $\mathbb{C}^7$, where

$$x = \tau_a(\rho), \quad y = \tau_b(\rho), \quad z = \tau_{ab}(\rho), \quad u = \tau_t(\rho), \quad v = \tau_{at}(\rho), \quad w = \tau_{bt}(\rho), \quad q = \tau_{abt}(\rho) \tag{3-1}$$

for $\rho \in R(M_\varphi)$. Consider the *restriction map*

$$r : X(M_\varphi) \to X(S), \quad (x, y, z, u, v, w, q) \mapsto (x, y, z).$$

Each representation $\rho \in R(M_\varphi)$ satisfies

$$\rho(t)^{-1}\rho(\gamma)\rho(t) = \rho(\varphi(\gamma)) \quad \text{for all } \gamma \in \pi_1(S).$$

Taking the traces on both sides,

$$\operatorname{tr}(\rho(\gamma)) = \operatorname{tr}\bigl(\rho(\varphi(\gamma))\bigr) \quad \text{for all } \gamma \in \pi_1(S).$$

Since χ_ρ in $X(S)$ is uniquely determined by $\operatorname{tr}\rho(a)$, $\operatorname{tr}\rho(b)$ and $\operatorname{tr}\rho(ab)$, the image of r satisfies

$$\begin{aligned}\operatorname{im}(r) \subseteq X_\varphi(S) := \big\{(\operatorname{tr}\rho(a), \operatorname{tr}\rho(b), \operatorname{tr}\rho(ab)) \mid \operatorname{tr}\rho(a) = \operatorname{tr}\rho(\varphi_*(a)),\ \operatorname{tr}\rho(b) = \operatorname{tr}\rho(\varphi_*(b)),\\ \operatorname{tr}(\rho(ab)) = \operatorname{tr}\rho(\varphi_*(ab)),\ \rho \in R(S)\big\}\\ \subseteq X(S).\end{aligned}$$

Since $\operatorname{tr}\rho(\varphi_*(a))$, $\operatorname{tr}\rho(\varphi_*(b))$ and $\operatorname{tr}\rho(\varphi_*(ab))$ can be written as polynomials with rational coefficients in $x = \operatorname{tr}\rho(a)$, $y = \operatorname{tr}\rho(b)$ and $z = \operatorname{tr}\rho(ab)$, and since φ_* is an automorphism of the free group in two letters, it follows that there is a polynomial automorphism $\overline{\varphi}\colon \mathbb{C}^3 \to \mathbb{C}^3$ induced by φ with the property that $X_\varphi(S)$ is the *set of fixed points of this polynomial automorphism*:

$$X_\varphi(S) = \{(x,y,z) \in \mathbb{C}^3 \mid (x,y,z) = \overline{\varphi}(x,y,z)\}. \tag{3-2}$$

Lemma 11 *We have* $\operatorname{im}(r) = X_\varphi(S)$. *Moreover, there are at most finitely many reducible characters in* $X_\varphi(S)$.

Proof By definition, $\operatorname{im}(r) \subseteq X_\varphi(S)$ and for the first statement, it remains to show that the map is surjective.

If $(x,y,z) \in X_\varphi(S)$ is the character of an irreducible representation, then $(x,y,z) \in \operatorname{im}(r)$ follows from the presentation of the group and the second statement in Lemma 4. We analyse this situation in more detail in Lemma 18.

Hence suppose $(x,y,z) \in X_\varphi(S)$ is the character of a reducible representation. Since this is also the character of an abelian representation, we may assume that it is the character of the representation ρ defined by

$$\rho(a) = \begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix}, \quad \rho(b) = \begin{pmatrix} p & 0 \\ 0 & p^{-1}\end{pmatrix}.$$

Since $\rho(\varphi_*(a))$ can be viewed as a word in these matrices, it is also diagonal. Similarly for $\rho(\varphi_*(b))$. Hence

$$\rho(\varphi_*(a)) = \begin{pmatrix} s_1 & 0 \\ 0 & s_1^{-1}\end{pmatrix}, \quad \rho(\varphi_*(b)) = \begin{pmatrix} p_1 & 0 \\ 0 & p_1^{-1}\end{pmatrix}.$$

Then the three trace conditions defining $X_\varphi(S)$ imply that we have the mutually exclusive cases,

(1) $(s_1, p_1) = (s, p)$ or
(2) $(s_1, p_1) = (s^{-1}, p^{-1})$ with either $s \neq s^{-1}$ or $p \neq p^{-1}$ (equivalently, $s \neq \pm 1$ or $p \neq \pm 1$).

In order to show that $(x,y,z) \in \operatorname{im}(r)$, we need to determine $T \in \mathrm{SL}(2,\mathbb{C})$ such that $T^{-1}\rho T = \rho\varphi_*$.

In the first case, this is satisfied by $T = \left(\begin{smallmatrix} m & 0 \\ 0 & m^{-1}\end{smallmatrix}\right)$ for arbitrary $m \in \mathbb{C}^*$. We note that we have $\rho(\varphi_*(\gamma)) = \rho(\gamma)$ for all $\gamma \in \pi_1(S)$, and the resulting representation of $\pi_1(M_\varphi)$ is abelian and hence reducible.

In the second case, this is satisfied by $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. We note that we have $\rho(\varphi_*(\gamma)) = \rho(\gamma)^{-1}$ for all $\gamma \in \pi_1(S)$, and the resulting representation of $\pi_1(M_\varphi)$ is irreducible and corresponds to a *binary dihedral representation* in $R(M_\varphi(\lambda)) \subset R(M_\varphi)$ (see [4, Section 4]).

Hence in either case, this shows that $(x,y,z) \in \operatorname{im}(r)$.

We now prove the second statement. The monodromy $\varphi_* = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in SL(2,\mathbb{Z})$ satisfies $|k_1+k_4|>2$. The pair of eigenvalues of the abelian representation ρ as above satisfies $(s^{k_1}p^{k_3}, s^{k_2}p^{k_4}) = (s^{\pm 1}, p^{\pm 1})$ since abelian representations factor through first homology and the latter is determined by φ_*. This implies that each of s and p raised to the power of $(k_1 \mp 1)(k_4 \mp 1) - k_2k_3$ equals one. There are finitely many solutions unless $(k_1 \mp 1)(k_4 \mp 1) = k_2k_3$. Using the determinant condition, this is equivalent with $k_1 + k_4 = \mp 2$, which contradicts the trace condition for φ_*. This proves the second statement. □

Scholion 12 *The irreducible characters in $X(M_\varphi)$ that map to reducible characters in $X_\varphi(S)$ are precisely the characters of the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$.*

3.2 The origin and the coordinate axes

For the automorphisms α and β^{-1} given in (2-6), the induced polynomial automorphisms are

$$\overline{\alpha}(x,y,z) = (x,z,xz-y) \quad \text{and} \quad \overline{\beta}^{-1}(x,y,z) = (z,y,yz-x),$$

respectively. Given the standard representative φ_* as in (2-8), it follows from Section 2.2 that the associated polynomial automorphism $\overline{\varphi}\colon \mathbb{C}^3 \to \mathbb{C}^3$ induced by φ factors in terms of $\overline{\iota}$, $\overline{\alpha}$ and $\overline{\beta}^{-1}$.

This implies that the origin $(0,0,0) \in X(S)$ is always contained in $X_\varphi(S)$ for any M_φ. Restricted to S, any representation with this character has image in $\mathrm{SL}(2,\mathbb{C})$ isomorphic to the quaternionic group, and image isomorphic to the Klein four group under the quotient map to $\mathrm{PSL}(2,\mathbb{C})$ (see (4-5) below).

The examples in Section 5.4 show that the component containing $(0,0,0)$ in $X_\varphi(S)$ may have arbitrarily large genus if $b_1(\varphi) \neq 3$. We now show that in the maximal rank case, at least one of the coordinate axes is always contained in $X_\varphi(S)$.

For all $x,y,z \in \mathbb{C}$, we have $\overline{\iota}(x,y,z) = (x,y,z)$ and

$$\begin{aligned} \overline{\alpha}(x,0,0) &= (x,0,0), & \overline{\beta}^{-1}(x,0,0) &= (0,0,-x), \\ \overline{\alpha}(0,y,0) &= (0,0,-y), & \overline{\beta}^{-1}(0,y,0) &= (0,y,0), \\ \overline{\alpha}(0,0,z) &= (0,z,0) & \overline{\beta}^{-1}(0,0,z) &= (z,0,0). \end{aligned}$$

Denote the x, y, and z coordinate axes in $\mathbb{C}^3$ by L_1, L_2 and L_3, respectively. Then $\overline{\alpha}$ and $\overline{\beta}^{-1}$ permute these three lines, with a sign change for exactly one of them. Indeed, we may assume that the permutation of the lines corresponds to the permutation of the indices of the lines induced by $[\alpha_2]$ and $[\beta_2]$ via the isomorphism $\mathrm{SL}(2,\mathbb{Z}_2) \cong \mathrm{Sym}(3)$.

Lemma 13 *If $H_1(M_\varphi,\mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then $X_\varphi(S)$ contains either exactly one or all three of L_1, L_2 and L_3. Moreover, $X_{\varphi^2}(S)$ contains all three lines.*

Proof Since $\overline{\iota}(x,y,z) = (x,y,z)$, it suffices to restrict to the case where $\operatorname{tr}\varphi_* > 0$. We may assume that φ_* is given as in (2-8). Recall from Section 2.3 that $H_1(M_\varphi,\mathbb{Z}_2) \cong \mathbb{Z}_2^3$ is equivalent to $o(\varphi_2) = 1$. Then

$$\mathrm{Id} = [\varphi_2] = [\alpha_2]^{a_1}[\beta_2]^{-b_1}[\alpha_2]^{a_2}[\beta_2]^{-b_2}\cdots[\alpha_2]^{a_n}[\beta_2]^{-b_n}.$$

It follows from the description of the action of α_2, β_2 on L_1, L_2 and L_3, that the induced polynomial automorphism $\overline{\varphi}$ stabilises each line. Hence, $\overline{\varphi}(L_i) = L_i$. Note that when $\overline{\alpha}$ or $\overline{\beta}^{-1}$ permutes the lines, the sign of exactly one line changes. Denote the number of sign changes of L_i by n_i. We have

$$\overline{\varphi}(x,0,0) = ((-1)^{n_1}x,0,0), \quad \overline{\varphi}(0,y,0) = (0,(-1)^{n_2}y,0), \quad \overline{\varphi}(0,0,z) = (0,0,(-1)^{n_3}z).$$

As $o(\varphi_2) = 1$, the number $n_1 + n_2 + n_3 = \sum_{i=1}^{n}(a_i + b_i)$ is also even. Hence at least one n_i is even and at least one coordinate is fixed by $\overline{\varphi}$. We also see that either exactly one n_i is even or all three n_i are even. This proves the first statement. The second statement follows from the observation that

$$\begin{aligned} \overline{\varphi}^2(x,0,0) &= ((-1)^{2n_1}x,0,0) = (x,0,0), \\ \overline{\varphi}^2(0,y,0) &= (0,(-1)^{2n_2}y,0) = (0,y,0), \\ \overline{\varphi}^2(0,0,z) &= (0,0,(-1)^{2n_3}z) = (0,0,z). \end{aligned}$$ □

If $b_1(M_\varphi) = 1$, then $[\varphi_2]$ has order 3. So $\overline{\varphi}$ does not stabilise any axis and $L_i \cap X_\varphi(S) = \{(0,0,0)\}$ for each i.

We also note that for any φ, $X_{\varphi^6}(S)$ contains all three lines.

As a last observation, we add that $(x,0,0) \in L_1$ is a reducible character if and only if $x^2 = 4$. In this case, the extension to $\pi_1(M_\varphi)$ is either reducible or binary dihedral according to Scholion 12.

3.3 Dimensions and cyclic covers

Proposition 14 *Let M_φ be a hyperbolic once-punctured torus bundle. Then every Zariski component of $X(M_\varphi)$ is one-dimensional.*

Proof We first note that since $H_1(M_\varphi, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Tor}(M_\varphi, \mathbb{Z})$, every component containing only reducible characters is one-dimensional and of genus zero.

Suppose $X \subset X(M_\varphi)$ is a component containing the character of the irreducible representation $\rho \in R(M_\varphi)$. Since M_φ does not contain a closed essential surface (see [8; 12]), the dimension of X is at most one.

We now appeal to a result due to Thurston (see Proposition 7) which implies that the dimension of X is at least one if $\rho(\pi_1(\partial M))$ is not contained in $\{\pm I\}$. Hence suppose that $\rho(\pi_1(\partial M))$ is contained in $\{\pm I\}$; such a representation is said to have *trivial peripheral holonomy*. Since the longitude is $a^{-1}b^{-1}ab$, we note that Lemma 4(1) implies that ρ is reducible if $\rho(t) = \pm I$ and $\rho(a^{-1}b^{-1}ab) = I$. Hence $\rho(a^{-1}b^{-1}ab) = -I$. An elementary calculation setting $\rho(a)$ and $\rho(b)$ as in (2-11) and solving $\rho(ab) = -\rho(ba)$ shows that the resulting representation satisfies $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b) = \operatorname{tr}\rho(ab) = 0$ and hence is unique up to conjugation.

In particular, ρ descends to an abelian irreducible representation $\bar{\rho}\colon \pi_1(\partial M) \to \mathrm{PSL}(2,\mathbb{C})$ with $\bar{\rho}(t) = \{\pm I\}$ and image isomorphic to the Klein four group, generated by the image of $\pi_1(S)$. It follows that $H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3$. In particular, Lemma 13 implies that the character of ρ is contained in at least one one-dimensional component containing irreducible representations. □

The natural n-fold cyclic covering $M_{\varphi^n} \to M_\varphi$ gives an embedding of $\pi_1(M_{\varphi^n})$ as a subgroup of index n in $\pi_1(M_\varphi)$. Hence the restriction map induces a regular map $X(M_\varphi) \to X(M_{\varphi^n})$. Example 16 below shows that this may not be surjective. We can say more about this map:

Proposition 15 *The natural n-fold cyclic covering $M_{\varphi^n} \to M_\varphi$ induces a regular map $X(M_\varphi) \to X(M_{\varphi^n})$ of degree at most two. If n is odd, then the degree is one. Moreover, the map takes canonical components to canonical components.*

Proof The first sentence follows directly from [4, Corollary 3.5], and the second from [4, Proposition 3.3]. The last sentence follows since the restriction of a discrete and faithful representation to a subgroup gives a discrete and faithful representation of that subgroup. □

We conclude with an example that ties the discussion in this section together and exhibits different behaviours.

Example 16 Let $\varphi = \alpha^2\beta^{-2}$. Then

$$[\varphi_*] = A^2B^{-2} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

and hence

$$H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3 \cong H_1(M_{\varphi^2}; \mathbb{Z}_2).$$

A computation with the set-up in the proof of Proposition 14 shows that $X(M_\varphi)$ contains no characters of representations with trivial peripheral holonomy. Indeed, $\overline{X}(M_\varphi)$ contains such a component, but it does not lift. This example fits in the family given in [19].

A calculation shows that $X_\varphi(S) = V \cup L_3$, where V is an irreducible curve, and $X(M_\varphi) = V' \cup L'_3$ also has two components. The preimage V' of V is the canonical curve and the preimage L'_3 of L_3 has the property that the trace of the meridian is identically zero on it (hence the meridian maps to an element of order four). All binary dihedral characters are contained in L'_3 and since they are simple points they are hence not contained in V'. Since these characters are the only branch points (as will be shown in general), it follows that $L'_3 \to L_3$ is a two-fold branched cover, and $V' \to V$ is a two-fold unbranched cover.

Now $X_{\varphi^2}(S) \supset X_\varphi(S) \cup L_1 \cup L_2 = V \cup L_1 \cup L_2 \cup L_3$. Hence this is an example where all three lines are contained in the fixed-point set. The preimage in $X(M_{\varphi^2})$ of V is the canonical curve, but the preimage of L_3 has two components L_3^+ and L_3^- that are characterised by whether the trace of the meridian is identically $+2$ or -2. The component L_3^- is the image of L'_3 under the map $X(M_\varphi) \to X(M_{\varphi^2})$ since the square of an element of order four has order two. Hence L_3^+ is not contained in the image of the map. We note that each map $L_3^- \to L_3$ and $L_3^+ \to L_3$ is one-to-one.

