AGT

Algebraic & Geometric Topology

Volume 25 (2025)

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Mitul Islam

Andrew Zimmer

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Algebraic & Geometric Topology 25:9 (2025) 5503–5539
DOI: 10.2140/agt.2025.25.5503
Published: 18 December 2025

# The structure of relatively hyperbolic groups in convex real projective geometry

MITUL ISLAM
ANDREW ZIMMER

We prove a general structure theorem for relatively hyperbolic groups (with arbitrary peripheral subgroups) acting naive convex cocompactly on properly convex domains in real projective space. We also establish a characterization of such groups in terms of the existence of an invariant collection of closed unbounded convex subsets with good isolation properties. This is a real projective analogue of results of Hindawi–Hruska–Kleiner for CAT(0) spaces. We also obtain an equivariant homeomorphism between the Bowditch boundary of the group and a quotient of the ideal boundary.

20F65, 20F67, 22E40, 57N16, 57S20

## 1 Introduction

Let $\mathbb{H}^d$ denote real hyperbolic $d$-space and recall that a discrete subgroup $\Gamma \leq \mathrm{Isom}(\mathbb{H}^d)$ is called *convex cocompact* if there exists a nonempty $\Gamma$-invariant geodesically convex closed subset $\mathcal{C} \subset \mathbb{H}^d$ where the quotient $\Gamma \backslash \mathcal{C}$ is compact. The Beltrami–Klein model realizes $\mathbb{H}^d$ as a properly convex domain $\mathbb{B}^d \subset \mathbb{P}(\mathbb{R}^{d+1})$ (namely the Euclidean unit ball in a standard affine chart) in such a way that the isometry group $\mathrm{Isom}(\mathbb{H}^d)$ coincides with the subgroup of $\mathrm{PGL}_{d+1}(\mathbb{R})$ which preserves $\mathbb{B}^d$ (namely $\mathrm{PO}(d,1)$ up to conjugation). Further, in this model, a subset being geodesically convex is equivalent to being convex in some (hence any) affine chart that contains $\mathbb{B}^d$.

The Beltrami–Klein model perspective allows one to naturally generalize the classical notion of convex cocompact groups. In particular, one can consider a general properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and the group $\mathrm{Aut}(\Omega) \leq \mathrm{PGL}_d(\mathbb{R})$ of projective automorphisms which preserve $\Omega$. Then a discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$ is called *naive convex cocompact* if there exists a nonempty $\Gamma$-invariant closed convex subset $\mathcal{C} \subset \Omega$ where the quotient $\Gamma \backslash \mathcal{C}$ is compact. In this case, we say that $(\Omega, \mathcal{C}, \Gamma)$ is a *naive convex cocompact triple*.

When the group $\Gamma$ is word hyperbolic (eg in the classical real hyperbolic setting), there is a close connection between this notion of naive convex cocompact groups and Anosov representations/higher Teichmüller theory; see [7; 27]. As one moves beyond the word hyperbolic case, the structure of these discrete groups becomes more mysterious.

In this paper, we study naive convex cocompact groups which are (intrinsically) relatively hyperbolic groups. This is a rich class with many examples, for instance where:

(a) $\Gamma$ is a projective reflection group generated by reflection along faces of a projective Coxeter polytope à la Vinberg and $\Gamma$ is irreducible as a Coxeter group; see [8; 19].

(b) $\Gamma$ is isomorphic to $\pi_1(M)$ where $M$ is a closed three-manifold such that each geometric component in its JSJ decomposition supports a hyperbolic structure; see [1; 17].

In both of these cases, the groups are relatively hyperbolic with respect to peripheral subgroups which are virtually abelian of rank at least two. However, there also exist examples where $\Gamma$ is relatively hyperbolic with respect to non-virtually abelian subgroups; see the discussion in [23, Section 2.6.3]. The primary goal of this paper is to describe the structure of such examples. One of our main results is proving that the relative hyperbolicity of $\Gamma$ is equivalent to the existence of a so-called *peripheral family*, a $\Gamma$-invariant collection of closed convex subsets with good isolation properties.

This investigation extends some recent work. Previously in [15], we considered the special case where the peripheral subgroups were virtually abelian of rank at least two. For such groups, we proved that relative hyperbolicity is equivalent to the existence of a collection of properly embedded simplices with good isolation properties. This geometric description is analogous to the case of CAT(0) spaces with isolated flats [13]. We also showed that the boundary of these simplices are, in a technical sense, the only places where the boundary is irregular; see [15, Theorem 1.19(6), Theorem 1.8(7) and (8)]. Shortly after, Weisman [23] considered convex cocompact groups (a more restrictive class than naive convex cocompact) who were relatively hyperbolic and whose peripheral subgroups were also convex cocompact, but not necessarily virtually abelian. For such groups he established a similar result about the irregular boundary points and also showed that the Bowditch boundary could be realized as a quotient of the boundary.

There are a number of other results in the literature concerning relatively hyperbolic groups acting on properly convex domains; see for instance [3; 4; 5; 6]. These results consider the case when $\Gamma \backslash \mathcal{C}$ is noncompact and characterize when $\Gamma$ is relatively hyperbolic with respect to the fundamental groups of the ends (under some geometric assumptions on the ends and $\mathcal{C}$). There is some similarity between the statements in this paper and the statements in [3; 4; 5], but to the best of our knowledge, there is no actual overlap between the results.

We extend the results in [15; 23] to the case of general peripheral subgroups and the case of general naive convex cocompact groups. The proofs build upon ideas from both papers.

We will now introduce the notation required to precisely state our main results. Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, we will let $\mathrm{d}_\Omega$ denote the Hilbert metric, which is a natural and classical $\mathrm{Aut}(\Omega)$-invariant, proper, and complete metric on $\Omega$; see Section 2.2 below. This allows us to speak of the diameter $\mathrm{diam}_\Omega(A)$ and $r$-open neighborhood $\mathcal{N}_\Omega(A, r) \subset \Omega$ of a subset $A \subset \Omega$ relative to the Hilbert metric.

**Definition 1.1** Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\mathcal{C} \subset \Omega$ is a closed[1] convex subset, $\Gamma \leq \mathrm{Aut}(\Omega)$, and $\mathcal{X}$ is a collection of closed unbounded convex subsets of $\mathcal{C}$. Then we say that:

(1) $\mathcal{X}$ is $\Gamma$*-invariant* if $\Gamma \cdot \mathcal{X} = \mathcal{X}$.

(2) $\mathcal{X}$ is *strongly isolated* if for every $r > 0$ there exists $D_1(r) > 0$ such that if $X_1, X_2 \in \mathcal{X}$ are distinct, then

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(X_1, r) \cap \mathcal{N}_\Omega(X_2, r)) \leq D_1(r).$$

(3) $\mathcal{X}$ *coarsely contains the properly embedded simplices of* $\mathcal{C}$ if there exists $D_2 > 0$ such that if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset \mathcal{N}_\Omega(X, D_2)$.

When $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ satisfies all three of the above conditions, we say that $\mathcal{X}$ is a *peripheral family of* $(\Omega, \mathcal{C}, \Gamma)$.

Given a closed convex subset $\mathcal{C} \subset \Omega$ of a properly convex domain, the *ideal boundary* of $\mathcal{C}$ is $\partial_i\, \mathcal{C} := \overline{\mathcal{C}} \cap \partial\Omega$ (see Section 2.3). Also, given $x \in \overline{\Omega}$ we will let $F_\Omega(x)$ denote the open face of $x$ in $\overline{\Omega}$ and given $A \subset \overline{\Omega}$ we will let $F_\Omega(A) = \bigcup_{x \in A} F_\Omega(x)$. Using these boundary faces and a peripheral family, one can define a natural quotient of the ideal boundary:

**Definition 1.2** Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$. Let

$$[\partial_i\, \mathcal{C}]_{\mathcal{X}} := \partial_i\, \mathcal{C}/\sim$$

be the topological quotient where $x \sim y$ if and only if

(1) $x, y \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$, or

(2) $F_\Omega(y) = F_\Omega(x)$.

**Remark 1.3** We remark that similar boundary quotients are considered by Choi [5] and Weisman [23].

We need one more piece of notation to state our main result: the *limit set* of a subgroup $G \leq \mathrm{Aut}(\Omega)$ is

$$\mathcal{L}_\Omega(G) := \partial\Omega \cap \bigcup_{p \in \Omega} \overline{G \cdot p}.$$

This definition ensures that $\mathcal{L}_\Omega(G)$ is base point independent — unlike real hyperbolic geometry, the accumulation points of a single orbit in $\Omega$ may depend on the base point.

Our main result — Theorem 1.4 below — is essentially an equivalence statement for a naive convex cocompact triple $(\Omega, \mathcal{C}, \Gamma)$: the relative hyperbolicity of the group $\Gamma$ is equivalent to the existence of a peripheral family $\mathcal{X}$. Theorem 1.4(1) shows that the group-theoretic property of relative hyperbolicity implies the geometric property of admitting a peripheral family. Theorem 1.4(2) shows the converse.

[1]We make it our convention that, "$Y \subset \Omega$ is a closed subset" always means that $Y$ is closed in the subspace topology inherited from $\Omega$.

**Theorem 1.4** *Suppose that* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex cocompact triple. Then the following hold*:

(1) *If* $\Gamma$ *is relatively hyperbolic with respect to* $\mathcal{P} = \{P_1, \dots, P_N\}$ *and* $X_j$ *is the closed convex hull of* $\mathcal{L}_\Omega(P_j) \cap \partial_i \mathcal{C}$ *in* $\Omega$, *then*

$$\mathcal{X} := \Gamma \cdot \{X_1, \dots, X_N\}$$

*is a peripheral family of* $(\Omega, \mathcal{C}, \Gamma)$.

(2) *If* $\mathcal{X}$ *is a peripheral family of* $(\Omega, \mathcal{C}, \Gamma)$ *and* $\mathcal{P} := \{P_1, \dots, P_N\}$ *is a set of representatives of the* $\Gamma$*-conjugacy classes in* $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$, *then* $\Gamma$ *is relatively hyperbolic with respect to* $\mathcal{P}$.

*Moreover*, *when either of the above conditions are satisfied*, *then*:

(a) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to* $\mathcal{X} = \Gamma \cdot \{X_1, \dots, X_N\}$.

(b) *There is a* $\Gamma$*-equivariant homeomorphism between the Bowditch boundary* $\partial(\Gamma, \mathcal{P})$ *and* $[\partial_i \mathcal{C}]_{\mathcal{X}}$.

(c) *Each* $(\Omega, X_j, P_j)$ *is a naive convex cocompact triple.*

(d) *There exists* $L > 0$ *such that if* $x \in \partial_i \mathcal{C}$ *and* $\mathrm{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L$, *then* $x \in F_\Omega(\partial_i X)$ *for some* $X \in \mathcal{X}$.

(e) *There exists* $R > 0$ *such that if* $X \in \mathcal{X}$ *and* $x \in \partial_i X$, *then*

$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i X,\ F_\Omega(x) \cap \partial_i \mathcal{C}) \leq R.$$

**Remark 1.5** Properties (d) and (e) are somewhat technical. Informally, property (d) states that any boundary face of $\Omega$ that $\partial_i \mathcal{C}$ intersects in a "large set" must intersect the ideal boundary of an element in $\mathcal{X}$. Property (e) informally states that if $X \in \mathcal{X}$, then $\partial_i X$ coarsely contains any boundary face of $\partial_i \mathcal{C}$ that it intersects.

Theorem 1.4(1) and (2) can be viewed as a real projective analogue of the results of Hruska–Kleiner and Hindawi–Hruska–Kleiner [13] in the setting of CAT(0)-geometry. This earlier work motivated the results in this paper, but the methods of proof are very different. In particular, we should also note that an old result of Kelly–Straus [18] says that a Hilbert geometry $(\Omega, \mathrm{d}_\Omega)$ is CAT(0) if and only if it is isometric to real hyperbolic $(d-1)$-space (in which case $\Omega$ coincides, up to a change of coordinates, with the Beltrami–Klein model of real hyperbolic $(d-1)$-space).

As mentioned above, in previous work [15] we proved a version of Theorem 1.4 in the special case when $\mathcal{X}$ consisted of properly embedded simplices of dimension at least two and the subgroups in $\mathcal{P}$ were virtually abelian of rank at least two. In this case, using the simple structure of simplices and abelian subgroups in naive convex cocompact groups (see [14]), one can weaken the strongly isolated assumption to only assuming that $\mathcal{X}$ is closed and discrete. In fact, a substantial portion of this earlier work was building a strongly isolated collection of properly embedded simplices from a closed and discrete collection. It seems unlikely to us that such a weakening is possible in the general case.

## 1.1 Convex cocompact groups

As mentioned above, the class of naive convex cocompact groups includes the more restrictive, but still interesting, class of convex cocompact groups. For this class of groups, Theorem 1.4 can be restated in a much simpler way.

Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and a discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$, let $\mathcal{C}_\Omega(\Gamma) \subset \Omega$ denote the closed convex hull of $\mathcal{L}_\Omega(\Gamma)$ in $\Omega$.

**Definition 1.6** A discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$ is called *convex cocompact* if $\mathcal{C}_\Omega(\Gamma)$ is nonempty and the quotient $\Gamma \backslash \mathcal{C}_\Omega(\Gamma)$ is compact.

Every convex cocompact subgroup is clearly naive convex cocompact, but the converse is not true; see [15, Section 2.3] for some examples. One key difference between the two definitions is that if $\Gamma \leq \mathrm{Aut}(\Omega)$ is convex cocompact, then any open boundary face of $\Omega$ which intersects $\partial_i \mathcal{C}_\Omega(\Gamma)$ is actually contained in $\partial_i \mathcal{C}_\Omega(\Gamma)$. On the other hand, if $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple, then it is possible for $\partial_i \mathcal{C}$ to intersect a boundary face of $\Omega$ in a small set. This seemingly small difference makes convex cocompact groups much easier to study.

For convex cocompact groups, Theorem 1.4(a)–(e) can be restated and expanded as follows.

**Theorem 1.7** (see Section 10) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact subgroup. If $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P} = \{P_1, \dots, P_N\}$, then*:

(a) *$(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to*

$$\mathcal{X} := \Gamma \cdot \{\mathcal{C}_\Omega(P_1), \dots, \mathcal{C}_\Omega(P_N)\}.$$

(b) *Let $[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ denote the quotient of $\partial_i \mathcal{C}_\Omega(\Gamma)$ obtained by collapsing each limit set $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ to a point (where $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$). Then there is a $\Gamma$-equivariant homeomorphism between the Bowditch boundary $\partial(\Gamma, \mathcal{P})$ and $[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$.*

(c) *Each $P_j$ is a convex cocompact subgroup of* $\mathrm{Aut}(\Omega)$.

(d) *If $x \in \partial_i \mathcal{C}_\Omega(\Gamma)$ is not a $\mathcal{C}^1$-smooth point of $\Omega$ (ie $\Omega$ does not have a unique supporting hyperplane at $x$), then $x \in \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

(e) *If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

We should note that the main new content of Theorem 1.7 is part (c). In particular:

(1) Part (a) is an immediate consequence of part (c).

(2) In [15] we previously proved parts (d) and (e) in the case where each $P_j$ is virtually abelian with rank at least two, and once part (c) is known, the same argument works in the more general setting.

(3) Weisman [23] established parts (b) and (e) with part (c) as an assumption.

In the context of convex cocompact groups, peripheral families can also be defined in terms of their boundary behavior.

