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A note on rational maps with three branching points on the Riemann sphere

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Studying the existence of rational maps with given branching datum is a classical problem in the field of complex analysis and algebraic geometry. This problem dates back to Hurwitz and remains open to this day. We utilize complex analysis to establish a property of rational maps with 3 branching points on the Riemann sphere.

Given two compact Riemann surfaces M and N, a pair $(d, \mathcal{D})$ of an integer $d \geq 2$ and a collection $\mathcal{D}$ of nontrivial partitions of d is called a candidate branching datum if it satisfies the Riemann–Hurwitz formula. A candidate branching datum is considered exceptional when there is no holomorphic map realization for it. As applications of our main theorem, we present some new types of exceptional branching datum which cover some previously known results. We also deduce the realizability of a certain type of candidate branching datum on the Riemann sphere.

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1 Introduction

Studying the existence of branched covers between two compact Riemann surfaces with given branching datum is an important problem in the field of complex analysis and algebraic geometry. This problem is commonly referred to as the Hurwitz existence problem. It asks whether, given two compact Riemann surfaces M and N, along with a collection $\mathcal{D}$ of partitions of a positive integer d, there exists a degree-d branched cover $f: M \to N$ with $\mathcal{D}$ as the branching datum.

Let M and N be compact Riemann surfaces, and let their respective Euler characteristics be denoted by $\chi(M)$ and $\chi(N)$. A smooth map $f: M \to N$ is a degree-d branched cover if for each $x \in N$ there is a partition $\mu(x) = [\alpha_1, \dots, \alpha_r]$ (here the square brackets are used to denote an unordered set with repetitions) of d such that, over a neighborhood of x in N, f is equivalent to the map $\tilde{f}: \{1, \dots, r\} \times \mathbb{D} \to \mathbb{D}$ where $\tilde{f}(j, z) = z^{\alpha_j}$ and x corresponds to 0 in the disc $\mathbb{D} = \{z \mid |z| < 1\} \subseteq \mathbb{C}$. The points $x \in N$ for which $\mu(x)$ is not the trivial partition $[1, 1, \dots, 1]$ of d constitute the branching set B_f of f, and it is a finite set. The collection $\mathcal{D} = \{\mu(x) \mid x \in B_f\}$ (with repetitions allowed) is called the branching datum of f. As is well known, the degree-d, Euler characteristics $\chi(M)$, $\chi(N)$ and the branching datum $\mathcal{D}$ of f should satisfy the Riemann–Hurwitz formula

$$\nu(\mathcal{D}) = d \cdot \chi(N) - \chi(M), \tag{1}$$

where $\nu(\mathcal{D})$ denotes the total branching of f, which is defined as follows. Let $B_f = \{x_1, \dots, x_n\} \subseteq N$ and $\mathcal{D} = \{[\alpha_1^1, \dots, \alpha_1^{r_1}], \dots, [\alpha_n^1, \dots, \alpha_n^{r_n}]\}$ represent the branching set and the branching datum of f, respectively. For each x_k, its preimage under f consists of a finite number of points $y_k^1, \dots, y_k^{r_k} \in M$, and near each y_k^j the map f is equivalent to $\widetilde{f}(z) = z^{\alpha_k^j}$. The integer α_k^j is usually referred to as the local degree or multiplicity of f at the point y_k^j. The total branching of f is then defined as $\nu(\mathcal{D}) = \sum_{k=1}^n \sum_{j=1}^{r_k} (\alpha_k^j - 1)$. Since

$$\sum_{j=1}^{r_k} \alpha_k^j = d \quad \text{for all } k = 1, \dots, n, \tag{2}$$

the Riemann–Hurwitz formula (1) can be expressed as

$$\sum_{k=1}^{n} (d - r_k) = d \cdot \chi(N) - \chi(M). \tag{3}$$

Given two compact Riemann surfaces M and N, a pair $(d, \mathcal{D})$ of an integer $d \geq 2$ and a collection $\mathcal{D}$ of nontrivial partitions of d is called a candidate branching datum if it satisfies the Riemann–Hurwitz formula. In his classical work [7], Hurwitz reduced the Hurwitz problem to a problem involving partitions realized by suitable permutations in symmetric groups. In [5], Edmonds, Kulkarni, and Stong proved that all candidate branching datum is realizable when $\chi(N) \leq 0$. However, when $N = S^2$ the problem becomes much more complex. It is well known that there exist candidate branching datum $(d, \mathcal{D})$ that cannot be realized by a branched cover, hence called exceptional. For example, this is the case for $d = 4$ and $\mathcal{D} = \{[3, 1], [2, 2], [2, 2]\}$. Characterizing all of such exceptional candidate branching datum remains an open problem to this day. In [25], Zheng determined all exceptional candidate branching datum with 3 branching points and degree $d \leq 22$ by computer for the cases where $M = N = S^2$ and $M = T^2$, $N = S^2$.