3.4 Reducible characters

Since $H_1(M_\varphi, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Tor}(M_\varphi, \mathbb{Z})$, it follows that $X^{\mathrm{red}}(M_\varphi)$ consists of finitely many affine lines.

From the set-up in the proof of Lemma 11, we see that the reducible characters in $X_\varphi(S)$ that are restrictions of reducible characters in $X(M_\varphi)$ are the same as characters of abelian representations ρ satisfying $\rho(\varphi_*(\gamma)) = \rho(\gamma)$ for all $\gamma \in \pi_1(S)$. Since the monodromy of M_φ is

$$\varphi_* = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in SL(2,\mathbb{Z}) \quad \text{with } |k_1 + k_4| > 2,$$

all possible pairs (s, p) of eigenvalues of $\rho(a)$ and $\rho(b)$ corresponding to a common invariant subspace are determined by

$$s = s^{k_1} p^{k_3}, \quad p = s^{k_2} p^{k_4}. \tag{3-3}$$

The (not necessarily distinct) components of $X^{\text{red}}(M_\varphi)$ corresponding to the finitely many solutions to these equations are given in our affine coordinates (3-1) by

$$X_{s,p} = \Big\{\big(s+s^{-1},\ p+p^{-1},\ sp+s^{-1}p^{-1},\ m+m^{-1},\ sm+s^{-1}m^{-1},\ pm+p^{-1}m^{-1},\ spm+s^{-1}p^{-1}m^{-1}\big) \ \Big|\ m \in \mathbb{C}^*\Big\}.$$

Here, we continue to use the convention

$$x = \tau_a(\rho) = s + s^{-1}, \quad y = \tau_b(\rho) = p + p^{-1} \quad \text{and} \quad u = \tau_t(\rho) = m + m^{-1}$$

from Section 3.1.

In particular, each component of $X^{\text{red}}(M_\varphi)$ has image a single point in $X_\varphi(S)$.

The proof of the following proposition is an application of the main result of [20] and generalises the example given in that paper by using fundamental results due to Fox [13]. We thank Michael Heusener and Joan Porti for encouraging correspondence, and refer the reader to [20] for the required definitions of twisted Alexander polynomials. The reader is especially encouraged to read [20, Examples 3.2, 4.4, 8.2] before going through the proof below.

Proposition 17 *For every component X_{s_0,p_0} of $X^{\text{red}}(M_\varphi)$, there exists $m_0 \in \mathbb{C}^*$ such that X_{s_0,p_0} intersects $X^{\text{irr}}(M_\varphi)$ at χ_{m_0}, where χ_{m_0} is the character of the abelian representation ρ_{m_0} with*

$$\rho_{m_0}(t) = \operatorname{Diag}(m_0, m_0^{-1}), \quad \rho_{m_0}(a) = \operatorname{Diag}(s_0, s_0^{-1}), \quad \rho_{m_0}(b) = \operatorname{Diag}(p_0, p_0^{-1}).$$

Moreover, χ_{m_0} is contained on exactly one curve C in $X^{\text{irr}}(M_\varphi)$, a smooth point of both X_{s_0,p_0} and C and the intersection at χ_{m_0} is transverse. Moreover, if $s_0, t_0 \in \{\pm 1\}$, then m_0 is the square root of a zero of the characteristic polynomial of φ_* and otherwise $m_0 = \pm 1$.

Proof Let $m = m_0^2$, $s = s_0^2$ and $p = p_0^2$. With this set up, the remainder of the proof applies verbatim to the $\mathrm{PSL}(2,\mathbb{C})$-character variety. We have a group homomorphism $\alpha\colon \pi_1(M_\varphi) \to \mathbb{C}^*$ with

$$\alpha(a) = s, \quad \alpha(b) = p, \quad \alpha(t) = m.$$

The pair (s, p) satisfies (3-3), that is, the restriction $\alpha|_{\mathrm{Tor}(M_\varphi,\mathbb{Z})}$ is induced by a group homomorphism $h : \langle a, b\rangle \to U(1)$ such that $h \circ \varphi = h$ and $h(a) = s$ and $h(b) = p$.

We apply the criterion of deformations of reducible characters in [20, Theorem 1.3]. Fix the canonical splitting $H_1(M_\varphi,\mathbb{Z}) \cong \mathbb{Z}\oplus \mathrm{Tor}(M_\varphi,\mathbb{Z}) \to \mathrm{Tor}(M_\varphi,\mathbb{Z})$ where $p(a)=a,\ p(b)=b$ and $p(t)=0$ and an element $\psi\in H^1(M_\varphi,\mathbb{Z})$ where $\psi(a)=\psi(b)=0$ and $\psi(t)=1$. To prove the statement, we only need to show that m is a simple zero of the twisted Alexander polynomial $\Delta^{\psi_\alpha}_{M_\varphi}$. A direct computation shows that the Jacobian matrix is

$$J_{M_\varphi}=\begin{pmatrix} t^{-1}(a^{-1}-1) & t^{-1}a^{-1}\Big(t\dfrac{\partial}{\partial a}\varphi(a)-1\Big) & t^{-1}a^{-1}t\dfrac{\partial}{\partial b}\varphi(a)\\ t^{-1}(b^{-1}-1) & t^{-1}b^{-1}t\dfrac{\partial}{\partial a}\varphi(b) & t^{-1}b^{-1}\Big(t\dfrac{\partial}{\partial b}\varphi(b)-1\Big)\end{pmatrix}.$$

Each $\varphi\in\mathrm{Aut}(\langle a,b\rangle)$ linearly extends to a ring homomorphism of $\mathbb{Z}\langle a,b\rangle$, also denoted by φ. Under the abelianisation homomorphism, it also induces a homomorphism $\varphi^{\mathrm{ab}}:\mathbb{Z}[a,b]\to\mathbb{Z}[a,b]$. The Jacobian of φ is

$$J_\varphi=\begin{pmatrix}\frac{\partial}{\partial a}\varphi(a) & \frac{\partial}{\partial b}\varphi(a)\\ \frac{\partial}{\partial a}\varphi(b) & \frac{\partial}{\partial b}\varphi(b)\end{pmatrix}.$$

Let J_φ^{ab} denotes its image under the abelianisation homomorphism. By the chain rule [13], $J_{\varphi_1\varphi_2}=\varphi_1(J_{\varphi_2})J_{\varphi_1}$ for any $\varphi_1,\varphi_2\in\mathrm{Aut}(F_2)$. Applying the abelianisation homomorphism and taking determinants on both sides, we have

$$\det J^{\mathrm{ab}}_{\varphi_1\varphi_2}=\det\varphi_1^{\mathrm{ab}}(J^{\mathrm{ab}}_{\varphi_2})\det J^{\mathrm{ab}}_{\varphi_1}=\varphi_1^{\mathrm{ab}}(\det J^{\mathrm{ab}}_{\varphi_2})\det J^{\mathrm{ab}}_{\varphi_1}.$$

We may assume that φ_* is given as in (2-8). Then $\det J^{\mathrm{ab}}_\alpha=\det J^{\mathrm{ab}}_{\beta^{-1}}=1$ implies that $\det J^{\mathrm{ab}}_\varphi=1$. Let J^h_φ denotes the image of J_φ under the map h. It is clear that $h(J^{\mathrm{ab}}_\varphi)=J^h_\varphi$. Hence $\det J^h_\varphi=1$.

By fundamental formula of free calculus [13], we have

$$\varphi(a)-1=\Big(\frac{\partial}{\partial a}\varphi(a)\Big)(a-1)+\Big(\frac{\partial}{\partial b}\varphi(a)\Big)(b-1).$$

Applying h on both sides, we have

$$\Big(h\Big(\frac{\partial}{\partial a}\varphi(a)\Big)-1\Big)(s-1)+h\Big(\frac{\partial}{\partial b}\varphi(a)\Big)(p-1)=0$$

since $h\circ\varphi=h$. By symmetry,

$$\Big(h\Big(\frac{\partial}{\partial b}\varphi(b)\Big)-1\Big)(p-1)+h\Big(\frac{\partial}{\partial a}\varphi(b)\Big)(s-1)=0.$$

Define $\psi_\alpha:\pi_1(M_\varphi)\to\mathbb{C}[x^{\pm1}]$, where $\psi_\alpha(\gamma)=\alpha(\gamma)x^{\psi(\gamma)}$. The Jacobian $J^{\psi_\alpha}_{M_\varphi}$ is given by $(J^{\psi_\alpha}_{M_\varphi})_{ij}=\psi_\alpha((J_{M_\varphi})_{ij})$. Since Alexander invariants are obtained from determinants, we may multiply each row by an unit and work with the following matrix:

$$\begin{pmatrix}1-s & h\Big(\dfrac{\partial\varphi(a)}{\partial a}\Big)x-1 & h\Big(\dfrac{\partial}{\partial b}\varphi(a)\Big)x\\ 1-p & h\Big(\dfrac{\partial\varphi(b)}{\partial a}\Big)x & h\Big(\dfrac{\partial}{\partial b}\varphi(b)\Big)x-1\end{pmatrix}.$$

We now have the following cases:

(1) If $h(a)=h(b)=1$, a direct computation shows that $J^h_\varphi=\varphi_*$. So $\Delta^{\psi_\alpha}_{M_\varphi}$ is the characteristic polynomial of φ_*, which has 2 distinct simple zeroes since the trace is distinct from ± 2. Hence m equals one of these roots.

(2) If $h(a)=1$ and $h(b)\neq 1$, from the above equations, we have

$$h\Big(\frac{\partial}{\partial b}\varphi(a)\Big)=0 \quad\text{and}\quad h\Big(\frac{\partial}{\partial b}\varphi(b)\Big)=1.$$

Combined with $\det J^h_\varphi=1$, we have $h\big(\frac{\partial}{\partial a}\varphi(a)\big)=1$ so $\Delta^{\psi_\alpha}_{M_\varphi}=x-1$. By symmetry, $\Delta^{\psi_\alpha}_{M_\varphi}=x-1$ if $h(a)\neq 1$ and $h(b)=1$. Hence in each of these cases, $m=1$.

(3) If $h(a)\neq 1$ and $h(b)\neq 1$, we have

$$\Big(h\Big(\frac{\partial}{\partial a}\varphi(a)\Big)-1\Big)\Big(h\Big(\frac{\partial}{\partial b}\varphi(b)\Big)-1\Big)=h\Big(\frac{\partial}{\partial b}\varphi(a)\Big)h\Big(\frac{\partial}{\partial a}\varphi(b)\Big)$$

by multiplying the two equations. Rearranging the above equation and substitute $\det J^h_\varphi=1$ into it, we have

$$h\Big(\frac{\partial}{\partial a}\varphi(a)\Big)+h\Big(\frac{\partial}{\partial b}\varphi(b)\Big)=2.$$

Then

$$h\Big(\frac{\partial}{\partial b}\varphi(b)\Big)-1=1-h\Big(\frac{\partial}{\partial a}\varphi(a)\Big)$$

and we can rewrite the second equation as

$$\Big(h\Big(\frac{\partial}{\partial a}\varphi(a)\Big)-1\Big)(p-1)=h\Big(\frac{\partial}{\partial a}\varphi(b)\Big)(s-1).$$

This gives us that the determinant of the first two columns of $J^{\psi_\alpha}_{M_\varphi}$ is $(1-p)(x-1)$. We can perform similar computation for the other two determinants which both have a factor $x-1$. This implies $\Delta^{\psi_\alpha}_{M_\varphi}=x-1$ and $m=1$. □

A direct consequence of the above proposition is that $r(X(M_\varphi))=r(X^{\mathrm{irr}}(M_\varphi))$. We now show that there are no ramification points of $r\colon X^{\mathrm{irr}}(M_\varphi)\to X_\varphi(S)$ at reducible characters contained on $X^{\mathrm{irr}}(M_\varphi)$. Consider the action of $H^1(M_\varphi,\mathbb{Z}_2)$ on $X^{\mathrm{red}}(M_\varphi)$. We first characterise the points in $X^{\mathrm{red}}(M_\varphi)$ with nontrivial stabilisers under this action. Suppose $h\in H^1(M_\varphi,\mathbb{Z}_2)$ with $h(t)=\epsilon_t,\ h(a)=\epsilon_a,\ h(b)=\epsilon_b$. Then a direct calculation shows that $S(h)\cap X^{\mathrm{red}}(M_\varphi)$ consists of all points in $X^{\mathrm{red}}(M_\varphi)$ satisfying

$$m^2=\epsilon_t,\quad s^2=\epsilon_a,\quad p^2=\epsilon_b.$$

In particular, the trace functions take values in $\{-2,0,2\}$.

We now focus on $S(\varepsilon)$. Recall that $\varepsilon(t)=-1,\ \varepsilon(a)=\varepsilon(b)=1$. In this case,

$$m^2=-1,\quad s^2=1,\quad p^2=1$$

and hence $s,p\in\{\pm 1\}$. If such a character is in the intersection of $X^{\mathrm{red}}(M_\varphi)\cap X^{\mathrm{irr}}(M_\varphi)$, then Lemma 5 implies that there is such a character of a nonabelian reducible representation ρ. Up to conjugacy and the action of ε on $R(M_\varphi)$, we may assume that

$$\rho(t)=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \rho(a)=\begin{pmatrix} s & u\\ 0 & s\end{pmatrix},\quad \rho(b)=\begin{pmatrix} p & v\\ 0 & p\end{pmatrix},$$

where $s,p\in\{\pm 1\}$ and $(u,v)\neq(0,0)$. Then

$$\begin{pmatrix} s & -u\\ 0 & s\end{pmatrix}=\rho(t^{-1}at)=\rho(\varphi(a))=\begin{pmatrix} s^{k_1}\epsilon_b^{k_3} & k_3s^{k_1}p^{k_3-1}v+k_1s^{k_1-1}p^{k_3}u\\ 0 & s^{k_1}p^{k_3}\end{pmatrix}$$

and

$$\begin{pmatrix} p & -v\\ 0 & p\end{pmatrix}=\rho(t^{-1}bt)=\rho(\varphi(b))=\begin{pmatrix} s^{k_2}p^{k_4} & k_4s^{k_2}p^{k_4-1}v+k_2s^{k_2-1}p^{k_4}u\\ 0 & s^{k_2}p^{k_4}\end{pmatrix}.$$

This implies in particular the equations $u=-sp(k_3v+k_1u)$ and $v=-sp(k_4v+k_2u)$. Now $sp=\pm 1$, and hence these two equations give

$$\text{transpose}[\varphi_*]\begin{pmatrix} u\\ v\end{pmatrix}=\begin{pmatrix} k_1 & k_3\\ k_2 & k_4\end{pmatrix}\begin{pmatrix} u\\ v\end{pmatrix}=\pm\begin{pmatrix} u\\ v\end{pmatrix}\neq\begin{pmatrix} 0\\ 0\end{pmatrix}.$$

But then φ_* has eigenvalue ± 1 and hence $\operatorname{tr}(\varphi_*)=\pm 2$, which is a contradiction. This shows that there are no ramification points of $r\colon X^{\text{irr}}(M_\varphi)\to X_\varphi(S)$ at reducible characters contained on $X^{\text{irr}}(M_\varphi)$.

3.5 Irreducible characters

The following lemma proves that every irreducible character in $X_\varphi(S)$ is the restriction of two distinct irreducible characters in $X(M_\varphi)$, and hence is not a branch point of $r\colon X(M_\varphi)\to X_\varphi(S)$. This also implies, via Proposition 17, that the degree of r is two on a Zariski dense subset. Moreover, Example 16 gives examples where these distinct characters lie in different Zariski components of $X(M_\varphi)$ and where they lie in the same Zariski component.