**Proposition 1.8** (see Section 10) *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact subgroup. If $\mathcal{X}$ is a $\Gamma$-invariant collection of closed unbounded convex subsets of $\Omega$, then the following are equivalent*:

(1) *$\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}_\Omega(\Gamma), \Gamma)$.*

(2) *$\mathcal{X}$ has the following properties*:

  (a) *$\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric $\mathrm{d}_\Omega$.*

  (b) *If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \varnothing$.*

  (c) *If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.*

*Moreover, when the above conditions are satisfied, then*

(1-1) $$[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}} = [\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{X}},$$

*where $\mathcal{P}$ is a set of representatives of the $\Gamma$-conjugacy classes in $\{\mathrm{Stab}_\Gamma(X) : X \in \mathcal{X}\}$.*

Previously, Weisman [23, Theorem 1.16] characterized when a convex cocompact subgroup is hyperbolic relative to a collection of convex cocompact subgroups in terms of the behavior of the limit sets of the subgroups. We note that combining Proposition 1.8 and Theorem 1.4 provides a "subset" version of this characterization.

## 1.2 The case of Gromov hyperbolic groups

A word hyperbolic group is relatively hyperbolic with respect to the empty set. Thus if $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\Gamma$ is a word hyperbolic group, then Theorem 1.4 holds with $\mathcal{X} = \varnothing$.

However, the choice of this peripheral family $\mathcal{X}$ is not canonical even for a convex cocompact group $\Gamma$ (unlike our work in [15] where $\mathcal{X}$ was the set of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimension at least two). For instance, suppose that $\Gamma := \langle a, b \rangle$ is a convex cocompact subgroup of $\mathrm{PSO}(2,1)$ that is isomorphic to a free group on two generators. Then $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P} := \{\langle g \rangle\}$ where $g := aba^{-1}b^{-1}$. Then $\mathcal{X} := \Gamma \cdot (g^+, g^-)$ where $(g^+, g^-)$ is the unique $g$-invariant projective line in the Beltrami–Klein model of $\mathbb{H}^2$. On the other hand, we can also choose $\mathcal{X} = \varnothing$ (when $\mathcal{P} = \varnothing$).

## 1.3 Outline of the paper and proofs

Sections 2 and 3 are expository. In Section 2 we recall the basic definitions and results about properly convex domains that we will require and in Section 3 we recall some properties of relatively hyperbolic spaces and groups.

Sections 4 through 9 are devoted to the proof of Theorem 1.4. The difficult part of Theorem 1.4 is part (2). Our general strategy is based on combining ideas from [15] and [23]. In particular, as in [23], we will use Yaman's characterization of relatively hyperbolicity to show that $\Gamma$ is relatively hyperbolic. We will verify the conditions in Yaman's theorem by further developing the ideas used in [15] to study closed and discrete collections of properly embedded simplices.

We should also note that Choi [5] (which predates the work of Weisman) used Yaman's characterization to verify that the fundamental groups of certain noncompact convex real projective manifolds were relatively hyperbolic with respect to the fundamental groups of their ends.

The arguments used to verify the conditions of Yaman's theorem in this paper are somewhat similar to the analogous ones in [5; 23]. But given the different setups, there does not seem to be an easy way to reduce the proofs in this paper to any lemmas in these two works.

The content of these sections is as follows:

(1) In Section 4 we prove a quantitative result which informally states that long line segments in an open boundary face (relative to the Hilbert metric of the face) imply the existence of nearby properly embedded simplices.

(2) In Section 5 we establish a number of useful properties of peripheral families.

(3) In Section 6 we prove that peripheral subgroups of a relatively hyperbolic naive convex cocompact subgroup are themselves naive convex cocompact.

(4) In Section 7 we prove a technical result which will allow us to show that nonconical limit points are bounded parabolic points. A key tool here is a "closest subset" projection map.

(5) In Section 8 we prove some basic properties of the quotient space $[\partial_i \mathcal{C}]_{\mathcal{X}}$, including a sufficient condition for a point to be a conical limit point.

(6) In Section 9 we put everything together and prove Theorem 1.4.

Finally, in Section 10, we use Theorem 1.4 to prove Theorem 1.7.

### Acknowledgements

M Islam thanks Louisiana State University for hospitality during a visit in March 2020 where work on this project started.

M Islam was partially supported by Emmy Noether Project 427903332 (funded by the DFG), DMS-1607260 (National Science Foundation), and Rackham Predoctoral Fellowship (University of Michigan). A Zimmer was partially supported by grants DMS-2105580 and DMS-2104381 from the National Science Foundation.

The authors thank the reviewers for their thorough reading, comments, and suggestions.

## 2 Preliminaries

### 2.1 Notation

If $V \subset \mathbb{R}^d$ is a nonzero linear subspace, we will let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^d)$ denote its projectivization. In most other cases, we will use $[o]$ to denote the projective equivalence class of an object $o$, for instance:

(1) If $v \in \mathbb{R}^d \setminus \{0\}$, then $[v]$ denotes the image of $v$ in $\mathbb{P}(\mathbb{R}^d)$.

(2) If $\phi \in \mathrm{GL}_d(\mathbb{R})$, then $[\phi]$ denotes the image of $\phi$ in $\mathrm{PGL}_d(\mathbb{R})$.

(3) If $T \in \mathrm{End}(\mathbb{R}^d) \setminus \{0\}$, then $[T]$ denotes the image of $T$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$.

We also standardize some metric notations. If $(X, \mathrm{d})$ is a metric space, $p \in X$, $A \subset X$, and $r > 0$, then:

(1) $\mathcal{N}_X(A, r) := \{x \in X : \mathrm{d}(x, A) < r\}$.

(2) $\mathcal{B}_X(p, r) := \{x \in X : \mathrm{d}(x, p) < r\}$.

(3) $\mathrm{diam}_X(A) := \sup\{\mathrm{d}(x, x') : x, x' \in A\}$.

## 2.2 Convexity and the Hilbert metric

A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is

(1) *properly convex* if there exists an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a bounded convex subset;

(2) a *properly convex domain* if $C$ is properly convex and open in $\mathbb{P}(\mathbb{R}^d)$.

Given a properly convex set $C \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset \overline{C}$, we define its *convex hull* as

$$\mathrm{ConvHull}_C(X) := \bigcap \big\{ Y : Y \text{ is a closed convex subset such that } X \subset Y \subset \overline{C} \big\}.$$

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$ there is no canonical line segment with endpoints $x$ and $y$, but we will use the following convention: if $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \overline{C}$, then (when the context is clear) we will let $[x, y]$ denote the closed line segment joining $x$ to $y$ which is contained in $\overline{C}$. In this case, we will also let $(x, y) = [x, y] \setminus \{x, y\}$, $[x, y) = [x, y] \setminus \{y\}$, and $(x, y] = [x, y] \setminus \{x\}$.

Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be a properly convex domain. If $x, y \in \Omega$ are distinct, let $[a, b] := \mathbb{P}(\mathrm{Span}\{x, y\}) \cap \overline{\Omega}$ where $a$ and $b$ are labeled such that $x \in [a, y]$ (ie the points are ordered $a$, $x$, $y$, $b$ along $[a, b]$). Then the *Hilbert distance* between $x$ and $y$ is defined to be

$$\mathrm{d}_\Omega(x, y) := \tfrac{1}{2} \log [a, x, y, b],$$

where

$$[a, x, y, b] := \frac{|x - b|\,|y - a|}{|x - a|\,|y - b|}$$

is the cross-ratio (here $|\cdot|$ is some (any) norm in some (any) affine chart which contains $a$, $x$, $y$, $b$). Then $(\Omega, \mathrm{d}_\Omega)$ is a complete geodesic metric space and $\mathrm{Aut}(\Omega)$ acts properly and by isometries on $\Omega$; see for instance [2, Section 28]. Further, the projective line segment $[x, y]$ is a geodesic for the Hilbert distance.

## 2.3 Boundaries and faces

Suppose that $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set. The *relative interior of* $C$, denoted by $\text{rel-int}(C)$, is the interior of $C$ in $\mathbb{P}(\mathrm{Span}\, C)$. We will say that $C$ is *open in its span* if $C = \text{rel-int}(C)$.

The *boundary of* $C$ is $\partial C := \overline{C} \setminus \text{rel-int}(C)$, the *ideal boundary of* $C$ is $\partial_i C := \partial C \setminus C$, and the *nonideal boundary of* $C$ is $\partial_n C := \partial C \cap C$.

**Definition 2.1** If $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set which is open in its span and $x \in \overline{C}$, let $F_C(x)$ denote the *open face* of $x$:

$$F_C(x) = \{x\} \cup \{y \in \overline{C} : \text{ there exists an open line segment in } \overline{C} \text{ containing } x \text{ and } y\}.$$

We have the following observation about faces; see [16, Appendix] for a proof.

**Observation 2.2** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.*

(1) $F_\Omega(x) = \Omega$ *when* $x \in \Omega$.

(2) $F_\Omega(x)$ *is open in its span.*

(3) $y \in F_\Omega(x)$ *if and only if* $x \in F_\Omega(y)$ *if and only if* $F_\Omega(x) = F_\Omega(y)$.

(4) *If* $x, y \in \overline{\Omega}$, $z \in (x, y)$, $p \in F_\Omega(x)$, *and* $q \in F_\Omega(y)$, *then*

$$(p, q) \subset F_\Omega(z).$$

*In particular,* $(p, q) \subset \Omega$ *if and only if* $(x, y) \subset \Omega$.

If $B \subset C \subset \mathbb{P}(\mathbb{R}^d)$ are properly convex sets, then we say that $B$ is *properly embedded* in $C$ if $B \hookrightarrow C$ is a proper map with respect to the subspace topology. Note that $B$ is properly embedded in $C$ if and only if $\partial_i B \subset \partial_i C$.

## 2.4 Limits of automorphisms

Every $T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$ induces a map

$$\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T) \to \mathbb{P}(\mathbb{R}^d)$$

defined by $x \mapsto T(x)$. We will frequently use the following observation.

**Observation 2.3** *If $\{T_n\}$ is a sequence in $\mathbb{P}(\text{End}(\mathbb{R}^d))$ converging to $T \in \mathbb{P}(\text{End}(\mathbb{R}^d))$, then*

$$T(x) = \lim_{n\to\infty} T_n(x)$$

*for all $x \in \mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$. The convergence is uniform on compact subsets of $\mathbb{P}(\mathbb{R}^d) \setminus \mathbb{P}(\ker T)$.*

We can view $\mathbb{P}(\text{End}(\mathbb{R}^d))$ as a compactification of $\text{PGL}_d(\mathbb{R})$ and then consider limits of automorphisms in this compactification.

**Proposition 2.4** [14, Proposition 5.6] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, and $\{g_n\}$ is a sequence in* $\text{Aut}(\Omega)$ *such that*:

(1) $g_n(p_0) \to x \in \partial\Omega$.

(2) $g_n^{-1}(p_0) \to y \in \partial\Omega$.

(3) $g_n \to T$ *in* $\mathbb{P}(\text{End}(\mathbb{R}^d))$.

*Then* image $T \subset$ Span $F_\Omega(x)$, $\mathbb{P}(\ker T) \cap \Omega = \varnothing$, *and* $y \in \mathbb{P}(\ker T)$.

In the case of "nontangential" convergence we can say more.

**Proposition 2.5** [14, Proposition 5.7] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $p_0 \in \Omega$, $x \in \partial\Omega$, $\{p_n\}$ is a sequence in $[p_0, x)$ converging to $x$, and $\{g_n\}$ is a sequence in $\mathrm{Aut}(\Omega)$ such that*

$$\sup_{n \geq 1} \mathrm{d}_\Omega(g_n(p_0), p_n) < +\infty.$$

*If $g_n \to T \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then $T(\Omega) = F_\Omega(x)$.*

Proposition 5.7 in [14] is stated differently and a proof of the statement above can be found in [28, Proposition 2.13].

## 2.5 Projective simplices

A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is called a *$k$-dimensional simplex* in $\mathbb{P}(\mathbb{R}^d)$ if there exists $g \in \mathrm{PGL}_d(\mathbb{R})$ such that

$$gS = \big\{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1, \ldots, x_{k+1} > 0\big\}.$$

In this case, we call the $k+1$ points

$$g^{-1}[1 : 0 : \cdots : 0],\ g^{-1}[0 : 1 : 0 : \cdots : 0],\ \ldots,\ g^{-1}[0 : \cdots : 0 : 1 : 0 : \cdots : 0] \in \partial S$$

the *vertices of $S$*.

The Hilbert metric on a simplex can be explicitly computed (see [11; 20, Proposition 1.7; 22]) and from this explicit form one obtains the following.

**Proposition 2.6** *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex, then $(S, \mathrm{d}_\Omega)$ is quasi-isometric to $\mathbb{R}^{\dim S}$.*

**Remark 2.7** By the definition of the Hilbert metric, $\mathrm{d}_S = \mathrm{d}_\Omega|_{S \times S}$. So the quasi-isometry constants only depend on $\dim S$.

## 2.6 The Hausdorff distance and local Hausdorff topology

When $(X, \mathrm{d})$ is a metric space, the *Hausdorff distance* between two subsets $A, B \subset X$ is defined by

$$\mathrm{d}_X^{\mathrm{Haus}}(A, B) = \max\Big\{\sup_{a \in A} \inf_{b \in B} \mathrm{d}(a, b),\ \sup_{b \in B} \inf_{a \in A} \mathrm{d}(a, b)\Big\}.$$

When $(X, \mathrm{d})$ is a complete metric space, $d_X^{\mathrm{Haus}}$ is a complete metric on the set of nonempty compact subsets of $X$.

The local Hausdorff topology is a natural topology on the set of nonempty closed sets in $X$. For a nonempty closed set $C_0$, a base point $x_0 \in X$, and $r_0, \epsilon_0 > 0$, define the set $U(C_0, x_0, r_0, \epsilon_0)$ to consist of all closed subsets $C \subset X$ where

$$\mathrm{d}_X^{\mathrm{Haus}}(C_0 \cap \mathcal{B}_X(x_0, r_0),\ C \cap \mathcal{B}_X(x_0, r_0)) < \epsilon_0.$$

The *local Hausdorff topology* on the set of nonempty closed subsets of $X$ is the topology generated by the sets $U(\cdot, \cdot, \cdot, \cdot)$. This is the same topology as the one called the *local Hausdorff convergence topology* in [16, Section 2.5].

## 2.7 Distance estimates for the Hilbert metric

The asymptotic behavior of the Hilbert distance connects naturally with the structure of open faces in the boundary.

**Proposition 2.8** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\{p_n\}$, $\{q_n\}$ are sequences in $\Omega$ where $p_n \to p \in \overline{\Omega}$ and $q_n \to q \in \overline{\Omega}$. If*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n) < +\infty,$$

*then $F_\Omega(p) = F_\Omega(q)$ and*

$$\mathrm{d}_{F_\Omega(p)}(p, q) \le \liminf_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n).$$

**Proof** This is a well-known property of the Hilbert metric, but we provide the proof for the readers' convenience.

Passing to a subsequence we can suppose that

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n) = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n).$$

Since $\mathrm{d}_\Omega$ is proper, if at least one of $p, q$ is in $\Omega$, then both must be in $\Omega$. Further, if $p, q \in \Omega$, then $F_\Omega(p) = F_\Omega(q) = \Omega$ and the continuity of the Hilbert metric implies that

$$\mathrm{d}_{F_\Omega(p)}(p, q) = \mathrm{d}_\Omega(p, q) = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n).$$

Hence we can assume that $p, q \in \partial\Omega$. It suffices to consider $p \neq q$ since otherwise $\mathrm{d}_{F_\Omega(p)}(p, q) = 0$.