Finding new types of exceptional candidate branching datum is of interest as it may provide insights towards establishing a universal criterion. However, the general pattern of realizable data remains unclear. Various approaches such as dessins d'enfant, Speiser graphs and the monodromy approach have been explored to attack the problem. For more details, see [1; 2; 4; 5; 6; 8; 9; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21; 22; 24; 25]. Particularly, we refer the reader to [4; 19; 21] for a review of available results and techniques. In [5], a conjecture proposing connections with number-theoretic facts has been put forward, which is supported by strong evidence in [15; 16].

Conjecture 1.1 (prime-degree conjecture) Suppose $(d, \mathcal{D})$ is a candidate branching datum. If d is a prime, then $(d, \mathcal{D})$ is realizable.

In [5], Edmonds, Kulkarni, and Stong reduced the prime-degree conjecture to the case of candidate branching datum with exactly 3 partitions. Up to now, this conjecture is still open. In this paper we investigate the properties of branched covers for the cases where $M = N = S^2$. Regarding $S^2 \cong \overline{\mathbb{C}}$ as the Riemann sphere, by Stoïlow's theorem (see [23] or [10]), the Hurwitz problem reduces to the existence of rational maps on $\overline{\mathbb{C}}$ with the given branching datum. Thus we can use complex analysis techniques to attack the problem. Motivation by the recent work of Song and Xu [22], we use their idea of expressing a rational map as a composition of a rational map and a power map. Our main result is as follows and can be viewed as a generalization of a result in [18, Theorems 1.4 and 1.5].

Theorem 1.2 *Suppose $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map of degree $d = rk$, $r \geq 2$, $k \geq 2$, with 3 branching points and associated branching datum*

$$\{\mu_1 = [\alpha_1, \dots, \alpha_A], \mu_2 = [rx_1, \dots, rx_B], \mu_3 = [ry_1, \dots, ry_C]\}.$$

Then, up to two Möbius transformations on $\overline{\mathbb{C}}$, there is a rational map $g: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree k such that $f = g^r$, with the property that its branching datum consists of $[x_1, \dots, x_B]$ and $[y_1, \dots, y_C]$, together with data arising by splitting the partition of $[\alpha_1, \dots, \alpha_A]$ into r partitions of k.

Remark In Theorem 1.2, the sentence "up to two Möbius transformations" means that $f = \varphi \circ g^k \circ \phi$, where $\varphi, \phi: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are two Möbius transformations.

Remark In Theorem 1.2, if $x_1 = \dots = x_B = 1$ or $y_1 = \dots = y_C = 1$ then the corresponding partition of k should be expunged from the branching datum of g.

Then, under the assumption of Theorem 1.2, one would have immediate corollaries for realizable data.

Corollary 1.3 *Each $\alpha_j \leq k$.*

Corollary 1.4 *If $A = 2$, then $r = 2$, $\mu_1 = [k, k]$.*

Corollary 1.5 *If $A = r$, then $\mu_1 = [k, \dots, k]$.*

As applications of these results, we offer new proofs of some of the results presented in [18; 26] and give some new type of exceptional candidate branching datum (Propositions 1.9 and 1.10).

Proposition 1.6 [26] *Suppose $d = 2k$ with $k \geq 2$. Then a candidate branching datum of the form*

$$(d, \{[k_1, k_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\})$$

is exceptional if $k_1 \neq k_2$. Suppose in addition $k \geq 3$. Then a candidate branching datum of the form

$$(d, \{[\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{j_1}, 2k - 2j_1], [\underbrace{2, \dots, 2}_{j_2}, 2k - 2j_2]\})$$

is exceptional if $j_1 \neq j_2$.

Proof Since $k_1 \neq k_2$ and $k_1 + k_2 = 2k$, $k_1, k_2 \neq k$. By Corollary 1.4, $(d, \mathcal{D})$ is exceptional.