Lemma 18 *For any irreducible character χ_ρ of S such that $\chi_\rho\in X_\varphi(S)$, there exists $T\in \mathrm{SL}(2,\mathbb{C})$ such that $T^{-1}\rho(\gamma)T=\rho(\varphi_*(\gamma))$ for all $\gamma\in\pi_1S$. Moreover, T is unique up to sign and at least one of $\operatorname{tr}T$, $\operatorname{tr}(\rho(a)T)$, $\operatorname{tr}(\rho(b)T)$ and $\operatorname{tr}(\rho(ab)T)$ is nonzero.*

Proof Let $\rho\in R(S)$ be an irreducible representation with $\chi_\rho\in X_\varphi(S)$. We define $\rho_1\in R(S)$ by $\rho_1(\gamma)=\rho(\varphi_*(\gamma))$ for $\gamma\in\pi_1(S)$. According to the definition of $X_\varphi(S)$, $\chi_\rho=\chi_{\rho_1}$. Since χ_ρ is the character of an irreducible representation, it follows that ρ and ρ_1 are conjugate. Hence there exists $T\in\mathrm{SL}(2,\mathbb{C})$ such that $T^{-1}\rho T=\rho_1$. Since χ is irreducible, the centraliser of ρ is $\pm E$, where E is the identity matrix. Hence T is unique up to sign. This proves the first part of the lemma, and it remains to show that at least one of the stated traces is nonzero.

Up to conjugation, we may assume that

$$\rho(a)=\begin{pmatrix} s & 0\\ 1 & s^{-1}\end{pmatrix},\quad \rho(b)=\begin{pmatrix} p & u\\ 0 & p^{-1}\end{pmatrix},$$

where $u \neq 0$. Since $\chi_\rho \in X_\varphi(S)$, we have

$$\rho(\varphi_*(a)) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s+s^{-1}-s_1 \end{pmatrix}, \quad \rho(\varphi_*(b)) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p+p^{-1}-p_1 \end{pmatrix}$$

for some $s_1, s_2, s_3, p_1, p_2, p_3 \in \mathbb{C}$. Suppose $T = \left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$ for some $m_1, m_1, m_3, m_4 \in \mathbb{C}$, and that $\operatorname{tr} T = \operatorname{tr}(\rho(a)T) = \operatorname{tr}(\rho(b)T) = \operatorname{tr}(\rho(ab)T) = 0$ holds. Then, we have

$$\begin{aligned} 0 &= \operatorname{tr} T = m_1 + m_4, \\ 0 &= \operatorname{tr}(\rho(a)T) = sm_1 + m_2 + s^{-1}m_4, \\ 0 &= \operatorname{tr}(\rho(b)T) = pm_1 + um_3 + p^{-1}m_4, \\ 0 &= \operatorname{tr}(\rho(ab)T) = spm_1 + pm_2 + usm_3 + (u + s^{-1}p^{-1})m_4. \end{aligned}$$

Note that the equations do not change if one replaces T with $-T$. From the first three equations, we have $m_4 = -m_1$, $m_2 = (s^{-1} - s)m_1$ and $m_3 = \frac{1}{u}(p^{-1} - p)m_1$, which gives

$$T = m_1 \begin{pmatrix} 1 & s^{-1} - s \\ \frac{p^{-1}-p}{u} & -1 \end{pmatrix}.$$

Substituting m_2, m_3, m_4 into the fourth equation, we have

$$(u + sp + s^{-1}p^{-1} - s^{-1}p - sp^{-1})m_1 = 0.$$

However, this implies the contradiction

$$\det T = -\frac{(u + sp + s^{-1}p^{-1} - s^{-1}p - sp^{-1})m_1}{u} = 0.$$

Hence at least one of $\operatorname{tr} T$, $\operatorname{tr}(\rho(a)T)$, $\operatorname{tr}(\rho(b)T)$ and $\operatorname{tr}(\rho(ab)T)$ is nonzero. □

Proposition 19 *The fibres of $r\colon X^{\mathrm{irr}}(M_\varphi) \to X_\varphi(S)$ are the orbits of ε and the branch points for r are contained in the set of reducible characters in $X_\varphi(S)$. Moreover, the ramification points are simple points of $X^{\mathrm{irr}}(M_\varphi)$ and are precisely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$.*

Proof The action of ε with respect to the affine coordinates (x, y, z, u, v, w, q) in $\mathbb{C}^7$ is given by the the involution $(x, y, z, u, v, w, q) \to (x, y, z, -u, -v, -w, -q)$. It was shown in Lemma 18 that the preimage under r of each irreducible character in $X_\varphi(S)$ is the orbit under this involution and has precisely two elements. It follows by continuity that each fixed point of ε is a ramification point for r. Hence the branch points for r are contained in the set of reducible characters in $X_\varphi(S)$.

It was shown in Section 3.4 that there are no ramification points of r in $X^{\mathrm{red}}(M_\varphi) \cap X^{\mathrm{irr}}(M_\varphi)$. Hence all ramification points correspond to irreducible characters in $X^{\mathrm{irr}}(M_\varphi)$ that restrict to reducible characters in $X_\varphi(S)$ and are fixed by ε. It was noted in Scholion 12 that the irreducible characters $X^{\mathrm{irr}}(M_\varphi)$ that restrict to reducible characters in $X_\varphi(S)$ are precisely the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$, and it is an elementary calculation to verify that they satisfy $u = v = w = q = 0$.

It is shown in [4, Proposition 5.3] that these binary dihedral characters are simple points of $X(M_\varphi)$. □

Proof of Theorem 1 First note that Proposition 17 implies that $r(X(M_\varphi)) = r(X^{\mathrm{irr}}(M_\varphi))$. The main statement of Theorem 1 is now the content of Propositions 14 and 19. The additional information about the number of branch points follows from [4], but the formula is different. We now justify our formulation. The number of binary dihedral characters in $X(M_\varphi)$ is given by [4, Propositions 5.3 and 4.5] as

$$\tfrac{1}{2}\big(|2+\mathrm{tr}(\varphi_*)| - \zeta_{\varphi_*}\big),$$

where $\zeta_{\varphi_*} \in \{1,2,4\}$ is the order of $\mathrm{Hom}(\,\mathrm{coker}(\varphi_* + 1_{H_1(S^1\times S^1)}),\ \mathbb{Z}_2)$. Here, the once-punctured torus is identified with the complement of a point in $S^1\times S^1$. A direct calculation shows that

$$\begin{aligned}\zeta_{\varphi_*} = 4 &\iff o(\varphi_2) = 1 \iff b_1(\varphi) = 3,\\ \zeta_{\varphi_*} = 2 &\iff o(\varphi_2) = 2 \iff b_1(\varphi) = 2,\\ \zeta_{\varphi_*} = 1 &\iff o(\varphi_2) = 3 \iff b_1(\varphi) = 1.\end{aligned}$$

Hence the identity $\zeta_{\varphi_*} = 2^{3-o(\varphi_2)} = 2^{b_1(\varphi)-1}$ gives

$$\tfrac{1}{2}\big(|2+\mathrm{tr}(\varphi_*)| - \zeta_{\varphi_*}\big) = \tfrac{1}{2}|2+\mathrm{tr}(\varphi_*)| - 2^{b_1(\varphi)-2}$$

as claimed in (1-1). □

The relationship between Euler characteristic and genus of curves gives the following.

Corollary 20 *Suppose $X_\varphi(S)$ and $X^{\mathrm{irr}}(M_\varphi)$ are irreducible, and that $X_\varphi(S)$ is a nonsingular affine curve. Let g_0 and g_1 denote the genera of the smooth projective models of $X^{\mathrm{irr}}(M_\varphi)$ and $X_\varphi(S)$, respectively. Then*

$$g_0 = 2g_1 - 1 + \frac{e_1 + e_\infty}{2},$$

where $e_1 = \frac{1}{2}|2+\mathrm{tr}(\varphi_*)| - 2^{b_1(\varphi)-2}$ is the number of branch points in $X_\varphi(S)$ and e_∞ is number of branch points at ideal points of $X_\varphi(S)$. In particular, if i_∞ is the total number of ideal points of $X_\varphi(S)$, then g_0 is bounded by

$$2g_1 - 1 + \frac{e_1}{2} \le g_0 \le 2g_1 - 1 + \frac{e_1}{2} + \frac{i_\infty}{2}.$$

4 Projective characters

Following the blueprint of Section 3, we now analyse the $\mathrm{PSL}(2,\mathbb{C})$-character varieties of once-punctured torus bundles. We make frequent use of Heusener and Porti [19] and refer the reader to original sources therein.

4.1 The restriction map

Let $\overline{X}_\varphi(S) = \{\chi_{\bar\rho} \in \overline{X}(S) \mid \chi_{\bar\rho} = \chi_{\bar\rho\varphi}\}$. We first show that the natural restriction map $\bar r : \overline{X}(M_\varphi) \to \overline{X}(S)$ again satisfies $\mathrm{im}(\bar r) = \overline{X}_\varphi(S)$. Recall that we have the four-fold branched covering map $X(S) \to \overline{X}(S)$ from Section 2.5,

$$X(S) \ni (x,y,z) \mapsto (x^2, y^2, z^2, xyz) \in \overline{X}(S). \tag{4-1}$$

Let $\bar{\rho}\colon \pi_1(M_\varphi) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. Let $T, A, B \in \mathrm{SL}(2,\mathbb{C})$ satisfying $\bar{\rho}(t) = \pm T$, $\bar{\rho}(a) = \pm A$, $\bar{\rho}(b) = \pm B$. To account for all possible choices of sign, we write $\varphi(a) = \varphi_a(a,b)$ for a fixed word given for $\varphi(a)$ in terms of the generators. Similarly for b. Then

$$T^{-1}AT = \varepsilon_a(A,B)\varphi_a(A,B), \tag{4-2}$$

$$T^{-1}BT = \varepsilon_b(A,B)\varphi_b(A,B), \tag{4-3}$$

where $\varepsilon_a(A,B), \varepsilon_b(A,B) \in \{\pm 1\}$. Then

$$T^{-1}ABT = \varepsilon_a(A,B)\varphi_a(A,B)\cdot\varepsilon_b(A,B)\varphi_b(A,B).$$

In particular, the product of the signs in the last coordinate of (4-1) equals one and hence $\mathrm{im}(\bar{r}) \subseteq \overline{X}_\varphi(S)$.

Conversely, let $\bar{\rho}\colon \pi_1(S) \to \pi_1(S)$ such that $\chi_{\bar{\rho}} \in \overline{X}_\varphi(S)$. Suppose $\bar{\rho}(a) = \pm A$ and $\bar{\rho}(b) = \pm B$, where $A, B \in \mathrm{SL}(2,\mathbb{C})$. Since $\chi_{\bar{\rho}} = \chi_{\bar{\rho}\varphi}$, the first three coordinates of the corresponding points in $\overline{X}(S)$ (with respect to the parametrisation in (4-1)) imply that there are $\varepsilon_a, \varepsilon_b, \varepsilon_{ab} \in \{\pm 1\}$ such that

$$\operatorname{tr} A = \varepsilon_a \operatorname{tr}(\varphi_a(A,B)), \quad \operatorname{tr} B = \varepsilon_b \operatorname{tr}(\varphi_b(A,B)), \quad \operatorname{tr} AB = \varepsilon_{ab} \operatorname{tr}(\varphi_a(A,B)\varphi_b(A,B)).$$

Now the fourth coordinate implies that $\varepsilon_a\varepsilon_b\varepsilon_{ab} = 1$ and hence $\varepsilon_{ab} = \varepsilon_a\varepsilon_b$. Hence

$$\operatorname{tr} A = \operatorname{tr}(\varepsilon_a\varphi_a(A,B)), \quad \operatorname{tr} B = \operatorname{tr}(\varepsilon_b\varphi_b(A,B)), \quad \operatorname{tr} AB = \operatorname{tr}(\varepsilon_a\varphi_a(A,B)\cdot\varepsilon_b\varphi_b(A,B)).$$

As in the proof of Lemma 11 there is $T \in \mathrm{SL}(2,\mathbb{C})$ with

$$T^{-1}AT = \varepsilon_a\varphi_a(A,B) \quad \text{and} \quad T^{-1}BT = \varepsilon_b\varphi_b(A,B).$$

This shows that $\chi_{\bar{\rho}} \in \mathrm{im}(\bar{r})$. Hence $\mathrm{im}(\bar{r}) = \overline{X}_\varphi(S)$ as claimed.

It follows as in the proof of Lemma 18 that if $\bar{\rho}\colon \pi_1(M_\varphi) \to \mathrm{PSL}(2,\mathbb{C})$ is irreducible, then

$$\big((\operatorname{tr}\bar{\rho}(t))^2,\ (\operatorname{tr}\bar{\rho}(ta))^2,\ (\operatorname{tr}\bar{\rho}(tb))^2,\ (\operatorname{tr}\bar{\rho}(tab))^2\big) \neq (0,0,0,0). \tag{4-4}$$

As in Section 3.4, it is easy to show that there are finitely many characters in $\overline{X}_\varphi(S)$ that arise from reducible representations of $\pi_1(M_\varphi)$. The proof of Proposition 17 shows that $\bar{r}\colon \overline{X}^{\mathrm{irr}}(M_\varphi) \to \overline{X}_\varphi(S)$ is surjective.

The proof of Corollary 3 and the statement of Proposition 14 now give the following:

Proposition 21 *Suppose M_φ is a hyperbolic once-punctured torus bundle with $b_1(\varphi) = 1$. Then we have the following commutative diagram of surjective maps whose degrees are indicated in the diagram*:

$$\begin{array}{ccccccc}
X^{\mathrm{irr}}(M_\varphi) & \xrightarrow[2:1]{r} & X_\varphi(S) & \subset & X(S) & \cong & \mathbb{C}^3 \\
{\scriptstyle q_1}\downarrow{\scriptstyle 2:1} & & {\scriptstyle q_2}\downarrow{\scriptstyle 1:1} & & {\scriptstyle q_2}\downarrow{\scriptstyle 4:1} & & \\
\overline{X}^{\mathrm{irr}}(M_\varphi) & \xrightarrow[1:1]{\bar{r}} & \overline{X}_\varphi(S) & \subset & \overline{X}(S) & &
\end{array}$$

In particular, every component of $\overline{X}^{\mathrm{irr}}(M_\varphi)$ is one-dimensional.

For the bundles with $b_1(\varphi) > 1$, we have the following to consider. Let $\bar\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation with $\chi_{\bar\rho} \in \overline{X}_\varphi(S)$. We call the characters in $\bar r^{-1}(\chi_{\bar\rho})$ *extensions* of $\chi_{\bar\rho}$, and similar for representations. Any two extensions of $\bar\rho$ to $\pi_1(M_\varphi)$ differ by an element in the centraliser $C(\mathrm{im}(\bar\rho))$. It is well-known (see [19]) that if $\bar\rho$ is irreducible, then the centraliser $C(\mathrm{im}(\bar\rho))$ is either trivial, or cyclic of order two, or isomorphic to the Klein four group. This is unlike the situation for $\mathrm{SL}(2,\mathbb{C})$, where the centraliser of an irreducible representation is always central. In terms of our coordinate system for $\overline{X}(S)$ these possibilities are given in Section 4.2. We start with the following preliminary observation; the group $H \le H^1(M_\varphi, \mathbb{Z}_2)$ in the statement was defined in Section 2.3 as those homomorphisms $h\colon \pi_1(M_\varphi) \to \mathbb{Z}_2$ satisfying $h(t) = 1$.

Lemma 22 *Suppose $\chi_\rho, \chi_\sigma \in X_\varphi(S)$ are in the same $H^1(S,\mathbb{Z}_2)$-orbit, and that χ_ρ is irreducible. If χ_ρ and χ_σ are in the same H-orbit, then $q_1(r^{-1}(\chi_\rho)) = q_1(r^{-1}(\chi_\sigma))$. If they are in distinct H-orbits, then $q_1(r^{-1}(\chi_\rho)) \neq q_1(r^{-1}(\chi_\sigma))$. Moreover, $q_1(r^{-1}(\chi_\rho))$ and $q_1(r^{-1}(\chi_\sigma))$ are the characters of representations into $\mathrm{PSL}(2,\mathbb{C})$ that agree on $\pi_1(S)$ and whose image of t differs by a nontrivial element in the centraliser of the image of $\pi_1(S)$.*

Proof Let $\rho\colon \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$ be a representation with character χ_ρ. Then there is $h \in H^1(S,\mathbb{Z}_2)$ such that the character of $\sigma(\gamma) = h(\gamma)\rho(\gamma)$ is χ_σ.