For each $n \ge 1$, let $a_n, b_n \in \partial\Omega$ be the points where $p_n, q_n \in (a_n, b_n)$ and labeled so that the points are in the order $a_n$, $p_n$, $q_n$, $b_n$ along $(a_n, b_n)$. Passing to a subsequence, we can assume that $a_n \to a$ and $b_n \to b$ in $\partial\Omega$. Then $p, q \in [a, b]$. Since $p \neq q$ and

$$\lim_{n\to\infty} \tfrac{1}{2} \log[a_n, p_n, q_n, b_n] = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n) < +\infty,$$

the definition of the cross-ratio implies that $a$, $p$, $q$, $b$ are all distinct. Hence $p, q \in (a, b)$ and so $F_\Omega(p) = F_\Omega(q)$. Finally, since $(a, b) \subset F_\Omega(p)$, the monotonicity of the Hilbert metric implies that

$$\mathrm{d}_{F_\Omega(p)}(p, q) \le \mathrm{d}_{(a,b)}(p, q) = \liminf_{n\to\infty} \mathrm{d}_\Omega(p_n, q_n).$$ □

We will frequently use the following fact about the Hausdorff distance between line segments.

**Proposition 2.9** [14, Proposition 5.3] *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $p_1, p_2, q_1, q_2 \in \overline{\Omega}$ are such that $(p_1, p_2) \subset \Omega$, $F_\Omega(p_1) = F_\Omega(q_1)$, and $F_\Omega(p_2) = F_\Omega(q_2)$, then*

$$\mathrm{d}_\Omega^{\mathrm{Haus}}\big([p_1, p_2] \cap \Omega, [q_1, q_2] \cap \Omega\big) \le \max\{\mathrm{d}_{F_\Omega(p_1)}(p_1, q_1), \mathrm{d}_{F_\Omega(p_2)}(p_2, q_2)\}.$$

Using induction, Proposition 2.9 can be upgraded as follows.

**Proposition 2.10** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $q_1, \dots, q_m \in \overline{\Omega}$, and*

$$z \in \text{rel-int}(\text{ConvHull}_\Omega(q_1, \dots, q_m)).$$

*If $p_1, \dots, p_m \in \overline{\Omega}$ and $F_\Omega(p_j) = F_\Omega(q_j)$ for all $1 \le j \le m$, then*

$$\mathrm{d}_F^{\mathrm{Haus}}\big(\text{ConvHull}_\Omega(q_1, \dots, q_m) \cap F,\ \text{ConvHull}_\Omega(p_1, \dots, p_m) \cap F\big) \le \max_{1 \le j \le m} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j),$$

*where $F := F_\Omega(z)$.*

**Proof** For $1 \le k \le m$, let

$$S_k := \text{rel-int}(\text{ConvHull}_\Omega(q_1, \dots, q_k)) \quad \text{and} \quad S'_k := \text{rel-int}(\text{ConvHull}_\Omega(p_1, \dots, p_k)).$$

Since $S_k, S'_k$ are open in their span, there exist faces $F_k, F'_k$ of $\Omega$ such that $S_k \subset F_k$ and $S'_k \subset F'_k$. For each $1 \le k \le m$, we claim that $F_k = F'_k$ and

$$\mathrm{d}_{F_k}^{\mathrm{Haus}}(S_k, S'_k) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

We induct on $k$. The base case $k = 1$ is by definition.

Fix $k > 1$. Then fix $w \in S_k$. We will show that $F_k = F'_k$ and

$$\mathrm{d}_{F_k}(w, S'_k) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j). \tag{2-1}$$

If $w = q_k$, then we must have $q_k \in S_{k-1}$. Then $q_k \in F_{k-1}$, which implies that $p_k \in F_{k-1}$. So $S_k \cup S'_k \subset F_{k-1}$ and $F_k = F_{k-1} = F'_k$ by the induction hypothesis. Further,

$$\mathrm{d}_{F_k}(w, S'_k) \le \mathrm{d}_{F_k}(q_k, p_k) = \mathrm{d}_{F_\Omega(q_k)}(q_k, p_k).$$

If $w \ne q_k$, then there exists $q' \in S_{k-1}$ such that $w \in (q', q_k)$. Indeed, if $q_k \in \overline{S}_{k-1}$, then $S_k = S_{k-1}$ and then this is obvious as we can extend $(q_k, w]$ beyond $w \in S_k$. Otherwise, if $q_k \notin \overline{S}_{k-1}$, then every point in $\overline{S}_k$ is a nontrivial convex combination of a point $q'' \in \overline{S}_{k-1}$ and $q_k$. Among all such points in $\overline{S}_k$, $S_k = \text{rel-int}(\overline{S}_k)$ is precisely the set of points for which $q'' \in S_{k-1}$. In particular, as $w \in S_k$, there exists $q' \in S_{k-1}$ such that $w \in (q_k, q')$.

Since $q' \in S_{k-1}$, by induction there exists $p' \in S'_{k-1}$ with

$$\mathrm{d}_{F_{k-1}}(q', p') \le \max_{1 \le j \le k-1} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

There are two cases to consider depending on whether $p'$ equals $p_k$ or not.

**Case 1** Assume $p' \neq p_k$. Then Observation 2.2(4) implies that $(q', q_k) \cup (p', p_k) \subset F_k$. Since $(p', p_k) \subset S_k$, we then have $F_k = F'_k$. Further, by Proposition 2.9,

$$\mathrm{d}_{F_k}(w, S'_k) \le \mathrm{d}^{\mathrm{Haus}}_{F_k}((q', q_k), (p', p_k)) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

**Case 2** Assume $p' = p_k$. Then $p_k = p' \in S'_{k-1} \subset F_{k-1}$, which implies that $q_k \in F_{k-1}$. Therefore $S_k \cup S'_k \subset F_{k-1}$ and $F_k = F_{k-1} = F'_k$ by the induction hypothesis. Further, by Proposition 2.9,

$$\mathrm{d}_{F_k}(w, S'_k) \le \mathrm{d}_{F_k}([q', q_k], [p_k, p_k]) \le \max(\mathrm{d}_{F_k}(q', p_k), \mathrm{d}_{F_k}(q_k, p_k)) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

Thus in all cases $F_k = F'_k$ and (2-1) holds. Since $w \in S_k$ was arbitrary, we see that

$$\sup_{w \in S_k} \mathrm{d}_{F_k}(w, S'_k) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

Repeating the same argument with $w' \in S'_k$ shows that

$$\sup_{w' \in S'_k} \mathrm{d}_{F_k}(w', S_k) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

Hence,

$$\mathrm{d}^{\mathrm{Haus}}_{F_k}(S_k, S'_k) \le \max_{1 \le j \le k} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

This completes the proof of the claim.

Now, to prove the proposition, we apply the claim when $k = m$. Since $z \in S_m$, we have $F = F_\Omega(z) = F_m$. By the claim, $S_m \cup S'_m \subset F_m$ and

$$d^{\mathrm{Haus}}_{F_m}(S_m, S'_m) \le \max_{1 \le j \le m} \mathrm{d}_{F_\Omega(q_j)}(q_j, p_j).$$

Finally, $S_m$ (resp. $S'_m$) is dense in $\mathrm{ConvHull}_\Omega(q_1, \dots, q_m) \cap F$ (resp. $\mathrm{ConvHull}_\Omega(p_1, \dots, p_m) \cap F$). So the result follows. □

## 3 Relatively hyperbolic spaces and groups

In this expository section we recall some basic properties of relatively hyperbolic groups and spaces. We define relative hyperbolic spaces and groups in terms of Druțu and Sapir's tree-graded spaces; see [10, Definition 2.1].

**Definition 3.1** (1) A complete geodesic metric space $(X, \mathrm{d})$ is said to be *relatively hyperbolic with respect to a collection of subsets* $\mathcal{S}$ if all its asymptotic cones, with respect to a fixed nonprincipal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of $\mathcal{S}$.

(2) A finitely generated group $G$ is said to be *relatively hyperbolic with respect to a family of subgroups* $\{H_1, \dots, H_N\}$ if the Cayley graph of $G$ with respect to some (hence any) finite set of generators is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G,\ i = 1, \dots, N\}$.

**Remark 3.2** These are one among several equivalent definitions of relatively hyperbolic spaces/groups; see [10] for more details.

Recall that if $(X, \mathrm{d})$ is a metric space, $A \subset X$, and $r > 0$, then

$$\mathcal{N}_X(A, r) := \{x \in X : \mathrm{d}(x, A) < r\}.$$

We will frequently use the following properties of relatively hyperbolic spaces. These results are taken from Druțu–Sapir [10].

**Theorem 3.3** *Suppose that $(X, \mathrm{d}_X)$ is relatively hyperbolic with respect to $\mathcal{S}$.*

(1) [10, Corollary 5.8] *For any $A \geq 1$ and $B \geq 0$ there exists $M = M(A, B)$ such that if $k \geq 2$ and $f : \mathbb{R}^k \to X$ is an $(A, B)$-quasi-isometric embedding, then there exists some $S \in \mathcal{S}$ such that*

$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S, M).$$

(2) [10, Lemma 4.15] *For any $A \geq 1$ and $B \geq 0$ there exists $t = t(A, B)$ such that if $S \in \mathcal{S}$, $r \geq 1$, and $\gamma : [0, T] \to X$ is an $(A, B)$-quasigeodesic with $\gamma(0), \gamma(T) \in \mathcal{N}_X(S, r)$, then*

$$\gamma([0, T]) \subset \mathcal{N}_X(S, tr).$$

(3) [10, Theorem 4.1] *For any $r > 0$ there exists $D = D(r) > 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then*

$$\mathrm{diam}_X(\mathcal{N}_X(S_1, r) \cap \mathcal{N}_X(S_2, r)) \leq D.$$

## 3.1 Immediate consequences in convex projective geometry

As an immediate consequence of the general theory of relatively hyperbolic spaces, we have the following.

**Proposition 3.4** *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ is a $\Gamma$-invariant collection of closed unbounded convex subsets of $\mathcal{C}$. If $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{X}$, then $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ (ie $\Gamma$-invariant, strongly isolated, and coarsely contains the properly embedded simplices of $\mathcal{C}$).*

**Proof** Theorem 3.3(3) implies that $\mathcal{X}$ is strongly isolated. Proposition 2.6 and Theorem 3.3(1) imply that $\mathcal{X}$ coarsely contains the properly embedded simplices of $\mathcal{C}$. □

## 3.2 Yaman's characterization

We recall Yaman's characterization of relatively hyperbolic groups in terms of convergence group actions [21; 26].

Recall that when $M$ is a nonempty compact Hausdorff metrizable space, a subgroup $G \leq \mathrm{Homeo}(M)$ is called a *convergence group* if for every sequence $\{g_n\}$ of distinct elements in $G$ there exists a subsequence $\{g_{n_j}\}$ and points $x, y \in M$ such that $g_{n_j}|_{M-\{y\}}$ converges locally uniformly to the constant map $x$. For such a subgroup:

(1) The *limit set* $\mathcal{L}(G) \subset M$ is the set of points $x \in M$ where there exist $y \in M$ and a sequence $\{g_n\}$ in $G$ where $g_n|_{M-\{y\}}$ converges locally uniformly to the constant map $x$.

(2) A point $x \in \mathcal{L}(G)$ is a *conical limit point* if there exist distinct points $a, b \in M$ and a sequence of elements $\{g_n\}$ in $G$ where $\lim_{n\to\infty} g_n(x) = a$ and $\lim_{n\to\infty} g_n(y) = b$ for all $y \in M - \{x\}$.

(3) A point $x \in \mathcal{L}(G)$ is a *parabolic point* if $\operatorname{Stab}_G(x)$ is infinite and nonloxodromic (ie $x$ is the unique fixed point in $\mathcal{L}(G)$ for every $g \in \operatorname{Stab}_G(x)$ of infinite order).

(4) A parabolic point $x \in \mathcal{L}(G)$ is a *bounded parabolic point* if $\operatorname{Stab}_G(x)$ acts cocompactly on $\mathcal{L}(G) - \{x\}$.

Finally, we say that a convergence group $G \leq \operatorname{Homeo}(M)$ is *geometrically finite* if $\mathcal{L}(G)$ is a nonempty perfect set (ie $\# \mathcal{L}(G) \geq 3$) and every point in $\mathcal{L}(G)$ is either a conical limit point or a bounded parabolic point.

**Theorem 3.5** (Yaman [26, Theorem 0.1]) *Suppose that $G \leq \operatorname{Homeo}(M)$ is a geometrically finite convergence group and $B \subset \mathcal{L}(G)$ is the set of bounded parabolic fixed points. If $B$ has finitely many $G$-orbits and $\operatorname{Stab}_G(b) := \{h \in G : h(b) = b\}$ is finitely generated for every $b \in B$, then*:

(1) *$G$ is relatively hyperbolic with respect to $\mathcal{P} = \{\operatorname{Stab}_G(p_1), \dots, \operatorname{Stab}_G(p_N)\}$ where $B = \bigsqcup_{i=1}^N G(p_i)$.*

(2) *The Bowditch boundary $\partial(G, \mathcal{P})$ is equivariantly homeomorphic to $\mathcal{L}(G)$.*

If $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P}$, then $\Gamma$ acts as a geometrically finite convergence group on the Bowditch boundary.

# 4 Finding properly embedded simplices

For convex cocompact groups, if there is a line segment in the ideal boundary of the convex hull, then there is a properly embedded simplex in the convex hull [7, Theorem 1.15]. This fails for naive convex cocompact subgroups; see [15, Section 2.3]. Instead we will show that if there exists a line segment in the ideal boundary which is sufficiently long (relative to the Hilbert metric of the face containing it), then there exists a properly embedded simplex.

In the following proposition, $\mathrm{d}_{\mathbb{P}}$ denotes some distance on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric.

**Proposition 4.1** *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $q \in \mathcal{C}$. For any $r, \epsilon > 0$ there exists $L = L(q, r, \epsilon) \geq 0$ with the following property*: *if*

(1) *$a, b \in \partial_i \mathcal{C}$ are contained in a boundary face $F$ of $\Omega$ with $\mathrm{d}_F(a, b) \geq L$,*

(2) *$m \in (a, b)$ is the midpoint of $[a, b]$ relative to $\mathrm{d}_F$, and*

(3) *$p \in \mathbb{P}(\operatorname{Span}\{a, b, q\}) \cap \mathcal{C}$ is sufficiently close to $m$ (in the metric $\mathrm{d}_{\mathbb{P}}$),*

*then there exists a properly embedded simplex $S \subset \mathcal{C}$ of dimension at least two with*

$$\mathbb{P}(\operatorname{Span}\{a, b, q\}) \cap \mathcal{B}_\Omega(p, r) \subset \mathcal{N}_\Omega(S, \epsilon).$$

**Proof** Suppose for a contradiction that the proposition is false for some choice of $r, \epsilon > 0$. Then for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \partial_i \mathcal{C}$ and a sequence $\{p_{n,j}\}_{j \geq 1}$ where:

(1) $a_n, b_n$ are contained in a boundary face $F_n$ and $\mathrm{d}_{F_n}(a_n, b_n) > n$.

(2) If $m_n \in (a_n, b_n)$ is the midpoint of $[a_n, b_n]$ relative to $\mathrm{d}_{F_n}$, then $m_n = \lim_{j \to \infty} p_{n,j}$.

(3) $p_{n,j} \in \mathbb{P}(\mathrm{Span}\{a_n, b_n, q\}) \cap \mathcal{C}$ for all $j \geq 1$.

(4) $\mathbb{P}(\mathrm{Span}\{a_n, b_n, q\}) \cap \mathcal{B}_\Omega(p_{n,j}, r)$ is not contained in the $\epsilon$-neighborhood of any properly embedded simplex in $\mathcal{C}$ of dimension at least two.