Suppose $k \geq 3$, and note that $j_1 + j_2 = k$ and $j_1 \neq j_2$. Then, $2j_1 < k$ or $2j_2 < k$. If $2j_1 < k$, then $2k - 2j_1 > k$; If $2j_2 < k$, then $2k - 2j_2 > k$. Thus, by Corollary 1.3, $(d, \mathcal{D})$ is exceptional. □

Proposition 1.7 [26] *Suppose $d = 3k$ with $k \geq 3$. Then a candidate branching datum of the form*

$$(d, \{[k-2, \underbrace{2, \dots, 2}_{k+1}], [\underbrace{3, \dots, 3}_{k}], [\underbrace{3, \dots, 3}_{k}]\})$$

is exceptional if k is odd.

Proof Since k is odd and 2 is even, there is just one odd in $[k-2, 2, \dots, 2]$. Thus $[k-2, 2, \dots, 2]$ cannot be split into 3 partitions of k. Therefore, by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. □

Proposition 1.8 [26] *Suppose $d = rk$ with $r \geq 2$, $k \geq 2$. Then a candidate branching datum of the form*

$$(d, \{[j_1, j_2, \underbrace{1, \dots, 1}_{(r-2)k}], [\underbrace{r, \dots, r}_{k}], [\underbrace{r, \dots, r}_{k}]\})$$

is exceptional if $j_1 \neq j_2$.

Proof Since $j_1 + j_2 = 2k$ and $j_1 \neq j_2$, it follows that either $j_1 > k$ or $j_2 > k$. Thus, by Corollary 1.3, $(d, \mathcal{D})$ is exceptional. □

Proposition 1.9 *Suppose $d = 3k$ with $k \geq 3$ and $k \equiv 1 \bmod 2$. Then a candidate branching datum of the form*

$$(d, \{[j_1, j_2, \underbrace{2, \dots, 2}_{k}], [\underbrace{3, \dots, 3}_{k}], [\underbrace{3, \dots, 3}_{k}]\})$$

is exceptional if j_1, j_2 have different parity.

Proof Since j_1, j_2 have different parity, there is just one odd in the partition

$$[j_1, j_2, \underbrace{2, \dots, 2}_{k}].$$

Since k is odd, $[k-2, 2, \dots, 2]$ cannot be split into 3 partitions of k. Thus by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. □

Proposition 1.10 *Suppose $d = 3k$ with $k = 2 + 3l$, $l \geq 1$. Then a candidate branching datum of the form*

$$\left(d, \{[\underbrace{3, \dots, 3}_{k-1}, \underbrace{1, 1, 1}_{3}], [\underbrace{3, \dots, 3}_{k}], [\underbrace{3, \dots, 3}_{k}]\}\right)$$

is exceptional.

Proof Since $k = 2 + 3l \geq 5$ and the number of 1's in

$$[\underbrace{3, \dots, 3}_{k-1}, \underbrace{1, 1, 1}_{3}]$$

equals 3,

$$[\underbrace{3, \dots, 3}_{k-1}, \underbrace{1, 1, 1}_{3}]$$

cannot be split into 3 partitions of k. Thus by Theorem 1.2, $(d, \mathcal{D})$ is exceptional. □

From the constructed exceptional datum mentioned above, we obtain an additional result regarding nonprime degrees on the case $M = N = S^2$, which has been proven in [5; 26] using different methods.

Corollary 1.11 [5; 26] *For every nonprime d, there exists at least one candidate branching datum $(d, \mathcal{D})$ that is exceptional.*

2 Proof of Theorem 1.2

The key to proving Theorem 1.2 is to use suitable Möbius transformations to simplify the representation of f. This simplification then facilitates the construction of the mapping g, derived from the transformed expression. The final step involves calculating the branching datum of g. To begin, we establish the following lemma.

Lemma 2.1 *Suppose $g\colon \mathbb{C} \to \mathbb{C}$ is a nonconstant holomorphic function. For a point $z_0 \in \mathbb{C}$, if $g(z_0) \neq 0$, then g and g^r, for every integer r, have the same local degree at z_0.*

Proof Call n the local degree of g at z_0. Then, knowing that $g(z_0) \neq 0$, the order-n Taylor expansion of g at z_0 has the form

$$g(z) = a + b(z - z_0)^n + \mathrm{o}((z - z_0)^n),$$

with $a, b \neq 0$. It follows that

$$g^r(z) = a^r + rab(z - z_0)^n + \mathrm{o}((z - z_0)^n),$$

and $rab \neq 0$, so g^r also has local degree n at z_0. □

Now, the proof of Theorem 1.2 is as follows.