Since $\chi_\rho \in X_\varphi(S)$, there is $T \in \mathrm{SL}(2,\mathbb{C})$ such that $T^{-1}\rho(\gamma)T = \rho(\varphi(\gamma))$ for all $\gamma \in \pi_1(S)$. In particular, ρ extends to a representation of $\pi_1(M_\varphi)$ into $\mathrm{SL}(2,\mathbb{C})$ by letting $\rho(t) = T$ or $\rho(t) = -T$ and these are the only extensions according to Lemma 18. Note that these extensions are in the same $\langle\varepsilon\rangle$-orbit.

If $h \in H$, then $h(\gamma) = h(\varphi(\gamma))$ for all $\gamma \in \pi_1(S)$. Hence

$$T^{-1}\sigma(\gamma)T = T^{-1}h(\gamma)\rho(\gamma)T = h(\gamma)T^{-1}\rho(\gamma)T = h(\gamma)\rho(\varphi(\gamma)) = h(\varphi(\gamma))\rho(\varphi(\gamma)) = \sigma(\varphi(\gamma))$$

for all $\gamma \in \pi_1(S)$. So as above, σ extends to a representation of $\pi_1(M_\varphi)$ by letting $\sigma(t) = T$ or $\sigma(t) = -T$. Hence the extensions of ρ are in the same $H^1(M_\varphi,\mathbb{Z}_2)$-orbit as the extensions of σ. This implies $q_1(r^{-1}(\chi_\rho)) = q_1(r^{-1}(\chi_\sigma))$.

Suppose there is no $h \in H^1(S,\mathbb{Z}_2)$ such that the character of $\sigma(\gamma) = h(\gamma)\rho(\gamma)$ is χ_σ. Then $q_1(r^{-1}(\chi_\rho)) \neq q_1(r^{-1}(\chi_\sigma))$. Since $q_2(\chi_\rho) = q_2(\chi_\sigma)$, we have $\bar r\big(q_1(r^{-1}(\chi_\rho))\big) = q_2(\chi_\rho) = q_2(\chi_\sigma) = \bar r\big(q_1(r^{-1}(\chi_\sigma))\big)$. So $q_1(r^{-1}(\chi_\rho))$ and $q_1(r^{-1}(\chi_\sigma))$ are the characters of nonconjugate representations that agree on $\pi_1(S)$. Hence the conclusion. □

4.2 The origin and three coordinate axes (revisited)

Let $\bar\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be an irreducible representation. We have $C(\mathrm{im}(\bar\rho)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if and only if $\chi_{\bar\rho} = (0,0,0,0) \in \overline{X}(S)$. For every φ, we have $(0,0,0) \in X_\varphi(S)$ and hence $(0,0,0,0) \in \overline{X}_\varphi(S)$. Up to conjugation, we may assume

$$\bar\rho(a) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \kappa_1, \quad \bar\rho(b) = \pm\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \kappa_2, \quad \bar\rho(ab) = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \kappa_3. \tag{4-5}$$

These Möbius transformations represent rotations by π with respective axes $[-i,i]$, $[-1,1]$ and $[0,\infty]$, and we have $C(\operatorname{im}(\bar\rho)) = \operatorname{im}(\bar\rho)$.

Lemma 23 *If $\bar\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is irreducible and $|C(\operatorname{im}(\bar\rho))| = 4$, then*

$$|\bar r^{-1}(\chi_{\bar\rho})| = 2^{b_1(\varphi)-1}.$$

Moreover, exactly one of the characters in $\bar r^{-1}(\chi_{\bar\rho})$ lifts to $X(M_\varphi)$. For each representation of $\pi_1(M_\varphi)$ with character in $\bar r^{-1}(\chi_{\bar\rho})$, the image of the longitude is always trivial; the image of the meridian has order two or four if $b_1(\varphi) = 2$ and it has order two or is trivial if $b_1(\varphi) = 3$.

Proof Since $(0,0,0) \in X_\varphi(S)$ is the unique preimage of $\chi_{\bar\rho}$ in $X_\varphi(S)$, it follows that any two lifts to $\mathrm{SL}(2,\mathbb{C})$ of the Klein four group are conjugate in $\mathrm{SL}(2,\mathbb{C})$. Hence the uniqueness up to sign in Lemma 18 implies that exactly one of the characters in $\bar r^{-1}(\chi_{\bar\rho})$ lifts to $X(M_\varphi)$.

Note that the image of the longitude is the commutator of κ_1 and κ_2 and hence trivial.

If $b_1(\varphi) = 3$, then we have $\varphi(\bar\rho(a)) = \bar\rho(a)$ and $\varphi(\bar\rho(b)) = \bar\rho(b)$ and hence $\bar\rho(t) \in C(\operatorname{im}(\bar\rho))$. This implies that $\bar\rho$ extends to $\pi_1(M_\varphi)$ with $\bar\rho(t)$ any element in the centraliser. The values taken in (4-4) are $(4,0,0,0)$, $(0,4,0,0)$, $(0,0,4,0)$, $(0,0,0,4)$. Hence, no two of these four extensions are conjugate, but all have image equal to $\operatorname{im}(\bar\rho)$. Example 16 shows that the character that lifts depends on φ. The four extensions are characterised by $\bar\rho(t) = 1$, $\bar\rho(t) = \bar\rho(a)$, $\bar\rho(t) = \bar\rho(b)$, and $\bar\rho(t) = \bar\rho(ab)$, respectively. This proves the claim about the order. We call these representations the *four extensions of the Klein four group* and denote them respectively by $\bar\rho_0$, $\bar\rho_1$, $\bar\rho_2$, $\bar\rho_3$ and their characters by $\bar\chi_0$, $\bar\chi_1$, $\bar\chi_2$, $\bar\chi_3$. Note that there is a natural action of the Klein four group on these characters defined by $\kappa_i \cdot \bar\chi_0 = \bar\chi_i$.

If $b_1(\varphi) = 2$, then φ interchanges two of $\bar\rho(a)$, $\bar\rho(b)$, $\bar\rho(ab)$ and fixes the third. Without loss of generality, assume $\varphi(\bar\rho(a)) = \bar\rho(b)$. Then a direct calculation shows that $\bar\rho$ extends to a representation of $\pi_1(M_\varphi)$ by letting

$$\bar\rho(t) = \pm\begin{pmatrix} 0 & \sqrt{i} \\ i\sqrt{i} & 0 \end{pmatrix}, \quad \bar\rho(a) = \kappa_1, \quad \bar\rho(b) = \kappa_2. \tag{4-6}$$

The remaining extensions are obtained from this by twisting by the elements κ_i. The above extension is conjugate to the twisted representation $t \mapsto \kappa_3\bar\rho(t)$ via κ_1, and their values taken in (4-4) are $(0,2,2,0)$. Similarly, the two representations obtained by twisting $\bar\rho$ by κ_1 or κ_2 are conjugate via κ_1, and their values taken in (4-4) are $(2,0,0,2)$; for instance

$$t \mapsto \kappa_1\bar\rho(t) = \pm\begin{pmatrix} i\sqrt{i} & 0 \\ 0 & -\sqrt{i} \end{pmatrix}.$$

Hence we obtain two nonconjugate extensions and the order of the meridian is as claimed.

If $b_1(\varphi) = 1$, then any extension lifts to $\mathrm{SL}(2,\mathbb{C})$ and hence any two extensions are conjugate. □

Let $\bar\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be an irreducible representation. We have $C(\mathrm{im}(\bar\rho)) \cong \mathbb{Z}_2$ if and only if $0 \neq \chi_{\bar\rho} \in \bar L_i$, where

$$\bar L_1 = \{(x^2,0,0,0) \mid x \in \mathbb{C}\} \subset \bar X(S),$$
$$\bar L_2 = \{(0,y^2,0,0) \mid y \in \mathbb{C}\} \subset \bar X(S),$$
$$\bar L_3 = \{(0,0,z^2,0) \mid z \in \mathbb{C}\} \subset \bar X(S)$$

are the images of the coordinate axes in $X(S)$. The points $(4,0,0,0)$, $(0,4,0,0)$ and $(0,0,4,0)$ are the only reducible characters on these lines. In $\bar R(S)$ we may choose for each $\bar L_i$ a curve of representations that have constant centraliser, and such that the curves meet in the representation given in (4-5).

Write $(4p^2,0,0,0) = (x,0,0,0) \in \bar L_1$ and let $q \in \mathbb{C}$ such that $p^2+q^2=1$. We may then choose

$$\bar\rho_1(a) = \begin{pmatrix} q & -p \\ p & q \end{pmatrix}\kappa_1, \quad \bar\rho_1(b) = \kappa_2, \quad \text{and hence} \quad C(\mathrm{im}(\bar\rho_1)) = \langle \kappa_1 \rangle. \tag{4-7}$$

Write $(0,-4p^2,0,0) = (0,y,0,0) \in \bar L_2$ and let $q \in \mathbb{C}$ such that $p^2-q^2=1$. We may then choose

$$\bar\rho_2(a) = \kappa_1, \quad \bar\rho_2(b) = \kappa_2\begin{pmatrix} q & p \\ p & q \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar\rho_3)) = \langle \kappa_2 \rangle. \tag{4-8}$$

Let $(0,0,(p+p^{-1})^2,0) = (0,0,z,0) \in \bar L_3$. We may choose

$$\bar\rho_3(a) = \kappa_1, \quad \bar\rho_3(b) = \kappa_2\begin{pmatrix} ip & 0 \\ 0 & -ip^{-1} \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar\rho_3)) = \langle \kappa_3 \rangle. \tag{4-9}$$

Lemma 24 *Suppose $\bar\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is irreducible. If $\chi_{\bar\rho} \in \mathrm{im}(\bar r)$ and $|C(\mathrm{im}(\bar\rho))| = 2$, then $|\bar r^{-1}(\chi_{\bar\rho})| = 2$ and $b_1(\varphi) > 1$. Moreover, either no character in $\bar r^{-1}(\chi_{\bar\rho})$ lifts to $X(M_\varphi)$; or exactly one lifts and $b_1(\varphi) = 3$; or two of them lift and $b_1(\varphi) = 2$.*

Proof We have $0 \neq \chi_{\bar\rho} \in \bar L_i$. Without loss of generality, we may assume $\bar L_i = \bar L_3$, and $\chi_{\bar\rho} = (0,0,z_0^2,0)$. Hence

$$(\operatorname{tr}\bar\rho(a))^2 = (\operatorname{tr}\bar\rho(b))^2 = 0 \neq (\operatorname{tr}\bar\rho(ab))^2$$

In L_3, this character has two pre-images, $\chi_+ = (0,0,z_0)$ and $\chi_- = (0,0,-z_0)$. Recall the action of $\bar\varphi$ on the coordinate axes. If $b_1(\varphi) = 1$, then the three axes are in the same orbit under $\bar\varphi$. Hence $z_0 \neq 0$, implies $b_1(\varphi) > 1$.

We next show that $|\bar r^{-1}(\chi_{\bar\rho})| = 2$. Up to conjugation, we have

$$\bar\rho(a) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar\rho(b) = \pm\begin{pmatrix} 0 & p \\ -p^{-1} & 0 \end{pmatrix}, \quad \text{and hence} \quad C(\mathrm{im}(\bar\rho)) = \left\langle \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle$$

where $z_0 = -(p+p^{-1}) \notin \{-2,0,2\}$ since $\bar\rho$ is reducible if and only if $z_0 = \pm 2$, and the centraliser has order four if and only if $z_0 = 0$. These two cases correspond to the characters $(x^2,y^2,z^2,xyz) = (0,0,4,0)$ and $(0,0,0,0)$, respectively, with respect to (2-14). Assume that $\bar\rho$ is the restriction of a representation of M_φ with $\bar\rho(t) = \pm T$ for some

$$T = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

We know that $|\bar{r}^{-1}(\chi_{\bar{\rho}})| \le |C(\mathrm{im}(\bar{\rho}))| = 2$. Suppose that $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 1$. Then the two possible extensions of the representation of S to M_φ defined by $\bar{\rho}(t) = \pm T$ or $\bar{\rho}(t) = \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)T$ are conjugate. In particular, their values taken in (4-4) are identical. This gives a system of 4 equations. A direct computation shows that $z_0 \notin \{-2, 0, 2\}$ implies that $m_1 = m_2 = m_3 = m_4 = 0$. This is a contradiction. Hence $|\bar{r}^{-1}(\chi_{\bar{\rho}})| = 2$.

If $b_1(\varphi) = 2$, then $\overline{\varphi}$ stabilises one axis and permutes the other two axes. Since $z_0 \neq 0$, it preserves L_3. We therefore have either $\overline{\varphi}(0,0,z) = (0,0,z)$ for each $z \in \mathbb{C}$ or $\overline{\varphi}(0,0,z) = (0,0,-z)$ for each $z \in \mathbb{C}$. In the second case, $\chi_{\bar{\rho}}$ does not have a preimage in $X_\varphi(S)$ and hence no character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts to $X(M_\varphi)$. In the first case, the points $(0,0,z_0)$ and $(0,0,-z_0)$ are both in $X_\varphi(S)$. Since $\overline{\varphi}$ stabilises L_3 and permutes L_1 and L_2, the group H is generated by $a \mapsto -1$, $b \mapsto -1$ and hence the points $(0,0,\pm z_0)$ are fixed by H and thus in distinct H-orbits. Hence their extensions map to the two distinct characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$.

If $b_1(\varphi) = 3$, we also have either $\overline{\varphi}(0,0,z) = (0,0,z)$ for each $z \in \mathbb{C}$ or $\overline{\varphi}(0,0,z) = (0,0,-z)$ for each $z \in \mathbb{C}$. As above, in the second case, neither character in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts. In the first case, the points $(0,0,z_0)$ and $(0,0,-z_0)$ are in $X_\varphi(S)$ and in the same H-orbit since $H = H^1(\pi_1(S), \mathbb{Z}_2)$. According to Lemma 22, the extensions of these characters are in the same $H^1(\pi_1(M), \mathbb{Z}_2)$-orbit and hence have the same image in $\bar{r}^{-1}(\chi_{\bar{\rho}})$. So exactly one of the characters in $\bar{r}^{-1}(\chi_{\bar{\rho}})$ lifts and the other does not lift. □

The proof of Lemma 24 supplied the missing details for the following:

Corollary 25 *Suppose M_φ is a hyperbolic once-punctured torus bundle. We have the following commutative diagram of maps whose degrees are indicated in the diagram*:

$$\begin{array}{ccccccc}
X^{\mathrm{irr}}(M_\varphi) & \xrightarrow[2:1]{r} & X_\varphi(S) & \subset & X(S) & \cong & \mathbb{C}^3 \\
{\scriptstyle q_1}\Big\downarrow {\scriptstyle 2^{b_1(\varphi)}:1} & & {\scriptstyle q_2}\Big\downarrow {\scriptstyle 2^{b_1(\varphi)-1}:1} & & {\scriptstyle q_2}\Big\downarrow {\scriptstyle 4:1} & & \\
\overline{X}^{\mathrm{irr}}(M_\varphi) & \xrightarrow{\bar{r}} & \overline{X}_\varphi(S) & \subset & \overline{X}(S) & &
\end{array}$$

Here, each of the maps r, q_1, q_2 is a branched covering map onto its image. The map q_1 is the quotient map associated with the action of $H^1(M_\varphi, \mathbb{Z}_2) = \langle\varepsilon\rangle \oplus H$, and the map q_2 is the quotient map associated with the action of $H^1(S, \mathbb{Z}_2)$ on $X(S)$. The map r is the quotient map associated with the action of $\langle\varepsilon\rangle$. The restriction of q_2 to $X_\varphi(S)$ is the quotient map associated with the action of H unless $b_1(\varphi) = 2$ and one of the coordinate axes is contained in $X_\varphi(S)$.