We first claim that for every $n$, there exists $j_n$ such that

$$\mathrm{d}_\Omega\big(p_{n,j_n}, (a_n, q] \cup [q, b_n)\big) \geq \tfrac{1}{2} n. \tag{4-1}$$

Fix $n$ and suppose not. Then for every $j \in \mathbb{N}$ we can find $a_{n,j} \in (a_n, q]$ and $b_{n,j} \in (b_n, q]$ with

$$\mathrm{d}_\Omega(p_{n,j}, \{a_{n,j}, b_{n,j}\}) \leq \tfrac{1}{2} n.$$

Since $\lim_{j \to \infty} p_{n,j} = m_n$ and $\mathrm{d}_\Omega$ is proper, we must have $\lim_{j \to \infty} a_{n,j} = a_n$ and $\lim_{j \to \infty} b_{n,j} = b_n$. Further, by the definition of the Hilbert metric,

$$\tfrac{1}{2} n \geq \limsup_{j \to \infty} \mathrm{d}_\Omega(p_{n,j}, \{a_{n,j}, b_{n,j}\}) \geq \mathrm{d}_{F_n}(m_n, \{a_n, b_n\}) > \tfrac{1}{2} n.$$

So we have a contradiction and hence for each $n$ such a $j_n$ exists.

After possibly passing to a subsequence we can find a sequence $\{\gamma_n\}$ in $\Gamma$ such that as $n \to \infty$,

$$\gamma_n(p_{n,j_n}) \to p_\infty \in \mathcal{C}, \quad \gamma_n(a_n) \to a_\infty \in \bar{\mathcal{C}}, \quad \gamma_n(b_n) \to b_\infty \in \bar{\mathcal{C}}, \quad \gamma_n(q) \to q_\infty \in \bar{\mathcal{C}}.$$

By construction, $[a_\infty, b_\infty] \subset \partial_i \mathcal{C}$, and by (4-1), we see that

$$[a_\infty, q_\infty] \cup [q_\infty, b_\infty] \subset \partial_i \mathcal{C}.$$

Hence $a_\infty, b_\infty, q_\infty$ are the vertices of a properly embedded two-dimensional simplex $S \subset \mathcal{C}$ which contains $p_\infty$. So for $n$ large, we have

$$\mathbb{P}\big(\mathrm{Span}\{\gamma_n(a_n), \gamma_n(b_n), \gamma_n(q)\}\big) \cap \mathcal{B}_\Omega(\gamma_n(p_{n,j_n}), r) \subset \mathcal{N}_\Omega(S, \epsilon),$$

which implies that

$$\mathbb{P}(\mathrm{Span}\{a_n, b_n, q\}) \cap \mathcal{B}_\Omega(p_{n,j_n}, r) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S, \epsilon).$$

Thus we have a contradiction. □

# 5 Properties of peripheral families

**Theorem 5.1** *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple and $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$ (ie $\mathcal{X}$ is $\Gamma$-invariant, strongly isolated, and coarsely contains all properly embedded simplices in $\mathcal{C}$). Then:*

(1) *$\Gamma$ has finitely many orbits in $\mathcal{X}$.*

(2) *If $X \in \mathcal{X}$, then $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $X$. In particular, $\mathrm{Stab}_\Gamma(X)$ is finitely generated.*

(3) *If $X_1, X_2 \in \mathcal{X}$ are distinct, then $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) = \varnothing$.*

(4) *There exists $L > 0$ such that if $x \in \partial_i \mathcal{C}$ and $\operatorname{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L$, then $x \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$.*

(5) *There exists $R > 0$ such that if $X \in \mathcal{X}$ and $x \in \partial_i X$, then*
$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}, F_\Omega(x) \cap \partial_i X) \leq R.$$

(6) *If we have $X \in \mathcal{X}$, $x \in \partial_i X$, and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$.*

(7) *If $[x, y] \subset \partial_i \mathcal{C}$, then either $F_\Omega(x) = F_\Omega(y)$, or there exists $X \in \mathcal{X}$ such that $x, y \in F_\Omega(\partial_i X)$.*

(8) *If $X \in \mathcal{X}$, then $\operatorname{Stab}_\Gamma(X)$ acts cocompactly on*
$$\operatorname{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$$
*and this set is nonempty.*

For the rest of the section, fix $\Omega$, $\mathcal{C}$, $\Gamma$, and $\mathcal{X}$ satisfying the hypotheses of the theorem.

Since $\mathcal{X}$ coarsely contains all properly embedded simplices in $\mathcal{C}$, there exists $D_2 > 0$ such that if $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with

$$S \subset \mathcal{N}_\Omega(X, D_2). \tag{5-1}$$

**Lemma 5.2** *If $K \subset \mathcal{C}$ is compact, then the set $\{X \in \mathcal{X} : X \cap K \neq \varnothing\}$ is finite.*

**Proof** Suppose not. Then there exist an infinite sequence of pairwise distinct elements $\{X_n\}$ in $\mathcal{X}$ where $X_n \cap K \neq \varnothing$ for all $n$. As each $X_n$ is unbounded, we can fix $k_n \in X_n \cap K$ and $x_n \in \partial_i X_n$. Passing to a subsequence we can suppose that $k_n \to k \in K$ and $x_n \to x \in \partial_i \mathcal{C}$. Then
$$\liminf_{n,\ell \to \infty} \operatorname{diam}_\Omega\bigl(\mathcal{N}_\Omega(X_n, 1) \cap \mathcal{N}_\Omega(X_\ell, 1)\bigr) \geq \liminf_{n,\ell \to \infty} \operatorname{diam}_\Omega\bigl(\mathcal{N}_\Omega([k_n, x_n), 1) \cap \mathcal{N}_\Omega([k_\ell, x_\ell), 1)\bigr) = \infty,$$
and so by the strong isolation property $X_n = X_\ell$ for sufficiently large $n, \ell$. So we have a contradiction. □

**Lemma 5.3** (part (1)) *$\Gamma$ has finitely many orbits in $\mathcal{X}$.*

**Proof** Since $\Gamma$ acts cocompactly on $\mathcal{C}$, there exists a compact set $K \subset \mathcal{C}$ such that $\Gamma \cdot K = \mathcal{C}$. Then for each $X \in \mathcal{X}$ there exists $g \in \Gamma$ such that $K \cap gX \neq \varnothing$. So by Lemma 5.2, there exist $X_1, \ldots, X_m$ such that $\mathcal{X} = \bigsqcup_{i=1}^m \Gamma \cdot X_i$. □

**Lemma 5.4** (part (2)) *If $X \in \mathcal{X}$, then $\operatorname{Stab}_\Gamma(X)$ acts cocompactly on $X$.*

**Proof** This argument is standard; see [12, Theorem 3.7; 13, Section 3.1; 25, Proposition 4.0.4].

Fix a compact set $K \subset \mathcal{C}$ such that $\Gamma \cdot K = \mathcal{C}$. Let
$$\mathcal{G} := \{g \in \Gamma : X \cap gK \neq \varnothing\}.$$
Then $X \subset \mathcal{G} \cdot K$. Also, by the previous lemma
$$\{g^{-1}X : g \in \mathcal{G}\} = \{h_1^{-1}X, \ldots, h_k^{-1}X\}$$

for some $h_1,\dots,h_k \in \mathcal{G}$. Notice that $g^{-1}X = h_j^{-1}X$ if and only if $gh_j^{-1} \in \mathrm{Stab}_\Gamma(X)$; hence

$$\mathcal{G} = \bigcup_{j=1}^{k} \mathrm{Stab}_\Gamma(X) h_j.$$

Finally, if $\hat{K} := \bigcup_{j=1}^k h_j K$, then $\hat{K}$ is compact and $X \subset \mathrm{Stab}_\Gamma(X)\cdot\hat{K}$. So $X = \mathrm{Stab}_\Gamma(X)\cdot(X\cap\hat{K})$. Since $X$ is closed, $X\cap\hat{K}$ is compact and thus $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $X$.

Now, since $X$ is convex, the metric space $(X, \mathrm{d}_\Omega)$ is geodesic. So by the fundamental lemma of geometric group theory, $\mathrm{Stab}_\Gamma(X)$ is finitely generated. □

**Lemma 5.5** (part (3)) *If $X_1, X_2 \in \mathcal{X}$ are distinct, then $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) = \varnothing$.*

**Proof** Suppose $X_1, X_2 \in \mathcal{X}$ and $x \in F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2)$. There exists $x_1 \in \partial_i X_1$ and $x_2 \in \partial_i X_2$ with

$$x_1, x_2 \in F_\Omega(x).$$

Fix $q_1 \in X_1$ and $q_2 \in X_2$. Then Proposition 2.9 implies that

$$\mathrm{d}_\Omega^{\mathrm{Haus}}([q_1,x_1),[q_2,x_2)) \le \max\{\mathrm{d}_\Omega(q_1,q_2), \mathrm{d}_{F_\Omega(x)}(x_1,x_2)\}.$$

So for $r > \max\{\mathrm{d}_\Omega(q_1,q_2), \mathrm{d}_{F_\Omega(x)}(x_1,x_2)\}$, we have

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(X_1,r)\cap\mathcal{N}_\Omega(X_2,r)) = \infty.$$

Since $\mathcal{X}$ is strongly isolated, then $X_1 = X_2$. □

Recall that $D_2$ is the constant in (5-1).

**Lemma 5.6** *There exists $L > 0$ such that if*

1. *$a, b \in \partial_i \mathcal{C}$ are in the same boundary face $F$ of $\Omega$,*
2. *$\mathrm{d}_F(a,b) \ge L$, and*
3. *$m$ is the midpoint of $[a,b]$ with respect to $\mathrm{d}_F$,*

*then there exists $X \in \mathcal{X}$ and $x \in \partial_i X \cap F$ such that*

$$\mathrm{d}_F(m,x) \le D_2 + 1.$$

**Proof** Since $\mathcal{X}$ is strongly isolated, there exists $r > 0$ such that

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(X_1, D_2+1)\cap\mathcal{N}_\Omega(X_2, D_2+1)) < r \tag{5-2}$$

for all distinct pairs $X_1, X_2 \in \mathcal{X}$. Fix $q \in \mathcal{C}$ and let $L > 0$ satisfy Proposition 4.1 for $q \in \mathcal{C}$ and constants $\epsilon = 1$ and $r$ as above. Now suppose

1. $a, b \in \partial_i \mathcal{C}$ are in the same boundary face $F$ of $\Omega$,
2. $\mathrm{d}_F(a,b) \ge L$, and
3. $m$ is the midpoint of $[a,b]$ with respect to $\mathrm{d}_F$.

By our choice of $L$, there exists $q' \in (m,q]$ such that if $p \in (m,q']$ then there exists some properly embedded simplex $S_p \subset \mathcal{C}$ of dimension at least two such that

$$\mathbb{P}(\mathrm{Span}\{a,b,q\}) \cap \mathcal{B}_\Omega(p,r) \subset \mathcal{N}_\Omega(S_p,1).$$

Then, by our choice of $D_2$ in (5-1), there exists $X_p \in \mathcal{X}$ such that

$$\mathbb{P}(\mathrm{Span}\{a,b,q\}) \cap \mathcal{B}_\Omega(p,r) \subset \mathcal{N}_\Omega(X_p, D_2+1).$$

We claim that $X_p$ does not depend on $p \in (m,q']$. To verify this it is enough to fix $p_1, p_2 \in (m,q']$ with $\mathrm{d}_\Omega(p_1,p_2) < r$ and show that $X_{p_1}$ and $X_{p_2}$ coincide. For such $p_1, p_2 \in (m,q']$ we have

$$\mathbb{P}(\mathrm{Span}\{a,b,q\}) \cap \mathcal{B}_\Omega(p_1,r) \cap \mathcal{B}_\Omega(p_2,r) \subset \mathcal{N}_\Omega(X_{p_1}, D_2+1) \cap \mathcal{N}_\Omega(X_{p_2}, D_2+1),$$

and

$$\mathrm{diam}_\Omega\big(\mathbb{P}(\mathrm{Span}\{a,b,q\}) \cap \mathcal{B}_\Omega(p_1,r) \cap \mathcal{B}_\Omega(p_2,r)\big) \geq r.$$

So $X_{p_1} = X_{p_2}$ by our choice of $r$; see (5-2).

Now let $X := X_p$ for $p \in (m,q']$. Then

$$(m,q'] \subset \mathcal{N}_\Omega(X, D_2+1)$$

which implies, by Proposition 2.8, that there exists $x \in \partial_i X \cap F$ such that

$$\mathrm{d}_F(m,x) \leq D_2 + 1. \qquad \square$$

**Lemma 5.7** (part (4)) *If $x \in \partial_i \mathcal{C}$ and*

$$\mathrm{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) \geq L,$$

*then $x \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$.*

**Proof** This follows immediately from Lemma 5.6. $\square$

**Lemma 5.8** (part (5)) *If $X \in \mathcal{X}$ and $x \in \partial_i X$, then*

$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}, F_\Omega(x) \cap \partial_i X) \leq \max\{L, 2D_2+2\}.$$

**Proof** Fix $X \in \mathcal{X}$ and $x \in \partial_i X$. Since $F_\Omega(x) \cap \partial_i X \subset F_\Omega(x) \cap \partial_i \mathcal{C}$, it suffices to show that

$$\sup_{y \in F_\Omega(x) \cap \partial_i \mathcal{C}} \mathrm{d}_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) \leq \max\{L, 2D_2+2\}.$$

Fix $y \in F_\Omega(x) \cap \partial_i \mathcal{C}$ and suppose for a contradiction that

$$\mathrm{d}_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) > \max\{L, 2D_2+2\}.$$

By changing $x$ we may suppose that

$$\mathrm{d}_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) = \mathrm{d}_{F_\Omega(x)}(y,x).$$

Let $m$ be the midpoint of $[x,y]$ with respect to $\mathrm{d}_{F_\Omega(x)}$. Then by Lemma 5.6 there exist $X' \in \mathcal{X}$ and $x' \in \partial_i X'$ such that

$$\mathrm{d}_{F_\Omega(x)}(m,x') \leq D_2 + 1.$$

However, then $x \in F_\Omega(\partial_i X) \cap F_\Omega(\partial_i X') \neq \varnothing$ and so by Lemma 5.5 we have $X = X'$. So

$$D_2 + 1 < \tfrac{1}{2} \mathrm{d}_{F_\Omega(x)}(y, F_\Omega(x) \cap \partial_i X) = \mathrm{d}_{F_\Omega(x)}(m, F_\Omega(x) \cap \partial_i X) \leq \mathrm{d}_{F_\Omega(x)}(m, x') \leq D_2 + 1,$$

and we have a contradiction. □

**Lemma 5.9** (part (7)) *If $[x, y] \subset \partial_i \mathcal{C}$, then either $F_\Omega(x) = F_\Omega(y)$, or there exists $X \in \mathcal{X}$ such that $x, y \in F_\Omega(\partial_i X)$.*

**Proof** We suppose that $F_\Omega(x) \neq F_\Omega(y)$ and show that $x, y \in F_\Omega(\partial_i X)$ for some $X \in \mathcal{X}$. Fix $z \in (x, y)$. Since $F_\Omega(x) \neq F_\Omega(y)$,

$$\operatorname{diam}_{F_\Omega(z)}((x, y)) = \infty.$$

So by Lemma 5.7 there exists $X \in \mathcal{X}$ with $z \in F_\Omega(\partial_i X)$. Then Lemma 5.8 implies that

$$(x, y) \subset \mathcal{N}_{F_\Omega(z)}(F_\Omega(z) \cap \partial_i X, \max\{L, 2D_2 + 2\}).$$

Finally Proposition 2.8 implies that $x, y \in F_\Omega(\partial_i X)$. □

**Lemma 5.10** (part (6)) *If we have $X \in \mathcal{X}$, $x \in \partial_i X$, and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$.*

**Proof** This follows immediately from Lemmas 5.9 and 5.5. □

**Observation 5.11** *If $X \in \mathcal{X}$, then $\mathrm{ConvHull}_\Omega(\partial_i X) \cap \Omega$ is nonempty.*

**Proof** By assumption $X$ is unbounded and so $\partial_i X \neq \varnothing$. Fix $w_2 \in \partial_i X$ and $w_1 \in X$. Also fix a sequence $\{w_n'\} \subset [w_1, w_2)$ such that $w_n' \to w_2$. As $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $X$, for each $n \geq 1$ there exists $\gamma_n \in \mathrm{Stab}_\Gamma(X)$ such that $\{\gamma_n w_n'\}$ lies in a compact subset of $X$. Up to passing to a subsequence, we can assume that $\gamma_n w_n' \to z' \in X$ and $\gamma_n w_i \to z_i \in \overline{X}$ for $i = 1, 2$. Then $z_2 \in \partial_i X$ as $\partial_i X$ is compact and $z_1 \in \partial_i X$ as $\mathrm{d}_\Omega(z', z_1) = \lim_{n\to\infty} \mathrm{d}_\Omega(\gamma_n w_n', \gamma_n w_1) = \infty$. Then $(z_1, z_2) \subset \Omega$, as $z' \in (z_1, z_2) \cap \Omega$. □

**Lemma 5.12** (part (8)) *If $X \in \mathcal{X}$, then $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed in $\mathbb{P}(\mathbb{R}^d)$ and $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on the nonempty set $\mathrm{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$.*

**Proof** We first verify that $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed. Suppose $\{x_n\}$ is a sequence in $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ converging to some $x \in \partial_i \mathcal{C}$ (note $\partial_i \mathcal{C}$ is closed). Then for each $n$ there exists $x_n' \in \partial_i X$ with $x_n \in F_\Omega(x_n')$. In particular, $[x_n, x_n'] \subset \partial_i \mathcal{C}$. Passing to a subsequence we can suppose that $x_n' \to x'$. Then $x' \in \partial_i X$ and $[x, x'] \subset \partial_i \mathcal{C}$. Then Lemma 5.10 implies that $x \in F_\Omega(\partial_i X)$. Hence $F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ is closed.