Since Möbius transformations act in a triply transitive fashion on $\overline{\mathbb{C}}$, one can assume that:

(i) 0 in the target $\overline{\mathbb{C}}$ is the branching point corresponding to the partition $[rx_1, \dots, rx_B]$.

(ii) ∞ in the target $\overline{\mathbb{C}}$ is the branching point corresponding to the partition $[ry_1, \dots, ry_C]$.

(iii) 1 in the target $\overline{\mathbb{C}}$ is not the branching point, and $f(\infty) = 1$.

Thus, the expression for f can be written as

$$f(z) = \frac{(z - z_1)^{rx_1}(z - z_2)^{rx_2} \cdots (z - z_B)^{rx_B}}{(z - w_1)^{ry_1}(z - w_2)^{ry_2} \cdots (z - w_C)^{ry_C}},$$

where $z_1, \dots, z_B, w_1, \dots, w_C \in \mathbb{C}$ are distinct complex numbers.

Define

$$g(z) = \frac{(z - z_1)^{x_1}(z - z_2)^{x_2} \cdots (z - z_B)^{x_B}}{(z - w_1)^{y_1}(z - w_2)^{y_2} \cdots (z - w_C)^{y_C}}.$$

Then g is a rational map with degree k, and $f = g^r$ on $\overline{\mathbb{C}}$. It is evident that 0 is a branching point of g if and only if $x_j \geq 2$ for some index j, and ∞ is a branching point of g if and only if $y_l \geq 2$ for some index l.

For any $z_0 \in \mathbb{C} \setminus \{z_1, \dots, z_B, w_1, \dots, w_C\}$, by Lemma 2.1, g and $f = g^r$ have the same local degree at z_0. Consequently, the branching datum of g is

$$\{[\alpha_1^1, \dots, \alpha_1^{l_1}], \dots, [\alpha_s^1, \dots, \alpha_s^{l_s}], [x_1, \dots, x_B], [y_1, \dots, y_C]\},$$

for some $1 \leq s \leq r$, where $\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}$ belong to $\alpha_1, \dots, \alpha_A$ which means that up to a permutation $(\alpha_1^1, \dots, \alpha_1^{l_1}, \dots, \alpha_s^1, \dots, \alpha_s^{l_s}, 1, \dots, 1) = (\alpha_1, \dots, \alpha_A)$.

In summary, we have completed the proof of the Theorem 1.2.

By Lemma 2.1, the proof of the following theorem is easy.

Theorem 2.2 *Suppose $g\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with degree $d \geq 3$ and branching datum*

$$\{[\alpha_1, \dots, \alpha_A], [\beta_1, \dots, \beta_B], [\gamma_1, \dots, \gamma_C]\}.$$

Then, for any integer $k \geq 2$, there is a Möbius transformation φ such that $f = (\varphi \circ g)^k$ is a rational map with degree kd and branching datum

$$\{[\alpha_1, \dots, \alpha_A, \underbrace{1, \dots, 1}_{(k-1)d}], [k\beta_1, \dots, k\beta_B], [k\gamma_1, \dots, k\gamma_C]\}.$$

3 Some existence results

We conclude the paper by presenting some existence results for a certain type of rational maps on $\overline{\mathbb{C}}$ with 3 branching points. First, we give a new proof of the following theorem which was proved in [5] and which is a special case in [4, Theorem 3.5].

Theorem 3.1 [5] *There exists a rational map $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum*

$$\{[\alpha_1, \alpha_2], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\},$$

where $k \geq 2$ and $\alpha_1 + \alpha_2 = 2k$, if and only if $\alpha_1 = \alpha_2 = k$.

Proof Necessity follows from Corollary 1.4. The proof of sufficiency is as follows.

Define $g(z) = (z^k - 1)/(z^k + 1)$. By direct calculation, the branching points of g are -1 and 1. Additionally, $g^{-1}(-1) = \{0\}$, $g^{-1}(1) = \{\infty\}$ and the branching datum of g is $\{[k], [k]\}$. Set $f = g^2$. By direct calculation, we find that the branching points of f are 1, 0, ∞, and the branching datum of f is

$$\{[k, k], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k}]\}. \qquad \square$$

Second, to give more existence results, we recall a result by Boccara [3].