Proof If $b_1(\varphi) = 1$ this is Proposition 21 and there is nothing to prove. If $b_1(\varphi) = 3$, we have $H = H^1(S, \mathbb{Z}_2)$ and there also is nothing to prove. Hence suppose $b_1(\varphi) = 2$ and that $\overline{L}_i$ is the unique coordinate axis in $\overline{X}_\varphi(S)$. It follows from Lemma 22 and the fact that all characters of representations with nontrivial centraliser are contained on the coordinate axes that if L_i is not contained in $X_\varphi(S)$, then q_2 is the quotient map of the H-action. If $L_i \subset X_\varphi(S)$, then for the same reasons, q_2 restricted to $\overline{X_\varphi(S) \setminus L_i}$ is the quotient map of the H-action. However, restricted to L_i, it is the quotient map associated with the sign change on L_i, this has degree two. □

Proposition 26 *Let M_φ be a hyperbolic once-punctured torus bundle. Let $\overline{X}$ be a Zariski component of $\overline{X}(M_\varphi)$ containing the character of an irreducible representation. If $\bar{r}\colon \overline{X} \to \overline{X}_\varphi(S)$ does not have degree one, then the degree is two and $b_1(\varphi) = 2$ and $\overline{X}$ is the preimage of one of the lines $\bar{L}_i \subset \overline{X}_\varphi(S)$.*

Proof The above classification of irreducible characters with nontrivial centraliser shows that

$$\bar{r}\colon \overline{X} \to \overline{X}_\varphi(S)$$

has degree one unless $\bar{L}_i \subset \overline{X}_\varphi(S)$ and $\overline{X} = \bar{r}^{-1}(\bar{L}_i)$ for some $i \in \{1, 2, 3\}$. In this case, the degree is two.

The conclusion for $b_1(\varphi) = 1$ now follows from the description of the action on the lines in the proof of Lemma 13, which gives $\bar{L}_i \cap \overline{X}_\varphi(S) = \{(0, 0, 0, 0)\}$ for each i. Alternatively, this follows from the fact that each representation lifts and from the uniqueness up to sign in Lemma 18.

Suppose $b_1(\varphi) = 3$ and that $\overline{X} = \bar{r}^{-1}(\bar{L}_i)$. It follows from Lemma 24 that $\overline{X} \to \bar{L}_i$ is a 2-fold (possibly branched) cover. Since $\bar{L}_i \cong \mathbb{C}$ and hence is simply connected it follows that the map $\overline{X} \to \bar{L}_i$ must have a branch point. Lemma 24 shows that there are only two potential branch points: the Klein four group or the reducible character on $\bar{L}_i$. It follows from the explicit description of the four extensions in the proof of Lemma 23 and the description of the covering map in the proof of Lemma 24 that the extensions of the Klein four group contained in $\bar{r}^{-1}(\bar{L}_i)$ are not ramification points. Hence it must be the reducible character on $\bar{L}_i$. Without loss of generality suppose $i = 3$ and use the parametrisation from (4-9):

$$\bar{\rho}(a) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\rho}(b) = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and let} \quad C = \pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{4-10}$$

Since $\bar{\rho}(a) = \bar{\rho}(b)$ and both have order two, the hypothesis that $b_1(\varphi) = 3$ implies that $\bar{\rho}(a)$ and $\bar{\rho}(t)$ commute. Hence there are two possibilities for $\bar{\rho}(t) = T_k$:

$$T_1 = \pm\begin{pmatrix} m_1 & m_2 \\ -m_2 & m_1 \end{pmatrix} \quad \text{or} \quad T_2 = \pm\begin{pmatrix} m_4 & m_3 \\ m_3 & -m_4 \end{pmatrix}, \tag{4-11}$$

where $m_1^2 + m_2^2 = 1$ or $m_3^2 + m_4^2 = -1$.

The representation defined by $\bar{\rho}_1(t) = T_1,\ \bar{\rho}_1(a) = \bar{\rho}_1(b) = \kappa_1$ satisfies $\mathrm{tr}[\bar{\rho}_1(t), \bar{\rho}_1(a)] = 2$ and hence is reducible. Indeed, this gives a 1-dimensional family of reducible representations with the quadruple in (4-4) equal to

$$\big(4m_1^2, 4(1-m_1^2), 4(1-m_1^2), 4m_1^2\big)$$

The representation defined by $\bar{\rho}_2(t) = T_2$, $\bar{\rho}_2(a) = \bar{\rho}_2(b) = \kappa_1$ satisfies $\mathrm{tr}[\bar{\rho}_2(t), \bar{\rho}_2(a)] = -2$ and hence is irreducible. For any choice of $m_3^2 + m_4^2 = -1$, the quadruple in (4-4) equals

$$(0, 0, 0, 0).$$

Indeed, this has image isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and lifts to a binary dihedral representation into $\mathrm{SL}(2, \mathbb{C})$.

The representations $\bar{\rho}_1$ and $\bar{\rho}_2$ are related by twisting by C (and possibly another element of the centraliser of $\bar{\rho}(a)$ that fixes the fixed points of $\bar{\rho}(a)$). In particular, the reducible character on $\bar{L}_i$ has exactly two preimages in $\overline{X} = \bar{r}^{-1}(\bar{L}_i)$. This shows that there are no ramification points. □

Examples for the case in Proposition 26 where the degree is two are given in Section 5.4.1. In these examples, one extension of the Klein four character is on this line, and the other extension is on the canonical component.

In the case of maximal rank, we can give a complete picture of the Zariski components that are pre-images of the lines:

Proposition 27 *If $H_1(M_\varphi, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, then $\overline{X}_\varphi(S)$ contains all three of $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$. Moreover, each line $\bar{L}_i$ has two pre-images $\bar{L}_i'$ and $\bar{L}_i''$ in $\overline{X}(M_\varphi)$ that are pairwise disjoint. There is a permutation $\sigma \in$* Sym(4) *such that the labels can be chosen such that $\chi_{\sigma(i)} \in \bar{L}_i''$ for $i \in \{1, 2, 3\}$, and $\chi_{\sigma(0)} = \bar{L}_1' \cap \bar{L}_2' \cap \bar{L}_3'$ is the only pairwise intersection point of the six preimages and is a character that lifts to* $\mathrm{SL}(2, \mathbb{C})$. *Up to passing to the degree two cover M_{φ^2}, we may assume that σ is the identity.*

Proof The fact that $\overline{X}_\varphi(S)$ contains all three of $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$ follows from the observation in the proof of Lemma 13, noting that the sign is now irrelevant, and the fact that irreducible representations of the fibre always extend. We know from Lemma 13 that $X_\varphi(S)$ either contains all three lines L_1, L_2, and L_3 or it contains exactly one of them.

First assume that $X_\varphi(S)$ contains all three lines. Then $X(M_\varphi)$ contains a character that extends the quaternionic group and is the common intersection of the pre-images L_1', L_2', and L_3' in $X(M_\varphi)$ of the three lines. Under the map to the $\mathrm{PSL}(2, \mathbb{C})$-character variety, these map to Zariski components $\bar{L}_1'$, $\bar{L}_2'$, $\bar{L}_3'$. Hence their common intersection is one of the extensions χ_j of the Klein four group.

Consider the line $\bar{L}_3$ and the parametrisation given in (4-9) that is consistent with (4-5). Each point in $\bar{L}_3$ has precisely two preimages in $\overline{X}(M_\varphi)$, and they are related by the action of κ_3. It follows that the preimage $\bar{r}^{-1}(\bar{L}_3)$ either has one or two Zariski components. Since the lifting obstruction is constant on topological components of the character variety and $\bar{r}^{-1}(\bar{L}_3)$ contains two distinct extensions of the Klein four group, namely χ_j and $\kappa_3 \cdot \chi_j$, it follows that $\bar{r}^{-1}(\bar{L}_3)$ is a disjoint union of two components $\bar{L}_3'$ and $\bar{L}_3''$ and we have $\chi_j \in \bar{L}_3'$ and $\kappa_3 \cdot \chi_j \in \bar{L}_3''$

The same argument applies to the lines $\bar{L}_1$ and $\bar{L}_2$ where the respective action with respect to the appropriate parametrisations consistent with (4-5) is given by κ_1 and κ_2. It now follows from the action of the Klein four group on the characters χ_0, χ_1, χ_2, χ_3 that no two of $\bar{L}_i''$ and $\bar{L}_k''$ have one of these characters in common. This implies that they are pairwise disjoint.

If $\chi_j \neq \chi_0$, then the meridian is mapped to an element of order two under the character corresponding to the triple intersection. Hence under the map $\overline{X}(M_\varphi) \to \overline{X}(M_{\varphi^2})$, the character $\chi_j \in \overline{X}(M_\varphi)$ is mapped to $\chi_0 \in \overline{X}(M_{\varphi^2})$, and each preimage of one of the lines $\bar{L}_i$ in $\overline{X}(M_\varphi)$ is mapped to the corresponding connected component of the preimage of $\bar{L}_i$ in $\overline{X}(M_{\varphi^2})$ that passes through $\chi_0 \in \overline{X}(M_{\varphi^2})$. This completes the proof of the lemma in the case where $X_\varphi(S)$ contains all three lines since $\kappa_i \cdot \chi_0 = \chi_i$.

Now suppose that $X_\varphi(S)$ only contains one of the three lines L_1, L_2, and L_3. Without loss of generality, we may assume this is L_1. Then in $\bar{X}(M_\varphi)$ there is a component $\bar{L}_1''$ that lifts to L_1. The same argument as above shows that $\bar{r}^{-1}(\bar{L}_1)$ has two disjoint components. Suppose $\chi_j \in \bar{L}_1'$. Then $\kappa_1 \cdot \chi_j \in \bar{L}_1''$. Since the lifting obstruction is constant on topological components, $\bar{r}^{-1}(\bar{L}_2)$ and $\bar{r}^{-1}(\bar{L}_3)$ do not pass through $\kappa_1 \cdot \chi_j$. Then $\bar{r}^{-1}(\bar{L}_2)$ also does not pass through $\kappa_2\kappa_1 \cdot \chi_j = \kappa_3 \cdot \chi_j$, and $\bar{r}^{-1}(\bar{L}_3)$ does not pass through $\kappa_3\kappa_1 \cdot \chi_j = \kappa_2 \cdot \chi_j$. Hence $\bar{r}^{-1}(\bar{L}_2)$ contains χ_j and $\kappa_2 \cdot \chi_j$; and $\bar{r}^{-1}(\bar{L}_3)$ contains χ_j and $\kappa_3 \cdot \chi_j$. This proves that each extensions of the Klein four group is contained in at least one of the pre-images of the lines $\bar{L}_i$. The proof is now completed by the observation in the proof of Proposition 26 that each $\bar{L}_i$ has preimage in $\bar{X}(M_\varphi)$ consisting of two disjoint Zariski components, each containing one of the two extensions of the Klein four group contained in $\bar{r}^{-1}(\bar{L}_i)$. The last statement now follows as in the previous paragraph. □

Remark 28 In the examples we computed explicitly, it is always that case that $\chi_0 = \bar{L}_1' \cap \bar{L}_2' \cap \bar{L}_3'$.

Example 29 We continue the example of $\varphi = \alpha^2\beta^{-2}$ discussed in Example 16. As $H_1(M_\varphi; \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, $\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \in \bar{X}_\varphi(S)$. A direct calculation shows that each line in $\bar{X}_\varphi(S)$ is the preimage of two Zariski components in $\bar{X}(M_\varphi)$ under the map $\bar{r}$. Consider the preimages $\bar{L}_3'$, $\bar{L}_3''$ of $\bar{L}_3$. Suppose that the component $L_3' \in X(M_\varphi)$ described in Example 16 is the preimage of the component $\bar{L}_3''$ in $\bar{X}(M_\varphi)$. The values in (4-4) on $\bar{L}_3''$ are $(0, 0, 0, 4 - z^2)$. Twisting by the generator of $C(\mathrm{im}(\bar{\rho}))$, we obtain another component $\bar{L}_3' \subset \bar{X}(M_\varphi)$ where the corresponding values are $(4, 0, 0, z^2)$. We may choose labels for the preimages of the other lines such that the values taken in (4-4) for $\bar{L}_1''$ are $(x^2, (x^2-2)^2, 0, 0)$, for $\bar{L}_1'$ are $(4 - x^2, x^2(4 - x^2), 0, 0)$, for $\bar{L}_2''$ are $(y^2, 0, (y^2-2)^2, 0)$, and for $\bar{L}_2'$ are $(4 - y^2, 0, y^2(4 - y^2), 0)$. Note that $\bar{L}_1'$, $\bar{L}_2'$ and $\bar{L}_3'$ intersect at the character with trivial peripheral holonomy, and each of the other 3 characters extending the Klein four group is contained in precisely one of $\bar{L}_1''$, $\bar{L}_2''$ and $\bar{L}_3''$.

4.3 Dimensions and lifting

As an application of the discussion in the previous section, we have the following:

Proposition 30 *Let M_φ be a hyperbolic once-punctured torus bundle. Then every Zariski component of $\bar{X}(M_\varphi)$ is at most one-dimensional. Moreover, if $\{\chi_{\bar{\rho}}\} \subset \bar{X}(M_\varphi)$ is a zero-dimensional component, then this character does not lift to* $\mathrm{SL}(2, \mathbb{C})$, $b_1(\varphi) \neq 1$, $\bar{\rho}\big(\mathrm{im}(\pi_1(T) \to \pi_1(M_\varphi))\big) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *and $\bar{\rho}$ is irreducible with trivial centraliser. Moreover, $\chi_{\bar{\rho}}$ maps to a character on a one-dimensional component in $\bar{X}(M_{\varphi^2})$.*

Proof The same arguments as in the proof of Proposition 14 show that every component containing only reducible characters is one-dimensional and of genus zero, and that the dimension of each Zariski component of $\bar{X}(M_\varphi)$ is at most one since M_φ does not contain a closed essential surface. Suppose X is a Zariski component containing the character of an irreducible representation $\bar{\rho}$. Proposition 8 implies that the dimension of X is at least one unless $\bar{\rho}\big(\mathrm{im}(\pi_1(T) \to \pi_1(M_\varphi))\big)$ is trivial or isomorphic with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If $\bar\rho\big(\mathrm{im}(\pi_1(T)\to\pi_1(M_\varphi))\big)$ is trivial, then $\operatorname{tr}\bar\rho(a^{-1}b^{-1}ab)=\pm 2$ since $a^{-1}b^{-1}ab$ is the longitude. If the trace equals $+2$, then $\bar\rho(a)$ and $\bar\rho(b)$ have a common fixed point on $P^1(\mathbb{C})$. But since $\rho(t)=\pm I$ this implies that the representation is reducible. Hence $\operatorname{tr}\bar\rho(a^{-1}b^{-1}ab)=-2$. In this case, $\bar\rho(\langle a,b\rangle)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$. It follows that $H_1(M_\varphi;\mathbb{Z}_2)\cong\mathbb{Z}_2^3$, and that $\chi_{\bar\rho}=\chi_0$ is the extension of the Klein four group with trivial peripheral holonomy. It now follows from Proposition 27 that this character is contained on a one-dimensional component.

Hence suppose that $\bar\rho\big(\mathrm{im}(\pi_1(T)\to\pi_1(M_\varphi))\big)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$. In this case, $\operatorname{tr}\bar\rho(a^{-1}b^{-1}ab)=0$ and therefore the representation restricted to $\pi_1(S)$ is irreducible and has trivial centraliser. Since the Klein four group lifts to the quaternionic group, this representation does not lift to $\mathrm{SL}(2,\mathbb{C})$. In particular, $b_1(\varphi)\neq 1$. Now under the map to $\overline{X}(M_{\varphi^2})$, $\chi_{\bar\rho}$ maps to a character with the property that the peripheral subgroup has image isomorphic with $\mathbb{Z}_2$. Hence Proposition 8 implies that the image lies on a one-dimensional component of $\overline{X}(M_{\varphi^2})$. □

Remark 31 We do not have examples of once-punctured torus bundles with zero-dimensional components in $\overline{X}(M_\varphi)$. The purported examples given in [2, Theorem 7.6] contradict Proposition 8 and a simple description of one-dimensional components containing them is given in Example 10.

Heusener and Porti [19, Section 4.2] give examples of hyperbolic once-punctured torus bundles that have arbitrarily many one-dimensional Zariski components in $\overline{X}(M_\varphi)$ that do not lift to $X(M_\varphi)$. All of these components are of genus zero. We remark that all these examples satisfy $b_1(\varphi)=3$. We give similar examples with $b_1(\varphi)=2$ in Section 5.4.1, and it was already remarked that if $b_1(\varphi)=1$, then every representation lifts.