Then the set

$$\mathcal{C}_X := \mathrm{ConvHull}_\Omega(F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}) \cap \Omega$$

is closed in $\mathcal{C}$. As $\mathrm{ConvHull}_\Omega(\partial_i X) \cap \Omega \subset \mathcal{C}_X$, Observation 5.11 implies that $\mathcal{C}_X$ is nonempty. As $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $X$, it suffices to show that $\mathcal{C}_X$ is contained in a bounded neighborhood of $X$. Fix $p \in \mathcal{C}_X$. Then there exist $p_1, \dots, p_\ell \in F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}$ such that $p \in \mathrm{ConvHull}_\Omega(p_1, \dots, p_\ell)$. Then by Lemma 5.8 there exist $p_1', \dots, p_\ell' \in \partial_i X$ such that

$$\mathrm{d}_{F_\Omega(p_j)}(p_j, p_j') \leq \max\{L, 2D_2 + 2\} \quad \text{for } j = 1, \dots, \ell.$$

Then by Proposition 2.10,

$$\mathrm{d}_\Omega(p,X) \le \mathrm{d}_\Omega^{\mathrm{Haus}}\big(\mathrm{ConvHull}_\Omega(p_1,\dots,p_\ell)\cap\Omega, \mathrm{ConvHull}_\Omega(p_1',\dots,p_\ell')\cap\Omega\big) \le \max_{1\le j\le \ell} \mathrm{d}_{F_\Omega(p_j)}(p_j,p_j') \le \max\{L, 2D_2+2\}.$$

Since $p\in\mathcal{C}_X$ was arbitrary, $\mathcal{C}_X \subset \overline{\mathcal{N}_\Omega(X,\max\{L,2D_2+2\})}$. □

## 6 Naive convex cocompactness of peripheral subgroups

**Proposition 6.1** *Suppose that* $(\Omega,\mathcal{C},\Gamma)$ *is a naive convex cocompact triple and* $\Gamma$ *is relatively hyperbolic with respect to* $\{P_1,\dots,P_N\}$*. For* $1\le j\le N$*, let*

$$X_j := \mathrm{ConvHull}_\Omega(\mathcal{L}_\Omega(P_j)\cap\partial_i\,\mathcal{C})\cap\Omega.$$

*Then*:

1. *Each* $(\Omega, X_j, P_j)$ *is a naive convex cocompact triple.*
2. $(\mathcal{C},\mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to* $\mathcal{X} := \Gamma\cdot\{X_1,\dots,X_N\}$*.*

The rest of the section is devoted to the proof of this proposition. Suppose $(\Omega,\mathcal{C},\Gamma)$ is a naive convex cocompact triple and $\Gamma$ is relatively hyperbolic with respect to $\{P_1,\dots,P_N\}$.

Fix a point $p_0\in\mathcal{C}$. The fundamental lemma of geometric group theory implies that $(\mathcal{C},\mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\Gamma\cdot\{P_1(p_0),\dots,P_N(p_0)\}$.

For each $1\le j\le N$, let $\mathcal{L}_j\subset\partial_i\,\mathcal{C}$ denote the accumulation points of the orbit $P_j(p_0)$ and let

$$\widehat{X}_j = \mathrm{ConvHull}_\Omega(\mathcal{L}_j)\cap\Omega.$$

**Lemma 6.2** $\widehat{X}_j$ *is nonempty and* $P_j$ *acts cocompactly on* $\widehat{X}_j$*.*

**Proof** For $n\ge 1$ let $Y_j^{(n)}$ be the points which are contained in the convex hull of at most $n$ points in $P_j(p_0)$. By Carathéodory's convex hull theorem, $Y_j^{(n)}=Y_j^{(d)}$ for all $n\ge d$. Further, the closure of $Y_j^{(d)}$ in $\Omega$ contains $\widehat{X}_j$.

By Theorem 3.3(2) there exists $t>0$ such that if $r\ge 1$ and $p,q\in\mathcal{C}\cap\mathcal{N}_\Omega(P_j(p_0),r)$, then we have $[p,q]\subset\mathcal{N}_\Omega(P_j(p_0),tr)$.

We claim that

$$Y_j^{(n)} \subset \mathcal{N}_\Omega(P_j(p_0),(n-1)t) \tag{6-1}$$

for all $n\ge 2$. The base case follows by our choice of $t$. Suppose (6-1) holds for some $n$. If $p\in Y_j^{(n+1)}$, then there exist $p_1\in P_j(p_0)$ and $p_2\in Y_j^{(n)}$ such that $p\in[p_1,p_2]$. Then, by induction there exists $q\in P_j(p_0)$ with $\mathrm{d}_\Omega(p_2,q)<(n-1)t$. Then

$$[p_1,q]\subset Y_j^{(2)}\subset\mathcal{N}_\Omega(P_j(p_0),t),$$

and by Proposition 2.9,

$$\mathrm{d}_\Omega(p,[p_1,q]) \leq \mathrm{d}_\Omega^{\mathrm{Haus}}([p_1,p_2],[p_1,q]) \leq \mathrm{d}_\Omega(p_2,q) < (n-1)t.$$

So $p \in \mathcal{N}_\Omega(P_j(p_0), nt)$. Since $p \in Y_j^{(n+1)}$ was arbitrary, this completes the induction step. Thus (6-1) holds for all $n \geq 2$.

Then

$$\widehat{X}_j \subset \overline{Y_j^{(d)}} \subset \overline{\mathcal{N}_\Omega(P_j(p_0),(d-1)t)},$$

and so

$$\widehat{X}_j = P_j \cdot (\overline{\mathcal{B}_\Omega(p_0,(d-1)t)} \cap \widehat{X}_j).$$

Thus $P_j$ acts cocompactly on $\widehat{X}_j$.

We now show that $\widehat{X}_j \neq \varnothing$. Since $P_j$ is infinite and acts properly on $\Omega$, there exists $\{g_k\}$ in $P_j$ such that

$$\lim_{k\to\infty} \mathrm{d}_\Omega(p_0, g_k(p_0)) = \infty.$$

Let $m_k$ be the midpoint of $[p_0, g_k(p_0)]$ with respect to $\mathrm{d}_\Omega$. By (6-1),

$$m_k \in Y_j^{(2)} \subset \mathcal{N}_\Omega(P_j(p_0), t).$$

Thus there exists $\{h_k\}$ in $P_j$ such that $h_k(m_k) \in \mathcal{B}_\Omega(p_0,t)$. Then, up to passing to a subsequence, we can assume that $h_k(m_k) \to m \in \Omega$ and $h_k[p_0, g_k(p_0)] \to [x,y]$. Then $x, y \in \mathcal{L}_j$ and so

$$m \in (x,y) \subset \widehat{X}_j. \qquad \square$$

The fundamental lemma of geometric group theory implies that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\Gamma \cdot \{\widehat{X}_1, \dots, \widehat{X}_N\}$. In particular, by Proposition 3.4 and Theorem 5.1(5) there exists $R > 0$ such that if $1 \leq j \leq N$ and $x \in \partial_i \widehat{X}_j$, then

$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C},\ F_\Omega(x) \cap \partial_i \widehat{X}_j) \leq R. \tag{6-2}$$

**Lemma 6.3** *If $1 \leq j \leq N$, then*

$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(\widehat{X}_j, X_j) \leq R.$$

*Hence $P_j$ acts cocompactly on $X_j$ and $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\Gamma \cdot \{X_1, \dots, X_N\}$.*

**Proof** First note that $\widehat{X}_j \subset X_j$, as $\mathcal{L}_j \subset \mathcal{L}_\Omega(P_j)$. By the previous lemma, $P_j$ acts cocompactly on $\widehat{X}_j$. Thus it suffices to show that

$$X_j \subset \overline{\mathcal{N}_\Omega(\widehat{X}_j, R)}.$$

Fix $q \in X_j$. Then there exist $q_1, \dots, q_\ell \in \mathcal{L}_\Omega(P_j) \cap \partial_i \mathcal{C}$ where

$$q \in \mathrm{ConvHull}_\Omega(q_1, \dots, q_\ell).$$

For each $1 \leq i \leq \ell$, there exist $p_i \in \Omega$ and a sequence $\{\gamma_{i,n}\}_{n\geq 1}$ in $P_j$ with $\gamma_{i,n}(p_i) \to q_i$. Passing to subsequences we can suppose that

$$\gamma_{i,n}(p_0) \to q_i' \in \mathcal{L}_j \subset \partial_i \widehat{X}_j,$$

for all $1 \leq i \leq \ell$. By Proposition 2.8, $F_\Omega(q_i) = F_\Omega(q_i')$. So by (6-2) there exist $q_i'' \in \partial_i \hat{X}_j$ such that $\mathrm{d}_{F_\Omega(q_i)}(q_i, q_i'') \leq R$. Then by Proposition 2.10,

$$\mathrm{d}_\Omega(q, \hat{X}_j) \leq \mathrm{d}_\Omega^{\mathrm{Haus}}\big(\mathrm{ConvHull}_\Omega(q_1,\dots,q_\ell) \cap \Omega, \mathrm{ConvHull}_\Omega(q_1'',\dots,q_\ell'') \cap \Omega\big) \leq \max_{1\leq i\leq \ell} \mathrm{d}_{F_\Omega(q_i)}(q_i, q_i'') \leq R. \quad \square$$

This finishes the proof of Proposition 6.1.

## 7 Cocompact boundary actions

Given a naive convex cocompact triple $(\Omega, X, G)$, it seems natural to ask if $G$ acts cocompactly on $\partial\Omega - F_\Omega(\partial_i X)$ or more generally on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$ when $\mathcal{C} \subset \Omega$ is a closed $G$-invariant convex subset containing $X$. The next theorem gives a sufficient condition for this to occur.

**Theorem 7.1** *Suppose that $(\Omega, X, G)$ is a naive convex cocompact triple and $\mathcal{C} \subset \Omega$ is a closed $G$-invariant convex subset containing $X$ with the property that if $x \in \partial_i X$ and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \subset F_\Omega(\partial_i X)$. Then $G$ acts cocompactly on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$.*

Letting $\mathcal{C} = \Omega$ and changing notation, we obtain:

**Corollary 7.2** *Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple with the property that if $x \in \partial_i \mathcal{C}$ and $[x, y] \subset \partial\Omega$, then $[x, y] \subset F_\Omega(\partial_i \mathcal{C})$. Then $\Gamma$ acts cocompactly on $\partial\Omega - F_\Omega(\partial_i \mathcal{C})$.*

**Remark 7.3** The main application of Theorem 7.1 will be in Section 9 as we now explain. Suppose that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex cocompact triple, $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$, and $X \in \mathcal{X}$. Then $(\Omega, X, \mathrm{Stab}_\Gamma(X))$ and $\mathcal{C}$ satisfy the conditions in Theorem 7.1; see Theorem 5.1(2) and (6). We will use this in the proof of Theorem 1.4(2). In particular, we use this to show that nonconical points are bounded parabolic points.

For the rest of the section, fix $\Omega$, $\mathcal{C}$, $X$, $G$ satisfying the hypotheses of Theorem 7.1. The key idea will be to study the following "closest point projection".

**Definition 7.4** Define $\pi_X : \bar{\mathcal{C}} \to \{\text{subsets of } X\}$ as follows:

- For $p \in \mathcal{C}$, let
$$\pi_X(p) := \{x \in X : \mathrm{d}_\Omega(p, x) = \mathrm{d}_\Omega(p, X)\}$$
denote the points in $X$ closest to $p$.
- For $y \in \partial_i \mathcal{C}$ let $\pi_X(y) \subset X$ denote the set of all points $x \in X$ where there exist sequences $\{p_n\}$ and $\{x_n\}$ such that $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \to y$, and $x_n \to x$.

We start by observing the following.

**Observation 7.5** (1) *If $p \in \mathcal{C}$, then $\pi_X(p)$ is a compact convex subset of $X$.*

(2) *If $p \in \mathcal{C}$ and $x \in \pi_X(p)$, then $x \in \pi_X(q)$ for all $q \in [x, p]$.*

(3) *If $g \in G$, then $\pi_X \circ g = g \circ \pi_X$.*

**Proof** Notice that if $p \in \mathcal{C}$, then

$$\pi_X(p) = X \cap \{x \in \Omega : \mathrm{d}_\Omega(p,x) \le \mathrm{d}_\Omega(p,X)\}.$$

Part (1) is then a consequence of the fact that closed metric balls in the Hilbert metric are convex and compact; see Proposition 2.9 above. Part (2) follows from the fact that $[x,p]$ can be parametrized to be a geodesic in the Hilbert metric and part (3) is by definition. □

**Lemma 7.6** *If $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$, then $\pi_X(y) \subset X$ is a nonempty bounded set.*

**Proof** It suffices to fix sequences $\{p_n\}$ and $\{x_n\}$ such that $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \to y$, and $x_n \to x \in \overline{X}$, then prove that $x \in X$.

Suppose for a contradiction that $x \in \partial_i X$. Since $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$ our hypothesis on $X$ says that $(x,y) \subset \Omega$. Fix $v \in (x,y)$. Then pick a sequence $v_n \in (x_n, p_n)$ with $v_n \to v$. Then $x_n \in \pi_X(v_n)$ by Observation 7.5(2). However, we then have

$$\mathrm{d}_\Omega(v,X) = \lim_{n\to\infty} \mathrm{d}_\Omega(v_n,X) = \lim_{n\to\infty} \mathrm{d}_\Omega(v_n,x_n) = \lim_{n\to\infty} \mathrm{d}_\Omega(v,x_n) = \infty,$$

and we have a contradiction. □

**Lemma 7.7** *If $y \in \partial_i \mathcal{C} \cap F_\Omega(\partial_i X)$, then $\pi_X(y) = \varnothing$.*

**Proof** Suppose for a contradiction that there exist sequences $\{p_n\}$ and $\{x_n\}$ where $p_n \in \mathcal{C}$, $x_n \in \pi_X(p_n)$, $p_n \to y$, and $x_n \to x \in X$.