Theorem 3.2 [3] *Let p and q be two positive integers. Consider a candidate branching datum*

$$\mathcal{D} = \{[\alpha_1, \dots, \alpha_p], [\beta_1, \dots, \beta_q], [p+q-1, 1, \dots, 1]\},$$

which contains 3 partitions of a positive integer d. Then there exists a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\mathcal{D}$ being its branching datum if and only if

$$p+q-1 \le \frac{d}{\mathrm{GCD}(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q)}.$$

Theorem 3.3 *Suppose that $k \ge 3$. There exists a degree-$2k$ rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum*

$$\{[\alpha_1, \alpha_2, \alpha_3], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-2}, 4]\}$$

if and only if one of the α_j's is k.

Proof **Sufficiency** Without loss of generality, we can suppose that $\alpha_1 = k$. By Theorem 3.2, there exists a rational map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum

$$\{[k], [\alpha_2, \alpha_3], [\underbrace{1, \dots, 1}_{k-2}, 2]\}.$$

Using a suitable Möbius transformation, we may suppose that:

(i) -1 is the branching point of g corresponding to the partition $[k]$.

(ii) 1 is the branching point of g corresponding to the partition $[\alpha_2, \alpha_3]$.

(iii) 0 is the branching point of g corresponding to the partition $[\underbrace{1, \dots, 1}_{k-2}, 2]$.

Then $f = g^2$ is the desired function.

Necessity Suppose $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with branching datum

$$\{[\alpha_1, \alpha_2, \alpha_3], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-2}, 4]\}.$$

Then, by Theorem 1.2, $[\alpha_1, \alpha_2, \alpha_3]$ can be split into 2 partitions of k. Thus one of the α_j's is k. □

In general, we obtain the following theorem.

Theorem 3.4 *Suppose that $k \ge 3$ and $x \ge 1$. There exists a degree-$2k$ rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum*

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+1}], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-x}, 2x]\}$$

if and only if $[\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$ can be split into two partitions of k and $k/\mathrm{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1}) \ge x$.

Proof **Necessity** Suppose $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map with branching datum

$$\{[\alpha_1, \alpha_2, \dots, \alpha_{x+1}], [\underbrace{2, \dots, 2}_{k}], [\underbrace{2, \dots, 2}_{k-x}, 2x]\}.$$

Then, by Theorem 1.2, there exists a rational map g of degree k with branching datum

$$\{[\alpha_1^1, \dots, \alpha_1^{r_1}], [\alpha_2^1, \dots, \alpha_2^{r_2}], [1, \dots, 1, x]\}$$

such that $f = g^2$, where $[\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2}] = [\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$. By Theorem 3.2,

$$\frac{k}{\mathrm{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1})} \ge x.$$

Sufficiency Suppose $[\alpha_1, \alpha_2, \dots, \alpha_{x+1}]$ can be split into two partitions of k. Denote the two partitions by $[\alpha_1^1, \dots, \alpha_1^{r_1}]$ and $[\alpha_2^1, \dots, \alpha_2^{r_2}]$. Since

$$\mathrm{GCD}(\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2}) = \mathrm{GCD}(\alpha_1, \alpha_2, \dots, \alpha_{x+1}),$$

we have

$$\frac{k}{\mathrm{GCD}(\alpha_1^1, \dots, \alpha_1^{r_1}, \alpha_2^1, \dots, \alpha_2^{r_2})} \ge x.$$

Thus by a result of Boccara [3], there exists a rational map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum

$$\{[\alpha_1^1, \dots, \alpha_1^{r_1}], [\alpha_2^1, \dots, \alpha_2^{r_2}], [1, \dots, 1, x]\}.$$

Using a suitable Möbius transformation, we may suppose that:

(i) -1 is the branching point of g corresponding to the partition $[\alpha_1^1, \dots, \alpha_1^{r_1}]$.

(ii) 1 is the branching point of g corresponding to the partition $[\alpha_2^1, \dots, \alpha_2^{r_2}]$.

(iii) 0 is the branching point of g corresponding to the partition $[1, \dots, 1, x]$.

Then $f = g^2$ is the desired function. □

Finally, we provide more explicit examples to explain our results.