Components containing only reducible characters have an analogous characterisation as given in Section 3.4. Moreover, the proof of Proposition 17 gives the analogous result for the $\mathrm{PSL}(2,\mathbb{C})$-character variety. We state this here for completeness:

Proposition 32 *For every Zariski component X of $\overline{X}^{\mathrm{red}}(M_\varphi)$, the restriction $\bar r\colon X\to\overline{X}_\varphi(S)$ is constant and $X\cap\overline{X}^{\mathrm{irr}}(M_\varphi)\neq\varnothing$. Each character $\bar\chi$ in the intersection is contained on exactly one curve C in $\overline{X}^{\mathrm{irr}}(M_\varphi)$, a smooth point of X and C and the intersection at $\bar\chi$ is transverse.*

4.4 An algebraic subset

For computations, it is useful to identify $\overline{X}_\varphi(S)$ with the image of a suitable algebraic subset of $X(S)$ under the four-fold branched covering map $X(S)\to\overline{X}(S)$. Let

(4-12) $$X^1_\varphi(S)=\{(x,y,z)\in\mathbb{C}^3\mid(\ \ x,-y,-z)=\bar\varphi(x,y,z)\},$$

(4-13) $$X^2_\varphi(S)=\{(x,y,z)\in\mathbb{C}^3\mid(-x,\ \ y,-z)=\bar\varphi(x,y,z)\},$$

(4-14) $$X^3_\varphi(S)=\{(x,y,z)\in\mathbb{C}^3\mid(-x,-y,\ \ z)=\bar\varphi(x,y,z)\}.$$

Note that the intersection of any two of the sets $X_\varphi(S)$, $X^1_\varphi(S)$, $X^2_\varphi(S)$, $X^3_\varphi(S)$ is contained in a coordinate axis. Recall that the natural map $q_2\colon X(S) \to \overline{X}(S)$ is the quotient map of the $H^1(S,\mathbb{Z}_2)$ action. The definitions imply that

$$q_2^{-1}(\overline{X}_\varphi(S)) = X_\varphi(S) \cup X^1_\varphi(S) \cup X^2_\varphi(S) \cup X^3_\varphi(S).$$

For each $(x,y,z) \in q_2^{-1}(\overline{X}_\varphi(S))$, there are $A, B \in \mathrm{SL}(2,\mathbb{C})$ with $\operatorname{tr} A = x$, $\operatorname{tr} B = y$ and $\operatorname{tr} AB = z$ and satisfying (4-2) and (4-3), where $\varepsilon_a, \varepsilon_b \in \{\pm 1\}$ are determined by

$$(\varepsilon_a x, \varepsilon_b y, \varepsilon_a \varepsilon_b z) = \overline{\varphi}(x,y,z).$$

The above sets therefore give us a simple way to compute the representations of $\pi_1(M_\varphi)$ into $\mathrm{PSL}(2,\mathbb{C})$. We now determine redundancies that arise from the action of the elements of $H^1(S,\mathbb{Z}_2)$ that are not in the group $H \le H^1(M_\varphi,\mathbb{Z}_2)$ that was defined in Section 2.3 as those homomorphisms $h\colon \pi_1(M_\varphi) \to \mathbb{Z}_2$ satisfying $h(t) = 1$.

View $h \in H^1(S,\mathbb{Z}_2)$ as a homomorphism $h\colon \pi_1(S) \to \{\pm 1\}$. Then $h \in H$ if and only if $h(a) = h(\varphi(a))$ and $h(b) = h(\varphi(b))$. Let $\bar\rho\colon \pi_1(M_\varphi) \to \mathrm{PSL}(2,\mathbb{C})$ and use the set up from (4-2) and (4-3). Then $\bar\rho$ lifts to $\mathrm{SL}(2,\mathbb{C})$ if and only if there is $h \in H^1(S,\mathbb{Z}_2)$ with

$$h(a) = h(\varphi(a))\varepsilon_{\boldsymbol{A}}, \quad h(b) = h(\varphi(b))\varepsilon_{\boldsymbol{B}}.$$

Note that the signs $\varepsilon_{\boldsymbol{A}}$ and $\varepsilon_{\boldsymbol{B}}$ are uniquely determined by $\bar\rho$ if $H = H^1(S,\mathbb{Z}_2)$ and otherwise they depend on the choice of the matrices A and B. The action of $h \in H^1(S,\mathbb{Z}_2)$ is given by

$$h \cdot (x,y,z) \mapsto (h(a)x, h(b)y, h(ab)z).$$

This induces a permutation of the sets $X_\varphi(S)$, $X^1_\varphi(S)$, $X^2_\varphi(S)$, and $X^3_\varphi(S)$. The permutation of the sets is determined by the action on the defining equations,

$$\overline{\varphi}(x,y,z) = \big(h(a)h(\varphi(a))\varepsilon_{\boldsymbol{A}} x,\ h(b)h(\varphi(b))\varepsilon_{\boldsymbol{B}} y,\ h(ab)h(\varphi(ab))\varepsilon_{\boldsymbol{A}}\varepsilon_{\boldsymbol{B}} z\big).$$

Each set is stabilised by $H \le H^1(S,\mathbb{Z}_2)$ and the complementary elements permute the sets $X_\varphi(S)$, $X^1_\varphi(S)$, $X^2_\varphi(S)$, $X^3_\varphi(S)$.

If $b_1(\varphi) = 3$, then $H = H^1(S,\mathbb{Z}_2)$ and hence each of the sets is fixed under this action. In particular, $\bar\rho$ does not lift if at least one of $\varepsilon_{\boldsymbol{A}}$ and $\varepsilon_{\boldsymbol{B}}$ equals -1. Define

$$X^\perp_\varphi(S) = X^1_\varphi(S) \cup X^2_\varphi(S) \cup X^3_\varphi(S).$$

If $b_1(\varphi) = 2$, then the action of $H^1(S,\mathbb{Z}_2)$ has two orbits, each containing two sets. Let

$$X^\perp_\varphi(S) = X^i_\varphi(S),$$

where $X^i_\varphi(S)$ is one of the sets not in the orbit of $X_\varphi(S)$.

If $b_1(\varphi) = 1$, then there is just one orbit and we define $X^\perp_\varphi(S) = \varnothing$. This observation can be viewed as an elementary proof of the fact that every representation into $\mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$ in this case.

These definitions imply that the map

$$X_\varphi(S) \cup X_\varphi^\perp(S) \to \overline{X}_\varphi(S)$$

is surjective, and corresponds to the quotient map of the action of $H^1(S, \mathbb{Z}_2)$ twisted via φ.

5 Examples

This section provides details about the three infinite families of once-punctured torus bundles mentioned in the introduction, as well as the beginnings of a census. We begin by stating some general results that will be used repeatedly. The main ingredients in the proofs are properties of a sequence of Fibonacci polynomials, and determining irreducibility and genus of a plane algebraic curve from the Newton polygon of a defining polynomial.

5.1 The Fibonacci polynomials

This section collects some facts about a family of recursive polynomials, which are used throughout the computation in [2]. The Fibonacci polynomials will be used to compute $X_\varphi(S)$ for our examples.

Definition 33 For every integer n, the n-th Fibonacci polynomial $f_n(u)$ is defined by the recursive relation

$$f_n(u) = u f_{n-1}(u) - f_{n-2}(u),$$

where $f_0(u) = 0$ and $f_1(u) = 1$.

A number of useful properties of f_n according to [2] are listed below.

Lemma 34 (1) *If $u = s + s^{-1}$, then*

$$f_n(u) = \begin{cases} \dfrac{s^n - s^{-n}}{s - s^{-1}} & \text{if } u \neq \pm 2, \\ n & \text{if } u = 2, \\ (-1)^{n+1} n & \text{if } u = -2. \end{cases}$$

(2) *If $n \neq 0$, the degree of $f_n(u)$ is $|n| - 1$.*

(3) *$f_n(u)$ is divisible by u if and only if n is even. If $n \neq 0$, $f_n(u)$ is not divisible by u^2.*

(4) *For any integer n, $f_n(u)$, $f_{n+1}(u) - f_n(u)$, $f_{n+1}(u) + f_n(u)$ and $f_{n+2}(u) - f_n(u)$ are separable except for $f_0(u) = 0$.*

(5) *$f_{n+2}(u) - f_{n+1}(u) = 0$ and $f_{n+1}(u) - f_n(u) = 0$ do not have a common root.*

Proof Part (1) follows from the defining relation. Parts (2), (3) and (4) follow from [2, Lemmas 4.3, 4.4 and 4.11]. A direct calculation using (1) results in (5). □

5.2 Genus and the Newton polygon

A classical link between the Newton polygon and the genus of an irreducible algebraic curve is known as Baker's formula:

Theorem 35 (Baker [1]) *Suppose that $F(x,y)=0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. The genus of X is at most the number of lattice points in the interior of the Newton polygon of F.*

Khovanskiĭ [21] showed that one generically has equality instead of an upper bound. We use a version of this result as implied by Beelen and Pellikaan [3]. A polynomial $F(x,y)\in\mathbb{C}[x,y]$ is said to be *nondegenerate* with respect to its Newton polygon if for every edge γ of its Newton Polygon with the corresponding polynomial F_γ, the ideal generated by F_γ, $x\dfrac{\partial F_\gamma}{\partial x}$ and $y\dfrac{\partial F_\gamma}{\partial y}$ has no zero in $(\mathbb{C}/\{0\})^2$.

Corollary 36 (Beelen–Pellikaan) *Suppose that $F(x,y)=0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. If F is nondegenerate with respect to its Newton polygon and it is smooth at every point $(x,y)\in X$ where $xy\neq 0$, then the genus of its nonsingular model is equal to the number of lattice points in the interior of its Newton polygon.*

Proof We show that this is implied by [3, Theorem 4.2]. The only statement that requires proof is that the smoothness hypothesis implies that the singular points of the homogeneous curve with equation $F^*(x,y,z)=0$ are among $(0:0:1)$, $(0:1:0)$ and $(1:0:0)$, where F^* denotes the homogenisation of F. If S is a singular point other than the three points, it is not in the line of infinity and corresponds to a singular point $s=(x,y)\in X$ such that $xy\neq 0$. This contradicts the smoothness condition. □

We will also appeal to the following result [5, Lemma 5.1]. The authors thank Michael Joswig for pointing them to this reference.

Lemma 37 (Castryck–Voight [5]) *Suppose that $F(x,y)=0$ defines an irreducible algebraic curve X in $\mathbb{C}^2$. Assume that F is nondegenerate with respect to its Newton polygon, and that there are at least two lattice points in the interior of the Newton polygon of F. Then X is hyperelliptic if and only if the interior lattice points of the Newton polygon are collinear.*

5.3 The family M_n

The genera of the Zariski components of the $\mathrm{SL}(2,\mathbb{C})$- and $\mathrm{PSL}(2,\mathbb{C})$-character varieties of the infinite family of once-punctured torus bundles M_n with monodromies $\varphi_n = AB^{n+2}$ were determined by Baker and Petersen [2, Theorem 5.1] via a birational isomorphism between $X^{\mathrm{irr}}(M_{\varphi_n})$ and a family of hyperelliptic curves. In this section, we compute the corresponding varieties $X_{\varphi_n}(S)$. Note that $\mathrm{tr}(\varphi_n)=-n$ and M_n is hyperbolic if $n\geq 3$ or $n\leq -3$. For computational simplicity, we only consider the case when $n\geq 3$ and n is odd, and so $b_1(\varphi_n)=1$.

The corresponding family of framings φ_n admit the form

$$\varphi_n = \begin{cases} a \to a(a^{-1}b^{-1})^{n+2}, \\ b \to ba. \end{cases}$$

By a direct computation as in Lemma 11, the binary dihedral characters for $X(M_n)$ are as follows.

Fact 38 For every odd integer $n \geq 3$, the ramification points of r are

$$\left(\beta^{-2k} + \beta^{2k}, \beta^{k} + \beta^{-k}, \beta^{k} + \beta^{-k}, 0, 0, 0, 0\right)$$

where $\beta = e^{\frac{2\pi i}{n-2}}$, $\beta^{2k} \neq 1$ and $k = 0, 1, \dots, n-3$. There are

$$\tfrac{1}{2}(n-3) = \tfrac{1}{2}|n-2| - \tfrac{1}{2} = \tfrac{1}{2}|\operatorname{tr}\varphi_n + 2| - 2^{b_1(\varphi_n)-2}$$

such binary dihedral characters in total, which agrees with (1-1).

5.3.1 Topology of $X_{\varphi_n}(S)$ In this subsection, we compute the fixed point set $X_{\varphi_n}(S)$. Substituting the automorphism φ_n into the definition of $X_{\varphi_n}(S)$ in Section 3.1, we obtain the three equations

$$\operatorname{tr}\rho(a) = \operatorname{tr}\rho(a(a^{-1}b^{-1})^{n+2}), \quad \operatorname{tr}\rho(b) = \operatorname{tr}\rho(ba), \quad \operatorname{tr}\rho(ab) = \operatorname{tr}\rho(a(a^{-1}b^{-1})^{n+1}).$$

The second equation gives $y = z$. Define polynomials $P_n(x, y, z) = \operatorname{tr}\rho(a(a^{-1}b^{-1})^n)$ for $n \in \mathbb{Z}$. Using the trace identities in Section 2, P_n satisfies the recursive relation $P_n(x, y, z) = zP_{n-1}(x, y, z) - P_{n-2}(x, y, z)$ with $P_0(x, y, z) = x$ and $P_1(x, y, z) = y$. Then $X_{\varphi_n}(S)$ is defined by $P_{n+2} - x$, $P_{n+1} - z$ and $y - z$. Recall the Fibonacci polynomial f_n in Section 5.1. We observe that

$$P_n(x, y, z) = yf_n(z) - xf_{n-1}(z) \quad \text{for } n \in \mathbb{Z}. \tag{5-1}$$

To simplify the defining equations of $X_{\varphi_n}(S)$, we use the idea of Buchberger's algorithm [6] which is used to compute the Gröbner basis of an ideal of a polynomial ring. Denote $X_{n+2} = P_{n+2} - x = P_{n+2} - P_0$, $X_{n+1} = P_{n+1} - P_1$. Buchberger's algorithm involves the so-called S-polynomials (the S stands for subtraction — one eliminates the leading terms of two polynomials). We apply the algorithm here and refer to [6] for definitions and details. We compute the S-polynomial of the first two polynomials recursively to summarise X_{n+2} and X_{n+1} into one single polynomial.

Lemma 39 *Define $X_{i+2} = zX_{i+1} - X_i$ for $i \geq 0$. When $n = 2k + 1$ is a positive odd integer, $\langle X_{n+2}, X_{n+1}\rangle = \langle P_{k+2} - P_{k+1}\rangle$. When $n = 2k$ is a positive even integer, $\langle X_{n+2}, X_{n+1}\rangle = \langle P_{k+2} - P_k\rangle$.*

Proof Using induction, we can prove that $X_i = P_i - P_{n+2-i}$ for $0 \leq i \leq n + 2$. Since X_{i+2} is a linear combination of X_{i+1} and X_i, $\langle X_{i+2}, X_{i+1}\rangle \subseteq \langle X_{i+1}, X_i\rangle$. By definition, $\langle X_{i+1}, X_i\rangle \subseteq \langle X_{i+2}, X_{i+1}\rangle$. Thus, $\langle X_{i+2}, X_{i+1}\rangle = \langle X_{i+1}, X_i\rangle$.

Hence when $n = 2k + 1$,

$$\langle X_{n+2}, X_{n+1}\rangle = \cdots = \langle X_{k+2}, X_{k+1}\rangle = \langle P_{k+2} - P_{k+1}\rangle.$$

The even case follows analogously. □

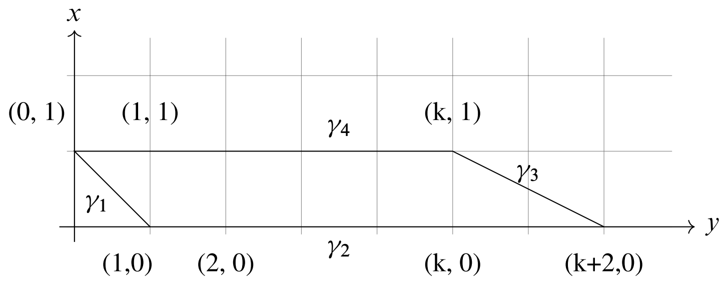

Figure 1: Newton polygon of g_k.