By hypothesis, there exists $w \in \partial_i X$ with $y \in F_\Omega(w)$. Fix $q \in (x,y)$ such that $\mathrm{d}_\Omega(q,x) > \mathrm{d}_{F_\Omega(w)}(y,w)$. Then fix $q_n \in (x_n,p_n)$ such that $q_n \to q$. By Observation 7.5(2),

$$\mathrm{d}_\Omega(q_n,X) = \mathrm{d}_\Omega(q_n,x_n),$$

and so

$$\mathrm{d}_\Omega(q,X) = \lim_{n\to\infty} \mathrm{d}_\Omega(q_n,X) = \mathrm{d}_\Omega(q,x) > \mathrm{d}_{F_\Omega(w)}(y,w).$$

However, this is impossible since Proposition 2.9 implies that

$$\mathrm{d}_\Omega(q,X) \le \mathrm{d}_\Omega(q,(x,w)) \le \mathrm{d}_\Omega^{\mathrm{Haus}}((x,y),(x,w)) \le \mathrm{d}_{F_\Omega(w)}(y,w).$$

So we have a contradiction. □

**Lemma 7.8** *If $K \subset X$ is compact, then the set*

$$\widehat{K} := \{y \in \partial_i \mathcal{C} : \pi_X(y) \cap K \neq \varnothing\}$$

*is compact.*

**Proof** Consider a sequence $\{y_n\}$ in $\widehat{K}$ where $y_n \to y \in \partial_i \mathcal{C}$. Fix $x_n \in \pi_X(y_n) \cap K$. Passing to a subsequence, we can suppose that $x_n \to x \in K$. By the definition of $\pi_X$, we can also find sequences $\{p_{n,m}\}$ and $\{x_{n,m}\}$ such that $p_{n,m} \in \mathcal{C}$, $x_{n,m} \in \pi_X(p_{n,m})$, and

$$y_n = \lim_{m\to\infty} p_{n,m} \quad \text{and} \quad x_n = \lim_{m\to\infty} x_{n,m}.$$

Then we can pick subsequences $\{p_{n_j,m_j}\}$ and $\{x_{n_j,m_j}\}$ such that

$$y = \lim_{j\to\infty} p_{n_j,m_j} \quad \text{and} \quad x = \lim_{j\to\infty} x_{n_j,m_j}.$$

So $x \in \pi_X(y) \cap K$ and hence $y \in \widehat{K}$. □

**Proof of Theorem 7.1** Fix a compact set $K \subset X$ such that $G \cdot K = X$. Then Lemmas 7.8 and 7.7 imply that

$$\widehat{K} := \{y \in \partial_i \mathcal{C} : \pi_X(y) \cap K \neq \varnothing\}$$

is compact and contained in $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$.

We claim that

$$G \cdot \widehat{K} = \partial_i \mathcal{C} - F_\Omega(\partial_i X).$$

Fix $y \in \partial_i \mathcal{C} - F_\Omega(\partial_i X)$. By Lemma 7.6 there exists $x \in \pi_X(y)$. There exists $g \in G$ such that $g(x) \in K$. Then $g(x) \in \pi_X(g(y))$ and so $g(y) \in \widehat{K}$. So $y \in G \cdot \widehat{K}$. Since $y$ was arbitrary, this proves the claim and the theorem. □

## 8 Basic properties of boundary quotients

For the rest of the section fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, a closed convex subset $\mathcal{C} \subset \Omega$, and a discrete subgroup $\Gamma \leq \mathrm{Aut}(\Omega)$ which preserves $\mathcal{C}$. We do not assume that $\Gamma$ acts cocompactly on $\mathcal{C}$.

Also fix a $\Gamma$-invariant equivalence relation $\sim$ on $\partial_i \mathcal{C}$ such that the set

$$R := \{(x, y) \in \partial_i \mathcal{C} \times \partial_i \mathcal{C} : x \sim y\}$$

is closed and $\#(\partial_i \mathcal{C}/{\sim}) \geq 3$. For each $x \in \partial_i \mathcal{C}$, let $[x]$ denote the equivalence class of $x$ and let

$$\mathcal{C}_x := \mathrm{ConvHull}_\Omega([x]) \cap \Omega. \tag{8-1}$$

Notice that it is possible for $\mathcal{C}_x$ to be empty.

We consider two conditions:

**Condition (1)** Whenever $x, y \in \partial_i \mathcal{C}$ and $[x, y] \subset \partial\Omega$, we have $x \sim y$.

**Condition (2)** There exist $r, D > 0$ such that if $x \not\sim y$, then

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(\mathcal{C}_x, r) \cap \mathcal{N}_\Omega(\mathcal{C}_y, r)) < D.$$

We will prove the following results about the quotient space $\partial_i \mathcal{C}/{\sim}$. These arguments are similar to analogous arguments of Choi [5] and Weisman [23].

**Proposition 8.1** *The quotient $\partial_i \mathcal{C}/{\sim}$ is a compact Hausdorff metrizable space.*

**Proposition 8.2** *If Condition (1) holds, then $\Gamma$ acts as a convergence group on $\partial_i \mathcal{C}/{\sim}$.*

To state the final result, we need a definition.

**Definition 8.3** A point $x \in \partial\Omega$ is a *uniformly conical limit point of $\Gamma$ acting on $\Omega$* if for any $p_0 \in \Omega$ the image of $[p_0, x)$ in $\Gamma\backslash\Omega$ is relatively compact.

**Proposition 8.4** *If Conditions (1) and (2) hold, $x \in \partial_i \mathcal{C}$ is a uniformly conical limit point of $\Gamma$ acting on $\Omega$, and $[x] \subset F_\Omega(x)$, then $[x] \in \partial_i \mathcal{C}/\sim$ is a conical limit point of $\Gamma$ (in the convergence group sense; see Section 3.2).*

## 8.1 Proof of Proposition 8.1

The proof is an exercise in point set topology. We include it for the convenience of the reader. We also note that similar arguments appear in [5, Proposition 10.3.8; 23, Proposition 8.10]. For the rest of this proof, let $\pi : \partial_i \mathcal{C} \to \partial_i \mathcal{C}/\sim$ denote the quotient map.

We begin the proof with an useful observation:

**Observation 8.5** *Let $U \subset \partial_i \mathcal{C}$ be an open set, $U^* := \{x \in \partial_i \mathcal{C} : [x] \subset U\}$, and $V := \pi(U^*)$. Then $U^*$ and $V$ are open subsets of $\partial_i \mathcal{C}$ and $\partial_i \mathcal{C}/\sim$ respectively.*

**Proof of Observation 8.5** If $U^*$ is empty, then $V$ is empty and the statement is true. So assume that $U^*$ is nonempty. To show that $U^*$ is open, we show that $\partial_i \mathcal{C} - U^*$ is closed. Consider any sequence $\{z_n\}$ in $\partial_i \mathcal{C} - U^*$ such that $z_n \to z \in \partial_i \mathcal{C}$. Since $z_n \notin U^*$, there exists $z_n' \notin U$ such that $(z_n, z_n') \in R$. As $\partial_i \mathcal{C} - U$ is compact, we can pass to a subsequence and assume that $z_n' \to z' \in \partial_i \mathcal{C} - U$. Since $R$ is closed and $(z_n, z_n') \to (z, z')$, we have $(z, z') \in R$. Thus, $z \in \partial_i \mathcal{C} - U^*$ and hence $\partial_i \mathcal{C} - U^*$ is closed.

Since $U^*$ is a union of equivalence classes, we have

$$\pi^{-1}(V) = \pi^{-1}(\pi(U^*)) = U^*,$$

and so, by the definition of the quotient topology, $V$ is open. □

**$\partial_i \mathcal{C}/\sim$ is Hausdorff** Since $\partial_i \mathcal{C}$ is a compact Hausdorff space and $R$ is closed, it is well known that $\partial_i \mathcal{C}/\sim$ is Hausdorff; eg [9, Proposition 1.4.4]. To see this, fix $x, y \in \partial_i \mathcal{C}$ such that $\pi(x) \neq \pi(y)$. Since $R$ is closed, $[w] = \{z \in \partial_i \mathcal{C} : (w, z) \in R\}$ is a closed, and hence compact, subset of $\partial_i \mathcal{C}$ for any $w \in \partial_i \mathcal{C}$. Then $[x]$ and $[y]$ are disjoint compact subsets of a Hausdorff space $\partial_i \mathcal{C}$. So we can separate $[x]$ and $[y]$ using disjoint open sets $U_x, U_y \subset \partial_i \mathcal{C}$.

Let $U_w^* := \{z \in \partial_i \mathcal{C} : [z] \subset U_w\}$ and $V_w := \pi(U_w^*)$ for $w = x, y$. By Observation 8.5, $U_w^*$ are disjoint open sets with $[w] \subset U_w$ for $w = x, y$. Then, by Observation 8.5, $V_x$ and $V_y$ are disjoint nonempty open subsets of $\partial_i \mathcal{C}/\sim$ that separate $\pi(x)$ and $\pi(y)$. So $\partial_i \mathcal{C}/\sim$ is Hausdorff.

**$\partial_i \mathcal{C}/\sim$ is compact and metrizable** Since $\partial_i \mathcal{C}/\sim$ is Hausdorff and $\partial_i \mathcal{C}/\sim = \pi(\partial_i \mathcal{C})$ is the continuous image of a compact metric space, it is compact and metrizable; eg [24, Corollary 23.2]. Once again, we include the argument.

Since $\partial_i \mathcal{C}$ is compact and $\pi$ is continuous, then $\partial_i \mathcal{C}/\sim$ is compact. Then, by the Uryshon metrization theorem, it suffices to show that $\partial_i \mathcal{C}/\sim$ has a countable basis to prove that it is metrizable. Since $\partial_i \mathcal{C}$ is a compact metrizable space, we can choose a countable basis $\mathcal{U} := \{U_n : n \in \mathbb{N}\}$ for $\partial_i \mathcal{C}$. Without loss of generality, we can assume that $\mathcal{U}$ is closed under finite unions. For $n \geq 1$, let

$$U_n^* := \{x \in \partial_i \mathcal{C} : [x] \subset U_n\} \quad \text{and} \quad V_n := \pi(U_n^*).$$

By Observation 8.5, these are open sets. Further, since any equivalence class is compact and $\mathcal{U}$ is closed under finite unions, for any $x \in \partial_i \mathcal{C}$ there exists $U_n$ with $[x] \subset U_n$. Then $x \in [x] \subset U_n^*$ and $[x] \in V_n$. Thus $\mathcal{U}^* := \{U_n^* : U_n^* \neq \varnothing, n \in \mathbb{N}\}$ is an open cover of $\partial_i \mathcal{C}$ and $\mathcal{V} := \{V_n : U_n^* \neq \varnothing, n \in \mathbb{N}\}$ is an open cover of $\partial_i \mathcal{C}/\sim$.

We will show that $\mathcal{V}$ a basis for the topology on $\partial_i \mathcal{C}/\sim$. Let $w \in W \subset \partial_i \mathcal{C}/\sim$ where $W$ is an open set. Then $\pi^{-1}(w) = [w]$ is an equivalence class and hence compact. Then, since $\mathcal{U}$ is closed under finite unions, there exists $U_n$ with $[w] \subset U_n \subset \pi^{-1}(W)$. Then $w \in V_n^* \subset W$. Thus $\mathcal{V}$ is a basis for the topology.

## 8.2 Proof of Proposition 8.2

We also note that similar arguments appear in [5, Theorem 10.3.1; 23, Proposition 8.8].

Notice that Condition (1) implies that

$$\partial_i \mathcal{C} \cap \overline{F_\Omega(x')} \subset [x']$$

for all $x' \in \partial_i \mathcal{C}$.

Suppose that $\{\gamma_n\}$ is a sequence of distinct elements in $\Gamma$. Fix $p_0 \in \mathcal{C}$. Passing to a subsequence we can suppose that $\gamma_n(p_0) \to x \in \partial_i \mathcal{C}$, $\gamma_n^{-1}(p_0) \to y \in \partial_i \mathcal{C}$, and $\gamma_n \to S$ in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then, by Observation 2.3,

$$\lim_{n\to\infty} \gamma_n(z) = S(z)$$

for all $z \in \mathbb{P}(\mathbb{R}^d) - \mathbb{P}(\ker S)$ and the convergence is locally uniform. By Proposition 2.4,

$$S(\overline{\Omega} - \mathbb{P}(\ker S)) \subset \overline{\Omega} \cap \mathrm{Span}\, F_\Omega(x) = \overline{F_\Omega(x)}.$$

Then

$$S(\overline{\mathcal{C}} - \mathbb{P}(\ker S)) \subset \partial_i \mathcal{C} \cap \overline{F_\Omega(x)} \subset [x].$$

Proposition 2.4 also implies that $\mathbb{P}(\ker S) \cap \Omega = \varnothing$ and $y \in \mathbb{P}(\ker S)$. So if $z \in \partial_i \mathcal{C} \cap \mathbb{P}(\ker S)$, then $[y, z] \subset \partial\Omega$ and then Condition (1) implies that $z \in [y]$. Hence

$$\gamma_n|_{(\partial_i \mathcal{C}/\sim)-[y]}$$

converges locally uniformly to the constant map $[x]$.

## 8.3 Proof of Proposition 8.4

We also note that similar arguments appear in [5, Theorem 10.3.1; 23, Proposition 8.17].

Recall the notation $\mathcal{C}_x$ from (8-1). We start with a lemma.

**Lemma 8.6** *If $y \in \partial_i \mathcal{C}$ and $z \in F_\Omega(\partial_i \mathcal{C}_y) \cap \partial_i \mathcal{C}$, then $z \sim y$.*

**Proof** By hypothesis, there exists some $z' \in \partial_i \mathcal{C}_y$ with $z \in F_\Omega(z')$.

Since $[y]$ is closed, the extreme points of $\mathrm{ConvHull}_\Omega([y])$ are contained in $[y]$. Then there exist $y_1, \dots, y_m \in [y]$ such that

$$z' \in \text{rel-int}\big(\mathrm{ConvHull}_\Omega(\{y_1, \dots, y_m\})\big).$$

Thus $y_1, \dots, y_m \in \overline{F_\Omega(z')}$. Then $[z, y_1] \subset \partial\Omega$ where $z, y_1 \in \partial_i \mathcal{C}$. Condition (1) implies that $z \sim y_1$. Thus $z \sim y$. □

Now suppose $x \in \partial_i \mathcal{C}$ is a uniform conical limit point of $\Gamma$ acting on $\Omega$. Fix $p_0 \in \mathcal{C}$ and a sequence $\{p_n\}$ in $[p_0, x)$ with $p_n \to x$.

By Condition (2), there exist $r, D > 0$ such that if $y_1 \not\sim y_2$, then

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(\mathcal{C}_{y_1}, r) \cap \mathcal{N}_\Omega(\mathcal{C}_{y_2}, r)) < D.$$

**Lemma 8.7** *For every $n \in \mathbb{N}$, there exists $q_n \in [p_n, x)$ such that*

$$\mathcal{B}_\Omega(q_n, 2D) \cap (p_0, x) \not\subset \mathcal{N}_\Omega(\mathcal{C}_y, r)$$

*for all $y \in \partial_i \mathcal{C}$.*

**Proof** Fix $n \in \mathbb{N}$ and suppose not. Then for every $q \in [p_n, x)$ there exists $y(q) \in \partial_i \mathcal{C}$ such that

$$\mathcal{B}_\Omega(q, 2D) \cap (p_0, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(q)}, r).$$

Note that this implies that $\mathcal{C}_{y(q)}$ is a nonempty set. We claim that $\mathcal{C}_{y(q)}$ does not depend on $q$. To show this it is enough to fix $q', q'' \in [p_n, x)$ with $\mathrm{d}_\Omega(q', q'') \le D$ and show that $\mathcal{C}_{y(q')} = \mathcal{C}_{y(q'')}$. In this case, $\mathcal{B}_\Omega(q', D) \subset \mathcal{B}_\Omega(q'', 2D)$; hence

$$\mathcal{B}_\Omega(q', D) \cap (p_0, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(q')}, r) \cap \mathcal{N}_\Omega(\mathcal{C}_{y(q'')}, r).$$

Then by our choice of $D$, we have $\mathcal{C}_{y(q')} = \mathcal{C}_{y(q'')}$. Thus $\mathcal{C}_{y(q)}$ does not depend on $q \in [p_n, x)$. Thus

$$(p_n, x) \subset \mathcal{N}_\Omega(\mathcal{C}_{y(p_n)}, r)$$

Then Proposition 2.8 implies that $x \in F_\Omega(\partial_i \mathcal{C}_{y(p_n)})$. So by the previous lemma $x \sim y(p_n)$. Then $\mathcal{C}_x = \mathcal{C}_{y(p_n)}$ is nonempty, which contradicts the assumption that $[x] \subset F_\Omega(x)$. □

We finish the proof of Proposition 8.4. Fix a sequence $\{q_n\}$ as in the above lemma. Since $x$ is a uniformly conical limit point, there exists a sequence $\{\gamma_n\}$ in $\Gamma$ such that $\{\gamma_n(q_n)\}$ is relatively compact in $\Omega$. Passing to a subsequence we can suppose that $\gamma_n(p_0) \to b$, $\gamma_n^{-1}(p_0) \to c$, and $\gamma_n(x) \to a$. By the proof of Proposition 8.2,

$$\gamma_n|_{(\partial_i \mathcal{C}/\sim) - [c]}$$

converges locally uniformly to the constant map $[b]$. Also $\gamma_n[x] \to [a]$. So, to show that $[x]$ is a conical limit point in the convergence group sense, we need to prove that $[x] = [c]$ and $[a] \ne [b]$.