Example 3.5 There exists a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branched data

$$\{[\alpha_1, \alpha_2, \alpha_3, 2, 2], [3, 3, 3], [3, 3, 3]\},$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 5$ and $\alpha_1 \ge \alpha_2 \ge \alpha_3$, if and only if $\alpha_1 = 3$.

Proof **Necessity** Since $\alpha_1 + \alpha_2 + \alpha_3 = 5$ and $\alpha_1 \ge \alpha_2 \ge \alpha_3$, it follows that $\alpha_1 = 2$ or 3. If $\alpha_1 = 2$, then $\alpha_2 = 2$ and $\alpha_3 = 1$. But $[\alpha_1, \alpha_2, \alpha_3, 2, 2] = [2, 2, 1, 2, 2]$ cannot be split into 3 partitions of 3. Thus $\alpha_1 = 3$.

Sufficiency If $\alpha_1 = 3$, then $\alpha_2 = \alpha_3 = 1$. By Theorem 3.2, there exists a rational map g with degree 3 and branching datum

$$\{[3], [2, 1], [2, 1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

(i) 1 is the branching point of g corresponding to the partition [3].

(ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 1].

(iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 1].

Then $f = g^3$ is a rational map with branching datum

$$\{[3,1,1,2,2],[3,3,3],[3,3,3]\}. \qquad \square$$

Example 3.6 There exists a rational map $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with branching datum

$$\{[\alpha_1,\alpha_2,\alpha_3,2,2,2,2],[3,3,3,3,3],[3,3,3,3,3]\},$$

where $\alpha_1+\alpha_2+\alpha_3 = 7$ and $\alpha_1 \ge \alpha_2 \ge \alpha_3$, if and only if $\alpha_1 = 5$ or $\alpha_1 = \alpha_2 = 3$.

Proof **Necessity** Since $\alpha_1+\alpha_2+\alpha_3 = 7$ and $\alpha_1 \ge \alpha_2 \ge \alpha_3$, it follows that $\alpha_1 = 3, 4$ or 5.

When $\alpha_1 = 3$, it follows that $(\alpha_2,\alpha_3) = (2,2)$ or $(3,1)$. When $\alpha_1 = 3, \alpha_2 = 2$ and $\alpha_3 = 2$, since $[3,2,2,2,2,2,2]$ cannot be split into 3 partitions of 5, we exclude this case. When $\alpha_1 = 3, \alpha_2 = 3$ and $\alpha_3 = 1$, it is evident that $[3,3,1,2,2,2,2]$ can be split into 3 partitions of 5.

When $\alpha_1 = 4$, since $\alpha_2+\alpha_3 = 3$, there is just one odd in $[4,\alpha_2,\alpha_3,2,2,2,2]$. Thus $[4,\alpha_2,\alpha_3,2,2,2,2]$ cannot be split into 3 partitions of 5. Thus we exclude this case.

When $\alpha_1 = 5$, it follows that $\alpha_2 = \alpha_3 = 1$. It is evident that $[5,1,1,2,2,2,2]$ can be split into 3 partitions of 5.

In summary, $\alpha_1 = 5$ or $\alpha_1 = \alpha_2 = 3$.

Sufficiency When $\alpha_1 = 5$, it follows that $\alpha_2 = \alpha_3 = 1$. By Theorem 3.2, there exists a rational map g with branching datum

$$\{[5],[2,2,1],[2,2,1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

(i) 1 is the branching point of g corresponding to the partition [5].

(ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 2, 1].

(iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition [2, 2, 1].

Then $f = g^3$ is a rational map with branching datum

$$\{[5,1,1,2,2,2,2],[3,3,3,3,3],[3,3,3,3,3]\}.$$

When $\alpha_1 = \alpha_2 = 3$, it follows that $\alpha_3 = 1$. By a result of Zheng [25] there exists a rational map g with branching datum

$$\{[3,2],[3,2],[2,2,1]\}.$$

Using a suitable Möbius transformation, we may suppose that:

(i) 1 is the branching point of g corresponding to the partition $[3,2]$.

(ii) $e^{\frac{2\pi}{3}i}$ is the branching point of g corresponding to the partition $[3,2]$.

(iii) $e^{\frac{4\pi}{3}i}$ is the branching point of g corresponding to the partition $[2,2,1]$.

Then $f = g^3$ is a rational map with branching datum

$$\{[3,3,1,2,2,2,2],[3,3,3,3,3],[3,3,3,3,3]\}. \qquad \square$$

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