When n is odd, $X_{\varphi_n}(S) = V(\langle P_{k+2} - P_{k+1}, y - z\rangle)$, where we use the notation $V(I)$ for the variety defined by the ideal I.. Under the restriction $(x, y, z) \mapsto (x, y)$, $X_{\varphi_n}(S)$ is homeomorphic to its image, denoted by U_n. Note that $U_n = V(\langle p_{k+2} - p_{k+1}\rangle)$ where $p_n(x, y) = P_n(x, y, y)$. Combining this with (5-1), U_n is defined by

$$g_k(x, y) := p_{k+2} - p_{k+1} = y(f_{k+2}(y) - f_{k+1}(y)) - x(f_{k+1}(y) - f_k(y)).$$

By Lemma 34, the Newton polygon of g_k is shown in Figure 1.

Lemma 40 $g_k(x, y)$ *is irreducible in* $\mathbb{C}[x, y]$ *for any positive integer* k.

Proof If $g_k(x, y)$ is reducible, then we can factorise it into $g_k(x, y) = s_k(x, y)t_k(x, y)$, where t_k, s_k are nonconstant polynomials in $\mathbb{C}[x, y]$. Since $g_k(x, y)$ is linear in x, without loss of generality, we assume that $t_k(x, y) = t_k(y)$ and $s_k(x, y) = x s_k^{(1)}(y) + s_k^{(2)}(y)$, where $s_k^{(i)}(y)$ and $t_k(y)$ are polynomials in $\mathbb{C}[y]$. Then

$$y(f_{k+2}(y) - f_{k+1}(y)) = s_k^{(2)}(y)t_k(y), \quad -(f_{k+1}(y) - f_k(y)) = s_k^{(1)}(y)t_k(y).$$

Since $t_k(y)$ is nonconstant, by the fundamental theorem of algebra, $t_n(y) = 0$ has a solution $\alpha \in \mathbb{C}$. Lemma 34 tells us that $f_{k+1}(y) - f_k(y)$ has a nonzero constant term, which implies that $\alpha \neq 0$. Then α is a common zero of $f_{k+2}(y) - f_{k+1}(y)$ and $f_{k+1}(y) - f_k(y)$. This contradicts Lemma 34(5). Hence $g_k(x, y)$ is irreducible. □

Lemma 41 *For* $n = 2k + 1 \geq 3$, *there is no singular point in* $U_n = V(g_k)$ *and* g_k *is nondegenerate.*

Proof Suppose (x, y) is a singular point. Then

$$y(f_{k+2}(y) - f_{k+1}(y)) - x(f_{k+1}(y) - f_k(y)) = 0, \quad \frac{\partial g_k}{\partial x}(x, y) = f_{k+1}(y) - f_k(y) = 0.$$

This clearly contradicts Lemma 34(5). The nondegeneracy condition for edges γ_1 and γ_3 in Figure 1 is obvious. For the edge γ_2, $F_{\gamma_2}(x, y) = y(f_{k+2}(y) - f_{k+1}(y))$. Suppose that $(x, y) \in (\mathbb{C}/\{0\})^2$ is a common zero of

$$F_{\gamma_2}, \quad x\frac{\partial F_{\gamma_2}}{\partial x} \quad \text{and} \quad y\frac{\partial F_{\gamma_2}}{\partial y}.$$

This results in

$$f_{k+2}(y) - f_{k+1}(y) = 0, \quad f'_{k+2}(y) - f'_{k+1}(y) = 0.$$

There is no such $y \neq 0$ satisfying the above two equations by Lemma 34(1). In detail, it is clear that $y = \pm 2$ are not zeros. When $y \neq \pm 2$, the first equation implies that $p^{2k+3} + 1 = 0$ and $p \neq -1$ given $y = p + p^{-1}$. Using the change of variable formula

$$\frac{df_n}{dp} = \frac{df_n}{dy}\frac{dy}{dp},$$

the second equation implies that

$$\frac{p^{-k}\big(k(p+1)(p^{2k+3}-1) - 1 - 2p + 2p^{2k+3} + p^{2k+4}\big)}{(p^2-1)(1+p)^2} = 0.$$

Since $y \neq 0$,

$$k(p+1)(p^{2k+3}-1) - 1 - 2p + 2p^{2k+3} + p^{2k+4} = 0.$$

Combining the above equation with $p^{2k+3} = -1$ implies $p = -1$, which is a contradiction. The nondegeneracy condition for edge γ_4 can be checked using a similar argument. □

Proposition 42 *If $n = 2k+1 \geq 3$, then the variety $\overline{X}^{\mathrm{irr}}(M_\varphi) \cong X_{\varphi_n}(S)$ is irreducible and a curve of genus 0.*

Proof *By Lemmas 40 and 41, $g_k(x, y)$ satisfies assumptions in Corollary 36, which implies that the genus of U_n is 0. As $U_n \cong X_{\varphi_n}(S)$, this proves the statement by Corollary 3.* □

According to the Newton polygon Figure 1 and the theory of Puiseux expansions, there are k ideal points such that $x \to \infty$, $y \to c$ (a nonzero constant) and one ideal point such that $x \to \infty$, $y \to \infty$. Hence there are $k + 1 = \frac{1}{2}(n+1)$ ideal points. Since the number $\frac{1}{2}(n-3)$ of branch points is positive if $n > 3$, the variety $X^{\mathrm{irr}}(M_n)$ is also irreducible, and the same can be confirmed by direct calculation if $n = 3$. Hence Corollary 20 implies:

Theorem 43 *Suppose that g_n denote the genus of $X^{\mathrm{irr}}(M_n)$. When n is a positive odd integer and $n \geq 3$, g_n is bounded by*

$$\tfrac{1}{4}(n-7) \leq g_n \leq \tfrac{1}{2}(n-3).$$

Baker and Petersen [2, Theorem 5.1] show that $g_n = \frac{1}{2}(n-3)$ for n odd. So there are $n-1$ branch points in total. Hence in addition to the $\frac{1}{2}(n-3)$ branch points at the binary dihedral characters in $X(M_\varphi(\lambda)) \subset X(M_\varphi)$, there are $\frac{1}{2}(n+1)$ branch points at ideal points. In particular, every ideal point is a branch point (and in the case $n = 3$ these are the only branch points).

5.4 The family N_n

Let N_n be the once-punctured torus bundle with monodromy $\psi_n = AB^{n+2}A$. Note that $\operatorname{tr}(\psi_n) = -2n-2$ and N_n is hyperbolic when $n \geq 1$ or $n \leq -3$. The corresponding family of framings ψ_n admit the form

$$\psi_n = \begin{cases} a \to a(a^{-1}b^{-1})^{n+2}, \\ b \to ba^2(a^{-1}b^{-1})^{n+2}. \end{cases}$$

We restrict to $n = 2k+1 \geq 1$ odd and hence $b_1(\psi_n) = 2$.

The corresponding fixed-point set $X_{\psi_n}(S)$ is defined by the equations

$$\begin{aligned} \operatorname{tr}\rho(a) &= \operatorname{tr}\rho(a(a^{-1}b^{-1})^{n+2}), \\ \operatorname{tr}\rho(b) &= \operatorname{tr}\rho(ba^2(a^{-1}b^{-1})^{n+2}), \\ \operatorname{tr}\rho(ab) &= \operatorname{tr}\rho(a(a^{-1}b^{-1})^{n+2}ba^2(a^{-1}b^{-1})^{n+2}). \end{aligned}$$

Using the trace identities in Section 2, the above equations can be simplified to $x = P_{n+2}(x,y,z)$, $y = P_{n+1}(x,y,z)$ and $z = xy - z$, where P_n is defined by (5-1). Then

$$X_{\psi_n}(S) = V(\langle P_{n+2} - x, P_{n+1} - y, xy - 2z \rangle).$$

By Lemma 39, $X_{\psi_n}(S) = V(\langle P_{k+2} - P_{k+1}, xy - 2z \rangle)$ when $n = 2k+1$.

If $x = 0$, then $z = 0$ by second equation, and then $0 = P_{k+2} - P_{k+1} = y(f_{k+2}(z) - f_{k+1}(z))$ which implies $y = 0$ by Lemma 34(3). In particular, there is a unique point in $X_{\psi_n}(S)$ where $x = 0$.

When $x \neq 0$, we substitute $y = 2z/x$ into $P_{k+2}(x,y,z) - P_{k+1}(x,y,z) = 0$ and multiply by x on both sides, giving

$$h_k(x,z) := 2(f_{k+2}(z) - f_{k+1}(z))z - (f_{k+1}(z) - f_k(z))x^2 = 0.$$

Let V_n be the variety in $\mathbb{C}^2$ defined by $h_k \in \mathbb{C}[x,z]$. Then $X_{\psi_n}(S)$ and V_n are birational via rational maps

$$(5\text{-}2) \qquad \begin{aligned} r_1 &\colon X_{\psi_n}(S) \dashrightarrow V_n, \quad (x,y,z) \mapsto (x,z), \\ r_2 &\colon V_n \dashrightarrow X_{\psi_n}(S), \quad (x,z) \mapsto (x, 2z/x, z). \end{aligned}$$

Lemma 44 *$h_k(x,z)$ is an irreducible polynomial in $\mathbb{C}[x,z]$ for every positive integer k.*

Proof If $h_k(x,z)$ is reducible, then we can factorise it into $h_k(x,z) = s_k(x,z)t_k(x,z)$, where t_k, s_k are nonconstant polynomials in $\mathbb{C}[x,z]$. Then the Newton polygon of $h_k(x,z)$ is the Minkowski sum of the Newton polygons of $s_k(x,z)$ and $t_k(x,z)$. The Newton polygon of $h_k(x,z)$ is shown in Figure 2. The *boundary vector sequence* of a Newton polygon is obtained by choosing an orientation of its boundary and cyclically listing the vectors between consecutive integer lattice points on the boundary. Travelling along the boundary of the Newton polygon of $h_k(x,z)$, we obtain the boundary vector sequence

$$v(h_k) = \big((1,-2),(1,0),\dots,(1,0),(-1,1),(-1,1),(-1,0),\dots,(-1,0)\big),$$

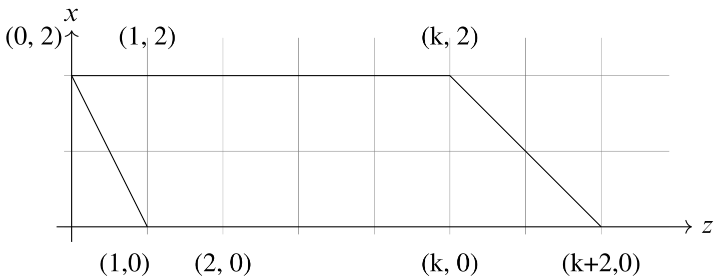

Figure 2: Newton polygon of $h_k(x,z)$.

where the number of repetitions of $(1,0)$ and $(-1,0)$ in $v(h_k)$ are $k+1$ and k, respectively. The sequence $v(h_k)$ can be partitioned into two disjoint nonempty subsequences $v(s_k)$ and $v(t_k)$, each of which sums to zero. Given $(1,-2)$, $(-1,1)$ and $(-1,1)$ are the only three vectors with nonzero second components, they must be in the same sequence. Without loss of generality, we assume that they are in $v(s_k)$. Noting that h_k has no linear term in x, we may write $t_k(x,z)=t_k(z)$ and $s_k(x,z)=x^2 s_k^{(1)}(z)+s_k^{(2)}(z)$, where $s_k^{(i)}(z)$ and $t_k(z)$ are polynomials in $\mathbb{C}[z]$. Hence

$$2(f_{k+2}(z)-f_{k+1}(z))z=s_k^{(2)}(z)t_k(z),\quad -(f_{k+1}(z)-f_k(z))=s_k^{(1)}(z)t_k(z).$$

The remainder of the proof is analogous to the proof of Lemma 40, where one obtains a contradiction to Lemma 34(5). □

Lemma 45 *When $n=2k+1\geq 1$, h_k is nondegenerate and there is no singular point in $V_n=V(h_k)$.*

Proof The proof is almost identical to Lemma 41. □

Combining Lemmas 44 and 45 with Corollary 36 and Lemma 37 implies:

Lemma 46 *The curve $V_n\cong X_{\psi_n}(S)$ is hyperelliptic of genus k for each $n=2k+1\geq 3$.*

5.4.1 PSL(2, ℂ)-character variety Since n is odd, $b_1(\psi_n)=2$. If a representation $\bar\rho\in\overline{R}(N_n)$ lifts to $\mathrm{SL}(2,\mathbb{C})$, then it generically has 4 lifts. Let $\overline{X}_0(N_n)$ be the subvariety of $\overline{X}^{\mathrm{irr}}(N_n)$ consisting of the characters of all $\mathrm{PSL}(2,\mathbb{C})$-representations that lift to representations into $\mathrm{SL}(2,\mathbb{C})$. By Corollary 2, $\overline{X}_0(N_n)\cong q_2(X_{\psi_n}(S))$. We have the following proposition.

Proposition 47 *For every positive odd integer $n=2k+1$, the subvariety $\overline{X}_0(N_n)$ is birational to the affine line.*

Proof Since $\overline{X}_0(N_n)$ is birational to $q_2(X_{\psi_n}(S))$, it suffices to show that the latter has genus 0.

Take a homomorphism $h\in\mathrm{Hom}(\pi_1(N_n),\mathbb{Z}_2)$. A direct calculation shows that either $(h(a),h(b))=(1,1)$ or $(h(a),h(b))=(-1,-1)$ since n is odd. Hence $q_2(X_{\psi_n}(S))$ is the quotient of $X_{\psi_n}(S)$ by the involution $(x,y,z)\to(-x,-y,z)$.

Recall that $X_{\psi_n}(S)$ is birational to its restriction V_n via the maps given in (5-2). Let $\overline{V}_n$ be the variety corresponding to the identification of V_n under the involution $(x,z)\to(-x,z)$. Letting $(X,Z)=(x^2,z)$, $\overline{V}_n$ is defined by

$$\bar{h}_k(X,Z):=2(f_{k+2}(Z)-f_{k+1}(Z))Z-(f_{k+1}(Z)-f_k(Z))X.$$

The genus of $\overline{V}_n$ is seen to be 0 by a simple modification of the argument of Proposition 42. □

For each irreducible $\mathrm{PSL}(2,\mathbb{C})$-representation $\bar{\rho}$ which does not lift, we may choose $T, A, B\in \mathrm{SL}(2,\mathbb{C})$ with the properties $\bar{\rho}(t)=\{\pm T\}$, $\bar{\rho}(a)=\{\pm A\}$, $\bar{\rho}(b)=\{\pm B\}$ and

$$T^{-1}AT=-\psi_n(A),\quad T^{-1}BT=\psi_n(B).$$

Using the convention from (4-12)–(4-14), the traces of these representatives under the restriction map r are contained in the set

$$X^2_{\psi_n}(S)=\{(x,y,z)\in\mathbb{C}^3\mid(-x,y,-z)=\overline{\psi}_n(x,y,z)\}=V(P_{n+2}+x,P_{n+1}-y,xy).$$

This implies that the components of $X^2_{\psi_n}(S)$ are

$$L_3=\{(0,0,z)\mid z\in\mathbb{C}\},\tag{5-3}$$

$$C_\xi=\{(0,y,\xi)\mid y\in\mathbb{C}\},\ \text{where } \xi \text{ is a root of } f_{k+2}(z)-f_{k+1}(z)=0,\tag{5-4}$$

$$C_\zeta=\{(x,0,\zeta)\mid x\in\mathbb{C}\},\ \text{where } \zeta \text{ is a root of } f_{k+1}(z)-f_k(z)=0.\tag{5-5}$$

For each of the components $C_z\subset X^2_{\psi_n}(S)$, the map $\overline{X}(N_n)\supset\bar{r}^{-1}q_2(C_z)\to\overline{X}_{\psi_n}(S)$ is a birational equivalence onto its image since the irreducible characters in (5-4) and (5-5) arise from $\mathrm{PSL}(2,\mathbb{C})$-representations of $\pi_1(S)$ with trivial centraliser.