Suppose for a contradiction that $[a]=[b]$. Since $\{\gamma_n(q_n)\}$ is relatively compact in $\Omega$ and $\gamma_n(q_n)\in(\gamma_n(x),\gamma_n(p_0))$, we have $(a,b)\subset\Omega$. Hence $(a,b)\subset\mathcal{C}_a$ and so for $n$ large,

$$\mathcal{B}_\Omega(\gamma_n(q_n),2D)\cap(\gamma_n(p_0),\gamma_n(x))\subset\mathcal{N}_\Omega(\mathcal{C}_a,r).$$

This implies that

$$\mathcal{B}_\Omega(q_n,2D)\cap(p_0,x)\subset\mathcal{N}_\Omega(\mathcal{C}_{\gamma_n^{-1}(a)},r),$$

which is a contradiction. So $[a]\neq[b]$.

Since $\gamma_n([x])\to[a]$ and $[a]\neq[b]$, we must have $[x]=[c]$.

# 9 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Suppose that $(\Omega,\mathcal{C},\Gamma)$ is a naive convex cocompact triple.

**Theorem 1.4(1)** Suppose that $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P}:=\{P_1,\dots,P_N\}$ and

$$X_j:=\mathrm{ConvHull}_\Omega(\mathcal{L}_\Omega(P_j)\cap\partial_i\,\mathcal{C})\cap\Omega,\quad\text{where } j=1,\dots,N.$$

Then Proposition 6.1 implies that $(\mathcal{C},\mathrm{d}_\Omega)$ is relatively hyperbolic with respect to

$$\mathcal{X}=\Gamma\cdot\{X_1,\dots,X_N\}.$$

Proposition 3.4 implies that $\mathcal{X}$ is a peripheral family of $(\Omega,\mathcal{C},\Gamma)$.

**Theorem 1.4(2)** Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega,\mathcal{C},\Gamma)$ and $\mathcal{P}$ is a set of representatives of the $\Gamma$-conjugacy classes in $\{\mathrm{Stab}_\Gamma(X):X\in\mathcal{X}\}$. Let

$$[\partial_i\,\mathcal{C}]_{\mathcal{X}}:=\partial_i\,\mathcal{C}/\sim$$

be as in Definition 1.2. We claim that the action of $\Gamma$ on $[\partial_i\,\mathcal{C}]_{\mathcal{X}}$ satisfies Theorem 3.5.

**Lemma 9.1** *The set $R:=\{(x,y)\in\partial_i\,\mathcal{C}\times\partial_i\,\mathcal{C}:x\sim y\}$ is closed.*

**Proof** Suppose that $\{(x_n,y_n)\}$ is a sequence in $R$ converging to $(x,y)$ in $\partial_i\,\mathcal{C}\times\partial_i\,\mathcal{C}$.

**Case 1** Assume $(x,y)\subset\Omega$. Then $(x_n,y_n)\subset\Omega$ for $n$ sufficiently large, and for such $n$ there exists $X_n\in\mathcal{X}$ with $x_n,y_n\in F_\Omega(\partial_i X_n)$. By Theorem 5.1(5) there is some $R>0$ such that for every $n$ there exist $x_n',y_n'\in\partial_i X_n$ where

$$\mathrm{d}_{F_\Omega(x_n)}(x_n,x_n')\le R\quad\text{and}\quad\mathrm{d}_{F_\Omega(y_n)}(y_n,y_n')\le R.$$

Then by Proposition 2.9,

$$\mathrm{d}_\Omega^{\mathrm{Haus}}((x_n,y_n),(x_n',y_n'))\le R.$$

Since $(x_n,y_n)\to(x,y)$, this implies that

$$\lim_{n,k\to\infty}\mathrm{diam}_\Omega(\mathcal{N}_\Omega(X_n,R+1)\cap\mathcal{N}_\Omega(X_k,R+1))=\infty.$$

Since $\mathcal{X}$ is strongly isolated, $X := X_n = X_k$ for $n, k$ sufficiently large. Then

$$(x, y) \subset \mathcal{N}_\Omega(X, R+1),$$

and so $x, y \in F_\Omega(\partial_i X)$ by Proposition 2.8. Thus $x \sim y$.

**Case 2** Assume $[x, y] \subset \partial\Omega$. Then $x \sim y$ by Theorem 5.1(7). □

Let

$$B := \{[F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] : X \in \mathcal{X}\} \subset [\partial_i \mathcal{C}]_{\mathcal{X}}.$$

**Lemma 9.2** (1) *$[\partial_i \mathcal{C}]_{\mathcal{X}}$ is a compact Hausdorff metrizable space.*

(2) *$\Gamma$ acts as a convergence group on $[\partial_i \mathcal{C}]_{\mathcal{X}}$.*

(3) *If $z \in [\partial_i \mathcal{C}]_{\mathcal{X}} - B$, then $z$ is a conical limit point of $\Gamma$.*

**Proof** We use the results of Section 8. Part (1) follows from the previous lemma and Proposition 8.1.

Notice that Theorem 5.1(7) implies that the quotient $[\partial_i \mathcal{C}]_{\mathcal{X}}$ satisfies Condition (1) from Section 8. We claim that the quotient $[\partial_i \mathcal{C}]_{\mathcal{X}}$ satisfies Condition (2) from Section 8. First notice that if $z \in [\partial_i \mathcal{C}]_{\mathcal{X}} - B$, then $\mathcal{C}_z = \varnothing$. So it suffices to consider elements in $B$. Further, if $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$, then by Theorem 5.1(2) and (8), $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on both $\mathcal{C}_b$ and $X$. So $\mathcal{C}_b$ is contained in a bounded neighborhood of $X$. By Theorem 5.1(1), we can choose this bound to be independent of $b$. Then since $\mathcal{X}$ is strongly isolated, $[\partial_i \mathcal{C}]_{\mathcal{X}}$ satisfies Condition (2).

Then $\Gamma$ acts as a convergence group on $[\partial_i \mathcal{C}]_{\mathcal{X}}$ by Proposition 8.2. If $z \in [\partial_i \mathcal{C}]_{\mathcal{X}} - B$, then we have $z = [F_\Omega(x) \cap \partial_i \mathcal{C}]$ for some $x \in \partial_i \mathcal{C}$ and so Proposition 8.4 implies that $z$ is a conical limit point of $\Gamma$. This proves parts (2) and (3). □

**Lemma 9.3** (1) *$\Gamma$ has finitely many orbits in $B$.*

(2) *If $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$, then $\mathrm{Stab}_\Gamma(b) = \mathrm{Stab}_\Gamma(X)$. In particular, $\mathrm{Stab}_\Gamma(b)$ is finitely generated.*

(3) *If $b \in B$, then $b$ is a bounded parabolic fixed point.*

**Proof** (1) This follows immediately from Theorem 5.1(1).

(2) It is clear that $\mathrm{Stab}_\Gamma(b) \supset \mathrm{Stab}_\Gamma(X)$. For the other inclusion, if $\gamma \in \mathrm{Stab}_\Gamma(b)$, then

$$F_\Omega(\partial_i(\gamma X)) = \gamma F_\Omega(\partial_i X) = F_\Omega(\partial_i X).$$

So $\gamma X = X$ by Theorem 5.1(3). Hence $\gamma \in \mathrm{Stab}_\Gamma(X)$ and so $\mathrm{Stab}_\Gamma(b) = \mathrm{Stab}_\Gamma(X)$. The "in particular" part then follows from Theorem 5.1(2).

(3) The proof is an application of Theorem 7.1, as outlined in Remark 7.3. We now provide the details.

Fix $b = [F_\Omega(\partial_i X) \cap \partial_i \mathcal{C}] \in B$. By Theorem 5.1(2), $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $X$. Furthermore, by Theorem 5.1(6), if $x \in \partial_i X$ and $[x, y] \subset \partial_i \mathcal{C}$, then $[x, y] \in F_\Omega(\partial_i X)$.

Then Theorem 7.1 implies that $\mathrm{Stab}_\Gamma(X)$ acts cocompactly on $\partial_i \mathcal{C} - F_\Omega(\partial_i X)$. Thus $\mathrm{Stab}_\Gamma(b)$ acts cocompactly on $[\partial_i \mathcal{C}]_{\mathcal{X}} - \{b\}$. □

Then Theorem 3.5 implies that $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P}$ and there exists a $\Gamma$-equivariant homeomorphism $\partial(\Gamma,\mathcal{P}) \to [\partial_i \mathcal{C}]_{\mathcal{X}}$.

**Theorem 1.4(a)–(e)** Now suppose that at least one of the conditions is satisfied. Proposition 6.1 implies parts (a) and (c). Part (b) was established in the proof of Theorem 1.4(2); see the last paragraph. Parts (d) and (e) follow from Theorem 5.1.

# 10 The convex cocompact case

We prove Theorem 1.7 and Proposition 1.8. As mentioned in the introduction, the key result which allows our results to simplify in the convex cocompact case is the following observation; see [7, Corollaries 4.10 and 4.13].

**Observation 10.1** *Suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact subgroup.*

(1) *If $x \in \partial\Omega$ and $F_\Omega(x) \cap \partial_i \mathcal{C}_\Omega(\Gamma) \neq \varnothing$, then $F_\Omega(x) \subset \partial_i \mathcal{C}_\Omega(\Gamma)$.*

(2) *$\mathcal{L}_\Omega(\Gamma) = \partial_i \mathcal{C}_\Omega(\Gamma)$. In particular, $\mathcal{L}_\Omega(\Gamma)$ is closed.*

**Proof** (1) Fix $x \in \partial\Omega$ with $F_\Omega(x) \cap \partial_i \mathcal{C}_\Omega(\Gamma) \neq \varnothing$. Then Proposition 2.5 implies that $F_\Omega(x) \subset \mathcal{L}_\Omega(\Gamma)$ and we have $\mathcal{L}_\Omega(\Gamma) \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ by the definition of $\mathcal{C}_\Omega(\Gamma)$.

(2) Since $\Gamma$ acts cocompactly on $\mathcal{C}$, we have $\partial_i \mathcal{C}_\Omega(\Gamma) \subset \mathcal{L}_\Omega(\Gamma)$ and the reverse inclusion is by definition of $\mathcal{C}_\Omega(\Gamma)$. □

## 10.1 Proof of Theorem 1.7

Suppose that $\Gamma \leq \mathrm{Aut}(\Omega)$ is a convex cocompact group such that $\Gamma$ is relatively hyperbolic with respect to $\mathcal{P} := \{P_1, \dots, P_N\}$. Then by definition $(\Omega, \mathcal{C}_\Omega(\Gamma), \Gamma)$ is a naive convex cocompact triple. Let $\mathcal{C} = \mathcal{C}_\Omega(\Gamma)$ and

$$\mathcal{X} := \Gamma \cdot \{\mathcal{C}_\Omega(P_1), \dots, \mathcal{C}_\Omega(P_N)\}.$$

**Lemma 10.2** (parts (a) and (c)) *Each $P_j$ is a convex cocompact subgroup of* $\mathrm{Aut}(\Omega)$ *and* $(\mathcal{C}, \mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to $\mathcal{X}$.*

**Proof** Since $\mathcal{L}_\Omega(P_j) \subset \mathcal{L}_\Omega(\Gamma) \subset \partial_i \mathcal{C}$, then

$$\mathcal{C}_\Omega(P_j) = \mathrm{ConvHull}_\Omega(\mathcal{L}_\Omega(P_j)) \cap \Omega = \mathrm{ConvHull}_\Omega(\mathcal{L}_\Omega(P_j) \cap \partial_i \mathcal{C}) \cap \Omega.$$

Proposition 6.1 implies that $P_j$ acts cocompactly on $\mathcal{C}_\Omega(P_j)$. Then $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{X}$. □

If $X \in \mathcal{X}$, then $X = \gamma\, \mathcal{C}_\Omega(P_j) = \mathcal{C}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $1 \leq j \leq N$. Then, since $P_j$ is a convex cocompact subgroup of $\mathrm{Aut}(\Omega)$, Observation 10.1 implies that

$$F_\Omega(\partial_i X) = \partial_i X = \mathcal{L}_\Omega(\gamma P_j \gamma^{-1}). \tag{10-1}$$

**Lemma 10.3** (part (e)) *If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

**Proof** Fix $x \in \text{rel-int}(\ell)$. Then $\dim F_\Omega(x) \geq 1$ and so, by Observation 10.1,

$$\operatorname{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i \mathcal{C}) = \operatorname{diam}_{F_\Omega(x)} F_\Omega(x) = \infty.$$

Thus by Theorem 1.4(d), there exist $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$ such that

$$x \in F_\Omega(\partial_i X),$$

where $X = \mathcal{C}_\Omega(\gamma P_j \gamma^{-1}) = \gamma\, \mathcal{C}_\Omega(P_j)$. Then (10-1) implies that

$$\text{rel-int}(\ell) \subset F_\Omega(x) \subset F_\Omega(\partial_i X) = \mathcal{L}_\Omega(\gamma P_j \gamma^{-1}).$$

Since $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ is closed (see Observation 10.1), this completes the proof. □

Recall that, by definition, $[\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}$ is obtained from $\partial_i \mathcal{C}_\Omega(\Gamma)$ by collapsing each $\mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ to a point. Hence (10-1) and Lemma 10.3 imply that

$$[\partial_i \mathcal{C}]_{\mathcal{X}} = [\partial_i \mathcal{C}]_{\mathcal{P}}. \tag{10-2}$$

**Lemma 10.4** (part (b)) *There is a $\Gamma$-equivariant homeomorphism*

$$\partial(\Gamma, \mathcal{P}) \to [\partial_i \mathcal{C}_\Omega(\Gamma)]_{\mathcal{P}}.$$

**Proof** This follows immediately from Theorem 1.4(b) and (10-2). □

To prove Theorem 1.7(d) we will use the following lemma.

**Lemma 10.5** [15, Lemma 15.5] *Assume that $x \in \partial_i \mathcal{C}$ is not a $C^1$-smooth point of $\partial\Omega$ and $q \in \mathcal{C}$. For any $r > 0$ and $\epsilon > 0$ there exists $q_{r,\epsilon} \in (x, q]$ with the following property: if $p \in (x, q_{r,\epsilon}]$, then there exists a properly embedded simplex $S = S(p) \subset \mathcal{C}$ of dimension at least two such that*

$$\mathcal{B}_\Omega(p, r) \cap (x, q] \subset \mathcal{N}_\Omega(S, \epsilon). \tag{10-3}$$

Let us now prove part (d).