Points in $X_{\psi_n}(S)$ satisfy $2z=xy$, and none of the roots ξ or ζ equal zero. Hence the only point in $X_{\psi_n}(S)\cap X^2_{\psi_n}(S)$ is $(0,0,0)$, and the existence of this intersection point is due to the fact that we are not working with $\overline{X}(N_n)$, but with $\overline{X}_{\psi_n}(S)$. The two representations of N_n predicted by Lemma 23 are given by letting

$$\bar{\rho}(a)=\pm\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \bar{\rho}(b)=\pm\begin{pmatrix}i&0\\0&-i\end{pmatrix}$$

and

$$T_1=\pm\begin{pmatrix}\frac{-i}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}\end{pmatrix},\quad T_2=\bar{\rho}(a)T_1=\pm\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}\\\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}.$$

Then $\bar{\rho}$ extends to a representation into $\mathrm{PSL}(2,\mathbb{C})$ by letting either $\bar{\rho}=T_1$ or $\bar{\rho}=T_2$, and the former lifts to $\mathrm{SL}(2,\mathbb{C})$ whilst the latter doesn't. Denote the characters in $\overline{X}(N_n)$ of the 2 extensions as $\bar{\chi}_1$ and $\bar{\chi}_2$. According to the discussion in Section 2.1, $\bar{\chi}_1$ and $\bar{\chi}_2$ are on different topological components of $\overline{X}(N_n)$. Note that $\bar{\chi}_1$ is in $\overline{X}_0(N_n)$ while $\bar{\chi}_2$ is in $\bar{r}^{-1}q_2(L_3)$.

For the component $L_3 \subset X^2_{\psi_n}(S)$, we follow the same parametrisation of $\bar{L}_3$ in (4-9). For any point $(0,0,(p+\frac{1}{p})^2,0) \in \bar{L}_3$, a direct computation shows that it extends to 2 characters in $\overline{X}(N_n)$ where the quadruples in (4-4) are

$$\left(2-p^{n+2}-\frac{1}{p^{n+2}},0,0,2-p^{n+4}-\frac{1}{p^{n+4}}\right) \quad\text{and}\quad \left(2+p^{n+2}+\frac{1}{p^{n+2}},0,0,2+p^{n+4}+\frac{1}{p^{n+4}}\right).$$

Thus $\bar{r}^{-1}q_2(L_3) \to \bar{L}_3$ is a 2-fold branched cover with the only ramification point $\bar{\chi}_2$.

5.5 The family L_n

Let L_n be the once-punctured torus bundle with monodromy $\omega_n = A^2B^{n+2}A$. Note that $\mathrm{tr}(\omega_n) = -3n-4$ so L_n is hyperbolic when n is positive. When n is odd, $b_1(\omega_n) = 1$ and hence $\overline{X}^{\mathrm{irr}}(L_n) \cong X_{\omega_n}(S)$ according to Corollary 3. We now show:

Theorem 48 *Suppose $n = 2k+1 \geq 3$ is odd. The* $\mathrm{PSL}(2,\mathbb{C})$*-character variety $\overline{X}^{\mathrm{irr}}(L_n)$ consists of $k+2$ components. The canonical curve $\overline{X}_0(L_n) \subset \overline{X}^{\mathrm{irr}}(L_n)$ is birationally equivalent to a hyperelliptic curve of genus k while all the other Zariski components are birational to the affine line.*

We remark that if $n = 1$, then $\overline{X}^{\mathrm{irr}}(L_n)$ has 3 components, each of genus zero.

The framing ω_n of L_n is

$$\omega_n = \begin{cases} a \to a(a^{-2}b^{-1})^{n+2}, \\ b \to ba^3(a^{-2}b^{-1})^{n+2}. \end{cases}$$

Then $X_{\omega_n}(S)$ is defined by

$$\begin{aligned} \mathrm{tr}\,\rho(a) &= \mathrm{tr}\,\rho(a(a^{-2}b^{-1})^{n+2}), \\ \mathrm{tr}\,\rho(b) &= \mathrm{tr}\,\rho(ba^3(a^{-2}b^{-1})^{n+2}), \\ \mathrm{tr}\,\rho(ab) &= \mathrm{tr}\,\rho(a(a^{-2}b^{-1})^{n+2}ba^3(a^{-2}b^{-1})^{n+2}). \end{aligned}$$

Using the trace identities in Section 2, the second equation is $\mathrm{tr}\,\rho(b) = \mathrm{tr}\,\rho(a(a^{-2}b^{-1})^{n+1})$ and the third equation is given by $z = xy - zx + y$. Let $Q_n(x,y,z) := \mathrm{tr}\,\rho(a(a^{-2}b^{-1})^n)$. We have $Q_0 = x$, $Q_1 = z$ and $Q_n = (xz-y)Q_{n-1} - Q_{n-2}$. Note that the third equation is equivalent to $(y-z)(x+1) = 0$ and hence naturally splits the variety $X_{\omega_n}(S)$ into the union of two algebraic sets:

$$X_{\omega_n}(S) = V(\langle Q_{n+2}-x, Q_{n+1}-y, x+1\rangle) \cup V(\langle Q_{n+2}-x, Q_{n+1}-y, y-z\rangle)$$

We denote the two algebraic sets in this union by R_n and W_n, respectively.

Lemma 49 *Each canonical component of $X^{\mathrm{irr}}(L_n)$ is contained in the preimage of W_n.*

Proof *We only need to show that each discrete and faithful character $\chi_{\rho_0} \in X^{\mathrm{irr}}(L_n)$ is not in the preimage of R_n. Clearly ρ_0 is not binary dihedral, so we can write $\rho_0(a)$ and $\rho_0(b)$ as in* (2-11)*. If $x = -1$, a simple calculation shows that $\rho_0(a)$ is of order* 3*, which contradicts the faithfulness of ρ_0 since a has infinite order in $\pi_1(M_{\omega_n})$.* □

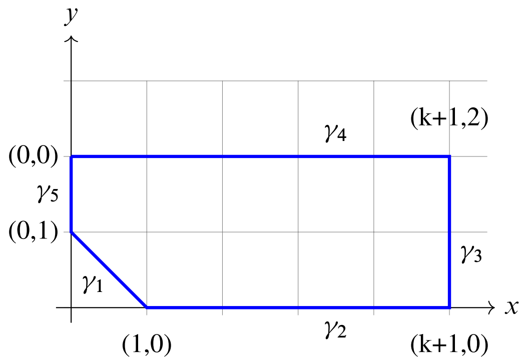

Figure 3: The Newton polygon of r_k.

We will analyse W_n and R_n separately. Before doing so, we define a family of auxiliary polynomials $Q'_n \in \mathbb{C}[x,y,z]$, which follow the same recursive relation as Q_n but with $Q'_0 = x$ and $Q'_1 = y$. When $n = 2k+1$ is odd, a similar argument as in Lemma 39 shows that

$$\langle Q_{n+2} - Q'_0, Q_{n+1} - Q'_1\rangle = \langle Q_{k+2} - Q'_{k+1}, Q_{k+1} - Q'_{k+2}\rangle.$$

In addition, we observe that

$$Q_n = z f_n(xz-y) - x f_{n-1}(xz-y) \quad \text{and} \quad Q'_n = y f_n(xz-y) - x f_{n-1}(xz-y).$$

5.5.1 The canonical component Under the map $(x,y,z) \mapsto (x-1,y)$, W_n is birationally equivalent with its image, denoted by $W'_n \subset \mathbb{C}^2$. As Q_n and Q'_n coincide when $y = z$, W'_n is defined by the single element $q_{k+2}(x,y) - q_{k+1}(x,y) \in \mathbb{C}[x,y]$, where

$$q_n(x,y) = y f_n(xy) - (x+1) f_{n-1}(xy).$$

Let $\mathscr{W}_n$ be the variety in $\mathbb{C}^2$ defined by the polynomial

$$r_k(x,y) := y^2(f_{k+2}(x) - f_{k+1}(x)) - (x+y)(f_{k+1}(x) - f_k(x)).$$

There exists a birational isomorphism between W'_n and $\mathscr{W}_n$ via

$$r_1 : W'_n \to \mathscr{W}_n, \quad (x,y) \mapsto (xy, y),$$
$$r_2 : \mathscr{W}_n \to W'_n, \quad (x,y) \mapsto (x/y, y).$$

Lemma 50 *$r_k(x,y)$ defined as above is an irreducible polynomial in $\mathbb{C}[x,y]$ for every positive odd integer $n = 2k+1 \geq 3$.*

Proof If $r_k(x,y)$ is reducible, then we can factorise it into $r_k(x,y) = s_k(x,y)t_k(x,y)$, where t_n, s_n are nonconstant polynomials in $\mathbb{C}[x,y]$. The Newton polygon of r_k is shown in Figure 3. The boundary vector sequence of $r_k(x,y)$ is

$$v(r_k) = \big((1,0),\dots,(1,0),(0,1),(0,1),(-1,0),(-1,0),\dots,(-1,0),(0,-1),(1,-1)\big),$$

where the number or repetitions of $(1,0)$ and $(-1,0)$ in the sequence are k and $k+1$, respectively. Given

$(0,1)$, $(0,-1)$ and $(1,-1)$ are the only vectors with nonzero vertical component, there are two cases to split $v(r_k)$ into two disjoint nonempty subsequences $v(s_k)$ and $v(t_k)$, each of which sums to zero.

(1) All the four vectors with nonzero vertical component are in the same subsequence. Without loss of generality, let $s_k(x,y) = s_k(x)$ and $t_k(x,y) = t_k^{(2)}(x)\ y^2 + t_k^{(1)}(x)\ y + t_k^{(0)}(x)$, where the superscript i indicates the degree in y that the polynomial $t_k^{(i)} \in \mathbb{C}[x]$ is a coefficient of. Then

$$\begin{aligned} f_{k+2}(x) - f_{k+1}(x) &= t_k^{(2)}(x) s_k(x), \\ -(f_{k+1}(x) - f_k(x)) &= t_k^{(1)}(x) s_k(x), \\ -x(f_{k+1}(x) - f_k(x)) &= t_k^{(0)}(x) s_k(x). \end{aligned}$$

This is impossible due to Lemma 34(5).

(2) Hence $(1,-1)$, $(0,1)$ belong to one sequence, and $(0,1)$, $(0,-1)$ belong to the other sequence. Let $r_k(x,y) = (a(x)y + b(x))(c(x)y + d(x))$ where a, b, c, d are polynomials in $\mathbb{C}[x]$. Then

$$\begin{aligned} f_{k+2}(x) - f_{k+1}(x) &= a(x)c(x), \\ -(f_{k+1}(x) - f_k(x)) &= a(x)d(x) + b(x)c(x), \\ -x(f_{k+1}(x) - f_k(x)) &= b(x)d(x). \end{aligned}$$

Without loss of generality, take a root α of $f_{k+1}(x) - f_k(x)$ which is also a root of $b(x)$. The third equation and Lemma 34(4) imply $d(\alpha) \neq 0$. The first equation and Lemma 34(5) imply $a(\alpha) \neq 0$. But this gives a contradiction to the second equation.

Hence $r_k(x,y)$ is irreducible. □

Remark 51 When $n = 1$, $W_1 = V(q_2 - q_1) = V(y+1) \cup V(xy - x - 1)$. Both components have genus 0.

Lemma 52 *When $n = 2k+1 \geq 3$, r_k is nondegenerate and there is no singular point in $V_n = V(r_k)$. Hence V_k is a hyperelliptic curve of genus k.*

Proof *The proof is straight forward using the same arguments as in the proof of Lemma 41.* □

5.5.2 Other components When $n = 2k+1$, $R_n = V(Q_{k+2} - Q'_{k+1}, Q_{k+1} - Q'_{k+2}, x+1)$ is birationally equivalent with its image, denoted by R'_n, under the restriction $(x,y,z) \mapsto (y,z)$. The variety R'_n is defined by $q'_{k+2} - q_{k+1}$ and $q_{k+2} - q'_{k+1}$, where

$$q_n(y,z) = z f_n(-z-y) + f_{n-1}(-z-y), \quad q'_n(y,z) = y f_n(-z-y) + f_{n-1}(-z-y).$$

By a direct computation using the definition of f_n,

$$R_n = \{(-1,-1)\} \cup V(f_{k+2}(-y-z) + f_{k+1}(-y-z)).$$

φ_*	$\mathrm{tr}(\varphi_*)$	$o(\varphi_2)$	$\#X_\varphi(S)$	genera
A^2B^3	-4	2	1	0
A^3B^2	-4	2	1	0
AB^5	-3	3	1	0
A^2B^4	-6	1	2	0, 0
A^3B^3	-7	3	3	0, 0, 0
A^4B^2	-6	1	2	0, 0

Table 1: Short words in A and B giving hyperbolic once-punctured torus bundles

The point $(-1,-1,-1)$ is already contained in W_n while the second algebraic set gives us $k+1$ components

$$\{(x,y,z) \mid y+z+\alpha=0, x+1=0\}$$

as α ranges over the $k+1$ distinct roots of $f_{k+2}(u)+f_{k+1}(u)=0$. All these components are birationally equivalent with affine lines.

5.6 Experimental results

As stated in Section 2.2, the choice of a monodromy as a positive word in A and B is not unique. Also, there is no simple criterion to ensure that the absolute value of the trace is greater than two. We list in Table 1 the shortest words with this property. The computations were executed with Singular [10] using the following code template, and $\#X_\varphi(S)$ denotes the number of Zariski components of $X_\varphi(S)$. The map m below represents $\overline{\varphi}$ and is given as a composition of maps alpha and betainv:

```
ring r=0,(x,y,z),dp;
map alpha=r,x,z,xz-y;
map betainv=r,z,y,yz-x;
map m = [composition of alpha and betainv];
ideal I = m[1]-x,m[2]-y,m[3]-z;
def S = absPrimdecGTZ(I);
setring S;
absolute_primes;
```

The representative

$$\pm A^{a_1}B^{-b_1}A^{a_2}B^{-b_2}\cdots A^{a_n}B^{-b_n},$$

where $n>0$, the a_i and b_i are positive integers, and the sign equals the sign of the trace of φ_*, allows us to build up a census of examples more efficiently. As explained in Section 2.4, if one is only interested in the topology of the fixed-point set, then it suffices to restrict to the case of positive trace. We summarise our computational results in Table 2.

φ_*	$\mathrm{tr}(\varphi_*)$	$o(\varphi_2)$	$\#X_\varphi(S)$	genera
AB^{-1}	3	3	1	0
AB^{-2}	4	2	1	0
A^2B^{-1}	4	2	1	0
AB^{-3}	5	3	1	0
A^2B^{-2}	6	1	2	0, 0
A^3B^{-1}	5	3	1	0
$AB^{-1}AB^{-1}$	7	3	3	0, 0, 0
AB^{-4}	6	2	2	0, 0
A^2B^{-3}	8	2	1	1
A^3B^{-2}	8	2	1	1
A^4B^{-1}	6	2	2	0, 0
$AB^{-1}AB^{-2}$	10	2	3	0, 0, 0
$AB^{-1}A^2B^{-1}$	10	2	3	0, 0, 0
AB^{-5}	7	3	1	0
A^2B^{-4}	10	1	2	1, 0
A^3B^{-3}	11	3	2	1, 0
A^4B^{-2}	10	1	2	1, 0
A^5B^{-1}	7	3	1	0
$AB^{-1}AB^{-3}$	13	3	3	1, 0, 0
$AB^{-1}A^2B^{-2}$	15	3	1	2
$AB^{-1}A^3B^{-1}$	13	3	3	1, 0, 0
$AB^{-2}AB^{-2}$	14	1	5	0, 0, 0, 0, 0
$AB^{-2}A^2B^{-1}$	15	3	1	2
$AB^{-1}AB^{-1}AB^{-1}$	18	1	7	0, 0, 0, 0, 0, 0, 0

Table 2: Short words in A and B^{-1} giving hyperbolic once-punctured torus bundles.

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