**Lemma 10.6** (part (d)) *If $x \in \partial_i \mathcal{C}_\Omega(\Gamma)$ is not a $C^1$-smooth point of $\Omega$, then $x \in \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$.*

**Proof** Fix $q \in \mathcal{C}$. By Theorem 1.4, $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}, \Gamma)$. So there exist constants $D_1, D_2 > 0$ such that:

(a) If $S \subset \mathcal{C}$ is a properly embedded simplex of dimension at least two, then there exists $X \in \mathcal{X}$ with $S \subset \mathcal{N}_\Omega(X, D_2)$.

(b) If $X_1, X_2 \in \mathcal{X}$ are distinct, then

$$\operatorname{diam}_\Omega(\mathcal{N}_\Omega(X_1, D_2 + 1) \cap \mathcal{N}_\Omega(X_2, D_2 + 1)) < D_1.$$

Let $q' \in (x,q]$ satisfy Lemma 10.5 for the constants $r = D_1 + 1$ and $\epsilon = 1$. Fix a sequence $\{p_n\}$ in $(x,q']$ such that $p_n \to x$ and $\mathrm{d}_\Omega(p_n, p_{n+1}) \le 1$. For each $n \ge 1$, fix $S_n \subset \mathcal{C}$ a properly embedded simplex of dimension at least two such that

$$\mathcal{B}_\Omega(p_n, D_1+1) \cap (x,q] \subset \mathcal{N}_\Omega(S_n, 1).$$

Then fix $X_n \in \mathcal{X}$ such that

$$S_n \subset \mathcal{N}_\Omega(X_n, D_2).$$

Then

$$\begin{aligned}\mathcal{B}_\Omega(p_n, D_1) \cap (x,q] &\subset \big(\mathcal{B}_\Omega(p_n, D_1+1) \cap \mathcal{B}_\Omega(p_{n+1}, D_1+1)\big) \cap (x,q] \\ &\subset \mathcal{N}_\Omega(X_n, D_2+1) \cap \mathcal{N}_\Omega(X_{n+1}, D_2+1).\end{aligned}$$

Since $\mathrm{diam}_\Omega(\mathcal{B}_\Omega(p_n, D_1) \cap (x,q]) \ge D_1$, this implies that there exists $X \in \mathcal{X}$ such that $X = X_n$ for all $n$ large enough. Then

$$(x,q'] \subset \mathcal{N}_\Omega(X, D_2+1).$$

Now $X = \mathcal{C}_\Omega(\gamma P_j \gamma^{-1})$ for some $\gamma \in \Gamma$ and $P_j \in \mathcal{P}$. Finally, Proposition 2.8 and (10-1) imply that $x \in \partial_i\, \mathcal{C}_\Omega(\gamma P_j \gamma^{-1}) = \mathcal{L}_\Omega(\gamma P_j \gamma^{-1})$. □

## 10.2 Proof of Proposition 1.8

Suppose that $\Gamma \le \mathrm{Aut}(\Omega)$ is a convex cocompact subgroup and $\mathcal{X}$ is a $\Gamma$-invariant collection of closed unbounded convex subsets of $\Omega$.

**(1) ⟹ (2)** Suppose that $\mathcal{X}$ is a peripheral family of $(\Omega, \mathcal{C}_\Omega(\Gamma), \Gamma)$.

**Lemma 10.7** *$\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric $\mathrm{d}_\Omega$.*

**Proof** This follows immediately from Lemma 5.2. □

**Lemma 10.8** *If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \varnothing$.*

**Proof** We prove the contrapositive. Suppose $x \in \partial_i X_1 \cap \partial_i X_2$. Fix $q_1 \in X_1$ and $q_2 \in X_2$. Then

$$\mathrm{d}_\Omega^{\mathrm{Haus}}((x,q_1],(x,q_2]) \le \mathrm{d}_\Omega(q_1,q_2)$$

by Proposition 2.9. So

$$(x,q_1] \subset \mathcal{N}_\Omega(X_1, \mathrm{d}_\Omega(q_1,q_2)+1) \cap \mathcal{N}_\Omega(X_2, \mathrm{d}_\Omega(q_1,q_2)+1),$$

and hence $X_1 = X_2$. □

**Lemma 10.9** *If $\ell \subset \partial_i\, \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.*

**Proof** Fix $x \in \text{rel-int}(\ell)$. Then $\dim F_\Omega(x) \ge 1$ and so by Observation 10.1,

$$\mathrm{diam}_{F_\Omega(x)}(F_\Omega(x) \cap \partial_i\, \mathcal{C}) = \mathrm{diam}_{F_\Omega(x)}\, F_\Omega(x) = \infty.$$

Thus by Theorem 5.1(4) there exists $X \in \mathcal{X}$ with $x \in F_\Omega(\partial_i X)$. We claim that $F_\Omega(x) \subset \partial_i X$ which will imply the lemma.

To show that $F_\Omega(x) \subset \partial_i X$, it suffices to fix an extreme point $e \in \partial F_\Omega(x)$ of $F_\Omega(x)$ and show that $\partial_i X$ contains $e$. By Observation 10.1 and Theorem 5.1(5) there exists $R > 0$ such that

$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(F_\Omega(x), \partial_i X \cap F_\Omega(x)) \le R.$$

Since $F_{F_\Omega(x)}(e) = \{e\}$, then Proposition 2.8 implies that $e \in \partial_i X$. □

**(2) ⟹ (1)** Suppose that $\mathcal{X}$ has the following properties:

(a) $\mathcal{X}$ is closed in the local Hausdorff topology induced by the Hilbert metric $\mathrm{d}_\Omega$.

(b) If $X_1, X_2 \in \mathcal{X}$ are distinct, then $\partial_i X_1 \cap \partial_i X_2 = \varnothing$.

(c) If $\ell \subset \partial_i \mathcal{C}_\Omega(\Gamma)$ is a nontrivial line segment, then $\ell \subset \partial_i X$ for some $X \in \mathcal{X}$.

**Lemma 10.10** *If $X_1, X_2 \in \mathcal{X}$ and $F_\Omega(\partial_i X_1) \cap F_\Omega(\partial_i X_2) \neq \varnothing$, then $X_1 = X_2$.*

**Proof** By hypothesis there exists $x_1 \in \partial_i X_1$ and $x_2 \in \partial_i X_2$ with $F_\Omega(x_1) = F_\Omega(x_2)$. If $x_1 = x_2$, then property (b) implies that $X_1 = X_2$. Otherwise, $[x_1, x_2] \subset F_\Omega(x_1) \subset \partial\Omega$ and so by property (c) there exists $X_3$ with $[x_1, x_2] \subset \partial_i X_3$. But then by property (b), $X_1 = X_3 = X_2$. □

**Lemma 10.11** *$\mathcal{X}$ is discrete in the local Hausdorff topology induced by the Hilbert metric $\mathrm{d}_\Omega$.*

**Proof** Fix a sequence $\{X_n\}$ in $\mathcal{X}$ converging to some closed subset $X$. Since $\mathcal{X}$ is closed, $X \in \mathcal{X}$. Fix $p \in X$ and $x \in \partial_i X$.

We claim that $X_n = X$ when $n$ is sufficiently large. Suppose not. Then after passing to a subsequence we can suppose that $X_n \neq X$ for all $n$. For each $n$ fix $p_n \in X_n$ such that $p_n \to p$. Passing to a tail of our sequence we can suppose that $\mathrm{d}_\Omega(p_n, p) < 1$ for all $n$. The previous lemma implies that $x \notin F_\Omega(\partial_i X_n)$ and so by Proposition 2.8,

$$\lim_{q \in [p_n, x), q \to x} \mathrm{d}_\Omega(q, X_n) = +\infty.$$

So for each $n$ there exists $q_n \in [p_n, x)$ with $\mathrm{d}_\Omega(q_n, X_n) = 1$. Notice that

$$\lim_{n\to\infty} \mathrm{d}_\Omega(q_n, X) \le \lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}_\Omega((x, p_n], (x, p]) \le \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, p) = 0$$

by Proposition 2.9. Then, since $X_n \to X$ in the local Hausdorff topology and $\mathrm{d}_\Omega(q_n, X_n) = 1$, the sequence $\{q_n\}$ must leave every compact subset of $\Omega$. So $q_n \to x$.

Since $q_n \in \mathcal{C}$, there exists $\{\gamma_n\}$ in $\Gamma$ such that $\{\gamma_n(q_n)\}$ is relatively compact in $\Omega$. Passing to a subsequence, we can suppose that $\gamma_n(q_n) \to q_\infty \in \mathcal{C}$, $\gamma_n(p) \to p_\infty \in \bar{\mathcal{C}}$, $\gamma_n(X) \to X_\infty$, and $\gamma_n(X_n) \to Y_\infty$. Since $q_n \to x$, we must have $p_\infty \in \partial_i \mathcal{C}$. Since $\mathcal{X}$ is $\Gamma$-invariant and closed in the local Hausdorff topology, $X_\infty, Y_\infty \in \mathcal{X}$.

By construction, $q_\infty \in X_\infty$ and $\mathrm{d}_\Omega(q_\infty, Y_\infty) = 1$. So $X_\infty \neq Y_\infty$. Also

$$\lim_{n\to\infty} \mathrm{d}_\Omega(\gamma_n(p), \gamma_n(p_n)) = \lim_{n\to\infty} \mathrm{d}_\Omega(p_n, p) = 0,$$

so Proposition 2.8 implies that $\gamma_n(p_n) \to p_\infty$. Then, by Proposition 2.9,

$$[q_\infty, p_\infty) \subset \mathcal{N}_\Omega(Y_\infty, 2),$$

since $[q_n, p_n] \subset \mathcal{N}_\Omega(X_n, 2)$ for all $n$. So Proposition 2.8 implies that $p_\infty \in F_\Omega(\partial_i Y_\infty)$. However, by construction, $p_\infty \in \partial_i X_\infty$ and so Lemma 10.10 implies that $X_\infty = Y_\infty$. So we have a contradiction. □

**Lemma 10.12** *$\mathcal{X}$ is strongly isolated.*

**Proof** Fix $r > 0$ and suppose for a contradiction that for every $n \in \mathbb{N}$ there exist $X_n, Y_n \in \mathcal{X}$ distinct such that

$$\operatorname{diam}_\Omega(\mathcal{N}_\Omega(X_n, r) \cap \mathcal{N}_\Omega(Y_n, r)) \geq n.$$

Proposition 2.9 implies that $C_n := \mathcal{N}_\Omega(X_n, r) \cap \mathcal{N}_\Omega(Y_n, r)$ is convex.

We claim that each $C_n$ is bounded in $(\mathcal{C}, \mathrm{d}_\Omega)$. If not, there exists $x \in \overline{C}_n \cap \partial\Omega$. Then Proposition 2.8 implies that $x \in F_\Omega(\partial_i X_n) \cap F_\Omega(\partial_i Y_n)$. So $X_n = Y_n$ by Lemma 10.10. Thus we have a contradiction and so each $C_n$ is bounded in $(\mathcal{C}, \mathrm{d}_\Omega)$.

For each $n$ let $(p_n, q_n) \subset C_n$ denote an open line segment with maximal length (with respect to the Hilbert metric). Then $\mathrm{d}_\Omega(p_n, q_n) \geq n$.

Now fix a sequence $\{\gamma_n\}$ in $\Gamma$ such that $\{\gamma_n(p_n)\}$ is relatively compact in $\Omega$. Passing to a subsequence, we can assume that $\gamma_n(p_n) \to p \in \Omega$, $\gamma_n(q_n) \to q \in \overline{\Omega}$, $\gamma_n(X_n) \to X$, and $\gamma_n(Y_n) \to Y$. Then $q \in \partial_i \mathcal{C}$, as $\mathrm{d}_\Omega(p, q_n) \to \infty$ and $q_n \in \mathcal{C}$. Furthermore,

$$(q, p) \subset \mathcal{N}_\Omega(X, r+1) \cap \mathcal{N}_\Omega(Y, r+1),$$

and $X, Y \in \mathcal{X}$. Proposition 2.8 implies that

$$q \in F_\Omega(\partial_i X) \cap F_\Omega(\partial_i Y).$$

So Lemma 10.10 implies that $X = Y$. Then Lemma 10.11 implies that

$$\gamma_n(X_n) = X = Y = \gamma_n(Y_n)$$

for $n$ large. So $X_n = Y_n$ for $n$ large and we have a contradiction. □

The following result implies that $\mathcal{X}$ coarsely contains the properly embedded simplices of $\mathcal{C}$.

**Lemma 10.13** *If $S \subset \mathcal{C}$ is a properly embedded simplex with dimension at least two*, *then there exists $X \in \mathcal{X}$ with $S \subset X$.*

**Proof** Fix a properly embedded simplex $S \subset \mathcal{C}$ with dimension at least two. Let $v_1, \dots, v_k$ denote the vertices of $S$. By property (c), for each $2 \leq j \leq k$ there exists some $X_j \in \mathcal{X}$ with $[v_1, v_j] \subset \partial_i X_j$. Then by property (b), $X_2 = X_3 = \cdots = X_k$. So by convexity $S \subset X_2$. □

**Proof of (1-1) in Proposition 1.8** Suppose that (1) and (2) hold, and let $\mathcal{P}$ be a set of representatives of the $\Gamma$-conjugacy classes in $\{\operatorname{Stab}_\Gamma(X) : X \in \mathcal{X}\}$. To show that the quotients $[\partial_i \mathcal{C}]_{\mathcal{X}}$ and $[\partial_i \mathcal{C}]_{\mathcal{P}}$ coincide it suffices to prove the following.

**Lemma 10.14** *If $X \in \mathcal{X}$, then $\partial_i X = F_\Omega(\partial_i X) = \mathcal{L}_\Omega(\mathrm{Stab}_\Gamma(X))$.*

**Proof** Let $P = \mathrm{Stab}_\Gamma(X)$.

We first observe that $\partial_i X = F_\Omega(\partial_i X)$. By definition $\partial_i X \subset F_\Omega(\partial_i X)$. For the other inclusion, fix $x \in F_\Omega(\partial_i X)$. Then fix $x' \in \partial_i X$ with $x \in F_\Omega(x')$. Observation 10.1 implies that $x \in \partial_i \mathcal{C}$. So, if $x' \neq x$, then there exists $Y \in \mathcal{X}$ with $[x', x] \subset \partial_i Y$. But then $\partial_i X \cap \partial_i Y \neq \varnothing$ which implies that $X = Y$. So $x \in \partial_i X$. Thus $\partial_i X = F_\Omega(\partial_i X)$.

Next we show that $\mathcal{L}_\Omega(P)$ is a subset of $\partial_i X$. Fix $x \in \mathcal{L}_\Omega(P)$, then there exist $p \in \Omega$ and a sequence $\{g_n\}$ in $P$ such that $g_n(p) \to x$. Fix $q \in X$. By passing to a subsequence we may suppose that $g_n(q) \to x' \in \partial_i X$. Then Proposition 2.8 implies that $x \in F_\Omega(x') \subset \partial_i X$.

Finally, Theorem 5.1(2) implies that $P$ acts cocompactly on $X$ and so $\partial_i X \subset \mathcal{L}_\Omega(P)$ by Proposition 2.5. □

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MI: *Mathematisches Institut*
*Heidelberg, Germany*

Current address: *Max Planck Institute for Mathematics in the Sciences*
*Leipzig, Germany*

AZ: *Department of Mathematics, University of Wisconsin–Madison*
*Madison, WI, United States*

`mitul.islam@gmail.com`, `amzimmer2@wisc.edu`



Received: 13 July 2023 Revised: 23 August 2024

ALGEBRAIC & GEOMETRIC TOPOLOGY
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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

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# ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 25 Issue 9 (pages 5175–5754) 2025

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